ABSTRACT

Title of Dissertation: POSITIVITY OF DIVISORS, GENERALIZED ABUNDANCE AND CANONICAL BUNDLE FORMULA

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Much of the work in this dissertation is centered around the generalized abundance conjecture of Lazić and Peternell [40]. It is a generalization of the clasical abundance conjecture- arguably one of the most important problems in the whole of algebraic geometry. I now briefly summarize the contents of this thesis.

In the first chapter, we collect some classical facts about linear systems on complex projective varieties. Then we recall some standard facts from birational geometry- in particular singularities of pairs, the cone, contraction and basepoint free theorems for klt pairs and the main conjectures of the minimal model program. We then turn our attention to some of the tools we use: canonical bundle formulas and the Nakayama-Zariski decomposition [46]. In the final section, we give a brief

summary of the results of this thesis.

Chapter 2 is mostly about homogeneous vector bundles on flag varieties. It covers the contents of [14]. I prove a global generation criteria for such bundles, namely that any such bundle which is nef is also globally generated. We then make a few results about the analogous question for bundles on abelian varieties.

Chapter 3 is about conjectures of Serrano and Campana, Chen and Peternell about (almost) strictly nef divisors. We also prove generalized abundance for a large class of uniruled varieties. This essentially covers the contents of [13]. Following methods of [6], in the final section, we show that generalized abundance unconditionally holds for klt pairs with big boundary (see Theorem 3.32). This result is new to the thesis.

Chapter 4 is about an attempt to generalize a classical theorem of Kawamata [33] to the setting of generalized klt pairs. In particular, we observe that Kawamata's theorem can be extended essentially to the setting of generalized abundance. The contents are essentially the same as the authors article [15]. Corollary 4.15 is new here and generalizes a previous result of [15].

In chapter 5, we give an inductive approach to generalized abundance using the nef dimension. This follows [16]. In particular, we observe some new situations in which generalized abundance holds in arbitrary dimension. Theorem 5.13 proves a variation of the original inductive statement of [16] and holds in the exact setting of generalized abundance.

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by

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Contents

Chapter 1

Introduction and background

An algebraic variety is a geometric object defined by vanishing of polynomials. The goal of algebraic geometry is to study the geometry of such objects.

The subject traces its roots to the nineteenth century Italian school consisting of Castelnuovo, Enriques and Severi and the German school consisting of Noether and Riemann. They used explicit geometric constructions to study the geometry of curves and surfaces. However, their techniques though extremely geometric, lacked rigour, resulting in some of their proofs being incorrect. This was fixed by the pioneering work of Oscar Zariski who created a firm algebraic foundation for the subject. Algebraic geometry was then revolutionised by the introduction of sheaf theory and sheaf cohomology by Serre in the 1950's and the language of schemes by Grothendieck in the 1960's. As it stands today, the subject has connections to almost all major areas of Mathematics.

The main problem of algebraic geometry is the following:

Problem: Classify all varieties up to isomorphism.

In practice, this seems like an unreachable goal. Indeed, this problem is very difficult even for curves. So one needs to try to solve a weaker problem. This leads to the subject of birational geometry. Two varieties are called birational (or birationally equivalent) if they are isomorphic "almost everywhere". The following is the main problem in this field:

Problem: Classify varieties upto birational equivalence.

In other words, for each variety, find a distinguished member in its birational equivalence class and study it.

Since any variety is birational to a smooth projective one, we can try to classify smooth projective varieties up to birational equivalence. In dimension one, each birational equivalence class contains only one smooth curve. So the search for a special representative is quickly over and we need to look at moduli spaces of smooth curves.

Birational geometry of surfaces: Let X be a smooth projective surface. If

X contains a (-1)-curve E, then there exists a birational morphism $f : X \to X_1$ such that $f(E) = pt$ and X_1 is also smooth projective. Now we replace X by X_1 and continue the process. After finitely many steps, we get a smooth surface Y which contains no (-1)-curves. In turns out that Y can only be $\mathbb{P}^2$, or a ruled surface over a curve or K_Y is nef. All this was essentially achieved by the Italian school.

Higher dimensions: In the early 1980's, Mori realized that the story for surfaces can be extended to higher dimensions by replacing the use of (-1)-curves by extremal rays and their contractions. Unlike the surface case, this process can lead us outside the realm of smooth varieties. Mori developed a very satisfactory structure theory for smooth threefolds and made major progress towards achieving one in higher dimensions. His results were extended to mildly singular varieties by Kawamata, Kollar, Reid Shokurov and others.

More recently, a huge breakthrough was achieved by Birkar, Cascini, Hacon and McKernan [6] who proved the existence of flips in all dimensions and the existence of minimal models for varieties of general type. They also showed that any uniruled variety is birational to a Mori fiber space (see Section 1.3 for relevant definitions). Since the appearance of [6], there has been an explosion of activity in the field. Some notable contributions include the study of moduli spaces of varieties of general type (due to Hacon, McKernan and Xu) and Fano varities and their K-stability (due to Xu and his school) and boundedness of Fano varieties (due to Birkar). There have been some advances on the more classical side - termination of flips and existence of minimal models. As for the abundance conjecture, to my knowledge, the only advance in the last two decades has been the proof of abundance in numerical dimension 0 by Nakayama. It seems abundance is the deepest problem in the field and would require entirely new techniques.

1.1 Preliminaries on linear systems

Let X be a normal complex projective variety and D a Cartier divisor on X.

Definition 1.1.1. D is called *nef* if $(D \cdot C) \geq 0$ for all curves $C \subset X$. If some multiple of $\mathcal{O}_X(D)$ is globally generated, then D is called *semiample*.

The *Kodaira dimension* (also called *Iitaka dimension*) of D is defined as $\kappa(D) := max \dim_{m \in \mathbb{N}}(\Phi_{|mD|}(X))$ where $\Phi_{|mD|} : X \dashrightarrow \mathbb{P}(H^0(X, mD))$ is the rational map defined by the complete linear system $|nD|$. If $\kappa(D) = \dim X$, then D is called *big*. This is equivalent to $h^0(mD) \sim m^{\dim X}$ for $m \gg 0$ or that there exists an ample divisor A and an effective divisor E such that $mD = A + E$ for some $m \in \mathbb{N}$. Note that if $D_1 \equiv D_2$ and D_1 is big, then so is D_2 ([38][Prop 2.2.22]). If D is big and $E \geq 0$, then $D + \epsilon E$ is big for all $0 < \epsilon \ll 1$.

Let $f : X \to Y$ is a morphism of normal projective varieties. Then $L \in \operatorname{Pic} X$ is called *f-nef* (Resp. *f-ample*) if L is nef (resp. ample) on every fiber of f. If L is big on a general fiber of f, then it is called *f-big*.

An $\mathbb{R}$-(Cartier) divisor is a real linear combination of (Cartier) divisors. An $\mathbb{R}$-divisor D is called *pseudoeffective* if it is a limit of effective $\mathbb{R}$ divisors. In case X is smooth and D is a $\mathbb{Z}$-divisor, this is equivalent to the existence of a singular Hermitian metric on $\mathcal{O}_X(D)$ with weight function ϕ such that the $(1,1)$-current $(i/\pi)\partial\overline{\partial}\phi \geq 0$ (see [18, Prop 4.2(a)]).

Definition 1.1.2. Given a smooth projective variety X and $L \in \operatorname{Pic} X$ such that $H^0(X, L)$ is nonzero, we define the *fixed part* of $|L|$ as

$$|L|_{fix} := max\{E \geq 0 | E \leq D \text{ for all } D \in |L|\}.$$

Let $Mv'(X)$ denote the convex cone in $N^1(X)$generated by the $c_1(L)$ of all $L \in \operatorname{Pic} X$ with $|L|_{fix} = 0$. Its closure, $\overline{Mv}(X)$ is called the *movable cone* and its interior $Mv(X)$ is called the *strictly movable cone*. An $\mathbb{R}$-divisor is called *movable* if $D \in \overline{Mv}(X)$.

Definition 1.1.3. Let X be a smooth projective variety and D a pseudoeffective $\mathbb{R}$-divisor. For any ample divisor A on X, define $\nu(D; A) := max\{k \in \mathbb{Z}_{\geq 0} | limsup_{m\to\infty} h^0(\lfloor mD \rfloor + A)/m^k > 0\}$ and the *numerical dimension* of D is defined as $\nu(D) := max_{A ample} \nu(D; A)$.

Unlike the Kodaira dimension, this is a numerical invariant of D. If D is nef, then it can be shown that $\nu(D) = max\{e | D^e \not\equiv 0\}$ and that $\nu(D) = \dim X$ iff D is big ([46, Chapter 5, Prop 2.7]). If $\kappa(D) = \nu(D)$, then D is called *abundant*.

1.2 Singularities of pairs and minimal model program

Definition 1.2.1 (Singularities of pairs)**.** Let (X, B) be a sub-pair consisting of a normal variety X and a $\mathbb{Q}$-divisor B on X such that $K_X + B$ is $\mathbb{Q}$-Cartier. It is called *sub-klt* if there exists a log resolution $Y \xrightarrow{\mu} X$ of (X, B) such that letting B_Y be defined by $K_Y + B_Y = \mu^*(K_X + B)$, all coefficients of B_Y are < 1 and *sub-lc* if all coefficients of B_Y are ≤ 1. If $B \geq 0$, we drop the prefix sub.

Definition 1.2.2 (Generalized pairs and their singularities(see [7]))**.** A *generalized sub-pair* $(X, B + \mathbf{M})$ consists of a normal projective variety X, a $\mathbb{Q}$-divisor B and a $\mathbb{Q}$-b-divisor $\mathbf{M}$ on X such that:

1. $K_X + B + \mathbf{M}_X$ is $\mathbb{Q}$-Cartier.
2. $\mathbf{M}$ is b-nef (i.e. the birational pushforward of a nef divisor).

When $B \geq 0$, we drop the prefix sub.

Let (X, B) be a generalized sub-pair and $Y \xrightarrow{\mu} X$ a higher birational model of X. Let B_Y be defined by $K_Y + B_Y + \mathbf{M}_Y = \mu^*(K_X + B + \mathbf{M}_X)$. We say that $(X, B + \mathbf{M})$ is *sub-generalized klt* (resp. *sub-generalized lc*) if every coefficient of B_Y is less than 1 (resp. less than or equal to 1). We will call them sub-gklt and sub-glc in short. The *generalized discrepancy b-divisor* $\mathbf{A}(X, B + \mathbf{M})$ is then defined by $\mathbf{A}_Y(X, B + \mathbf{M}) = -B_Y$. Note that $\mathbf{M}_Y$ does not contribute to the singularities of

$(Y, B_Y + \mathbf{M})$ once it descends to a nef divisor on Y, i.e. $\mathbf{A}(Y, B_Y + \mathbf{M}_Y) = \mathbf{A}(Y, B_Y)$ and thus $(Y, B_Y + \mathbf{M}_Y)$ is gklt (resp. glc) iff (Y, B_Y) is klt (resp. lc).

Let (X, B) be a klt pair. If $K_X + B$ is not nef, then there exists a curve class R such that $(K_X + B) \cdot R < 0$. Then there exists a contraction morphism $\phi = cont_R : X \to Y$ such that for a curve C on X, $\phi(C) = pt$ iff $[C] \in R$ (see [36, Theorem 3.7]). There are three possibilities for ϕ:

1. ϕ is birational and the exceptional locus (i.e. the locus where ϕ is not an isomorphism) is a divisor (called a*divisorial contraction*).

2. ϕ is birational and the exceptional locus is of codimension at least 2 (called a *small contraction*)

3. $\dim X > \dim Y$ (called a *Fano contraction* or a *Mori fiber space*. This happens if X is uniruled, for example).

Now we describe the steps of the minimal model program (MMP).

1. Start with a klt pair (X, B). If $K_X + B$ is nef, then stop. Then the *abundance conjecture* predicts that $K_X + B$ is semiample. Otherwise there exists some $\phi = cont_R : X \to Y$.

2. If ϕ is a divisorial contraction, the replace (X, B) by $(Y, \phi_* B)$ and go back to step 1.

3. If ϕ is a small contraction, then we are in trouble because $K_Y + \phi_* B$ needn't be $\mathbb{Q}$-Cartier (Note that this can happen only in dimension atleast 3). In this case,

we can replace (X, B) with a pair (X', B') where X' is birational to X and isomorphic in codimension 1. X and X' are related by a birational map called a *flip*, which exist by the work of Birkar, Cascini, Hacon and McKernan [6]. Replace (X, B) by (X', B') where B' is the proper transform of B and go back to step 1.

4. If $\dim Y < \dim X$, then stop.

It is expected that this process terminates with a pair (X_{min}, B_{min}) such that $K_{X_{min}} + B_{min}$ is nef (and semiample). For this, it is necessary to show that there is no infinite sequence of flips (called *termination of flips* conjecture). This, along with the abundance conjecture would imply that any variety X with K_X pseudoeffective is birational to a variety X_{min} (called the *minimal model* of X) with $K_{X_{min}}$ semimaple. All these are known in the following cases:

1. $\dim X \leq 3$ (classical). Partial results are known for fourfolds.

2. $\nu(K_X + B) = 0$ [46] and others.

3. B is big [6].

If $K_X + B$ is not pseudoeffective then by [6, Corollary 1.3.3], X is connected to a Mori fiber fiber space via an MMP.

Theorem 1.1. *(Kawamata-Shokurov's Basepoint free theorem, see [36]) Let (X, B) be a klt pair and D a nef $\mathbb{Q}$-Cartier divisor on X such that $aD - (K_X + B)$ is nef* *and big for some $a > 0$. Then D is semiample.*

Finally, we recall the classical canonical bundle formula of Kawamata and Ambro, which will be our key technical tool in later chapters.

Theorem 1.2. *(Canonical bundle formula) Let (X, B) be a sub-klt pair, $f : (X, B) \to Y$ a contraction such that $K_X + B \sim_{\mathbb{Q}} f^*(N)$ for some $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor N on Y and rank $f_*\mathcal{O}_X(\lceil -B \rceil) = 1$ (such a fibration is called klt-trivial). Then we can write $N \sim_{\mathbb{Q}} K_Y + B_Y + M_Y$ where $B_Y = \Sigma \delta_l Q_l$, $1 - \delta_l := sup\{t | (X, B + tf^*Q_l)$ is sub-lc over the generic point of $Q_l\}$ where we have*

1. *M_Y is birational pushforward of a nef divisor(b-nef).*

2. *If B restricted to the general fiber of f is effective, then M_Y is a birational pushforward of a nef and abundant divisor.*

There are also other versions of canonical bundle formulas derived from the above version due to Fujino and Mori [26] for parabolic fibrations and for generalized pairs due to Filipazzi [22].

This concludes a brief summary of the minimal model program relevant to this work.

1.3 Nakayama-Zariski decomposition

The main reference for this section is [46]. Let X be a smooth projective variety and let B be a big $\mathbb{R}$-divisor on X (an $\mathbb{R}$-divisor is just a real linear combination of Weil divisors). The linear system $|B|$ is the set of all effective $\mathbb{R}$-divisors linearly equivalent to B. Let $|B|_{\mathbb{Q}}$ (resp. $|B|_{num}$) be the set of effective $\mathbb{R}$-divisors Δ such that $\Delta \sim_{\mathbb{Q}} B$ (resp. $\Delta \equiv B$). Note that by definition, we have $|B| = |\lfloor B \rfloor| + \langle B \rangle$ and $|B|_{\mathbb{Q}} = \cup_{m\in\mathbb{N}}(1/m)|mB|$.

For a prime divisor Γ on X, we define $\sigma_\Gamma(B)_{\mathbb{Q}} := inf\{mult_\Gamma\Delta | \Delta \in |B|_{\mathbb{Q}}\}$ and $\sigma_\Gamma(B) := inf\{mult_\Gamma\Delta|\Delta \in |B|_{num}\}$.

Then it is easy to check that $\sigma_\Gamma(B)_{\mathbb{Q}}$ and $\sigma_\Gamma(B)$ both satisfy the very useful triangle inequality

$$\sigma_\Gamma(B_1 + B_2)_* \leq \sigma_\Gamma(B_1)_* + \sigma_\Gamma(B_2)_*$$

where $* = \mathbb{Q}$ or $\emptyset$. Note that $\sigma_\Gamma(tB) = t\sigma_\Gamma(B)$ for all $t > 0$. We have the following important set of basic properties of σ_Γ (see Lemma 1.4, Chapter 3 of [46]):

Lemma 1.3. *Let B be a big R divisor and Σ a prime divisor.*

1. *$\sigma_\Gamma(A)_* = 0$ for $* = \mathbb{Q}$ or $\emptyset$ and any ample divisor A.*

2. *$lim_{\epsilon\to 0}\sigma_\Gamma(B + \epsilon A) = \sigma_\Gamma(B)$ for any ample divisor A.*

3. *$\sigma_\Gamma(B)_{\mathbb{Q}} = \sigma_\Gamma(B)$. (Thus σ_Γ yields the same result using linear instead of numerical equivalence).*

Definition 1.3.1. For a pseudoeffective $\mathbb{R}$-divisor$D$ on a smooth projective variety X and a prime divisor Γ on X, we define $\sigma_\Gamma(D) := lim_{\epsilon\to 0}\sigma_\Gamma(D+\epsilon A)$, where A is any ample divisor on X. This does not depend on the choice of A. Define

$$N_\sigma(D) := \Sigma_\Gamma \sigma_\Gamma(D)\Gamma \text{ (note that this is a finite sum) and } P_\sigma(D) := D - N_\sigma(D).$$

The decomposition $D = P_\sigma(D) + N_\sigma(D)$ is called the *Nakayama-Zariski decomposition* of D. It generalizes the classical Zariski decomposition for surfaces [53, Theorem 7.7].

We summarize some of its important properties here.

Proposition 1.4. *([46, Chapter 3, Prop 1.14])*

1. *$N_\sigma(D) = 0$ iff D is movable.*

2. *If $D - \Delta$ is movable for some $\Delta \geq 0$, then $\Delta \geq N_\sigma(D)$*

Lemma 1.5.

1. *([46, Chapter 3, Lemma 1.8]) If $0 \leq Q \leq N_\sigma(D)$ then $N_\sigma(D - Q) = N_\sigma(D) - Q$. In particular, $P_\sigma(D)$ is always movable.*

2. *([10, proof of Theorem 5.5]) $\kappa(D) = \kappa(P_\sigma(D))$.*

3. *([46, Chapter 5, Corollary 1.12]) $D \equiv N_\sigma(D)$ iff $\nu(D) = 0$.*

4. *([46, Chapter 5, Prop 2.7]) $\nu(D) = \nu(P_\sigma(D))$*

Definition 1.3.2. Let $f : X \to Y$ be a surjective contraction between normal projective varieties. A divisor D on X is called *f-horizontal* if $f(Supp(D)) = Y$ and *f-vertical* otherwise. An f-vertical divisor D is called *f-exceptional* if $\operatorname{codim} f(Supp(D)) \geq 2$ and more generally, *f-degenerate* if for each prime divisor P on Y, there exists a prime divisor Q on X such that $f(Q) = P$ and $Q \not\subset Supp(D)$.

f-degenerate divisors behave well with respect to P_σ:

Lemma 1.6. *([28, Lemma 2.16], [46]) Let $f : X \to Y$ from a smooth projective variety to a normal projective variety. Let E be an effective f-degenerate divisor on X. The for any pseudoeffective $\mathbb{R}$-Cartier divisor L on Y, we have $P_\sigma(f^*L + E) = P_\sigma(f^*L)$.*

Lemma 1.7. *([46, Chapter 3, Cor 5.17]) Let $f : X \to Y$ be a surjective contraction between smooth projective varieties. If $P_\sigma(D)$ is nef, then $f^*P_\sigma(D) = P_\sigma(f^*(D))$.*

1.4 Discussion of previous work and summary of results

Let G be a complex semisimple Lie group and P a parabolic subgroup. Then the flag variety $X := G/P$ is Fano (i.e $-K_X$ is ample). In chapter 2, we study the positivity of homogeneous vector bundles on such varieties. Such bundles correspond to a representation of P. We prove that

Theorem 1.8. *(= Theorem 2.1) If a homogeneous vector bundle $\mathcal{E} = G \times^P E_0$ on*

G/P is nef, then it is globally generated.

The idea is to look at the splitting of $\mathcal{E}|_{C(\alpha_i)}, i \notin I$, where $C(\alpha_i) \cong \mathbb{P}^1$ corresponds to the simple positive roots $\pm\alpha_i$ and I is the subset of simple positive roots defining the parabolic P and then then apply a global generation criterion due to Dennis Snow. Note that the same result holds and the proof works under the weaker assumption that $\mathcal{E}$ is nef when restricted to the torus invariant curves $C(\alpha_i)$. An immediate corollary is that homogeneous strictly nef vector bundles on complex flag varieties are always ample.

The rest of the thesis is devoted to abundance type problems for certain adjoint divisors. Here I quickly recall the relevant conjectures and their interconnections. To begin, recall the classical abundance conjecture:

Conjecture 1.9. *Let (X, B) be a projective klt pair such that $K_X + B$ is nef. Then $K_X + B$ is semiample.*

In [40], the authors propose the following generalization:

Conjecture 1.10. *(Generalized Abundance) Let (X, B) be a projective klt pair with K_X+B pseudoeffective and L a nef Cartier divisor on X such that K_X+B+L is also nef. Then there exists a semiample $\mathbb{Q}$-divisor M on X such that $K_X + B + L \equiv M$.*

The case $L = 0$ is the abundance conjecture. Indeed, by [12, Corollary 3.5], if $K_X + B$ is numerically equivalent to a semiample divisor, then it is actually semiample. The case $K_X + B \sim 0$ is the semiampleness conjecture for Calabi-Yau varieties (also known as the SYZ conjecture). This is not known even for threefolds. So generalized abundance is much harder than even the abundance conjecture. Generalized abundance is known to hold for surface pairs [40, Corollary C] and for threefold pairs with $\kappa(K_X + B) > 0$ [40, Corollary D]. In the case of nonuniruled varieties, the following conjecture of Serrano [48] is a special case of this conjecture:

Conjecture 1.11. *Let X be a smooth projective variety and $L \in \operatorname{Pic} X$ be strictly nef. Then $K_X + tL$ is ample for all $t > \dim X + 1$.*

This was known to hold for surfaces and many threefolds [48, Theorem 2.3, Theorem 4.4], [11, Theorem 0.4] and for varieties X with $\kappa(X) \geq \dim X - 2$. The case of Calabi-Yau threefolds is very difficult and wide open. Campana, Chen and Peternell [11] propose the following generalization of Serrano's conjecture. Recall that an almost strictly nef divisor (ASN) is a birational pullback of a strictly nef divisor.

Conjecture 1.12. *([11, Conjecture 2.2]) Let X be a smooth projective variety and $L \in \operatorname{Pic} X$ be ASN. Then $K_X + tL$ is big for all $t > \dim X + 1$.*

Now, I summarize the main results of chapter 3. It roughly covers the contents of [13]. I show that

Theorem 1.13. *(=Corollary 3.27 and Corollary 3.29) Conjecture 1.12 holds if X is a Gorenstein surface or if $\kappa(aK_X + bL) \geq \dim X - 2$ for some $a \geq 0$.*

Though generalized abundance may appear like a conjecture for nonuniruled varieties, I show that it still holds for a large class of uniruled varieties:

Theorem 1.14. *(= Theorem 3.16) Let X be a smooth projective variety admitting a contraction morphism $\phi : X \to Y$ to a variety Y of general type such that $-K_X$ is ample on a general fiber of ϕ (for example, a Mori fiber space). If $L \in \operatorname{Pic} X$ is nef such that $K_X + L$ is also nef, then $K_X + L$ is semiample.*

Using results and techniques of [6], I then show that generalized abundance holds for klt pairs with big boundary:

Theorem 1.15. *(= Theorem 3.32) Let (X, B) be a klt pair with B big and $L \in \operatorname{Pic} X$ nef. Then there exists a birational morphism $\mu : \tilde{X} \to X$ from a smooth projective variety such that $P_\sigma(\mu^*(K_X + B + L))$ is semiample.*

This result hasn't appeared elsewhere and is new to the thesis.

Now I summarize the contents of chapter 4. This roughly covers the contents of [15]. In the 1980's, Kawamata [33] proved that for a klt pair (X, B) if $K_X + B$ is nef and abundant, then it is semiample. This is considered an important result from the point of view of the abundance conjecture. Using the canonical bundle formula of Kawamata and Ambro, Fujino [24] proved a generalized basepoint free theorem theorem which implies this result. It is natural to wonder if one can prove such results for generalized pairs. Using techniques similar to Fujino, I show that

Theorem 1.16. *(= Corollary 4.15) Let $(X, B + \mathbf{M})$ be a generalized klt pair, H a* *nef $\mathbb{Q}$-Cartier divisor on X such that $H-(K_X+B+\mathbf{M}_X)$ is nef and abundant. Assume further that $\mathbf{M}$ is b-abundant, $\kappa(aH-(K_X+B+\mathbf{M}_X))\geq 0$ and $\nu(aH-(K_X+B+\mathbf{M}_X))=\nu(H-(K_X+B+\mathbf{M}_X))$ for some $a\in\mathbb{Q}_{>1}$. Then H is semiample.*

This result is new and generalizes [15, Corollary 12]. I then prove an analogue of Kawamata's theorem in the setting of generalized abundance assuming $\kappa(K_X+B)\geq 0$.

Theorem 1.17. *(= Corollary 4.21) Let (X,B) be a klt pair with $\kappa(K_X+B)\geq 0$, H and L nef $\mathbb{Q}$-Cartier divisors on X such that $H-(K_X+B+L)$ is nef and abundant. Assume further that $\kappa(aH-(K_X+B+L))\geq 0$ and $\nu(aH-(K_X+B+L))=\nu(H-(K_X+B+L))$ for some $a\in\mathbb{Q}_{>1}$. Then H is semiample.*

Now I briefly mention the contents of chapter 5. In [3], Ambro gave an "inductive" approach to the abundance conjecture in terms of the nef dimension of K_X+B. It is natural to wonder if one can prove similar results for generalized abundance. I show that this is indeed the case. More precisely

Theorem 1.18. *(= Theorem 5.2) Let (X,B) be an n-dimensional klt pair with $K_X+B\geq 0$ and let $L\in\operatorname{Pic}X$ be nef such that K_X+B+L is nef with nef dimension $n(K_X+B+L)=d$.*
Assume termination of klt flips [35, Conjecture 5-1-13] in dimension d, abundance conjecture [36, Conjecture 3.12] in dimension $\leq d$, semiampleness conjecture [40, page 2] in dimension $\leq d$ and generalized non-vanishing conjecture [40, page 2] in dimension $\leq d-1$.

Then $K_X + B + L \equiv M$ for some semiample divisor M.

This has the following pleasant corollary:

Corollary 1.19. *(= Corollary 5.3) Generalized abundance holds (in any dimension) in the following two cases:*

1. *$n(K_X + B + L) = 2$ and $K_X + B \geq 0$, or*

2. *$n(K_X + B + L) = 3$ and $\kappa(K_X + B) > 0$.*

Note that any divisor of nef dimension 1 is already numerically equivalent to a semiample divisor by [5, 2.4.4]. Thus generalized abundance also holds in this case.

Note that our hypothesis is slightly stronger than Lazic and Peternell's since we assume effectivity of K_X+B instead of pseudoeffectivity (though conjecturally, they are expected to be the same). In the case $K_X + B$ is merely pseudoeffective, we still have the following

Theorem 1.20. *(= Theorem 5.13) Let (X, B) be a klt pair of dimension d with $K_X + B$ pseudoeffective and let $L \in \operatorname{Pic} X$ be nef such that $K_X + B + L$ is also nef with $n(K_X + B + L) < d$.*
Assume termination of klt flips [35, Conjecture 5-1-13] in dimension $< d$, abundance conjecture [36, Conjecture 3.12] in dimension $< d$, semiampleness conjecture [40, page 2] in dimension $< d$ and generalized non-vanishing conjecture [40, page 2] in dimension $< d$.
Then $K_X + B + L \equiv M$ for some semiample divisor M.

Though the above statement is not explicitly stated in [16], the proof uses mostly the same ideas.

Chapter 2

Positivity of vector bundles

The main reference for this section is [14].

A line bundle L on a projective variety X is called *strictly nef* if $\deg(L|_C) > 0$ for all integral curves $C \subset X$. A vector bundle $\mathcal{E}$ on X is called strictly nef if the corresponding hyperplane bundle $\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)$ is strictly nef on $\mathbb{P}\mathcal{E}$, where $\mathbb{P}\mathcal{E}$ is the projective bundle over X associated to $\mathcal{E}$. There are old examples of Mumford and Ramanujam (example 10.6, page 56 and example 10.8, page 57 in [29]) which show that a strictly nef (even effective) line bundle needn't be ample. However, it is known that strictly nef line bundles on homogeneous varieties are ample. For abelian varieties, this is Prop 1.4 in [48], whereas for flag varieties (being Fano), it follows from the basepoint free theorem (Theorem 3.3 in [36]): if L is a strictly nef line bundle on a smooth Fano variety X, then $L - K_X$ is ample, hence L is semiample. Now, it is easy to see that L must be ample: replacing L by a positive tensor power, we may assume that L is globally generated. Let $\phi : X \to Y$ be the morphism given by the linear system $|L|$. Then $L = \phi^*(A)$ for some ample $A \in \operatorname{Pic} Y$. If $\phi(C) = pt$ for some curve $C \in X$, then $L|_C = \phi^*(A)|_C = \mathcal{O}_C$ which can't be since L is strictly nef. Thus ϕ is finite and L, being a finite pullback of an ample divisor is also ample.

Given all this, it is natural to wonder what happens for strictly nef bundles of higher rank on homogeneous varieties. We will see in Section 1 that strictly nef bundles on a flag variety which are homogeneous, are indeed ample. In Section 2, we will consider question 2.5 in [9] which asks whether for vector bundles on an abelian variety, ampleness on curves implies ampleness.

2.1 Homogeneous bundles on flag varieties

Let G be a complex semisimple Lie group, B be a fixed Borel subgroup containing a maximal torus T. Let R be the set of roots of G with respect to T and let R_- denote the set of roots in the Lie algebra of B, the so-called negative roots. Let P be a parabolic subgroup defined by a subset I of the set of simple positive roots $\{\alpha_1, ..., \alpha_l\}$. For any root α of $\mathfrak{g}$ with respect to $\mathfrak{t}$, we can choose a representative X_α of the one-dimensional root space $\mathfrak{g}_\alpha$ such that if $Z_\alpha := [X_\alpha, X_{-\alpha}]$, then $\alpha(Z_\alpha) = 2$. Let $\lambda_1, ..., \lambda_l$ be the *fundamental weights*, i.e., such that $\langle \lambda_i, \alpha_j \rangle = \delta_{ij}$. Here $\langle , \rangle$ denotes the canonical pairing between roots and weights (see page 6 of [50]). Then any arbitrary weight λ can be written as $\lambda = n_1\lambda_1 + ... + n_l\lambda_l$ where $n_i \in \mathbb{Z}\ \forall i$. If

$n_i \geq 0\ \forall i$, then we say λ is *dominant*. Let Λ^+ denote the set of dominant weights. There exists a partial order on the weights: $\mu < \lambda$ if $\lambda - \mu$ is a positive (possibly 0) linear combination of dominant weights. Weights maximal with respect to this partial order will be called *maximal* weights.

Let $\mathcal{E} = G \times^P E_0$ be a homegeneous vector bundle on X=G/P given by a P module E_0. In what follows, $\Lambda(E_0)$ will denote the weights of E_0 and $\Lambda_{\max}(E_0)$ will denote the subset of maximal weights. Here are a few facts we will be using:

(A) Corollary 11.5 on page 47 of [50] says: Let $\mathcal{E} = G \times^P E_0$ be a homogeneous vector bundle on $\mathrm{X} = G/P$ where P is a parabolic subgroup of G. If $\Lambda_{\max}(E_0) \subset \Lambda^+$, then $\mathcal{E}$ is generated by its global sections.

(B) We will also need the following well known fact: If $L(\lambda) \in \operatorname{Pic} G/P$ is the homogeneous line bundle corresponding to the weight λ (see beginning of page 18 of [50] for details on the correspondence) and if $C(\alpha) \cong \mathbb{P}^1$ is the rational curve in G/P corresponding to the roots $\pm\alpha$ (see the proof of Prop 4.4 in page 231 of [32] for details. As a flag variety, $C(\alpha) = S_\alpha/B_\alpha$, where $\mathfrak{s}_\alpha := \mathbb{C}Z_\alpha + \mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}$ and $\mathfrak{b}_\alpha := \mathbb{C}Z_\alpha + \mathfrak{g}_{-\alpha}$.), then $\deg L(\lambda)|_{C(\alpha)} = \langle \lambda, \alpha \rangle$.

Theorem 2.1. *If a homogeneous vector bundle $\mathcal{E}$ on G/P is strictly nef, then it is globally generated and hence ample.*

Proof. Let $\mathcal{E} = G \times^P E_0$ be given by the P module E_0 and P correspond to the index set I. In what follows, we say $j \notin I$ to mean $\alpha_j \notin I$. Let $\Lambda(E_0) = \{\Lambda_1, ..., \Lambda_r\}$ counted with multiplicities, where $\Lambda_k = \sum_{i=1}^{l} n_{i(k)}\lambda_i$. The main idea is to understand the splitting of $\mathcal{E}|_{C(\alpha_j)} \forall j \notin I$. If $\mathfrak{t}$ denotes the Lie algebra of the maximal torus in X and $\mathfrak{t}_{C(\alpha_j)}$ that of the torus in $C_{(\alpha_j)} \forall j \notin I$, then $\mathfrak{t}_{C(\alpha_j)} = \mathbb{C}(Z_{\alpha_j})$. Since $\forall k = 1, ..., l$, the fundamental weights λ_k are linear functionals $\lambda_k : \mathfrak{t} \to \mathbb{C}$, the restriction $\lambda_k|_{\mathfrak{t}_{C(\alpha_j)}}$ is determined by $\lambda_k(Z_{\alpha_j}) = \langle \lambda_k, \alpha_j \rangle = \delta_{kj}$. Thus $\Lambda(E_0|_{C(\alpha_j)}) = \{n_{j(1)}\lambda_j, ..., n_{j(r)}\lambda_j\}$. Suppose $\mathcal{E}|_{C_{(\alpha_j)}}$ gives a representation $\mathbb{C}Z_{\alpha_j} \to M_r(\mathbb{C})$ where r is the rank of $\mathcal{E}$. $\mathbb{C}Z_{\alpha_j}$ being one dimensional, this splits as a direct sum of one dimensional representations given by these weights. Thus $\mathcal{E}|_{C(\alpha_j)} = \mathcal{L}(n_{j(1)}\lambda_j)|_{C(\alpha_j)} \oplus ... \oplus \mathcal{L}(n_{j(r)}\lambda_j)|_{C(\alpha_j)}$, where the direct summands denote the restrictions of the line bundles on X corresponding to the indicated weights. This is $\cong \mathcal{O}(n_{j(1)}) \oplus .... \oplus \mathcal{O}(n_{j(r)})$ $\forall j \notin I$ by (B) above. But since all line bundle quotients of a strictly nef bundle restricted to a curve have positive degree (see Prop 2.1 in [42]), thus $n_{j(k)} > 0$ $\forall j \notin I$, $\forall k = 1, ..., r$. Now consider the projection $G/B \xrightarrow{\pi} G/P$. Since $\pi(C(\alpha_j)) = pt$ $\forall j \in I$, $(\pi^*\mathcal{E})|_{C(\alpha_j)}$ is trivial $\forall j \in I$ which means that $n_{j(k)} = 0$ $\forall j \in I$, $\forall k = 1, ..., r$ arguing as above. Thus $n_{j(k)} \geq 0$ $\forall j$ and for all weights Λ_k, $k = 1, ..., r$ and thus $\mathcal{E}$ is spanned by (A) above. Now by the canonical surjection $\pi^*\mathcal{E} \to \mathcal{O}_{\mathbb{P}\mathcal{E}}(1)$, $\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)$ is generated by its global sections and strictly nef, hence ample.

□

Remark 1: In view of Proposition 6.1.8 (ii) of [39], the arguments of the above proof show the more general fact that homogeneous bundles on G/P which are nef on the finite set of curves $C(\alpha_j)$ for $j \notin I$ are globally generated.

Remark 2: The above theorem has also been proved independently around the same time by [9]. (See Theorem 3.1 in [9].)

Remark 3: It is worth noting that the above theorem is actually false without the homogeneity assumption: there do exist nef vector bundles on $\mathbb{P}^2$ which are not globally generated (see [47, Proposition 2]).

2.2 Ampleness for bundles on curves

In this section, we consider question 2.5 of [9] which asks (in analogy with the line bundle case): Is a vector bundle on an abelian variety ample if its restriction to every curve is ample? We have the following lemma whose proof is an easy application of Hartshorne's ampleness criterion for vector bundles on a smooth projective curve (Theorem 2.4 in [30])

Lemma 1. Let $\mathcal{E}$ be a vector bundle on a projective variety X. Then $\mathcal{E}$ is ample when restricted to curves iff $\forall$ finite morphisms $f : C \to X$ where C is a smooth projective curve, all vector bundle quotients of $f^*(\mathcal{E})$ are of positive degree.

Proof. $\Longleftarrow$: If $C \subset X$ is a curve, consider its normalization $\tilde{C} \xrightarrow{f} C \subset X$. Then

$f^*(\mathcal{E})$ is ample on $\tilde{C}$ by Hartshorne's criterion. Hence $\mathcal{E}|_C$ is also ample by Prop 6.1.8 (iii) in [39].

$\Longrightarrow$: Let $f : C \to X$ be a finite morphism from a smooth projective curve and let $C' = f(C)$. Then $\mathcal{E}|_{C'}$ is ample, thus $f^*(\mathcal{E}|_{C'})$ is ample on C. By Hartshorne's ampleness criterion on C, we are done. □

Corollary 1. Let $X \xrightarrow{\pi} Y$ be a finite morphism of projective varieties, $\mathcal{E}$ be a bundle on Y that is ample when restricted to curves. Then $\pi^*(\mathcal{E})$ is also ample when restricted to curves.

Proof. Let $C \xrightarrow{f} X$ be a finite morphism, where C is a smooth projective curve. Let $f^*\pi^*(\mathcal{E}) \to Q \to 0$ be a quotient bundle. Now $\pi \circ f : C \to Y$ is finite and $\deg(Q) > 0$ by ampleness of $f^* \circ \pi^*(\mathcal{E})$. Thus $\pi^*(\mathcal{E})$ is ample by above lemma. □

Remark 3: Let X be an n-dimensional projective variety. Then X admits a finite surjective morphism, say $f : X \to \mathbb{P}^n$. If E is a vector bundle on $\mathbb{P}^n$ which is ample on curves, so is $f^*(E)$ by the above corollary. Moreover, by Prop 6.1.8 (iii) in [39], E is ample iff $f^*(E)$ is. Thus, if there exists a vector bundle on $\mathbb{P}^n$ for some n, which is ample on curves, but not ample, that would negatively answer the question of [9].

Chapter 3

Strictly nef divisors and generalized abundance

The reference for the contents of this section is [13]. The main result of the last section is new and hasn't appeared elsewhere.

A Cartier divisor D on a projective variety X is called *strictly nef* if $D \cdot C > 0$ for all effective curves $C \subset X$. Early on, there were only two known examples of such divisors which are not ample: one due to Mumford of a divisor on a ruled surface which has Iitaka dimenion $-\infty$ and one due to Ramanujam of a divisor on a threefold which is nef and big. Afterwards, Mumford's example was generalized to higher dimensions by Subramanian ([51]) to construct such divisors on projective bundles of arbitrary rank over curves. Later on, Mehta and Subramanian ([45]) constructed such examples in finite characteristic.

If X is a projective variety of dimension d and D is any strictly nef divisor on X, then the Cone Theorem shows that the adjoint divisor $K_X + tD$ is again strictly nef for all $t > d+1$ if X is smooth and for all $t > 2d$ if X has log terminal singularities (see Lemma 6). It is natural to wonder about the ampleness of these adjoint divisors. In [48], Serrano conjectures that in fact:

Conjecture 3.1. *If X is a smooth projective d-dimensional variety and $D \in \operatorname{Pic} X$ is strictly nef, then $K_X + tD$ is ample for all $t > d+1$.*

Again the Cone Theorem shows that Serrano's Conjecture is equivalent to the assertion that $K_X^{\perp} \cap D^{\perp} = 0$ in $\overline{NE}(X)$.

Note that if a strictly nef divisor L is semiample, then it is in fact ample. Indeed, replacing L by a positive tensor power, we may assume that L is globally generated. Let $\phi : X \to Y$ be the morphism given by the linear system $|L|$. Then $L = \phi^*(A)$ for some ample $A \in \operatorname{Pic} Y$. If $\phi(C) = pt$ for some curve $C \subset X$, then $L|_C = \phi^*(A)|_C = \mathcal{O}_C$ which can not be since L is strictly nef. Thus ϕ is finite and L, being a finite pullback of an ample divisor is also ample.

Serrano proves his conjecture for Gorenstein surfaces and for most smooth threefolds. His ideas use specific results about classification of surfaces and threefold extremal contractions. This is followed by Campana, Chen and Peternell [11] who prove the conjecture for varieties X with Kodaira dimension atleast $d-2$. The general case seems to be hard and related to the main conjectures of the Minimal Model Program. In fact, for non uniruled varieties, it is a special case of the Generalized Abundance Conjecture (see [40, page 2]).

Conjecture 3.2. *(Generalized Abundance) Let (X, Δ) be a klt pair with $K_X + \Delta$ pseudoeffective and L a nef Cartier divisor on X such that $K_X + \Delta + L$ is also nef. Then there exists a semiample $\mathbb{Q}$-divisor M on X such that $K_X + \Delta + L \equiv M$.*

While this conjecture is expected to fail for uniruled varieties (see Remark 3.17), the following result produces a large class of uniruled varieties on which generalized abundance holds:

Theorem 3.3. *Let X be a smooth projective variety admitting a surjective morphism $X \xrightarrow{\phi} Y$ with connected fibers such that $-K_X$ is ample on a general fiber of ϕ and Y is of general type. Suppose that $L, K_X + L \in \operatorname{Pic} X$ are both nef. Then $K_X + L$ is semiample.*

In the situation of the above theorem, it is interesting to ask how far one can relax the restriction on ω_Y being big. For example, can we say anything interesting about $K_X + L$ if $\kappa(Y) \geq 1$?

One interesting question about moving strictly nef divisors themselves is: how far are they from being big? While we do not know the answer in general, we show that if $\kappa(L) \geq \dim(X) - 2$, then L is big (see Proposition 3.21). In the fourth section, I prove the surface case of a conjecture of Campana, Chen and Peternell ([11, Conjecture 2.2]), which is a generalization of Serrano's conjecture. Here we briefly recall the result. A line bundle $L \in \operatorname{Pic} X$ is called *almost strictly nef* if there is a birational morphism $\pi : X \to Y$ and $M \in \operatorname{Pic} Y$ strictly nef such that $L = \pi^*(M)$.

Theorem 3.4. *Let S be a Gorenstein surface and $L \in \operatorname{Pic} S$ almost strictly nef. Then $K_S + tL$ is big for all $t > 3$.*

As a corollary, we show that this conjecture holds for varieties X with $\kappa(X) \geq \dim(X) - 2$ (see Corollary 3.29). In the fourth section, using a conjecture of Segre, Harbourne, Gimigliano and Hirschowitz (called the SHGH conjecture in short; see [21, Conjecture 3.1]) on the expected dimension of certain linear systems on blow-ups of $\mathbb{P}^2$, we produce a series of examples of strictly nef non-ample divisors on surfaces of arbitrary Kodaira dimension. More precisely, we show:

Theorem 3.5. *Suppose SHGH conjecture holds. Then for any projective surface S, there exists a surface S' birational to S admitting a strictly nef non ample divisor.*

The following is a supplementary result on generalized abundance which appears in the final section of this chapter. It is independent of the other results presented here.

Theorem 3.6. *Let (X, B) be a klt pair with B big and $L \in \operatorname{Pic} X$ nef. Then there* *exists a birational morphism $\mu : \tilde{X} \to X$ from a smooth projective variety such that $P_\sigma(\mu^*(K_X + B + L))$ is semiample.*

3.1 The results we will be using

The following lemmas will be used throughout this chapter. The first one is due to Serrano and the second is a modification of one of his results.

Lemma 3.7. *([48, Lemma 1.1]) Let X be a projective variety of dimension d and $L \in \operatorname{Pic} X$ strictly nef . Then $K_X + tL$ is strictly nef for all $t > d+1$ if X is smooth and for all $t > 2d$ if X has klt singularities.*

Proof. For the smooth case, see [48, Lemma 1.1]. If X has klt singularities, it follows from the Cone Theorem that $K_X + 2dL$ is nef. Thus $K_X + tL$ is strictly nef for all $t > 2d$. □

The following result gives a numerical criterion for generalized abundance to fail. In view of the above lemma, it generalizes [48, Lemma 1.3].

Lemma 3.8. *Let X be a projective variety with at worst klt singularities. Suppose that $L, K_X + L \in \operatorname{Pic} X$ are both nef. If $K_X + L$ is not semiample, then $K_X^d = K_X^{d-1} \cdot L = \cdots = L^d = 0$ where $d = \dim X$.*

Proof. If $(K_X + 2L)^d > 0$, then $K_X + 2L = 2(K_X + L) - K_X$ is nef and big and thus $K_X + L$ is semiample by the basepoint free theorem (see [36, Theorem 3.3]) which contradicts our assumption. Thus $(K_X + 2L)^d = 0$. Since $K_X + L, L$ are both nef, this implies $(K_X+L)^i \cdot L^{d-i} = 0$ for all i and from this, we deduce that $K_X^i \cdot L^{d-i} = 0$ for all i . □

3.2 Strictly nef divisors and generalized abundance

Lemma 3.9. *Let X be a smooth projective variety of dimension n, and let $L \in \operatorname{Pic} X$ be strictly nef. Assume that $|aK_X+bL|$ contains a smooth divisor D for some $a, b \in \mathbb{Z}$ and that Serrano's conjecture holds for D. Then $K_X + tL$ is ample for all $t > n+1$.*

Proof. Let $D \in |aK_X + bL|$ be smooth. Then $K_D + tL|_D$ is ample for all $t > n$ by assumption. This implies that

$$L|_D \cdot (K_D + tL|_D)^{n-2} = L \cdot D \cdot (K_X + D + tL)^{n-2} > 0$$

and hence

$$L \cdot (aK_X + bL) \cdot (K_X + aK_X + (b+t)L)^{n-2} > 0$$

for all $t > n$. Now we can appeal to [48, Lemma 1.3]. □

Proposition 3.10. *Let X be a smooth projective variety with $n = \dim X \geq 4$ and let $L \in \operatorname{Pic} X$ be strictly nef. Assume that the linear system $|aK_X + bL|$ is basepoint free with $\kappa(|aK_X + bL|) \geq n-3$ for some $a > 0$. Then $K_X + tL$ is ample for all $t > n+1$.*

Proof. For the sake of illustration, let us consider a few lower dimensional cases. Suppose $n = 4$, $\kappa(|aK_X + bL|) \geq 1$. Let $D \in |aK_X + bL|$ be smooth. Then

$$aK_D + bL|_D = a(K_X + D)|_D + bL|_D$$

$$= (aK_X + bL)|_D + a(aK_X + bL)|_D$$

is globally generated. This $\cong \mathcal{O}_D$ if $\kappa(|aK_X + bL|) = 1$. Thus $K_D \equiv_{\mathbb{Q}} \pm L|_D$. Thus $\pm K_D$ is strictly nef, hence ample (by abundance for 3-folds and [48, Theorem 3.9] respectively) and Serrano's Conjecture holds for D. If $\kappa(|aK_X + bL|) > 1$, then $\kappa(|aK_D + bL|_D|) \geq 1$ and Serrano's Conjecture holds on D by [48, Proposition 3.1]. Now we are done for the case $n = 4$ by the above Lemma.

Now suppose $n = 5$, $\kappa(|aK_X + bL|) \geq 2$, $D \in |aK_X + bL|)$ smooth. Then $|aK_D + bL|_D|$ is basepoint free and

$$|aK_D + bL|_D| \supset |aK_X + bL||_D + |a(aK_X + bL)||_D \tag{3.11}$$

and both the pieces have Iitaka dimension ≥ 1. Thus $\kappa(|aK_D + bL|_D|) \geq 1$ and we are reduced to the above case.

The general case proceeds by induction as in the above cases once we note that $\kappa(|aK_D + bL|_D|) \geq \kappa(|aK_X + bL|) - 1$ for all $D \in |aK_X + bL|$ smooth by (4.9).

□

Proposition 3.12. *Serrano's conjecture for Calabi-Yau varieties implies Serrano's conjecture for varieties X with $\pm K_X$ semiample.*

Proof. Suppose $L \in \operatorname{Pic} X$ is strictly nef. Let

$$\phi = \phi_{|\pm mK_X|} : X \to Y$$

be the Kodaira fibration, so $\pm mK_X = \phi^*(A)$ for some $A \in \operatorname{Pic} Y$ very ample. Let $F \subset X$ be a general fiber of ϕ. Then

$$\pm mK_F = \pm mK_X|_F = (\phi^* A)|_F = \mathcal{O}_F.$$

Let $f : \hat{F} \to F$ be an associated unramified cyclic cover (see [36, Definition 2.49]). Then we have

$$f^*(K_F) \cong \mathcal{O}_{\hat{F}} \cong K_{\hat{F}}.$$

If $\hat{L} := f^*(L|_F)$, then $\hat{L} = f^*(K_F + L|_F)$ is ample by assumption. Thus $K_F + L|_F$ is also ample, so $K_X + L$ is ϕ-big. Thus for $t \gg 0$, $K_X + tL + \phi^* A$ is nef and big by [11, Lemma 2.5] and we are done by [48, Lemma 1.3].

$\square$

The following simple remark will be used several times in the sequel:

Remark 3.13. If $L \in \operatorname{Pic} X$ is nef such that $K_X + L$ is also nef (this happens for example if L is a large multiple of a strictly nef divisor on a smooth projective variety) and $\phi : X \to Y$ an extremal contraction with some fiber F, then $L|_F$ is ample: We know that $-K_X|_F$ is ample and $(K_X + L)|_F$ is nef, thus $L|_F$ is ample.

Proposition 3.14. *Let $X = Bl_p(X') \xrightarrow{\pi} X'$ be the blow-up of a smooth projective variety X' at a point p. Assume that $K_{X'}$ is pseudoeffective and suppose that $L, K_X +$*

$L \in \operatorname{Pic} X$ are both nef. Then $K_X + L$ is semiample.

Proof. Let $n = \dim X$. Then $K_X \cdot L^{n-1} = (\pi^*(K_{X'}) + E) \cdot L^{n-1} > 0$ by Remark 3.13 and assumption. Thus we are done by Lemma 3.8.

$\square$

Remark 3.15. Since for all $n \geq 1, H^0(nK_X) = H^0(nK_{X'})$ (see [38, Example 2.1.16]), the above corollary proves Generalized Abundance for all smooth projective varieties which are obtained by blowing up points on smooth projective varieties X with K_X $\mathbb{Q}$-effective.

Though Generalized Abundance does not always hold for uniruled varieties (see Remark 3.17), in the following situation, it does:

Theorem 3.16. *Suppose that X is a smooth projective variety admitting a surjective morphism $X \xrightarrow{\phi} Y$, with connected fibers such that $-K_X$ is ample on a general fiber of ϕ (for example, ϕ could be a Fano contraction) and Y is of general type. Suppose that L and $K_X + L$ are both nef divisors on X. Then $K_X + tL$ is big if $t \gg 0$. In particular, $K_X + L$ is semiample.*

Proof. There exists a birational morphism $\tau : Y^{'} \to Y$, where $Y^{'}$ is smooth projective such that letting $X^{'}$ denote the desingularization of the main component of $X \times_Y Y^{'}$ and $\phi^{'} : X^{'} \to Y^{'}$, $\tau^{'} : X^{'} \to X$ the induced morphisms, any $\phi^{'}$-exceptional divisor is also τ'-exceptional. Indeed, one can choose τ to be a birational flattening of $\phi : X \to Y$. Let $L^{'} := \tau^{'*}(L)$. Then $L^{'}$ is also ample on a general fiber of $\phi^{'}$. Indeed, letting $U \subset Y$ be a non empty open set contained in the smooth locus of ϕ

over which $-K_X$ is ample and letting $U^{'} := \tau^{-1}(U)$, $L^{'}$ is ample over $U^{'}$.

Then by [48, Lemma 4.2], $[\phi^{'}_*(\omega_{X^{'}/Y^{'}} \otimes rL^{'}))^{\otimes s}]^{**} \otimes \omega_{Y^{'}} \otimes G^{\otimes(m+1)}$ is generically spanned for all $r, s > 0$, $G \in \operatorname{Pic} Y^{'}$ very ample where $m = \dim Y$. Now since $L^{'}$ is ample on a general fiber of $\phi^{'}$, the natural map

$$(\phi^{'}_*(\omega_{X^{'}/Y^{'}} \otimes rL^{'}))^{\otimes s} \to \phi^{'}_*(s(\omega_{X^{'}/Y^{'}} \otimes rL^{'}))$$

is generically surjective if $r \gg 0$. Thus we can conclude that

$$[\phi^{'}_*(s(\omega_{X^{'}/Y^{'}} \otimes rL^{'})]^{**} \otimes \omega_{Y^{'}} \otimes G^{\otimes(m+1)}$$

is also generically spanned for all $r, s > 0$. By a theorem of Nakayama (see [24, Theorem 1.2], [46, Lemma 5.10, page 107]), this equals

$$\phi^{'}_*(s(\omega_{X^{'}/Y^{'}} \otimes rL^{'} \otimes E)) \otimes \omega_{Y^{'}} \otimes G^{\otimes(m+1)}$$

for some effective $\phi^{'}$-exceptional (hence also $\tau^{'}$-exceptional) divisor E. This equals

$$\phi^{'}_*(s(\omega_{X^{'}} \otimes rL^{'} \otimes E)) \otimes (1-s)\omega_{Y^{'}} \otimes G^{\otimes(m+1)}.$$

Now fix t_0 sufficiently large such that there exists $D \geq 0$ such that $D \in |t_0 K_{Y^{'}} - G|$, so that $(m+1)(D+G) \in |t_0(m+1)K_{Y^{'}}|$. Then

$$\phi^{'}_*(s(\omega_{X^{'}} \otimes rL^{'} \otimes E)) \otimes (1-s)\omega_{Y^{'}} \otimes \mathcal{O}_{Y^{'}}((m+1)(D+G))$$

$$\cong \phi^{'}_*(s(\omega_{X^{'}} \otimes rL^{'} \otimes E)) \otimes (t_0(m+1)+1-s)\omega_{Y^{'}} =: \mathcal{F}$$

is also generically spannned. Thus there exists $0 \neq \sigma \in H^0(\mathcal{F})$ and

$$0 \neq \tau \in H^0((s-1-t_0(m+1))K_{Y^{'}} \otimes \mathcal{O}_{Y^{'}}(-G)) \text{ for } s \gg 0$$

since $K_{Y'}$ is big. Then

$$0 \neq \sigma \otimes \tau \in H^0(\phi'_*(s(\omega_{X'} \otimes rL' \otimes E)) \otimes \mathcal{O}_{Y'}(-G))$$

and thus $\kappa(s(\omega_{X'} \otimes rL' \otimes E) \otimes \phi'^*(-G)) \geq 0$.

Now the Easy addition theorem (see [46, Theorem 3.13, page 51]) implies that

$$\omega_{X'} \otimes rL' \otimes E = \tau'^*(\omega_X \otimes rL) \otimes \mathcal{O}_{X'}(E + E')$$

is big for $r \gg 0$ (where $E, E' \geq 0$ are τ'-exceptional divisors). Then

$$\tau'_*(\tau'^*(\omega_X \otimes rL) \otimes \mathcal{O}_{X'}(E + E')) = \omega_X \otimes rL$$

is big for $r \gg 0$. Thus, $(K_X + rL)^n > 0$ and by Lemma 3.8, $K_X + L$ is semiample. $\square$

Remark 3.17. Generalized Abundance can fail without any assumption on ω_Y as the following example (see [49, example 1.1]) shows :

Let C be a smooth elliptic curve and let $\mathcal{E}$ be a rank 2 bundle on C given by a non-trivial extension

$$0 \to \mathcal{O}_C \xrightarrow{\sigma} \mathcal{E} \to \mathcal{O}_C \to 0$$

corresponding to a nonzero element $\zeta \in h^1(\mathcal{O}_C)$. Let $X = \mathbb{P}\mathcal{E} \xrightarrow{\pi} C$ be the associated ruled surface, $L = \mathcal{O}_{\mathbb{P}\mathcal{E}}(3)$ and $C_0 \subset X$ a section of π with $\mathcal{O}_X(C_0) = \mathcal{O}_{\mathbb{P}\mathcal{E}}(1)$. Then, since $K_X = \mathcal{O}_{\mathbb{P}\mathcal{E}}(-2)$, both $L, K_X + L = \mathcal{O}_{\mathbb{P}\mathcal{E}}(1)$ are nef. We will show that $\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)$ is not semiample and that $\kappa(\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)) = 0$.

Proof. We first show that $\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)$ can not be semiample. Suppose it is. Then we can choose $n \gg 0$ such that:

$$h^0(\mathcal{O}_{\mathbb{P}\mathcal{E}}(n)) > 2 \tag{3.18}$$

and there exists

$$nC_0 \neq Y \in |\mathcal{O}_{\mathbb{P}\mathcal{E}}(n)|$$

which is a smooth elliptic curve. Consider the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}\mathcal{E}}(n-Y) = \mathcal{O}_{\mathbb{P}\mathcal{E}} \to \mathcal{O}_{\mathbb{P}\mathcal{E}}(n) \to \mathcal{O}_Y(n) \to 0 \tag{3.19}$$

on X. Now $C_0^2 = 0$ implies that $\deg(\mathcal{O}_Y(n)) = 0$ and hence $h^0(\mathcal{O}_Y(n)) \leq 1$ which gives $h^0(\mathcal{O}_{\mathbb{P}\mathcal{E}}(n)) \leq 2$ by (3.19) thus contradicting (3.18). This proves that $\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)$ is not semiample .

Now we show that $\kappa(\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)) = 0$. First note that $h^0(\mathcal{E}) = 1$: this follows from the cohomology exact sequence

$$0 \to H^0(\mathcal{O}_C) \to H^0(\mathcal{E}) \to H^0(\mathcal{O}_C) \xrightarrow{\phi} H^1(\mathcal{O}_C)$$

where the extension class $\phi(1) = \zeta \in H^1(\mathcal{O}_C)$ giving $\mathcal{E}$ being nonzero means that ϕ is an isomorphism and thus $h^0(\mathcal{E}) = 1$.

If $\kappa(\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)) > 0$, choose n to be the smallest integer such that $h^0(\mathcal{O}_{\mathbb{P}\mathcal{E}}(n)) \geq 2$. Thus we have two distinct effective divisors $nC_0, D \in |\mathcal{O}_{\mathbb{P}\mathcal{E}}(n)|$ which can not have any C_0 component in common by minimality of n. Now $(nC_0 \cdot D) = 0$ implies that $|\mathcal{O}_{\mathbb{P}\mathcal{E}}(n)|$ is basepoint free, which can not be as we saw above. □

Corollary 3.20. *Let $X = Bl_Z(Y) \xrightarrow{\pi} Y$ be the blow-up of a smooth projective variety Y with $\omega_Y \cong \mathcal{O}_Y$ along a smooth subvariety $Z \subset Y$ of general type such that the conormal bundle $F := \mathcal{N}^*_{Z/Y}$ is nef(i.e. $\mathcal{O}_{\mathbb{P}F}(1)$ is nef). If $L, K_X + L \in \operatorname{Pic} X$ are both nef, then $K_X + L$ is semiample.*

Proof. Let E denote the exceptional divisor of π and let $L \in \operatorname{Pic} X$ be as above. Recall that $\pi|_E : E \to Z$ can be identified with the natural projection $\mathbb{P}F \to Z$ under which $E|_E$ identifies with $\mathcal{O}_{\mathbb{P}F}(-1) =: \mathcal{O}_E(-1)$. Let c be the codimension of Z in Y. Then

$$K_E + tL|_E = (K_X + E)|_E + tL|_E$$

$$= tL|_E + \mathcal{O}_E(-c)$$

is nef and big for all $t \gg 0$ by Theorem 3.16. Now $\mathcal{O}_E(1)$ being nef forces $L|_E$ to be nef and big. Then if $n = \dim X$,

$$E \cdot L^{n-1} = K_X \cdot L^{n-1} > 0$$

and we are done by Lemma 3.8. □

3.3 Almost strictly nef divisors

$L \in \operatorname{Pic} X$ is called *almost strictly nef* (ASN) if there is a birational morphism $\pi : X \to Y$ to some projective variety Y and $M \in \operatorname{Pic} Y$ strictly nef such that

$\pi^*M = L$. We have the following interesting property of moving ASN divisors:

Proposition 3.21. *Let X be a projective variety of dimension n and let $L \in \operatorname{Pic} X$ be almost strictly nef (ASN) with $\kappa(L) \geq n-2$. Then L is big.*

Proof. Let the following be a resolution of indeterminacy of the Iitaka fibration $\phi = \phi_{|mL|}$ of L:

$$\begin{tikzcd} X \arrow[r, dotted, "\phi"] & Y \\ \hat{X} \arrow[u, "\pi"] \arrow[ur, "\hat{\phi}"] & \end{tikzcd}$$

So $\pi^*(mL) = \hat{\phi}^*(A) + E'$ for some $A \in \operatorname{Pic} Y$ ample and E' effective. Let $\hat{L} = \pi^*L$. Let $F \subset \hat{X}$ be a general fiber of $\hat{\phi}$. Then $\dim F \leq 2$. An effective strictly nef divisor on a surface is ample, so we may assume $n \geq 3$. Then $\hat{L}|_F$ is ASN. Consider the exact sequence

$$0 \to m\hat{L} \otimes I_F \to m\hat{L} \to m\hat{L}|_F \to 0.$$

If $H^0(m\hat{L}|_F) = 0$, then $H^0(m\hat{L} \otimes I_F) = H^0(m\hat{L})$ which can only happen if $F \subset Bs|m\hat{L}| = E'$. This is impossible since F is a general fiber. Thus $m\hat{L}|_F$ is ASN and effective. Since $\dim F \leq 2$, $\hat{L}|_F$ is big. Now $\hat{L}$ being $\hat{\phi}$-big,

$$\hat{L} + \hat{\phi}^*(A) = (m+1)\hat{L} - E'$$

is big by [11, lemma 2.5]. Thus $\hat{L}$ is also big.

$\square$

Remark 3.22. Thus Serrano's conjecture holds for all strictly nef line bundles $L \in \operatorname{Pic} X$ with $\kappa(L) \geq \dim X - 2$ by [48, Lemma 1.3].

Remark 3.23. I do not know if it is possible for an ASN $L \in \operatorname{Pic} X$ to have $0 \leq \kappa(L) < \dim(X) - 2$.

Campana, Chen and Peternell conjecture that if X is a smooth projective variety and $L \in \operatorname{Pic} X$ is almost strictly nef, then $K_X + tL$ is big for all $t > \dim(X) + 1$ ([11, Conjecture 2.2]). We prove this conjecture when X is a surface.

Theorem 3.24. *Let S be a smooth projective surface, $L \in \operatorname{Pic} S$ almost strictly nef. Then $K_S + tL$ is big for all $t > 3$.*

Proof. If $\kappa(S) = 0$, this is [11, Proposition 2.3]. We will treat the other cases:

Case 1 : $\kappa(S) > 0$: Let $S \xrightarrow{\pi} S'$ be the minimal model for S. Note that if $\pi^*(K_{S'}) + tL$ is big , then so is $K_S + tL$ and we are done. Otherwise, if $\pi^*(K_{S'}) + tL$ is not big, then $(\pi^* K_{S'} + tL)^2 = 0$, thus

$$\pi^* K_{S'}^2 = \pi^*(K_{S'}) \cdot L = L^2 = 0,$$

which by Hodge Index Theorem, implies that $\pi^* K_{S'} = cL$ for some $c > 0$. Since $\pi^* K_{S'}$ is semiample, hence L and $\pi^* K_{S'}$ are both big. Thus $K_S = \pi^* K_{S'} + E$ (where E is some effective divisor) is big. Hence $K_S + tL$ is also big.

Case 2 : $\kappa(S) = -\infty$: Suppose we have $\pi : S \to S_0$ birational, S_0 normal projective such that $L = \pi^* M$, where $M \in \operatorname{Pic}(S_0)$ is strictly nef. Note: we may assume that S_0 is singular, because if S_0 is smooth, then $K_{S_0} + tM$ is ample for all $t > 3$ by [48, Proposition 2.1] which would imply that $K_S + tL$ is big and we are done. Then π can be factorized as $\pi : S \xrightarrow{p} S' \xrightarrow{f} S_0$ such that S' is smooth and f does not contract any (-1)- curves. $S^{'}$ can be constructed as follows: let S_1 be a smooth surface obtained by contracting some (-1) curve (if any) in $Ex(\pi)$. Repeat this process to the induced morphism $\pi_1 : S_1 \to S_0$. Since the Picard rank drops at each step, the process eventually ends with $f : S^{'} \to S_0$ as above. Now

$$K_S + tL = (p)^*(K_{S'} + tL^{'}) + \text{ some effective divisor},$$

where $L^{'} = f^*(M)$. Hence it is enough to show that $K_{S'} + tL^{'}$ is big. Thus replacing S with S' and L with $f^*(M)$, we may assume that π does not contract any (-1) curves. Moreover, since $\kappa(S) = -\infty$, S is obtained from a projective bundle over a smooth curve C by a sequence of blow-ups, giving $\phi : S \to C$ or $S = \mathbb{P}^2$. The Theorem is clear in the latter case.

With this in mind, we claim that $K_S + tL$ is nef for all $t \geq 3$. Indeed, if $D \subset S$ is a curve, then

$$D \equiv \Sigma_{i=1}^m a_i C_i + N, a_i \geq 0$$

for all $i, N \in \overline{NE}(S)$ with $(N \cdot K_S) \geq 0$ and $C_i \subset S$ extremal rational curves with $0 > (K_S \cdot C_i) \geq -3$ for all i. Then by adjunction, $C_i^2 = -1, 0$ or 1 for all i (This follows from classification of extremal contractions on surfaces, see [44, Theorem 1-4-8]). Since $\pi : S \to S_0$ does not contract any such curves and $L = \pi^*(M)$, M strictly nef, thus $(L \cdot C_i) \geq 1$ for all i and thus $(K_S + tL \cdot C_i) \geq 0$ for all $t \geq 3$. Therefore, $(K_S + tL \cdot D) \geq 0$ for all $t \geq 3$, which proves the claim.

Thus if K_S+tL is not big for some $t>3$, then $(K_S+tL)^2=(K_S+3L+(t-3)L)^2=0$. Since K_S+3L and L are both nef, this means $(K_S+3L)^2=L^2=0$, so

$$K_S^2=K_S\cdot L=L^2=0 \tag{3.25}$$

By Hodge Index Theorem,

$$-K_S=L^{\otimes m} \tag{3.26}$$

for some $0<m\leq 3$.

Thus K_S+tL is almost strictly nef for all $t>3$. If $t\gg 0$, then K_S+tL is very ample on the fibers of $\phi: S\to C$. Since C is a smooth curve, $\phi_*(\mathcal{O}_S(K_S+tL))$ being torsion free, is a bundle of rank >1. The remainder of the proof will be divided into three cases:

Case 2.a): $g(C)>1$: Then by Theorem 3.16, K_S+tL is big for all $t\gg 0$.

Case 2.b): $g(C)=0$: i.e. $C\cong\mathbb{P}^1$. Then it follows from [48, Lemma 4.2] that

$$\phi_*((\omega_{S/\mathbb{P}^1}\otimes L^{\otimes N})^{\otimes s})$$

is generically spanned for all $s>0$, i.e.

$$(\phi_*(\omega_S\otimes L^{\otimes N}))\otimes\mathcal{O}_{\mathbb{P}^1}(-2)$$

and hence $\phi_*((N-m)L)$ is generically spanned. If $N \gg m$, then $\phi_*((N-m)L)$ has rank > 1 and hence $h^0 > 1$ and L is almost strictly nef. Then L is big as above and we done by (4.20).

Case 2.c): $g(C) = 1$. In this case, C is an elliptic curve and we can argue as in the proof of [11, Theorem 3.1] (the proof given there actually uses only nefness of $L, K_X + tL$ and relative bigness of $K_X + tL$) to show that $h^0(S, a(K_S + tL) \otimes \phi^* P) \neq 0$ for some $a > 0$ and $P \in \mathrm{Pic}^0(C)$. Thus $a(K_S + tL) \equiv D \geq 0$ and it is also nef. Now D is not big iff $D^2 = 0$ iff D is numerically equivalent to a π-exceptional curve which can not be since D is nef (recall $-K_S$ and L were positively proportional). Thus $D^2 > 0$ and we are done as before. This finishes the proof. □

Corollary 3.27. *Let S be a Gorenstein surface and $L \in \mathrm{Pic}\, S$ almost strictly nef (ASN). Then $K_S + tL$ is big for all $t > 3$.*

Proof. We follow the arguments of [48, Theorem 2.3]. Consider

$$\pi : R \xrightarrow{f} T \xrightarrow{j} S$$

where T is the normalization of S and R is a desingularization of T. Then by Theorem 3.24, $K_R + t\pi^* L$ is big for all $t > 3$. We use the following result of Kawamata:

Proposition 3.28. *(See [34, Proposition 7]) Let X and Y be Gorenstein varieties, $j : Y' \to Y$ the normalization of Y and let $f : X \to Y'$ be a proper birational* *morphism. Then for all $m \in \mathbb{N}$, there exists a natural injective homomorphism $f_*(\omega_X^{\otimes m}) \hookrightarrow j^*(\omega_Y^{\otimes m})$.*

This gives us an injection

$$f_*(r(K_R + m\pi^* L)) = f_*(rK_R) \otimes j^*(rmL) \hookrightarrow j^*(rK_S + mrL).$$

Let $m \geq 4$. Then

$$h^0(r(K_R + m\pi^* L)) \sim r^2$$

for all $r \gg 0$ and thus

$$h^0(j^*(rK_S + mrL)) \sim r^2$$

if $r \gg 0$. Now there exists an exact sequence

$$0 \to \mathcal{O}_S \to j_* \mathcal{O}_T \to \mathcal{F} \to 0$$

where $\mathcal{F}$ is an $\mathcal{O}_S$-module supported on a closed subvariety. Tensoring with $\mathcal{O}_S(r(K_S + mL))$ and taking H^0, this gives:

$$0 \to H^0(\mathcal{O}_S(r(K_S{+}mL))) \to H^0(j_* j^*(r(K_S{+}mL))) \to H^0(\mathcal{F} \otimes \mathcal{O}_S(r(K_S{+}mL))) \to .$$

The middle term $\sim r^2$ and the right term $\lesssim r$ for all $r \gg 0$. Thus

$$h^0(r(K_S + mL)) \sim r^2$$

for all $r \gg 0$.

$\square$

Corollary 3.29. *Let X be a smooth projective variety of dimension n and let $L \in \operatorname{Pic} X$ be almost strictly nef (ASN) with $\kappa(aK_X + bL) \geq n-2$ for some $a \geq 0$. Then $K_X + tL$ is big for all $t \gg 0$.*

Proof. The proof is quite similar in spirit to that of Proposition 3.21 and [11, Theorem 2.6], so the details are left out. Letting $M = aK_X + bL$, we consider a smooth resolution of indeterminacy $\pi : \hat{X} \to X$ of the Iitaka fibration $\phi = \phi_{|mM|}$. Let $\hat{\phi} : \hat{X} \to Y$ be the composite. Let $F \subset \hat{X}$ be a general fiber of $\hat{\phi}$ and $\hat{L} = \pi^* L$. Suppose $K_{\hat{X}} = \pi^* K_X + E'$ and

$$\pi^*(mM) = \hat{\phi}^*(A) + E,$$

$A \in \operatorname{Pic} Y$ ample and $E, E' \geq 0$. Then we can show that $K_{\hat{X}} + t\hat{L}$ is $\hat{\phi}$-big by Theorem 3.24 and pseudoeffective, thus $K_{\hat{X}} + t\hat{L} + \hat{\phi}^*(A)$ is big for all $t \gg 0$ (see [11, Lemma 2.5]) and this will show that $K_{\hat{X}} + t\hat{L}$ is big for all $t \gg 0$. Hence, so is $K_X + tL = \pi_*(K_{\hat{X}} + t\hat{L})$. $\square$

3.4 Examples of strictly nef non-ample divisors

In this section, we prove Theorem 5. Our calculations are in the spirit of [37, example 3.3].

First, we need to state the following conjecture (see [21, Conjecture 3.1] and the references listed there):

Conjecture 3.30. *(SHGH) Fix integers $d, m_1, \dots m_r \geq 0$, $r > 0$. Let $X = Bl_{p_1,\dots p_r}(\mathbb{P}^2) \xrightarrow{\pi} \mathbb{P}^2$ be a blow-up of $\mathbb{P}^2$ at r general points and $E_1, \dots, E_r$ the exceptional curves. Suppose the linear system $L := |d\pi^*\mathcal{O}_{\mathbb{P}^2}(1) - m_1E_1 - \cdots - m_rE_r|$ contains a reduced curve. Then* $\dim L = \binom{d+2}{2} - 1 - \Sigma_i\binom{m_i+1}{2}$ *(The quantity on the right hand side is often called the expected dimension of L).*

Now fix any integer $d \geq 4$. Let $p_1, \dots, p_{d^2} \in \mathbb{P}^2$ be general points and $X = Bl_{p_1,\dots,p_{d^2}}\mathbb{P}^2 \xrightarrow{\pi} \mathbb{P}^2$ be the blow-up with exceptional divisors $E_1, \dots, E_{d^2}$. Let $l = \pi^*(\mathcal{O}_{\mathbb{P}^2}(1))$ be the pulled back hyperplane class. Let

$$L = dl - E_1 - \cdots - E_{d^2}.$$

Then clearly $L^2 = 0$ and $L \cdot E_i = 1$ for all i. Now $d \geq 4$ and the generality of the p_i ensures that $\kappa(L) = -\infty$. Let $C = ml - \Sigma_{i=1}^{d^2} r_iE_i$ be the proper transform of a curve of degree m having singularities of orders $r_i - 1$ at p_i. Note that $(m, r_1, \dots, r_{d^2})$ are the coordinates of C on $\mathrm{Pic}(X) \otimes \mathbb{R} \cong \mathbb{R}^{d^2+1}$. Since for testing strict nefness of L we may assume that C is reduced, the SHGH conjecture implies that the expected dimension of the linear system $|mL - \Sigma_{i=1}^{d^2} r_iE_i|$ which is

$$\binom{m+2}{2} - 1 - \Sigma_1^{d^2}\binom{r_i+1}{2}$$

is nonnegative which simplifies to

$$\rho(m)^2 := m^2 + 3m - d^2/4 \geq \Sigma_1^{d^2}(r_i + 1/2)^2.$$

For m fixed, this is the equation of a sphere with center

$$A = (-1/2, \dots, -1/2) \in \mathbb{R}^{d^2}$$

and radius

$$\rho(m) = \sqrt{m^2 + 3m - d^2/4}$$

in $\mathbb{R}^{d^2}$. Now $L \cdot C = dm - \Sigma_1^{d^2} r_i$. For m fixed, consider the hyperplane

$$H = (r_1 + \cdots + r_{d^2} - dm = 0) \subset \mathbb{R}^{d^2}.$$

Notice that $A \in H_{<0}$ and

$$AH = d/2 + m =: \beta(m).$$

In fact, the whole sphere above is contained in $H_{<0}$ because

$$\rho(m)^2 = m^2 + 3m - d^2/4 \leq \beta(m)^2 = m^2 + md + d^2/4$$

for all $d \geq 4$ and thus $L \cdot C > 0$.

Now let S be a projective surface. It admits a finite morphism $f : S \to \mathbb{P}^2$ which, suppose, is of degree m. Since the points $p_1, \ldots, p_{d^2} \in \mathbb{P}^2$ are general, we may assume that they are not in the branch locus of f. Let $\{q_1, \ldots, q_{md^2}\} = f^{-1}\{p_1, \ldots, p_{d^2}\}$. Then f extends to a finite morphism $f' : S' := Bl_{q_1,\ldots,q_{md^2}}(S) \to X$ which is the base change of f via π. Now $f'^*(L) \in \mathrm{Pic}(S')$ is strictly nef and non-ample.

3.5 Generalized abundance for klt pairs with big boundary

This is a supplementary section whose contents are independent of the rest of this chapter. First, we recall the following theorem of [6]:

Theorem 3.31. *([6, Corollary 3.9.2])Let (X, B) be a klt pair with B big. If K_X+B is nef, then it is semiample.*

Proof. I give a self contained short proof here. The technique (which probably is standard in the field) will be used again in this section.

Since B is big, for $\alpha \gg 0$, so is $K_X + \alpha B$. Fix such an α. We can write $K_X + \alpha B \sim_{\mathbb{R}} A + E$ where A is ample and $E \geq 0$. Then for $\epsilon > 0$, we have

$$\epsilon(K_X + B) = \epsilon(K_X + \alpha B) + (\epsilon - \epsilon\alpha)B = \epsilon(A + E) + \epsilon(1 - \alpha)B.$$

and thus

$$(1 + \epsilon)(K_X + B) = K_X + \epsilon A + (1 + \epsilon(1 - \alpha))B + \epsilon E$$

Let $\Delta = (1 + \epsilon(1 - \alpha))B + \epsilon A$. Now for ϵ sufficiently small, (X, Δ) is klt. Then

$$(1 + \epsilon)(K_X + B) - (K_X + \Delta) \sim_{\mathbb{R}} \epsilon A$$

and $K_X + B$ is semiample by the usual basepoint free theorem

$\square$

We now use this to prove the corresponding theorem for generalized abundance.

Theorem 3.32. *Let (X, B) be a klt pair with B big and $L \in \operatorname{Pic} X$ nef. Then there exists a birational morphism $\mu : \tilde{X} \to X$ from a smooth projective variety such that $P_\sigma(\mu^*(K_X + B + L))$ is semiample.*

Proof. Suppose first that $K_X + B + L$ is nef. Then as above, for $\alpha \gg 0$, we can write $K_X + \alpha B + L \sim A + E$ where A is ample and $E \geq 0$ and similarly, we have

$$(1 + \epsilon)(K_X + B + L) \sim K_X + (1 + \epsilon(1 - \alpha))B + (L + \epsilon A) + \epsilon E \sim K_X + \Delta$$

where (X, Δ) is klt. Indeed, $(X, (1+\epsilon(1-\alpha))B+\epsilon E)$ is klt if ϵ is sufficiently small. Since $L + \epsilon A$ is ample, we can choose a divisor in its linear system with arbitrarily small coefficients and adding that divisor preserves the klt property. Now we are done by the above theorem theorem since Δ is big.

If $K_X + B + L$ is not nef, let $A \geq 0$ be an ample divisor and run an MMP on $K_X + B + L$ with scaling of A. This is also an MMP on $K_X + \Delta$ with scaling of $(1 + \epsilon)A$. Since Δ is big, by [6], this terminates with some minimal model (X', Δ') such that $K_{X'} + \Delta'$ is semiample. Let $X \xleftarrow{\mu} \tilde{X} \xrightarrow{\pi} X'$ be a common resolution. Then $\mu^*(K_X + \Delta) = \pi^*(K_{X'} + \Delta') + E$ where $E \geq 0$ is π-exceptional. Then $P_\sigma(\mu^*(K_X + B + L)) = P_\sigma(\mu^*(K_X + \Delta)) = P_\sigma(\pi^*(K_{X'} + \Delta')) = \pi^*(K_{X'} + \Delta')$ is semiample. $\square$

Chapter 4

Nef and abundant divisors and canonical bundle formula

The reference for this section is [15]. Corollary 4.15 is new and hasn't appeared elsewhere.

In the 1980's Kawamata proved that if (X, B) is a klt pair such that $K_X + B$ is nef and abundant, then it is semiample. Thus to prove the abundance conjecture, it is enough to show that $K_X + B$ is abundant. This leads to the following strong form of the abundance conjecture:

Conjecture 4.1. *(Strong abundance) If (X, B) is a klt pair, then $K_X + B$ is always abundant.*

It is known that this is equivalent to the existence of good minimal models (see [28, Theorem 4.3]).

This chapter will be concerned with the following possible generalization of Kawamata's theorem:

Question 4.2. *Let (X, B) be a klt pair and $L \in \operatorname{Pic} X$ nef such that $K_X + B + L$ is nef and abundant, then is $K_X + B + L$ semiample?*

This is clearly related to the following conjecture of Lazić and Peternell [40].

Conjecture 4.3. *(Generalized Abundance) Let (X, B) be a klt pair with $K_X + B$ pseudoeffective and L a nef Cartier divisor on X such that $K_X + B + L$ is also nef. Then there exists a semiample $\mathbb{Q}$-divisor M on X such that $K_X + B + L \equiv M$.*

Note that we are not assuming pseudoeffectivity of $(K_X + B)$ in our question. There are two reasons why one might expect this to be true. Firstly, in [13], the author has shown that generalized abundance does continue to hold for many uniruled varieties. Secondly, if $L \in \operatorname{Pic} X$ is a large multiple of a strictly nef divisor, then under the above assumptions, one can show that $K_X + B + L$ is ample. So it is natural to ask for a version in which the pseudoeffectivity of $K_X + B$ is not required. The surface case of this question is very easily seen to be true. We now state our main results which settle certain special cases. Essentially, they show that Question 1 has a positive answer in the following two situations:

1. If L is a birational pushforward of a semiample divisor (in fact, it does not need to be nef here).

2. If $K_X + B \geq 0$.

Theorem 4.4. *Let $(X, B + \mathbf{M})$ be a generalized klt pair, H a nef $\mathbb{Q}$-Cartier divisor on X such that $H - (K_X + B + \mathbf{M}_X)$ is nef and abundant. Assume further that $\mathbf{M}$ is b-semiample, $\kappa(aH - (K_X + B + \mathbf{M}_X)) \geq 0$ and $\nu(aH - (K_X + B + \mathbf{M}_X)) = \nu(H - (K_X + B + \mathbf{M}_X))$ for some $a \in \mathbb{Q}_{>1}$. Then H is semiample.*

The case $\mathbf{M} = 0$ of the above theorem is due to Kawamata [33] and Fujino [24]. Our approach of using canonical bundle formula is similar to that of Fujino's. We later generalize this result to the case $\mathbf{M}$ is b-nef and b-abundant (i.e. birational pushforward of a nef and abundant divisor).

Theorem 4.5. *Let (X, B) be a klt pair with $\kappa(K_X + B) \geq 0$, H and L nef $\mathbb{Q}$-Cartier divisors on X such that $H - (K_X + B + L)$ is nef and abundant. Assume further that $\kappa(aH - (K_X + B + L)) \geq 0$ and $\nu(aH - (K_X + B + L)) = \nu(H - (K_X + B + L))$ for some $a \in \mathbb{Q}_{>1}$. Then H is semiample.*

This will be obtained as a consequence of a generalized canonical bundle formula (Theorem 4.16) to be proved in the third section.

Remark 4.6. Thus in particular, if (X, B) is a klt pair with $K_X + B$ effective and if $L \in \operatorname{Pic} X$ is nef such that $K_X + B + L$ nef and abundant, then $K_X + B + L$ is semiample. The non-vanishing conjecture ([40, page 1]) predicts that assuming $K_X + B$ pseudoeffective would suffice for this to hold.

4.1 Preliminaries

We will always work with $\mathbb{Q}$-divisors.

Definition 4.1.1 (Abundant divisors)**.** Let X be a normal projective variety and D be a nef $\mathbb{Q}$-divisor with Iitaka dimension $\kappa(X, D)$. The *numerical dimension* of D is defined as $\nu(X, D) = max\{e|D^e \not\equiv 0\}$. In general, $\kappa(X, D) \leq \nu(X, D)$. D is *abundant* if $\kappa(X, D) = \nu(X, D)$. Examples include big nef divisors, semiample divisors and their pullbacks.

Definition 4.1.2 (b-divisors (see [17]))**.** Let X be a normal variety. A *b-divisor* $\mathbf{D}$ on X is a collection of Weil divisors $\mathbf{D}_Y$ on higher birational models $Y \to X$ of X that are compatible under pushforward. $\mathbf{D}_Y$ is called the *trace* of $\mathbf{D}$ on Y. The *Cartier closure* of a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D on X is the b-$\mathbb{Q}$ divisor $\overline{\mathbf{D}}$ with trace $\overline{\mathbf{D}}_Y := \rho^*(D)$ on a birational model $\rho : Y \to X$. A b-$\mathbb{Q}$-divisor $\mathbf{D}$ on a normal variety X is *b-$\mathbb{Q}$-Cartier* if it is the Cartier closure of a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on some birational model of X; in this situation, we say that $\mathbf{D}$ *descends* to Y. $\mathbf{D}$ is called *b-nef* (resp. *b-semiample*) if it is descends to a nef (resp. semiample) $\mathbb{Q}$-divisor on some birational model over X. Let (X, B) be a sub-pair consisting of a normal variety X and a $\mathbb{Q}$-divisor B on X such that $K_X + B$ is $\mathbb{Q}$-Cartier. The *discrepancy b-divisor* $\mathbf{A} = \mathbf{A}(X, B)$ is defined by $K_Y = f^*(K_X + B) + \mathbf{A}_Y$ for all models $Y \xrightarrow{f} X$. (X, B) is called *sub-klt* (resp *sub-lc*) if all coefficients of $\mathbf{A}_Y(X, B)$ are > -1 (resp. ≥ -1) for some (and hence, by [36, Lemma 2.30], for all) birational models $f : Y \to X$. Thus in particular, if X is smooth, all coefficents of B are less than 1 and B has normal crossing support, then (X, B) is sub-klt. In case $B \geq 0$, we drop the prefix sub.

Definition 4.1.3 (Generalized pairs and their singularities(see [7]))**.** A *generalized sub-pair* $(X, B + \mathbf{M})$ consists of a normal projective variety X, a $\mathbb{Q}$-divisor B and a $\mathbb{Q}$-b-divisor $\mathbf{M}$ on X such that:

1. $K_X + B + \mathbf{M}_X$ is $\mathbb{Q}$-Cartier.

2. $\mathbf{M}$ is b-nef.

When $B \geq 0$, we drop the prefix sub.

Let (X, B) be a generalized sub-pair and $Y \xrightarrow{\mu} X$ a higher birational model of X. Let B_Y be defined by $K_Y + B_Y + \mathbf{M}_Y = \mu^*(K_X + B + \mathbf{M}_X)$. We say that $(X, B+\mathbf{M})$ is *sub-generalized klt* (resp. *sub-generalized lc*) if every coefficient of B_Y is less than 1 (resp. less than or equal to 1). We will call them sub-gklt and sub-glc in short. The *generalized discrepancy b-divisor* $\mathbf{A}(X, B + \mathbf{M})$ is then defined by $\mathbf{A}_Y(X, B + \mathbf{M}) = -B_Y$. Note that $\mathbf{M}_Y$ does not contribute to the singularities of $(Y, B_Y + \mathbf{M})$ once it descends to a nef divisor on Y, i.e. $\mathbf{A}(Y, B_Y + \mathbf{M}_Y) = \mathbf{A}(Y, B_Y)$ and thus $(Y, B_Y + \mathbf{M}_Y)$ is gklt (resp. glc) iff (Y, B_Y) is klt (resp. lc).

Definition 4.1.4 (Generalized klt-trivial fibrations)**.** Let (X, B) be a pair consisting of a normal variety X and a $\mathbb{Q}$-divisor B on X. A morphism $f : (X, B) \to Y$, Y normal is *generalized klt-trivial* if

1. f is surjective with connected fibers,
2. (X, B) is sub-klt over the generic point of Y,
3. there exists a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D on Y and a nef $\mathbb{Q}$-divisor L on X such that $K_X + B + L \sim_{\mathbb{Q}} f^*D$
4. there exists a log-resolution $\pi^{'} : (X^{'}, B_{X'}) \to (X, B)$ such that if $f^{'} : (X^{'}, B_{X'}) \to Y$ is the induced morphism, then rank $f^{'}_*\mathcal{O}_{X'}(\lceil -B_{X'} \rceil) = 1$.

The following fact seems to be well-known to the experts. I will include a proof for the convenience of readers and for want of a reference.

Lemma 4.7. *Let $f : (X, B) \to Y$ be a generalized klt-trivial fibration, $\sigma : Y^{'} \to Y$ a generically finite morphism from a normal variety $Y^{'}$. Let $X^{'}$ denote a desingularization of the main component of $X \times_Y Y^{'}$ and let $f^{'} : (X^{'}, B_{X'}) \to Y^{'}$ be the induced morphism. Then $f^{'}$ is generalized klt-trivial.*

Proof. Only item (4) in the above definition needs to be verified. Since $(X^{'}, B_{X'})$ is sub-klt, the rank is surely ≥ 1. We only need to show it is ≤ 1 and may assume that X itself is smooth. Let $\pi : (X^{'}, B_{X'}) \to (X, B)$ be the induced morphism.

1. Suppose first that σ is birational. Then there exists an effective, integral, π-exceptional divisor $E^{'}$ on $X^{'}$ such that $K_{X'} = \pi^*(K_X) + E^{'}$. The condition $K_{X'} + B_{X'} = \pi^*(K_X + B)$ then gives $B_{X'} + E^{'} = \pi^* B_X$. If we can show that $\sigma \circ f^{'} : X^{'} \to Y^{'}$ satisfies rank $(\sigma \circ f^{'})_*\mathcal{O}_{X'}(\lceil -B_{X'} \rceil) = 1$, then rank

$f'_*\mathcal{O}_{X'}(\lceil -B_{X'} \rceil) = 1$ also since $\sigma \circ f'$ and f' have the same general fiber. Thus we may assume $\sigma = id$ and thus $f \circ \pi = f'$. Now,

$$f'_*\mathcal{O}_{X'}(\lceil -B_{X'} \rceil) = f'_*\mathcal{O}_{X'}(\lceil -\pi^* B_X + E' \rceil)$$

$$= f'_*\mathcal{O}_{X'}(\lceil -\pi^* B_X \rceil + E') \text{ (since } E' \text{ is integral)}$$

$$\subset f'_*\mathcal{O}_{X'}(\pi^* \lceil -B_X \rceil + E')$$

$$= f_*\pi_*\mathcal{O}_{X'}(\pi^*(\lceil -B_X \rceil) + E') = f_*\mathcal{O}_X(\lceil -B_X \rceil) \otimes \pi_*\mathcal{O}_{X'}(E'))$$

$$= f_*\mathcal{O}_X(\lceil -B_X \rceil) \text{ (by Hartog's theorem since } E' \text{ is } \pi\text{-exceptional).}$$

Thus rank $f'_*\mathcal{O}_{X'}(\lceil -B_{X'} \rceil) \leq 1$.

2. Now suppose that σ is finite. In this case also, there exists an integral divisor $R' \geq 0$ such that $K_{X'} = \pi^*(K_X) + R'$ (Supp (R') is the ramification locus of π). This gives $B_{X'} + R' = \pi^* B_X$. To show that rank $f'_*\mathcal{O}_{X'}(\lceil -B_{X'} \rceil) = 1$, it suffices to show that R' is not f'-horizontal. Now [31, Chapter 3, Proposition 10.1 (b)] implies that $f^{-1}(Sm(\sigma)) \subset Sm(\pi)$ (Sm denotes the smooth locus of the respective morphism) or in other words, $f(Branch(\pi)) = f(\pi(R')) \subset Branch(\sigma)$. Thus R' can not be f'-horizontal.

□

4.2 Nef and abundant divisors

We start off by proving two easy results. They provide positive evidence for Question 1. Recall that an *almost strictly nef* (ASN) divisor means a birational pullback of a strictly nef Cartier divisor. See [13], [54], [43] for some recent developments around ASN divisors.

Lemma 4.8. *Let X be a normal projective variety, $M \in \operatorname{Pic} X$ be almost strictly nef (ASN) and abundant. Then M is big.*

Proof. By [33, Proposition 2.1], there exist morphisms $X \xleftarrow{\mu} X' \xrightarrow{f} Y$, where μ is birational, f a contraction and $X^{'}$ and Y are smooth and projective such that

$$M^{'} := \mu^* M = f^*(M_Y) \tag{4.9}$$

for some $M_Y \in \operatorname{Pic} Y$ nef and big. If $\dim Y = \dim X$, then f is generically finite and M is already big. So assume that $\dim Y < \dim X$. By definition of ASN, there exists $X \xrightarrow{\pi} X_0$ birational and $M_0 \in \operatorname{Pic} X_0$ strictly nef such that $\pi^*(M_0) = M$. Let $E \subset X$ be the exceptional locus of π. Then for all curves $C \not\subset E$, $(M \cdot C) = (\pi^*(M_0) \cdot C) > 0$. If $F \subset X^{'}$ is a general fiber of f, then $F \not\subset E$. Then there is a curve $C \subset F$ such that $C \not\subset E$ and thus $(M \cdot C) > 0$. This is not possible by (4.9). Therefore, $\dim Y = \dim X$ and M is big. $\square$

Corollary 4.10. *Let (X, B) be a klt pair, $L \in \operatorname{Pic} X$ strictly nef such that $K_X + B + t_0 L$ is abundant for some $t_0 > 2 \cdot \dim X$. Then $K_X + B + tL$ is ample for all $t > 2 \cdot \dim X$.*

Proof. $K_X + B + t_0L$ is strictly nef by the cone theorem and abundant, hence big by above lemma. Then $2(K_X + B + t_0L) - (K_X + B)$ is nef and big and thus $K_X + B + t_0L$ is semiample by the basepoint free theorem. Now it is easy to see that a strictly nef and semiample divisor is ample, thus $K_X + B + t_0L$ is ample. The conclusion now follows from [48, Lemma 1.3] (or rather its log version which has the same proof). □

The following generalization of the basepoint-free theorem will be indispensible for us. It can be proved by a minor modification of the usual arguments ([36, page 78]). See [2, Theorem 2.1] for the proof of a much more general statement.

Theorem 4.11. *Let (X, B) be a sub-klt pair, H a nef $\mathbb{Q}$-Cartier divisor on X. Assume that*

1. *$rH - (K_X + B)$ is nef and big for some $r \in \mathbb{N}$.*

2. *$H^0(\mathcal{O}_X(\lceil \mathbf{A}(X, B) \rceil + j\overline{\mathbf{H}})) \subset H^0(\mathcal{O}_X(jH))$ for all $j \in \mathbb{N}$.*

Then H is semiample.

Lemma 4.12. *Let $(X, B+\mathbf{M})$ be a sub-generalized klt pair, H nef $\mathbb{Q}$-Cartier divisor on X such that $H - (K_X + B + \mathbf{M}_X)$ is nef and abundant. Suppose also that $\kappa(aH-(K_X+B+\mathbf{M}_X)) \geq 0$ and $\nu(aH-(K_X+B+\mathbf{M}_X)) = \nu(H-(K_X+B+\mathbf{M}_X))$ for some $a \in \mathbb{Q}_{>1}$. Let*

$$(X, B + \mathbf{M}_X) \xleftarrow{\mu} (Y, B_Y + \mathbf{M}_Y) \xrightarrow{f} Z$$

be the standard diagram induced by the Iitaka fibration of $H - (K_X + B + \mathbf{M}_X)$ *(see [33, Proposition 2.1]). Then we have the following:*

1. *Suppose that the moduli b-divisor* $\mathbf{M}_Z$ *induced on* Z *by the generalized sub-pair* $(Y, B_Y + \mathbf{M}_Y)$ *is b-nef. Then* H *is semiample.*

2. *Suppose* $(X, B + \mathbf{M})$ *is generalized klt. Then rank* $f_*\mathcal{O}_Y(\lceil -B_Y \rceil) = 1$.

Proof. We argue along the lines of [24].

Recall first that [33, Proposition 2.1,2.3] says that there exist smooth projective varieties Y and Z and morphisms

$$(X, B + \mathbf{M}_X) \xleftarrow{\mu} (Y, B_Y + \mathbf{M}_Y) \xrightarrow{f} Z$$

where μ is birational, f is a surjective morphism with connected fibers birational to the Iitaka fibration of $H - (K_X + B + \mathbf{M}_X)$, $\mu^*(K_X + B + \mathbf{M}_X) = K_Y + B_Y + \mathbf{M}_Y$, where B_Y is defined by $\mu_*(\mathbf{M}_Y) = \mathbf{M}_X$. We may assume that $\mathbf{M}_Y$ descends to a nef Cartier divisor on Y (possibly after replacing Y by a higher model) and there exists $D_Z \in \mathrm{Pic}(Z)$ nef and big such that, setting $H_Y = \mu^*(H)$, we have

$$\mu^*(H - (K_X + B + \mathbf{M}_X)) = H_Y - (K_Y + B_Y + \mathbf{M}_Y) \sim_{\mathbb{Q}} f^*(D_Z).$$

Also, there exists $H_Z \in \mathrm{Pic}(Z)$ nef such that $H_Y \sim_{\mathbb{Q}} f^*(H_Z)$. Note that (Y, B_Y) is sub-klt and

$$\mathbf{A}_Y(Y, B_Y + \mathbf{M}_Y) = -B_Y$$

As noted above, there exists a nef and big divisor D_Z and a nef divisor H_Z on Z such that $K_Y + B_Y + \mathbf{M}_Y = f^*(H_Z - D_Z)$. Now by assumption, we can write $H_Z - D_Z \sim_{\mathbb{Q}} K_Z + B_Z + \mathbf{M}_Z$ such that

1. $\mathbf{M}_Z$ is $\mathbb{Q}$-Cartier and b-nef.

2. (Z, B_Z) is sub-klt. B_Z is defined as follows. Let $D \subset Z$ be any prime divisor. Let

$$a_D := sup\{t|(Y, B_Y + tf^*(D)) \text{ is sub-lc over the generic point of } D\}$$

and define $B_Z = \Sigma(1 - a_D)D$. We note that replacing Y and Z by higher models, we may assume that $(Y, B_Y) \xrightarrow{f} (Z, B_Z)$ satisfies standard SNC assumptions (see [22, Definition 4.4]). Then the definition of a_D requires that all coefficients of $B_Y + tf^*(D)$ are ≤ 1. Now, (Y, B_Y) being sub-klt implies that all coefficients of B_Y are < 1 and thus that $a_D > 0$ for all D. (Z, B_Z) is thus sub-klt.

Now let $(Z^{'}, B_{Z^{'}}) \to (Z, B_Z)$ be a higher model such that $\mathbf{M}_{Z^{'}}$ is nef. Then $(Z^{'}, B_{Z^{'}})$ is of course also sub-klt. Let $Y^{'}$ be the normalization of the main component of $Y \times_Z Z^{'} \to Z^{'}$. Replacing Y and Z by $Y^{'}$ and $Z^{'}$, we may assume that M_Z is nef on Z. We need:

Lemma 4.13. *(See [17, Lemma 9.2.2]) Let $f : (Y, B_Y) \to Z$ be a fibration such that (Y, B_Y) is sub-klt over the generic point of Z. Let $\mathbf{B}$ be the induced discriminant*

$\mathbb{Q}$-*b-divisor of* Z. *Then* $\mathcal{O}_Z(\lceil -\mathbf{B} \rceil) \subset f_*\mathcal{O}_Y(\lceil \mathbf{A}(Y, B_Y) \rceil)$.

As in the proof of [17, Proposition 9.2.3], this shows that $\mathcal{O}_Z(\lceil -\mathbf{B} + \mathbf{D} \rceil) \subset f_*\mathcal{O}_Y(\lceil \mathbf{A}(Y, B_Y) + f^*\mathbf{D} \rceil)$ holds for every $\mathbb{Q}$-b-Cartier $\mathbb{Q}$-b-divisor $\mathbf{D}$ on Y: we can replace Z by a higher model Z' on which $\mathbf{D}$ is $\mathbb{Q}$-Cartier and replace Y by the normalization of the main component of $Y \times_Z Z'$ as before. Then $(Y, B_Y - f^*D)$ is sub-klt over the generic point of Z with discriminant $\mathbb{Q}$-b-divisor $\mathbf{B} - \mathbf{D}$ and

$$\mathbf{A}(Y, B_Y - f^*D) = \mathbf{A}(Y, B_Y) + f^*(D).$$

Now the above lemma applied to $f : (Y, B_Y - f^*D) \to Z$ gives

$$\mathcal{O}_Z(\lceil -\mathbf{B} + \mathbf{D} \rceil) \subset f_*\mathcal{O}_Y(\lceil \mathbf{A}(Y, B_Y) + f^*\mathbf{D} \rceil)$$

as claimed. Thus

$$\mathcal{O}_Z(\lceil \mathbf{A}(Z, B_Z) \rceil + j\overline{\mathbf{H}_Z}) \subset f_*\mathcal{O}_Y(\lceil \mathbf{A}(Y, B_Y) \rceil + j\overline{\mathbf{H}_Y})$$

for all $j \in \mathbb{Z}$. Taking H^0 gives

$$H^0(Z, \mathcal{O}_Z(\lceil \mathbf{A}(Z, B_Z) \rceil + j\overline{\mathbf{H}_Z})) \subset H^0(Y, \mathcal{O}_Y(\lceil \mathbf{A}(Y, B_Y) \rceil + j\overline{\mathbf{H}_Y}))$$

$$= H^0(X, \mu_*(\mathcal{O}_Y(\lceil \mathbf{A}(Y, B_Y) \rceil + j\overline{\mathbf{H}_Y})) = H^0(X, \lceil A(X, B) \rceil + jH)$$

$$= H^0(X, \mathcal{O}_X(jH)) = H^0(Y, \mathcal{O}_Y(jH_Y))$$

$$= H^0(Y, \mathcal{O}_Y(jf^*(H_Z))) = H^0(Z, \mathcal{O}_Z(jH_Z)).$$

Thus H_Z is semiample by Theorem 4.11. This proves (1).

Now we prove (2). Clearly rank $f_*\mathcal{O}_Y(\lceil -B_Y\rceil) \geq 1$ since (Y, B_Y) is sub-klt.

Take $A \in \operatorname{Pic} Z$ sufficiently ample such that $f_*\mathcal{O}_Y(\lceil -B_Y\rceil) \otimes A$ is globally generated. Since D_Z is big,

$$f_*\mathcal{O}_Y(\lceil -B_Y\rceil) \otimes A \subset f_*\mathcal{O}_Y(\lceil -B_Y\rceil) \otimes \mathcal{O}_Z(mD_Z)$$

if $m \gg 0$. Now note that since $B_Y = \mu_*^{-1}(B) + F$ where F is μ-exceptional, $B \geq 0$ and

$$0 \leq \lceil -B_Y\rceil = \lceil \mu_*^{-1}(-B)\rceil + \lceil -F\rceil,$$

we can conclude that $\lceil \mu_*^{-1}(-B)\rceil = 0$ and hence $\lceil -B_Y\rceil$ is an effective μ-exceptional divisor. Now

$$H^0(Z, f_*\mathcal{O}_Y(\lceil -B_Y\rceil) \otimes A) \subset H^0(Z, f_*\mathcal{O}_Y(\lceil -B_Y\rceil) \otimes \mathcal{O}_Z(mD_Z)) =$$
$$H^0(Y, \mathcal{O}_Y(\lceil -B_Y\rceil \otimes f^*(mD_Z)).$$

We have

$$H^0(Y, \mathcal{O}_Y(\lceil -B_Y\rceil \otimes \mathcal{O}_Y(f^*(mD_Z))) = H^0(\mathcal{O}_Y(E) \otimes f^*(mD_Z))$$

where $E \geq 0$ is μ-exceptional. This equals

$$H^0(Y, \mathcal{O}_Y(E) \otimes \mu^*(m(H - (K_X + B + \mathbf{M}_X)))) =$$
$$H^0(X, m(H - (K_X + B + \mathbf{M}_X))) = H^0(Z, mD_Z).$$

This shows that rank $f_*\mathcal{O}_Y(\lceil -B_Y\rceil) = 1$.

$\square$

Corollary 4.14. *Let $(X, B+\mathbf{M})$ be a generalized klt pair, H a nef $\mathbb{Q}$-Cartier divisor on X such that $H-(K_X+B+\mathbf{M}_X)$ is nef and abundant. Assume further that $\mathbf{M}$ is b-semiample, $\kappa(aH-(K_X+B+\mathbf{M}_X)) \geq 0$ and $\nu(aH-(K_X+B+\mathbf{M}_X)) = \nu(H-(K_X+B+\mathbf{M}_X))$ for some $a \in \mathbb{Q}_{>1}$. Then H is semiample.*

Proof. As in the proof of the previous Lemma, there exist morphisms

$$(X, B+\mathbf{M}_X) \xleftarrow{\mu} (Y, B_Y+\mathbf{M}_Y) \xrightarrow{f} Z$$

where $\mathbf{M}_Y$ is semiample, Y and Z are both smooth, there exists $D_Z \in \operatorname{Pic} Z$ and nef and big, $H_Z \in \operatorname{Pic} Z$ such that $H_Y \sim_{\mathbb{Q}} f^*(H_Z)$,

$$\mu^*(H-(K_X+B+\mathbf{M}_X)) = H_Y-(K_Y+B_Y+\mathbf{M}_Y) \sim_{\mathbb{Q}} f^*(D_Z)$$

and

$$\mu^*(K_X+B_X+\mathbf{M}_X) = K_Y+B_Y+\mathbf{M}_Y \sim_{\mathbb{Q}} f^*(H_Z-D_Z)$$

such that $\operatorname{Supp}(B_Y)$ is SNC. Note: $\lceil -B_Y \rceil \geq 0$. Since $\mathbf{M}_Y$ is semiample, for m sufficiently large and divisible, there exists $H = D_1+\cdots+D_k \in |m\mathbf{M}_Y|$ such that D_i is smooth for all i, $D_i \cap D_j = \varnothing$ for all $i \neq j$ and $\operatorname{Supp} H \cup \operatorname{Supp} B_Y$ is SNC. By taking $m \gg 0$, $(1/m)H \in |\mathbf{M}_Y|_{\mathbb{Q}}$ can be made arbitarily small, thus $(Y, B_Y+(1/m)H)$ is sub-klt. Now consider the induced fibration $(Y, B_Y+(1/m)H) \xrightarrow{f} Z$. Then

$$f_*\mathcal{O}_Y(\lceil -(B_Y+(1/m)H) \rceil) = f_*\mathcal{O}_Y(\lceil -B_Y \rceil)$$

since H can be chosen to have no components in common with B_Y. By Lemma 4.12(2), we have rank $f_*\mathcal{O}_Y(\lceil -B_Y \rceil) = 1$. Thus $(Y, B_Y+1/mH) \xrightarrow{f} Z$ is a klt-trivial fibration. If B_Z and $\mathbf{M}_Z$ denote the induced discriminant and the moduli divisors on Z, then by [1, Theorem 2.7], $H_Z - D_Z = K_Z + B_Z + \mathbf{M}_Z$ where $\mathbf{M}_Z$ is b-nef. Thus we are done by the $\mathbf{M} = 0$ case of the Lemma 4.12(1) (which is essentially worked out in [24]).

□

Corollary 4.15. *Let $(X, B+\mathbf{M})$ be a generalized klt pair, H a nef $\mathbb{Q}$-Cartier divisor on X such that $H - (K_X + B + \mathbf{M}_X)$ is nef and abundant. Assume further that $\mathbf{M}$ is b-abundant, $\kappa(aH - (K_X + B + \mathbf{M}_X)) \geq 0$ and $\nu(aH - (K_X + B + \mathbf{M}_X)) = \nu(H - (K_X + B + \mathbf{M}_X))$ for some $a \in \mathbb{Q}_{>1}$. Then H is semiample.*

Proof. In the standard diagram, $(X, B + \mathbf{M}_X) \stackrel{\mu}{\leftarrow} (Y, B_Y + \mathbf{M}_Y) \stackrel{f}{\rightarrow} Z$, replacing Y with a suffiently high model, we may assume that $\mathbf{M}_Y$ descends to a nef and abundant divisor on Y. As above, we write

$$\mu^*(K_X + B_X + \mathbf{M}_X) = K_Y + B_Y + \mathbf{M}_Y \sim_{\mathbb{Q}} f^*(H_Z - D_Z)$$

Since $\mathbf{M}_Y$ is nef and abundant, we have a Kawamata diagram for $\mathbf{M}_Y$: $Y \stackrel{\pi}{\leftarrow} Y' \stackrel{g}{\rightarrow} S$ such that for all $n \gg 0$, there exists and ample divisor A_n and an effective divisor E on S such that $\pi^*\mathbf{M}_Y \sim g^*(A_n + (1/n)E)$. Replacing Y with Y' we write $\mathbf{M}_Y \sim g^*(A_n + (1/n)E)$.

Since $-B_Y \geq -(B_Y + (1/n)g^*E)$, we have

$$1 = rank f_*\mathcal{O}_Y(\lceil -B_Y \rceil) \geq rank f_*\mathcal{O}_Y(\lceil -(B_Y + (1/n)g^*E) \rceil) > 0$$

where the last inequality holds since $(Y, B_Y + (1/n)g^*E)$ is sub-klt for $n \gg 0$. Hence we conclude that $rank f_*\mathcal{O}_Y(\lceil -(B_Y + (1/n)g^*E)\rceil = 1$. Now we argue as above. Since f^*A_n is semiample, there exists $D \in |f^*A_n|_{\mathbb{Q}}$ such that D has arbitrarily small coefficients and has no components in common with $Supp(B_Y + (1/n)g^*E)$. In particular $(Y, B_Y + (1/n)g^*E + D)$ is sub-klt and

$$f_*\mathcal{O}_Y(\lceil -(B_Y + (1/n)g^*E)\rceil) = f_*\mathcal{O}_Y(\lceil -(B_Y + (1/n)g^*E + D)\rceil).$$

Thus $(Y, B_Y + (1/n)g^*E + D) \xrightarrow{f} Z$ is a klt-trivial fibration. Let $\mathbf{M}_Z$ and B_Z be the induced discriminant and moduli divisors on Z. Then $\mathbf{M}_Z$ is b-nef by [1]. We conclude by Lemma 4.12.

□

The following is a generalized canonical bundle formula which is used for proving theorem 4.5. It may also be of independent interest.

Theorem 4.16. *Let (Y, B) be a sub klt pair, $L \in \operatorname{Pic} X$ nef and $f : Y \to Z$ be a contraction such that $K_Y + B + L \sim_{\mathbb{Q},f} 0$ and rank $f_*\mathcal{O}_Y(\lceil \mathbf{A}(Y, B)\rceil) = 1$. Suppose $K_Y + B \geq_{\mathbb{Q}} 0$. Then the moduli b-divisor $\mathbf{M}_Z$ induced on Z is b-nef.*

Proof. We write

$$K_Y + B_Y + L_Y \sim_{\mathbb{Q}} f^*(D_Z) = f^*(K_Z + B_Z + \mathbf{M}_Z). \tag{4.17}$$

as before. Let $F \subset Y$ be a general fiber of f. Then $(K_Y + B_Y)|_F \geq 0$. Suppose $(K_Y + B_Y)|_F >_{\mathbb{Q}} 0$. Then there exists a curve $C \subset F$ such that $((K_Y + B_Y)|_F \cdot C) > 0$ and thus $((K_Y + B_Y + L_Y)|_F \cdot C) > 0$. This contradicts (5.2). Thus $(K_Y + B_Y)|_F \sim_{\mathbb{Q}} 0$, $L|_F \sim_{\mathbb{Q}} 0$, where F is a general fiber of f. Let $d \in \mathbb{N}$ such that $\mathcal{O}_F(dL_Y|_F) \cong \mathcal{O}_F$.

Now by [40, Lemma 3.1], after replacing Y and Z with higher models, we can assume that L_Y is numerically trivial on every fiber of f. By performing a flattening base change, we can replace Y and Z again by higher models and assume that f is flat. Note that $\mathcal{O}_F(dL_Y|_F) \cong \mathcal{O}_F$ is still true for a general fiber F. Then by the semicontinuity theorem,

$$z \to h^0(\mathcal{O}(dL_Y)|_{Y_z})$$

is an upper semicontinuous function on Z. Thus

$$\{z \in Z | h^0(\mathcal{O}(dL_Y|_{Y_z})) \geq 1\}$$

is closed in Z. Since $\mathcal{O}_Y(dL_Y)$ is trivial on a general fiber and numerically trivial on all fibers, this shows that $\mathcal{O}_Y(dL_Y|_{Y_z}) \cong \mathcal{O}_{Y_z}$ for all $z \in Z$. Now there exists a finite morphism $\theta : Z^{'} \to Z$, from a smooth projective variety $Z^{'}$ such that letting $Y^{'}$ denote a desingularization of the main component of $Y \times_Z Z^{'}$, the induced morphism $f^{'} : Y^{'} \to Z^{'}$ is semistable in codimension 1. Since $\mathcal{O}_{Y'}(dL_{Y'})$ is trivial on all fibers of $f^{'}$ and $f^{'}$ is flat with reduced connected fibers over a large open set, there exists $P_{Z'} \in \operatorname{Pic}(Z^{'})$ nef such that $f^{'*}(P_{Z'}) = \mathcal{O}_{Y'}(dL_{Y'})$ (same arguments as [31, Chapter 3, Exercise 12.4]). Let $\alpha : Z^{''} \to Z^{'}$ be a Bloch-Gieseker cover with $Z^{''}$ smooth such that $\alpha^*(P_{Z'}) = L_{Z''}^{\otimes d}$ for some $L_{Z''} \in \operatorname{Pic}(Z^{''})$ (see [38, Theorem 4.1.10]). Then $L_{Y''} = f^{''*}(L_{Z''})$ where $f^{''} : Y^{''} \to Z^{''}$ is obtained by base changing f.

We now argue along the lines of [1, Lemma 5.2(5)] to show that $\mathbf{M}_Z$ is b-nef. Let $b \in \mathbb{N}$ smallest such that $b(K_{Y''} + B_{Y''} + L_{Y''}) \sim_{f''} 0$. There exists $\phi \in K(Y^{''})^*$

such that $K_{Y''} + B_{Y''} + L_{Y''} + 1/b(\phi) = f''^*(K_{Z''} + B_{Z''} + \mathbf{M}_{Z''})$. Let V'' be a desingularization of the normalization of Y'' in $K(Y'')(\phi^{1/b})$. Note that V'' is irreducible by the minimality of b. Let

$$\pi'' : (V'', B_{V''}) \xrightarrow{\nu''} (Y'', B_{Y''}) \xrightarrow{f''} Z''$$

be the induced set-up.

Let $\Sigma_{f''} \subset Z''$ be the maximal closed subvariety such that f'' is smooth over $Z'' \setminus \Sigma_{f''}$. Replacing Z'' by a log-resolution, we can assume that $\Sigma_{f''}$ has SNC support. Also by [1, Proposition 5.4], there exists $\tau : \overline{Z'} \to Z''$ finite such that $\overline{Z'}$ admits an SNC divisor supporting $\tau^{-1}(\Sigma_{f''})$ and the locus where τ is not etale such that the induced morphism $(\overline{V'}, B_{\overline{V'}}) \xrightarrow{\overline{\pi'}} \overline{Z'}$ is semistable in codimension 1. Then $\overline{\pi'}_*(\omega_{\overline{V'}/\overline{Z'}})$ is nef ([1, Theorem 4.4]). If $\mathbf{M}_{\overline{Z'}}$ is the induced moduli b-divisor on $\overline{Z'}$, then $\tau^*\mathbf{M}_{Z''} = \mathbf{M}_{\overline{Z'}}$ ([1, Proposition 5.5]) and $(\theta \circ \alpha)^*(\mathbf{M}_{Z''}) = \mathbf{M}_Z$ ([1, Lemma 5.1]). So it suffices to show that $\mathbf{M}_{\overline{Z'}}$ is nef. The category of generalized klt-trivial fibrations is closed under generically finite base changes (Lemma 5). Thus $\overline{f'} : (\overline{Y'}, B_{\overline{Y'}}) \to \overline{Z'}$ is again generalized klt-trivial.

To simplify notation, we thus replace $\overline{V'}, \overline{Y'}, \overline{Z'}, \overline{\nu'}, \overline{\pi'}, \overline{f'}$ by V, Y, Z, ν, π, f respectively.

Then $\pi_*(\omega_{V/Z})$

$$= f_*(\bigoplus_{i=0}^{b-1} \mathcal{O}_Y(K_Y + \lceil i/b(\phi)\rceil) \otimes f^*\omega_Z^{-1}) \otimes f_*(\bigoplus_{i=0}^{b-1} \mathcal{O}_Y(\lfloor -i/b(\phi)\rfloor))$$

([52, Lemma 2.3]). We have

$$f_*(\bigoplus_{i=0}^{b-1} \mathcal{O}_Y(\lfloor -i/b(\phi)\rfloor)) = \mathcal{O}_Z$$

and

$$i/b(\phi) = i(-K_{Y/Z} - B_Y - L_Y + f^*(B_Z + \mathbf{M}_Z)).$$

from which it follows that

$$\pi_*(\omega_{V/Z})$$

$$= \bigoplus_{i=0}^{b-1} f_*\mathcal{O}_Y(\lceil (1-i)K_{Y/Z} - iB_Y - iL_Y + if^*(B_Z + \mathbf{M}_Z)\rceil)$$

where the i-th summand on the right hand side is the eigensheaf corresponding to the eigenvalue ζ^i, where ζ is a primitive b-th root of unity, the action being that of $\mathbb{Z}/(b)$ on $K(V)$ and ψ is a rational function on V such that $\psi^b = \phi$ (see [20, page 23-24] for more details). By [1, Theorem 4.4], $\pi_*(\omega_{V/Z})$ is a nef vector bundle and hence so are the direct summands. Then

$$K_V + B_V + (\psi) = \pi^*(K_Z + B_Z + \mathbf{M}_Z - L_Z). \tag{4.18}$$

Let

$$\mathcal{L} = f_*\mathcal{O}_Y(\lceil -B_Y - L_Y + f^*(B_Z + \mathbf{M}_Z)\rceil)$$

be the ζ-eigensheaf. Then $\mathcal{L} \in \operatorname{Pic} Z$ (it has rank 1 since rank $f_*\mathcal{O}_Y(\lceil -B_Y\rceil) = 1$ and L_Y is trivial on all fibers).

Claim 4.19. There exists $U \subset Z$ large open such that $(-B_V + \pi^*B_Z)|_{\pi^{-1}(U)} \geq 0$.

Proof. By (4.18), we see that $(K_{V/Z})^h + B_V^h + (\psi)^h = 0$ (^{h} denotes the horizontal part) and hence that B_V^h is a $\mathbb{Z}$-divisor. (V, B_V) being klt, $\lceil -B_V^h \rceil = -B_V^h \geq 0$. Let $B_V^v = \Sigma_j d_j P_j$ (^{v} denotes the vertical part). We can ignore those P_j such that codim $(\pi(P_j)) \geq 2$ and consider only those that map to some prime component of B_Z. Recall that $B_Z = \Sigma_l \delta_l Q_l$ where

$$1 - \delta_l = sup\{t | (Y, B_Y + t\pi^*(Q_l)) \text{ is sub-lc over the generic point of } Q_l\}.$$

which translates to $\Sigma_j d_j P_j + \pi^*(\Sigma_l (1-\delta_l) Q_l) \leq 1$. Fix l and consider those P_j which map to Q_l. Since π is semistable in codimension 1, this leads to $max_{P_j \to Q_l}(d_j) + 1 - \delta_l \leq 1$ with equality holding for some j. Thus $-d_j + \delta_l \geq 0$ for all j such that $\pi(P_j) = Q_l$. This proves the claim. $\square$

Also note the above proof shows $(-B_V + \pi^* B_Z)$ supports no fibers over codimension 1 points of Z. Let $H \subset V$ be a general fiber of π. Then $(\psi|_H) + K_H = -B_H \geq 0$ (Note everything is integral here). Thus ψ is a rational section of $\pi_*(\omega_{V/Z})$. Since $\psi \to \zeta \cdot \psi$ we conclude that ψ is a rational section of $\mathcal{L}$. Let $\mathcal{K}(Z)$ be the (constant) sheaf of rational functions on Z. Considering the injection $\mathcal{L} \hookrightarrow \mathcal{K}(Z)$ that sends a rational section e of $\mathcal{L}$ to e/ψ shows that $\mathcal{L} \subset \mathcal{K}(Z) \cdot \psi$.

The above claim and (4.18) show that there exists $U \subset Z$ large open such that $\mathcal{O}_Z(\mathbf{M}_Z - L_Z) \cdot \psi|_U \subset \pi_*(\omega_{V/Z})|_U$ and thus $\mathcal{O}_Z(\mathbf{M}_Z - L_Z) \cdot \psi|_U \subset \mathcal{L}|_U$. Conversely, let $a \cdot \psi$ be a section of $\mathcal{L}$, $a \in K(Z)$. Then it is a section of $\pi_*(\omega_{V/Z})$, so $(\pi^*(a) \cdot$

$\psi) + K_{V/Z} \geq 0$. Since

$$(\pi^*(a) \cdot \psi) + K_{V/Z} = \pi^*((a) + \mathbf{M}_Z - L_Z) + (-B_V + \pi^* B_Z) \tag{4.20}$$

we have $\pi^*((a) + \mathbf{M}_Z - L_Z) + (-B_V + \pi^* B_Z) \geq 0$. Since $-B_V + \pi^* B_Z$ contains no fibers over codimension 1 points of Z, this implies that $(a) + \mathbf{M}_Z - L_Z \geq 0$ and hence that $\mathcal{L} \subset \mathcal{O}_Z(\mathbf{M}_Z - L_Z) \cdot \psi$. We therefore conclude that

$$\mathcal{L}|_U = \mathcal{O}_Z(\mathbf{M}_Z - L_Z) \cdot \psi|_U.$$

Since U is large and $\mathcal{L}$ is a line bundle, $\mathcal{L} = \mathcal{O}_Z(\mathbf{M}_Z - L_Z) \cdot \psi$. Since $\mathcal{L}$ and L_Z are nef, we conclude that $\mathbf{M}_Z$ is nef.

□

Corollary 4.21. *Let (X, B) be a klt pair with $\kappa(K_X + B) \geq 0$, H and L nef $\mathbb{Q}$-Cartier divisors on X such that $H - (K_X + B + L)$ is nef and abundant. Assume further that $\kappa(aH - (K_X + B + L)) \geq 0$ and $\nu(aH - (K_X + B + L)) = \nu(H - (K_X + B + L))$ for some $a \in \mathbb{Q}_{>1}$. Then H is semiample.*

Proof. Let $(X, B + L) \xleftarrow{\mu} (Y, B_Y + L_Y) \xrightarrow{f} Z$ be the standard diagram as before. Then rank $f_*\mathcal{O}_Y(\lceil -B_Y \rceil) = 1$ by Lemma 4.12(2). Now the conclusion follows from Lemma 4.12(1) and the above theorem.

□

Chapter 5

Inductive approach to generalized abundance using nef reduction

Recall the following conjecture of [40]:

Conjecture 5.1. *(Generalized Abundance) Let (X, B) be a klt pair with $K_X + B$ pseudoeffective. If L is a nef divisor on X such that $K_X + B + L$ is also nef, then $K_X + B + L \equiv M$ for some semiample $\mathbb{Q}$-divisor M.*

Lazić and Peternell [40] show that this conjecture holds in case $\dim X = 2$ and for $\dim X = 3$ if $\kappa(K_X + B) > 0$ (see [40, Corollary C,D]). The main purpose of this chapter is to prove the following result. It gives a higher dimensional version of their result.

Theorem 5.2. *Let (X, B) be an n-dimensional klt pair with $K_X + B \geq 0$ and let $L \in \operatorname{Pic} X$ be nef such that K_X+B+L is nef with nef dimension $n(K_X+B+L) = d$.*

Assume termination of klt flips [35, Conjecture 5-1-13] in dimension d, abundance conjecture [36, Conjecture 3.12] in dimension $\leq d$, semiampleness conjecture [40, page 2] in dimension $\leq d$ and generalized non-vanishing conjecture [40, page 2] in dimension $\leq d-1$.

Then $K_X + B + L \equiv M$ for some semiample divisor M.

Note that the phrase "Conjecture $(*)$ holds in dimension d (resp. $\leq d$)" above means that $(*)$ holds for all klt pairs (X, B) such that $\dim X = d$ (resp. $\leq d$).

In case $L = 0$, such a theorem has been proved by [1]. The main ingredients of our proof are the canonical bundle formula for parabolic fibrations [26] and Nakayama-Zariski decomposition [46]. We have the following corollary

Corollary 5.3. *Generalized abundance holds (in any dimension) in the following two cases:*

1. *$n(K_X + B + L) = 2$ and $K_X + B \geq 0$, or*

2. *$n(K_X + B + L) = 3$ and $\kappa(K_X + B) > 0$.*

Note that any divisor of nef dimension 1 is already numerically equivalent to a semiample divisor by [5, 2.4.4]. Thus generalized abundance also holds in this case.

5.1 Preliminaries

Definition 5.1.1 (Singularities of pairs)**.** Let (X, B) be a sub-pair consisting of a normal variety X and a $\mathbb{Q}$-divisor B on X such that $K_X + B$ is $\mathbb{Q}$-Cartier. It is called *sub-klt* if there exists a log resolution $Y \xrightarrow{\mu} X$ of (X, B) such that letting B_Y be defined by $K_Y + B_Y = \mu^*(K_X + B)$, all coefficients of B_Y are < 1 and *sub-lc* if all coefficients of B_Y are ≤ 1. If $B \geq 0$, we drop the prefix sub.

Definition 5.1.2 (Nef reduction and nef dimension [5])**.** Let X be a normal projective variety and let $L \in \operatorname{Pic} X$ be nef. Then there exists a dominant rational map $\phi : X \dashrightarrow Y$ with connected fibers which is proper and regular over an open subset of Y (i.e. there exists $V \subset Y$ nonempty open such ϕ restricts to a proper morphism $\phi|_{\phi^{-1}(V)} : \phi^{-1}(V) \to V$) where Y is also normal projective such that

1. If $F \subset X$ is a general compact fiber of ϕ with $\dim F = \dim X - \dim Y$, then $L|_F \equiv 0$.

2. If $x \in X$ is a very general point and $C \subset X$ a curve passing x such that $\dim(\phi(C)) > 0$, then $(L \cdot C) > 0$.

ϕ is called the **nef reduction map** of L and $\dim Y$ the **nef dimension** $n(L)$ of L.

Remark 5.4. Note that if $\phi : X \dashrightarrow Y$ is proper and regular over an open subset of Y, then there exists a resolution $\hat{\phi} : \hat{X} \to Y$ of ϕ such that the exceptional divisor

$Ex(\hat{X}/X)$ is $\hat{\phi}$-vertical. For example, $\hat{X}$ can be chosen to be the closure of the graph of ϕ.

Notation 5.5. *For a pseudoeffective divisor D on a smooth projective variety X, $P_\sigma(D)$ and $N_\sigma(D)$ will denote the positive and negative parts of the Nakayama-Zariski decomposition of D (see [46, Chapter 3] for details). $\kappa(D)$ will denote the Iitaka dimension and $\nu(D)$ the numerical dimension (see [28, Definition 2.10]) of D.*

5.2 Inductive approach to generalized abundance

Proof of Theorem 5.2. We will follow some ideas of [1, Theorem 5.1] and [28, Lemma 4.4]. Let Φ be the nef reduction map of $K_X + B + L$. Let $\hat{X}$ be the normalization of the closure of the graph of Φ. We have an induced commutative diagram:

$$\begin{tikzcd} \hat{X} \arrow[d, "\beta"] \arrow[dr, "\phi"] & \\ X \arrow[r, dashed, "\Phi"] & Y \end{tikzcd}$$

Note that since Φ is proper and regular over an open subset of Y, $Ex(\beta)$ is ϕ-vertical. We will make base changes that preserve this property. Let $F \subset \hat{X}$ be a general fiber of ϕ. Then

$$(K_{\hat{X}} + B_{\hat{X}} + L_{\hat{X}})|_F \equiv 0. \tag{5.6}$$

Here $L_{\hat{X}} := \beta^* L$ and $B_{\hat{X}}$ is defined by $K_{\hat{X}} + B_{\hat{X}} = \beta^*(K_X + B)$. Now since $K_{\hat{X}} + B_{\hat{X}} \geq 0$, we have $K_F + B_F \geq 0$ which implies $K_F + B_F \sim_{\mathbb{Q}} 0$. Indeed, suppose that $K_F + B_F > 0$. Then there exists a curve $C \subset F$ such that $((K_F + B_F) \cdot C) > 0$.

Then $((K_F + B_F + L_F) \cdot C) > 0$. This contradicts (5.6). We also conclude that $L_{\hat{X}}|_F \equiv 0$.

By [40, Lemma 3.1], there exists $\sigma : Y' \to Y$ birational from a smooth projective variety Y' such that letting X' denote the normalization of the main component of $\hat{X} \times_Y Y' \to Y'$ and $\phi' : X' \to Y'$ the induced morphism, there exists a nef $\mathbb{Q}$-Cartier divisor $L_{Y'}$ on Y' such that $L_{X'} \equiv \phi'^*(L_{Y'})$. Letting F' denote a general fiber of ϕ', note that $(B^-_{X'})|_{F'} \sim 0$. Thus

$$(K_{X'} + B_{X'})|_{F'} \sim (K_{X'} + B^+_{X'})|_{F'} \sim 0.$$

Then by [26, Section 4], there exists a birational morphism $\widetilde{Y} \to Y'$, $\widetilde{X}$ birational to the main component of $X' \times_{Y'} \widetilde{Y}$ where $\widetilde{X}$ and $\widetilde{Y}$ are both smooth projective such that letting $\widetilde{\phi} : \widetilde{X} \to \widetilde{Y}$ denote the induced morphism, there exists a $\mathbb{Q}$-divisor Δ on $\widetilde{X}$ such that $\widetilde{\phi} : (\widetilde{X}, B^+_{\widetilde{X}} + \Delta) \to \widetilde{Y}$ is a klt trivial fibration ([23, Definition 2.1]) where Δ^+ is exceptional over $\widetilde{Y}$ and X' and $\widetilde{\phi}_* \mathcal{O}_{\widetilde{X}}(\lfloor l\Delta^- \rfloor) \cong \mathcal{O}_{\widetilde{Y}}$ for all $l \in \mathbb{N}$.

$$\begin{tikzcd}
\widetilde{X} \arrow[r, "\pi"] \arrow[d, "\widetilde{\phi}"] & X' \arrow[r] \arrow[d, "\phi'"] & \hat{X} \arrow[r, "\beta"] \arrow[d, "\phi"] & X \arrow[dl, dotted] \\
\widetilde{Y} \arrow[r] & Y' \arrow[r] & Y &
\end{tikzcd}$$

By [28, Lemma 2.15], the last condition implies that the vertical part $(\Delta^-)^v$ is $\widetilde{\phi}$-degenerate (see [28, Definition 2.14]). Letting $B_{\widetilde{Y}}$ and $M_{\widetilde{Y}}$ denote the induced discriminant and moduli divisors on $\widetilde{Y}$, we have

$$K_{\widetilde{X}} + B^+_{\widetilde{X}} + \Delta^+ - \Delta^- \sim_{\mathbb{Q}} \widetilde{\phi}^*(K_{\widetilde{Y}} + B_{\widetilde{Y}} + M_{\widetilde{Y}}).$$

Since $(\Delta^-)^v$ is $\widetilde{\phi}$-degenerate, it follows from the definition of $B_{\widetilde{Y}}$ that $B_{\widetilde{Y}} \geq 0$. Since $K_{\widetilde{X}} + B^+_{\widetilde{X}} = \pi^*(K_{X'} + B^+_{X'}) + E$, where $E \geq 0$ is π-exceptional, if $\widetilde{F} \subset \widetilde{X}$ is a general fiber of $\widetilde{\phi}$, $(K_{\widetilde{X}} + B^+_{\widetilde{X}})|_{\widetilde{F}} = E|_{\widetilde{F}}$. Note that $E|_{\widetilde{F}}$ is an exceptional divisor for the induced birational morphism $\pi|_{\widetilde{F}} : \widetilde{F} \to F'$. So $\kappa((K_{\widetilde{X}} + B^+_{\widetilde{X}})|_{\widetilde{F}}) = \nu((K_{\widetilde{X}} + B^+_{\widetilde{X}})|_{\widetilde{F}}) = 0$. Thus we can run a relative MMP with ample scaling over $\widetilde{Y}$ to get

$$\begin{tikzcd} \widetilde{X} \arrow[r, dotted, "\psi"] \arrow[d, "\widetilde{\phi}"] & \widetilde{X}_m \arrow[dl, "\widetilde{\phi}_m"] \\ \widetilde{Y} & \end{tikzcd}$$

such that letting $K_{\widetilde{X}_m} + B^+_{\widetilde{X}_m} = \psi_*(K_{\widetilde{X}} + B^+_{\widetilde{X}})$ and $\Delta_{\widetilde{X}_m} := \psi_*\Delta$, we have $(K_{\widetilde{X}_m} + B^+_{\widetilde{X}_m})|_{\widetilde{F}_m} \sim_{\mathbb{Q}} 0$, $\widetilde{F}_m$ being the general fiber of $\widetilde{\phi}_m$ (see the first paragraph of the proof of [28, Lemma 4.4] for details). Consider the induced klt-trivial fibration

$$\widetilde{\phi}_m : (\widetilde{X}_m, B_{\widetilde{X}_m} + \Delta^+_{\widetilde{X}_m} - \Delta^-_{\widetilde{X}_m}) \to \widetilde{Y}.$$

Note: the fact that $\widetilde{\phi}_m$ is klt-trivial can be shown by choosing a smooth resolution of indeterminacy $\widetilde{X} \xleftarrow{p} W \xrightarrow{q} \widetilde{X}_m$ of ψ and using the fact that $p^*(K_{\widetilde{X}} + B^+_{\widetilde{X}}) = q^*(K_{\widetilde{X}_m} + B^+_{\widetilde{X}_m}) + E$ where $E \geq 0$ is q-exceptional. By [1, Lemma 2.6], $\widetilde{\phi}_m$ and $\widetilde{\phi}$ induce the same discriminant and moduli divisors on $\widetilde{Y}$. In the case of $\widetilde{\phi}_m$, $(\Delta^-_{\widetilde{X}_m})|_{\widetilde{F}_m} \sim_{\mathbb{Q}} 0$, thus by [2, Theorem 3.3], $M_{\widetilde{Y}}$ is b-nef and b-good. Hence, by the arguments in the proof of [2, Theorem 4.1], we can write

$$K_{\widetilde{X}} + B^+_{\widetilde{X}} + \Delta^+ - \Delta^- \sim_{\mathbb{Q}} \widetilde{\phi}^*(K_{\widetilde{Y}} + \Delta_{\widetilde{Y}}).$$

where $(\widetilde{Y}, \Delta_{\widetilde{Y}})$ is klt. We write this as

$$K_{\widetilde{X}} + B^+_{\widetilde{X}} + \Delta^+ - (\Delta^-)^h \sim_{\mathbb{Q}} \widetilde{\phi}^*(K_{\widetilde{Y}} + \Delta_{\widetilde{Y}}) + (\Delta^-)^v \tag{5.7}$$

where ^{h} and ^{v} denote the horizontal and vertical parts with respect to $\widetilde{\phi}$ respectively. As observed above, $(K_{\widetilde{X}} + B^+_{\widetilde{X}})|_{\widetilde{F}}$ is $\pi|_{\widetilde{F}}$-exceptional and $(K_{\widetilde{X}} + B^+_{\widetilde{X}})|_{\widetilde{F}} \sim_{\mathbb{Q}} (\Delta^-)^h|_{\widetilde{F}}$. Thus $(\Delta^-)^h|_{\widetilde{F}} = N_\sigma((K_{\widetilde{X}} + B^+_{\widetilde{X}})|_{\widetilde{F}})(= (K_{\widetilde{X}} + B^+_{\widetilde{X}})|_{\widetilde{F}})$. Now it follows from the definition of N_σ that

$$N_\sigma((K_{\widetilde{X}} + B^+_{\widetilde{X}})|_{\widetilde{F}}) \leq N_\sigma(K_{\widetilde{X}} + B^+_{\widetilde{X}})|_{\widetilde{F}}. \tag{5.8}$$

Since $\widetilde{F}$ is a general fiber and $(\Delta^-)^h$ has no vertical components, it follows that

$$(\Delta^-)^h \leq N_\sigma(K_{\widetilde{X}} + B^+_{\widetilde{X}}) \tag{5.9}$$

Since $P_\sigma(K_{\widetilde{X}} + B^+_{\widetilde{X}})$ is effective, it follows that $K_{\widetilde{X}} + B^+_{\widetilde{X}} - (\Delta^-)^h \geq 0$. By [46, Lemma 1.8], (5.9) implies that $N_\sigma(K_{\widetilde{X}} + B^+_{\widetilde{X}} - (\Delta)^h) = N_\sigma(K_{\widetilde{X}} + B^+_{\widetilde{X}}) - (\Delta^-)^h$ and thus $P_\sigma(K_{\widetilde{X}} + B^+_{\widetilde{X}} - (\Delta^-)^h) = P_\sigma(K_{\widetilde{X}} + B^+_{\widetilde{X}})$. Now we take P_σ in (5.7). Since $\kappa(D) = \kappa(P_\sigma(D))$ for any pseudoeffective divisor D ([28, Lemma 2.9]), the fact that Δ^+ and $B^-_{\widetilde{X}}$ are exceptional over X, the $\widetilde{\phi}$-degeneracy of $(\Delta^-)^v$ and [28, Lemma 2.16] imply that

$$\kappa(K_X + B) = \kappa(K_{\widetilde{Y}} + \Delta_{\widetilde{Y}}) \tag{5.10}$$

Let $\mu : \widetilde{X} \to X$ be the induced birational morphism. Then (5.7) gives

$$K_{\widetilde{X}} + B^+_{\widetilde{X}} + L_{\widetilde{X}} + \Delta^+ - (\Delta^-)^h \equiv \widetilde{\phi}^*(K_{\widetilde{Y}} + \Delta_{\widetilde{Y}} + L_{\widetilde{Y}}) + (\Delta^-)^v \tag{5.11}$$

where Δ^+ is exceptional over X' and $\widetilde{Y}$ and $(\Delta^-)^v$ is $\widetilde{\phi}$-degenerate. Since $L_{\widetilde{X}}|_{\widetilde{F}} \equiv 0$, as observed above, we have

$$N_\sigma((K_{\widetilde{X}} + B^+_{\widetilde{X}} + L_{\widetilde{X}})|_{\widetilde{F}}) = N_\sigma((K_{\widetilde{X}} + B^+_{\widetilde{X}})|_{\widetilde{F}}) \geq (\Delta^-)^h|_{\widetilde{F}} \text{ implying}$$

$$(\Delta^-)^h \leq N_\sigma((K_{\widetilde{X}} + B^+_{\widetilde{X}} + L_{\widetilde{X}})$$

and exactly as before, we conclude that

$$P_\sigma(K_{\widetilde{X}} + B^+_{\widetilde{X}} + L_{\widetilde{X}} + \Delta^+ - (\Delta^-)^h) = P_\sigma(K_{\widetilde{X}} + B^+_{\widetilde{X}} + L_{\widetilde{X}} + \Delta^+).$$

But the latter equals

$$P_\sigma(\mu^*(K_X + B + L) + B^-_{\widetilde{X}} + \Delta^+) = P_\sigma(\mu^*(K_X + B + L)) = \mu^*(K_X + B + L)$$

since Δ^+ and $B^-_{\widetilde{X}}$ are exceptional over X and $K_X + B + L$ is nef. Now taking P_σ in (5.11) and using [28, Lemma 2.16], we get

$$\mu^*(K_X + B + L) = P_\sigma(\widetilde{\phi}^*(K_{\widetilde{Y}} + \Delta_{\widetilde{Y}} + L_{\widetilde{Y}})).$$

By [40, Theorem 5.3], there exists a smooth projective $\overline{Y}$ and $w : \overline{Y} \to \widetilde{Y}$ birational such that $P_\sigma(w^*(K_{\widetilde{Y}} + \Delta_{\widetilde{Y}} + L_{\widetilde{Y}}))$ is numerically equivalent to a semiample divisor. Letting $\overline{X}$ denote a desingularization of the main component of $\widetilde{X} \times_{\widetilde{Y}} \overline{Y}$ and v : $\overline{X} \to \widetilde{X}$, $\overline{\phi} : \overline{X} \to \overline{Y}$ the induced morphisms, we have

$$\overline{\phi}^* P_\sigma(w^*(K_{\widetilde{Y}} + \Delta_{\widetilde{Y}} + L_{\widetilde{Y}})) = P_\sigma(\overline{\phi}^* w^*(K_{\widetilde{Y}} + \Delta_{\widetilde{Y}} + L_{\widetilde{Y}})) \text{ (since}$$

$$P_\sigma(w^*(K_{\widetilde{Y}} + \Delta_{\widetilde{Y}} + L_{\overline{Y}})) \text{ is nef; see [40, Lemma 2.5])}$$

$$= P_\sigma(v^*\widetilde{\phi}^*(K_{\widetilde{Y}} + \Delta_{\widetilde{Y}} + L_{\widetilde{Y}})) = v^* P_\sigma(\widetilde{\phi}^*(K_{\widetilde{Y}} + \Delta_{\widetilde{Y}} + L_{\widetilde{Y}})) \text{ (since}$$

$$P_\sigma(\widetilde{\phi}^*(K_{\widetilde{Y}} + \Delta_{\widetilde{Y}} + L_{\widetilde{Y}})) \text{ is nef)} = v^*\mu^*(K_X + B + L)\ (*)$$

is numerically equivalent to a semiample divisor. Thus $K_X + B + L$ is numerically equivalent to a semiample divisor.

□

Remark 5.12. The above result can also be proved by using the MMP instead of the canonical bundle formula. See for example the proof of [41, Theorem 3.5].

Proof of Corollary 5.3.

1. If $n(K_X+B+L) = 2$, then $\dim \widetilde{Y} = 2$ and $K_{\widetilde{Y}}+\Delta_{\widetilde{Y}} \geq 0$ by (5.10). Termination of flips for threefold pairs and abundance for surface and threefold pairs are classical. Semiampleness and generalized non-vanishing conjectures hold for surface pairs by [40, Corollary C, page 4]. Thus the case $n(K_X + B + L) = 2$ i.e. when $\dim \widetilde{Y} = 2$ is immediate.

2. If $n(K_X + B + L) = 3$, then $\dim \widetilde{Y} = 3$ and by (5.10), $\kappa(K_{\widetilde{Y}} + \Delta_{\widetilde{Y}}) > 0$. Then by [40, Remark 5.4], $P_\sigma(w^*(K_{\widetilde{Y}} + \Delta_{\widetilde{Y}} + L_{\widetilde{Y}}))$ is numerically equivalent to a semiample divisor for some birational morphism $w : \overline{Y} \to \widetilde{Y}$ from a smooth projective variety $\overline{Y}$. Then so is $K_X + B + L$ by $(*)$ above.

□

In case $K_X + B$ is only pseudoeffective, essentially the same arguments as above can be used to prove the following statement:

Theorem 5.13. *Let (X, B) be a klt pair of dimension d with K_X+B pseudoeffective and let $L \in \operatorname{Pic} X$ be nef such that $K_X + B + L$ is also nef with $n(K_X + B + L) < d$. Assume termination of klt flips [35, Conjecture 5-1-13] in dimension $< d$, abundance* *conjecture [36, Conjecture 3.12] in dimension* $< d$*, semiampleness conjecture [40, page 2] in dimension* $< d$ *and generalized non-vanishing conjecture [40, page 2] in dimension* $< d$*.*

Then $K_X + B + L \equiv M$ *for some semiample divisor* M*.*

Proof. (Sketch) The idea is to consider the nef reduction map of $K_X + B + mL$ for some $m > 2d$. By [40, Lemma 2.11], we have $n(K_X + B + mL) < d$. By [40, Lemma 2.8], on the general fiber of the nef reduction map, $K_X + B$ is numerically trivial, hence linearly trivial by [4, Theorem 0.1]. L is also numerically trivial on the general fiber and we can argue as above.

□

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