ABSTRACT

Title of dissertation: ESSAYS ON AUCTION THEORY AND APPLICATION

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This dissertation contributes to auction theory with application of the theory to the analysis of some real-life problem.

In Chapter 1, I study the problem of competition between contest designers where they offer differentiated prizes to a group of contestants with some minimal effort requirements. The equilibrium among contestants is either a separating equilibrium, where strong contestants participating in high-prize contest and weak contestants in low-prize contest, or a mixing equilibrium, where strong players participate in high-prize contest with probability 1, middle-type players randomize between the two contests, and weak players go to low-prize contest with certainty. I then solve an equilibrium of contest designers where one designer's choice of minimal effort level is assumed to be non-strategic. Finally, I provide conditions such that the assumed non-strategic choice of minimal effort level is optimal and thus characterize at least part of the equilibrium set, which expands the knowledge on competing auctions.

In Chapter 2, I apply auction theory to analyze the effect of a merger on firms' research and development (R&D) investment. There is a substantial literature on the effects of mergers on product prices, but the effects of mergers on other outcomes, such as R&D investment spending, are less studied. I develop a model for evaluating the likely effects of a merger (or joint research venture) on the R&D efforts of competing firms. The R&D process is modeled as an all-pay contest (auction) among firms, with the payoff from investment going to the firm that invests the largest amount. I provide an explicit characterization of the equilibrium in a multi-player asymmetric all-pay contest model. The equilibrium solution then is applied through simulation to calibrate the effects of mergers on firms' R&D efforts and efficiency as well as on social welfare. I find that each firm is expected to exert more efforts after a merger, but if there are only few firms premerger, a merger reduces total R&D effort. A merger may also cause inefficiency, but the loss in efficiency is low. My results also show that net surplus increases after a merger if the number of firms is small.

In Chapter 3, I study a problem of sequential auctions and extend the standard model of sequential second-price auctions to a dynamic game with an infinite horizon with one new buyer entering the auction every period. I first derive properties of the symmetric and stationary equilibrium, where buyers bid according to their private valuation less a pivotal continuation value, and I also show that the price path in such equilibrium is weakly decreasing. Imposing preconsistent beliefs, I give the conditions under which a stationary equilibrium exists.

ESSAYS ON AUCTION THEORY AND APPLICATION

by

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Dissertation submitted to the Faculty of the Graduate School of the
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Dedication

To my wife, Dr. Yunyao Li, my son, Timothy, and my parents.

Acknowledgments

Looking back the five years in the graduate school, I consider myself extremely lucky. I got frustrated for many times, but still manage to present something at last. At the juncture of entering a new stage of my life, I would like to express my gratitude to all the people who have offered help in the process of this thesis; thank you so much for making this thesis possible.

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My family members deserve my deepest thanks. Special thanks to my wife, Dr. Yunyao Li, who, busy on her post-doc researches, also takes care of every aspect of the family during my job-seeking and thesis-writing seasons. Days would be unimaginable without you. Little Timothy, especially your laughter, is another source of encouragement.

Last but not least, I would like to give thanks to all the church members around me. You are the God-given rod and staff with whom I pass through valleys of life. My heart is so comforted by your prayers and support. Finally, thanks be to the Lord, the real Comforter, the One who lifts up my head.

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Chapter 1: Simultaneous Contests

1.1 Introduction

Contests are situations in which agents spend resources in order to win one or more prizes. A leading feature is that, regardless of winning or not, all contestants bear some costs. Most of the literature on contest has considered the case where agents compete for a unique prize, such as Tullock (1980) [13], Varian (1980) [14], Dasgupta (1986) [4] and many others. But the prevalence of contests with multiple prizes is obvious. Employees spend effort in order to be promoted in organizational hierarchies, which often consist of several types of well-defined positions; athletes compete for gold, silver, and bronze medals, or for monetary prizes; young musicians compete for the first, second, and third prizes; students compete for grades in tests and courses.

Several papers study contest models with multiple prizes under complete information setting. Broecker's (1990) [2] model of credit markets has several features of an all-pay auction with as many prizes as contestants. Wilson (1979) [15] and Anton and Yao (1992) [1] study split-award auctions where several bidders can win. More recently, Moldovanu and Sela (2001, 2006) [9, 10] study multi-prize contests with incomplete information. They show that the winner-takes-all structure (i.e.

only one big prize) in which the highest-effort contestant wins the prize is usually the optimal architecture for maximizing expected total effort.

However, these papers assume that all contestants are not constrained from participating in a contest. In real-life contests, contestants usually face some prerequisites for participation. For example, researchers at universities are required to achieve a minimal quality and quantity of output in order to be promoted. Similarly, entry in professional sport competitions is often restricted, whereby only contestants who have achieved a certain predefined minimal requirement are allowed to compete. Such constraints can have significant effect on the optimal allocation of prizes. Megidish and Sela (2013) [8] show that if the exogenous minimal effort constraint is sufficiently high, the expected total effort in a random contest is higher than in a winner-take-all contest.

Although a grand contest where all contestants participate is efficient in a private-value setting, it might not be always welcome. There are also other practices where prizes are offered in multiple contests, which I refer as simultaneous contests[1]. For instance, authorities may want to encourage weak competitors and give them the chance to win some of the prizes. This type of mechanism commonly appears in sport competitions where strong teams compete in a high league while weak teams compete in a lower one. If we limit the players to participate only in one contest, then the intuition about the bidders' behavior will be ambiguous. A high ability (type) contestant may believe he should participate in the low prize contest since as

[1] Contests need not be held at the same time. As long as contestants are allowed to participate in only one of the contests, they can be considered as held at the same time.

a strong player, the probability of winning is high. On the other hand, a low ability contestant may take a chance and participate in the high prize contest hoping to be the sole participant and win the high prize. It is interesting to study the choice of contest in addition to the choice of effort level.

The literature on competing auctions has treated this kind of problems. McAfee (1993) [7] and Peters (1997) [12] consider cases of a limiting number of sellers, while Burguet and Sákovics (1999) [3] and Hernando-Veciana (2005) [6] investigate the strategic interactions among a finite number of sellers. In these models, sellers offer mechanisms or reserve prices and buyers choose among the auction sites. Gavious (2009) [5] works on a related but different scenario where one auctioneer sells two differentiated objects in two second-price auctions. Gavious' work is a benchmark for my analysis and I provide a richer characterization of contestants' equilibrium in this chapter. In addition, this chapter is one of the few studies of competing auctions that provide the equilibrium between differentiated auctioneers (or contest designers in the context of this chapter).

This chapter is organized as follows. Section 2 introduces the setting of the model. Section 3 studies the equilibrium among contestants given the minimal effort requirements for participation. Section 4 studies the optimal choice of minimal effort requirements by contest designers, who expect that contestants behave according to the equilibrium in section 3. A brief conclusion is presented in section 5. Proofs can be found in the appendix.

1.2 The Model

Consider two contests offered by L and H (hereafter contest L and H) with $n \geq 3$ contestants. The prize of contest L is normalized to 1, and the prize of contest H is assumed to be $a > 1$. The prize of a contest is awarded to the one who makes the highest effort in that contest. Assume that if contestant i makes an effort x_i, he bears a cost of $\frac{x_i}{c_i}$, where c_i is the ability (or type) of contestant i. Assume that abilities are drawn independently of each other from an interval $[\underline{c}, \bar{c}]$ ($\bar{c} > \underline{c} \geq 0$[2]) according to a distribution function F with a continuous density $f > 0$. Also assume that abilities are private information, but the distribution is common knowledge.

Assume that there is a minimal effort requirement $r \geq 0$ associated with contest H. Since the support of contestant types, $[\underline{c}, \bar{c}]$, is general, an assumption of 0 effort requirement in contest L does not create any qualitative difference[3] but will simplify the notation later. Then, a contestant's strategy consists of two parts: the contest in which he chooses to compete and the level of effort to exert in that contest.

[2]In the case of $\underline{c} = 0$, the cost for player of type 0 is not defined. However, since it is the weakest type, it is safe to assume the type 0 player always exerts 0 effort and expects to receive 0 payoff. Details are discussed in the proof of Lemma 1.1 in appendix.

[3]A positive minimal effort requirement in contest L only blocks out contestants with low abilities from any contest.

1.3 Contestant's Equilibrium

1.3.1 Separating Equilibrium

I first consider an equilibrium where contestants are clearly partitioned, i.e. strong (high type) contestants participating in contest H and weak (low type) contestants in contest L. On one hand, my analysis in this section is based on that of Gavious' (2009), where he proves the existence of such equilibrium for some "appropriately" chosen reserve prices in H and L auctions. On the other hand, I extend his result further to show the existence of such equilibrium for an arbitrary minimal effort requirement (equivalent to reserve price in regular auction setting). Before presenting any result, it is useful to define c^* such that $aF^{n-1}(c^*) = 1$ first.

Proposition 1.1. *If r satisfies the condition*

$$c^* - \int_{\underline{c}}^{c^*} \Big(1 - F(c^*) + F(t)\Big)^{n-1} dt \leq r \leq a\bar{c} - \int_{\underline{c}}^{\bar{c}} F^{n-1}(t)dt$$

then there exists $\bar{s} \in [c^*, \bar{c}]$ such that the strategy

$$\sigma(c) = \begin{cases} \{L, b_L(c)\} & \text{if } \underline{c} \leq c < \bar{s} \\ \{H, b_H(c)\} & \text{if } \bar{s} \leq c \leq \bar{c} \end{cases} \tag{1.1}$$

forms an equilibrium for the contestants, where

$$\begin{aligned} b_L(c) &= \int_{\underline{c}}^{c} td\Big(1-F(\bar{s})+F(t)\Big)^{n-1} + \underline{c}\Big(1-F(\bar{s})\Big)^{n-1} \\ b_H(c) &= a\int_{\bar{s}}^{c} tdF^{n-1}(t)+r \end{aligned}$$

and $\bar{s}$ is uniquely determined by the equation

$$\int_{\underline{c}}^{\bar{s}} \Big(1-F(\bar{s})+F(t)\Big)^{n-1} dt = a\bar{s}F^{n-1}(\bar{s}) - r. \tag{1.2}$$

The pivotal type $\bar{s}$ is the type of contestant who is indifferent between participating in contest L and winning with certainty, or participating in contest H and winning if he is the only player in contest H. The condition which determines c^* is very meaningful. For any $\bar{s} \geq c^*$, we have $aF^{n-1}(\bar{s}) \geq 1$. This expression is equivalent to $a\bar{s}F^{n-1}(\bar{s}) \geq \bar{s}$, which indicates that the expected utility for a contestant with a cutoff ability when he participates in contestant L is less than his utility when he participates in contest H. Moreover, in light of Myerson (1981) [11], the derivative of the expected payoff is equal to the probability of winning. Thus, the condition that $aF^{n-1}(\bar{s}) \geq 1$ can be also interpreted as that the marginal utility of the cutoff type player is higher in contestant H than in contestant L. Figure 1.1 illustrates the effort functions in the case of uniform distribution on [0,1] and parameters $a = 6$, $n = 4$ and $\bar{s} = 0.7$.

Based on the proof of Proposition 1.1, we know that in equilibrium, r cannot be too large, because if $r > a\bar{c} - \int_{\underline{c}}^{\bar{c}} F^{n-1}(t)dt$, none of the players will participate

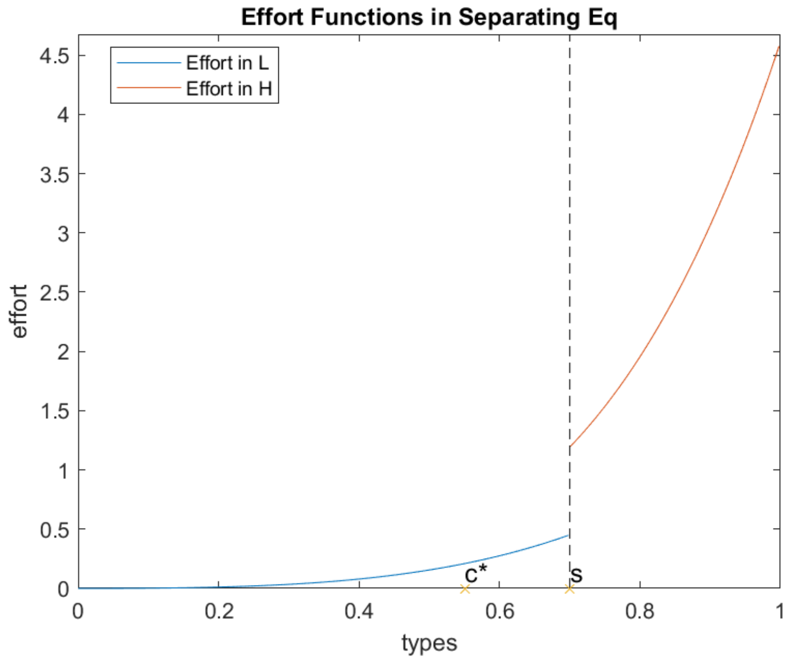

Figure 1.1: Effort Functions in Separating Equilibrium

in contest H. Since contestants endogenously choose which contest to participate, we should find out how the cutoff type changes with respect to the minimal effort and the value of prize factor.

Corollary 1.1. *The pivotal type $\bar{s}$ increases with r but decreases with a.*

It is intuitive that increasing r will increase $\bar{s}$. Corollary 1.1 also shows that increasing a will decrease $\bar{s}$ since more contestants closer to the pivotal type will be better off by participating in contest H and thus decreases $\bar{s}$.

1.3.2 Equilibrium with Mixed Strategy

In the previous section, I constructed equilibrium for some values of the minimal effort requirement. For other values, a full separation of the players is not guaranteed. Instead, certain types of contestants may want to employ mixed strategies in equilibrium. Before going into the details of any mixed strategy equilibrium, I would like to present a result which relates the mixed strategies to the cutoff value c^*.

Theorem 1.1. *In equilibrium, the upper bound of the type of contestants who may take mixed strategies is c^*. If the type of a player lies in the mixing interval, he participates in contest L with probability α and in contest H with probability $1-\alpha$, where*

$$\alpha = \frac{1}{F(c^*)+1} = \frac{a^{\frac{1}{n-1}}}{a^{\frac{1}{n-1}}+1}.$$

Moreover, the mixing interval is convex.

Proof of Theorem 1.1. Assume contestant with type c participates in contest L with probability $p(c)$ and in contest H with probability $1-p(c)$, where $0 \leq p(c) \leq 1$. If $p(c)$ equals to 0 or 1, it means that this player actually employs a pure strategy. Furthermore, suppose x is the lowest type of players who randomizes between contest L and contest H. Then the expected payoff of player with type c ($\geq x$) in contest j can be characterized by $U^j(c) = \int_x^c \mathrm{Pr}^j(t)dt + U^j(x)$, where $\mathrm{Pr}^j(t)$ ($j = L, H$) is the probability that a type t player will win in contest j (See Myerson (1981) [11]).

In a mixed-strategy equilibrium, the player should be indifferent between the two contests, i.e. $U^L(c) = \int_x^c \mathrm{Pr}^L(t)dt + U^L(x) = U^H(v) = a\int_x^c \mathrm{Pr}^H(t)dt + U^H(x)$ for every c in the mixing interval. One of the necessary conditions is that the rate at which U^L and U^H increase must also be the same. That is, we require $\mathrm{Pr}^L(c) = a\mathrm{Pr}^H(c)$ for all c in the mixing interval. The probabilities are computed as below:

$$
\begin{aligned}
\Pr^L(c) &= \left(1 - \int_c^{\bar{c}} p(t)f(t)dt\right)^{n-1} \\
\Pr^H(c) &= \left(1 - \int_c^{\bar{c}} (1-p(t))f(t)dt\right)^{n-1}
\end{aligned}
$$

Then, the following equation must hold for all c in the mixing interval:

$$
1 - \int_c^{\bar{c}} p(t)f(t)dt = a^{\frac{1}{n-1}}\left(1 - \int_c^{\bar{c}} (1-p(t))f(t)dt\right)
$$

which, after rearranging all terms to the right-hand side, is equivalent to

$$
a^{\frac{1}{n-1}} - 1 - \int_c^{\bar{c}} \left(a^{\frac{1}{n-1}} - (a^{\frac{1}{n-1}}+1)p(t)\right)f(t)dt = 0 \tag{1.3}
$$

Taking derivative on both sides with respect to c, we immediately find that

$$
p(c) = \frac{a^{\frac{1}{n-1}}}{a^{\frac{1}{n-1}}+1},
$$

for all c in the mixing interval, which is defined as α. This means that if a contestant randomizes between the two contests, he has to do it according to the probability profile $(\alpha, 1-\alpha)$.

However, the upper bound of the mixing interval must be lower than $\bar{c}$; otherwise, the left-hand side of equation (1.3) is positive, while the right-hand side is

0. Observe that for $s \geq c^*$,

$$
\begin{aligned}
& \int_s^{\bar{c}} \left(a^{\frac{1}{n-1}} - (a^{\frac{1}{n-1}} + 1)p(t)\right) f(t)dt \\
\leq & \int_s^{\bar{c}} a^{\frac{1}{n-1}} f(t)dt \\
= & \ a^{\frac{1}{n-1}} (F(\bar{c}) - F(s)) \\
\leq & \ a^{\frac{1}{n-1}} (F(\bar{c}) - F(c^*)) \\
= & \ a^{\frac{1}{n-1}} - 1.
\end{aligned}
$$

The first inequality uses the non-negativity of $p(t)$, the second inequality is from $s \geq c^*$ and the last equality follows the definition of $\bar{c}$ and c^*. Combined with equation (1.3), this implies that $p(t)$ has to be 0 for all $t > c^*$ and that the upper bound of the types of randomizing contestant is c^*. In fact, as suggested by the inequality, those whose types are above c^* prefer to compete in contest H. To sum up,

$$
p(t) = \begin{cases} \frac{a^{\frac{1}{n-1}}}{a^{\frac{1}{n-1}}+1} & \text{if } x \leq t \leq c^* \\ 0 & \text{if } c^* < t \leq 1. \end{cases}
$$

For the last part of the theorem, suppose there were a "hole", say (y, z), in the interval. Then, the expected payoff would grow at different rates in the two contests as a function of player's type, and thus y and z could not be both indifferent. This implies that if there is some positive measure of players who utilize mixing strategies, then the mixing interval should be a single large interval rather than several disconnected small intervals. $\square$

According to Theorem 1.1, the probability profile of a player who randomizes between the two contests is independent of his type as long as he is in the mixing interval. The intuition for this, as can be seen in the proof, is that the returns, and thus the marginal returns, of participating in both contests should be the same. Moreover, the rate at which the marginal return changes should also be the same, which is independent of the type of the randomizing contestants.

Based on Theorem 1.1, it is expected that in equilibrium, high type players go to contest H with probability 1, middle type players randomize between the two contests with a profile $(\alpha, 1-\alpha)$, and low type players go to contest L with certainty. The following proposition makes a full characterization of the equilibrium with mixed strategy (which I refer to as mixing equilibrium).

Proposition 1.2. *If* r *satisfies*[4]

$$a\underline{c}\Big(\alpha F(c^*) + (1-\alpha)F(\underline{c})\Big)^{n-1} \leq r < c^* - \int_{\underline{c}}^{c^*} \Big(1 - F(c^*) + F(t)\Big)^{n-1} dt$$

then, there exists $\underline{s} \in [\underline{c}, c^*)$ *such that the strategy*

$$\sigma(c) = \begin{cases} \{L, \beta_L(c)\}, & \text{if } \underline{c} \leq c < \underline{s} \\ \{(\alpha, 1-\alpha), (\beta_{ML}(c), \beta_{MH}(c))\} & \text{if } \underline{s} \leq c < c^* \\ \{H, \beta_H(c)\}, & \text{if } c^* \leq c \leq \bar{c} \end{cases} \tag{1.4}$$

[4]When $0 \leq r < a\underline{c}\Big(\alpha F(c^*) + (1-\alpha)F(\underline{c})\Big)^{n-1}$, the solution to equation (1.5) is negative, which means $\underline{s} = \underline{c}$. In this case, the minimal effort requirement is not binding and the mixing interval starts from the lower bound $\underline{c}$ to c^*.

forms an equilibrium for the contestants, where $\alpha = \frac{1}{F(c^*)+1}$,

$$\begin{aligned}
\beta_L(c) &= \int_{\underline{c}}^{c} td\Big(1-\alpha F(c^*)-(1-\alpha)F(\underline{s})+F(t)\Big)^{n-1} \\
&\quad +\underline{c}\Big(1-\alpha F(c^*)-(1-\alpha)F(\underline{s})\Big)^{n-1} \\
\beta_{ML}(c) &= \int_{\underline{s}}^{c} td\Big(1-\alpha F(c^*)+\alpha F(t)\Big)^{n-1}+\beta_L(\underline{s}) \\
\beta_{MH}(c) &= a\int_{\underline{s}}^{c} td\Big(\alpha F(c^*)+(1-\alpha)F(t)\Big)^{n-1}+r \\
\beta_H(c) &= a\int_{c^*}^{c} tdF^{n-1}(t)+\beta_{MH}(c^*)
\end{aligned}$$

and $\underline{s}$ *is uniquely determined by*

$$\begin{aligned}
&\int_{\underline{c}}^{\underline{s}}\Big(1-\alpha F(c^*)-(1-\alpha)F(\underline{s})+F(t)\Big)^{n-1}dt \\
&= a\underline{s}\Big(\alpha F(c^*)+(1-\alpha)F(\underline{s})\Big)^{n-1}-r.
\end{aligned} \tag{1.5}$$

The cutoff type $\underline{s}$ is the lowest type of contestant who is indifferent between participating in contest L or contest H. The next corollary shows how the cutoff value $\underline{s}$ is affected by the minimal effort and the prize factor. The behavior of $\underline{s}$ and the intuition of its behavior are the same as $\bar{s}$ in the previous section. Figure 1.2 illustrates the effort functions in the case of uniform distribution on [0,1] and parameters $a = 6$, $n = 4$ and $\underline{s} = 0.3$.

Corollary 1.2. $\underline{s}$ *increases with* r *but decreases with* a.

One further observation is that randomizing players will participate in contest L with probability $\alpha = \frac{a^{\frac{1}{n-1}}}{a^{\frac{1}{n-1}}+1} > \frac{1}{2}$, which indicates that such players are more

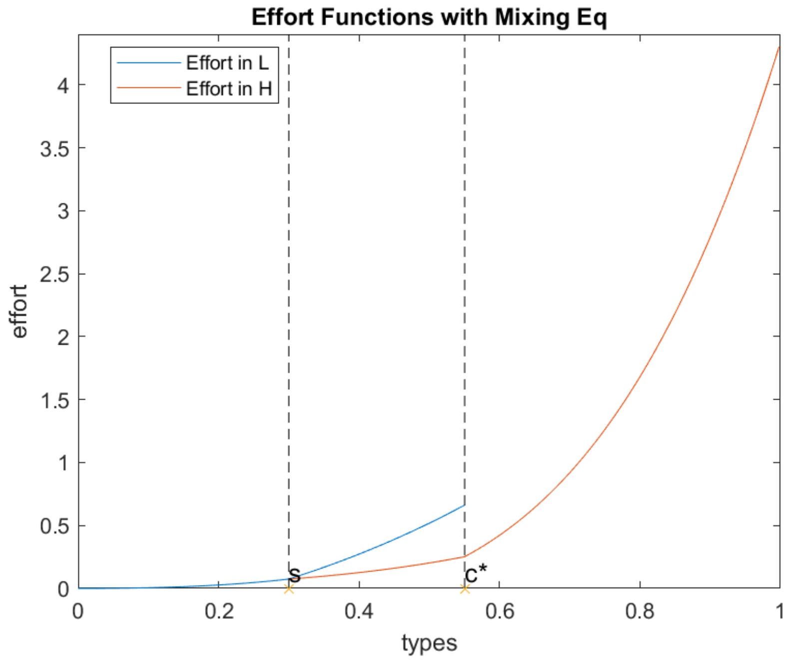

Figure 1.2: Effort Functions in Mixing Equilibrium

likely to participate in contest L than in contest H.

1.4 Designers' Problem

Now that the contestants' strategy has been characterized for any reasonable minimal effort requirement, let us turn to the designers' problems. Suppose that designers maximize the expected total effort in their respective contest by setting the minimal effort requirements.

1.4.1 Contest H

First consider designer H. Assume that the distribution and the support of contestant types is such that the optimal minimal effort requirement of contest L

is at a corner[5] (which is normalized to 0 to be consistent with previous analysis). According to conditions (1.2) and (1.5), there is a one-to-one mapping from r to either $\bar{s}$ or $\underline{s}$. Therefore, the problem of choosing an optimal r is equivalent to choosing a proper cutoff value s ($\bar{s}$ or $\underline{s}$).

The problem of H is to maximize

$$TE_H = \begin{cases} n\Big(\int_s^{\bar{c}} b_H(c)f(c)dc\Big), & \text{if } s \in [c^*, \bar{c}] \\ n\Big((1-\alpha)\int_s^{c^*} \beta_{MH}(c)f(c)dc + \int_{c^*}^{\bar{c}} \beta_H(c)f(c)dc\Big), & \text{if } s \in [\underline{c}, c^*). \end{cases}$$

Undoubtedly, TE_H is continuous at c^* because as s approaches c^*, $\beta_H(c^*-) = b_H(c^*)$. Therefore, there exists a maximizer of TE_H according to extreme value theorem.

Proposition 1.3. *TE_H is maximized at an inner point s that satisfies either*

$$\begin{aligned} & (1-F(s))(aF^{n-1}(s)-1) \\ + \ & (n-1)(1-F(s))f(s)\int_{\underline{c}}^{s}\Big(1-F(s)+F(t)\Big)^{n-2}dt \\ = \ & f(s)\left[asF^{n-1}(s) - \int_{\underline{c}}^{s}\Big(1-F(s)+F(t)\Big)^{n-1}dt\right] \end{aligned}$$

or

$$\begin{aligned} & \Big(\alpha-(1-\alpha)F(s)\Big)(n-1)\int_{\underline{c}}^{s}\Big(\alpha-(1-\alpha)F(s)+F(t)\Big)^{n-2}dt \\ = \ & s\alpha^{n-1}(1+F^{n-1}(s)) - \int_{\underline{c}}^{s}\Big(\alpha-(1-\alpha)F(s)+F(t)\Big)^{n-1}dt. \end{aligned}$$

[5] A full solution where both H and L chooses their minimal effort requirements is too difficult at the current stage.

The conditions in Proposition 1.3 are the first-order conditions of TE_H in the separating equilibrium and in the mixing equilibrium respectively. Intuitively, the higher a is, the lower c^* will be and the more likely that the optimal s will be in the region of a separating equilibrium.

Example 1.1. *Suppose there are 4 contestants and their abilities are distributed according to $F(c) = c$. If $a = 6$, then $c^* = 0.55$ and optimal $s^* = 0.588$, which implies a separating equilibrium among contestants. Figure 1.3 shows the total effort in contest H as a function of s for these parameter values.*

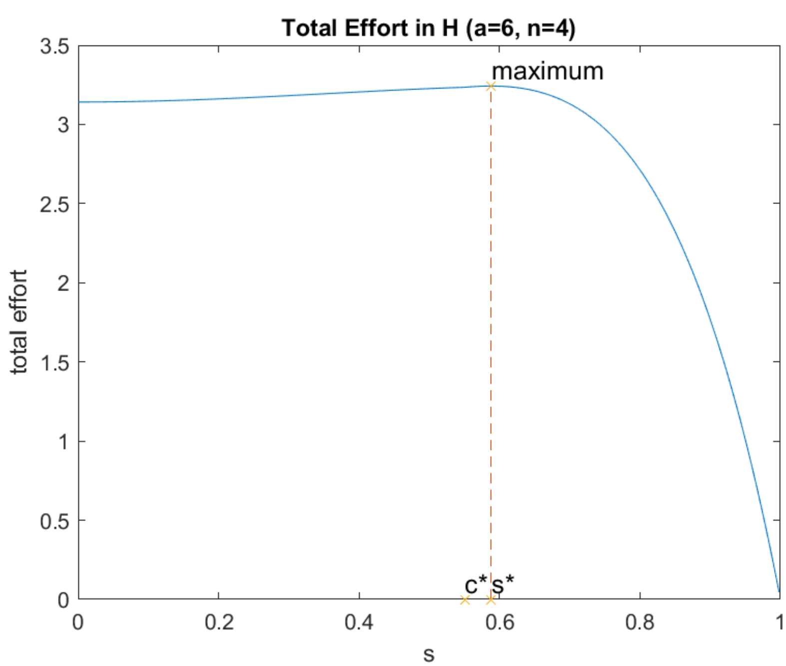

Figure 1.3: Total Effort Example 1

Example 1.2. *Suppose there are 4 contestants and their abilities are distributed according to $F(c) = c$. If $a = 2$, then $c^* = 0.794$ and the optimal $s^* = 0.516$, which*

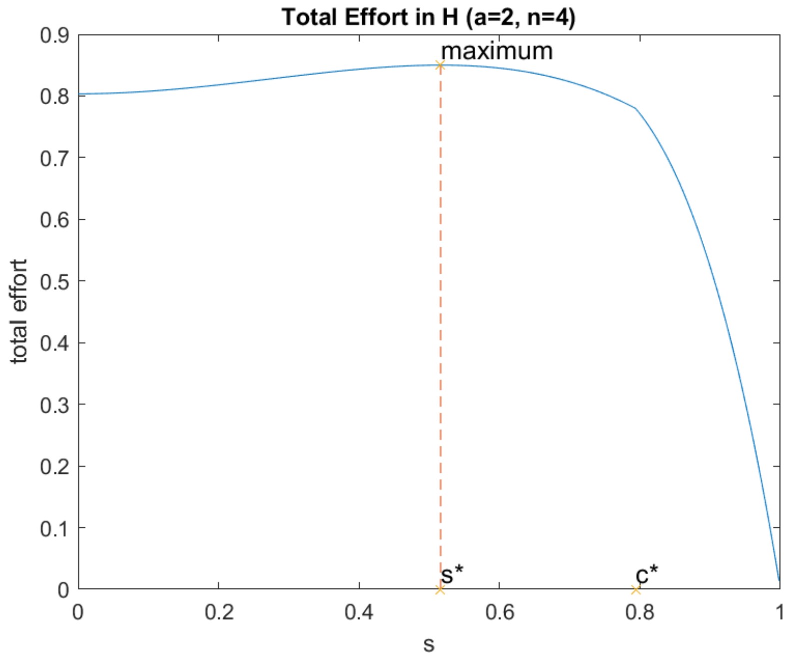

Figure 1.4: Total Effort Example 2

corresponds to a mixing equilibrium among contestants. Figure 1.4 shows the total effort in contest H in these parameter values.

1.4.2 Contest L

In this part, I revisit the assumption that the optimal minimal effort in contest L is at a corner and give conditions where the corner solution is optimal.

The idea is to mimic the analysis of virtual value in standard auctions. Designer L does not have incentive to exclude any contestant if the "virtual value" under the current setting is non-negative for all types. For a fixed pivotal type s, the "virtual value" takes the form of $t - \frac{F(s)-F(t)}{f(t)}$[6]. However, s in general is affected

[6] This is for the case of separating equilibrium. For the case of mixing equilibrium, a slight difference need to be made. See the proof of Proposition 1.4 for detail.

by the choice of minimal effort level in L as well. The next proposition addresses the concern of endogeneity and shows that a regular condition of a commonly defined virtual value function, along with proper a boundary condition, is sufficient for the optimality at corner.

Proposition 1.4. *The minimal effort requirement at a corner is optimal for L if the virtual value function $t - \frac{1-F(t)}{f(t)}$ is increasing and $\underline{c}f(\underline{c}) - 1 \geq 0$.*

The results of Propositions 1.3 and 1.4 are important. They provide conditions that characterize at least part of the equilibrium set of the competing auction (contest) problem, which brings us more closer to the full solution of such kind of problem.

1.5 Concluding Remarks

I study the problem of competing contests with differentiated prizes. Given the condition of minimal effort requirements, equilibrium is either a separating equilibrium, where strong contestants participating in contest H and weak contestants in contest L, or a mixing equilibrium, where strong players participate in contest H with probability 1, middle-type players randomize between the two contests, and weak players go to contest L with certainty. I also solve an equilibrium of contest designers where one designer's choice of minimal effort level is assumed to be non-strategic. A full solution, though desirable, is too computationally involved at the current stage. Finally, I investigate the conditions such that the assumed non-strategic choice of minimal effort level is optimal and thus characterize at least part of the equilibrium set, which expands the knowledge of competing auctions.

1.6 Appendix A: Proofs

Lemma 1.1. *Consider a simple contest where n players compete for a single prize V. Suppose an effort x causes a cost $\frac{x}{t}$ to a contestant with type t, the types are distributed on $[\underline{t}, \bar{t}]$ and let $G(u) = \Pr(t \leq u)$. Then the equilibrium strategy $b(t)$ and expected utility $U(t)$ for a type t player are*

$$\begin{aligned} b(t) &= V \int_{\underline{t}}^{t} s dG^{n-1}(s) + V \underline{t} G^{n-1}(\underline{t}) \\ U(t) &= \frac{V}{t} \int_{\underline{t}}^{t} G^{n-1}(s) ds. \end{aligned}$$

Lemma 1.1 is a derivation from Myerson (1981) [11] for the specific parameterization. Nevertheless, a proof is still included, which serves as a quick reference for latter results.

Proof. The player with type t act as if his type is s to maximize the expected utility,

$$\max_s U(t, s) = V G^{n-1}(s) - \frac{b(s)}{t}.$$

In equilibrium, the problem must be solved by $s = t$. Then, the calculation yields

$$b(t) = V \int_{\underline{t}}^{t} s dG^{n-1}(s) + k$$

where k is a constant. A reasonable k is such that the expected utility of the weakest contestant (type $\underline{t}$) is 0. Otherwise, if the weakest contestant has positive expected payoff, he will have incentive to deviate by exerting an effort ε higher than his close neighbor.

Next, let us check that a sufficient second-order condition is satisfied,

$$U_{12}(t,s) = -b'(s) < 0$$

because $b(\cdot)$ is strictly increasing.

Then, plug in the expression of b and abuse the notation a little bit,

$$\begin{aligned}
U(t) = U(t,t) &= VG^{n-1}(t) - \frac{b(t)-k}{t} \\
&= VG^{n-1}(t) - \frac{k}{t} - \frac{V}{t}\int_{\underline{t}}^{t} s dG^{n-1}(s) \\
&= VG^{n-1}(t) - \frac{k}{t} \\
&\quad -\frac{V}{t}\Big(tG^{n-1}(t) - \underline{t}G^{n-1}(\underline{t}) - \int_{\underline{t}}^{t} G^{n-1}(s)ds\Big) \\
&= \frac{V}{t}\int_{\underline{t}}^{t} G^{n-1}(s)ds + \frac{V\underline{t}G^{n-1}(\underline{t})}{t} - \frac{k}{t}
\end{aligned}$$

where the third equality comes from integration by parts. Since we require $U(\underline{t}) = 0$, $k = V\underline{t}G^{n-1}(\underline{t})$ and thus $U(t) = \frac{V}{t}\int_{\underline{t}}^{t} G^{n-1}(s)ds$.

Finally, let me show that the assumption that type 0 contestant makes an effort of 0 and gains 0 expected utility is still consistent. $b(t)$ is well-defined and is continuous at $t = 0$ because

$$b(0) = 0 = \lim_{t\to 0+} b(t)$$

Define $U(0)=0$. According to L'Hospital's rule,

$$\lim_{t\to 0+} U(t) = \lim_{t\to 0+} \frac{V}{t}\int_0^t G^{n-1}(s)ds = \lim_{t\to 0+} \frac{G^{n-1}(t)}{1} = 0$$

and thus $U(t)$ is also continuous at $t=0$. $\square$

Proof of Proposition 1.1. Let $G_1(c) = \int_{\underline{c}}^{c} \Big(1 - F(c) + F(t)\Big)^{n-1} dt - acF^{n-1}(c) + r$. Using the fact that $aF^{n-1}(c^*) = 1$ and $F(1) = 1$,

$$\begin{aligned} G_1(c^*) &= \int_{\underline{c}}^{c^*} \Big(1 - F(c^*) + F(t)\Big)^{n-1} dt - c^* + r \geq 0 \\ G_1(\bar{c}) &= \int_{\underline{c}}^{\bar{c}} F^{n-1}(t)dt - a + r \leq 0 \end{aligned}$$

Therefore, the existence of $\bar{s}$ is guaranteed by the continuity of $G_1(c)$. Moreover,

$$G_1'(c) = [1 - aF^{n-1}(c)] - (n-1)f(c)[acF^{n-2}(c) + \int_{\underline{c}}^{c} (1 - F(c) + F(t))^{n-2} dt]$$

Since $aF^{n-1}(\bar{s}) \geq aF^{n-1}(c^*) = 1$, $G_1'(c) < 0$. This completes the proof that the cutoff value $\bar{s}$ is unique.

The results from Lemma 1.1 can be carried over to establish the functional form of b_L and b_H with proper accommodation to the context. For b_L, $\Big(1 - F(\bar{s}) + F(t)\Big)^{n-1}$ is the probability that a type t contestant would win in contest L, which happens when those with types in $[t, \bar{s}]$ are absent, and the constant for the weakest type is $\underline{c}\Big(1 - F(\bar{s})\Big)^{n-1}$. For b_H, it is adjusted with the observations that the prize is multiplied by a and that the lowest type in contest H should exert an effort equal to r.

It is left to show that no player will defect from one contest to the other. Let $U^j(t|c)$ be a contestant's expected utility given that his type is c, he acts as if his type is t and he participates in contest $j = L, H$ and the other $n-1$ contestants playing according to strategy (1.1). Incentive compatibility requires that

$$U^L(c|c \leq \bar{s}) \geq U^H(\hat{c}|c \leq \bar{s}) \tag{1.6}$$

$$U^L(\hat{c}|c \geq \bar{s}) \leq U^H(c|c \geq \bar{s}). \tag{1.7}$$

Using the results from Lemma 1.1 again,

$$\begin{aligned} U^L(c|c \leq \bar{s}) &= \frac{1}{c}\int_{\underline{c}}^{c} \Big(1 - F(\bar{s}) + F(t)\Big)^{n-1} dt \\ U^H(c|c \geq \bar{s}) &= const + \frac{a}{c}\int_{\bar{s}}^{c} F^{n-1}(t)dt \\ &= aF^{n-1}(\bar{s}) - \frac{r}{\bar{s}} + \frac{a}{c}\int_{\bar{s}}^{c} F^{n-1}(t)dt. \end{aligned}$$

The constant $aF^{n-1}(\bar{s}) - \frac{r}{\bar{s}}$ appears because a player's utility in contest H is not zero even if his type is $\bar{s}$. Then, using condition (1.2) and $aF^{n-1}(\bar{s}) \geq aF^{n-1}(c^*) = 1$,

$$\begin{aligned}
& cU^L(\hat{c} \leq \bar{s} | c \geq \bar{s}) \\
= \; & c\Big(1 - F(\bar{s}) + F(\hat{c})\Big)^{n-1} - b_L(\hat{c}) \\
= \; & \hat{c}U^L(\hat{c} \leq \bar{s} | \hat{c} \leq \bar{s}) + (c - \hat{c})\Big(1 - F(\bar{s}) + F(\hat{c})\Big)^{n-1} \\
= \; & \int_{\underline{c}}^{\hat{c}} \Big(1 - F(\bar{s}) + F(t)\Big)^{n-1} dt + (c - \hat{c})\Big(1 - F(\bar{s}) + F(\hat{c})\Big)^{n-1} \\
\leq \; & \int_{\underline{c}}^{\bar{s}} \Big(1 - F(\bar{s}) + F(t)\Big)^{n-1} dt + (c - \bar{s})\Big(1 - F(\bar{s}) + F(\hat{c})\Big)^{n-1} \\
\leq \; & a\bar{s}F^{n-1}(\bar{s}) - r + c - \bar{s} \\
\leq \; & a\bar{s}F^{n-1}(\bar{s}) - r + (c - \bar{s})aF^{n-1}(\bar{s}) \\
\leq \; & a\bar{s}F^{n-1}(\bar{s}) - r + a\int_{\bar{s}}^{c} F^{n-1}(t)dt \\
\leq \; & c(aF^{n-1}(\bar{s}) - \frac{r}{\bar{s}}) + a\int_{\bar{s}}^{c} F^{n-1}(t)dt \\
= \; & cU^H(c | c \geq \bar{s})
\end{aligned}$$

and thus (1.7) is satisfied. For (1.6), notice that

$$U^H(\hat{c} > \bar{s} | c \leq \bar{s}) = aF^{n-1}(\hat{c}) - \frac{b_H(\hat{c})}{c}$$

with a derivative

$$\frac{\partial U^H(\hat{c} > \bar{s} | c \leq \bar{s})}{\partial \hat{c}} = (1 - \frac{\hat{c}}{c})(n-1)F^{n-2}(\hat{c})f(\hat{c}) < 0.$$

Therefore,

$$U^H(\hat{c} > \bar{s}|c \leq \bar{s}) \leq U^H(\hat{c} = \bar{s}|c \leq \bar{s}) = aF^{n-1}(\bar{s}) - \frac{r}{c}.$$

To complete the proof, it remains to show that

$$U^L(c|c \leq \bar{s}) = \frac{1}{c}\int_{\underline{c}}^{c} \Big(1 - F(\bar{s}) + F(t)\Big)^{n-1} dt \geq aF^{n-1}(\bar{s}) - \frac{r}{c}$$

or, equivalently,

$$\int_{\underline{c}}^{c} \Big(1 - F(\bar{s}) + F(t)\Big)^{n-1} dt \geq acF^{n-1}(\bar{s}) - r.$$

Define

$$h(c) = \int_{\underline{c}}^{c} \Big(1 - F(\bar{s}) + F(t)\Big)^{n-1} dt - acF^{n-1}(\bar{s}) + r$$

and observe that $h(\underline{c}) = r > 0$ and that $h(s) = 0$ by condition (1.2). Since

$$h'(c) = \Big(1 - F(\bar{s}) + F(c)\Big)^{n-1} - aF^{n-1}(\bar{s}) \leq 1 - 1 = 0,$$

$h(c)$ is non-increasing and thus the condition $h(c) \geq 0$ is satisfied. □

Proof of Corollary 1.1. Differentiating (1.2) with respect to r, we get

$$\begin{aligned} &\Big(1 - \int_{\underline{c}}^{\bar{s}} (n-1)(1 - F(\bar{s}) + F(t))^{n-2} f(\bar{s}) dt\Big) \frac{\partial \bar{s}}{\partial r} \\ = &\Big(aF^{n-1}(\bar{s}) + a\bar{s}(n-1)F^{n-2}(\bar{s})f(\bar{s})\Big)\frac{\partial \bar{s}}{\partial r} - 1 \end{aligned}$$

Since $aF^{n-1}(\bar{s}) > 1$, $\frac{\partial \bar{s}}{\partial r} > 0$.

Similarly, differentiating (1.2) with respect to a,

$$\begin{aligned}
&\Big(1-\int_{\underline{c}}^{\bar{s}}(n-1)(1-F(\bar{s})+F(t))^{n-2}f(\bar{s})dt\Big)\frac{\partial\bar{s}}{\partial a}\\
= \;& \bar{s}F^{n-1}(\bar{s})+\Big(aF^{n-1}(\bar{s})+a\bar{s}(n-1)F^{n-2}(\bar{s})f(\bar{s})\Big)\frac{\partial\bar{s}}{\partial a}
\end{aligned}$$

it can be seen directly that $\frac{\partial\bar{s}}{\partial a}<0$. □

Proof of Proposition 1.2. Similar to the proof of Proposition 1.1, let

$$\begin{aligned}
G_2(c)= \quad & \int_{\underline{c}}^{c}\Big(1-\alpha F(c^*)-(1-\alpha)F(c)+F(t)\Big)^{n-1}dt\\
& -ac\Big(\alpha F(c^*)+(1-\alpha)F(c)\Big)^{n-1}+r
\end{aligned}$$

When $0\leq r<c^*-\int_{\underline{c}}^{c^*}(1-F(c^*)+F(t))^{n-1}dt$,

$$\begin{aligned}
G_2(\underline{c}) &= r\geq 0\\
G_2(c^*) &= \int_{\underline{c}}^{c^*}\Big(1-F(c^*)+F(t)\Big)^{n-1}dt-ac^*F^{n-1}(c^*)+r<0.
\end{aligned}$$

The existence of $\underline{s}$ is, thus, guaranteed by the continuity of $G_2(c)$. Moreover,

$$\begin{aligned}
G_2'(c) &= \Big(1-\alpha F(c^*)+\alpha F(c)\Big)^{n-1}\\
&- (n-1)(1-\alpha)f(c)\int_{\underline{c}}^{c}\Big(1-\alpha F(c^*)-(1-\alpha)F(c)+F(t)\Big)^{n-2}dt\\
&- a\Big(\alpha F(c^*)+(1-\alpha)F(c)\Big)^{n-1}\\
&- (n-1)(1-\alpha)f(c)ac\Big(\alpha F(c^*)+(1-\alpha)F(c)\Big)^{n-2}.
\end{aligned}$$

The third term negates the first term, because $1-\alpha F(c^*) = a^{\frac{1}{n-1}}\alpha F(c^*)$ and $\alpha = a^{\frac{1}{n-1}}(1-\alpha)$, and the second and forth terms are obviously negative. So $G_2'(c) < 0$, which indicates that $\underline{s}$ is unique.

Now, given the existence of $\underline{s}$, the construction in Theorem 1.1 indicates that contestants with $c \in [\underline{s}, c^*]$ are indeed indifferent between entering contest H and contest L. For players whose $c > c^*$,

$$
\begin{aligned}
U^H(c|c>c^*) &= U^H(c^*) + a\int_{c^*}^{c} \mathrm{Pr}^H(t)dt = U^H(c^*) + a\int_{c^*}^{c} F^{n-1}(t)dt \\
U^L(c|c>c^*) &= U^L(c^*) + \int_{c^*}^{c} \mathrm{Pr}^L(t)dt = U^H(c^*) + (c-c^*).
\end{aligned}
$$

Since $aF^{n-1}(c) > 1$ for all $c > c^*$, $a\int_{c^*}^{c} F^{n-1}(t)dt > a(c-c^*)F^{n-1}(c^*) = c - c^*$. Thus, we have established first part of the entering strategies of contestants whose types are above the cutoff point $\underline{s}$.

For players with type $c < \underline{s}$, we have

$$
\begin{aligned}
U^L(c|c<\underline{s}) &= \int_{\underline{c}}^{c} \Big(1-\alpha F(c^*) - (1-\alpha)F(\underline{s}) + F(t)\Big)^{n-1} dt \\
&= U^L(\underline{s}) - \int_{c}^{\underline{s}} \Big(1-\alpha F(c^*) - (1-\alpha)F(\underline{s}) + F(t)\Big)^{n-1} dt \\
U^H(c|c<\underline{s}) &= ac\Big(\alpha F(c^*) + (1-\alpha)F(\underline{s})\Big)^{n-1} - r \\
&= U^H(\underline{s}) - a(\underline{s}-c)\Big(\alpha F(c^*) + (1-\alpha)F(\underline{s})\Big)^{n-1}.
\end{aligned}
$$

Actually, using $F(c) < F(\underline{s})$, $1-\alpha F(c^*) = a^{\frac{1}{n-1}}\alpha F(c^*)$ and $\alpha = a^{\frac{1}{n-1}}(1-\alpha)$, we can derive

$$\begin{aligned}
& \int_{c}^{\underline{s}} \Big(1 - \alpha F(c^*) - (1-\alpha)F(\underline{s}) + F(t)\Big)^{n-1} dt \\
< \;& (\underline{s} - c)\Big(1 - \alpha F(c^*) + \alpha F(\underline{s})\Big)^{n-1} \\
= \;& (\underline{s} - c)a\Big(\alpha F(c^*) + (1-\alpha)F(\underline{s})\Big)^{n-1}
\end{aligned}$$

Therefore, $U^L(c|c < \underline{s}) > U^H(c|c < \underline{s})$, which means players whose types $c < \underline{s}$ participate in contest L definitely.

The functional form of β_L, β_{ML}, β_{MH} and β_H can be derived using Lemma 1.1 again with proper accommodations. For β_L, the probability of that a type $t \in [\underline{c}, \underline{s}]$ contestant would win in contest L is

$$\begin{aligned}
& \Big(1 - \alpha F(c^*) - (1-\alpha)F(\underline{s}) + F(t)\Big)^{n-1} \\
= \;& \Big(1 - \alpha\big(F(c^*) - F(\underline{s})\big) - \big(F(\underline{s}) - F(t)\big)\Big)^{n-1}
\end{aligned}$$

which happens when those with types in $[t, \underline{s}]$ are absent and those with types in $[\underline{s}, c^*]$ participates in contest H. The constant for the weakest type in this case is $\underline{c}\Big(1 - \alpha F(c^*) - (1-\alpha)F(\underline{s})\Big)^{n-1}$. For β_{ML}, the probability of winning in contest L is $\Big(1 - \alpha F(c^*) + \alpha F(t)\Big)^{n-1}$ for a contestant with type $t \in [\underline{s}, c^*]$ with the restriction that $\beta_{ML}(\underline{s}) = \lim_{c \to \underline{s}} \beta_L(c)$. For β_{MH}, the probability of winning in contest H is

$$\Big(1 - (1-\alpha)\big(F(c^*) - F(t)\big)\Big)^{n-1} = \Big(\alpha F(c^*) + (1-\alpha)F(t)\Big)^{n-1}$$

with the lowest type exerting an effort equal to r. For β_H, it is adjusted with the restriction that $\beta_H(c^*) = \lim_{c\to c^*} \beta_M(c)$. $\square$

Proof of Corollary 1.2. Differentiating (1.5) with respect to r,

$$
\begin{aligned}
& \frac{\partial \underline{s}}{\partial r}\Bigg[\Big(1-\alpha F(c^*)+\alpha F(\underline{s})\Big)^{n-1} \\
& -(n-1)(1-\alpha)f(\underline{s})\int_{\underline{c}}^{\underline{s}}\Big(1-\alpha F(c^*)-(1-\alpha)F(\underline{s})+F(t)\Big)^{n-2}dt\Bigg] \\
= \;& \frac{\partial \underline{s}}{\partial r}\Bigg[a\Big(\alpha F(c^*)+(1-\alpha)F(\underline{s})\Big)^{n-1} \\
& +(n-1)(1-\alpha)f(\underline{s})a\underline{s}\Big(\alpha F(c^*)+(1-\alpha)F(\underline{s})\Big)^{n-2}\Bigg]-1.
\end{aligned}
$$

Since $1-\alpha F(c^*) = a^{\frac{1}{n-1}}\alpha F(c^*)$,

$$
\Big(1-\alpha F(c^*)+\alpha F(\underline{s})\Big)^{n-1} = a\Big(\alpha F(c^*)+(1-\alpha)F(\underline{s})\Big)^{n-1}
$$

and thus $\frac{\partial \underline{s}}{\partial r} > 0$.

Before doing the same thing for a, observe that $1-\alpha F(c^*) = \alpha = a^{\frac{1}{n-1}}(1-\alpha)$. Equation (1.5) is thus equivalent to

$$
\int_{\underline{c}}^{\underline{s}}\Big(\alpha-(1-\alpha)F(\underline{s})+F(t)\Big)^{n-1}dt = \underline{s}\Big(\alpha+\alpha F(\underline{s})\Big)^{n-1}-r
$$

Then, taking derivative with respect to a and rearranging terms,

$$
(1+F(\underline{s}))\frac{\partial \alpha}{\partial a} = (1-\alpha)f(\underline{s})\frac{\partial \underline{s}}{\partial a}
$$

Because $\frac{\partial \alpha}{\partial a} < 0$, $\frac{\partial \underline{s}}{\partial a} < 0$ as well. $\square$

Proof of Proposition 1.3. According to the extreme value theorem, TE_H must attain a maximum because it is a real-valued, continuous function on $[\underline{c}, \bar{c}]$. A necessary condition for maximizer is that it shall be a root of the first-order condition or be at the end points.

When $s \in [c^*, \bar{c}]$,

$$
\begin{aligned}
TE_H &= n\left(\int_s^{\bar{c}} b_H(c) f(c) dc\right) \\
&= n\int_s^{\bar{c}} \left(acF^{n-1}(c) - a\int_s^c F^{n-1}(t)dt\right) f(c)dc \\
&\quad -n(1-F(s))\int_{\underline{c}}^s \Big(1-F(s)+F(t)\Big)^{n-1} dt
\end{aligned}
$$

where the last equality uses the expression of r from equation (1.2) and the observation that $\int_0^s \Big(1-F(s)+F(t)\Big)^{n-1} dt$ is a constant for any given s. Taking derivative of TE_H with respect to s, after simplification,

$$
\begin{aligned}
\frac{\partial TE_H}{\partial s} &= n(1-F(s))(aF^{n-1}(s)-1) \\
&\quad +n(n-1)(1-F(s))f(s)\int_{\underline{c}}^s \Big(1-F(s)+F(t)\Big)^{n-2} dt \\
&\quad +nf(s)\left[\int_{\underline{c}}^s \Big(1-F(s)+F(t)\Big)^{n-1} dt - asF^{n-1}(s)\right] \\
&= n(n-1)(1-F(s))f(s)\int_{\underline{c}}^s \Big(1-F(s)+F(t)\Big)^{n-2} dt \\
&\quad +n(1-F(s))(aF^{n-1}(s)-1) - nf(s)r.
\end{aligned}
$$

Therefore, $\frac{\partial TE_H}{\partial s}$ is negative at the upper bound $\bar{c}$.

Similarly, when $s \in [\underline{c}, c^*)$,

$$TE_H = n\Bigg((1-\alpha)\int_s^{c^*} \beta_{MH}(c)f(c)dc + \int_{c^*}^{\bar{c}} \beta_H(c)f(c)dc\Bigg).$$

Then,

$$\begin{aligned}
\frac{\partial TE_H}{\partial s} &= (1-\alpha)n\Big[-\beta_{MH}(s)f(s) + \int_s^{c^*} \frac{\partial \beta_{MH}(c)}{\partial s} f(c)dc\Big] \\
&\quad +(1-F(c^*))\frac{\partial \beta_{MH}(c^*)}{\partial s} \\
&= (1-\alpha)nf(s)\Big[-s\alpha^{n-1}(1+F^{n-1}(s)) + \int_{\underline{c}}^{s} \Big(\alpha-(1-\alpha)F(s) \\
&\quad +F(t)\Big)^{n-1} dt \\
&\quad +\Big(\alpha-(1-\alpha)F(s)\Big)(n-1)\int_{\underline{c}}^{s}\Big(\alpha-(1-\alpha)F(s)+F(t)\Big)^{n-2}dt\Big] \\
&\propto \Big(\alpha-(1-\alpha)F(s)\Big)(n-1)\int_{\underline{c}}^{s}\Big(\alpha-(1-\alpha)F(s)+F(t)\Big)^{n-2}dt - r
\end{aligned}$$

and thus $\frac{\partial TE_H}{\partial s}$ is positive at the infimum of s such that $r = 0$.

It can also be shown that

$$\frac{\partial TE_H}{\partial s}|_{s=c^*-} = (1-\alpha)\frac{\partial TE_H}{\partial s}|_{s=c^*+}$$

which implies that c^* is a maximizer only if the first-order condition is equal to 0 at c^*, although $\frac{\partial TE_H}{\partial s}$ is generally not continuous at $s = c^*$. □

Proof of Proposition 1.4. Consider a more general minimal effort requirement in contest L, r_L. Let the lower bound of contests who participate in contest L be c_L[7].

[7] The notation is purposely chosen to be different from $\underline{c}$. In previous sections, c_L is identical to $\underline{c}$ because r_L is set at 0. However, contestant with type $\underline{c}$ may not want to participate in contest

In a separating equilibrium,

$$r_L = c_L\Big(1 - F(s) + F(c_L)\Big)^{n-1}. \tag{1.8}$$

Then, the total effort in contest L is

$$\begin{aligned}
TE_L &= n\int_{c_L}^{s} b_L(c)f(c)dc \\
&= n\Bigg[\int_{c_L}^{s}\int_{c_L}^{c} td\Big(1 - F(s) + F(t)\Big)^{n-1} f(c)dc \\
&\quad + \int_{c_L}^{s} c_L\Big(1 - F(s) + F(c_L)\Big)^{n-1} f(c)dc\Bigg] \\
&= n\Bigg[\int_{c_L}^{s}\Big(\int_{t}^{s} f(c)dc\Big)td\Big(1 - F(s) + F(t)\Big)^{n-1} \\
&\quad + c_L\Big(1 - F(s) + F(c_L)\Big)^{n-1}\Big(F(s) - F(c_L)\Big)\Bigg] \\
&= n\Bigg[\int_{c_L}^{s}\Big(F(s) - F(t)\Big)td\Big(1 - F(s) + F(t)\Big)^{n-1} \\
&\quad + c_L\Big(1 - F(s) + F(c_L)\Big)^{n-1}\Big(F(s) - F(c_L)\Big)\Bigg] \\
&= n\Big(F(s) - F(t)\Big)t\Big(1 - F(s) + F(t)\Big)^{n-1}\Big|_{c_L}^{s} \\
&\quad + nc_L\Big(1 - F(s) + F(c_L)\Big)^{n-1}\Big(F(s) - F(c_L)\Big) \\
&\quad - n\int_{c_L}^{s}\Big(1 - F(s) + F(t)\Big)^{n-1}\Big(F(s) - F(t) - tf(t)\Big)dt \\
&= n\int_{c_L}^{s}\Big(t - \frac{F(s) - F(t)}{f(t)}\Big)\Big(1 - F(s) + F(t)\Big)^{n-1} f(t)dt
\end{aligned}$$

where the third equality comes from reversing the order of integration and the fifth equality uses integration by parts.

L for a positive minimal effort r_L.

Assume a regularity condition hold that $\psi(t) = t - \frac{F(s)-F(t)}{f(t)}$ is increasing in $[\underline{c}, s]$. To maximize expected effort, L would like to award the prize to the player with the highest type (lowest cost) above an established cutoff value t^* such that $\psi^{-1}(t^*) = 0$. Therefore, if $\underline{c} \geq t^*$, it is of L's interest not to exclude any contestant from participating in contest L. Since the "virtual values" are increasing, as assumed, it suffices to have $\psi(\underline{c}) = \underline{c} - \frac{F(s)}{f(\underline{c})} \geq 0$ because $F(\underline{c}) = 0$.

Now consider the conditions in Proposition 1.4 that a standard virtual value function $t - \frac{1-F(t)}{f(t)}$ is increasing and that $\underline{c}f(\underline{c}) - 1 \geq 0$. Observe that

$$\begin{aligned}\frac{\partial \psi(t)}{\partial t} &= 1 - \frac{-f^2(t) - [F(s) - F(t)]f'(t)}{f^2(t)} \\ &= 2 - \frac{[F(s) - F(t)]f'(t)}{f^2(t)}.\end{aligned}$$

When $f'(t) \geq 0$,

$$\frac{\partial \psi(t)}{\partial t} \geq 2 - \frac{[1 - F(t)]f'(t)}{f^2(t)} > 0$$

because $t - \frac{1-F(t)}{f(t)}$ is increasing. When $f'(t) < 0$,

$$\frac{\partial \psi(t)}{\partial t} \geq 2 > 0$$

because $F(s) - F(t)$ is non-negative in $[\underline{c}, s]$. Moreover,

$$\underline{c} - \frac{F(s)}{f(\underline{c})} \geq \underline{c} - \frac{1}{f(\underline{c})} \geq 0.$$

Therefore, the two conditions in Proposition 1.4 are sufficient for the conditions prescribed for optimality at corner in the case of a separating equilibrium.

Likewise, in a mixing equilibrium,

$$
\begin{aligned}
TE_L &= n\Big(\int_{\underline{c}}^{s} \beta_L(c)f(c)dc + \alpha\int_{s}^{c^*} \beta_{ML}(c)f(c)dc\Big) \\
&= n\int_{\underline{c}}^{s}\int_{\underline{c}}^{c} td\Big(1-\alpha F(c^*)-(1-\alpha)F(s)+F(t)\Big)^{n-1} f(c)dc \\
&\quad +n\underline{c}\Big(1-\alpha F(c^*)-(1-\alpha)F(s)\Big)^{n-1}F(s) \\
&\quad +\alpha n\Big[\int_{s}^{c^*}\int_{s}^{c} td\Big(1-\alpha F(c^*)+\alpha F(t)\Big)^{n-1} f(c)dc \\
&\quad +\Big(F(c^*)-F(s)\Big)\int_{\underline{c}}^{s} td\Big(1-\alpha F(c^*)-(1-\alpha)F(s)+F(t)\Big)^{n-1} \\
&\quad +n\underline{c}\Big(1-\alpha F(c^*)-(1-\alpha)F(s)\Big)^{n-1}\Big(F(c^*)-F(s)\Big)\Big] \\
&= n\int_{\underline{c}}^{s}\Big(\int_{t}^{s} f(c)dc\Big)td\Big(1-\alpha F(c^*)-(1-\alpha)F(s)+F(t)\Big)^{n-1} \\
&\quad +n\underline{c}\Big(1-\alpha F(c^*)-(1-\alpha)F(s)\Big)^{n-1}\Big(\alpha F(c^*)+(1-\alpha)F(s)\Big) \\
&\quad +\alpha n\Big[\int_{s}^{c^*}\Big(\int_{t}^{c^*} f(c)dc\Big)td\Big(1-\alpha F(c^*)+\alpha F(t)\Big)^{n-1} \\
&\quad +\Big(F(c^*)-F(s)\Big)\int_{\underline{c}}^{s} td\Big(1-\alpha F(c^*)-(1-\alpha)F(s)+F(t)\Big)^{n-1} \\
&= n\int_{\underline{c}}^{s}\Big(F(s)-F(t)\Big)td\Big(1-\alpha F(c^*)-(1-\alpha)F(s)+F(t)\Big)^{n-1} \\
&\quad +n\underline{c}\Big(1-\alpha F(c^*)-(1-\alpha)F(s)\Big)^{n-1}\Big(\alpha F(c^*)+(1-\alpha)F(s)\Big) \\
&\quad +n\alpha\Big[\int_{s}^{c^*}\Big(F(c^*)-F(t)\Big)td\Big(1-\alpha F(c^*)+\alpha F(t)\Big)^{n-1} \\
&\quad +\Big(F(c^*)-F(s)\Big)\int_{\underline{c}}^{s} td\Big(1-\alpha F(c^*)-(1-\alpha)F(s)+F(t)\Big)^{n-1}\Big].
\end{aligned}
$$

Combine the first and third terms,

$$\begin{aligned}
TE_L &= n\int_{\underline{c}}^{s}\Big(\alpha F(c^*)+(1-\alpha)F(s)-F(t)\Big)td\Big(1-\alpha F(c^*)-(1-\alpha)F(s)+F(t)\Big)^{n-1} \\
&\quad +n\underline{c}\Big(1-\alpha F(c^*)-(1-\alpha)F(s)\Big)^{n-1}\Big(\alpha F(c^*)+(1-\alpha)F(s)\Big) \\
&\quad +n\alpha\int_{s}^{c^*}\Big(F(c^*)-F(t)\Big)td\Big(1-\alpha F(c^*)+\alpha F(t)\Big)^{n-1}.
\end{aligned}$$

Use integration by parts again,

$$\begin{aligned}
& TE_L \\
= \ & n\Big(\alpha F(c^*)+(1-\alpha)F(s)-F(t)\Big)t\Big(1-\alpha F(c^*)-(1-\alpha)F(s)+F(t)\Big)^{n-1}\Big|_{\underline{c}}^{s} \\
- \ & n\int_{\underline{c}}^{s}\Big(1-\alpha F(c^*)-(1-\alpha)F(s)+F(t)\Big)^{n-1}\Big(\alpha F(c^*)+(1-\alpha)F(s)-F(t)-tf(t)\Big)dt \\
+ \ & n\underline{c}\Big(1-\alpha F(c^*)-(1-\alpha)F(s)\Big)^{n-1}\Big(\alpha F(c^*)+(1-\alpha)F(s)\Big) \\
+ \ & n\alpha\Big(F(c^*)-F(t)\Big)t\Big(1-\alpha F(c^*)+\alpha F(t)\Big)^{n-1}\Big|_{s}^{c^*} \\
- \ & n\alpha\int_{s}^{c^*}\Big(1-\alpha F(c^*)+\alpha F(t)\Big)^{n-1}\Big(F(c^*)-F(t)-tf(t)\Big)dt \\
= \ & n\int_{\underline{c}}^{s}\Big(t-\frac{\alpha F(c^*)+(1-\alpha)F(s)-F(t)}{f(t)}\Big)\Big(1-\alpha F(c^*)-(1-\alpha)F(s)+F(t)\Big)^{n-1}f(t)dt \\
+ \ & n\alpha\int_{s}^{c^*}\Big(t-\frac{F(c^*)-F(t)}{f(t)}\Big)\Big(1-\alpha F(c^*)+\alpha F(t)\Big)^{n-1}f(t)dt.
\end{aligned}$$

The conditions for L not to exclude any contestant are $t-\frac{\alpha F(c^*)+(1-\alpha)F(s)-F(t)}{f(t)} \geq 0$ for all $t \in [\underline{c}, s]$ and $t-\frac{F(c^*)-F(t)}{f(t)} \geq 0$ for all $t \in [s, c^*]$. It suffices to have $t-\frac{\alpha F(c^*)+(1-\alpha)F(s)-F(t)}{f(t)}$ increasing in $[\underline{c}, s]$ with $\underline{c}-\frac{\alpha F(c^*)+(1-\alpha)F(s)}{f(\underline{c})} \geq 0$, and $t-\frac{F(c^*)-F(t)}{f(t)}$ increasing in $[s, c^*]$ with $s-\frac{F(c^*)-F(s)}{f(s)} \geq 0$.

Similar to the case in separating equilibrium, the two conditions in Proposition

1.4 are also sufficient for those in the previous paragraph. □

1.7 Appendix B: A Numerical Example

In this part, I will present some formula assuming a uniform distribution on $[x, x+1]$. Thus, $F(t) = t - x$ and $f(t) = 1$.

First, consider those in contest H. When $s \geq c^*$,

$$\begin{aligned} b_H(c) &= a\int_s^c td(t-x)^{n-1} + as(s-x)^{n-1} - \int_x^s (1-s+t)^{n-1}dt \\ &= \frac{n-1}{n}a(c-x)^n + \frac{1}{n}a(s-x)^n + ax(c-x)^{n-1} - \frac{1}{n} + \frac{1}{n}(1-s+x)^n \end{aligned}$$

and

$$\begin{aligned} TE_H &= n\int_s^{x+1} b_H(c)dc \\ &= \frac{n-1}{n+1}a[1-(s-x)^{n+1}] + (1+x-s)(a(s-x)^n - 1) \\ &\quad +ax[1-(s-x)^n] + (1-s+x)^{n+1}. \end{aligned}$$

Taking derivative,

$$\frac{\partial TE_H}{\partial s} = 1 + an(1-x)(s-x)^{n-1} - 2an(s-x)^n - (n+1)(1-s+x)^n.$$

When $s < c^*$, notice that $a(c^* - x)^{n-1} = 1$ and $a^{\frac{1}{n-1}}(1-\alpha) = \alpha$. Then,

$$\begin{aligned}
\beta_{MH}(c) &= a\int_s^c td(\alpha c^* + (1-\alpha)t - x)^{n-1} + as(\alpha c^* + (1-\alpha)s - x)^{n-1} \\
&\quad - \int_x^s (1 - \alpha c^* - (1-\alpha)s + t)^{n-1} dt \\
&= \frac{n-1}{n}\alpha^{n-1}(1+c-x)^n - \alpha^{n-1}(1-x)(1+c-x)^{n-1} \\
&\quad + \frac{1}{n}[\alpha^{n-1}(1-\alpha)(1+s-x)^n + (\alpha - (1-\alpha)(s-x))^n].
\end{aligned}$$

Using the fact that $\alpha(1 + c^* - x) = 1$,

$$\beta_{MH}(c^*) = c^* - \frac{1 + c^* - x}{n} + \frac{\alpha^{n-1}}{n}(1-\alpha)(1+s-x)^n + \frac{1}{n}(\alpha - (1-\alpha)(s-x))^n$$

and thus

$$\begin{aligned}
\beta_H(c) &= a\int_{c^*}^c td(t-x)^{n-1} + \beta_{MH}(c^*) \\
&= \frac{n-1}{n}a(c-x)^n + ax(c-x)^{n-1} - \frac{1}{n} \\
&\quad + \frac{\alpha^{n-1}}{n}(1-\alpha)(1+s-x)^n + \frac{1}{n}(\alpha - (1-\alpha)(s-x))^n.
\end{aligned}$$

Total effort can be calculated as

$$
\begin{aligned}
TE_H &= n\Bigg((1-\alpha)\int_s^{c^*}\beta_{MH}(c)dc + \int_{c^*}^{x+1}\beta_H(c)dc\Bigg)\\
&= (1-\alpha)\alpha^{n-1}\Big\{\frac{n-1}{n+1}[(1+c^*-x)^{n+1}-(1+s-x)^{n+1}]\\
&\qquad -(1-x)[(1+c^*-x)^n-(1+s-x)^n]\Big\}\\
&\quad +\frac{n-1}{n+1}a[1-(c^*-x)^{n+1}]+ax[1-(c^*-x)^n]\\
&\quad -(x+1-c^*)+(\alpha-(1-\alpha)(s-x))^{n+1}\\
&\quad +(1-\alpha)\alpha^{n-1}(\alpha-(1-\alpha)(s-x))(1+s-x)^n
\end{aligned}
$$

and its derivative is

$$
\begin{aligned}
\frac{\partial TE_H}{\partial s} &= (1-\alpha)[(n+1)\alpha^n(1+s-x)^n - n\alpha^{n-1}(2s-x)(1+s-x)^{n-1}\\
&\quad -(n+1)(\alpha-(1-\alpha)(s-x))^n.
\end{aligned}
$$

Then, consider formula in contest L. When $s \geq c^*$,

$$
\begin{aligned}
b_L(c) &= \int_x^s td(1-s+t)^{n-1}\\
&= (n-1)\int_x^s t(1-s+t)^{n-2}dt\\
&= \frac{n-1}{n}[(1-s+c)^n-(1-s+x)^n]\\
&\quad -(1-s)[(1-s+c)^{n-1}-(1-s+x)^{n-1}]
\end{aligned}
$$

and

$$\begin{aligned}
TE_L &= n\int_x^s b_L(c)dc \\
&= \frac{n-1}{n+1}[1-(1-s+x)^{n+1}]-(n-1)(s-x)(1-s+x)^n \\
&\quad -(1-s)[1-(1-s+x)^n]+n(1-s)(s-x)(1-s+x)^{n-1}.
\end{aligned}$$

When $s < c^*$,

$$\begin{aligned}
\beta_L(c) &= \int_x^c td(1-\alpha c^*-(1-\alpha)s+t)^{n-1} \\
&= \frac{n-1}{n}[(\alpha-(1-\alpha)s+c)^n-(\alpha-(1-\alpha)s+x)^n] \\
&\quad -(\alpha-(1-\alpha)s)[(\alpha-(1-\alpha)s+c)^{n-1}-(\alpha-(1-\alpha)s+x)^{n-1}]
\end{aligned}$$

$$\begin{aligned}
\beta_{ML}(c) &= \int_s^c td(1-\alpha c^*+\alpha t)^{n-1}+\beta_L(s) \\
&= \frac{n-1}{n}[(\alpha-(1-\alpha)s+c)^n-(\alpha-(1-\alpha)s+x)^n] \\
&\quad -\alpha^{n-1}(1+c)^{n-1}+\frac{1}{n}\alpha^{n-1}(1-\alpha)(1+s)^n \\
&\quad -(\alpha-(1-\alpha)s)(\alpha-(1-\alpha)s+x)^{n-1}
\end{aligned}$$

and thus

$$\begin{aligned}
TE_L &= n\left(\int_x^s \beta_L(c)dc + \alpha\int_s^{c^*} \beta_{ML}(c)dc\right) \\
&= \frac{n-1}{n+1}[1+c^* - (1-\alpha)\alpha^n(1+s)^{n+1}] - 1 + c^*\alpha^n(1+s)^n \\
&\quad + n(\alpha-(1-\alpha)s)(\alpha-(1-\alpha)s+x)^{n-1} \\
&\quad - (n-1)(\alpha-(1-\alpha)s+x)^n \\
&\quad + \frac{n(n-1)}{n+1}(\alpha-(1-\alpha)s+x)^{n+1} \\
&\quad - (n+1)(\alpha-(1-\alpha)s)(\alpha-(1-\alpha)s+x)^n.
\end{aligned}$$

For formula with a more general uniform distribution on $[\underline{c}, \bar{c}]$, simply replace s, c and x with

$$s' = \frac{s}{\bar{c}-\underline{c}}, \quad c' = \frac{c}{\bar{c}-\underline{c}} \quad \text{and} \quad x' = \frac{\underline{c}}{\bar{c}-\underline{c}}$$

respectively in the equations above.

Chapter 2: Mergers in R&D Races

2.1 Introduction

Whether proposed mergers would reduce the level of competition is the focal concern of antitrust reviews. There are a huge literature on modeling and estimating the impacts of a merger, yet price effect is their only subject. Depending on specific market characteristics, competition in developing new products through research and development (R&D) activities can also be a determinant factor for consideration. Recent merger proposals have successfully drawn the attention of antitrust agencies and researchers to this question[1].

Unlike pricing, R&D usually requires substantial investments and often takes up many years. Thus, firms in concentrated markets are more likely to invest in R&D because they can appropriate more returns from innovation. Then, extending further, if there are even fewer rivals, they may want to do more R&D since more returns can be captured. On the other hand, in concentrated industries, firms have

[1]For example, in a number of recent high-profile cases (AT&T/T-Mobile, Applied Materials/Tokyo Electron, and Halliburton/Baker Hughes), the Department of Justice expressed concerns about the loss of innovation competition resulting from a merger between competitors. In like manner, the proposal of John Deere/Precision Planting in high-speed precision planting market was terminated (May 2017). Proposals in the chemical and seed market also raised similar concerns, yet all were approved (ChemChina/Syngenta in May 2017, Dow/DuPont in August 2017 and Bayer/Monsanto in March 2018).

existing products and economic profits associated with them. They invest in R&D partly to protect their existing positions, and also to give themselves a chance to leap ahead of rivals with a major innovation. Once the number of rivals is reduced, thereby reducing research competition among rivals, firms may be less incentivized to invest in research and innovation. Thus, the effects of a merger that would reduce the number of active rivals in a concentrated market is ambiguous, which is confirmed by a growing body of empirical investigations on mergers and R&D. For example, Ornaghi (2009) [34] and Stiebale and Reize (2011) [41] find a decrease in innovative effort after mergers, while Bertrand (2009) [17] and Stiebale (2013) [40] report a significant increase in R&D intensity after mergers.

Theoretical works are even fewer, as MacDonald has concluded in his report (2016) [31] that there is still a very limited base on which such effects can be estimated. To my knowledge, only Davidson and Ferrett (2007) [22], Phillips and Zhdanov (2013) [36], Motta and Tarantino (2017) [33], and Federico, Langus and Valletti (2017) [25] have provided theoretical insights on this topic. In Davidson and Ferrett (2007) [22], firms first decide the level of process R&D, which reduces production cost, and then compete in the product market. They show that when the degree of R&D complementarity is non-trivial, a merger encourages insiders (merger participants) to invest more in R&D and benefits insiders with a lower cost on product market, regardless of the strategic variable in market competition (price vs quantity). Motta and Tarantino (2017) [33] also study a Bertrand game with cost-reducing investment. Under a variety of cases, they show that absent efficiencies, a horizontal merger reduces innovation and suppresses price competition between them.

Though process innovation, analyzed by the previous two papers, is important in certain industries, the competition among firms does not go beyond their current products. On the other hand, my focus in this chapter is on product innovation, which is related to firms' future products, and is more aligned with the concerns of recent merger proposals. For product innovation, Phillips and Zhdanov (2013) [36] study the incentives of merger under a setting of Bertrand competition in differentiated goods. They show that large firms may optimally decide to let small firms conduct R&D and then acquire these small innovative companies. However, they assume that R&D inputs by different firms are the same and that the probability of successful innovation is evenly distributed among firms who conduct R&D. In other words, firms in their model either maintain the R&D effort or quit R&D activity completely. The lack of flexibility in adjusting R&D effort makes their model not well suited for evaluating merger effects. Moreover, they restrict the range of capital ratios of different firms and thus their results do not carry over to mergers among firms of similar size.

In this chapter, I develop an alternative model, a model of contest, for evaluating the likely effects of a merger on R&D investment of competing firms. Such a modeling choice of R&D, which overcomes the weaknesses aforementioned, is not uncommon in the literature (see, for example, Dasgupta and Stiglitz, 1980 [23]; Fudenberg et al., 1983 [26]; Harris and Vickers, 1985 [28]; and Leininger, 1991 [30]), but is rare in merger analysis. In this sense, my model is more close to Federico, Langus, and Valletti (2017) [25] and, similar to theirs, my model is appropriate to the situation where all firms are effective innovators. However, my model is different from theirs in several ways. First, their model is a game of complete information and firms (except the merged firms) are symmetric and thus make the same R&D effort. My model is an incomplete information game. Although firms are symmetric ex ante, they may make different R&D effort in the interim stage, which makes my model closer to an empirical application. Second, the probability of winning in their model is a function of the firm's own effort, while in my model, the winning probability generally depends on the efforts of other firms as well. The technique used to solve for equilibrium is different. Last but not least, in spite of some common results on firms' expected effort, we diverge on the welfare effect of a merger. They posit that consumers are worse off after the merger. Firms in their model do not differ except the status of being merged or not, and thus, a merger always decreases the probability of price competition. But, it is also reasonable to posit that firms in their model employ similar research strategies. As argued by Dasgupta and Stiglitz (1980) [23], if firms tend to imitate each other's research strategy, much of R&D expenditure may be essentially duplicative, and consequently socially wasteful (p. 267). Therefore, welfare analysis shall not neglect the firms' side because they are part of an economy too. My results focus on this aspect and show that if the number of firms is small, due to a significant reduction in total R&D effort, the net surplus (which is the innovator's realized profit subtracted by all firms' R&D effort) tends to increase after a merger. This also suggests an argument put forth earlier by Fullerton and McAfee (1999) [27] and Che and Gale (2003) [20] that the optimal number of competitors is two.

2.1.1 Other Related Literature

My work is directly related to those on asymmetric all-pay contest/auction, because asymmetry is always involved in merger analysis regardless of modeling choice. If firms are symmetric pre-merger, then they must be asymmetric post-merger, let alone they might be asymmetric at first. Only recently, Siegel (2009, 2010) [37, 38] has contributed significantly to the understanding of asymmetries in complete information all-pay auctions[2]. The incomplete information case is considerably less well-understood, mostly due to the difficulty in obtaining explicit solutions. Most research in the latter area considers only 2 players. Amann and Leininger (1996) citeAmann show existence and uniqueness of equilibrium under independent private value setting with continuous signals, while Szech (2011) [42] studies such model with discrete signals. Siegel (2014) [39] also studies discrete signals game, where signals can be correlated and values interdependent, and makes connections between incomplete and complete information games. The only exception is Parreiras and Rubinchik (2009) [35]. They model a contest among many asymmetric players and prove the existence of a unique equilibrium. Although my theoretical results nest within theirs, their model is too abstract for easy use within application. Instead, I offer an explicit characterization of a multi-player asymmetric all-pay contest/auction model for applications, which is my second contribution in this chapter.

In addition, this chapter can be seen as a continuation of the works on mergers

[2] A comprehensive survey of earlier studies on all-pay contests or all-pay auctions can be found in the book of Konrad (2009) [29].

in auction markets. Waehrer (1999) [45] and Dalkir, Logan, and Masson (2000) [21] and Thomas (2004) [43] examine mergers in asymmetric first-price auctions. Brannman and Froeb (2000) [19], Tschantz, Crooke, and Froeb (2000) [44] and Waehrer and Perry (2003) [46] examine mergers in asymmetric second-price auctions. More recently, Mares and Shor (2008) [32] examine mergers in asymmetric common-value auctions. My results, thus, supplement this strand of literature with an analysis on mergers in all-pay auctions.

The rest of this chapter is organized as follow. Section 2 introduces the asymmetric all-pay contest model in the language of a R&D race and characterizes the unique perfect Bayesian equilibrium. Its application to merger analysis is demonstrated in Section 3 and section 4 presents the simulated merger effects. Section 5 concludes. All proof are in the appendices.

2.2 The Model

Consider n $(= n_1 + n_2)$ firms in a R&D race (or a technology procurement). Following the assumptions in Dasgupta and Stiglitz (1980) [4], let firms follow the same research strategy toward some patentable characteritics so that the firm who exerts the most effort will invent first and win the R&D race[3] and let the winner capture all benefits that are to be had among firms (i.e. the winner takes all). Each firm possesses a private signal concerning the potential profit[4]. For simplicity, let

[3]Fudenberg et al. (1983) [26] also assume that an higher level of effort leads to an innovation sooner, though their effort levels are discrete. In this sense, it is also the firm with the most effort that is the first to invent and realizes profits.

[4]As discussed by Bhattacharya (2016) [18], firms' different profits can be stemmed from their differences in delivering a patent to a commercial good, or can be their different assessments of a

each firm's (potential) profit be equal to their signal. Firms decide their investment levels (which I call effort later on) simultaneously based on the observed signals.

Suppose firms are of two types, where n_1 are type 1 and n_2 are type 2. The signals of type 1 firms follow a distribution function F and the signals of type 2 firms follow a (different) distribution function G. Assume further that all signals are independent, that F and G are both continuously differentiable functions on a common support $[0, 1]$, and that the corresponding density functions are continuous and are bounded away from zero for all values in $[0, 1]$[5]. Finally, assume that everything described so far is common knowledge except each firm's private signal.

Each firm decides how much effort to exert in R&D activities simultaneously. Suppose the strategies are symmetric within each type. Specifically, denote $a_i = \alpha(x_i)$, the effort a type 1 firm with signal x_i will exert, and $b_j = \beta(y_j)$ for a type 2 firm with signal y_j. Then, the expected payoffs for firms of each type can be written as

$$\begin{aligned}\Pi_1(a_i, x_i; \alpha, \beta) &= x_i F^{n_1-1}(\alpha^{-1}(a_i)) G^{n_2}(\beta^{-1}(a_i)) - a_i,\\ \Pi_2(b_j, y_j; \alpha, \beta) &= y_j F^{n_1}(\alpha^{-1}(b_j)) G^{n_2-1}(\beta^{-1}(b_j)) - b_j.\end{aligned}$$

The problem is usually solved through first-order conditions, which typically forms a system of differential equations. To simplify the analysis, I follow Amann and Leininger (1996) [16] and define $k(x) = \beta^{-1}(\alpha(x))$, which maps the signal of

potential procurement contract.

[5] This last assumption is the sufficient condition for the uniqueness of equilibrium, according to Parreiras and Rubinchik (2009) [35].

a type 1 firm into the signal of a type 2 firm who exerts the same effort. $k(x)$ is well defined on (0,1][6] and maps [0,1] to [0,1]. Then, the unique equilibrium[7] can be characterized as follow

Theorem 2.1 (Unique Equilibrium)**.** *The following strategies form the unique perfect Bayesian equilibrium*

$$\begin{aligned}
\alpha(x) &= \int_{\max\{k^{-1}(0)\}}^{x} k(t)d[F^{n_1}(t)G^{n_2-1}(k(t))] \\
\beta(x) &= \alpha(k^{-1}(x))
\end{aligned}$$

where $k(x)$ is the solution to the following ordinary differential equation with boundary condition $k(1) = 1$,

$$k(x)[F^{n_1}(x)G^{n_2-1}(k(x))]' = x[F^{n_1-1}(x)G^{n_2}(k(x))]'.$$

The proof essentially transforms the system of differential equations (first-order conditions) into an ordinary differential equation using the defined mapping k. The boundary condition of $k(1) = 1$ simply means that firms with the highest signal exert the same level of effort regardless of their types.

[6] $k(x)$ may have a mass at 0. See the proof of Theorem 2.1 in Appendix A for more detail.

[7] This result is nested in Parreiras and Rubinchik (2009) [35] whose model is more general yet too abstract. For application purpose, I explicitly solve the equilibrium strategies for a case of restricted asymmetry.

2.3 Application to Merger Analysis

In this section, I follow the methodology of Dalkir, Logan and Masson (2000) [21]. Suppose each firm has some signal draws from a common distribution function and selects the highest one. Premerger I define the number of draws by firm i as q_i. The merger of two firms, i and j, are modeled as a single firm with a total number of draws $q_{merger} = q_i + q_j$. If all firms have the same number of draws prior to merger, the post merger number of draws will differ among firms. This necessitates the analysis of asymmetric model in the previous section. If firms are asymmetric premerger, they may or may not be symmetric post merger.

Let us start with the number of firms equal to the number of i.i.d signal draws, which is denoted as n. Mergers are thus a "regrouping" of the signals among the rest of firms. In a typical two-firm merger, the merged firm with two signals faces $n-2$ rivals each with a single signal. To simplify the analytical work (not simulation yet), assume all signals follows a cumulative distribution $F(x) = x^a$ $(a > 0)$ on [0,1].

2.3.1 The symmetric market

I review the symmetric case first in order to have easier comparison with the asymmetric case later. Using the language in the previous section, $F(x) = G(x) = x^a$ and $n_1 + n_2 = n$ fully depict the situation. Therefore, the optimal effort of a firm with signal x is

$$\int_0^x t dt^{(n-1)a} = \frac{(n-1)a}{(n-1)a+1} x^{(n-1)a+1}.$$

Accordingly, the expected effort of a firm is

$$\int_0^1 \frac{(n-1)a}{(n-1)a+1} x^{(n-1)a+1} dx^a = \frac{(n-1)a^2}{[(n-1)a+1](na+1)}.$$

2.3.2 The asymmetric market

In a two-firm merger, there are $n_1 = n - 2$ unmerged firm. Their number and distribution of signals remain unchanged, i.e. $q_i = 1$ and $F(x) = x^a$. For the merged firm, it has $q_{merger} = 2$ signals each following F, and thus the highest signal follows $G(x) = F^2(x) = x^{2a}$. Using the result of Theorem 2.1, $k(x)$ can be solved analytically,

$$k(x) = \begin{cases} 1 + \frac{1}{2}\ln x & \text{if } n = 3 \\ 1 + \frac{1}{n-3}(1 - x^{-\frac{n-3}{2}}) & \text{if } n \geq 3 \end{cases}$$

which is the solution for uniform distribution ($a = 1$). Solutions for other values of a and the detailed derivation is omitted for expositional easiness and they can be found in Appendix B. Based on the expression of $k(x)$, I can calculate the equilibrium effort $\alpha(x)$ and $\beta(x)$.

In Figure 2.1, I illustrate the equilibrium effort using uniform distribution ($a = 1$) for the case of 4[8] premerger firms (postmerger, there are two unmerged firms and one merged firm). It is shown that the merged firm always exerts positive effort, while the unmerged firms do not exert any effort for small signals, which is 0.25 and below in this case. However, when signals are relatively large (0.39 and above), unmerged firms exert a higher effort than the merged firm. This kind

[8]The number of firms does not qualitatively change the shape of equilibrium strategies.

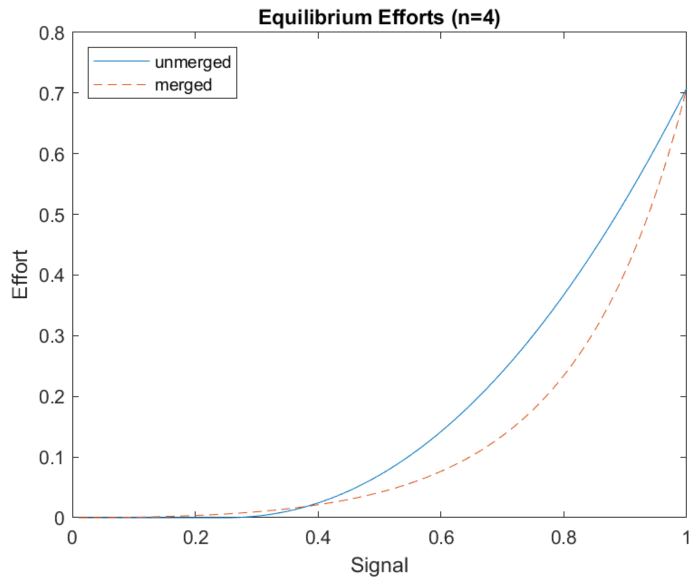

Figure 2.1: Equilibrium Efforts when 4 Premerger Firms

of strategies seems strange at first glance, as compared with their counterpart in other asymmetric auction formats where the strategies of different types do not cross in the interior of bidding interval. The core intuition behind is as follows. The unmerged firms are "weaker" than the merged firm in the sense of first-order stochastic dominance. When the signal is low, weaker firms know that they are not likely to win the competition and thus avoid this irreversible effort. On the other hand, when the signal is high, it is in the interest of weak firms to exert high effort as the chance of winning from doing so is sufficiently high. The reason for this is that although the strong firm is fully aware of the equilibrium strategy of weak firms, the likelihood of her weak rivals having high signals and exert aggressive efforts is rather small, and thus, the strong firm almost "overlooks" the weak firms.

2.4 Calibrating the Effects of a Merger

Before calibrating the model, I would like to discuss some features of the model. Firstly, my model implies that there will be a winner anyway[9]. Then the effort of all firms (including the winner) is a "waste" from the viewpoint of a social planner. The ideal situation would be that the winner realizes all profits from innovation while every firm spend little R&D input. Nevertheless, governments (or procurers) may not have access to firms' signals and thus have to encourage innovations and award patents/contracts through such costly races. Therefore, it is important to calibrate the total effort level in the industry, in addition to the effort of individual firms. It seems that merger would reduce total effort and cause less waste. Nevertheless, it shall not be taken as an implication of this model that monopoly is the best market structure. A monopolist in my model would exert ε effort, which is not desirable because the probability of innovation is also negligible. Moreover, potential entrants, who are absent in my model, shall play a more significant role in a monopoly market[10].

Secondly, inefficiency is present whenever there is asymmetry. By inefficiency, I mean that the firm with the highest signal in the market does not win the R&D race. As can be inferred from Figure 2.1, inefficiency may arise in two scenarios. One is that the merged firm does not possess the highest signal but happens to win

[9] This seems strange, as innovation is not deterministic. But my model does not impose time constraints on firms. There will be some firm who makes an innovation at last and this firm is the winner.

[10] For example, Harris and Vickers (1985) [28] and Leininger (1991) [30] study patent races between incumbent and challenger under complete information and in their environments, both firms' efforts are socially waste and such waste is inevitable.

because all signals of unmerged firms are low and fall inside a neighbourhood of the no-effort region. The other case, which is more likely, is that the merged firm does have the highest signal, yet she loses the race because some unmerged firm exerts a higher level of effort.

Finally, based on the previous two arguments, a merger appears to raise a tradeoff between a reduction in total effort and an increase in inefficiency. Thus, the overall effect, which is the realized profit (the winning signal) subtracted by the total effort, is ambiguous and is worthy of simulation. I label this overall effect as the net surplus on the firms' side from the R&D race.

2.4.1 Calibration of merger effects

Calibration of the model is conducted through numerical calculation and simulation. Except where otherwise stated, I use uniform distribution for simulation. In all cases, I only consider a merger of 2 firms.

Table 2.1 shows the baseline results. The first column is the market structure, depicting the number of firms both premerger and post-merger, and column 2 is the number of signals per firm. Column 3 is the ex ante expected effort of each firm, which adds up to be the expected total effort presented in column 4. It shows that after a merger, each firm is expected to exert more effort, no matter merged or not. On an aggregate level, post-merger total effort increases faster in the number of premerger firms. A merger reduces total effort as long as the number of firms is small, and the cutoff number here is 6, which is a close description of a concentrated

(1) Market structure		(2) q_i	(3) Expected effort	(4) Total effort expected	(5) Total effort simulated	(6) Winner's signal	(7) Ineff. rate	(8) Net surplus
pre-merger	$n = 3$	1	0.167	0.5	0.513	0.755	0	0.242
					(0.333)	(0.194)		(0.199)
post-merger	$n_1 = 1$	1	0.190	0.453	0.420	0.750	0.106	0.330
	$n_2 = 1$	2	0.263		(0.249)	(0.198)		(0.137)
post - pre					-0.093	-0.005		0.088
t-stat					-17.08	-2.387		47.40
pre-merger	$n = 4$	1	0.150	0.6	0.601	0.800	0	0.199
					(0.395)	(0.164)		(0.278)
post-merger	$n_1 = 2$	1	0.171	0.590	0.549	0.795	0.106	0.246
	$n_2 = 1$	2	0.248		(0.347)	(0.168)		(0.237)
post - pre					-0.052	-0.005		0.047
t-stat					-5.95	-3.035		11.08
pre-merger	$n = 5$	1	0.133	0.667	0.646	0.828	0	0.182
					(0.437)	(0.144)		(0.332)
post-merger	$n_1 = 3$	1	0.148	0.666	0.613	0.825	0.077	0.212
	$n_2 = 1$	2	0.221		(0.400)	(0.148)		(0.297)
post - pre					-0.033	-0.003		0.03
t-stat					-2.976	-2.036		4.831
pre-merger	$n = 6$	1	0.119	0.714	0.726	0.859	0	0.134
					(0.480)	(0.120)		(0.392)
post-merger	$n_1 = 4$	1	0.130	0.716	0.705	0.856	0.092	0.150
	$n_2 = 1$	2	0.198		(0.459)	(0.127)		(0.371)
post - pre					-0.019	-0.003		0.016
t-stat					-1.467	-3.887		1.812
pre-merger	$n = 7$	1	0.107	0.750	0.752	0.878	0	0.126
					(0.501)	(0.106)		(0.425)
post-merger	$n_1 = 5$	1	0.115	0.753	0.732	0.876	0.064	0.144
	$n_2 = 1$	2	0.178		(0.493)	(0.110)		(0.417)
post - pre					-0.020	-0.002		0.018
t-stat					-1.242	-2.752		1.552

Note: The t-stat measures the significance of the difference of post-merger and pre-merger efforts. A common threshold for significance at 5% is 1.96.

Table 2.1: Effects of Merger

industry. Column 5 records the simulated total effort.

The results in the last 3 columns are derived through simulation for 1000 times. The numbers in column 6 is the average winning signals, or the average realized profit, with the standard deviation recorded in parenthesis. According to the values presented, inefficiency does exist, and the average realized profit is significantly lower after a merger, but the magnitude of decrease is small (less than 1%). The frequency of inefficient outcome is shown in column 7 (around 10%).

The last column measures the integrated effect on the net surplus. It shows that in all cases[11], a merger increases the net surplus and that the increase is statistically significant if there is a small number of firms premerger. This is aligned with previous findings because the reduction of total effort outweighs the decrease in profit when the number of firms is small. It may further imply that two firms is optimal in industries which are innovation-driven and require intensive R&D investment. Such a result, distinct from the implication of the innovation theory of harm as summarized in Federico (2017) [24], calls for an attention to the welfare analysis where firms shall also be included as part of the economy.

Table 2.2 presents a comparison between symmetric and asymmetric mergers through which triopoly becomes duopoly. The first row, which can be found in Table 2.1 as well, is the case of symmetric firms becoming asymmetric after a merger. The second row shows the case where asymmetric firms merging to become symmetric. The third row shows the case where firms become even more asymmetric after

[11] The net surplus does not always increase after a merger. If the number of firms is large enough, for example 70, the loss in efficiency is dominant, and thus reduces the net surplus.

(1) Market structure		(2) q_i	(3) Expected effort	(4) Total effort	(5) Winner's signal	(6) Ineff. rate	(7) Net surplus
pre-merger	$n = 3$	1	0.167	0.5	0.755	0	0.242
					(0.194)		(0.199)
post-merger	$n_1 = 1$	1	0.190	0.453	0.750	0.106	0.330
	$n_2 = 1$	2	0.263		(0.198)		(0.137)
post - pre					-0.005		0.088
t-stat					-2.387		47.40
pre-merger	$n_1 = 2$	1	0.171	0.590	0.795	0.106	0.246
	$n_2 = 1$	2	0.248		(0.168)		(0.237)
post-merger	$n = 2$	2	0.267	0.534	0.800	0	0.264
					(0.164)		(0.159)
post - pre					0.005		0.018
t-stat					0.673		1.99
pre-merger	$n_1 = 2$	1	0.171	0.590	0.795	0.106	0.246
	$n_2 = 1$	2	0.248		(0.168)		(0.237)
post-merger	$n_1 = 1$	1	0.193	0.504	0.784	0.177	0.359
	$n_2 = 1$	3	0.311		(0.176)		(0.167)
post - pre					-0.011		0.113
t-stat					-1.430		12.33

Table 2.2: Symmetry vs. Asymmetry

a merger. Please note that I omit the simulated total effort, because it can be inferred directly from winner's signal and net surplus. Since the number of firms is still small, merger leads to higher expected effort level per firm and reduces total effort as expected. Moreover, a net surplus gain is realized in all cases.

2.4.2 Sensitivity

One calibration issue is whether it is sensitive to the chosen density function, i.e. the uniform distribution. Without a detailed analysis, I examine the sensitivity with an example. Suppose each firm has two signals from a uniform distribution. Then the highest signal does not follow uniform. Table 2.3 presents a contrast of calibrated results for triopoly merging to be duopoly. The case of uniform distribution is in the first row, and the new model the last row.

(1) Market structure		(2) q_i	(3) Expected effort	(4) Total effort	(5) Winner's signal	(6) Ineff. rate	(7) Net surplus
pre-merger	$n = 3$	1	0.167	0.5	0.755	0	0.242
					(0.194)		(0.199)
post-merger	$n_1 = 1$	1	0.190	0.453	0.750	0.106	0.330
	$n_2 = 1$	2	0.263		(0.198)		(0.137)
post - pre					-0.005		0.088
t-stat					-2.387		47.40
pre-merger	$n = 3$	2	0.229	0.686	0.863	0	0.155
					(0.119)		(0.325)
post-merger	$n_1 = 1$	2	0.256	0.530	0.857	0.121	0.308
	$n_2 = 1$	4	0.274		(0.124)		(273)
post - pre					-0.006		0.153
t-stat					-6.262		26.82

Table 2.3: Uniform vs. Non-uniform Distribution

The two sets of results do not exhibit qualitative differences. Conclusions in previous part can be carried over to the non-uniform case, which is a desirable property since close model shall have close outcomes. It may be extended further that if firms have similar initial shares of signals, merger would yield to similar consequences based on my model.

2.5 Conclusion

This chapter develops a model for analyzing mergers in markets where, apart from price competition, R&D investment decision also plays an important role. R&D investment is modeled as an all-pay contest, and I give an explicit characterization of the unique solution to this multi-player asymmetric all-pay contest. Simulation shows that each firm is expected to exert more effort after a merger, but the total effort will be lower after a merger if the industry is concentrated premerger. Merger may cause inefficiency, but the loss is not large. As an overall estimate of merger effects, the net surplus tends to increase after merger if the number of firms is small.

As an early attempt to analyze merger effects on R&D investment using an incomplete information contest model, there are several directions to extend current work. Firstly, in my model, the winner takes all potential profits, while efforts by the other firms are simply wasted. But it may not be so in real life. Firms, though falling behind, may come up with some second-best substitutes and share a fraction of the signaled profits. It would be interesting to explore models that incorporate the multi-prize feature into merger analysis.

Secondly, the current model considers R&D investment for only one round. It is reasonable to extend it to multiple rounds as R&D projects can be progressive. Moreover, it is possible for a temporary laggard to catch up and even surpass the leader. Thus, to develop methods for estimating merger effects in the long run is important, yet challenging as well.

Thirdly, current analysis is under the independent private value framework. Since my model in this chapter is more appropriately applied to industries with high concentration, firms' profit signals or their costs for R&D activity are likely to be correlated or affiliated. How merger outcomes would change when affiliation is taken into consideration remains an open question.

Lastly, the focus of this chapter is the likely effects of merger on R&D investment decisions, while the more direct effects on price competition is captured by the signals for simplicity. It is, thus, worthwhile to develop a comprehensive model which unifies both price and R&D competitions. All of these, however, are left for future research.

2.6 Appendix A: Proof of Theorem 2.1

The existence and uniqueness of equilibrium is proved in Parreiras and Rubinchik (2009) [35] in a more general sense. It is then sufficient for me to prove the equilibrium using necessary conditions.

Before proving, I would like to define the effort distribution function $M(a) := F(\alpha^{-1}(a))$ and $N(b) := G(\beta^{-1}(b))$ as in Amann and Leininger (1996) [16]. I present some general results concerning the equilibrium below. These are of the same essence as those in Amann and Leininger (1996) [16], but the proofs are different from theirs.

Lemma 2.1 (Common Support)**.** *$supp(M) = supp(N)$.*

Proof. Suppose $a = \alpha(x) \in supp(M)$, yet $a \notin supp(N)$. Then there is an open neighborhood of a, $U(a)$, such that for all $a' \in U(a)$, $N(a') = N(a)$. Suppose $N(a) > 0$. Then if type 1 firms all lower their effort from a to $a' < a$ but $a' \in U(a)$, their probability of winning does not change, while they may save cost through reducing effort from a to a'. This improves type 1 firms' payoffs and, thus, a cannot be optimal, which contradicts $a = \alpha(x)$. Consequently, $N(a) = 0$.

A similar argument holds for type 2 firms. □

Lemma 2.2 (Full Support)**.**

$$\begin{aligned} supp(M) &= [0, \max_{x\in[0,1]} \alpha(x)], \\ supp(N) &= [0, \max_{y\in[0,1]} \beta(x)]. \end{aligned}$$

Proof. Suppose there is a "hole" (s, t), $0 < s < t < \max_{y\in[0,1]} \beta(y)$, over which M is constant, while s and t belong to the support of M. Then, by Lemma 2.1, N must be constant over (s, t). Since $N(s) = N(t)$, it can never be optimal for type 2 firms to exert effort $b = t$ by the same argument as in Lemma 2.1. Hence, such a hole in the interior of $[0, \max_{x\in[0,1]} \alpha(x)]$ cannot exist. Neither can it exist in the interior of $[0, \max_{y\in[0,1]} \beta(y)]$. □

Lemma 2.3 (Monotonicity). *Let $x > x'$, $a = \alpha(x)$, and $a' = \alpha(x')$. Then $N(a) \geq N(a')$.*

Proof. By definition of equilibrium, we must have

$$\begin{aligned} \Pi_1(a, x; \alpha, \beta) &\geq \Pi_1(a', x; \alpha, \beta), \\ \Pi_1(a', x'; \alpha, \beta) &\geq \Pi_1(a, x'; \alpha, \beta). \end{aligned}$$

Plug in the expression of Π_1 and rearrange the terms, we get

$$(xM^{n_1-1}(a) - x'M^{n_1-1}(a'))(N^{n_2}(a) - N^{n_2}(a')) \geq 0.$$

By definition of effort distribution, $M(a) = F(\alpha^{-1}(a)) = F(x) > F(x') = M(a')$. Then the first term is positive and, thus, the second term must be non-negative, which implies $N(a) \geq N(a')$. □

Lemma 2.4 (No Atoms). *M is continuous on $[0, \beta(1)]$ with $\beta(1) \leq 1$, N is continuous on $[0, \alpha(1)]$ with $\alpha(1) \leq 1$.*

Proof. Suppose N is not continuous at z; i.e., let $z \in (0, \alpha(1)]$ and $\delta > 0$ such that $N(z) > N(z - \varepsilon) + \delta$ for all $\varepsilon < \varepsilon_1(z, \delta)$.

Using the monotonicity in Lemma 2.3, we have

$$\begin{aligned}
& \Pi_1(z,x;\alpha,\beta) - \Pi_1(z-\frac{\varepsilon}{2},x;\alpha,\beta) \\
= \ & xM^{n_1-1}(z)N^{n_2}(z) - z - [xM^{n_1-1}(z-\frac{\varepsilon}{2})N^{n_2}(z-\frac{\varepsilon}{2}) - (z-\frac{\varepsilon}{2})] \\
> \ & xM^{n_1-1}(z)[N^{n_2}(z) - N^{n_2}(z-\frac{\varepsilon}{2})] - \frac{\varepsilon}{2} \\
> \ & xM^{n_1-1}(z)[N^{n_2}(z) - (N(z)-\delta)^{n_2}] - \frac{\varepsilon}{2} \\
> \ & \frac{\varepsilon}{2}
\end{aligned}$$

for any $\varepsilon < xM^{n_1-1}(z)[N^{n_2}(z) - (N(z)-\delta)^{n_2}] \equiv \varepsilon_2(x,z,\delta)$.

If we define $\bar{\varepsilon} = \min\{\varepsilon_1(z,\delta), \varepsilon_2(x,z,\delta)\}$, then for all $\varepsilon < \bar{\varepsilon}$,

$$\Pi_1(z,x;\alpha,\beta) - \Pi_1(z-\frac{\varepsilon}{2},x;\alpha,\beta) > \frac{\varepsilon}{2} > 0,$$

which means that type 1 firm will not exert any effort in the range of $[z-\bar{\varepsilon}, z]$. As a consequent, $M(\cdot)$ is constant for $[z-\bar{\varepsilon}, z]$. However, by the same argument as in Lemma 2.1, z cannot be a best response for type 2 firms, which is in contradiction to $z = \alpha(x)$ for some $x \in [0,1]$. Thus, N must be continuous.

A symmetric argument works for M. □

Lemma 2.5. *If $F(0) = G(0) = 0$, then* $\min\{M(0), N(0)\} = 0$.

Proof. Suppose $M(0) = s > 0$ and $N(0) = t > 0$. Then for any $x \neq 0$, take some

$\epsilon < \frac{n-1}{n} x s^{n_1-1} t^{n_2}$,

$$\begin{aligned}
\Pi_1(\epsilon, x; \alpha, \beta) &= x M^{n_1-1}(\epsilon) N^{n2}(\epsilon) \\
&\geq x s^{n_1-1} t^{n_2} - \epsilon \\
&> \frac{1}{n} x s^{n_1-1} t^{n_2} \\
&= \Pi_1(0, x; \alpha, \beta).
\end{aligned}$$

However, this means that for almost any type 1 firm (except those with signal 0), a strictly positive effort would yield a better payoff than no effort, which is a contradiction to equilibrium. Thus, either $M(0) = 0$ or $N(0) = 0$, whichever completes the proof. □

Lemmas 2.1, 2.2 and 2.3 together implies that $\alpha(1) = \beta(1)$. Also, based on the 5 Lemmas, $k(x)$ is well defined on (0,1] and maps [0,1] to [0,1]. Moreover, $k(x)$ is strictly increasing except possibly on $k^{-1}(0)$.

We are now ready to prove Theorem 2.1.

Proof of Theorem 2.1. The expected payoffs for firms of each type are

$$\begin{aligned}
\Pi_1(a, x; \alpha, \beta) &= x F^{n_1-1}(\alpha^{-1}(a)) G^{n_2}(\beta^{-1}(a)) - a \\
\Pi_2(b, y; \alpha, \beta) &= y F^{n_1}(\alpha^{-1}(b)) G^{n_2-1}(\beta^{-1}(b)) - b
\end{aligned}$$

where I will suppress subscripts i and j for succinctness. Let f and g be the density function of F and G respectively. Then, the first-order conditions are

$$\begin{aligned} & x\Big((n_1-1)F^{n_1-2}(x)f(x)G^{n_2}(\beta^{-1}(\alpha(x)))\tfrac{1}{\alpha'(x)} \\ + \quad & n_2F^{n_1-1}(x)G^{n_2-1}(\beta^{-1}(\alpha(x)))g(\beta^{-1}(\alpha(x)))\tfrac{1}{\beta'(\beta^{-1}(\alpha(x)))}\Big) \quad = 1 \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} & y\Big(n_1F^{n_1-1}(\alpha^{-1}(\beta(y)))f(\alpha^{-1}(\beta(y))G^{n_2-1}(y)\tfrac{1}{\alpha'(\alpha^{-1}(\beta(y))} \\ + \quad & (n_2-1)F^{n_1}(\alpha^{-1}(\beta(y))G^{n_2-2}(\beta^{-1}(y)g(y)\tfrac{1}{\beta'(y)}\Big) \quad = 1. \end{aligned} \tag{2.2}$$

Given the definition of $k(x)$, we have $\beta(k(x)) = \alpha(x)$ and

$$k'(x) = (\beta^{-1})'(\alpha(x))\alpha'(x) = \frac{\alpha'(x)}{\beta'(\beta^{-1}(\alpha(x)))}.$$

Then, equation (2.1) can be rewritten as

$$\begin{aligned} \alpha'(x) &= x\Big((n_1-1)F^{n_1-2}(x)f(x)G^{n_2}(k(x)) \\ &\quad +n_2F^{n_1-1}(x)G^{n_2-1}(k(x))g(k(x))k'(x)\Big) \\ &= x[F^{n_1-1}(x)G^{n_2}(k(x))]'. \end{aligned}$$

Let $y = k(x)$ in equation (2.2) and observe that $\alpha^{-1}(\beta(k(x))) = x$,

$$\begin{aligned}\alpha'(x) &= k(x)\Big(n_1 F^{n_1-1}(x) f(x) G^{n_2-1}(k(x)) \\ &\quad +(n_2-1)F^{n_1}(x)G^{n_2-2}(k(x))g(k(x))k'(x)\Big) \\ &= k(x)[F^{n_1}(x)G^{n_2-1}(k(x))]'.\end{aligned}$$

Therefore, a necessary condition for $k(x)$ is such that

$$k(x)[F^{n_1}(x)G^{n_2-1}(k(x))]' = x[F^{n_1-1}(x)G^{n_2}(k(x))]'.$$

This is an ordinary first-order differential equation, which admits a unique solution with boundary condition $k(1) = 1$.

Then, $k(x)$ yields the unique equilibrium strategies

$$\begin{aligned}\alpha(x) &= \int_{\max\{k^{-1}(0)\}}^{x} k(t)d[F^{n_1}(t)G^{n_2-1}(k(t))] \\ \beta(x) &= \alpha(k^{-1}(x))\end{aligned}$$

where $\alpha(x) = 0$ if and only if $x \in k^{-1}(0)$ by Lemma 2.5. □

2.7 Appendix B: Derivation in Merger Analysis

In this section, I derive the analytical solution to the model introduced in Merger Analysis. The boundary condition that $k(1) = 1$ is implied in each of the following solutions.

When the signals follow $F(x) = x^a$, after a merger, there are $n_1 = n - 2$ unmerged firms whose signals follow distribution F. For the merged firm, its highest signal follows $G(x) = F^2(x) = x^{2a}$. Then, according to Theorem 2.1,

$$k(x)[(x^a)^{n-2}]' = x[(x^a)^{n-3}k^{2a}(x)]'$$

which is, after simplification,

$$(n-2)x^{a-1} = (n-3)k^{2a-1}(x) + 2xk^{2a-2}(x)k'(x). \tag{2.3}$$

Several special cases shall be addressed before I give a general solution.

When $a = 1$, equation (2.3) becomes

$$k'(x) + \frac{n-3}{x}k(x) = \frac{n-2}{x}.$$

The solution to this first order linear differential equation is

$$k(x) = \begin{cases} 1 + \frac{1}{2}\ln x & \text{if } n = 3 \\ 1 + \frac{1}{n-3}(1 - x^{-\frac{n-3}{2}}) & \text{if } n \geq 3. \end{cases}$$

When $a = \frac{1}{2}$, equation (2.3) becomes

$$\frac{k'(x)}{k(x)} + \frac{n-3}{2x} = \frac{n-2}{2}x^{-\frac{3}{2}}$$

which is equivalent to

$$d\ln(k(x)) = \Big(\frac{n-2}{2}x^{-\frac{3}{2}} - \frac{n-3}{2x}\Big)dx.$$

Therefore, the solution is

$$k(x) = x^{-\frac{n-3}{2}}e^{(n-2)(1-x^{-\frac{1}{2}})}.$$

When $a \neq 1$ **or** $\frac{1}{2}$, equation (2.3) is a Bernoulli differential equation. Define $z = k^{1-(2-2a)} = k^{2a-1}$ to transform it into a linear differential equation such that

$$z' + \frac{(2a-1)(n-3)}{2x}z = \frac{(2a-1)(n-2)}{2}x^{a-2}.$$

The solution to z is

$$z(x) = \frac{(2a-1)(n-2)}{(2a-1)(n-2)-1}x^{a-1} - \frac{1}{(2a-1)(n-2)-1}x^{-\frac{(2a-1)(n-3)}{2}}$$

and then

$$k(x) = [z(x)]^{\frac{1}{2a-1}}.$$

Chapter 3: Sequential Auctions with New Entrants

3.1 Introduction

Auctions are usually practiced in a sequential manner and, in a variety of context, auctions are expected to be held periodically even without an end. One of the examples is the monthly car plate auction in Shanghai, China. The city government issues thousands of car plates every month and allocates these car plates among hundreds of thousands of buyers through a modified pay-as-bid auction. A natural question arises: is such a mechanism efficient? That is, is a buyer with higher valuation more likely to get an object? Moreover, is a buyer with higher valuation more likely to get an object earlier? To answer these questions, I investigate a simplified version of this practice by a model of sequential second-price auctions on an infinite time horizon.

The bulk literature on sequential auctions stems from a seminal paper by Milgrom and Weber (2000) [66][1]. They derive the symmetric equilibrium bidding strategies in both sequential first-price and second-price auctions and prove that in an independent private value setting where buyers have unit demand, the sequence of prices forms a martingale. However, an observation by Ashenfelter (1989) [47]

[1] It was written in 1982 and was not published until 2000.

indicates that prices in sequential auctions for identical items (rare wines) do not remain constant but rather follow a declining path. Such a phenomenon, named "declining price anomaly", was also empirically supported by other authors[2].

Afterwards, many research on sequential auctions has been devoted to compromising the anomaly by modifying Milgrom and Weber's model in different aspects. McAfee and Vincent (1993) [61] introduce risk preferences, Bernhardt and Scoones (1994) consider stochastically equivalent goods, Menezes and Monteiro (1997) [63] study the effect of participation fees, Jeitschko (1999) [59] investigates uncertainty on the supply side, and Gale and Stegeman (2001) [54] add asymmetry among bidders into sequential auctions with complete information. More recently, Mezzetti (2011) [64] shows a different kind of risk aversion, called "aversion to price risk", can explain declining prices in sequential auctions. Hu and Zou (2015) [57] further generalize the result in Mezzetti (2011) [64] by considering a more general utility function. Ghosh and Liu (2017) [55] let the bidders to form beliefs on the number of inactive bidders. With a richer set of beliefs, they show that the equilibrium depends on the entire history of prices and that the equilibrium generates a downward price trend in expectation.

Although substantial progress has been made, all the works mentioned above on sequential auctions are still restricted to only finite periods and a fixed population of bidders. Nonetheless, sequential auctions may express certain dynamic features and the knowledge of sequential auctions so far is very limited. There are three

[2] See Ashenfelter and Genovese (1992) [48], McAfee and Vincent (1993) [61], Beggs and Graddy (1997) [50], Van den Berg et al. (2001) [70] and Lambson and Thurston (2006) [60].

papers in the literature that address the dynamics of an infinite-horizon model of sequential auctions. McAfee and Vincent (1997) [62] focus on the problem of an auctioneer who sells a single good optimally by setting a sequence of reserve prices. It is not a trivial possibility that, in the presence of a reserve price, the auction ends with no sale. When no sale occurs, the auctioneer tends to sell the good again with a new and lower reserve price. They characterize the Perfect Bayesian Equilibrium in both sequentially optimal second-price and first-price auctions.

Instead of selling only one good, I consider the case where the auctioneer offers one good in every period. In this sense, this chapter is more closely related to those of Said (2011, 2012) [68, 69]. His first paper (Said, 2011 [68]) shows the equilibrium bidding strategy in a sequential second-price auctions of stochastically equivalent objects. In that model, buyers draw new private valuations whenever a seller arrives. This assumption, however, does not carry to my model as I study the cases where buyers' private valuations are persistent over periods.

The second paper by Said (2012) [69] is more general and comprehensive. He examines an environment where seller offers a random number of objects each period and where, with entry and exit, the number of buyers whose valuations are persistent is also random in each period. He proves that a dynamic VCG mechanism is efficient, and he also proposes an indirect implementation of the dynamic VCG mechanism through sequential ascending auctions. The underlying thought is that when a bidder quits the current auction, his valuation is indirectly revealed to all remaining ones, which allows the remaining buyers to update their beliefs and bidding strategies, and thus, the indirect mechanism may achieve an outcome that is equivalent to the direct mechanism.

In spite of the elegant theoretical results, such efficient mechanisms are rarely observed due to the complicated calculation of payoffs or optimal bidding strategies. This chapter models a more common form of sequential auctions by extending the original Milgrom and Weber's model to an infinite time line. The dynamic feature is captured by admitting one additional bidder after each auction. With a mild restriction on the discount factor, I establish a symmetric and stationary equilibrium where buyers behave according to their private valuation less a pivotal continuation value, the value of participating in future auctions. I also show that the price path in such equilibrium is weakly decreasing, and thus contribute to the literature on sequential auctions and the explanation of declining price anomaly.

The rest of the chapter is organized as follows. Section 2 describes the setting of the model. Section 3 first demonstrates properties of the stationary equilibrium of the sequential second-price auctions model, then provides conditions for the existence of equilibrium, and finally discusses the model with that of sequential bargaining. The equilibrium results are extended to sequential auctions of other formats in section 4. Comments and conclusions are presented in section 5. All proofs are in the appendix.

3.2 The Model

Consider a discrete and infinite time line where an auction is held in each period. There is a large pool of risk-neutral buyers with unit demand. Each buyer has a valuation (type) x_i, which follows a commonly known distribution function F on $[0, 1]$ with a continuous density function f. The valuation is private knowledge and does not change across time. Finally, assume that buyers discount the future with a factor $\delta \in (0, 1)$.

In each period, the auctioneer offers an identical object through a sealed-bid second-price auction. After an auction, the winner leaves the auction with the object while the rest of the buyers remain on site. Before the next auction begins, a new buyer arrives at the auction site and the auctioneer announces the winning bid[3] of the last auction. With a new buyer coming in, the number of competitors in each auction, denoted as n, is kept constant[4]. With the announcement of winning bid, the new entrant is not informationally disadvantaged compared with those remainders. This allows all buyers to update their belief on the distribution of competitors, which is important to form the stationary equilibrium in this chapter. Learning in sequential auctions is also studied by Jeitschko (1998) [58] and Ghosh and Liu (2017) [55].

Since the sequential second-price auctions model is a dynamic game of incomplete information, the equilibrium concept I use is that of perfect Bayesian equilibrium. This solution concept requires that behaviors be rational with respect to agents' beliefs, and that agent's beliefs be updated according to Bayes' rule wherever possible.

[3]Here I assume that the winning bid rather than the winning price is disclosed. In sequential second-price auctions, the announced winning price reveals one (the highest) of the valuations among the remaining bidders, which introduces asymmetry among bidders and thus complicates the model.

[4]This assumption allows me to avoid the effect of changing n when deriving the value functions.

To solve this infinite-horizon model, I take the approach in Fudenberg, Levine and Tirole (1985) [52] where they first assume the existence of an equilibrium and then compute it from the differential equations resulting from a dynamic programming method. In the same spirit, I restrict my attention to perfect Bayesian equilibrium that is stationary and symmetric. By stationarity I mean that the strategies used by buyers can depend only on his private information.

3.3 Continuation Value

With the winning bid made public, the winner's valuation, as well as the distribution of the remaining $n-1$ bidders, can be inferred provided that all buyers follow the same monotonic bidding strategy. Let y be the minimum of all previously announced winning valuations[5], which is sufficient to be an upper bound of the types of remaining buyers. The bidding strategy may be a function of both the buyer's private type and the upper bound of the types of remainders, i.e. $b = b(x, y)$. Also denote $V(x, y)$ the expected value in this scenario, i.e. the expected value to a buyer with valuation x when the types of all the other bidders in the auction are capped by y.

Suppose the remaining buyers hold the following beliefs:

(B1) Distribution all remainders are distributed on the truncated interval $[0, y]$, while the new entrant's valuation is a draw from the entire support.

(B2) Strategy the new entrant uses the same bidding strategy as the remaining

[5] In the case there is no previous auction, let $y = 1$.

buyers.

Then, the Bellman Equation for a remaining buyer with valuation x_i is

$$\begin{aligned} V(x_i, y) &= \max_{b_i} E[x_i - \max b_{-i} | b_i > \max b_{-i}] F(b^{-1}(b_i)) \left(\tfrac{F(b^{-1}(b_i))}{F(y)}\right)^{n-2} \\ &+ \delta \textstyle\int_{b^{-1}(b_i)}^{y} V(x_i, y') d\tfrac{F^{n-1}(y')}{F^{n-2}(y)} \\ &+ \delta V(x_i, y)(1 - F(y)) \end{aligned} \tag{3.1}$$

where b_{-i} means the bid made by all bidders other than i. The first term describes the expected payoff when the buyer wins the current auction. The second term reflects the continuation value if the winner in the current auction has a valuation $y' \in (x_i, y)$, implying that the newcomer's valuation is below y, and the last term shows the continuation value if the new entrant's valuation is above y.

The equilibrium bidding strategy is not obvious with respect to the Bellman equation (3.1) per se. But there is a common conclusion in the literature on sequential auctions that a buyer's bidding strategy in a sequential second-price auction is the buyer's private value less his continuation value. Therefore, the crux of the problem is to figure out the continuation value $V(x, y)$.

Let us first consider a new entrant in some period. The current auction ends with two possible outcomes: either the newcomer wins the object and receives a payoff of $x_j - \max b_{-j}$, or he loses and enters the next period as one of the remaining buyers with some "announced" highest type in the next auction $y' \in (x_j, y)$. In the latter case, the problem of the new entrant will not be different from other remaining buyers. This means that the new entrant's continuation value can also be characterized by V, which implies that the new entrant's bidding strategy is also the same as the remaining bidders.

Now that all buyers, either the remainders or the new entrant, employ the same bidding strategy, we may proceed to calculate the expected payoffs in a stationary equilibrium with a further assumption on the bidding function.

Proposition 3.1 (Equilibrium payoff)**.** *Suppose there is a symmetric, monotonic and stationary equilibrium where b is a function of x only. Then, the expected payoff, or the continuation value, V satisfies*

$$V(x,y) = \int_0^x \frac{P^{n-1}(t)}{P^{n-2}(y)} dt$$

where

$$P(x) = \frac{F(x)}{1-\delta+\delta F(x)}.$$

Proposition 3.1 provides a succinct expression for the value function $V(x,y)$. Recall that in any standard auction, the derivative of the value function (with respect to the private valuation) is the probability of winning. Thus, we may conjecture that $V_1(x,y)$ also measures some probability of winning. To see this, let us first look into $P(x)$. Observe that

$$P(x) = F(x) + \delta(1-F(x))\Big[F(x) + \delta(1-F(x))[\cdots]\Big],$$

which is the sum of a discounted flow of probabilities that a pivotal buyer would win. This means that $P(x)$ is the probability of winning against a flow of new entrants. In other words, if we consider the flow of new entrants as one buyer whose valuation is drawn from the entire support, then $P(x)$ is simply the probability of winning against this "one player". Thus, the role of $P(x)$ in an infinite sequence of auctions is just like the role of $F(x)$ in a one-shot auction. Consequently, given the information structure that there is one new entrant whose valuation is drawn from the entire support and the other $n-2$ remaining buyers whose valuations are less than y, the probability that a buyer with valuation x wins in the auction is

$$P(x)\Big(\frac{P(x)}{P(y)}\Big)^{n-2} = V_1(x,y)$$

and, not surprisingly, $P(x) = V_1(x,x)$. It is also intuitive that $V_1(x,y)$, the probability of winning, is increasing in x, the buyer's own valuation, and decreasing in y, the last winning valuation, which indicates the level of competition in the current auction.

3.4 Equilibrium

Now that some features of the stationary equilibrium have been characterized, we turn to explore the conditions under which such an equilibrium exists. A candidate strategy for the equilibrium, which is aligned with Proposition 3.1, is proposed:

(S) Consider a buyer with valuation x_i. Let $b(x_i) = x_i - \delta V(x_i, x_i)$, which is the buyer's valuation less his pivotal continuation value.

Noticed that in the proposed strategy (**S**), a new entrant employs the same strategy as the remaining bidders. Such argument is straightforward if the new entrant has a valuation below y, since his continuation value is the same as the remaining oners. If the new entrant has a valuation above y, which means that he has the highest valuation in the current auction, any bid above $b(y)$ could secure the good with the same payment, given the strategies of the remainders fixed. Therefore, a sincere bid[6] that follows the same functional form as described would not harm the new entrant.

Let us then proceed to verify the monotonicity of the bidding strategy as a first step of the proof of equilibrium.

Proposition 3.2 (Monotonicity)**.** *The bidding function $b(x)$ is increasing in x.*

A direct implication of Proposition 3.2 concerns the evolution of prices in this sequence of auctions.

Corollary 3.1 (Declining prices)**.** *In the infinite-horizon sequential second-price auctions, the price path is non-increasing.*

The underlying thought of Corollary 3.1 is straightforward. Consider some period with the lowest historical winning valuation as y. If the new entrant's valuation is above y, then the winning price in the current period remains the same as in the previous one. If the new entrant's valuation is below y, which means every

[6] I avoid using the term of "continuation value" to describe the strategy for this particular kind of new entrant, because strictly speaking, such new entrant wins the current auction in equilibrium and does not continue in the game. The possibility that such new entrant would like to obtain the good later at a lower price is discussed in detail in the proof of Theorem 3.1.

buyer's valuation is less than y, the winning price in the current period has to be lower than the last one according to Proposition 3.2.

Since bidding strategies b is strictly increasing, behavior along the equilibrium path is perfectly separating, which implies that Bayesian updating fully determines beliefs. To determine optimality off the equilibrium path, which is not necessarily stationary, we need to consider the behavior of buyers after a deviation. Since such post-deviation histories are zero probability events, off-equilibrium beliefs can be chosen arbitrarily. Therefore, suppose that, after a deviation, buyers still believe that the deviating one is behaving in accordance with b.

This specification of off-equilibrium beliefs is particular. These beliefs are consistent with Bayes' rule even after zero probability histories, which is equivalent to the condition of preconsistency in Hendon, Jacobsen and Sloth (1996) [56][7]. They argue that preconsistency is sufficient to apply one-shot deviation principle in an extensive form game of incomplete information. Perea (2002) [67] shows that a weaker condition, called updating consistency, is both sufficient and necessary for the one-shot deviation principle. Then, the following result can be established.

Theorem 3.1 (Equilibrium)**.** *There exists $\bar{\delta} < 1$ such that $\forall \delta \in [0, \bar{\delta}]$, the strategy (**S**) and the beliefs (**B**) form a stationary perfect Bayesian equilibrium in the infinite-horizon sequential second-price auctions.*

Intuitively, given the expectation of declining prices, the only possible devi-

[7] These beliefs also satisfy the "no-signaling-what-you-don't-know condition" in Fudenberg and Tirole (1991) [53]. This means that a conditional probability system could be constructed for this equilibrium such that it satisfies Fudenberg and Tirole's conditions for perfect extended Bayesian equilibrium. The set of all such equilibria coincides with the set of sequential equilibria in finite games.

ation from the equilibrium is a strategic delay of serious bidding, i.e. shading the bid in one period in order to win an item later with a (possibly) lower price. If the discount factor is equal to 1, which means waiting is at no cost, everyone has an incentive to deviate. On the other hand, if the discount factor is 0, which makes it a one-time game in nature, it is optimal for everyone to adhere to the prescribed strategy. Therefore, the equilibrium is supported by a range of discount factor that is bounded away from 1[8]. A numerical exercise shows that if F is a uniform distribution on $[0, 1]$, $\bar{\delta}$ is around 0.92 for $n = 10$[9].

This part shall be concluded with some comments related to the literature on durable good monopoly and sequential bargaining with the interpretation that a single seller faces a continuum of infinitesimal buyers. Firstly, the features of the current model correspond to the Coasian dynamics (Coase, 1972 [51]). Needless to say, both models exhibit a decreasing sequence of prices, which is established upon a stationarity assumption on strategies (Ausubel and Deneckere, 1989 [49]). Moreover, the periodic efficiency sets forth the skimming property that higher-valued buyers buy earlier.

Secondly, there is a discrepancy when the discount factor δ becomes close to 1. In Fudenburg, Levine, and Tirole (1985) and others, perfect Bayesian equilibrium always exists, while stationary equilibrium does not exist in the current model. This is becuase buyers no longer have incentives to separate themselves. When δ

[8]In practice, some time is required between two consecutive auctions for organization and for buyers to show up. It is reasonable to skip the discussion for equilibrium at δ close to 1, though this range of δ may be of some theoretical interest.

[9]Fixing uniform distribution, further experiments suggest that $\bar{\delta}$ decreases as n becomes larger and that given n, higher valuation supports a higher δ. The latter statement indicates that stationary equilibrium exists at earlier stage even though discount factor is large.

approaches to 1, it becomes costless to wait and mimic lower-valued buyers and thus stationary equilibrium as described may not exist.

3.5 Efficiency

At last, we come to the efficiency argument of the sequential auctions. Before that, it is helpful to first define an efficiency concept.

Definition 3.1. A multi-period mechanism is *periodically efficient* if the allocation in each period is efficient.

Periodic efficiency is different from full efficiency, as the latter requires that the agent with higher valuation receives a good with higher probability and, particularly, receives a good sooner. In a dynamic model with entry and exit, it is possible that a winner, or even a loser, in a later period values the good more than some winner in a previous period. Generally, full efficiency is hard, if not impossible, to attain in an infinite-horizon model. Thus, I would rather use the notion of periodic efficiency in this chapter.

Corollary 3.2 (Efficiency)**.** *The infinite-horizon sequential second-price auctions is periodically efficient.*

The efficiency property is a direct result of the monotonicity as presented in Proposition 3.2. This means in the infinite-horizon auctions, the winner in each period is the highest-valued among all buyers in that period. Moreover, given the fact that there is only one new entrant in each period, it may be inferred that if a good is awarded to a higher-valued buyer, then this buyer must be absent in all previous periods, and that the losers do not have a valuation higher than any previous winner. This indicates that for this particular model, periodic efficiency is equivalent to full efficiency.

3.6 Conclusion

This chapter supplements the literature on sequential auctions and declining price anomaly. My model extends the standard sequential auctions to an infinite horizon and introduces a dynamic population where one new buyer enters the auction in every period. Symmetric and stationary equilibria are derived, where buyers bid according to their private valuation less a pivotal continuation value. It is also shown that in the stationary equilibrium, the price path is non-increasing.

The current analysis has several directions for further development. A natural extension is to study a similar kind of sequential auctions, i.e. sequential pay-as-bid auctions and sequential uniform-price auctions, where in every period multiple units are sold and the same number of new buyers enter. However, in such settings, the remainders' inference of the distribution of opponents' valuation are not proportional to that of new buyers.

Other possibilities include studying an optimal design for the auctioneer in this setting. Alternative lines of research may be dropping the assumption of unit demand or allowing buyers to exit. These, however, introduce additional inter-temporal considerations, as the expected payoffs are no longer identical functions of individual valuations. These questions, however, are left for future work.

3.7 Appendix: Proofs

Proof of Proposition 3.1. First, use the assumption that b is a function of x only and rewrite the Bellman Equation (3.1) as

$$\begin{aligned} & F^{n-2}(y)(1-\delta+\delta F(y))V(x,y) \\ = \; & \max_{b_i} E[x_i - \max b_{-i} | b_i > \max b_{-i}] F^{n-1}(b^{-1}(b_i)) \\ + \; & \delta \int_{b^{-1}(b_i)}^{y} V(x,y') dF^{n-1}(y'). \end{aligned}$$

Apply the Envelope Theorem by taking partial derivative with respect to y,

$$\begin{aligned} & (n-2)F^{n-3}(y)f(y)(1-\delta+\delta F(y))V(x,y) \\ + \; & F^{n-2}(y)\delta f(y)V(x.y) \\ + \; & F^{n-2}(y)(1-\delta+\delta F(y))V_2(x,y) \\ = \; & \delta V(x,y)(n-1)F^{n-2}(y)f(y) \end{aligned}$$

where the subscript V_2 means the partial derivative of V with respect to the second argument. After simplification, it turns out that

$$\frac{V_2(x,y)}{V(x,y)} = -\frac{(n-2)(1-\delta)f(y)}{F(y)(1-\delta+\delta F(y))}.$$

Notice that the right-hand side depends only on y not on x. Therefore, one solution of $V(x, y)$ is such that $V(x, y)$ be decomposed to two separate functions of x and y respectively. Let $V(x, y) = \phi(x)\psi(y)$. Then

$$\frac{\psi'(y)}{\psi(y)} = \frac{V_2(x, y)}{V(x, y)} = -\frac{(n-2)(1-\delta)f(y)}{F(y)(1-\delta+\delta F(y))}.$$

Integrating both sides with respect to y yields

$$\begin{aligned}
\log(\psi(y)) &= -\int_{-\infty}^{y} \frac{(n-2)(1-\delta)f(t)dt}{F(t)(1-\delta+\delta F(t))} \\
&= -(n-2)\int_{-\infty}^{y} \frac{(1-\delta)dF(t)}{F(t)(1-\delta+\delta F(t))} \\
&= -(n-2)\int_{-\infty}^{y} \Big(\frac{1}{F(t)} - \frac{\delta}{1-\delta+\delta F(t)}\Big)dF(t) \\
&= -(n-2)[\log(F(y)) - \log(1-\delta+\delta F(y))] \\
&= -(n-2)\log\frac{F(y)}{1-\delta+\delta F(y)}.
\end{aligned}$$

Therefore,

$$\psi(y) = \Big(\frac{1-\delta+\delta F(y)}{F(y)}\Big)^{n-2}.$$

Now apply the Envelope Theorem again by taking partial derivative with respect to x, we have

$$F^{n-2}(y)(1-\delta+\delta F(y))\phi'(x)\psi(y) = F^{n-1}(x) + \delta\int_{x}^{y} \phi'(x)\psi(y')dF^{n-1}(y').$$

Plugging in $\psi(y)$, it follows directly that

$$\phi'(x) = \Big(\frac{F(x)}{1-\delta+\delta F(x)}\Big)^{n-1}.$$

Given the common structure of both $\phi'(x)$ and $\psi(y)$, denote

$$P(x) = \frac{F(x)}{1-\delta+\delta F(x)}.$$

Then, $V_1(x, y) = \phi'(x)\psi(y) = P^{n-1}(x)/P^{n-2}(y)$. Impose a reasonable boundary condition that a buyer with valuation 0 has expected payoff 0, i.e. $V(0, y) = 0$ for all y, we have

$$V(x, y) = \int_0^x \frac{P^{n-1}(t)}{P^{n-2}(y)} dt.$$

□

Proof of Proposition 3.2. Taking derivative of b,

$$\begin{aligned} b'(x) &= \frac{d(x-\delta V(x,x))}{dx} \\ &= 1-\delta P(x) + \delta\frac{(n-2)P'(x)\int_0^x P^{n-1}(t)dt}{P^{n-1}(x)} \\ &= \frac{1-\delta}{1-\delta+\delta F(x)} + \frac{\delta(n-2)P'(x)\int_0^x P^{n-1}(t)dt}{P^{n-1}(x)} > 0 \end{aligned}$$

because

$$
\begin{aligned}
P(x) &= \frac{F(x)}{1-\delta+\delta F(x)} > 0, \\
P'(x) &= \frac{(1-\delta)f(x)}{[1-\delta+\delta F(x)]^2} \geq 0.
\end{aligned}
$$

$\square$

Proof of Theorem 3.1. Denote $s_i = b(x_i) = x_i - \delta V(x_i, x_i)$. Suppose other buyers also follow this bidding strategy. Let $\hat{b}_j = \max_{j \neq i} b_j$, i.e. the highest bid among the rest of buyers, and denote $\hat{b}_j = \hat{x}_j - \delta V(\hat{x}_j, \hat{x}_j)$, where $\hat{x}_j$ is then the highest valuation among the rest of buyers.

The table below shows the payoffs of buyer x_i in different situations. For example, the grid in column 1 row 1 is the expected payoff when buyer x_i bids b_i', higher than s_i, while the highest bid of the others is even higher.

Highest Bid of Others	Bid of Buyer x_i		
	$b_i' > s_i$	$b_i = s_i$	$b_i'' < s_i$
$\hat{b}_j > b_i'$	$\delta V(x_i, \hat{x}_j)$	$\delta V(x_i, \hat{x}_j)$	$\delta V(x_i, \hat{x}_j)$
$b_i' > \hat{b}_j > s_i$	$x_i - \hat{b}_j$	$\delta V(x_i, \hat{x}_j)$	-
$s_i > \hat{b}_j > b_i''$	-	$x_i - \hat{b}_j$	$\delta V(x_i, \hat{x}_j)$
$b_i'' > \hat{b}_j$	$x_i - \hat{b}_j$	$x_i - \hat{b}_j$	$x_i - \hat{b}_j$

Table 3.1: Pay-offs in Sequential Second-Price Auctions

Then, it suffices to show that $x_i - \hat{b}_j < \delta V(x_i, \hat{x}_j)$ if $b_i' > \hat{b}_j > s_i$ and that $x_i - \hat{b}_j > \delta V(x_i, \hat{x}_j)$ if $s_i > \hat{b}_j > b_i''$, which is equivalent to show that $x - \delta V(x, y)$ is increasing in x when $x < y$. This is straightforward if we take derivative with respect to x,

$$1 - \delta V_1(x, y) = 1 - \delta \frac{P^{n-1}(x)}{P^{n-2}(y)} > 1 - \delta P(y) = \frac{1-\delta}{1-\delta+\delta F(y)} > 0.$$

Given the structure of Bellman equation (3.1), if a buyer aims to win the good in current period, then he cannot do better than adhering to strategy b provided that others follow the same strategy. However, expecting a declining price, one may want to wait for some later period to win the good with a low price.

Apply the one-shot deviation principle (Hendon, Jacobsen and Sloth, 1996 [56] and Perea, 2000 [67]). Consider a buyer whose valuation is x. It is an interesting case only if x is the highest among the current n buyers (otherwise, the bidder x does not matter in the current period). Suppose bidder x underbids in one period and makes a winning bid in the next period. The expected winning price in the current auction is then $b(E(X_{(2)}|X_{(1)} = x))$, where $X_{(k)}$ means the k-th highest order statistic of a sample of n. The expected winning price in the next period is $b(E(X_{(3)}|X_{(1)} = x))$. Therefore, to prevent any attempt for such strategic delay, it is sufficient that for all x,

$$x - b(E(X_{(2)}|X_{(1)} = x)) \geq \delta[x - b(E(X_{(3)}|X_{(1)} = x))]. \tag{3.2}$$

Denote $g(x, \delta) = x - b(E(X_{(2)}|X_{(1)} = x)) - \delta(x - b(E(X_{(2)}|X_{(1)} = x)))$. Then, $b(x) < x$ for all x implies $g(x, 0) > 0$ and $b(x)$ being increasing implies $g(x, 1) < 0$. By a continuity argument, for a fixed x, there exists $\delta_x \in (0, 1)$ such that $g(x, \delta) \geq 0$

for all $\delta \leq \delta_x$. Now define

$$\bar{\delta} = \inf_{0 \leq x \leq 1} \{\delta_x\}.$$

Then, for all $\delta \leq \bar{\delta}$, condition (3.2) always holds and thus strategic delay is prevented in the equilibrium described as Proposition 3.1.

On the other hand, according to the definition of $\bar{\delta}$, there is x_δ such that $g(x_\delta, \delta) < 0$ for $\delta > \bar{\delta}$. This means condition (3.2) is also necessary for the existence of stationary equilibrium. $\square$

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