

ABSTRACT

Title of Dissertation: **PRECISION CALCULATIONS OF PARTON PHYSICS
UNDER LARGE MOMENTUM EFFECTIVE THEORY**

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Parton distributions are effective field theory (EFT) descriptions of the momentum distributions of quarks and gluons in a hadron moving at asymptotically large momentum. They serve as essential inputs for precision tests of the Standard Model (SM) and for searches for physics beyond the SM, as well as key probes of hadron structure. Large Momentum Effective Theory (LaMET) provides a systematic EFT framework to compute parton distributions through a large-momentum expansion and perturbative matching of Euclidean correlators accessible in lattice quantum chromodynamics (QCD).

This dissertation provides a brief review of LaMET and presents two of the author's PhD works within this framework. The first is an effort to control the precision in LaMET calculations: the large-distance asymptotic analysis of the Fourier transform. The second implements the LaMET analysis of the unpolarized $u - d$ proton PDF, which serves as a benchmark to calibrate the analysis procedure and demonstrate precision-control capability. These studies constitute important steps toward precision determinations of parton distributions within the LaMET framework.

PRECISION CALCULATIONS OF PARTON PHYSICS
UNDER LARGE MOMENTUM EFFECTIVE THEORY

by

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List of Abbreviations

SM	Standard Model
BSM	beyond Standard Model
LHC	Large Hadron Collider
EIC	Electron-Ion Collider
PDF	Parton Distribution Function
QCD	quantum chromodynamics
LaMET	Large Momentum Effective Theory
EFT	effective field theory
IR	infrared
UV	ultraviolet
OPE	operator product expansion
NLO	next-to-leading-order
NNLO	next-to-next-to-leading order
DA	distribution amplitude
GPD	generalized parton distribution
TMD	transverse-momentum dependent
DPD	double parton distribution
SDE	short distance expansion
DIS	deeply inelastic scattering
DY	Drell-Yan
LFTD	light-front Tamm-Dancoff
SCET	soft-collinear effective theory
HQET	heavy quark effective theory
CG	Clebsch-Gordan
IBP	integration-by-part
MI	master integral
DGLAP	Dokshitzer-Gribov-Lipatov-Altarelli-Parisi
RG	renormalization group
RGR	renormalization group resummation
TR	threshold resummation
LRR	leading renormalon resummation
FT	Fourier transform
QFT	quantum field theory
WF	wavefunction
CG	Coulomb gauge
KELI	kinematically-enhanced lattice interpolator
GEVP	generalized eigenvalue problem
LA	leading asymptotics
NLA	next-to-leading asymptotics

PV	principal value
EMC	European Muon Collaboration
ANL	Argonne National Lab
BNL	Brookhaven National Lab
MSU	Michigan State University
ETMC	Extended Twisted Mass Collaboration
LPC	Lattice Parton Collaboration
CLQCD	China Lattice QCD

Chapter 1: Introduction

Nature is governed by four fundamental forces: gravitation, electromagnetism, the weak interaction, and the strong interaction. This dissertation is devoted to the study of the strong interaction. This force binds nucleons (protons and neutrons) together to form a nucleus, as depicted in the left panel of Fig. 1.1. It also determines the internal structures of each nucleon, such as the distributions of quarks and gluons within it, shown in the right panel of Fig. 1.1.

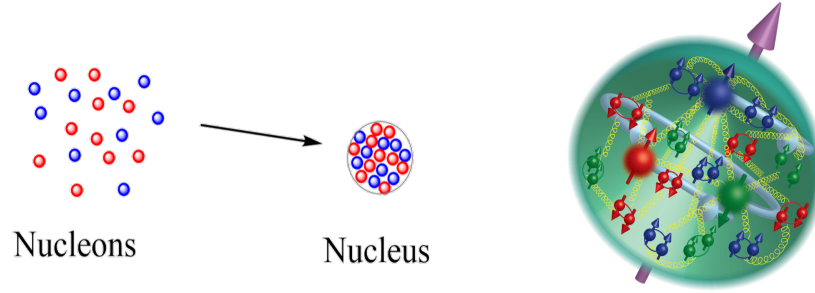


Figure 1.1: Phenomena related to strong interaction. Left panel: the binding of nucleons into a nucleus [4]. Right panel: the distributions of quarks and gluons inside a nucleon [5].

1.1 Quantum chromodynamics

The fundamental theory of the strong interaction is quantum chromodynamics (QCD), an $SU(3)$ non-abelian gauge theory describing the dynamics of quarks and gluons. Its validity is firmly established by a wide range of experimental observations:

- Bjorken scaling. In 1969, Richard Feynman proposed the parton model for high-energy collisions, which emphasized the importance of the infinite-momentum frame [51]. In this frame, relativistic time dilation effectively freezes the internal motion of the proton constituents, while the proton charge distribution is Lorentz contracted along the direction of motion. As a result, an incident particle scatters instantaneously and incoherently from individual point-like con-

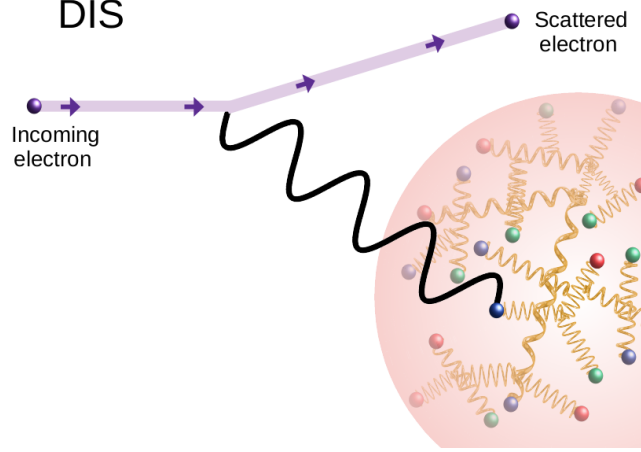


Figure 1.2: Schematic illustration of initial scattering in DIS, in Ref. [6].

stituents in the proton, referred to as partons.

Building on this picture, Bjorken and Paschos applied the parton model to deeply inelastic scattering (DIS) [52], where a lepton probes the proton through the exchange of a highly virtual photon, which can be viewed as the scattering between the lepton and an individual parton inside the proton, as illustrated in Fig. 1.2. They predicted the Bjorken scaling behavior, where the properly normalized differential cross section depends primarily on the dimensionless Bjorken variable x_B , which is consistent with experimental data when the photon virtuality Q^2 does not vary too much.

This observation implies asymptotic freedom, which states that a higher energy scale corresponds to a smaller coupling constant. Among the four-dimensional quantum field theories that satisfy Lorentz symmetry, discretized symmetries (C, P, T), and renormalizability, the non-abelian gauge theory exhibits asymptotic freedom [53, 54].

- The quark wavefunction of Ω^- baryon, the e^+e^- annihilation into hadrons, and the electromagnetic decay of the neutral pion. To explain these phenomena, a quantum number known as “color” is introduced, which transforms under the $SU(3)$ gauge group [55].

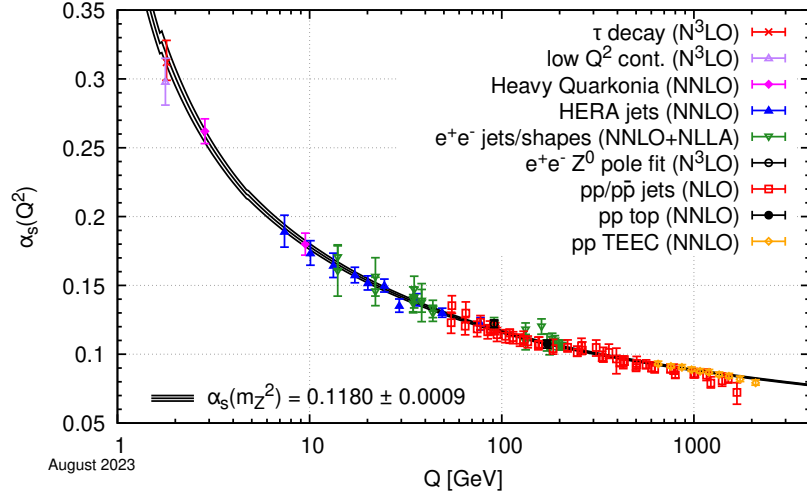


Figure 1.3: The QCD running coupling α_s as a function of the energy scale Q . The black curves are computed at five loops and use the PDG average as input: $\alpha_s(m_Z^2) = 0.1180 \pm 0.0009$. The data points are from experimental measurements. This plot is taken from Ref. [7].

• ...

One of the central challenges in QCD is to explain and predict hadronic properties at the infrared (IR) scale, ~ 1 fm or ~ 0.2 GeV, such as mass and spin, from the underlying quark and gluon dynamics at the ultraviolet (UV) scale, $\lesssim 0.1$ fm or $\gtrsim 2$ GeV. This difficulty arises from the running of the coupling constant α_s , which becomes large at IR scales, as illustrated in Fig. 1.3, rendering perturbative expansions in α_s ineffective. In this regime, a hadron behaves as a strongly interacting many-body system, analogous to systems encountered in condensed matter physics [56], and is therefore inherently challenging to study.

1.2 Parton distribution functions

To shed light on this challenge, one can probe the internal structure of hadrons via parton distribution functions (PDFs), which depend on momentum fraction x and energy resolution Q . They are formally defined using light-cone correlators [57] to effectively describe [58] momentum distributions

of quarks and gluons in a hadron moving at asymptotically large momentum. On the other hand, the global information encoded in PDFs is related to hadronic properties; for example, the first moments of unpolarized PDFs are related to hadron masses, and the zeroth moments of helicity PDFs are related to hadron spins. In this sense, PDFs serve as a bridge between the fundamental quark–gluon dynamics and observable hadronic properties.

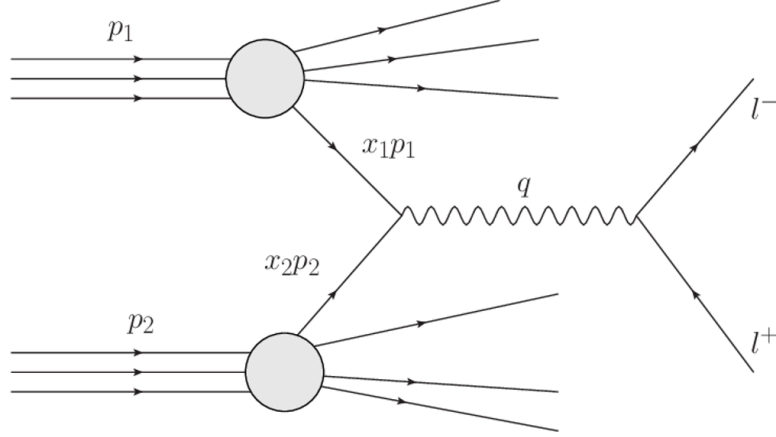


Figure 1.4: The DY process. The plot is from Ref. [8].

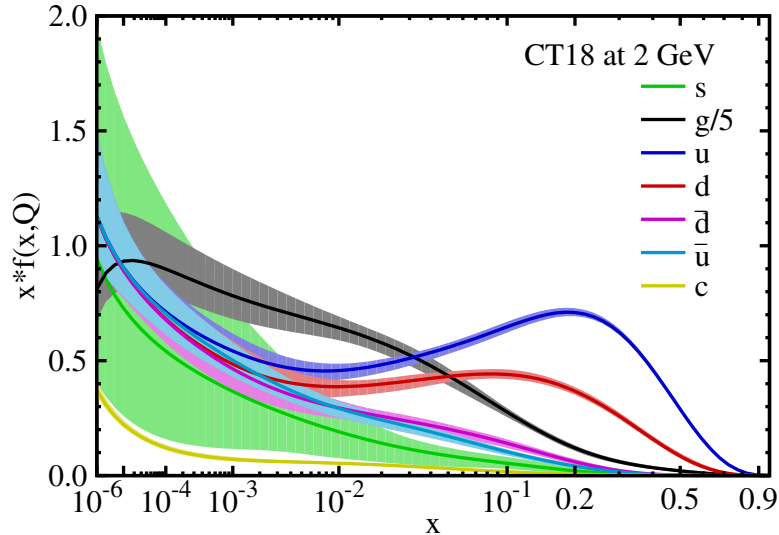


Figure 1.5: The CT18 [9] unpolarized proton PDFs at scale $Q = 2$ GeV for various quark flavors $u, \bar{u}, d, \bar{d}, s = \bar{s}, c = \bar{c}$ and gluon g .

PDFs can be extracted from experimental processes [9, 59–64], such as deeply inelastic scat-

tering (DIS) or Drell-Yan (DY). In DIS, as illustrated in Fig. 1.2, the cross section depends on the momentum carried by the struck parton, whose distribution is effectively described by the PDF. In DY, two protons collide, which can be viewed as the scattering between partons from each proton, as illustrated in Fig. 1.4; the cross section is therefore also sensitive to the PDF. An example of experimentally extracted PDFs is shown in Fig. 1.5.

1.3 Lattice QCD calculations of PDFs

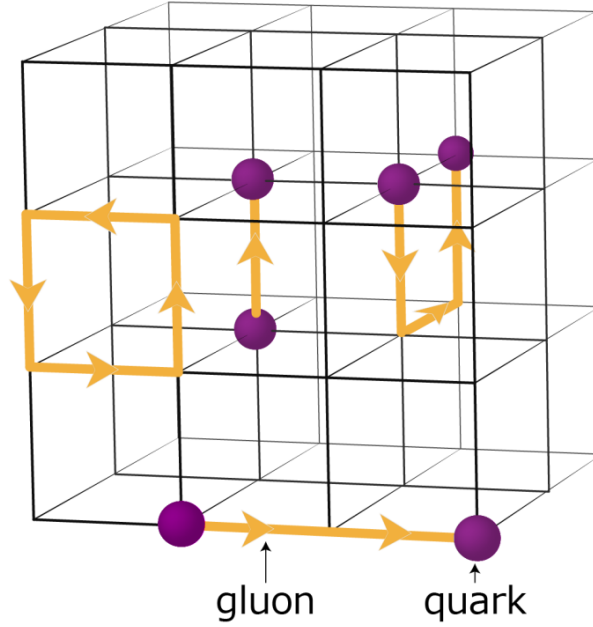


Figure 1.6: Lattice QCD discretization illustration. Quark fields lie on the grid points, and gluons are represented by links connecting two points. The figure and caption are taken from Ref. [10].

On the other hand, PDFs can be calculated from the first principles of QCD, such as lattice QCD. While the original QCD is formulated in continuous four-dimensional Minkowski spacetime, lattice QCD performs numerical simulations in a discretized four-dimensional Euclidean spacetime. In this framework, quark fields reside on lattice sites, and gluon fields are represented as gauge links

connecting neighboring sites, as illustrated in Fig. 1.6.

The moments of PDFs can be calculated from matrix elements of local quark bilinear operators with covariant derivatives [65–78], as well as through the short-distance factorization of nonlocal operators [15, 79–83]. In practice, current lattice QCD techniques allow the determination of only a few low-order moments.

The x -dependence of PDFs in the moderate- x region can be directly calculated within the framework of Large Momentum Effective Theory (LaMET) [11, 58, 84–87]. In this approach, one starts from Euclidean correlators of large-momentum hadrons, which are accessible in lattice QCD. These quantities can then be related to PDFs through a systematic large-momentum expansion and perturbative matching.

Significant progress has been made within the LaMET framework in calculating PDFs [12, 16, 17, 44, 47–50, 83, 88–145] and other parton observables [23–29, 36, 40, 123, 130, 146, 146–227]. A representative example is the unpolarized $u - d$ proton PDF, which is precisely determined from experimental data and thus serves as a benchmark for validating and calibrating LaMET analysis. As shown in Fig. 4.8, the LaMET result is consistent with experimental PDFs within $\sim 1\sigma$, demonstrating LaMET precision control capability. Another example is the pion valence PDF, which is less constrained experimentally; in this case, LaMET provides complementary information. As illustrated in Fig. 2.5, the uncertainty of the LaMET prediction is smaller than the spread among experimental extractions, indicating that the LaMET result outperforms experimental PDFs in this case.

Considerable research opportunities remain within the LaMET framework. Further efforts are needed to improve precision control, particularly in enhancing statistical precision and mitigating excited-state contamination in lattice data generation. In addition, it is important to extend LaMET to a broader range of parton observables, including gluon PDFs, heavy-meson DAs, baryon DAs, and

DPDs, many of which are still in their early stages of development. It is also of interest to explore novel quasi-correlator constructions, such as the Coulomb gauge approach [44, 218], correlators of heavy–light currents [228], and correlators of light–light currents [80, 229, 230].

1.4 Contributions and outlines

The author of this dissertation has contributed to several works during PhD within the LaMET framework, which can be broadly categorized into LaMET precision control and lattice QCD calculations. The precision-control studies involve renormalization [168, 231, 232], Fourier transform [133, 233], and perturbative matching [17, 118, 122, 123, 154, 232]. The lattice QCD calculations cover collinear PDFs [12], DAs [23, 162, 171], and TMDs [25, 26, 210, 213, 215, 220, 223].

The rest of this dissertation is organized as follows. Chapter 2 provides a pedagogical introduction to LaMET, which will also appear in the Encyclopedia of Nuclear Physics. Chapter 3 presents the large-distance asymptotic analysis, an approach to improving the precision control in LaMET calculations, and is based on the author’s research article Ref. [233]. Chapter 4 focuses on the LaMET analysis of the unpolarized $u - d$ proton PDF, which serves as a benchmark for validating the framework; this work is currently unpublished. The dissertation concludes with Chapter 5.

Chapter 2: What is LaMET?

Parton physics arises from the momentum distributions of quarks and gluons, including varieties of one-dimensional longitudinal-momentum distributions and wave function distribution amplitudes, and three-dimensional generalized parton distributions and transverse-momentum-dependent distributions, etc. The large-momentum effective theory (LaMET) is an effective framework for *ab initio* calculations of parton distributions through a systematic expansion of the large hadron momentum P^z , utilizing Euclidean correlators accessible in lattice quantum chromodynamics simulations. In this article, we provide basic arguments and framework for large momentum expansion, showcase its applications to various parton physics, and highlight some technical advances for precision calculations.

2.1 Introduction

Partons are effective degrees of freedom (collinear quark and gluon modes with infinite longitudinal momentum) useful in describing hadrons travelling at the speed of light [51, 52, 58]. Parton distributions, including parton distribution functions (PDFs) [57, 234], distribution amplitudes (DAs) [235–240], generalized parton distributions (GPDs) [241–246], and transverse-momentum-dependent (TMD) distributions [247–255], are related to the momentum distributions of quarks and gluons in a hadron at asymptotically large momentum. They are important standard model inputs in the search for beyond-standard-model physics at the Large Hadron Collider (LHC) [256, 257] or the future Electron-Ion Collider (EIC) [258, 259]. They also provide valuable information about the internal structures of a hadron and shed light on the origin of its mass [260, 261] and spin [241, 262]. These distributions can be extracted from experimental data through global fits such as ATLASpdf21 [59], NNPDF4.0 [60], MSHT20 [61], CT18 [9], ABMP16 [62], HERAPDF2.0 [63], and JR14 [64]. An example of one-dimensional proton PDFs including quarks and gluons is shown in Fig. 1.5. This ar-

ticle focuses on their theoretical definition and first-principles calculations, which can be essential for improving the precision of beyond-standard-model searches and accessing aspects of hadron structure beyond current experimental reach.

The parton distributions are traditionally defined through light-cone correlators [57, 263, 264], which have a direct momentum-density or particle number interpretation [265]. Moreover, the light-cone distributions are universal across multiple experiments since the high-energy physics is factorized out, thereby enhancing the predictive power of quantum chromodynamics (QCD).

Despite their conceptual simplicity, parton distributions are difficult to calculate directly from first principles. Light-cone correlators are, in principle, best calculated in light-front quantization, which, however, introduces additional divergences, known as light-front singularities, which are challenging to regulate systematically and require an infinite number of counterterms in 3+1 dimensional field theories [58, 266, 267]. Moreover, they involve real-time dependence and therefore cannot be directly evaluated in lattice QCD, which is formulated in Euclidean time.

Large-momentum effective theory (LaMET) [11, 58, 84–87] provides a framework to overcome these difficulties. Light-cone singularities are regulated by a finite but large hadron momentum, and the resulting finite-momentum correlators are related to their light-cone counterparts through a systematic expansion in $1/P^z$ and perturbative matching. The Lorentz boost can be applied to the external states instead of the operators, leading to equal-time correlators that are accessible in lattice QCD. Because the ultraviolet physics of partons and finite-momentum quarks and gluons are different, LaMET is an effective field theory (EFT) that enables the calculation of parton distributions in the moderate- x region, $\Lambda_{\text{QCD}}/2P^z \lesssim x \lesssim 1 - \Lambda_{\text{QCD}}/2P^z$, using Euclidean correlators.

LaMET has been applied to a wide range of parton physics, including total gluon helicity [268–271], quark collinear PDFs [12, 16, 17, 44, 47–50, 83, 88–134, 272], gluon collinear PDFs [135–145],

light-meson DAs [23, 146–159], heavy-meson DAs [160–166, 273], baryon DAs [167–174], GPDs [24, 130, 146, 175–196], TMDs [25–29, 36, 40, 123, 197–224, 274], and double parton distributions (DPDs) [225–227]. Comprehensive overviews of LaMET are provided in Refs. [275–277]. A comparison between the LaMET expansion used for direct parton calculations and the standard short-distance expansion (SDE) approach applied to parton moments is presented in Ref. [86].

In recent years, considerable progress has been made in controlling the precision of LaMET calculations, at various steps of the analysis procedure, including lattice data generation [44, 189, 218, 278, 279], renormalization [94, 216, 219, 231, 232, 280–287], Fourier transform [16, 133, 158, 159, 233, 287] and perturbative matching [17, 115, 118, 122, 123, 154, 156, 157, 194, 232, 288, 289]. Currently, the uncertainties in the last three steps are well under control and are limited to a few percent in the moderate- x region. In contrast, more effort is needed in lattice data generation, especially for the excited state control of nucleon matrix elements, potentially using the methods discussed in Refs. [290–296]. Also see Ref. [22] for a short review on LaMET precision control.

The remaining parts of this article are organized as follows. Sec. 2.2 discusses the conceptual foundations of parton physics and LaMET. Sec. 2.3 presents some of the recent lattice QCD calculations under LaMET. Sec. 2.4 reviews the theoretical developments that consolidate the foundation, improve the precision or extend the range of LaMET applications. Sec. 2.5 introduces the relevant numerical methods that enhance statistical precision or reduce excited state contaminations for generating LaMET correlators in lattice QCD. Conclusions are given in Sec. 2.6.

2.2 Nature of Partons and Large-Momentum Expansion

This section discusses the following topics: partons are effective degrees of freedom that describe high-energy scattering processes in the infinite-momentum limit or light-front frame; LaMET provides an alternative formulation of parton physics in Euclidean field theory; under the LaMET expansion, the light-cone parton distributions can be systematically calculated from the Euclidean momentum distributions accessible in lattice QCD. The contents of this section closely follow Refs. [11, 58, 84–87].

2.2.1 Partons as EFT Degrees of Freedom in QCD Factorization

An effective field theory (EFT) retains specific degrees of freedom (dofs) from the full theory, such as QCD, while integrating out the remaining ones. It is typically constructed based on scale separation, power counting of an expansion parameter, and symmetry constraints. A common mathematical structure in an EFT is an asymptotic expansion, similar to a Taylor expansion, which enables controlled precision for a small expansion parameter.

The asymptotic freedom of QCD enables perturbative calculations of ultraviolet (UV) modes at a scale much larger than $\Lambda_{\text{QCD}} \sim 300 \text{ MeV}$. However, almost all experimental processes, such as deep-inelastic scattering (DIS) and Drell-Yan (DY), involve not only UV physics but also infrared (IR) modes at the scale Λ_{QCD} (hadron structure), which cannot be calculated in perturbation theory. Therefore, one performs a scale separation between the UV and IR modes, where the former are calculated in perturbation theory, and the latter are absorbed into the non-perturbative matrix elements.

For example, a quark field can be decomposed into high (UV) modes and low (IR) modes,

$$\psi = \psi_{\text{high}} + \psi_{\text{low}} . \quad (2.1)$$

In this sense, the application of perturbative QCD and QCD factorization can be seen as an EFT.

Consider, for example, the DIS, a high-energy collision between a lepton and a hadron, mediated by a highly-virtual photon. A schematic illustration of DIS process is displayed in Fig. 1.2. According to Feynman's parton model [51, 52], this process can be viewed as the impulse interaction between the lepton and a quark parton, where the latter is distributed with different momentum fraction x within the hadron. The tree-level diagrams for the cross section are depicted in Fig. 2.1, where P denotes the hadron momentum, k the parton, and q the virtual photon. $Q^2 = -q^2$ is the virtuality of the photon, and $x_B = Q^2/(2P \cdot q)$ is the Bjorken scaling variable. The structure function $F_2(Q, x_B)$ in the cross section is factorized into the UV matching kernel $C_2(Q/\mu, x_B/x)$ and the IR light-cone parton distribution function (PDF) $f(x, \mu)$ in the large Q^2 (Bjorken) limit [57, 263, 264],

$$F_2(Q, x_B) = \int_{x_B}^1 dx C_2\left(\frac{Q}{\mu}, \frac{x_B}{x}\right) f(x, \mu) + O\left(\frac{\Lambda_{\text{QCD}}^2}{Q^2}\right) , \quad (2.2)$$

where the summation over quark flavor and gluon is omitted for simplicity. The PDF is formally defined via the light-cone correlator,

$$f(x, \mu) \equiv \int_{-\infty}^{+\infty} \frac{d\xi^-}{4\pi} e^{-ixP^+\xi^-} \langle H(P) | \bar{\psi}_p(\xi^-) \gamma^+ U(\xi^-, 0) \psi_p(0) | H(P) \rangle . \quad (2.3)$$

Here, we introduce the light-cone coordinates $p = (1, 0, 0, 1)/\sqrt{2}$ and $n = (1, 0, 0, -1)/\sqrt{2}$, and define $\xi^+ = n \cdot \xi = (\xi^t + \xi^z)/\sqrt{2}$ and $\xi^- = p \cdot \xi = (\xi^t - \xi^z)/\sqrt{2}$. The low mode is restricted to the

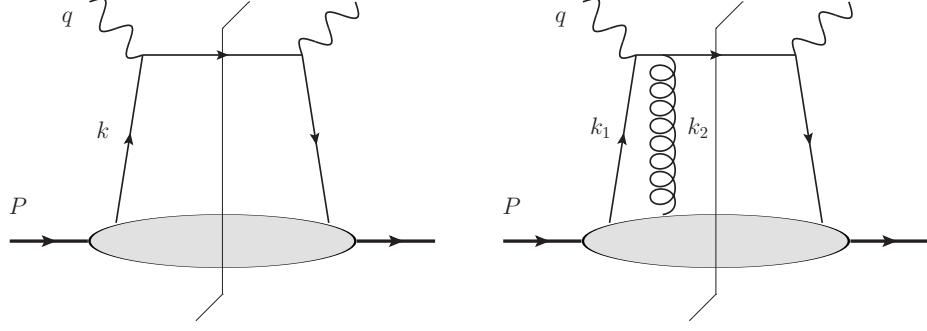


Figure 2.1: Tree-level cross-section diagrams for deep-inelastic scattering (DIS) process.

collinear mode along the light-cone p direction, $\psi_{\text{low}} = \psi_p + \dots$, where the momentum mode power counting is $k \sim \alpha p + \beta n + k_\perp$ with $\alpha \sim Q$, $\beta \sim \Lambda_{\text{QCD}}^2/Q$ and $k_\perp \sim \Lambda_{\text{QCD}}$, and the leading-order equation of motion is $\not{p}\psi_p = 0$. The quark and anti-quark fields are separated along the light cone by the distance ξ^- and connected by a straight-line Wilson line $U(\xi^-, 0)$ to preserve gauge invariance. The Dirac structure γ^+ is for the unpolarized projection. The external state $|H(P)\rangle$ is an unpolarized hadron with momentum P , while the light-cone PDF $f(x, \mu)$ is independent of P due to boost invariant of the light-cone correlator. The momentum fraction is $x = k^+/P^+$, the parton light-front momentum divided by that of the hadron. The variable μ is the renormalization scale, often chosen in the $\overline{\text{MS}}$ scheme.

The field ψ_p belongs to the quark p -collinear sector of the soft-collinear effective theory (SCET) [297, 298],

$$\mathcal{L}_{q,p}^{(0)} = \bar{\psi}_p \left[i p \cdot D + (i \not{D}_\perp) \frac{1}{2 i n \cdot D} (i \not{D}_\perp) \right] \not{n} \psi_p, \quad (2.4)$$

which is an EFT that describes the low-energy dofs in high-energy QCD processes in terms of light-cone correlators.

For more complicated QCD processes, one can introduce other parton observables, and construct

effective lagrangian using power counting, and symmetry constraints [297, 298].

2.2.2 QCD Factorization with Longitudinal Momentum Distribution

Refs. [58, 87] use DIS as an example to demonstrate that high-energy processes can also be factorized in terms of Euclidean correlators of QCD fields, namely, the relevant cross sections or amplitudes can be factorized in terms of momentum distributions of quarks and gluons.

Consider the DIS in the Breit frame, with the following virtual photon and hadron momenta,

$$q^\mu = (0, 0, 0, -Q) ,$$

$$P^\mu = M\gamma v^\mu , \quad \gamma = \sqrt{1 + \frac{Q^2}{4x_B^2 M^2}} , \quad (2.5)$$

where $v^\mu = (1, 0, 0, v)$ and $v^\mu v_\mu = 1/\gamma^2$. The Bjorken limit, $Q \rightarrow \infty$ (or $\gamma \rightarrow \infty$) for a fixed x_B , corresponds to a large momentum hadron with $P^z \gg \Lambda_{\text{QCD}}$.

The tree-level diagram without gluon insertions is shown in the left panel of Fig. 2.1. By restricting the low mode to the collinear mode along the hadron velocity v , $\psi_{\text{low}} = \psi_v + \dots$, performing the corresponding power counting, $k^\mu = \alpha v^\mu + \beta \bar{v}^\mu + k_\perp^\mu$ with $\bar{v}^\mu = (v, 0, 0, -1)$, $\alpha \sim \gamma \Lambda_{\text{QCD}}$, $\beta \sim \Lambda_{\text{QCD}}/\gamma$, and $k_\perp \sim \Lambda_{\text{QCD}}$, and imposing the leading order equation of motion (EOM), $\not{v}\psi_v = 0$, one can show that the DIS process can be factorized in terms of the following Euclidean distribution,

$$\int \frac{dz}{4\pi} e^{iyP^z z} \langle H(P) | \bar{\psi}_v(z) \gamma^z \psi_v(0) | H(P) \rangle . \quad (2.6)$$

The key step is that the k^0 component of the quark propagator can be eliminated by the EOM $\not{v}\psi_v = 0$, so that the k^0 can be integrated out, resulting in an equal-time correlator.

The above derivation can be repeated for diagrams in which the quark field propagates in the hadron's background gluon field, denoted as A_v^μ . One example involving a single gluon insertion is depicted in the right panel of Fig. 2.1. By using the power counting of A_v^μ including $A_v^0 \sim A_v^z$, and the EOM of the quark field $\not{v}\psi_v = 0$, one obtains that effectively $A_v = -2A_v^z\gamma^z$. After adding all the quark eikonal contributions, the Euclidean distribution becomes,

$$\tilde{f}(y, P^z) \equiv \int \frac{dz}{4\pi} e^{iyP^z z} \langle H(P) | \bar{\psi}_v(z) \gamma^z U(z, 0) \psi_v(0) | H(P) \rangle, \quad (2.7)$$

where $U(z, 0)$ is a straight line gauge link in z -direction to keep the non-local Euclidean correlator gauge invariant. $\tilde{f}(y, P^z)$ is called quasi-PDF [84, 85] or longitudinal momentum distribution (LMD) [87].

The above discussions can be formulated into an EFT that captures the low-energy modes in high-energy QCD processes, which contains only Euclidean correlators of collinear fields of velocity v . The leading lagrangian for the quark v -collinear sector,

$$\mathcal{L}_{q,v}^{(0)} = \bar{\psi}_v \left[iv \cdot D + \frac{i\bar{v} \cdot D}{2\gamma^2} + (i\not{D}_\perp) \frac{1}{2i\bar{v} \cdot D} (i\not{D}_\perp) \right] \not{v} \psi_v, \quad (2.8)$$

where $\bar{v} = (v, 0, 0, -1)/2v$ and $v_\mu \bar{v}^\mu = 1$. Because the power counting is carried out in both $1/\gamma$ and $1/\bar{v} \cdot D$, this EFT can be referred to as a large-velocity/rapidity effective theory. In practical calculations, however, the expansion is organized in powers of the inverse hadron momentum, and the framework is therefore known as large momentum effective theory (LaMET) [85].

In the limit $\gamma \rightarrow \infty$ or $v \rightarrow c$, this LaMET lagrangian, Eq. (2.8), formally reduces to the SCET lagrangian, Eq. (2.4). Nonetheless, there remain several essential differences:

- LaMET does not have light-cone singularities. In the light-cone parton distributions [57, 297, 298], by taking the $P^z \rightarrow \infty$ before imposing UV cutoff Λ_{UV} , namely $P^z \gg \Lambda_{UV} \rightarrow \infty$, additional UV/IR divergences (light-cone singularities) are introduced. LaMET takes the physical limit $\Lambda_{UV} \gg P^z \rightarrow \infty$, where the light-cone singularities are naturally regularized by a finite large momentum P^z .
- LaMET expansion can be accomplished in an Euclidean formulation of QCD. The momentum distributions do not have real-time dependence, which allows for direct non-perturbative calculations in lattice QCD.
- The momentum distributions are frame dependent, and the momentum P^z dependence follows the renormalization group (RG) equation [85]. As mentioned before, the light-cone divergence is effectively regularized by the finite large momentum P^z , which means the P^z evolution in the momentum distributions is the same as the RG evolution for the light-cone distributions. For collinear PDFs, the P^z evolution is identical to the DGLAP evolution. For TMDs, the P^z evolution is also related to rapidity evolution. Therefore, one can implement a global fit program with experimental data in terms of momentum distributions at a particular P^z , analogous to light-cone distributions at a certain μ or rapidity scale.

In practical applications of the LaMET expansion, the relevant collinear modes are selected through external hadron states. Moreover, for momentum distributions, the UV physics of the effective lagrangian in Eq. (2.8) is the same as the full theory. Therefore, one can make computations just with the full lagrangian, with the EFT power counting applied for constructing interpolating fields for high-momentum hadrons.

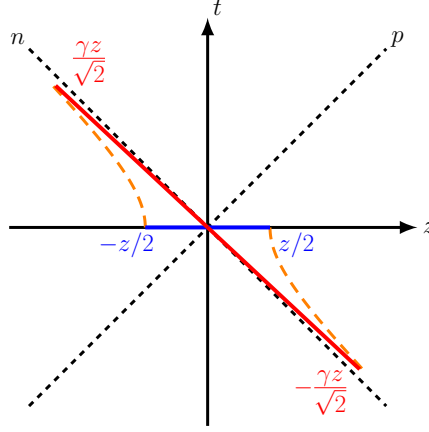


Figure 2.2: Two different formulations of parton physics, longitudinal momentum distribution (blue) and light-cone distribution (red), are connected through a Lorentz boost at the operator level. The plot is taken from Ref. [11].

2.2.3 The Relation between Parton and Momentum Distributions

At the operator level, without considering perturbative corrections, one can formally show the equivalence between the $P^z \rightarrow \infty$ limit of the LMD and the light-cone PDF through the Lorentz boost. Express the large momentum hadron as $|H(P)\rangle = \Lambda(P)|H_{\text{res}}\rangle$, where $|H_{\text{res}}\rangle$ is the hadron at rest and $\Lambda(P)$ is a large Lorentz boost operator. The matrix element in LMD in Eq. (2.7) can be written as,

$$\langle H_{\text{res}} | \Lambda^\dagger(P) \bar{\psi}_v(z) \gamma^z U(z, 0) \psi_v(0) \Lambda(P) | H_{\text{res}} \rangle . \quad (2.9)$$

Implementing the Lorentz boost on the non-local quark bilinear operator and taking the $P^z \rightarrow \infty$ limit, one obtains the light-cone correlator in Eq. (2.3). A diagrammatic illustration of the Lorentz boost picture is shown in Fig. 2.2.

Since QCD is an asymptotically free theory with non-trivial logarithmic pre-asymptotics, the

$P^z \rightarrow \infty$ limit cannot be naively taken. As shown in the one-loop perturbative result for LMD [88],

$$\tilde{f}(y, P^z) = \delta(y - 1) + \frac{\alpha}{2\pi} \frac{1 + y^2}{1 - y} \theta(y - 1) \theta(y) \ln(P^z/m) + \dots, \quad (2.10)$$

the logarithm in P^z indicates that the $P^z \rightarrow \infty$ limit is not analytic. However, the coefficient in front of $\ln(P^z/m)$ is the Dokshitzer–Gribov–Lipatov–Altarelli–Parisi (DGLAP) kernel, meaning that the IR divergence regularized by a finite quark mass m is the same as the light-cone PDF. Therefore, the light-cone parton physics corresponds to taking $P^z \rightarrow \infty$ before the UV cutoff Λ_{UV} is imposed, which is similar to taking $m_Q \rightarrow \infty$ before implementing renormalization in heavy quark effective theory (HQET) [299–309]. In this sense, partons are IR EFT dofs of LMD, and the large P^z expansion of LMD in Eq. (2.7) is connected to the light-cone PDF in Eq. (2.3) through a matching relation or factorization formula,

$$\tilde{f}(y, P^z) = \int_{-\infty}^{+\infty} \frac{dx}{|x|} \tilde{C}\left(\frac{y}{x}, \frac{\mu}{xP^z}\right) f(x, \mu) + O\left(\frac{\Lambda_{\text{QCD}}^2}{y^2 P_z^2}, \frac{\Lambda_{\text{QCD}}^2}{(1-y)^2 P_z^2}\right), \quad (2.11)$$

where the UV difference between $\tilde{f}$ and f is absorbed into the perturbative matching kernel $\tilde{C}$.

On the other hand, since the two EFTs, SCET in Eq. (2.4) and LaMET in Eq. (2.8), are equivalent for IR physics, PDF in Eq. (2.3) and LMD in Eq. (2.7) can be naturally connected through a LaMET expansion,

$$f(x, \mu) = \int_{-\infty}^{+\infty} \frac{dy}{|y|} C\left(\frac{x}{y}, \frac{\mu}{yP^z}\right) \tilde{f}(y, P^z) + O\left(\frac{\Lambda_{\text{QCD}}^2}{x^2 P_z^2}, \frac{\Lambda_{\text{QCD}}^2}{(1-x)^2 P_z^2}\right), \quad (2.12)$$

where C is the perturbative matching kernel that takes into account the UV non-commuting effect between $P^z \rightarrow \infty$ and $\Lambda_{\text{UV}} \rightarrow \infty$. Operationally, C can be obtained from a forest formula similar to

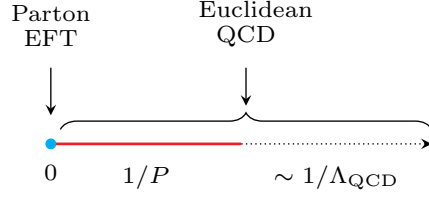


Figure 2.3: Interpretation of LaMET by analogy with critical phenomena. The parton EFT corresponds to infinite momentum limit at the critical point. The Euclidean QCD is related to finite large momentum, a domain in the neighborhood of the critical point. They are connected at large momentum expansion by perturbative matching involving a momentum RG evolution. The plot is from Ref. [11].

the BPHZ renormalization [310, 311] or from inverting $\tilde{C}$ in Eq. (2.11) order by order perturbatively. This LaMET expansion can be used to calculate the PDF with controlled precision in the moderate x range for $\Lambda_{\text{QCD}}/2P^z \lesssim x \lesssim 1 - \Lambda_{\text{QCD}}/2P^z$ from the LMD, that is accessible in lattice QCD.

In fact, all light-cone parton distributions (including PDFs, DAs, GPDs, TMDs...), whose UV divergences are properly regularized in a perturbative renormalization scheme or a rapidity scheme, can be calculated through large momentum expansion and perturbative matching from the Euclidean momentum distributions that are available from lattice QCD. The equivalence of the IR physics between light-cone distributions and Euclidean momentum distributions can be shown to all orders using the reduced-diagram approach [57]. Their UV difference can be calculated in perturbation theory and factorized out into a matching kernel.

We may interpret the physics of LaMET by analogy with critical phenomena [11]. In the phase diagram of a statistical system, the critical point is the endpoint of a phase equilibrium curve. The critical point is generally considered a singular point, where the correlation length tends to infinity. The physics near the critical point is universal across a class of systems, where the exponents of the scaling laws can be studied using the RG equation. As for parton physics, the infinite-momentum limit or light-cone correlator is a critical point that is elegant and beautiful due to momentum independence

and scaling symmetry but singular and difficult to handle mathematically. LaMET, a finite large-momentum framework, focuses on a domain close to the critical point (not exactly at the critical point), thereby yielding well-defined objects and enabling direct calculations. Similarly, the P^z -evolution of LMD near the critical point can be studied with the RG equation. A schematic diagram of the above discussions is depicted in Fig. 2.3.

2.3 Lattice Calculations of Various Parton Physics under LaMET

This section presents some examples of the state-of-the-art lattice QCD calculations for parton distributions using LaMET expansions. More comprehensive reviews can be found here [255, 275–277, 312–317].

2.3.1 Parton Distribution Functions

The collinear parton distribution functions (PDFs) are EFT descriptions of longitudinal momentum distributions (LMDs) of quarks and gluons in a hadron in the infinite momentum limit. They have been widely studied under LaMET [12, 16, 17, 44, 49, 50, 83, 88–145]. Some PDFs, like the unpolarized proton $u - d$ PDF, have been determined with high precision from experimental measurements and therefore provide a benchmark to validate and calibrate LaMET analysis procedure. Other PDFs, including the pion valence, the nucleon transversity, and the gluon PDFs, are not well constrained by experimental data; in these situations, LaMET can provide complementary information.

The nucleon transversity PDF has been obtained within the LaMET framework by the Lattice Parton Collaboration (LPC) [12], using CLS ensembles [318] with four lattice spacings, $a = 0.098, 0.085, 0.064$, and 0.049 fm, a range of pion masses from 354 to 222 MeV, and several hadron

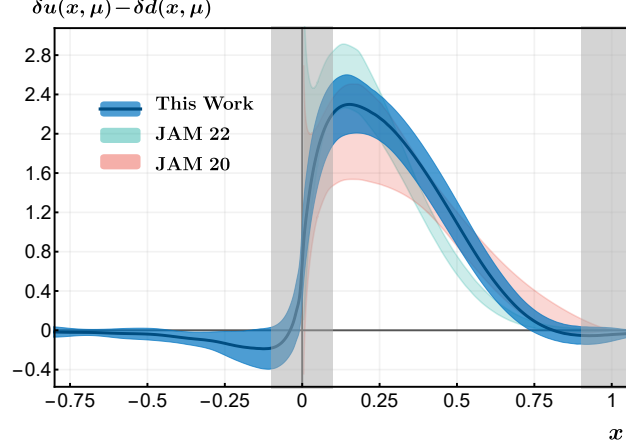


Figure 2.4: The nucleon isovector transversity PDFs. The blue band is calculated from lattice QCD under LaMET [12], extrapolated to continuum limit, physical pion mass, and infinite momentum limit, compared with JAM20 (red band) [13] and JAM22 (green band) [14] global fits. The plot is from Ref. [12].

momenta up to 2.8 GeV. The lattice matrix elements are renormalized in the hybrid scheme [287] with self renormalization [231]. The final distribution is extrapolated to the continuum limit, to the physical pion mass, and to infinite momentum. This result is displayed as the blue band in Fig. 2.4, and is compared with the JAM global analysis with experimental data [13, 14]. In this case, the LaMET PDF achieves a precision that is comparable to, or even better than, phenomenological PDFs. While lattice systematics have already been carefully controlled in Ref. [12], the study can be improved by incorporating the most recent techniques for theoretical precision control, such as DGLAP logarithm resummation [118] and leading renormalon resummation [232], as implemented in Ref. [121], together with threshold logarithm resummation [17, 115, 122, 123] and large-distance asymptotic analysis [233]. In addition, the lattice signal itself can be enhanced by employing the recently proposed kinematically enhanced lattice interpolators (KELI) [279].

The most recent pion valence PDF calculated by Argonne National Lab/Brookhaven National Lab (ANL/BNL) Collaboration [15, 16, 22] is shown as the green band in Fig. 2.5. This result incorporates a sophisticated treatment of theoretical systematics, including two-loop perturbative match-

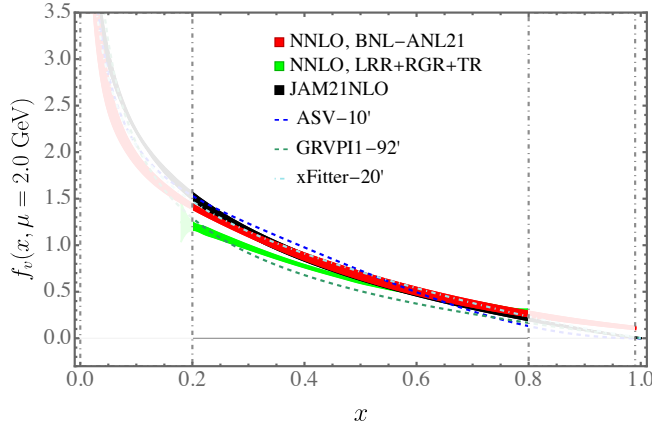


Figure 2.5: The pion valence PDFs. The red and green bands are LaMET PDFs with lattice spacing $a = 0.04$ fm and pion mass $m_\pi = 300$ MeV at momentum $P^z = 1.94$ GeV [15, 16], where the red band is the first NNLO result obtained by ANL-BNL21 [16] and the green band is an updated result implementing the resummation techniques [17]. The other curves are experimental PDFs by GRVP11-92' [18], ASV-10' [19], xFitter-20' [20], and JAM21NLO [21]. The plot comes from Ref. [22].

ing [288, 289], leading renormalon resummation [232], DGLAP logarithm resummation [118], and threshold logarithm resummation [17, 115, 122, 123]. The LaMET PDF uncertainty is smaller than the spread among different experimental PDFs. Further work is required to control the continuum limit, the physical-pion-mass limit, and the infinite-momentum limit.

2.3.2 Distribution Amplitudes

The distribution amplitudes (DAs) are related to the probability amplitudes of how the longitudinal momentum of a hadron is distributed among quarks in its leading Fock state. They appear in the B meson weak decays [319, 320] that provide useful information on CP violation and Cabibbo-Kobayashi-Maskawa matrix, and play an important role in search for beyond-standard-model physics. They are also inputs in the study of pion elastic form factors [236], the pion-photon transition form factor [321–323], and hard exclusive meson production [324]. The studies of DAs under LaMET include light-meson DAs [23, 146–159], heavy-meson DAs [160–166, 273], and baryon DAs [167–174].

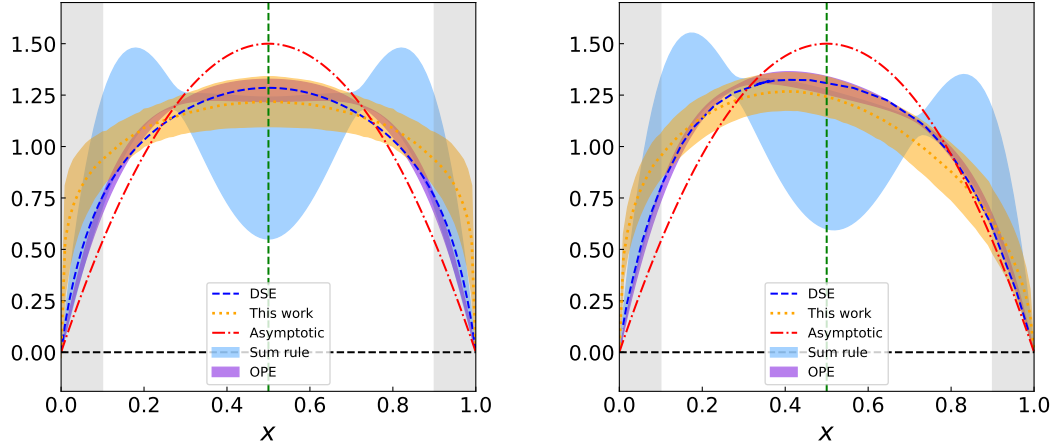


Figure 2.6: The LCDAs for π (top) and K (bottom). The yellow bands denote the LaMET calculations [23]. For the kaon, x is the momentum fraction carried by the light quark. The plots are from Ref. [23].

The LPC work [23] reports a lattice QCD study of the pion and kaon DAs using MILC ensembles [325, 326], with three lattice spacings, the physical pion mass, and multiple hadron momenta. The final results, depicted as yellow bands in Fig. 2.6, are obtained with a systematic treatment of lattice artifacts, involving hybrid renormalization, physical pion mass, continuum limit, and infinite-momentum extrapolation. Theoretical uncertainties can be further controlled by incorporating two-loop perturbative matching [157], renormalization group resummation [154], leading renormalon resummation [154], threshold resummation [156], and large-distance asymptotic analysis [233]. Moreover, data quality can be further improved using KELI [279].

2.3.3 Generalized Parton Distributions

The generalized parton distributions (GPDs) are defined through off-forward matrix elements between hadronic states carrying different momenta. They depend on three kinematic variables: the average parton momentum fraction x , the skewness parameter ξ characterizing the longitudinal mo-

momentum transfer, and the Lorentz-invariant momentum transfer squared Q^2 . GPDs encode information on the nucleon spin structure [241, 327] and enable a three-dimensional tomography of the nucleon. Experimentally, they can be accessed via deeply virtual Compton scattering and deeply virtual meson production [241, 328]. Because the operators that define GPDs are identical to those used for PDFs, the LaMET framework can be straightforwardly generalized to compute GPDs [24, 130, 146, 175–196].

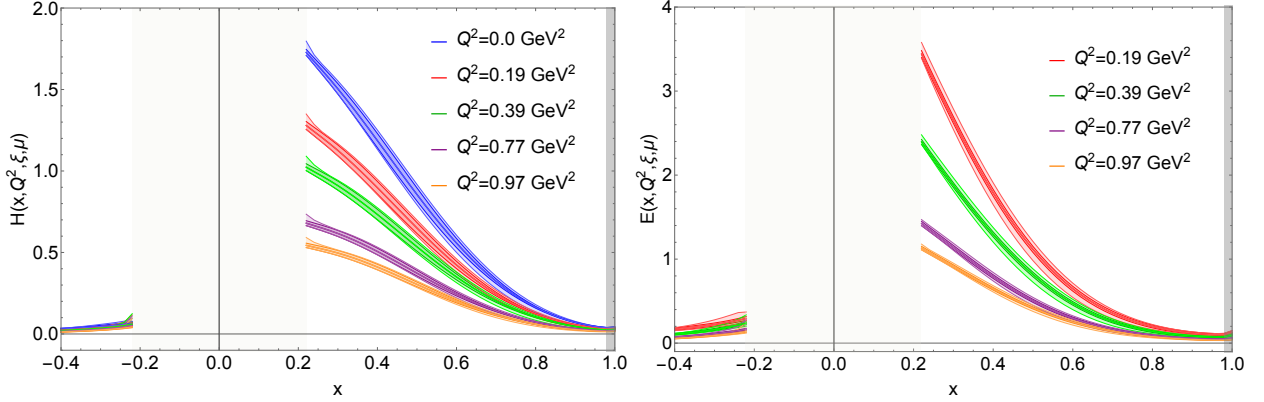


Figure 2.7: The unpolarized nucleon GPDs calculated under LaMET [24]. The skewness is $\xi = 0$ and perturbative matching is NNLO+LRR+RGR. The inner error bands are statistical, and the outer error bands are combined statistical and systematic errors. The perturbative matching breaks down for $|x| \lesssim 0.2$, which is shaded in light gray. The plots are from Ref. [24].

A LaMET calculation performed by Michigan State University (MSU) [24] utilizes the MILC ensembles [325, 326], with a physical pion mass and a lattice spacing of $a = 0.09$ fm. For zero skewness cases, NNLO perturbative matching [288, 289], DGLAP log resummation [118], and leading renormalon resummation [232] have been implemented, as illustrated in Fig. 2.7. To achieve a more precise control of theoretical uncertainties, one may include threshold resummation [194] and large-distance asymptotic analysis [233]. In addition, lattice systematic effects, such as the continuum and infinite-momentum extrapolations, can be studied. The recently introduced KELI method [279] provides a way to improve the quality of GPD data.

2.3.4 Transverse Momentum Dependent Distributions

TMDPDFs denote transverse-momentum-dependent parton distribution functions. In theoretical analysis, the dependence on the transverse momentum $k_\perp$ is typically converted into a dependence on the impact parameter $b_\perp$. TMDs have been studied extensively under LaMET [25–29, 36, 40, 123, 197–224, 274].

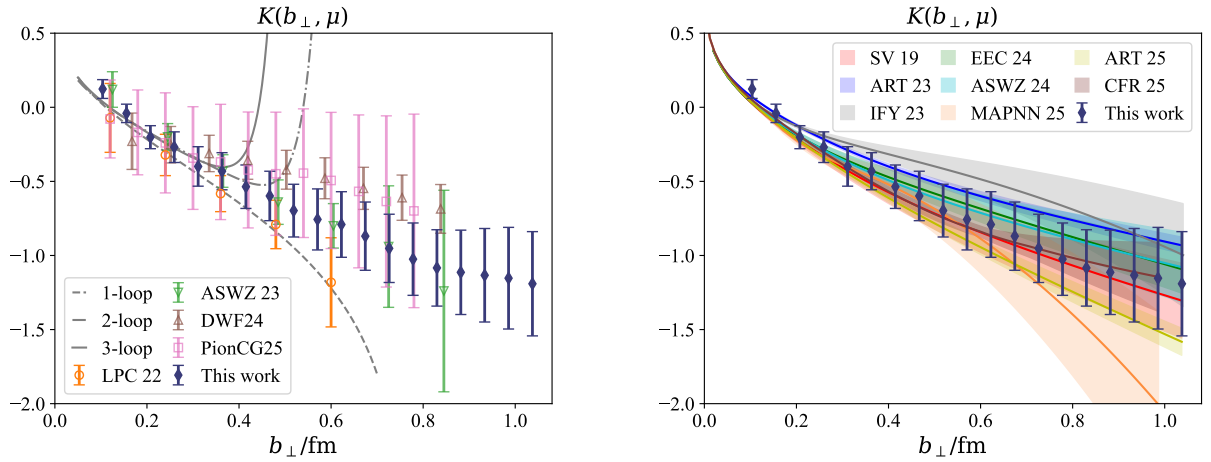


Figure 2.8: Left panel: comparison of the Collins-Soper kernels calculated under LaMET, including LPC25 [25] (labeled as “This work”), LPC22 [26], ASWZ23 [27], DWF24 [28], PionCG25 [29], as well as the perturbative calculations up to three loops [30, 31]. Right panel: comparison of LPC25 [25] with phenomenological extractions from SV19 [32], ART23 [33], IFY23 [34], EEC24 [35], ASWZ24 [36], MAPNN25 [37], ART25 [38], CFR25 [39]. The plots are taken from Ref. [25].

A key characteristic of the TMDPDF is the rapidity evolution, described by the Collins–Soper (CS) kernel. The CS kernel carries the dependence on $b_\perp$ dependence, which behaves as perturbative $\ln(b_\perp \mu)$ when $b_\perp \ll 1/\Lambda_{\text{QCD}}$, and turning nonperturbative once $b_\perp \sim 1/\Lambda_{\text{QCD}}$. The CS kernel has been computed by the MIT group [36, 204], LPC [25, 26, 215], and ANL/BNL collaboration [28, 29]. A plot from Ref. [25] comparing recent lattice QCD calculations as well as phenomenological results is presented in Fig. 2.8. Another crucial input in LaMET quasi-TMD factorization is the reduced soft factor, which encodes soft radiation emitted by color sources separated by $b_\perp$. The first lattice

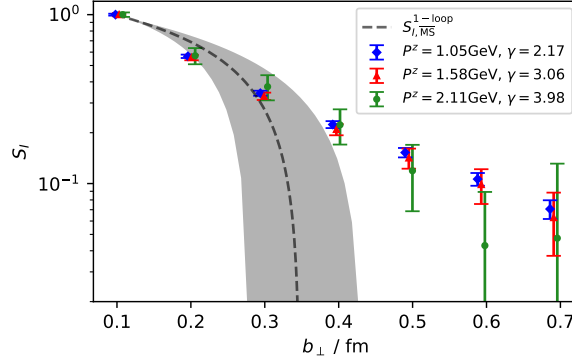


Figure 2.9: The intrinsic soft factor calculated and presented in Ref. [40]. The lattice QCD calculations with different momenta are displayed as blue, red, and green points. The dashed curve shows the result of the 1-loop calculation, with the strong coupling constant $\alpha_s(1/b_\perp)$.

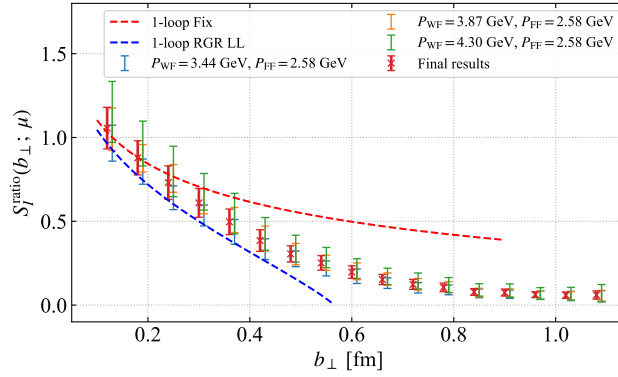


Figure 2.10: Intrinsic soft function in Coulomb gauge method at $\mu = 2$ GeV. The LaMET results extracted at different pairs of hadron momenta are shown as blue, orange, and green points [29]. The final results evaluated as a correlated average over the momentum pairs are shown as red points with error bars [29]. The corresponding one-loop fixed-order perturbative results are the red dashed line, and the one-loop RG-resummed (RGR) perturbative results are the blue dashed line. The plots are taken from Ref. [29].

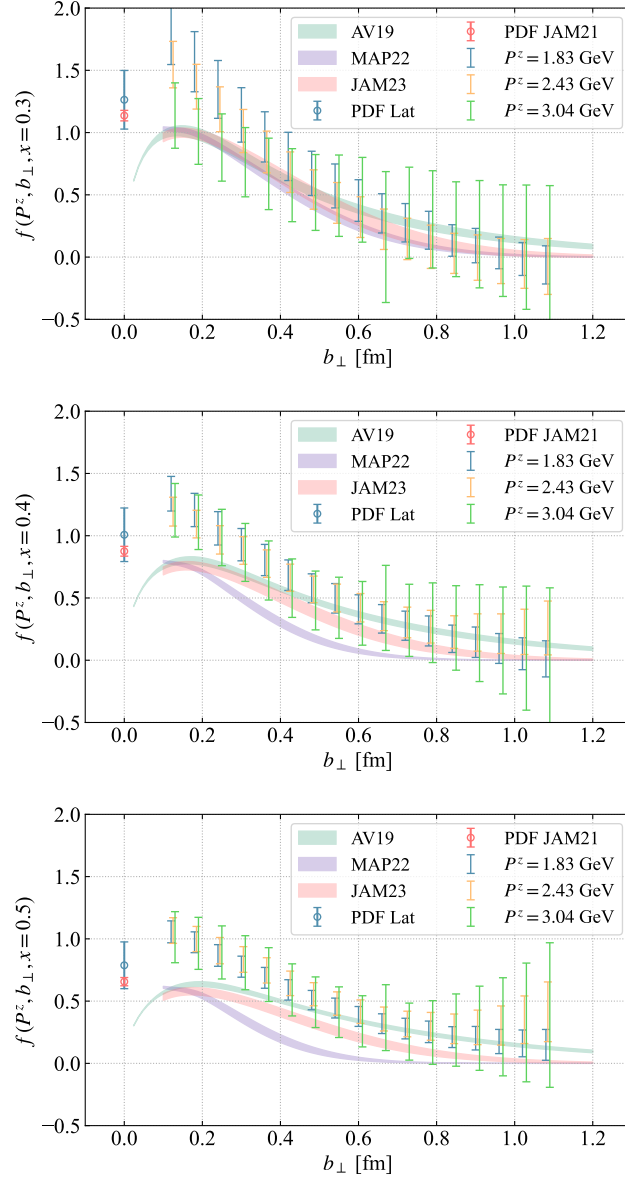


Figure 2.11: Unpolarized light-cone TMDPDF of pion at $x = 0.3$, $x = 0.4$ and $x = 0.5$ in $b_\perp$ space up to $b_\perp \gtrsim 1$ fm. The LaMET calculations from different hadron momenta P^z are displayed as blue, yellow, and green data points [29], which are illustrated alongside the phenomenological results from AV19 [41], MAP22 [42] and JAM23 [43]. In addition, the collinear PDFs from lattice calculation in CG (PDF Lat) [44] and global fit (PDF JAM21) [21] are plotted for comparison. The plots are taken from Ref. [29].

determination of the reduced soft factor was carried out by LPC [40] shown in Fig. 2.9, and the most up-to-date result is calculated by the ANL/BNL collaboration [29] displayed in Fig. 2.10.

The first lattice calculation of the TMDPDF was carried out by the LPC [210]. A more recent determination by the ANL/BNL collaboration [29] is displayed in Fig. 2.11. This study employs lattice QCD ensembles generated by the HotQCD Collaboration [329], with a pion mass of 300 MeV, a fine lattice spacing of $a = 0.06$ fm, and several hadron momenta reaching up to 3 GeV. To enhance the signal, both the Coulomb gauge method [218] and the KELI method [279] are utilized. In addition, RG resummation of the hard kernel is implemented to better constrain the theoretical uncertainties. Further work is needed on the physical pion mass, discretization effects, and the linear renormalon [123].

2.4 Theoretical Foundations, Precision Control and Generalizations

The applications of LaMET depend on two important theoretical understandings: matching IR physics of quasi- and light-cone distributions to all orders in perturbation theory, and the renormalization of UV divergences in quasi-correlators. Together, they justify the basic idea and enable its practical implementation.

Further developments of LaMET have mainly taken place in three directions: improving theoretical precision using standard QFT tools, extending the framework to a wider range of parton physics, and constructing novel quasi-correlators within the same universality class.

2.4.1 Matching IR Physics

A central requirement for LaMET to work is that at leading in $1/P^z$ quasi-distributions reproduce the same IR physics as light-cone distributions, while their UV differences can be factorized

into perturbatively calculable matching coefficients. This implies that the quasi-PDF $\tilde{f}(y, P^z)$ defined in Eq. (2.7), at the leading twist in the large P^z expansion, should be factorized into the light-cone PDF $f(x, \mu)$ defined in Eq. (2.3) regarding the IR collinear mode and a perturbative matching kernel $\tilde{C}(y/x, \mu/xP^z)$ containing only the UV scales, as shown in Eq. (2.11). The validity of this factorization must be established to all orders in perturbation theory. This is nontrivial because the quasi-PDF involves a spacelike gauge link, unlike the lightlike or timelike Wilson lines in standard factorization frameworks, such as Libby-Sterman analysis [57] or SCET [297, 298].

The essential structure of this factorization can already be seen at one-loop, where the IR behavior is identical while UV differences can be factorized out. This was first done in Ref. [88] in a cutoff scheme and was then confirmed in Refs. [103, 106] in the $\overline{\text{MS}}$ scheme. In Ref. [88], the collinear divergence is regulated by the quark mass m and the UV divergence is regulated by a transverse momentum cut-off μ . The one-loop result for quasi-PDF for $x \neq 1$, under the expansion $\mu \gg P^z \gg m$, takes the form,

$$\tilde{f}^{(1)}(x) = \frac{\alpha C_F}{2\pi} \begin{cases} \frac{1+x^2}{1-x} \ln \frac{x}{x-1} + 1 + \frac{\mu}{(1-x)^2 P^z}, & x > 1, \\ \frac{1+x^2}{1-x} \ln \frac{4(P^z)^2}{m^2} + \frac{1+x^2}{1-x} \ln \frac{x}{1-x} - \frac{4x}{1-x} + 1 + \frac{\mu}{(1-x)^2 P^z}, & 0 < x < 1, \\ \frac{1+x^2}{1-x} \ln \frac{x-1}{x} - 1 + \frac{\mu}{(1-x)^2 P^z}, & x < 0. \end{cases} \quad (2.13)$$

The one-loop result for light-cone PDF for $x \neq 1$ corresponds to another expansion $P^z \gg \mu \gg m$,

$$f^{(1)}(x) = \frac{\alpha C_F}{2\pi} \begin{cases} 0, & x > 1 \text{ or } x < 0, \\ \frac{1+x^2}{1-x} \ln \frac{\mu^2}{m^2} - \frac{1+x^2}{1-x} \ln(1-x)^2 - \frac{2x}{1-x}, & 0 < x < 1. \end{cases} \quad (2.14)$$

Clearly, the IR collinear mode, $\sim \frac{\alpha C_F}{2\pi} \frac{1+x^2}{1-x} \ln \frac{1}{m^2}$, is the same between light-cone and quasi PDFs. Their UV differences, involving scales μ and P^z , can be factorized into a perturbative matching kernel (or into a renormalization factor for some terms). In particular, the light-cone singularity in the light-cone PDF behaving as $\sim \frac{\alpha C_F}{2\pi} \frac{1+x^2}{1-x} \ln \mu^2$ is now properly regularized by the finite large momentum P^z in the quasi-PDF appearing as $\sim \frac{\alpha C_F}{2\pi} \frac{1+x^2}{1-x} \ln P_z^2$, and their relation can be obtained through perturbative matching. Therefore, the LaMET factorization Eq. (2.11) has been verified up to one-loop order.

The all-order justification has been studied using several approaches. One approach is the operator product expansion (OPE), which expands a correlator in terms of local operators and Wilson coefficients, studies the symmetry properties of local operators, and sums the series to obtain the factorization theorem for the correlator. The basic idea of applying OPE to LaMET was first discussed in Ref. [84], and comprehensive derivations can be found in Refs. [103, 106, 230]. One important step is the application of Lorentz symmetry to the quark bilinear derivative operators, which indicates that the IR physics is independent of gauge link directions.

An alternative diagrammatic approach is the Libby Sterman analysis, or reduced diagram analysis [57], which identifies the pinch-pole singularities by solving the Landau equations, obtains the leading-region diagram by power counting, and factorizes out the IR divergences using ladder decomposition. An all-order justification in the reduced diagram analysis was first given in Ref. [330], and a similar diagrammatic proof was then outlined in Ref. [276]. One crucial point in the proof is that $A^0 \sim A^z$ for a collinear gluon [58], which means the spacelike gauge link has the necessary component to reproduce the collinear mode.

In summary, the LaMET factorization is supported by multiple approaches, including SCET in Refs. [194, 331], establishing the theoretical foundation for the large-momentum expansion and

perturbative matching in LaMET.

2.4.2 Renormalization of UV Divergences

The quasi-correlators in LaMET contain UV divergences. For example, the lattice QCD quasi-PDF matrix element exhibits both linear and logarithmic divergences in lattice spacing a ,

$$\tilde{h}^B(z, P^z, a) \equiv \langle H(P) | \bar{\psi}(z) \gamma^t U(z, 0) \psi(0) | H(P) \rangle . \quad (2.15)$$

A key requirement for implementing LaMET with lattice QCD is that these UV divergences can be factorized and removed through a well-defined renormalization procedure. The main challenge in establishing this renormalization arises from the linear divergence associated with the spacelike gauge link, which is absent in the traditional building blocks of QCD factorization theorems.

Evidence of these UV divergences is seen both in perturbative calculations and in lattice data. In perturbation theory, the one-loop result in Eq. (2.13) exhibits explicit UV sensitivity. The term $\sim \mu / [(1-x)^2 P^z]$ indicates the existence of linear divergence, which corresponds to $\sim z/a$ in coordinate space lattice data. The $x \rightarrow \infty$ asymptotics $\sim -3/(2|x|)$ originates from the Fourier transform of $\sim 3/2 \ln(z^2 \mu^2)$, which implies the existence of logarithmic divergence $\sim 3/2 \ln(z^2/a^2)$ in lattice data for $z \gg a$. Correspondingly, lattice data show strong dependence on lattice spacing a , as exemplified in Fig. 2 of Ref. [231], which confirms the presence of these divergences. These UV divergences, especially the linear divergence, must be properly eliminated before taking the continuum limit. Meanwhile, one needs to convert the lattice scheme into a perturbatively well-defined scheme for subsequent steps, such as perturbative matching.¹

¹Moreover, a mismatch arises at $z \rightarrow 0$ between continuum perturbation theory and lattice data: the former approaches zero due to a positive anomalous dimension $\sim 3/2$, whereas the latter tends to one, the conserved vector charge. This

It has been shown that these UV divergences are multiplicatively renormalizable [280–282, 332], allowing the quasi-correlator to be factorized into a UV renormalization factor and a finite matrix element. In Ref. [280], the gauge link is represented by the auxiliary heavy quark. The UV divergence of the heavy-light current yields the logarithmic divergence, which is factorized out as in other local operators. The self-energy of the auxiliary heavy quark is related to the linear divergence, which behaves as an overall exponential factor $\sim e^{-\delta m(a)|z|}$. Since these UV divergences are factorized out, a multiplicative renormalization method becomes possible.

A commonly used non-perturbative renormalization scheme is the RI/MOM scheme [94, 283–285], which was adopted in early LaMET studies. The key object in the RI/MOM scheme is the RI/MOM factor, $Z^{RI}(z, \mu_R, a) \propto \text{Tr} [\gamma_t \langle q | \bar{\psi}(z) \gamma^t U(z, 0) \psi(0) | q \rangle] \big|_{p^2 = -\mu_R^2, p_z = p_t = 0}$, which is expected to share the same linear and logarithmic divergences as the hadronic matrix element in Eq. (2.15) because of the same operator. The external state $|q\rangle$ is an off-shell quark state with momentum p , where the off-shellness $-p^2 \gg \Lambda_{\text{QCD}}$ remains a perturbative scale. The renormalized matrix element in the RI/MOM scheme is the ratio of the hadron matrix element to the RI/MOM factor, $\tilde{h}^B(z, P^z, a)/Z^{RI}(z, \mu_R, a)$, where the a -dependence is canceled and the scale μ_R is perturbatively controllable. However, additional non-perturbative IR effects are introduced at large $z \sim 1/\Lambda_{\text{QCD}}$, which is confirmed with lattice data, as exemplified in Fig. 11 in Ref. [231]. This issue is solved with hybrid renormalization [287].

The hybrid renormalization [287] with the self-renormalization [231] is now the standard method for LaMET. The motivation for hybrid renormalization is to implement an almost minimal UV subtraction scheme without introducing additional IR behavior and to mitigate the short-distance dis-

discrepancy is caused by different orders of limits, $a \ll z \rightarrow 0$ for continuum and $z \ll a \rightarrow 0$ for lattice. This mismatch issue will be mitigated by taking the ratio of large-momentum to zero-momentum matrix elements at small z in the hybrid renormalization [287].

crepancies between lattice and continuum theories arising from discretization effects. The hybrid renormalized matrix element is [287],

$$\tilde{h}^R(z, P^z) = \frac{\tilde{h}^B(z, P^z, a)}{\tilde{h}^B(z, 0, a)} \theta[z_s - |z|] + \frac{\tilde{h}^B(z, P^z, a)}{Z_R(z, a, \mu)} \frac{Z_R(z_s, a, \mu)}{\tilde{h}^B(z_s, 0, a)} \theta[|z| - z_s] . \quad (2.16)$$

The quantity $\tilde{h}^B(z, 0, a)$ is the zero momentum matrix element generated on lattice. The factor $Z_R(z, a, \mu)$, converting lattice data to $\overline{\text{MS}}$ scheme, can be extracted using self renormalization [231]. The hybrid cutoff z_s is also chosen within the window of $a < z_s \ll 1/\Lambda_{\text{QCD}}$, so the z_s dependence can be described by continuum perturbation theory. For $|z| > z_s$, it is the same as the $\overline{\text{MS}}$ scheme up to an overall constant², which purely subtracts off the UV divergences without introducing additional IR effects. For $|z| < z_s$, the hybrid renormalization is the same as the ratio method, which eliminates the UV divergences as well as part of the discretization effects. This standard method has been adopted in various LaMET calculations [12, 23–25, 50, 118, 124, 125, 127, 133, 141–145, 154, 156, 162, 165, 168, 170, 172, 173, 232]. The generalization of hybrid renormalization to baryon DA has been studied in Ref. [168] and the quark-singlet case in Ref. [126]. In addition to self-renormalization, hybrid renormalization can be combined with other approaches, such as the static heavy-quark potential [16] or gradient flow [173].

To summarize, the UV divergences have been shown theoretically to factorize out, thereby enabling a multiplicative renormalization method. The hybrid renormalization with the self renormalization method has been proposed for practical lattice calculations, which is now a standard method for LaMET. These developments establish the theoretical foundation for the renormalization of lattice matrix elements in LaMET. For future work, one can conduct a more careful analysis for $z \sim a$, which

²The factor $Z_R(z_s, a, \mu)/\tilde{h}^B(z_s, 0, a)$ is introduced to make the curve continuous at $|z| = z_s$.

may involve lattice perturbation theory [93, 283, 333, 334] and cubic group symmetry [335, 336].

2.4.3 Precision Control in Collinear PDFs

Building upon the theoretical foundation described above, the extraction of light-cone parton distributions in LaMET proceeds through several steps:

- Calculation of lattice QCD matrix elements;
- Renormalization of quasi-correlators, implementing hybrid renormalization [287];
- Fourier transformation [233] to momentum space, yielding quasi-distributions;
- Perturbative matching, relating quasi-distributions to light-cone distributions.

This subsection discusses the control of theoretical uncertainties, involving perturbative matching, renormalization, and Fourier transform, taking the collinear quark PDF as an example. They are studied under asymptotic expansions near $P^z \rightarrow \infty$, $a \rightarrow 0$ or $z \rightarrow \infty$. The theoretical uncertainties of other parton physics will be presented in Sec. 2.4.4, and the numerical precision related to lattice data generation will be discussed in Sec. 2.5.

Two-loop calculations for the matching kernel C for quark PDF in Eq. (2.12) have been done in Refs. [288, 289]³. These calculations are typically carried out using integration-by-parts (IBPs) relations to reduce a large set of loop integrals into linear combinations of a small set of master integrals (MIs), and then evaluating the MIs using dimensional recurrence relations, as in Ref. [288], or differential equations, as in Ref. [289]. The numerical effects of two-loop matching have been tested on the pion valence PDF in Refs. [16, 118], showing that the results are almost consistent with one-loop matching, except for small deviations.

³The two-loop calculation for quasi-DA can be found in Ref. [157].

The matching kernel C for quark PDF contains the Dokshitzer–Gribov–Lipatov–Altarelli–Parisi (DGLAP) logs, which appear as $\frac{\alpha C_F}{2\pi} \frac{1+x^2}{1-x} \ln \frac{\mu^2}{4x^2 P_z^2}$ at one-loop by comparing Eqs. (2.13)(2.14), and behave as $\sim \alpha^n \ln^n [4x^2 P_z^2/\mu^2]$, $n = 0, 1, 2, 3 \dots$ up to all-orders based on the renormalization group (RG) equation. These logarithms have appreciable numerical effects if $2|x|P^z$ deviates from μ , such as in the small x region, which requires resummation using the RG equation to maintain perturbative accuracy. The renormalization group resummation (RGR) has been done in Ref. [118]⁴, which demonstrates the breakdown of LaMET matching for $2|x|P^z \sim \Lambda_{\text{QCD}}$ and controllable perturbative corrections in the moderate x range.

In addition to DGLAP logs, the matching kernel C for quark PDF also contains the threshold logs, behaving as $\sim \alpha^n \ln^{2n} [4(1-x)^2 P_z^2/\mu^2]$, $n = 0, 1, 2, 3 \dots$ up to all orders. They mostly impact the accuracy in the large x region, where $2|1-x|P^z$ deviates from μ . The resummation of threshold logs (TR) has been studied in Refs. [17, 115, 122, 123]^{5 6}, which indicates the breakdown of LaMET matching for $2|1-x|P^z \sim \Lambda_{\text{QCD}}$ and reliable prediction in the moderate x range, e.g. $2|1-x|P^z \gtrsim 1 \text{ GeV}$ and $2|x|P^z \gtrsim 1 \text{ GeV}$.

The linear renormalon involves both matching and renormalization. The matching kernel C in Eq. (2.12) contains the linear renormalon series [337–340], $\sim \alpha^{n+1} \beta_0^n n!$, from the gauge link self energy [98, 109, 232], which means the perturbation series is divergent because $n!$ increases faster than any exponential- n behavior. One can regulate this renormalon divergence in the Borel plane by choosing a contour, and the ambiguity among various choices amounts to a linear power correction, $O(z\Lambda_{\text{QCD}})$ in coordinate space or $O(\Lambda_{\text{QCD}}/xP^z, \Lambda_{\text{QCD}}/(1-x)P^z)$ in momentum space. This linear power correction conspires with the mass renormalization ambiguity in the renormalization of linear

⁴The renormalization group resummation (RGR) for quasi-DA can be found in Ref. [154].

⁵The threshold resummation (TR) for quasi-DA can be found in Ref. [156] and for quasi-GPD in Ref. [194].

⁶Ref. [123] establishes the fully factorized form of the threshold asymptotics at twist-two level in the coordinate and Mellin space

divergence. If the linear divergence in the quasi-PDF matrix element and the linear renormalon series in the matching kernel are regularized in the same renormalon scheme, this linear power correction disappears in Eq. (2.12) and linear power accuracy is achieved for LaMET expansion. Ref. [232] discussed the above concepts, including renormalon, Borel plane, UV-IR conspiracy, and power accuracy, and presented the method to achieve linear power accuracy in LaMET, called leading renormalon resummation (LRR). In LRR, the linear renormalon series in the renormalized matrix element and matching kernel are summed and regularized in the same renormalon scheme, the PV prescription. Ref. [232] also performed numerical tests, demonstrating large uncertainty if the renormalon is not under control, and showing good perturbative convergence between NLO and NNLO after LRR. Further studies of the renormalon effect under conformal mapping [128] and in the gradient-flow framework [131] have also been conducted.

Before carrying out the matching, one must first perform the Fourier transform (FT), which in principle requires lattice data over an infinite z -range, including the large-distance region (e.g. $\gtrsim 1$ fm). In principle, confinement implies that the correlator should fall off exponentially at large separations, so the long-distance tail is expected to make only a small contribution to the Fourier transform. However, in actual lattice calculations, data at large z suffer from exponentially worsening signal-to-noise ratios and thus cannot be reliably used directly. To maintain control over the FT accuracy, a large-distance asymptotic analysis is employed: the noisy large-distance data are discarded, the lattice results are fitted in a suitable region, extrapolated to large z , and then used to perform the FT. This requires determining the large-distance asymptotic forms used to fit and extrapolate lattice data. Ref. [287] proposed a phenomenological parametrization inspired by the Regge and threshold limits of momentum-space PDFs. Ref. [16] later derived the asymptotic behavior using HQET and dispersive analysis, assuming no prefactors in the HQET reduction, and connected and constant form

factors. In addition, that work established an upper limit on the FT uncertainty by approximating the extrapolated function as a flat curve within one oscillatory period. Recently, Ref. [133] argued that LaMET is fundamentally a forward-problem formalism, which is systematically improvable without encountering an ill-posed inverse problem. It further emphasized that a physics-based asymptotic extrapolation yields the most reliable estimates of FT error. It also noted that there are and will be more high-quality lattice data for robust asymptotic analysis. These points responded to the concerns from Refs. [341, 342]. Ref. [233] performed more sophisticated derivations grounded in fundamental properties of QFT without model assumptions, presented all-order mathematical structures of the large distance behaviors involving exponential and polynomial dependences, discussed the accuracy counting principle for truncation to a certain fixed order, and demonstrated controlled precision of the Fourier transform.

These theoretical developments establish quantitative control of theoretical uncertainties in LaMET analysis, reaching a few-percent level in the moderate- x region with the current available momentum $2 \sim 3$ GeV on lattice. These precision calculation techniques have already been applied to several lattice QCD calculations on quark collinear PDFs [12, 16, 44, 50, 117, 119, 121, 124, 125, 134].

2.4.4 Developments for Other Parton Physics

LaMET has been applied to a wide range of parton physics, including DA, GPD, TMD, and gluon PDF, in addition to the quark collinear PDF previously mentioned. Compared to Sec. 2.3 on lattice QCD calculations, this subsection mainly reviews their theoretical developments, which may involve the establishment of factorization formulas, the design of renormalization programs, and the precision control of theoretical uncertainties.

For the light-meson DA, the one-loop matching coefficient was derived in Refs. [146, 149, 150], and the two-loop in Ref. [157]. The hybrid renormalization with the self-renormalization was first applied in Ref. [23]. The RGR and LRR were first implemented in Ref. [154]. The threshold resummation was developed in Ref. [156]. The lattice QCD calculations on light-meson DAs can be found here [23, 147, 148, 151–156].

For the heavy-meson DA, initial attempts [160, 161] used an off-light-cone gauge link, which requires lattice implementations of moving HQET and may suffer from poor signal-to-noise issues. Later on, Refs. [162, 165] proposed a two-step matching framework under the scale separation $P^z \gg m_Q \gg \Lambda_{\text{QCD}}$, which can be realized on lattice. The power corrections under this framework have been studied in Refs. [163, 166]. The heavy quark spin symmetry in the heavy-meson DA has been analyzed in Ref. [164]. A recent work [273] makes critical improvements, such as multiple lattice ensembles, high-precision simulation techniques, and renormalon resummation, to achieve a more rigorous, precise, and comprehensive determination of heavy meson LCDAs.

For the baryon DA, the one-loop matching kernel was calculated in momentum space [167] and coordinate space [169]. The hybrid renormalization method was proposed in Ref. [168], tested with lattice data in Ref. [172], and combined with the gradient flow method in Ref. [173]. Ref. [170] applied the previously developed procedure to all leading-twist light-cone DAs (LCDAs) of lowest-lying octet and decuplet baryons. Ref. [174] studied the LCDA for a heavy baryon and derived a factorization formula connecting the leading twist QCD LCDA to the boosted HQET LCDA.

As for the LaMET application to GPD, the one-loop matching kernel has been studied for the twist-2 case in Refs. [139, 146, 177, 186, 187, 343]. The studies regarding the twist-3 contributions can be found in Refs. [180, 181, 190]. The renormalon effect in GPD has been studied in Ref. [193]. The RGR and LRR have been implemented in Refs. [24, 192]. The TR was derived using SCET and tested

with GPD model in Ref. [194]. A unified neural network framework combining LaMET and SDE was presented in Ref. [130]. Here are the lattice QCD calculations on GPD [178, 179, 182, 183, 185, 185, 188, 189, 191, 192, 195, 196].

Considerable efforts have been devoted to establishing the theoretical foundation for TMD physics under LaMET, where the construction of the soft function is a crucial step. The early idea is the bent soft function [197], which has been confirmed to produce the correct IR logs at one-loop [200], but is expected to break down at higher orders [202]. Another idea, the naive rectangle-shaped Wilson loop [199, 200], cannot produce the correct IR physics. Although the desired soft factor was not found at that stage, Ref. [198] proposed extracting the Collins-Soper (CS) kernel from the ratio of TMDPDFs, which does not require the exact format of the soft factor.

Refs. [202, 203, 205] established the theoretical foundations of LaMET quasi-TMD factorizations and proposed lattice calculations of the soft factor. It was realized that the bent soft factor cannot capture the right IR physics, while the correct TMD factorization requires a so-called reduced soft factor⁷ with a different two-loop anomalous dimension than the bent version. Based on the analysis of certain Wilson-loop averages in various perturbation theories, the reduced soft factor can be extracted through form factors in moving HQET, with space-like kinematics. On the other hand, the light meson formalism allows the reduced soft factor to be extracted from a light-meson form-factor and the quasi-TMD wavefunction (WF). This reduced soft factor also appears in the factorizations of the semi-inclusive deep-inelastic scattering (DIS) and Drell-Yan (DY) cross sections in terms of the quasi-TMDPDF. Based on this framework, many lattice QCD calculations on TMD have been accomplished [25–27, 36, 40, 201, 204, 210, 212, 213, 215, 216, 220, 221, 223].

Other important theoretical works for TMD involve perturbative calculations [122, 208, 209,

⁷The reduced soft factor is sometimes called “intrinsic soft factor” in literature.

214, 224], renormalization [207, 219], time-ordering issue [217], diagrammatic justification of factorization [206], power correction [27, 123, 211], and novel operator construction [218]. Ref. [208] conducted a complete one-loop analysis for TMDWF, soft factors, and hard kernel, which further confirmed the proposal in Refs. [202, 205]. Refs. [122, 214] extracted the two-loop hard kernel for the quasi-TMD factorization. Refs. [209, 224] performed one-loop calculations for gluon TMDPDFs under LaMET. The renormalization of quasi-TMDs on the lattice was further analyzed in Refs. [207, 219]. The time-ordering issues of TMD soft factors have been studied in Ref. [217]. The all-order justification using Libby Stermann analysis has been presented in Ref. [206]. Ref. [123] explored in the TMD expansion ($b_\perp P^z \gg 1$) the linear renormalon and power correction, a proper treatment of which could reduce the undesired imaginary part in the extraction of the CS kernel for large $b_\perp$. Ref. [27] added the perturbative power correction to the hard kernel to reduce the imaginary part for small $b_\perp$. The quasi-TMD factorization with power corrections has been studied in Refs. [211, 274]. Ref. [218] proposed the Coulomb gauge method to resolve the signal-to-noise issue arising from the long staple-shaped gauge link, which will be discussed in detail in Sec. 2.4.5.

For the gluon PDF, the renormalization was studied in Refs. [136, 137]. The one-loop matching kernel was calculated in Refs. [135, 139]. Several lattice QCD calculations on gluon PDF have already been performed [138, 140–145].

These developments demonstrate the broad applicability of the LaMET to a diverse range of parton observables. Several parton observables are in the early stages, and many research opportunities remain.

2.4.5 Universality and Coulomb Gauge Correlators

The construction of appropriate Euclidean correlators is crucial for lattice implementations of LaMET. From the theoretical perspective, the large momentum limit is a critical point that can be approached from multiple “directions”, namely different operator constructions in a universality class. In practice, other operators might improve the signal in lattice calculations. The original proposal for the quasi-PDF matrix element in Eq. (2.15) includes a gauge link that introduces a linear divergence, thereby suppressing the matrix element and the signal at large distances. This motivates the exploration of alternative operator constructions beyond the conventional gauge-link formulation; some of these will be discussed in this subsection.

The main idea of the Coulomb gauge (CG) method was proposed in Refs. [44, 218], where the gauge link $U(z, 0)$ in Eq. (2.15) is replaced by a pair of Dirac dressing factors $U_C^\dagger(z)U_C(0)$,

$$\left\langle H(P) \left| \bar{\psi}(z) U_C^\dagger(z) \gamma^t U_C(0) \psi(0) \right| H(P) \right\rangle. \quad (2.17)$$

Here, U_C is a gauge transformation that rotates a gauge potential $\vec{A}$ to Coulomb gauge. Eq. (2.17) is a gauge invariant construction. If the gauge field A is already under Coulomb gauge, dressing factors in Eq. (2.17) disappear.

A central question is whether the CG method captures the desired light-cone parton physics. Ref. [218] shows that $U_C^\dagger(z)U_C(0)$ under an infinite Lorentz boost becomes a lightlike gauge link, and derives the factorization of CG quasi-TMD using SCET. Ref. [44] calculates the one-loop matching kernel that contains the desired DGLAP evolution and demonstrates the consistent numerical results for the pion valence light-cone PDF between the CG method and the original gauge link method.

These results provide evidence that the new operator construction in Eq. (2.17) reproduces the same IR collinear physics as the light-cone correlator and thus belongs to the same universality class as the original construction in Eq. (2.15), although further investigations are still needed.

The CG method offers several advantages and also introduces new theoretical questions. The CG method in Eq. (2.17) effectively distributes the color flux over a three-dimensional (3D) spatial region, in contrast to the one-dimensional (1D) gauge-link structure in Eq. (2.15). Therefore, the CG method is advantageous for suppressing UV fluctuations, as they are smeared over the 3D volume. For example, the CG method does not exhibit the linear divergence present in the original gauge-link construction, and yields good signals even at large distances, which is extremely helpful for calculating TMDs and gluon PDFs. Several state-of-the-art lattice QCD calculations have already been conducted using the CG method, including quark collinear PDFs [44, 134], TMDs [28, 29, 222], and gluon PDF [141]. On the other hand, the CG method may raise interesting IR sensitivity and open up research opportunities, involving Gribov copies [129], IR renormalon [123], and time-reversal odd effects.

In addition to the CG method, the universality class of quasi-correlators could include the correlator of the light-light currents [80, 229, 230, 272], $\bar{\psi}(z)\gamma^\mu\psi(z)\bar{\psi}(0)\gamma^\nu\psi(0)$, the correlator of the heavy-light currents [228], $\bar{\psi}(z)\gamma^\mu Q(z)\bar{Q}(0)\gamma^\nu\psi(0)$, the correlator replacing intermediate quarks into scalars [344, 345], and quark bilinear operators in any physical gauge [346]. They approach the same IR physics in the large-momentum limit and could be combined into the LaMET framework to calculate x -dependent parton physics. A systematic comparison of their perturbative calculations and lattice implementations remains an important direction for future investigation.

2.5 Numerical methods

The development of numerical techniques is essential for LaMET applications, primarily for enhancing statistical precision or controlling excited states. These techniques are related to the lattice data generation step mentioned in Sec. 2.4.3. The section reviews several important methods that have already had or will have an impact on LaMET implementations. These methods themselves contain interesting physical insights, and some of them influence not only LaMET but also other lattice QCD calculations.

LaMET requires hadronic matrix elements, while lattice QCD directly calculates vacuum matrix elements. They are connected by hadron interpolation operators and Euclidean-time evolutions. For example, the hadronic matrix element in Eq. (2.15) is obtained through a coordinate space reduction formula,

$$\begin{aligned} \frac{C_3(t_{\text{sep}}, t, z, P^z)}{C_2(t_{\text{sep}}, P^z)} \bigg|_{t_{\text{sep}}-t \rightarrow +\infty, t \rightarrow +\infty} &= \langle H(P) | \bar{\psi}(z) \gamma^t U(z, 0) \psi(0) | H(P) \rangle \\ &+ O\left(e^{-\Delta E(t_{\text{sep}}-t)}, e^{-\Delta E t}, e^{-\Delta E t_{\text{sep}}}\right), \end{aligned} \quad (2.18)$$

where the three-point (3pt) function C_3 and the two-point (2pt) function C_2 are vacuum correlation functions that are directly measurable in lattice QCD,

$$\begin{aligned} C_3(t_{\text{sep}}, t, z, P^z) &\equiv \sum_{\vec{y}, \vec{x}} e^{-iy^z P^z} \langle \Omega | O_H(t_{\text{sep}}, \vec{y}) O_C(t, \vec{x}, z) O_H^\dagger(0, 0) | \Omega \rangle, \\ C_2(t_{\text{sep}}, P^z) &\equiv \sum_{\vec{y}} e^{-iy^z P^z} \langle \Omega | O_H(t_{\text{sep}}, \vec{y}) O_H^\dagger(0, 0) | \Omega \rangle. \end{aligned}$$

Here, t_{sep}, t are Euclidean times on lattice, $|\Omega\rangle$ is the QCD vacuum, $O_C(t, \vec{x}, z) \equiv \bar{\psi}(t(\vec{x}+\vec{z}))\gamma^t U(t(\vec{x}+\vec{z}), t\vec{x})\psi(t\vec{x})$ is the quasi-PDF operator, and O_H is the hadron interpolation operator with the desired quantum numbers of the targeted hadron [1]. However, O_H overlaps not only with the targeted ground-state hadron but also with several excited states. The ground-state matrix element in Eq. (2.15) is obtained in the asymptotic limit of large Euclidean time separations, where the ratio in Eq. (2.18) isolates the desired matrix element, up to excited-state contaminations denoted as $O(e^{-\Delta E(t_{\text{sep}}-t)}, e^{-\Delta E t}, e^{-\Delta E t_{\text{sep}}})$ which are present at finite time separations in practical lattice QCD calculations but approach zero asymptotically as time separations increase.

A central question is finding an appropriate hadron interpolator O_H to improve the signal or to control excited states. This is a common issue in lattice QCD calculations. However, it becomes more important for LaMET applications; as P^z increases, the signal deteriorates due to the higher-frequency mode $e^{-iy^z P^z}$, and the excited states decay more slowly because of the smaller energy gap ΔE . Several approaches to address this issue will be presented in Secs. 2.5.1, 2.5.2, 2.5.3, 2.5.4, and 2.5.5, which are improvements compared to the traditional point-source interpolators, where the pion and nucleon cases take the forms,

$$\begin{aligned} O_\pi(t, \vec{x}) &= \bar{u}(t\vec{x})\gamma^5 d(t\vec{x}) , \\ O_N(t, \vec{x}) &= \epsilon_{abc} P_+ u(t\vec{x})_a (u(t\vec{x})_b^T C \gamma_5 d(t\vec{x})_c) , \end{aligned} \quad (2.19)$$

where $P_+ = (1 + \gamma^t)/2$ is positive parity projection operator.

Another important aspect is improving the signal associated with the current operator O_C . On lattice, smearing techniques, such as HYP or APE smearing [347], are typically adopted to reduce UV fluctuations and improve the signal. However, applying smearing to the current operator modifies its

UV properties, and the perturbative control of this modification remains unclear. The gradient flow method [131, 173, 348–361] provides a perturbatively controllable way to reduce UV fluctuations, as will be discussed in Sec. 2.5.6.

2.5.1 Momentum smearing source

The momentum smearing source [278] introduces a spatial phase on a Gaussian profile [362, 363] in O_H to improve the overlap with a large momentum hadron. Namely, the point-source quark field $q(t\vec{x}) = u(t\vec{x})$ or $d(t\vec{x})$ in Eq. (2.19) is replaced by

$$(F_{\vec{k}}q)[t\vec{x}] = \sum_{\vec{y}} e^{i\vec{k}\cdot(\vec{x}-\vec{y})} f(\vec{x} - \vec{y}) G(t\vec{x}, t\vec{y}) q(t\vec{y}) . \quad (2.20)$$

The function G is a gauge transporter. The Gaussian wave packet $f(\vec{x} - \vec{y}) = f_0 \exp(-|\vec{x} - \vec{y}|^2/2\sigma^2)$ depends on the parameter σ , which is related to the size of the ground state wavefunction, that can be tested and optimized using 2pt and effective mass. The oscillatory factor $e^{i\vec{k}\cdot(\vec{x}-\vec{y})}$ is to improve the alignment with the high-frequency mode in the large momentum projection. For a free theory, where $G(t\vec{x}, t\vec{y}) = 1$, one can show that the optimized $\vec{k} = \vec{P}/2$ for a pion and $\vec{k} = \vec{P}/3$ for a nucleon. In practical lattice QCD calculations, one can test with 2pts to find the optimized $\vec{k}$.

This method significantly improves statistical precision in lattice calculations; for example, Ref. [92] shows that, to achieve the same precision, the momentum smearing source requires only 150 measurements, while the traditional Gaussian smearing source requires 30000 measurements. The method has been widely adopted in LaMET studies.

2.5.2 Kinematically-enhanced lattice interpolator

The kinematically-enhanced lattice interpolator (KELI) [279] optimizes the Dirac structure in O_H , compared with the momentum smearing source that modifies the spatial distribution. For example, it replaces γ^5 in Eq. (2.19) with $\gamma^5\gamma^\mu$, where μ can be chosen as t , z or $+$. Under the large-rapidity power counting discussed around Eq. (2.8), KELI provides the leading contribution for a boosted hadron, namely $\bar{\psi}_v\gamma^5\gamma^\mu\psi_v \sim \gamma$, while the original operator in Eq. (2.19) is a higher order correction. It has already been applied in several state-of-the-art lattice calculations [25, 29, 144, 172, 364, 365], despite being proposed only recently.

2.5.3 Distillation

The distillation method [366] sums up a number of low eigenmodes of the Laplacian operator as the source to construct the hadron interpolator O_H , in order to improve the signal or reduce excited states. By introducing some phases to the distillation eigenmodes [367], this method could be generalized to large-momentum hadrons. The distillation has already been applied in a few works [144, 368] involving large-momentum matrix elements. This subsection provides a brief introduction to distillation based on Refs. [366–369].

Distillation starts from solving eigenvalues and eigenvectors of the Laplacian operator,

$$-\nabla^2(t; \vec{x}, \vec{y})_{a,b} \xi_b^{(k)}(t\vec{y}) = \lambda^{(k)}(t) \xi_a^{(k)}(t\vec{x}) . \quad (2.21)$$

The variable t is the time, and $\vec{x}, \vec{y}$ are spatial locations. The labels a and b denote the color indices.

The upper index k is the label of distillation eigenvalue under the ascending order $\lambda^{(1)}(t) < \lambda^{(2)}(t) < \dots$

$\lambda^{(3)}(t) < \dots$. The indices $\vec{y}$ and b are summed while t and k are not. The Laplacian operator is a gauge covariant operator,

$$\nabla^2(t; \vec{x}, \vec{y})_{a,b} = \sum_{\mu=1,2,3} (\delta_{\vec{x}, \vec{y}+\hat{\mu}} U_{ab}(t\vec{x}, t(\vec{y}+\hat{\mu})) + \delta_{\vec{x}, \vec{y}-\hat{\mu}} U_{ab}(t\vec{x}, t(\vec{y}-\hat{\mu})) - 2\delta_{\vec{x}, \vec{y}} \delta_{a,b}) . \quad (2.22)$$

For each t , the total number of distillation vectors is $L^3 \times N_c$. The gauge transformation of the eigenvector ξ is the same as a quark field. We then can define the smearing operator (or called distillation kernel in literature),

$$\square_{a,b}(t; \vec{x}, \vec{y}) = \sum_{i=1}^{N_D} \xi_a^{(i)}(t\vec{x}) \xi_b^{(i)\dagger}(t\vec{y}) , \quad (2.23)$$

where we usually choose the low modes for smearing purposes, meaning that $N_D \ll L^3 \times N_c$. The distillation kernel shares the same gauge transformation as a gauge link $U(t\vec{x}, t\vec{y})$ and Dirac dressing factors $U_C^\dagger(t\vec{x}) U_C(t\vec{y})$. By applying the distillation kernel on the original quark propagator S , one can define the smeared (distillated) propagator,

$$S_{\alpha a, \beta b}(t_1 \vec{x}, t_2 \vec{y}) \rightarrow \square_{a, a'}(t_1; \vec{x}, \vec{x}') S_{\alpha a', \beta b'}(t_1 \vec{x}', t_2 \vec{y}') \square_{b', b}(t_2; \vec{y}', \vec{y}) , \quad (2.24)$$

where repeated indices $\vec{x}', \vec{y}', a', b'$ are summed up, while t_1 and t_2 are not. (All indices are preserved in this paragraph.)

Distillation can be applied to 2pts. Recall the original 2pt for a pion,

$$C_{2,\pi^+}(t, \vec{p}) = - \sum_{\vec{x}} e^{-i\vec{p}\cdot\vec{x}} \langle \Omega | \bar{d}(t\vec{x}) \gamma_5 u(t\vec{x}) \bar{u}(0\vec{0}) \gamma_5 d(0\vec{0}) | \Omega \rangle \quad (2.25)$$

$$= \sum_{\vec{x}} e^{-i\vec{p}\cdot\vec{x}} \langle \Omega | \text{Tr}[\gamma_5 S(t\vec{x}; 0\vec{0}) \gamma_5 S(0\vec{0}; t\vec{x})] | \Omega \rangle . \quad (2.26)$$

One can use Eq. (2.24) to obtain the distilled 2pt,

$$\begin{aligned} C_{2,\pi^+}(t, \vec{p}) &\rightarrow \sum_{\vec{x}} e^{-i\vec{p}\cdot\vec{x}} \langle \Omega | \text{Tr}[\gamma_5 \square(t; \vec{x}, \vec{w}'') S(t\vec{w}'', 0\vec{w}') \square(0; \vec{w}', 0) \gamma_5 \square(0; 0, \vec{v}') S(0\vec{v}', t\vec{v}'') \square(t; \vec{v}'', \vec{x})] | \Omega \rangle \\ &= \sum_{\vec{x}} e^{-i\vec{p}\cdot\vec{x}} \sum_{i,j,k,l=1}^{N_D} \langle \Omega | \text{Tr}[\xi_l^\dagger(t\vec{x}) \gamma_5 \xi_i(t\vec{x}) \xi_i^\dagger(t\vec{w}'') S(t\vec{w}'', 0\vec{w}') \xi_j(0\vec{w}') \xi_j^\dagger(0\vec{0}) \gamma_5 \xi_k(0\vec{0}) \xi_k^\dagger(0\vec{v}') S(0\vec{v}', t\vec{v}'') \xi_l(t\vec{v}'')] | \Omega \rangle \end{aligned} \quad (2.27)$$

One can then introduce the basic (gauge invariant) building blocks,

- **Perambulator:** $\tau_{\alpha\beta}^{(i,j)}(t, t') = \xi_{a'}^{(i)\dagger}(t\vec{w}'') S_{\alpha a', \beta b'}(t\vec{w}'', t'\vec{w}') \xi_{b'}^{(j)}(t'\vec{w}') ;$
- **Annihilation operator:** $\Phi_{\Gamma, \alpha\beta}^{(l,i)}(t, \vec{p}) = \sum_{\vec{x}} e^{-i\vec{p}\cdot\vec{x}} \xi_a^{(l)\dagger}(t\vec{x}) \Gamma_{\alpha\beta} \xi_a^{(i)}(t\vec{x}) ;$
- **Creation operator:** Conjugate of the annihilation operator.

The distilled 2pt becomes,

$$C_{2,\pi^+}(t, \vec{p}) \rightarrow \sum_{i,j,k,l=1}^{N_D} \langle \Omega | \text{Tr}[\Phi_{\gamma_5}^{(l,i)}(t, \vec{p}) \tau_{ij}(t, 0) \Phi_{\gamma_5}^{(j,k)\dagger}(0, \vec{p}) \tau_{kl}(0, t)] | \Omega \rangle . \quad (2.28)$$

To obtain the perambulators $\tau_{ij}(t, 0)$ in rank N_D distillation space, one needs to do the inversions of Dirac operator for N_D times. However, once perambulators are generated, they can be saved for future use since they are universal among a class of functions. Calculating $\Phi_{\Gamma}^{(l,i)}(t, \vec{p})$ is not time consuming,

and one can reuse the same set of perambulators for 2pts with different sinks and sources. Therefore, it is easy to generate datasets with various sinks and sources, which makes it convenient to implement the generalized eigenvalue problem (GEVP). (In this paragraph, color and Dirac indices are omitted in C_{2,π^+} . All indices are preserved in the definitions of the perambulator and annihilation operator.)

Distillation can also be used to calculate 3pts. Take pion isovector quasi-PDF matrix element as an example,

$$C_{3,(u-d)/\pi^+}(t_{\text{sep}}, t, z, \vec{p}) = \langle \Omega | O_{\pi^+}(t_{\text{sep}}, \vec{p}) O_{u-d}(t, z) O_{\pi^+}^\dagger(0, \vec{p}) | \Omega \rangle \quad (2.29)$$

where $O_{u-d}(t, z) = \sum_{\vec{y}} \bar{u}(t(\vec{z} + \vec{y})) \gamma^t U(t(\vec{z} + \vec{y}), t\vec{y}) u(t\vec{y}) - \bar{d}(t(\vec{z} + \vec{y})) \gamma^t U(t(\vec{z} + \vec{y}), t\vec{y}) d(t\vec{y})$ and $O_{\pi^+}(t, \vec{p}) = \sum_{\vec{x}} e^{-i\vec{x} \cdot \vec{p}} \bar{d}(t\vec{v}) \square(t; \vec{v}, \vec{x}) \Gamma \square(t; \vec{x}, \vec{w}) u(t\vec{w})$. In distillation space, it becomes

$$C_{3,(u-d)/\pi^+}(t_{\text{sep}}, t, z, \vec{p}) = \text{Tr} \langle \Omega | \Phi_{\gamma^5}(t_{\text{sep}}, \vec{p}) J(t_{\text{sep}}, t, z) \Phi_{\gamma^5}^\dagger(0, \vec{p}) \tau(0, t_{\text{sep}}) | \Omega \rangle - \dots, \quad (2.30)$$

where the generalized perambulator is,

$$J_{\alpha\rho}^{(k_1, k_2)}(t_{\text{sep}}, t, z) = \xi_a^{(k_1)\dagger}(t_{\text{sep}} \vec{x}_1) S_{\alpha a, \beta b}(t_{\text{sep}} \vec{x}_1, t(\vec{z} + \vec{y})) \gamma_t^{\beta\gamma} U_{bc}(t(\vec{z} + \vec{y}), t\vec{y}) S_{\gamma c, \rho d}(t\vec{y}, 0 \vec{x}_2) \xi_d^{(k_2)}(0 \vec{x}_2), \quad (2.31)$$

where the repeated indices $\vec{x}_1, \vec{y}, \vec{x}_2, a, b, c, d, \beta, \gamma$ are summed up. The generalized perambulator contains the quark fields from the current, which, in principle, should be preserved in full modes and cannot be smeared arbitrarily. The question is how to evaluate the generalized perambulator. The Had-Struc collaboration computes the generalized perambulator using the sequential source method [368], which is a universal method but requires inverting the Dirac operator for each current separately. The

CLQCD collaboration proposes the blending method [369], which does not require additional inversions for different currents, and has been successfully applied to nucleon local operators such as g_A , g_S , and g_T [292]. Generalizing the blending method to quasi-PDFs requires further investigation.

Here we summarize the advantages of distillation:

- Statistical precision: distillation includes more statistics compared to a point source or a wall source. It uses all spatial locations compared to a single point in a point source; It uses N_D modes compared to a single mode in a wall source.
- Computer time: once perambulators are saved, they can be imported to compute a large number of quantities. One does not need to perform inversions repeatedly.
- Storage: saving perambulators takes less storage than saving all-to-all propagators.
- Control of excited states: smaller N_D corresponds to fewer excited states. (However, a smaller N_D also leads to a worse signal. The choice of N_D is a balance of excited state and signal.)
- More convenient (not necessary) for implementing GEVP.
- Gauge invariant building blocks: it can be easily transformed to a different gauge fixing.

We expect distillation to have more applications in LaMET calculations in the future. For quark operators, one needs to find a more efficient way to compute the generalized perambulator, perhaps by exploring the blending method for LaMET quantities. For gluon operators, one need not consider the generalized perambulator, and we expect distillation to become useful with current methods.

2.5.4 GEVP and current enhanced excited states

The generalized eigenvalue problem (GEVP) [290, 370] reduces the excited states using an optimized hadron interpolator O_H^{opt} obtained by diagonalizing the transfer matrix e^{-Ht} in a truncated Hilbert space. It has already been applied to large-momentum matrix elements [371–373], though not analyzed under the LaMET framework.

GEVP starts from a matrix of 2pts,

$$C_{2,ij}(t) = \langle \Omega | O_i(t) O_j^\dagger(0) | \Omega \rangle, \quad \text{where } i, j = 1 - N. \quad (2.32)$$

This is the transfer matrix e^{-Ht} in a truncated Hilbert space, where H is the Hamiltonian whose eigenstates are physical states in QCD. By diagonalizing this matrix,

$$C_{2,ij}(t') V_j^\alpha(t', t_0) = \lambda^\alpha(t', t_0) C_{2,ij}(t_0) V_j^\alpha(t', t_0), \quad (2.33)$$

we obtain the optimized hadron interpolator for the ground state,

$$O_H^{\text{opt}} = \sum_i V_i^{H*}(t', t_0) O_i. \quad (2.34)$$

Then we use the optimized interpolator to calculate 3pt and 2pt,

$$C_3(t_{\text{sep}}, t) = \langle \Omega | O_H^{\text{opt}}(t_{\text{sep}}) O_C(t) O_H^{\text{opt}\dagger}(0) | \Omega \rangle = \sum_{m,n} e^{-E_m(t_{\text{sep}}-t)} e^{-E_n t} \langle \Omega | O_H^{\text{opt}} | m \rangle \langle m | O_C | n \rangle \langle n | O_H^{\text{opt}\dagger} | \Omega \rangle , \quad (2.35)$$

$$C_2(t_{\text{sep}}) = \langle \Omega | O_H^{\text{opt}}(t_{\text{sep}}) O_H^{\text{opt}\dagger}(0) | \Omega \rangle = \sum_n e^{-E_n t_{\text{sep}}} \langle \Omega | O_H^{\text{opt}} | n \rangle \langle n | O_H^{\text{opt}\dagger} | \Omega \rangle . \quad (2.36)$$

The ground-state matrix element is extracted from the ratio of 3pt to 2pt, just like Eq. (2.18),

$$\frac{C_3(t_{\text{sep}}, t)}{C_2(t_{\text{sep}})} = \langle H | O_C | H \rangle + \dots \quad (2.37)$$

where ... denotes the excited state contamination, and the question is how large it is.

Ref. [290], focusing on the wavefunction amplitude $\langle n | O_H^{\text{opt}\dagger} | \Omega \rangle$ and particular choices of t', t_0 , shows the excited state contamination “...” is at the order of,

$$O(e^{-(E_{N+1}-E_1)t_0}) , \quad (2.38)$$

which means that GEVP enlarges the mass gap, thereby reducing excited states. In this view, GEVP is a large-time asymptotic analysis that systematically approaches the desired ground-state matrix element as the basis dimension N increases.

Ref. [291], studying the effects from the current transfer matrix $\langle m | O_C | n \rangle$ ⁸, argues that some of the excited states are enhanced by the current O_C by a factor related to the volume. Therefore, one should design particular strategies to eliminate these current-enhanced excited states. This is

⁸Actually, the author directly studies a 3pt in the type of $\langle H_N H_M O_C H_N^\dagger \rangle$, but the finding from this 3pt can be extended to a current transfer matrix $\langle m | O_C | n \rangle$ once a large volume power counting has been established.

intrinsically related to a large volume power counting.

We expect that GEVP and current-enhanced method will have more applications to LaMET calculations, especially for the excited state control of nucleon matrix elements.

2.5.5 Lanczos algorithm

The Lanczos algorithm [293–295, 374–376] is a model-independent approach for extracting ground-state matrix elements and providing controlled error estimates for excited-state contamination. It constructs an optimized hadron interpolator O_H by summing Euclidean time series with weights determined by the algorithm. Error estimates under different levels of assumptions can be obtained within this framework [294, 295]. Several lattice groups, such as LPC and ANL/BNL, are planning to adopt this method in LaMET calculations.

2.5.6 Gradient flow

The gradient flow [131, 173, 348–361] provides a perturbatively controllable approach to suppress UV fluctuations of the current O_C . This can be understood from an EFT perspective. The original theory requires the UV cutoff to be much larger than the physical scale, $\Lambda_{UV} \gg \Lambda_{\text{phy}}$, where the UV cutoff is related to the running coupling, mass, and anomalous dimension. The gradient flow introduces a flow time τ that acts as an additional scale to regulate UV fluctuations associated with anomalous dimensions, while the UV cutoff Λ_{UV} continues to govern renormalization effects related to the running coupling and mass. This leads to the scale separation $\Lambda_{UV} \gg 1/\sqrt{8\tau} \gg \Lambda_{\text{phy}}$, which suppresses UV fluctuations without modifying the desired long-distance physics. The τ dependence can be calculated perturbatively, and Ref. [355] provides a detailed formulation of the corresponding

Feynman rules. The gradient flow method has been explored in LaMET applications through both perturbative studies and lattice implementations [[131](#), [141–143](#), [145](#), [173](#), [357](#)].

2.6 Conclusion and outlook

In recent years, considerable progress has been made in LaMET, involving theoretical development, numerical methods, and lattice calculations. The theoretical foundation is now well established, innovative numerical techniques continue to extend the precision frontier, and some state-of-the-art calculations even outperform the global fit results. Future efforts can focus on improving precision control, particularly in controlling excited-state contaminations in nucleon matrix elements during lattice data generation. In addition, significant opportunities remain to apply LaMET to a broader range of parton observables and to explore novel quasi-correlator constructions.

Chapter 3: Large distance asymptotic analysis

One step in LaMET analysis is the Fourier transform of the lattice correlator from coordinate space to momentum space, which formally requires correlator values over an infinite range of spatial separations [287]. In principle, lattice data should decay exponentially at large distances (e.g. ~ 1 fm), implying that contributions from the long-distance tail (e.g. $\gtrsim 1$ fm) to FT should be negligible and that the FT precision should be well controlled. In practice, lattice data become increasingly noisy at large distances and cannot be directly used for FT. To control the FT precision, we adopt the large-distance asymptotic analysis, which discards noisy data at large distances, performs the physical extrapolation of the correlators to large distances, and computes the FT using the extrapolated form.

The discussion therefore turns to the theoretical understanding of the large-distance asymptotic behavior of Euclidean correlators for physical extrapolation, as well as to the quantification of the corresponding FT uncertainty arising from asymptotic analysis. In Ref. [287], motivated by the Regge and threshold limits of momentum-space PDFs, a phenomenological parametrization of the large-distance asymptotic form was proposed and has since been widely adopted in LaMET calculations. Ref. [16] subsequently provided a derivation of the asymptotic form based on heavy quark effective theory (HQET) and dispersive analysis, under the assumptions of constant form factors and the validity of analytic continuation. That work also derived an upper bound for the FT uncertainty by approximating the extrapolated function as a flat curve within one oscillatory period. More recently, Ref. [133] argued that LaMET is fundamentally a forward-problem formalism, which is systematically improvable without encountering an ill-posed inverse problem. It further emphasized that a physics-based asymptotic extrapolation yields the most reliable estimates of FT error. It also noted that there are and will be more high-quality lattice data for robust asymptotic analysis. These points directly responded to the concerns from Refs. [341, 342].

This work extends and complements the previous papers in several key aspects, leading to more sophisticated derivations and results on large-distance asymptotic forms as well as FT precision estimates. First, we carefully study the HQET reduction of the coordinate space propagator into the gauge link, which involves additional exponential behavior related to the heavy quark pole mass m_Q , and polynomial z -dependence from the “kinematic energy term” $D_\perp^2/(2m_Q)$. Second, we explore in the dispersive analysis the form factor connectedness [265], which affects the phases of the large-distance asymptotic forms, thereby explaining why the lattice data for quasi-DA matrix elements exhibit stronger oscillations than most quasi-PDFs. Third, we further investigate the exponential decay of spacelike correlators from the viewpoints of locality and analyticity. We prove explicit lower bounds for the decay speed consistent with the dispersive analysis. Fourth, we perform two-loop calculations in the ϕ^3 theory as supporting evidence for the dispersive analysis. Fifth, we provide all-order mathematical structures for large-distance asymptotic expansions, which are series of exponential decays, each exponential decay multiplied by a series of polynomial dependences. We also discuss the accuracy-counting principles for truncation up to a given order for practical lattice data analysis. Finally, we derive upper bounds on the FT uncertainty by considering three phases that primarily affect different end-point regions.

Thanks to all the efforts mentioned above, we now have a solid theoretical foundation for large-distance asymptotic forms, grounded in the fundamental properties of quantum field theory. Moreover, we accurately determine the large-distance behavior, including the physical origins of mass gaps in the exponential decay, the powers of polynomial dependences, and the overall phases that contain the external momentum. Last but not least, we have sophisticated FT precision control for both moderate y range and endpoint regions. Under the current lattice status, the FT uncertainty is just a few percent in the moderate y range and slightly larger but still under control at the endpoints.

The rest of the chapter is organized as follows. Sec. 3.1 summarizes the main results in this work for the convenience of lattice practitioners. In Sec. 3.2, we present the theoretical derivations of large distance asymptotic forms, based on HQET reduction, dispersive analysis, Lorentz symmetry, and heavy-flavor spectra, taking unpolarized quasi-PDF as an example. Sec. 3.3 generalizes these derivations to other quasi-correlators. In Sec. 3.4, we implement the derived large distance behaviors for the Fourier transform of lattice data to demonstrate controlled precision. We conclude this work in Sec. 3.5. Some supporting evidence is provided in Appendix A.

3.1 Summary of main results

We summarize the large-distance expansions of various quasi-correlators up to leading asymptotics (LA) and next-to-leading asymptotics (NLA). They are derived using the methods in Sec. 3.2, including HQET reduction in Sec. 3.2.1, dispersive analysis in Sec. 3.2.2, Lorentz symmetry in Sec. 3.2.3, and accuracy-counting principles in Sec. 3.2.4.

- Pion quark quasi-PDF matrix element $\tilde{h}(z, P^z) = \langle \pi(P) | \bar{\psi}(z) U(z, 0) \gamma^t \psi(0) | \pi(P) \rangle$ where ψ and $\bar{\psi}$ denote the quark and anti-quark fields separated by a spatial distance z , $U(z, 0)$ is a straight Wilson link to preserve the gauge invariance, and $|\pi(P)\rangle$ is an external pion state with four-momentum $P = (\sqrt{m_\pi^2 + P_z^2}, 0, 0, P^z)$. This matrix element is expanded at $z \rightarrow \infty$ with fixed large P^z , up to LA and NLA, respectively,

$$\tilde{h}^{\text{LA}}(z, P^z) = [A_2 e^{i \text{sign}(z) \phi_2} + A_1 e^{i \text{sign}(z) \phi_1} e^{-i P^z z} + A_3 e^{i \text{sign}(z) \phi_3} e^{i P^z z}] e^{-\Lambda |z|}, \quad (3.1)$$

$$\begin{aligned} \tilde{h}^{\text{NLA}}(z, P^z) = & \left[A_2 e^{i \text{sign}(z) \phi_2} + A_1 e^{i \text{sign}(z) \phi_1} e^{-i P^z z} + A_3 e^{i \text{sign}(z) \phi_3} e^{i P^z z} \right] e^{-\Lambda |z|} \\ & + \left[A'_2 e^{i \text{sign}(z) \phi'_2} + A'_1 e^{i \text{sign}(z) \phi'_1} e^{-i P^z z} + A'_3 e^{i \text{sign}(z) \phi'_3} e^{i P^z z} \right] \frac{e^{-\Lambda |z|}}{|z|}, \end{aligned} \quad (3.2)$$

where A_i , A'_i , ϕ_i , ϕ'_i and Λ are real parameters to be fitted with lattice data.

If ψ and $\bar{\psi}$ are light quarks, the mass gap Λ is related to the binding energy of a vector or pseudoscalar heavy-light meson ($Q\bar{q}$), which is ~ 500 MeV estimated using physical states as discussed in Sec. 3.2.4. On lattice, the mass gap may deviate from the physical value up to $O(\Lambda_{\text{QCD}})$ due to unphysical pion mass, discretization effect, or the choice of renormalon scheme. So one can fit it from lattice data under a physics-motivated constraint such as $[500 \text{ MeV} - O(\Lambda_{\text{QCD}}), 500 \text{ MeV} + O(\Lambda_{\text{QCD}})]$. If ψ and $\bar{\psi}$ are heavy quarks, the mass gap can be estimated analogously from the spectrum of the corresponding heavy-heavy meson.

For valence combination, the parameters are constrained by isospin symmetry: $\phi_2 = \phi'_2 = 0$, $\phi_3 = -\phi_1$, $\phi'_3 = -\phi'_1$, $A_1 = A_3$, and $A'_1 = A'_3$. For sea quarks (including light and heavy quarks), $A_1 = A_3 = A'_1 = A'_3 = 0$ since the corresponding mass gaps are associated with four-quark states with the combined quantum numbers of a light meson and a heavy meson, and are therefore larger than those of a single heavy meson.

The general mathematical structure of the above results can be found in Eq. (3.44), the accuracy counting for the pion valence PDF in Sec. 3.2.4, and the numerical tests in Sec. 3.4.2.1.

- Nucleon quark unpolarized quasi-PDF $\tilde{h}(z, P^z) = \langle N(P) | \bar{\psi}(z) U(z, 0) \gamma^t \psi(0) | N(P) \rangle$,

$$\tilde{h}^{\text{LA}}(z, P^z) = A_2 e^{i \phi_2 \text{sign}(z)} e^{-\Lambda |z|}, \quad (3.3)$$

$$\tilde{h}^{\text{NLA}}(z, P^z) = \left[A_2 e^{i\phi_2 \text{sign}(z)} + \frac{A'_2 e^{i\phi'_2 \text{sign}(z)}}{|z|} \right] e^{-\Lambda|z|}, \quad (3.4)$$

where the mass gap Λ here has a similar physical origin as the pion quark quasi-PDF case, so does the Λ in all the other asymptotic forms in this section. The above formats apply to valence and sea quarks. More detailed discussions can be found in Eq. (3.44), Sec. 3.2.4, and Sec. 3.4.2.2.

- Nucleon quark transversity quasi-PDF

$$\tilde{h}(z, P^z) = \langle N(P S_\perp) | \bar{\psi}(z) \gamma^t \gamma^\perp \gamma^5 U(z, 0) \psi(0) | N(P S_\perp) \rangle,$$

and helicity quasi-PDF

$$\tilde{h}(z, P^z) = \langle N(P S_L) | \bar{\psi}(z) \gamma^z \gamma^5 U(z, 0) \psi(0) | N(P S_L) \rangle.$$

Their large distance asymptotic expansions truncated up to LA and NLA are,

$$\tilde{h}^{\text{LA}}(z, P^z) = A_2 e^{i\phi_2 \text{sign}(z)} e^{-\Lambda|z|}, \quad (3.5)$$

$$\tilde{h}^{\text{NLA}}(z, P^z) = \left[A_2 e^{i\phi_2 \text{sign}(z)} + \frac{A'_2 e^{i\phi'_2 \text{sign}(z)}}{|z|} \right] e^{-\Lambda|z|}. \quad (3.6)$$

Theoretical derivations can be found in Sec. 3.3.1 and numerical tests on nucleon quark transversity quasi-PDF in Sec. 3.4.2.3.

- Meson quasi-DA $\tilde{h}(z, P^z) = \langle M(P) | \bar{\psi}_1(z) U(z, 0) \gamma^z \gamma^5 \psi_2(0) | \Omega \rangle$,

$$\tilde{h}^{\text{LA}}(z, P^z) = [A_1 e^{i\phi_1 \text{sign}(z)} e^{-izP^z} + A_2 e^{i\phi_2 \text{sign}(z)}] e^{-\Lambda|z|}, \quad (3.7)$$

$$\begin{aligned} \tilde{h}^{\text{NLA}}(z, P^z) = & \left[(A_1 e^{i\phi_1 \text{sign}(z)} e^{-izP^z} + A_2 e^{i\phi_2 \text{sign}(z)}) \right. \\ & \left. + \left(A'_1 e^{i\phi'_1 \text{sign}(z)} e^{-izP^z} + A'_2 e^{i\phi'_2 \text{sign}(z)} \right) \frac{1}{|z|} \right] e^{-\Lambda|z|}. \end{aligned} \quad (3.8)$$

If both ψ_1 and ψ_2 are light quarks with similar masses, all terms should be kept. In particular, $A_1 = A_2$, $\phi_1 = -\phi_2$, $A'_1 = A'_2$ and $\phi'_1 = -\phi'_2$ for pion quasi-DA. If ψ_1 is a light quark and ψ_2 is a heavy quark, $A_1 = A'_1 = 0$. If ψ_1 is a heavy quark and ψ_2 is a light quark, $A_2 = A'_2 = 0$. In all the situations mentioned here, Λ is the binding energy of a pseudoscalar heavy-light meson, which also appears in the pion quark quasi-PDF case. Theoretical derivations can be found in Sec. 3.3.2 and numerical tests on pion quasi-DA in Sec. 3.4.2.4.

- Pion quark quasi-GPD $\tilde{h}(z, P^z, P'^z) = \langle \pi(P) | \bar{\psi}(z) U(z, 0) \gamma^t \psi(0) | \pi(P') \rangle$,

$$\begin{aligned} \tilde{h}^{\text{LA}}(z, P^z, P'^z) = & \left[A_1 e^{i\phi_1 \text{sign}(z)} e^{-izP^z} + A_3 e^{i\phi_3 \text{sign}(z)} e^{izP'^z} \right. \\ & \left. + A_2 e^{i\phi_2 \text{sign}(z)} + \tilde{A}_2 e^{i\tilde{\phi}_2 \text{sign}(z)} e^{-i(P^z - P'^z)z} \right] e^{-\Lambda|z|}, \end{aligned} \quad (3.9)$$

$$\begin{aligned}
\tilde{h}^{\text{NLA}}(z, P^z, P'^z) = & \left[A_1 e^{i\phi_1 \text{sign}(z)} e^{-izP^z} + A_3 e^{i\phi_3 \text{sign}(z)} e^{izP'^z} \right. \\
& \left. + A_2 e^{i\phi_2 \text{sign}(z)} + \tilde{A}_2 e^{i\tilde{\phi}_2 \text{sign}(z)} e^{-i(P^z - P'^z)z} \right] e^{-\Lambda|z|} \\
& + \left[A'_1 e^{i\phi'_1 \text{sign}(z)} e^{-izP^z} + A'_3 e^{i\phi'_3 \text{sign}(z)} e^{izP'^z} \right. \\
& \left. + A'_2 e^{i\phi'_2 \text{sign}(z)} + \tilde{A}'_2 e^{i\tilde{\phi}'_2 \text{sign}(z)} e^{-i(P^z - P'^z)z} \right] \frac{e^{-\Lambda|z|}}{|z|} . \quad (3.10)
\end{aligned}$$

For valence combination, all terms are preserved. For sea quarks (including light and heavy quarks), $A_1 = A_3 = A'_1 = A'_3 = 0$ for the same reason as the pion quasi-PDF case. More details can be found in Sec. 3.3.3.

- Nucleon quark quasi-GPD, including the unpolarized case

$$\tilde{h}(z, P^z, P'^z) = \langle N(P) | \bar{\psi}(z) U(z, 0) \gamma^t \psi(0) | N(P') \rangle ,$$

and polarized case

$$\tilde{h}(z, P^z, P'^z) = \langle N(PS) | \bar{\psi}(z) U(z, 0) \gamma^z \gamma^5 \psi(0) | N(P'S') \rangle .$$

The large distance expansions up to LA and NLA are,

$$\tilde{h}^{\text{LA}}(z, P^z, P'^z) = \left[A_2 e^{i\phi_2 \text{sign}(z)} + \tilde{A}_2 e^{i\tilde{\phi}_2 \text{sign}(z)} e^{-i(P^z - P'^z)z} \right] e^{-\Lambda|z|} , \quad (3.11)$$

$$\begin{aligned}
\tilde{h}^{\text{NLA}}(z, P^z, P'^z) = & \left[A_2 e^{i\phi_2 \text{sign}(z)} + \tilde{A}_2 e^{i\tilde{\phi}_2 \text{sign}(z)} e^{-i(P^z - P'^z)z} \right. \\
& \left. + \left(A'_2 e^{i\phi'_2 \text{sign}(z)} + \tilde{A}'_2 e^{i\tilde{\phi}'_2 \text{sign}(z)} e^{-i(P^z - P'^z)z} \right) \frac{1}{|z|} \right] e^{-\Lambda|z|} . \quad (3.12)
\end{aligned}$$

More discussions can be found in Sec. 3.3.3.

Using the above large distance expansions and following the procedure and criteria in Sec. 3.4.1.1, we demonstrate the controlled precision of asymptotic analysis, through the analytically-derived upper error bound in Eq. (3.99)¹, and numerical tests among various lattice quasi-correlators in Sec. 3.4.2. With the current quality of lattice data, the uncertainty from asymptotic analysis is just a few percent compared to the center value at $O(1)$ in the moderate y range. The precision can even be improved in the near future with recently proposed techniques [44, 218, 279].

3.2 Large distance behavior of unpolarized quark quasi-PDF

In this section, we derive the large distance asymptotic behavior² of the unpolarized quark quasi-PDF matrix element [58, 84, 85, 275, 276],

$$\langle H(P) | \bar{\psi}(z) U(z, 0) \gamma^t \psi(0) | H(P) \rangle, \quad (3.13)$$

where $|H(P)\rangle$ denotes a boosted hadron with momentum $P = \left(\sqrt{m_H^2 + P_z^2}, 0, 0, P^z \right)$ and mass m_H , where P^z is much larger than m_H . $\psi, \bar{\psi}$ are quark and anti-quark fields separated by the spatial distance z , and $U(z, 0)$ is a spacelike Wilson line to ensure the gauge invariance.

Understanding this correlator's large-distance behavior is essential because it provides a physical constraint for controlling the precision of the Fourier transformation. The quasi-correlator Eq. (3.13) is one of the key objects in LaMET for parton physics calculations. It can be calculated non-perturbatively in lattice QCD, renormalized using standard self-renormalization and hybrid schemes [231, 287],

¹It applies to all the quasi-correlators discussed in Sec. 3.1 except for GPD.

²In LaMET applications, the large distance behavior is referred to as the expansion at $z \rightarrow \infty$ for fixed large P^z .

Fourier transformed to momentum space, and finally perturbatively matched to the unpolarized quark light-cone PDF. The Fourier transform, however, formally requires input over an infinite spatial range. In practice, lattice data at large separations (e.g. $\gtrsim 1$ fm) suffer from exponentially deteriorating signal-to-noise ratios, making them unusable directly. Our goal in this section is therefore to derive the theoretical large-distance asymptotic form, which will serve as a physical constraint enabling precision control of the Fourier transformation in the next section.

Our methods are summarized here and will be detailed in the following subsections. First, we represent the gauge link with dynamical heavy quarks through the coordinate-space HQET reduction. Then, a dispersive analysis along the timelike separation with physical states is performed to obtain the large-time asymptotic form. Next, using Lorentz symmetry, this form is converted to the spacelike separation, which corresponds to the Euclidean correlator simulated in lattice QCD. Finally, we discuss the accuracy-counting principles for the large-distance expansions, including the definitions of leading asymptotics (LA) and next-to-leading asymptotics (NLA). The methods here apply to many other quasi-correlators shown in Secs. [3.3.1](#), [3.3.2](#), [3.3.3](#) and [3.3.4](#).

3.2.1 Matching between heavy quark and gauge link

In this subsection, we represent the gauge link with QCD heavy quarks (not auxiliary heavy quarks), for the convenience of dispersive analysis in the next subsection.

A natural approach to deriving the large-distance expansion is dispersive analysis, which inserts a complete basis of intermediate particles into the correlator and relates the mass gaps of exponential decays to the spectrum of these particles. However, it is challenging to perform dispersive analysis directly with quasi-PDF because of the gauge link – one has to insert colored states, which are not the

physical states of QCD. To avoid this issue, we represent the gauge link with dynamical heavy quarks. For the convenience of follow-up discussions, we will start from a general situation that applies to either spacelike (as in Eq. (3.13)) or timelike separation.

According to the heavy quark effective theory (HQET) [299–309], the momentum space heavy quark with multiple gluon insertions is eikonalized into a gauge link along its worldline. As shown in Appendix A.1, we derive a novel reduction formula for the coordinate space heavy quark propagator $Q(x)\bar{Q}(0)$ in the heavy pole mass limit $m_Q \rightarrow +\infty$,

$$Q(x)\bar{Q}(0) \rightarrow U(x,0)D_Q(x^2, m_Q^2) \frac{i\not{x} + \sqrt{-x^2}}{2\sqrt{-x^2}} \left[1 + O\left(\frac{1}{m_Q\sqrt{-x^2}}, \frac{\Lambda_{\text{QCD}}}{m_Q}\right) \right], \quad (3.14)$$

where x is a Lorentz coordinate that can be either timelike or spacelike. When x is timelike, we take $x = (t, 0, 0, 0)$ with $t > 0$. When it is spacelike, we use $x = (0, 0, 0, z)$. $U(x,0)$ is a straight-line gauge link connecting 0 to x and encodes the color structure of the heavy quark propagator. The scalar function $D_Q(x^2, m_Q^2)$ contains the kinematic structure,

$$D_Q(x^2, m_Q^2) = \frac{m_Q^3}{2\sqrt{2}\pi^{3/2}} \frac{e^{-m_Q\sqrt{-x^2}}}{(m_Q\sqrt{-x^2})^{3/2}}. \quad (3.15)$$

The prescription of $\sqrt{-x^2}$ depends on the time ordering. For the time-ordered correlator, one needs $\sqrt{-x^2 + i0^+}$ [377–379].

Therefore, the coordinate space current-current correlator,

$$M_{\Gamma_1\Gamma_2}(x, P, \mu) \equiv \langle H(P) | \bar{\psi}(x)\Gamma_1 Q(x) \bar{Q}(0)\Gamma_2 \psi(0) | H(P) \rangle_c, \quad (3.16)$$

can be matched onto the gauge link object,

$$\tilde{h}_{\Gamma_1\Gamma_2}(x, P, \mu) \equiv \langle H(P) | \bar{\psi}(x) \Gamma_1 U(x, 0) \frac{i\not{x} + \sqrt{-x^2}}{2\sqrt{-x^2}} \Gamma_2 \psi(0) | H(P) \rangle_c, \quad (3.17)$$

in the heavy quark pole mass expansion $m_Q \rightarrow +\infty$,

$$M_{\Gamma_1\Gamma_2}(x, P, \mu) = H_{\Gamma_1\Gamma_2, \hat{x}} \left(\frac{m_Q^2}{\mu^2} \right) D_Q(x^2, m_Q^2) \tilde{h}_{\Gamma_1\Gamma_2}(x, P, \mu) \left[1 + O \left(\frac{1}{m_Q \sqrt{-x^2}}, \frac{\Lambda_{\text{QCD}}}{m_Q} \right) \right]. \quad (3.18)$$

Here Γ_1 and Γ_2 are Dirac Gamma matrices, and the kinematic scalar factor D_Q is defined in Eq. (3.15).

The hard coefficient $H_{\Gamma_1\Gamma_2, \hat{x}}(m_Q^2/\mu^2)$ is perturbatively calculable and depends on the gauge link direction $\hat{x} \equiv ix/\sqrt{-x^2}$. All matrix elements are renormalized in the $\overline{\text{MS}}$ scheme at scale μ^3 . The subscript “ c ” indicates the connected contribution,

$$\begin{aligned} & \langle H(P) | \bar{\psi}(x) \Gamma_1 Q(x) \bar{Q}(0) \Gamma_2 \psi(0) | H(P) \rangle_c \\ & \equiv \langle H(P) | \bar{\psi}(x) \Gamma_1 Q(x) \bar{Q}(0) \Gamma_2 \psi(0) | H(P) \rangle - \langle H(P) | H(P) \rangle \langle \Omega | \bar{\psi}(x) \Gamma_1 Q(x) \bar{Q}(0) \Gamma_2 \psi(0) | \Omega \rangle. \end{aligned} \quad (3.19)$$

Here $|\Omega\rangle$ denotes the QCD vacuum. This matching relation shows that the current-current correlator $M_{\Gamma_1\Gamma_2}(x, P)$ and the quasi-PDF matrix element $\tilde{h}_{\Gamma_1\Gamma_2}(x, P)$ exhibit different exponential and polynomial large distance behaviors. These differences are captured by the kinematic factor D_Q in Eq. (3.15), indicating that the gauge link is a more efficient approach for transporting quantum correlations over large distances. Intuitively, the heavy-quark propagator transports quantum correlations via diffusion

³In the following discussions, we suppress the explicit scale μ dependence for simplicity.

in all directions, whereas the gauge link is more like a direct transport along its path.

The coordinate-space HQET reduction formulas in Eqs. (3.14) and (3.18) are justified in the timelike case through an expansion-by-region analysis in Appendix A.1.1 and perturbative calculations up to one-loop in Appendix A.1.2. It is crucial to keep the “kinematic energy term” $D_\perp^2/(2m_Q)$, which provides the correct polynomial- t dependence. Given the HQET reduction in the timelike region, in Appendix A.1.3, we use techniques in complex analysis to show that the reduction also holds in the spacelike region.

Different choices of Γ_1, Γ_2 correspond to different partonic polarizations. In the following subsections, we focus on the unpolarized combination,

$$M_{\{\Gamma_x, I\}}(x, P) = M_{\Gamma_x, I}(x, P) + M_{I, \Gamma_x}(x, P) , \quad (3.20)$$

where the gamma matrix Γ_x is defined as,

$$\Gamma_x \equiv \begin{cases} \gamma^z , & x = (t, 0, 0, 0) \\ \gamma^t , & x = (0, 0, 0, z) \end{cases} . \quad (3.21)$$

We will begin with the timelike separation and convert the result to the spacelike case at the end. According to Eq. (B4) of Ref. [16], this gamma matrix construction involves only a single Lorentz scalar function, which simplifies the conversion between timelike and spacelike correlators, as will be discussed in Sec. 3.2.3. Finally, in the spacelike case, $M_{\{\Gamma_x, I\}}(x, P)$ can be matched to the standard quasi-PDF matrix element $\langle H(P) | \bar{\psi}(z) U(z, 0) \gamma^t \psi(0) | H(P) \rangle$.

3.2.2 Dispersive analysis considering form factor connectedness

In this subsection, we start with timelike separation for clear physical interpretations and derive the large-time behavior using dispersive analysis, considering three types of form-factor representations regarding connectedness.

To analyze the large time behaviors ($x = (t, 0, 0, 0)$ with $t \rightarrow +\infty$) of Eq. (3.16) for the unpolarized case, we insert a complete basis of eigenstates of QCD Hamiltonian,

$$M_{\{\Gamma_x, I\}}(x, P) = \int d\Gamma_X(k) e^{i(P^t - k^t)t} \langle H(P) | \bar{\psi} \gamma^z Q | X(k) \rangle \langle X(k) | \bar{Q} \psi | H(P) \rangle|_c + (\gamma^z \leftrightarrow I) , \quad (3.22)$$

where $X(k)$ denotes a multi-particle state,

$$X(k) \equiv n_1(k_1) n_2(k_2) \dots n_N(k_N) , \quad (3.23)$$

where $n_i(k_i)$ is a single particle state with species n_i and momentum k_i , and the total momentum $k = \sum_{i \in X} k_i$. The notation “ $i \in X$ ” means i is a single particle in the multi-particle state X . The Lorentz indices and permutation signatures are omitted for simplicity. $\int d\Gamma_X(k)$ is the Lorentz invariant phase space,

$$\int d\Gamma_X(k) \equiv \frac{1}{S_X} \Pi_{i \in X} \int \frac{d^3 \vec{k}_i}{(2\pi)^3} \frac{1}{2\sqrt{\vec{k}_i^2 + m_i^2}} , \quad (3.24)$$

where S_X is the symmetry factor for the identical particles, and m_i is the mass of the single particle state “ i ”.

Although the full correlator $M_{\{\Gamma_x, I\}}(x, P)$ is connected, the form factors contain disconnected

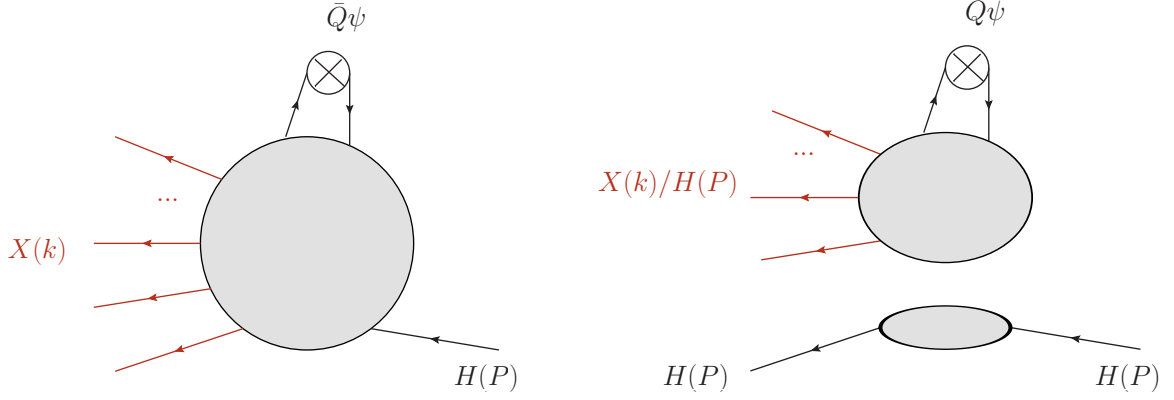


Figure 3.1: Connected and disconnected contributions to the form factor $\langle X(k)|\bar{Q}\psi|H(P)\rangle$. The left panel denotes the diagram $\langle X(k)|\bar{Q}\psi|H(P)\rangle_C$ where $\bar{Q}\psi$ and $|H(P)\rangle$ are connected. The right panel represents the contribution $\langle X(k)|\bar{Q}\psi|H(P)\rangle_{DC}$ for $\bar{Q}\psi$ and $|H(P)\rangle$ being disconnected. $H(P)$ is the external hadron of quasi-correlator and $X(k)$ is the inserted multi-particle state.

contributions [57, 380],

$$\langle X(k)|\bar{Q}\psi|H(P)\rangle = \langle X(k)|\bar{Q}\psi|H(P)\rangle_C + \langle X(k)|\bar{Q}\psi|H(P)\rangle_{DC} , \quad (3.25)$$

where the matrix element $\langle X(k)|\bar{Q}\psi|H(P)\rangle_{DC}$ is defined as $\bar{Q}\psi$ and $|H(P)\rangle$ being disconnected, shown as the right panel of Fig. 3.1,

$$\begin{aligned} \langle X(k)|\bar{Q}\psi|H(P)\rangle_{DC} &\equiv \sum_{n \in X} \langle X(k)/n(q) | \bar{Q}\psi | \Omega \rangle \langle n(q) | H(P) \rangle \\ &= \sum_{n \in X} \langle X(k)/n(q) | \bar{Q}\psi | \Omega \rangle (2\pi)^3 2\sqrt{\vec{P}^2 + m_H^2} \delta^3(\vec{q} - \vec{P}) \delta_{n,H} , \end{aligned} \quad (3.26)$$

where $X(k)/n(q)$ denotes the state $X(k)$ eliminating the single particle $n(q)$. The matrix element $\langle X(k)|\bar{Q}\psi|H(P)\rangle_C$ is the remaining part where $\bar{Q}\psi$ and $|H(P)\rangle$ are connected, denoted as the left

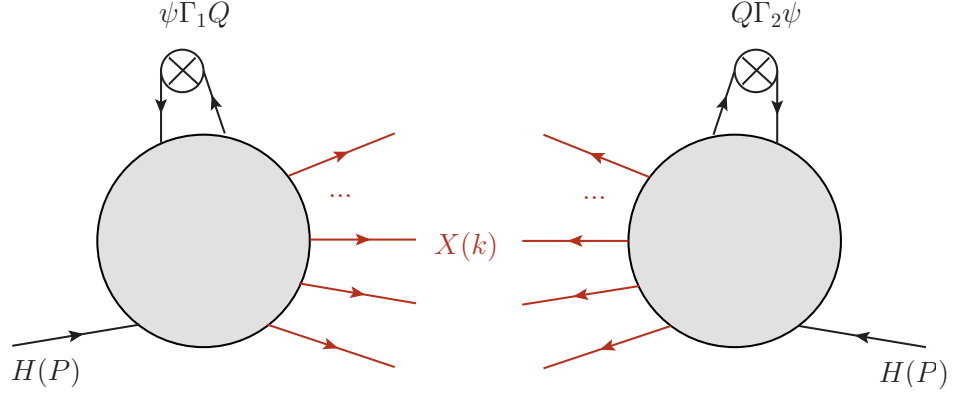


Figure 3.2: Form factor representation M_X , where both form factors are connected. For the unpolarized case, we sum up $(\Gamma_1, \Gamma_2) = (\gamma^z, I)$ and $(\Gamma_1, \Gamma_2) = (I, \gamma^z)$.

panel of Fig. 3.1. Eq. (3.22) is decomposed based on form factor connectedness,

$$M_{\{\Gamma_x, I\}}(x, P) = M_X(x, P) + M_Y(x, P) + M_Z(x, P) . \quad (3.27)$$

In $M_X(x, P)$, both form factors are connected, as shown in Fig. 3.2. In $M_Y(x, P)$, one form factor is connected while the other is disconnected, as depicted in Fig. 3.4. Finally, in $M_Z(x, P)$, both form factors are disconnected, as illustrated in Fig. 3.3.

Start from $M_X(x, P)$ in Fig. 3.2,

$$M_X(x, P) = e^{iP^t t} \sum_X \int d\Gamma_X(k) e^{-ik^t t} \langle H(P) | \bar{\psi} \gamma^z Q | X(k) \rangle_C \langle X(k) | \bar{Q} \psi | H(P) \rangle_C + (\gamma^z \leftrightarrow I) . \quad (3.28)$$

Express it in terms of the multiplication of single-particle states,

$$\begin{aligned}
M_X(x, P) &= e^{iP^t t} \sum_X \frac{1}{S_X} \left(\Pi_{i \in X} \int \frac{d^3 \vec{k}_i}{(2\pi)^3} \frac{1}{2\sqrt{\vec{k}_i^2 + m_i^2}} e^{-i\sqrt{\vec{k}_i^2 + m_i^2} t} \right) \\
&\times \langle H(P) | \bar{\psi} \gamma^z Q | \Pi_{i \in X} n_i(k_i) \rangle_C \langle \Pi_{i \in X} n_i(k_i) | \bar{Q} \psi | H(P) \rangle_C + (\gamma^z \leftrightarrow I) . \quad (3.29)
\end{aligned}$$

See the meaning of “ $i \in X$ ” below Eq. (3.23). The large-time asymptotic form can be calculated using the stationary-phase analysis [381]. In the $t \rightarrow +\infty$ limit, the phase factor $e^{-i\sqrt{\vec{k}_i^2 + m_i^2} t}$ is highly oscillatory, leading to strong cancellations in the phase space integral, except for a domain near the stationary-phase point $\vec{k}_i^2 \sim 0$, where we perform the expansion,

$$\begin{aligned}
M_X(x, P)_{t \rightarrow +\infty} &= e^{iP^t t} \sum_X \frac{1}{S_X} \left(\Pi_{i \in X} \int \frac{d^3 \vec{k}_i}{(2\pi)^3} \frac{1}{2m_i} e^{-im_i t} e^{-i\frac{\vec{k}_i^2}{2m_i} t} \right) \\
&\times \langle H(P) | \bar{\psi} \gamma^z Q | \Pi_{i \in X} n_i(m_i \hat{x}) \rangle_C \langle \Pi_{i \in X} n_i(m_i \hat{x}) | \bar{Q} \psi | H(P) \rangle_C \\
&\times \left[1 + O(k_i^z) + O(\vec{P} \cdot \vec{k}_i) + O(\vec{k}_i \cdot \vec{k}_j)_{j \neq i} + O(\vec{k}_i^2) \right] + (\gamma^z \leftrightarrow I) \\
&= e^{iP^t t} \sum_X \frac{1}{S_X} \left(\Pi_{i \in X} \frac{1}{2m_i} \frac{m_i^3}{2\sqrt{2}\pi^{3/2}} \frac{e^{-im_i t}}{(im_i t)^{3/2}} \right) \\
&\times \langle H(P) | \bar{\psi} \gamma^z Q | \Pi_{i \in X} n_i(m_i \hat{x}) \rangle_C \langle \Pi_{i \in X} n_i(m_i \hat{x}) | \bar{Q} \psi | H(P) \rangle_C \left[1 + O\left(\frac{1}{t}\right) \right] + (\gamma^z \leftrightarrow I) . \quad (3.30)
\end{aligned}$$

See the definition of $\hat{x}$ below Eq. (3.18). The result has a clear physical interpretation, with a simple power counting rule. The dominant contribution arises from the momentum mode $k_i \sim (m_i, \sqrt{m_i/t}, \sqrt{m_i/t}, \sqrt{m_i/t})$, which is almost like a particle at rest except for slight spatial quantum fluctuations. The time component gives the exponential behavior $e^{-im_i t}$, which is the phase for a particle at rest, and the spatial components provide the polynomial dependence $1/t^{3/2}$, which is related to the size of quantum fluctuations.

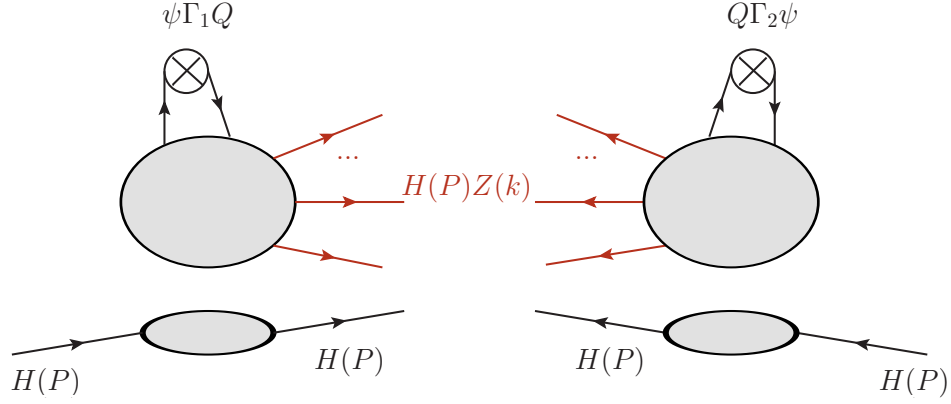


Figure 3.3: Form factor representation M_Z , where both form factors are disconnected.

The form factors in Eq. (3.30) imply that the mass gaps $m_X \equiv \sum_{i \in X} m_i$ in the large time expansion can be identified with the spectrum of states projected by the common quantum numbers of $\bar{Q}\psi|H(P)\rangle$ and $\bar{Q}\gamma^z\psi|H(P)\rangle$.

The higher order correction is $O(1/t)$ suppressed instead of $O(1/\sqrt{t})$, as a consequence of rotational symmetry. In the expansion around the stationary phase point $\vec{k}_i^2 \sim 0$, there are $O(k_i^z)$, $O(\vec{P} \cdot \vec{k}_i)$ and $O(\vec{k}_i \cdot \vec{k}_j)_{j \neq i}$ corrections originating from the form factors. In particular, $O(k_i^z)$ and $O(\vec{P} \cdot \vec{k}_i)$ could potentially lead to $O(1/\sqrt{t})$ by power counting. However, these corrections vanish in the phase space integral because the factor $e^{-i\frac{\vec{k}_i^2}{2m_i}t}$ is isotropic. In contrast, the $O(\vec{k}_i^2)$ term survives, yielding the $O(1/t)$ correction. Generally speaking, only corrections with even powers of spatial momenta contribute, leading to an integer-power series in t .

For $M_Z(x, P)$ in Fig. 3.3 where both form factors are disconnected, integrating out the delta functions from Eq. (3.26) yields,

$$M_Z(x, P) = e^{-iP^t t} \sum_Z \int d\Gamma_Z(k) e^{-ik^t t} \langle \Omega | \bar{\psi} \gamma^z Q | H(P) Z(k) \rangle \langle H(P) Z(k) | \bar{Q} \psi | \Omega \rangle + (\gamma^z \leftrightarrow I). \quad (3.31)$$

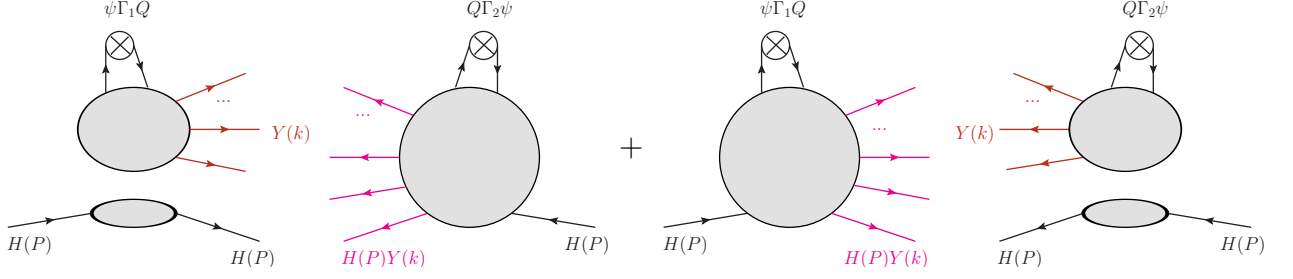


Figure 3.4: Form factor representation M_Y , where one form factor is disconnected, and the other one is connected. There are two terms in this representation.

Following a similar logic as $M_X(x, P)$, we obtain the large time asymptotic form,

$$M_Z(x, P)_{t \rightarrow +\infty} = e^{-iP^t t} \sum_Z \frac{1}{S_Z} \left(\Pi_{i \in Z} \frac{1}{2m_i} \frac{m_i^3}{2\sqrt{2}\pi^{3/2}} \frac{e^{-im_i t}}{(im_i t)^{3/2}} \right) \\ \times \langle \Omega | \bar{\psi} \gamma^z Q | H(P) \Pi_{i \in Z} n_i(m_i \hat{x}) \rangle \langle H(P) \Pi_{i \in Z} n_i(m_i \hat{x}) | \bar{Q} \psi | \Omega \rangle \left[1 + O\left(\frac{1}{t}\right) \right] + (\gamma^z \leftrightarrow I) . \quad (3.32)$$

The result is similar to M_X except for the overall phase factor and the physical origins of mass gaps. In M_X , the external hadron $H(P)$ appears as an incoming state, giving rise to the phase factor $e^{iP^t t}$. In contrast, for M_Z , after integrating out the disconnected contributions, the external hadron $H(P)$ effectively appears as an outgoing state, leading to the phase factor $e^{-iP^t t}$. The mass gap $m_Z \equiv \sum_{i \in Z} m_i$ in M_Z is related to the state Z projected by the common quantum numbers of $\bar{Q}\psi | \bar{H}(P) \rangle$ and $\bar{Q}\gamma^z \psi | \bar{H}(P) \rangle$, where $\bar{H}(P)$ denotes the anti-particle of $H(P)$.

As for $M_Y(x, P)$ in Fig. 3.4, where one form factor is connected and the other one is not, integrating out the delta function in Eq. (3.26) leads to

$$M_Y(x, P) = \sum_Y \int d\Gamma_Y(k) e^{-ik^t t} \left[\langle \Omega | \bar{\psi} \gamma^z Q | Y(k) \rangle \langle H(P) Y(k) | \bar{Q} \psi | H(P) \rangle_C \right. \\ \left. + \langle H(P) | \bar{\psi} \gamma^z Q | H(P) Y(k) \rangle_C \langle Y(k) | \bar{Q} \psi | \Omega \rangle \right] + (\gamma^z \leftrightarrow I) . \quad (3.33)$$

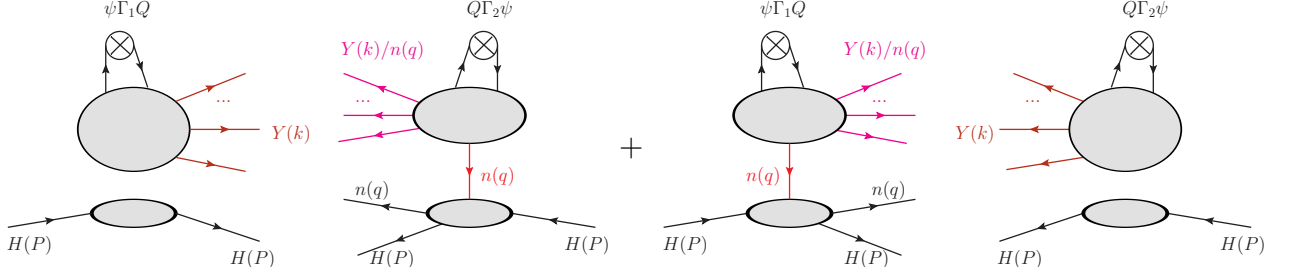


Figure 3.5: One-particle resonance diagram $M_{Y,fs}$ (or called forward singularity diagram), belonging to M_Y in Fig. 3.4. The multi-particle state $Y(k)$ contains the single particle $n(q)$, and see the definition of $Y(k)/n(q)$ below Eq. (3.26). For the unpolarized case, we sum up $(\Gamma_1, \Gamma_2) = (\gamma^z, I)$ and $(\Gamma_1, \Gamma_2) = (I, \gamma^z)$. For moving intermediate states, allowed by the quantum numbers of the operators, the propagator $n(q)$ and the inserted single-particle $n(q)$ can be of the same species. Constrained by momentum conservation, the propagator $n(q)$ is on-shell.

The form factors $\langle H(P)Y(k)|\bar{Q}\psi|H(P)\rangle_C$ and $\langle H(P)|\bar{\psi}\gamma^z Q|H(P)Y(k)\rangle_C$ contain divergences associated with one-particle resonance poles [382]. Fig. 3.5 displays the one-particle resonance diagram $M_{Y,fs}$, as part of the contributions to the form factor representation M_Y . Since $\bar{Q}\psi$ and $\bar{Q}\gamma^z\psi$ have the common quantum numbers to interpolate the same state as long as $\vec{k} \neq 0$ [279], the species of the propagator $n(q)$ can be the same as the inserted single-particle $n(q)$. Constrained by momentum conservation, the propagator momentum q is the same as this inserted on-shell state $n(q)$. Therefore, the propagator $n(q)$ is on-shell, and the form factor becomes divergent.

To study this divergence, we introduce the off-forward regulator, which has been discussed in Sec. 6.11.6 in Ref. [57],

$$\langle H(P)|\bar{\psi}(x)\gamma^z Q(x)\bar{Q}(0)\psi(0)|H(P)\rangle_c \rightarrow \langle H(P)|\bar{\psi}(x)\gamma^z Q(x)\bar{Q}(0)\psi(0)|H(P')\rangle_c, \quad (3.34)$$

and the one-particle resonance diagram $M_{Y,fs}$ becomes Fig. 3.6. The propagator $n(q + P - P')$ or $n(q + P' - P)$ now becomes off-shell, which means the off-forwardness regularizes the divergence.

Since each term becomes singular in the forward limit $P' \rightarrow P$, this divergence is called the *for-*

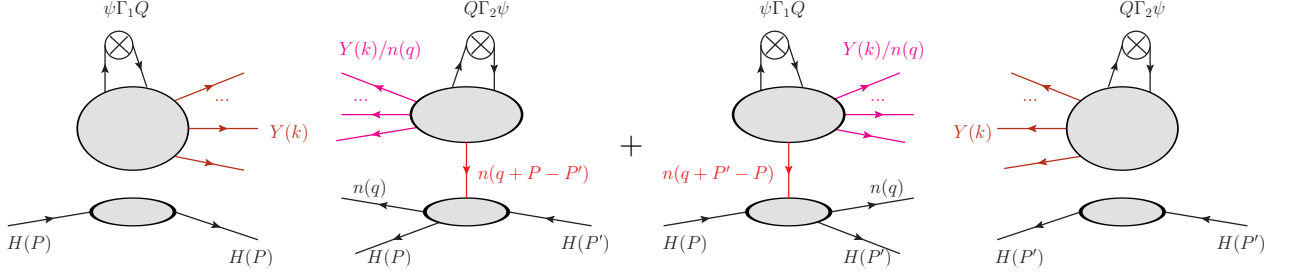


Figure 3.6: Off-forward regulator for the forward singularity. The propagators $n(q + P - P')$ and $n(q + P' - P)$ are off-shell under the off-forwardness.

ward singularity in our language. Using the methods in Ref. [382], the forward singularity diagram $M_{Y,fs}$ can be reduced into the single-particle propagator, two-particle forward scattering amplitude $T(n(q), H(P))$ ⁴, and wavefunction amplitudes, in the domain $P' \sim P$,

$$\begin{aligned}
 M_{Y,fs}(x, P, P') &= \sum_Y \sum_{n(q) \in Y} \int d\Gamma_Y(k) e^{-ik^t t} \left[\langle \Omega | \bar{\psi} \gamma^z Q | Y(k) \rangle \frac{T(n(q), H(P)) \langle Y(k) | \bar{Q} \psi | \Omega \rangle}{(q + P - P')^2 - m_n^2 + i0^+} \right. \\
 &\quad \left. + e^{i(P^t - P'^t)t} \frac{\langle \Omega | \bar{\psi} \gamma^z Q | Y(k) \rangle T(n(q), H(P))}{(q + P' - P)^2 - m_n^2 - i0^+} \langle Y(k) | \bar{Q} \psi | \Omega \rangle \right] + O \left[(P'^t - P^t)^0 \right] + (\gamma^z \leftrightarrow I).
 \end{aligned} \tag{3.35}$$

Here $Y(k)$ denotes a multi-particle state containing a single particle $n(q)$ with on-shell momentum $q^2 = m_n^2$. The notation $O \left[(P'^t - P^t)^0 \right]$ represents part of the regular contributions in the forward limit, while the other parts are retained explicitly in the propagators and phase $e^{i(P^t - P'^t)t}$ for later convenience.

We provide a heuristic argument for the cancellation of the forward singularity between the two terms in Eq. (3.35) or Fig. 3.6. The propagators $n(q + P - P')$ or $n(q + P' - P)$ are factorized out

⁴In this work, the amplitude T does not include the Dirac delta function of momentum conservation, which differs from the convention used in Ref. [65].

from the remaining parts labeled as g_1 and g_2 ,

$$\begin{aligned}
M_{Y,fs}(x, P, P') &= \int d^3\vec{q} \left[\frac{g_1(P, P', \vec{q}, t)}{(q + P - P')^2 - m_n^2 + i0^+} + \frac{g_2(P, P', \vec{q}, t)}{(q + P' - P)^2 - m_n^2 - i0^+} \right] \\
&= \int d^3\vec{q} \frac{1}{E_{\vec{q}} + P^t - P'^t - E_{\vec{q}+\vec{P}-\vec{P}'} + i0^+} \\
&\quad \times \left[\frac{g_1(P, P', \vec{q}, t)}{E_{\vec{q}} + P^t - P'^t + E_{\vec{q}+\vec{P}-\vec{P}'}} - \frac{g_2(P, P', \vec{q} - \vec{P}' + \vec{P}, t)}{E_{\vec{q}+\vec{P}-\vec{P}'} + P'^t - P^t + E_{\vec{q}}} \right], \tag{3.36}
\end{aligned}$$

where $E_{\vec{q}} \equiv \sqrt{\vec{q}^2 + m_n^2}$. Due to the symmetric structures of the diagrams, g_1 and g_2 are the same in the forward limit $P' \rightarrow P$. Provided that g_1 and g_2 are analytic in the neighborhood of $P'^t \sim P^t$, the above expression is finite in the $P' \rightarrow P$ limit, which means the forward singularity is canceled.

Although the forward singularity is canceled, it could have an impact on the polynomial- t dependence. After taking the forward limit, the forward singularity diagram $M_{Y,fs}(x, P)$ becomes

$$\begin{aligned}
M_{Y,fs}(x, P) &= \sum_Y \sum_{n(q) \in Y} \int d\Gamma_Y(k) e^{-ik^t t} \left[\frac{i(P^t - P'^t) \textcolor{red}{t}}{(q + P' - P)^2 - m_n^2} \right]_{P' \rightarrow P} \\
&\quad \times \langle \Omega | \bar{\psi} \gamma^z Q | Y(k) \rangle T(n(q), H(P)) \langle Y(k) | \bar{Q} \psi | \Omega \rangle \left[1 + O\left(\frac{(P'^t - P^t)^0}{t}\right) \right] + (\gamma^z \leftrightarrow I), \tag{3.37}
\end{aligned}$$

where the factor $\textcolor{red}{t}$ arises from the expansion of the phase $e^{i(P^t - P'^t)t}$ in Eq. (3.35) in the forward limit.

The notation $O\left((P'^t - P^t)^0/t\right)$ denotes other regular contributions without the linear t enhancement.

However, the above linear- t enhancement does not contribute to the leading asymptotics of the large time expansion, which corresponds to the zero-momentum intermediate states, that cannot be simultaneously projected by the operators $\bar{Q}\psi$ and $\bar{Q}\gamma^z\psi$. To demonstrate this, we calculate the large

time limit of Eq. (3.37) using stationary phase analysis as before,

$$\begin{aligned}
M_{Y,fs}(x, P)_{t \rightarrow +\infty} &= \sum_Y \sum_{n(q) \in Y} \frac{1}{S_Y} \left(\Pi_{i \in Y} \frac{1}{2m_i} \frac{m_i^3}{2\sqrt{2}\pi^{3/2}} \frac{e^{-im_i t}}{(im_i t)^{3/2}} \right) \\
&\times T(n(m_n \hat{x}), H(P)) \left[\frac{i(P^t - P'^t)t}{(m_n \hat{x} + P' - P)^2 - m_n^2} \right]_{P'^t \rightarrow P^t} \\
&\times \langle \Omega | \bar{\psi} \gamma^z Q | \Pi_{i \in Y} n_i(m_i \hat{x}) \rangle \langle \Pi_{i \in Y} n_i(m_i \hat{x}) | \bar{Q} \psi | \Omega \rangle \\
&\times \left[1 + O\left(\frac{1}{t}\right) \right] + (\gamma^z \leftrightarrow I), \tag{3.38}
\end{aligned}$$

where the intermediate states Y are at rest, which can be determined from the quantum numbers of the rest states, as shown in Tab. 6.1 in Ref. [1]. Based on the form factor $\langle \Pi_{i \in Y} n_i(m_i \hat{x}) | \bar{Q} \psi | \Omega \rangle$, Y has the quantum number of a scalar meson. According to $\langle \Omega | \bar{\psi} \gamma^z Q | \Pi_{i \in Y} n_i(m_i \hat{x}) \rangle$, Y has the quantum number of a vector meson. As a result, Y does not exist, and the linear- t enhanced contribution vanishes.

Therefore, the large time behavior of M_Y can be calculated using similar methods as M_X or M_Z ,

$$\begin{aligned}
M_Y(x, P)_{t \rightarrow +\infty} &= \sum_Y \frac{1}{S_Y} \left(\Pi_{i \in Y} \frac{1}{2m_i} \frac{m_i^3}{2\sqrt{2}\pi^{3/2}} \frac{e^{-im_i t}}{(im_i t)^{3/2}} \right) \\
&\times [\langle \Omega | \bar{\psi} \gamma^z Q | \Pi_{i \in Y} n_i(m_i \hat{x}) \rangle \langle H(P) \Pi_{i \in Y} n_i(m_i \hat{x}) | \bar{Q} \psi | H(P) \rangle_C \\
&\quad + \langle H(P) | \bar{\psi} \gamma^z Q | H(P) \Pi_{i \in Y} n_i(m_i \hat{x}) \rangle_C \langle \Pi_{i \in Y} n_i(m_i \hat{x}) | \bar{Q} \psi | \Omega \rangle] \\
&\times \left[1 + O\left(\frac{1}{t}\right) \right] + (\gamma^z \leftrightarrow I). \tag{3.39}
\end{aligned}$$

Compared to M_X and M_Z , there is no overall phase $e^{iP^t t}$ or $e^{-iP^t t}$. The mass gap $m_Y \equiv \sum_{i \in Y} m_i$ has a different physical origin, related to a state Y projected by the quantum numbers of $\bar{Q} \psi | \Omega \rangle$ or $\bar{Q} \gamma^z \psi | \Omega \rangle$.

The derived large-time asymptotic forms here are supported by the perturbative calculations up

to two-loop in ϕ^3 theory in Appendix A.2.2, including the methods to determine the physical origins of mass gaps, the influence of form factor connectedness on the phases, and the power-counting of polynomial- t dependence for M_X and M_Z . Moreover, the cancellation and impact of forward singularities have been demonstrated in ϕ^3 theory through perturbative calculations in Appendices A.2.3 and A.2.4 and all-order discussions in Appendix A.2.5.

The discussion of forward singularities in this subsection can be straightforwardly generalized to other “forward” quasi-correlators, in particular for utilizing Eq. (3.38) to determine whether the large time expansion of Y terms exhibits the enhanced polynomial dependence. Starting from a general operator displacement in a forward matrix element,

$$\langle H(P) | O_1(x) O_2(0) | H(P) \rangle , \quad (3.40)$$

the factor $\langle \Omega | \bar{\psi} \gamma^z Q | \Pi_{i \in Y} n_i(m_i \hat{x}) \rangle \langle \Pi_{i \in Y} n_i(m_i \hat{x}) | \bar{Q} \psi | \Omega \rangle$ appearing in Eq. (3.38) is replaced by

$$\langle \Omega | O_1 | \Pi_{i \in Y} n_i(m_i \hat{x}) \rangle \langle \Pi_{i \in Y} n_i(m_i \hat{x}) | O_2 | \Omega \rangle . \quad (3.41)$$

Consequently, if O_1 and O_2 have overlap with the same rest-frame intermediate state, as in the examples of ϕ^3 theory discussed in Appendix A.2 and the gluon PDF analyzed in Sec. 3.3.4, the polynomial dependence of the Y term is enhanced in the large time expansion. By contrast, when O_1 and O_2 do not share overlap with the same rest state, as in the unpolarized quark PDF considered in this subsection and the polarized cases studied in Sec. 3.3.1, such an enhancement does not occur. We emphasize that these examples correspond to the specific choices of Dirac gamma matrices commonly employed in lattice QCD calculations. For alternative choices, different conclusions may

follow; for instance, the correlator $\langle H(P) | \bar{\psi}(z) U(z, 0) \gamma^z \psi(0) | H(P) \rangle$ exhibits polynomial enhancement compared to $\langle H(P) | \bar{\psi}(z) U(z, 0) \gamma^t \psi(0) | H(P) \rangle$.

3.2.3 Conversion to Euclidean correlator using Lorentz symmetry

In this subsection, we summarize the large-time asymptotic expansions derived in the previous subsection and convert the results to the spacelike separation using Lorentz symmetry.

Combining M_X in Eq. (3.30), M_Y in Eq. (3.39), and M_Z in Eq. (3.32), the large-time asymptotic expansion of the current-current correlator takes the form,

$$\begin{aligned} M_{\{\Gamma_x, I\}}(x, P)_{\sqrt{-x^2} \rightarrow \infty} = & \sum_{X, l=0} \frac{e^{iP \cdot x} e^{-m_X \sqrt{-x^2}}}{(\sqrt{-x^2})^{\frac{3}{2}N_X + l}} C_{X, l} \left(P, \left\{ k_j = \frac{ixm_j}{\sqrt{-x^2}} \middle| j \in X \right\} \right) \\ & + \sum_{Y, l=0} \frac{e^{-m_Y \sqrt{-x^2}}}{(\sqrt{-x^2})^{\frac{3}{2}N_Y + l}} C_{Y, l} \left(P, \left\{ k_j = \frac{ixm_j}{\sqrt{-x^2}} \middle| j \in Y \right\} \right) \\ & + \sum_{Z, l=0} \frac{e^{-iP \cdot x} e^{-m_Z \sqrt{-x^2}}}{(\sqrt{-x^2})^{\frac{3}{2}N_Z + l}} C_{Z, l} \left(P, \left\{ k_j = \frac{ixm_j}{\sqrt{-x^2}} \middle| j \in Z \right\} \right), \quad (3.42) \end{aligned}$$

where m_X is the mass of the state X projected by common quantum numbers of $\bar{Q}\psi|H(P)\rangle$ and $\bar{Q}\gamma^z\psi|H(P)\rangle$, m_Y is for the state Y with quantum numbers of $\bar{Q}\psi|\Omega\rangle$ or $\bar{Q}\gamma^z\psi|\Omega\rangle$, and m_Z is for the state Z with common quantum numbers of $\bar{Q}\psi|\bar{H}(P)\rangle$ and $\bar{Q}\gamma^z\psi|\bar{H}(P)\rangle$. The quantities N_X , N_Y , and N_Z denote the numbers of single-particles in the corresponding multi-particle states X , Y , and Z , respectively. The coefficients $C_{X, l}$, $C_{Y, l}$, and $C_{Z, l}$ depend on the momenta of the external hadron and inserted particles.

The asymptotic expansion in Eq. (3.42) is written in a Lorentz invariant form, which corresponds to a single Lorentz scalar function by construction in Eq. (3.21). The phase factors $e^{iP \cdot x}$ and $e^{-iP \cdot x}$ are Lorentz invariant. In the exponential factors, the separation necessarily appears as the Lorentz-

invariant quantity $\sqrt{-x^2}$, since the associated mass gaps are Lorentz invariant. See the discussions on the prescription of $\sqrt{-x^2}$ below Eq. (3.15). We also write the polynomial dependence in terms of $\sqrt{-x^2}$. While alternative parameterizations of the polynomial factors are possible, any such differences can be absorbed into a redefinition of the coefficients. These coefficients depend on the momenta of the inserted particles and can, in general, be expanded along the “velocity” $ix/\sqrt{-x^2}$ like moving HQET.

Based on HQET reduction in Eq. (3.18), the asymptotic expansion of quasi-PDF matrix element is,

$$\begin{aligned} \tilde{h}_{\{\Gamma_x, \Gamma\}}(x, P)_{\sqrt{-x^2} \rightarrow \infty} = & \sum_{X,l=0} \frac{e^{iP \cdot x} e^{-(m_X - m_Q)\sqrt{-x^2}}}{(\sqrt{-x^2})^{\frac{3}{2}(N_X - 1) + l}} a_{X,l} \left(P, \left\{ k_j = \frac{ixm_j}{\sqrt{-x^2}} \middle| j \in X \right\} \right) \\ & + \sum_{Y,l=0} \frac{e^{-(m_Y - m_Q)\sqrt{-x^2}}}{(\sqrt{-x^2})^{\frac{3}{2}(N_Y - 1) + l}} a_{Y,l} \left(P, \left\{ k_j = \frac{ixm_j}{\sqrt{-x^2}} \middle| j \in Y \right\} \right) \\ & + \sum_{Z,l=0} \frac{e^{-iP \cdot x} e^{-(m_Z - m_Q)\sqrt{-x^2}}}{(\sqrt{-x^2})^{\frac{3}{2}(N_Z - 1) + l}} a_{Z,l} \left(P, \left\{ k_j = \frac{ixm_j}{\sqrt{-x^2}} \middle| j \in Z \right\} \right), \quad (3.43) \end{aligned}$$

which differs from $M_{\{\Gamma_x, \Gamma\}}(x, P)_{\sqrt{-x^2} \rightarrow \infty}$ by exponential and polynomial dependences in the kinematic factor D_Q in Eq. (3.15).

Using Lorentz symmetry and setting $x = (0, 0, 0, z)$, we obtain the large-distance expansion for

the spacelike correlator in $\overline{\text{MS}}$ scheme,

$$\begin{aligned}
\tilde{h}_{\{\gamma^t, I\}}(z, P^z) &= \langle H(P) | \bar{\psi}(z) U(z, 0) \gamma^t \psi(0) | H(P) \rangle_c \\
&\stackrel{z \rightarrow \infty}{=} \sum_{X, l=0} \frac{e^{-iP^z z} e^{-\Lambda_X |z|}}{|z|^{\frac{3}{2}(N_X-1)+l}} A_{X,l}(P^z) e^{i \text{sign}(z) \phi_{X,l}(P^z)} \\
&\quad + \sum_{Y, l=0} \frac{e^{-\Lambda_Y |z|}}{|z|^{\frac{3}{2}(N_Y-1)+l}} A_{Y,l}(P^z) e^{i \text{sign}(z) \phi_{Y,l}(P^z)} \\
&\quad + \sum_{Z, l=0} \frac{e^{iP^z z} e^{-\Lambda_Z |z|}}{|z|^{\frac{3}{2}(N_Z-1)+l}} A_{Z,l}(P^z) e^{i \text{sign}(z) \phi_{Z,l}(P^z)}, \tag{3.44}
\end{aligned}$$

where $\Lambda_X = m_X - m_Q$ is the binding energy of the state X projected by the common quantum numbers of $\bar{Q}\psi|H(P)\rangle$ and $\bar{Q}\gamma^z\psi|H(P)\rangle$, $\Lambda_Y = m_Y - m_Q$ is for the state Y with quantum numbers of $\bar{Q}\psi|\Omega\rangle$ or $\bar{Q}\gamma^z\psi|\Omega\rangle$, and $\Lambda_Z = m_Z - m_Q$ is for the state Z of the common quantum numbers of $\bar{Q}\psi|\bar{H}(P)\rangle$ and $\bar{Q}\gamma^z\psi|\bar{H}(P)\rangle$.⁵

These binding energies Λ_i are the $O(\Lambda_{\text{QCD}})$ corrections in the HQET expansion of physical masses m_i of heavy-flavor systems,

$$m_i = m_Q(\tau) + \Lambda_i(\tau) + O\left(\frac{\Lambda_{\text{QCD}}^2}{m_Q}\right), \tag{3.45}$$

where τ denotes the renormalon regularization scheme of the heavy quark pole mass [383–386]. While the individual components $m_Q(\tau)$ and $\Lambda_i(\tau)$ depend on τ , the physical masses m_i are renormalon-scheme independent. In lattice QCD calculations of quasi-PDF matrix elements, the heavy quark pole mass m_Q is effectively replaced by the lattice cutoff $1/a$. In this case, the binding energies appear as

⁵Because of Lorentz invariance of mass gaps, we determine the spectra in the spacelike case using the quantum numbers of physical states in the timelike case. This approach is more convenient than studying the spacelike case directly, which involves unphysical states with imaginary spatial momenta shown in Eq. (3.43), although such an approach is in principle possible via complex Lorentz transformations [377].

the $O(\Lambda_{\text{QCD}})$ corrections in the small a expansion of bare mass gaps $m_i^B(a)$ of quasi-correlators,

$$m_i^B(a) = \left[\frac{\sum_{n=0} r_n \alpha_{\text{lat}}^{n+1}(a)}{a} \right] (\tau) + \Lambda_i(\tau) + O(a\Lambda_{\text{QCD}}^2) . \quad (3.46)$$

The linear divergences depend on the renormalon scheme τ [98, 232], and its τ -dependence is compensated by that of the binding energies. Under the same choice of renormalon scheme τ (as well as physical pion mass and other possible conditions in lattice setup), these binding energies are expected to be consistent between nature and lattice QCD. In the following discussions, we will adopt the principal value (PV) prescription for τ [232].

The polynomial powers in z are affected by N_X , N_Y and N_Z defined below Eq. (3.42). In the spacelike separation $x = (0, 0, 0, z)$, the coefficients $a_{X,l}$, $a_{Y,l}$ and $a_{Z,l}$ appearing in Eq. (3.43) generally acquire phases that depend on the sign of z . These phases are made explicit in Eq. (3.44) through the factors $e^{i \text{sign}(z) \phi_{X,l}(P^z)}$, $e^{i \text{sign}(z) \phi_{Y,l}(P^z)}$ and $e^{i \text{sign}(z) \phi_{Z,l}(P^z)}$. Both the amplitudes A_i and phases ϕ_i depend on the external momentum P^z in general.

Eq. (3.44) applies directly to the hybrid renormalized matrix elements [287], since the hybrid scheme differs from the $\overline{\text{MS}}$ scheme only by an overall constant factor at large distances. Eq. (3.44) can also be generalized to bare matrix elements if one replaces the parameters $\{\Lambda_X, \Lambda_Y, \Lambda_Z\}$ into the bare mass gaps $\{m_X^B(a), m_Y^B(a), m_Z^B(a)\}$. However, it does not apply to the so-called ratio method [387], because the leading exponential decay cancels in the ratio of the large momentum to the zero momentum matrix elements. The follow-up discussions in this paper are based on the $\overline{\text{MS}}$ or hybrid scheme [287].

The connection between timelike and spacelike correlators using Lorentz symmetry has been verified up to two-loop in ϕ^3 theory in Eqs. (A.44) and (A.45).

3.2.4 Accuracy-counting of large distance expansion

In this subsection, we discuss the accuracy-counting principles of the large-distance expansion in Eq. (3.44), and define the leading asymptotics (LA) and the next-to-leading asymptotics (NLA). These discussions are closely connected to the data analysis in the next section, where the large-distance expansion is truncated up to a certain accuracy.

Eq. (3.44) is a double expansion structure, which is a series of exponential- z decays, each of which is multiplied by a series of polynomial- z dependences. The accuracy in the large distance limit ($z \rightarrow \infty$) should be counted in this way:

- Combine the three spectra $\{\Lambda_X, \Lambda_Y, \Lambda_Z\}$, and rank the mass gaps in this joint spectrum, $\Lambda_0 < \Lambda_1 < \Lambda_2 < \Lambda_3 < \dots$, which provides the leading exponential decay $e^{-\Lambda_0|z|}$, next-to-leading exponential decay $e^{-\Lambda_1|z|}$, next-to-next-to-leading exponential decay $e^{-\Lambda_2|z|}$, etc. An illustrative example is shown below,

$$\begin{aligned}
 \tilde{h}_{\{\gamma^t, I\}}(z, P^z) &\stackrel{z \rightarrow \infty}{\sim} e^{-\Lambda_0|z|} \left(a_{0,0} + \frac{a_{0,1}}{|z|} + \frac{a_{0,2}}{z^2} + \dots \right) \\
 &+ e^{-\Lambda_1|z|} \left(a_{1,0} + \frac{a_{1,1}}{|z|} + \frac{a_{1,2}}{z^2} + \dots \right) \\
 &+ e^{-\Lambda_2|z|} \left(a_{2,0} + \frac{a_{2,1}}{|z|} + \frac{a_{2,2}}{z^2} + \dots \right) \\
 &+ \dots
 \end{aligned} \tag{3.47}$$

Note that the powers of the polynomial- z dependences could differ among real cases according to the discussions around Eqs. (3.40) and (3.41). The phase factors, irrelevant to the discussions here, are absorbed into those coefficients. The momentum P^z dependence in those coefficients

is omitted for simplicity.

- Count the polynomial accuracy within each exponential decay. For example, in the leading exponential decay, count the leading polynomial accuracy $a_{0,0}$, next-to-leading polynomial accuracy $a_{0,1}/|z|$, etc.
- One cannot achieve the next-to-leading exponential accuracy until the polynomial accuracy in the leading exponential decay is entirely under control. This accuracy-counting principle is similar to the operator product expansion (OPE) [388], where one cannot achieve power accuracy until the logarithmic accuracy in the leading twist is fully under control.
- Focus on the leading exponential decay (associated with the smallest mass gap) and count its polynomial accuracy, where we call the leading polynomial accuracy the *leading asymptotics* (LA) and the next-to-leading polynomial accuracy the *next-to-leading asymptotics* (NLA).

It is convenient to use physical heavy hadron masses as references when applying the above accuracy-counting principles to lattice QCD quasi-correlators, particularly for identifying the smallest mass gap that governs the leading exponential decay used to construct LA and NLA, provided that the following conditions are met:

- The higher power corrections $O(\Lambda_{\text{QCD}}^2/m_Q)$ in Eq. (3.45) and $O(\Lambda_{\text{QCD}}^2 a)$ in Eq. (3.46) are negligible. To satisfy this condition, m_Q and $1/a$ should be large enough compared to Λ_{QCD} . Under current lattice techniques, $a = 0.03 - 0.1$ fm, much smaller than $1/\Lambda_{\text{QCD}} \sim 0.7 - 1$ fm, thereby suppressing the power corrections $O(\Lambda_{\text{QCD}}^2 a)$ in Eq. (3.46). As for heavy-flavor systems in nature, we will choose the spectra of bottom mesons and baryons in Tabs. 3.1 and 3.2 as references, since the bottom quark pole mass is estimated as $m_Q \sim 4.8$ GeV [386], large enough to suppress

State	$I(J^P)$	Γ	Particle and Mass
Scalar (u/d)	$1/2(0^+)$	I, γ^t	B_0^* (5756 MeV)
Pseudo-scalar (u/d)	$1/2(0^-)$	$\gamma^5, \gamma^t \gamma^5$	$B^\pm$ (5279.41 ± 0.07 MeV)
Vector (u/d)	$1/2(1^-)$	$\gamma^i, \gamma^t \gamma^i$	B^* (5324.75 ± 0.20 MeV)
Axial or Tensor (u/d)	$1/2(1^+)$	$\gamma^i \gamma^5$ or $\gamma^i \gamma^j$	B_1 ($5726.0^{+2.5}_{-2.7}$ MeV)
Pseudo-scalar (s)	$0(0^-)$	$\gamma^5, \gamma^t \gamma^5$	B_s^0 (5366.91 ± 0.11 MeV)
Vector (s)	$0(1^-)$	$\gamma^i, \gamma^t \gamma^i$	B_s^* (5415.4 ± 1.4 MeV)

Table 3.1: The B mesons of various spins and parities. The “ Γ ” here denotes the Dirac structures in the meson interpolators $\bar{b}\Gamma q$ for particles at rest [1], where $q = u/d$ or s . The Dirac structures for moving particles can be obtained using the Lorentz boost. “ i ” and “ j ” denote the spatial indices. The masses of pseudo-scalar, vector, and axial mesons are obtained from Particle Data Group [2]. The mass of the scalar meson is calculated using heavy meson chiral perturbation theory in Ref. [3]. We only present the lowest mass for each category.

the power corrections $O(\Lambda_{\text{QCD}}^2/m_Q)$ in Eq. (3.45).

- The unphysical pion masses in lattice setups do not alter the ranking of the mass gaps, or some mass gaps are pretty close to each other so that their differences can be ignored. To estimate the effects of unphysical pion masses, we will compare the bottom hadrons with s and u/d quarks. This is reasonable since the kaon and pion masses in nature are ~ 500 MeV and ~ 130 MeV, respectively, which roughly cover the pion mass variation range $\sim 100 - 400$ MeV in practical lattice simulations.

Take the pion valence quasi-PDF matrix element as an example,

$$\langle \pi(P) | [\bar{u}(z)U(z,0)\gamma^t u(0) - \bar{d}(z)U(z,0)\gamma^t d(0)] | \pi(P) \rangle, \quad (3.48)$$

which is a particular case of Eq. (3.44). Based on the quantum numbers listed below Eq. (3.44) and the masses of B mesons in Tab. 3.1, the lightest Y corresponds to a heavy-light vector meson ($Q\bar{q}$, $1/2(1^-)$, 5325 MeV if estimated using B^*), and X or Z is related to a heavy-light pseudo-scalar meson ($Q\bar{q}$,

$1/2(0^-)$, 5279 MeV if estimated using $B^\pm$). The two masses differ by only a small amount ~ 46 MeV. On lattice, due to the unphysical pion masses, those values could differ from the physical values, by ~ 87 MeV if estimated using the difference between B_s^0 and $B^\pm$. Therefore, we are unsure whether lattice artifacts could alter the ranking of these two mass gaps. On the other hand, this fine mass split effect would not affect the practical data analysis and precision control, as we will see in the next section. Therefore, our strategy is to treat the two mass gaps as the same when constructing the LA and NLA, and keep all relevant terms to be fitted with lattice data,

$$\tilde{h}^{\text{LA}}(z, P^z) = [A_2 + 2A_1 \cos(\phi - zP^z)] e^{-\Lambda|z|}, \quad (3.49)$$

$$\tilde{h}^{\text{NLA}}(z, P^z) = \left[A_2 + \frac{A'_2}{|z|} + 2A_1 \cos(\phi - zP^z) + \frac{2A'_1 \cos(\phi' - zP^z)}{|z|} \right] e^{-\Lambda|z|}, \quad (3.50)$$

where all the parameters A_2 , A_1 , ϕ , Λ , A'_2 , A'_1 and ϕ' are real numbers. The A_2 and A'_2 terms arise from the state Y , where the phases vanish due to the isospin symmetry. The combination of X and Z terms gives the A_1 and A'_1 terms here. The mass gap $\Lambda \sim 400 - 600$ MeV if estimated using vector and pseudo scalar B-meson masses $\sim 5279 - 5415$ MeV and bottom quark pole mass in PV prescription ~ 4836 MeV [386].

Consider the isovector quark quasi-PDF of an unpolarized nucleon,

$$\langle N(P) | [\bar{u}(z)U(z, 0)\gamma^t u(0) - \bar{d}(z)U(z, 0)\gamma^t d(0)] | N(P) \rangle, \quad (3.51)$$

as another example of Eq. (3.44). The leading exponential decay can be determined by comparing the quantum numbers below Eq. (3.44) and the masses of bottom quark systems in Tabs. 3.1 and 3.2.

$I(J^P)$	Particle and Mass
$0(1/2^+)$	Λ_b^0 (5619.57 ± 0.16 MeV)
$0(1/2^-)$	$\Lambda_b(5912)^0$ (5912.16 ± 0.16 MeV)
$0(3/2^+)$	$\Lambda_b(6146)^0$ (6146.2 ± 0.4 MeV)
$0(3/2^-)$	$\Lambda_b(5920)^0$ (5920.07 ± 0.16 MeV)
$1(1/2^+)$	Σ_b^+ (5810.56 ± 0.25 MeV)
$1(3/2^+)$	Σ_b^{*+} (5830.32 ± 0.27 MeV)
$1/2(1/2^+)$	Ξ_b^0 (5791.7 ± 0.4 MeV)
$1/2(3/2^+)$	$\Xi_b(5945)^0$ (5952.3 ± 0.6 MeV)

Table 3.2: The bottom baryons of various isospins, spins, and parities obtained from Particle Data Group [2]. For each set of quantum numbers, only the lightest particle is presented.

The smallest mass gap of the Y terms is $\Lambda \sim 0.5$ GeV, the binding energy of a heavy-light vector meson ($Q\bar{q}$, 1^-), following a similar logic as the pion valence PDF. The smallest mass gaps of X and Z terms are related to binding energies of heavy baryon (Qqq , $1/2^+$) and heavy anti-pentaquark ($Q\bar{q}\bar{q}\bar{q}\bar{q}$), respectively, which are ~ 0.9 GeV and ~ 1.5 GeV, if estimated using the average of $\Lambda_b^0 \sim 5.6$ GeV and $\Xi_b^0 \sim 5.8$ GeV in Tab. 3.2 and the baryon-meson molecule ~ 6.3 GeV, subtracted by the bottom quark pole mass in the “PV-prescription” ~ 4.8 GeV [386]. At large distances, the X and Z terms are exponentially suppressed compared to the Y terms (e.g. $e^{-(0.9-0.5)\text{GeV}\times 1.5\text{fm}} = 0.0476$ and $e^{-(1.5-0.5)\text{GeV}\times 1.5\text{fm}} = 0.0005$). Therefore, the LA and NLA formulas are the Y terms,

$$\tilde{h}^{\text{LA}}(z, P^z) = A e^{i\phi \text{sign}(z)} e^{-\Lambda|z|}, \quad (3.52)$$

$$\tilde{h}^{\text{NLA}}(z, P^z) = \left[A e^{i\phi \text{sign}(z)} + \frac{A' e^{i\phi' \text{sign}(z)}}{|z|} \right] e^{-\Lambda|z|}, \quad (3.53)$$

where the parameters A , ϕ , Λ , A' and ϕ' are real numbers.

3.3 Generalizations to other quasi-correlators

This section generalizes the methods discussed in Sec. 3.2 to many other quasi-correlators, such as the proton quark polarized PDF in Sec. 3.3.1, meson DA in Sec. 3.3.2, quark GPD in Sec. 3.3.3, and gluon PDF in Sec. 3.3.4.

3.3.1 Nucleon quark transversity and helicity PDFs

The nucleon quark transversity quasi-PDF matrix element is constructed as [12, 116],

$$\langle N(PS_\perp) | \bar{\psi}(z) \gamma^t \gamma^\perp \gamma^5 U(z, 0) \psi(0) | N(PS_\perp) \rangle_c, \quad (3.54)$$

where $|N(PS_\perp)\rangle$ denotes a proton with momentum $P = (\sqrt{m_N^2 + P_z^2}, 0, 0, P^z)$ with $P^z \gg m_N$ and transverse polarization $S_\perp$ in x or y direction. Following the similar logic as Sec. 3.2 but replacing Eq. (3.21) into

$$\Gamma_x \equiv \begin{cases} \gamma^z \gamma^\perp \gamma^5, & x = (t, 0, 0, 0) \\ \gamma^t \gamma^\perp \gamma^5, & x = (0, 0, 0, z) \end{cases}, \quad (3.55)$$

we obtain this correlator's large distance expansion up to all orders in $\overline{\text{MS}}$ or hybrid scheme,

$$\begin{aligned}
& \langle N(PS_\perp) | \bar{\psi}(z) \gamma^t \gamma^\perp \gamma^5 U(z, 0) \psi(0) | N(PS_\perp) \rangle_c \\
& \stackrel{z \rightarrow \infty}{=} \sum_{X,l=0} \frac{e^{-iPz} e^{-\Lambda_X |z|}}{|z|^{\frac{3}{2}(N_X-1)+l}} A_{X,l} e^{i \text{sign}(z) \phi_{X,l}} \\
& + \sum_{Y,l=0} \frac{e^{-\Lambda_Y |z|}}{|z|^{\frac{3}{2}(N_Y-1)+l}} A_{Y,l} e^{i \text{sign}(z) \phi_{Y,l}} \\
& + \sum_{Z,l=0} \frac{e^{iPz} e^{-\Lambda_Z |z|}}{|z|^{\frac{3}{2}(N_Z-1)+l}} A_{Z,l} e^{i \text{sign}(z) \phi_{Z,l}} , \tag{3.56}
\end{aligned}$$

where $\Lambda_X = m_X - m_Q$ is the binding energy of the state X projected by the common quantum numbers of $\bar{Q}\psi |N(PS_\perp)\rangle$ and $\bar{Q}\gamma^z \gamma^\perp \gamma^5 \psi |N(PS_\perp)\rangle$, $\Lambda_Y = m_Y - m_Q$ is for the state Y with quantum numbers of $\bar{Q}\psi |\Omega\rangle$ or $\bar{Q}\gamma^z \gamma^\perp \gamma^5 \psi |\Omega\rangle$, and $\Lambda_Z = m_Z - m_Q$ is for the state Z with common quantum numbers of $\bar{Q}\psi |\bar{N}(PS_\perp)\rangle$ and $\bar{Q}\gamma^z \gamma^\perp \gamma^5 \psi |\bar{N}(PS_\perp)\rangle$. N_X , N_Y , and N_Z denote the numbers of single-particles in the multi-particle states X , Y and Z , respectively. The P and $S_\perp$ dependences in the coefficients and phases are omitted for simplicity.

We repeat this logic for nucleon quark helicity quasi-PDF [116],

$$\langle N(PS_L) | \bar{\psi}(z) \gamma^z \gamma^5 U(z, 0) \psi(0) | N(PS_L) \rangle_c , \tag{3.57}$$

except for replacing Eq. (3.21) into

$$\Gamma_x \equiv \begin{cases} \gamma^t \gamma^5 , & x = (t, 0, 0, 0) \\ \gamma^z \gamma^5 , & x = (0, 0, 0, z) \end{cases} . \tag{3.58}$$

The same mathematical structure as Eq. (3.56) can be obtained, although the values of mass gaps,

coefficients, and phases could differ. In this case, the state X is projected by the common quantum numbers of $\bar{Q}\psi|N(PS_L)\rangle$ and $\bar{Q}\gamma^t\gamma^5\psi|N(PS_L)\rangle$, the state Y with quantum numbers of $\bar{Q}\psi|\Omega\rangle$ or $\bar{Q}\gamma^t\gamma^5\psi|\Omega\rangle$, and the state Z with common quantum numbers of $\bar{Q}\psi|N(PS_L)\rangle$ and $\bar{Q}\gamma^t\gamma^5\psi|N(PS_L)\rangle$.

Similar to the unpolarized case in Eq. (3.44), the polarized case in Eq. (3.56) does not receive the linear z enhancement from the forward singularity as well. This is because the forward singularity in the large distance limit is also forbidden by the quantum numbers of the operators for the polarized case. See the discussions between and around Eqs. (3.33) and (3.41) on the origin and impact of forward singularity.

Following the accuracy-counting principles in Sec. 3.2.4, we truncate the large distance expansion in Eq. (3.56) up to LA and NLA,

$$\tilde{h}^{\text{LA}}(z, P^z) = A e^{i\phi \text{sign}(z)} e^{-\Lambda|z|}, \quad (3.59)$$

$$\tilde{h}^{\text{NLA}}(z, P^z) = \left[A e^{i\phi \text{sign}(z)} + \frac{A' e^{i\phi' \text{sign}(z)}}{|z|} \right] e^{-\Lambda|z|}, \quad (3.60)$$

which apply to both transversity and helicity. Λ is the binding energy of a vector (for transversity) or pseudo-scalar (for helicity) heavy-light meson if ψ is a light quark, or a heavy-heavy meson if ψ is a heavy quark.

3.3.2 Meson DA

The quasi-DA matrix element for a pseudo scalar meson [23, 147],

$$\langle M(P) | \bar{\psi}_1(z) U(z, 0) \gamma^z \gamma^5 \psi_2(0) | \Omega \rangle_c, \quad (3.61)$$

where ψ_1 and ψ_2 are quark fields under different flavors. Following the methods in Sec. 3.2, we obtain the all-order large distance asymptotic form in $\overline{\text{MS}}$ or hybrid scheme,

$$\begin{aligned} & \langle M(P) | \bar{\psi}_1(z) U(z, 0) \gamma^z \gamma^5 \psi_2(0) | \Omega \rangle_c \\ & \stackrel{z \rightarrow \infty}{=} \sum_{X,l=0} \frac{e^{-iPz} e^{-\Lambda_X |z|}}{|z|^{\frac{3}{2}(N_X-1)+l}} A_{X,l} e^{i \text{sign}(z) \phi_{X,l}} \\ & + \sum_{Y,l=0} \frac{e^{-\Lambda_Y |z|}}{|z|^{\frac{3}{2}(N_Y-1)+l}} A_{Y,l} e^{i \text{sign}(z) \phi_{Y,l}}, \end{aligned} \quad (3.62)$$

where $\Lambda_X = m_X - m_Q$ is the binding energy of the state X with the quantum numbers of $\bar{Q}\psi_2|\Omega\rangle$ or $\bar{Q}\gamma^t\gamma^5\psi_2|\Omega\rangle$, and $\Lambda_Y = m_Y - m_Q$ is for the state Y with the quantum numbers of $\bar{Q}\psi_1|\Omega\rangle$ or $\bar{Q}\gamma^t\gamma^5\psi_1|\Omega\rangle$. See the definitions of N_X and N_Y below Eq. (3.56). Compared with the quasi-PDF in Eq. (3.44), the DA lacks the Z term in which both form factors are disconnected.

Based on the accuracy counting principles in Sec. 3.2.4, we truncate the large distance expansion in Eq. (3.62) up to LA and NLA,

$$\tilde{h}^{\text{LA}}(z, P^z) = [A_1 e^{i\phi_1 \text{sign}(z)} e^{-izP^z} + A_2 e^{i\phi_2 \text{sign}(z)}] e^{-\Lambda|z|}, \quad (3.63)$$

$$\begin{aligned} \tilde{h}^{\text{NLA}}(z, P^z) = & \left[(A_1 e^{i\phi_1 \text{sign}(z)} e^{-izP^z} + A_2 e^{i\phi_2 \text{sign}(z)}) \right. \\ & \left. + (A'_1 e^{i\phi'_1 \text{sign}(z)} e^{-izP^z} + A'_2 e^{i\phi'_2 \text{sign}(z)}) \frac{1}{|z|} \right] e^{-\Lambda|z|}. \end{aligned} \quad (3.64)$$

The above asymptotic forms apply to light meson DAs. If both ψ_1 and ψ_2 are light quarks with similar masses, all terms should be kept. Λ corresponds to the binding energy of a pseudoscalar heavy-light meson. In particular, $A_1 = A_2$, $\phi_1 = -\phi_2$, $A'_1 = A'_2$ and $\phi'_1 = -\phi'_2$ for pion quasi-DA. They can be generalized to heavy-light meson DAs [160–166], which have been hot topics recently. If ψ_1 is a light quark and ψ_2 is a heavy quark, the state X is heavier than Y and the leading exponential decay arises from the Y term, which means $A_1 = A'_1 = 0$. If ψ_1 is a heavy quark and ψ_2 is a light quark, $A_2 = A'_2 = 0$.

3.3.3 Quark GPD

The quark unpolarized quasi-GPD [146, 189],

$$\langle H(P) | \bar{\psi}(z) U(z, 0) \gamma^t \psi(0) | H(P') \rangle_c, \quad (3.65)$$

whose all-order large distance asymptotic expansion in $\overline{\text{MS}}$ or hybrid scheme can be obtained following a similar derivation process as Sec. 3.2,

$$\begin{aligned}
& \langle H(P) | \bar{\psi}(z) U(z, 0) \gamma^t \psi(0) | H(P') \rangle_c \\
& \stackrel{z \rightarrow \infty}{=} \sum_{X,l=0} \frac{e^{-iP^z z} e^{-\Lambda_X |z|}}{|z|^{\frac{3}{2}(N_X-1)+l}} A_{X,l} e^{i \text{sign}(z) \phi_{X,l}} \\
& + \sum_{Y,l=0} \frac{e^{-\Lambda_Y |z|}}{|z|^{\frac{3}{2}(N_Y-1)+l}} \left[A_{Y,l} e^{i \text{sign}(z) \phi_{Y,l}} + \tilde{A}_{Y,l} e^{i \text{sign}(z) \tilde{\phi}_{Y,l}} e^{-i(P^z - P'^z)z} \right] \\
& + \sum_{Z,l=0} \frac{e^{iP'^z z} e^{-\Lambda_Z |z|}}{|z|^{\frac{3}{2}(N_Z-1)+l}} A_{Z,l} e^{i \text{sign}(z) \phi_{Z,l}} , \tag{3.66}
\end{aligned}$$

where the binding energies have the same physical origins as Eq. (3.44). See the definitions of N_X , N_Y and N_Z below Eq. (3.56). The forward singularity discussed between and around Eqs. (3.33) and (3.41) does not exist in the off-forward GPD, which means there is no enhanced polynomial z -dependence in the Y term.

For pion external state, we truncate the large distance expansion in Eq. (3.66) up to LA and NLA following Sec. 3.2.4,

$$\begin{aligned}
\tilde{h}^{\text{LA}}(z, P^z, P'^z) = & \left[A_1 e^{i\phi_1 \text{sign}(z)} e^{-izP^z} + A_3 e^{i\phi_3 \text{sign}(z)} e^{izP'^z} \right. \\
& \left. + A_2 e^{i\phi_2 \text{sign}(z)} + \tilde{A}_2 e^{i\tilde{\phi}_2 \text{sign}(z)} e^{-i(P^z - P'^z)z} \right] e^{-\Lambda|z|} , \tag{3.67}
\end{aligned}$$

$$\begin{aligned}
\tilde{h}^{\text{NLA}}(z, P^z, P'^z) = & \left[A_1 e^{i\phi_1 \text{sign}(z)} e^{-izP^z} + A_3 e^{i\phi_3 \text{sign}(z)} e^{izP'^z} \right. \\
& \left. + A_2 e^{i\phi_2 \text{sign}(z)} + \tilde{A}_2 e^{i\tilde{\phi}_2 \text{sign}(z)} e^{-i(P^z - P'^z)z} \right] e^{-\Lambda|z|} \\
& + \left[A'_1 e^{i\phi'_1 \text{sign}(z)} e^{-izP^z} + A'_3 e^{i\phi'_3 \text{sign}(z)} e^{izP'^z} \right. \\
& \left. + A'_2 e^{i\phi'_2 \text{sign}(z)} + \tilde{A}'_2 e^{i\tilde{\phi}'_2 \text{sign}(z)} e^{-i(P^z - P'^z)z} \right] \frac{e^{-\Lambda|z|}}{|z|}. \quad (3.68)
\end{aligned}$$

For valence combination, all terms are preserved for the same reason as the pion valence quasi-PDF discussed in Sec. 3.2.4. For sea quarks (including light and heavy quarks), $A_1 = A_3 = A'_1 = A'_3 = 0$ since their mass gaps are associated with four-quark states that have the combined quantum numbers of a light meson and a heavy meson and are therefore larger than those of a single heavy meson.

For proton external state, the LA and NLA of Eq. (3.66) can be obtained following Sec. 3.2.4,

$$\tilde{h}^{\text{LA}}(z, P^z, P'^z) = \left[A_2 e^{i\phi_2 \text{sign}(z)} + \tilde{A}_2 e^{i\tilde{\phi}_2 \text{sign}(z)} e^{-i(P^z - P'^z)z} \right] e^{-\Lambda|z|}, \quad (3.69)$$

$$\begin{aligned}
\tilde{h}^{\text{NLA}}(z, P^z, P'^z) = & \left[A_2 e^{i\phi_2 \text{sign}(z)} + \tilde{A}_2 e^{i\tilde{\phi}_2 \text{sign}(z)} e^{-i(P^z - P'^z)z} \right. \\
& \left. + \left(A'_2 e^{i\phi'_2 \text{sign}(z)} + \tilde{A}'_2 e^{i\tilde{\phi}'_2 \text{sign}(z)} e^{-i(P^z - P'^z)z} \right) \frac{1}{|z|} \right] e^{-\Lambda|z|}, \quad (3.70)
\end{aligned}$$

where the leading exponential decay arises from the Y term for the same reason as the proton quark unpolarized quasi-PDF discussed in Sec. 3.2.4.

As for the proton polarized quasi-GPD [146, 189],

$$\langle N(PS) | \bar{\psi}(z) U(z, 0) \gamma^z \gamma^5 \psi(0) | N(P'S') \rangle_c, \quad (3.71)$$

we follow the methods in Sec. 3.2 to derive its large distance expansion, which has the same mathematical structure as Eq. (3.66) though the values of mass gaps, coefficients and phases can be different. The physical origins of binding energies are the same as the proton helicity quasi-PDF in Eq. (3.57). Following Sec. 3.2.4, the LA and NLA have the same mathematical formats as Eqs. (3.69) and (3.70), respectively.

3.3.4 Gluon PDF

In this section, we derive the large distance expansion up to all orders for the gluon quasi-PDF matrix element using the adjoint heavy quark. However, its truncations up to LA and NLA are presented as conjectures because the spectra of the exotic hadrons have not been fully understood⁶.

Consider the gluon quasi-PDF matrix element in the adjoint representation [137],

$$\langle H(P) | F_a^{z\mu}(z) L_{ab}(z, 0) F_{b,\mu}^z(0) | H(P) \rangle_c . \quad (3.72)$$

For the convenience of dispersive analysis, we introduce the auxiliary heavy quark in the adjoint representation $\mathcal{Q}$ with trivial spin degrees of freedom, following a similar method as Ref. [137],

$$\mathcal{L} = \mathcal{L}_{\text{QCD}} + \bar{\mathcal{Q}}(x) \left(in \cdot D - m_{\mathcal{Q}} - \frac{D_{\perp}^2}{2m_{\mathcal{Q}}} \right) \mathcal{Q}(x) , \quad (3.73)$$

where the kinematic energy term $D_{\perp}^2/(2m_{\mathcal{Q}})$ is retained to ensure the correct polynomial dependence.

⁶at least by the authors

Then, the adjoint gauge link can be represented by adjoint heavy quarks,

$$\langle H(P) | F_a^{z\mu}(x) \mathcal{Q}_a(x) \bar{\mathcal{Q}}_b(0) F_{b,\mu}^z(0) | H(P) \rangle_c . \quad (3.74)$$

We follow a similar logic as Sec. 3.2 to derive the all-order large distance expansion for the gluon quasi-PDF matrix element in $\overline{\text{MS}}$ or hybrid scheme,

$$\begin{aligned} & \langle H(P) | F_a^{z\mu}(z) L_{ab}(z, 0) F_{b,\mu}^z(0) | H(P) \rangle_c \\ & \stackrel{z \rightarrow \infty}{=} \sum_{X,l=0} \frac{e^{-iP^z z} e^{-\Lambda_X |z|}}{|z|^{\frac{3}{2}(N_X-1)+l}} A_{X,l} e^{i \text{sign}(z) \phi_{X,l}} \\ & + \sum_{Y,l=0} \frac{e^{-\Lambda_Y |z|}}{|z|^{\frac{3}{2}(N_Y-1)+l-1}} A_{Y,l} e^{i \text{sign}(z) \phi_{Y,l}} \\ & + \sum_{Z,l=0} \frac{e^{iP^z z} e^{-\Lambda_Z |z|}}{|z|^{\frac{3}{2}(N_Z-1)+l}} A_{Z,l} e^{i \text{sign}(z) \phi_{Z,l}} , \end{aligned} \quad (3.75)$$

where $\Lambda_X = m_X - m_Q$ is the binding energy of the state X with the quantum numbers of $\bar{\mathcal{Q}}_b F_{b,\mu}^t | H(P) \rangle$, $\Lambda_Y = m_Y - m_Q$ is for the state Y of $\bar{\mathcal{Q}}_b F_{b,\mu}^t | \Omega \rangle$, and $\Lambda_Z = m_Z - m_Q$ for $\bar{\mathcal{Q}}_b F_{b,\mu}^t | \bar{H}(P) \rangle$. According to Ref. [389], an adjoint heavy quark has the same color quantum number as a pair of fundamental heavy quark and anti-quark under repulsive color-Coulomb potential, which means $\bar{\mathcal{Q}}_b F_{b,\mu}^t | \Omega \rangle$ can produce the quarkonium-hybrid ($\bar{Q}Qg$). Also see the definitions of N_X , N_Y and N_Z below Eq. (3.56). Since

the gluon quasi-PDF matrix element is purely real, the above expression can be further simplified,

$$\begin{aligned}
& \langle H(P) | F_a^{z\mu}(z) L_{ab}(z, 0) F_{b,\mu}^z(0) | H(P) \rangle_c \\
& \stackrel{z \rightarrow \infty}{=} \sum_{X,l=0} \frac{e^{-\Lambda_X |z|}}{|z|^{\frac{3}{2}(N_X-1)+l}} A_{X,l} \cos[-P^z z + \text{sign}(z) \phi_{X,l}] \\
& + \sum_{Y,l=0} \frac{e^{-\Lambda_Y |z|}}{|z|^{\frac{3}{2}(N_Y-1)+l-1}} A_{Y,l}, \tag{3.76}
\end{aligned}$$

where the X and Z terms are combined into a single term, and the phase of the Y term does not exist. Compared to the unpolarized quark quasi PDF case in Eq. (3.44), the Y terms in the gluon PDF manifest the linear z enhancement as the remnant of the forward singularity. This is because the symmetric structure of the gluon quasi-correlator has the desired quantum numbers to preserve the forward singularity diagram in the large time limit, according to the discussions around Eqs. (3.40) and (3.41).

The next thing is to truncate the large distance expansion in Eq. (3.76) up to LA and NLA following the accuracy-counting principles in Sec. 3.2.4, which requires ranking the spectrum of $\{\bar{Q}_b F_{b,\mu}^t | H(P) \rangle, \bar{Q}_b F_{b,\mu}^t | \Omega \rangle\}$ and finding the lowest mass gap. However, it is challenging to rank this spectrum because it involves hybrids ($\bar{Q}Qg$), tetraquarks ($\bar{Q}Q\bar{q}q$), and pentaquarks ($\bar{Q}Qqqq$), which are exotic hadrons under investigation from both experimental and theoretical perspectives [390–396]. Moreover, the introduced auxiliary heavy-quark field could alter the original QCD spectrum, further complicating this problem.

Here, we propose the following conjecture regarding the spectrum of $\{\bar{Q}_b F_{b,\mu}^t | H(P) \rangle, \bar{Q}_b F_{b,\mu}^t | \Omega \rangle\}$ that the lightest hybrid ($\bar{Q}Qg$) and tetraquark ($\bar{Q}Q\bar{q}q$) share similar masses, which are smaller than the lightest pentaquark ($\bar{Q}Qqqq$). Based on this conjecture, we derive the LA and NLA.

For light baryon external states, $\bar{Q}_b F_{b,\mu}^t |H(P)\rangle$ corresponds to a heavy pentaquark ($\bar{Q}Qqqq$), which has higher energy than a tetraquark ($\bar{Q}Q\bar{q}q$) or hybrid ($\bar{Q}Qg$) generated by $\bar{Q}_b F_{b,\mu}^t |\Omega\rangle$. Therefore, the leading exponential decay comes from the Y term,

$$\tilde{h}^{\text{LA}}(z, P^z) = A|z|e^{-\Lambda|z|}, \quad (3.77)$$

$$\tilde{h}^{\text{NLA}}(z, P^z) = [A|z| + A']e^{-\Lambda|z|}, \quad (3.78)$$

where A , A' , and Λ are real parameters to be fitted with lattice data.

For light meson external states, $\bar{Q}_b F_{b,\mu}^t |H(P)\rangle$ corresponds to a tetraquark ($\bar{Q}Q\bar{q}q$), whose mass is similar to a hybrid ($\bar{Q}Qg$) produced by $\bar{Q}_b F_{b,\mu}^t |\Omega\rangle$. Therefore, all three types of terms could potentially contribute to the leading exponential decay. At LA, one only considers the Y term due to the enhancement of polynomial- z dependence, while at NLA, one should consider all three types,

$$\tilde{h}^{\text{LA}}(z, P^z) = A_2|z|e^{-\Lambda|z|}, \quad (3.79)$$

$$\tilde{h}^{\text{NLA}}(z, P^z) = [A_2|z| + A'_2 + 2A_1 \cos(\phi - zP^z)]e^{-\Lambda|z|}, \quad (3.80)$$

where the real parameters A_2 , Λ , A'_2 , A_1 , and ϕ can be fitted from lattice data.

For heavy baryon or meson external states, the LA and NLA forms should share the same mathematical structures as Eqs. (3.77) and (3.78), respectively.

There are two ways to examine the conjecture presented above. The first is to perform a de-

tailed analysis of the spectra of hybrids, tetraquarks, and pentaquarks in lattice QCD, considering the auxiliary field. The second is to validate the fit formulas directly using gluon quasi-PDF data.

3.4 Precision control of large distance asymptotic analysis

In this section, we apply the previously obtained large distance asymptotic forms as physical constraints on the Fourier transform (FT) of the lattice data to control its precision, a process called large distance asymptotic analysis. At present, the uncertainty from this asymptotic analysis is only a few percent in the intermediate y region, and slightly larger but still well controlled near the endpoints.

3.4.1 General discussions of asymptotic analysis

In this subsection, we discuss the general procedure and criteria for asymptotic analysis and derive upper error bounds for FT using analytical methods.

3.4.1.1 Procedure and criteria

The procedure of asymptotic analysis,

- Determine the large distance asymptotic forms $\tilde{h}_{\text{asym}}(z, P^z)$ up to certain accuracy (e.g. LA or NLA) according to the discussions in Sec. 3.2.4.
- Fit the asymptotic form $\tilde{h}_{\text{asym}}(z, P^z)$ with lattice data $\tilde{h}_{\text{dat}}(z, P^z)$ within $[z_{\min}, z_{\max}]$.
- Perform the Fourier transform (FT) using lattice data $\tilde{h}_{\text{dat}}(z, P^z)$ for $|z| < z_L$ and fitted asymp-

otic form $\tilde{h}_{\text{asym}}(z, P^z)$ for $|z| > z_L$:

$$\tilde{f}(y, P^z) = P^z \int_{|z| < z_L} \frac{dz}{2\pi} e^{izP^zy} \tilde{h}_{\text{dat}}(z, P^z) + P^z \int_{|z| > z_L} \frac{dz}{2\pi} e^{izP^zy} \tilde{h}_{\text{asym}}(z, P^z) . \quad (3.81)$$

General guidelines for choosing fit ranges, parameter settings, and truncation orders of asymptotic forms,

- For lattice data to be reliably described by the asymptotic expansion, $z_{\min}$ should be large enough compared to the mass gap Λ of exponential decay,

$$z_{\min} \gtrsim \frac{1}{\Lambda} . \quad (3.82)$$

This is confirmed by a simple example in ϕ^3 theory shown in Fig. A.7. For most light-quark bilinear quasi-correlators, Λ can be estimated as the binding energy of heavy-light meson $\sim 400 - 600$ MeV as discussed in Sec. 3.2.4, and the appropriate z for asymptotic analysis should be no less than ~ 0.5 fm.

- For $z_{\min} \sim 2/\Lambda$, $\sim 3/\Lambda$, or larger, the LA alone could provide a good description of the lattice data. As illustrated in Fig. A.7, the deviation between LA and the full result is only a few percent at $z \sim 2/m$. However, when $z_{\min} \sim 1/\Lambda$, the inclusion of NLA may be necessary to achieve comparable precision.
- The mass gaps Λ appearing in the asymptotic form can be loosely constrained around their physical expectations to improve the stability and robustness of the fits.
- The switching point z_L can be chosen between $z_{\min}$ and $z_{\max}$ to ensure a smooth transition from

lattice data to asymptotic form.

3.4.1.2 FT precision in the moderate y range

Following the above procedure and criteria, we derive the upper error bounds of FT from asymptotic analysis for the moderate y range, which is related to the uncertainty of asymptotic forms at z_L , that originates from the statistical uncertainty of the fit with data, and systematic uncertainty from the omitted higher order terms in the large distance expansion.

For the convenience of our discussions, we first consider the Y terms, that appear in Eq. (3.44) or other quasi correlators in Secs. 3.3.1, 3.3.2, and 3.3.4, up to arbitrary exponential and polynomial accuracy⁷,

$$\tilde{h}_Y(z) = \sum_{Y,l} \frac{e^{-\Lambda_Y|z|} A_{Y,l} e^{i \text{sign}(z) \phi_{Y,l}}}{|z|^{\frac{3}{2}(N_Y-1)+l}}. \quad (3.83)$$

The contribution from this extrapolated form to quasi-PDF is,

$$\begin{aligned} \tilde{f}_Y(y, P^z) &= P^z \int_{|z|>z_L} \frac{dz}{2\pi} e^{iyP^z z} \tilde{h}_Y(z) \\ &= \sum_{Y,l} \frac{A_{Y,l} P^z}{2\pi z_L^{\frac{3}{2}(N_Y-1)+l-1}} \left[e^{i\phi_{Y,l}} E_{\frac{3}{2}(N_Y-1)+l}(z_L(-iP^z y + \Lambda_Y)) \right. \\ &\quad \left. + e^{-i\phi_{Y,l}} E_{\frac{3}{2}(N_Y-1)+l}(z_L(iP^z y + \Lambda_Y)) \right], \end{aligned} \quad (3.84)$$

⁷For simplicity, the dependence on P^z in the function $\tilde{h}$ is suppressed in this and the following subsections.

where E denotes the exponential integral function. The large momentum limit gives,

$$\begin{aligned}\tilde{f}_Y(y, P^z)_{P^z \rightarrow \infty} &= \frac{-1}{\pi y} \sum_{Y,l} \frac{A_{Y,l} e^{-z_L \Lambda_Y}}{z_L^{\frac{3}{2}(N_Y-1)+l}} [\cos(\phi_{Y,l}) \sin(P^z y z_L) + \sin(\phi_{Y,l}) \cos(P^z y z_L)] \\ &= - \frac{\text{Re} [\tilde{h}_Y(z_L)] \sin(P^z y z_L) + \text{Im} [\tilde{h}_Y(z_L)] \cos(P^z y z_L)}{\pi y} .\end{aligned}\quad (3.85)$$

Part of the errors of $\tilde{f}_Y(y, P^z)_{P^z \rightarrow \infty}$ arise from the statistical uncertainties from the fit, denoted as $\delta^{\text{stat}} \text{Re} \tilde{h}_Y(z_L)$ and $\delta^{\text{stat}} \text{Im} \tilde{h}_Y(z_L)$,

$$\begin{aligned}\delta^{\text{stat}} \tilde{f}_Y(y, P^z)_{P^z \rightarrow \infty} &= \frac{1}{\pi |y|} \sqrt{v^T K v} < \frac{1}{\pi |y|} \sqrt{\text{Tr}[K]} \\ &= \frac{1}{\pi |y|} \sqrt{\left(\delta^{\text{stat}} \text{Re} \tilde{h}_Y(z_L)\right)^2 + \left(\delta^{\text{stat}} \text{Im} \tilde{h}_Y(z_L)\right)^2} ,\end{aligned}\quad (3.86)$$

where K is the covariance matrix of $\text{Re} [\tilde{h}_Y(z_L)]$ and $\text{Im} [\tilde{h}_Y(z_L)]$, and the vector $v = (\sin(P^z y z_L), \cos(P^z y z_L))$. The other part comes from the omitted higher order terms for truncating the large distance expansion up to certain accuracy, denoted as $\delta^{\text{sys}} \text{Re} \tilde{h}_Y(z_L)$ and $\delta^{\text{sys}} \text{Im} \tilde{h}_Y(z_L)$,

$$\begin{aligned}\delta^{\text{sys}} \tilde{f}_Y(y, P^z)_{P^z \rightarrow \infty} &= \frac{1}{\pi |y|} \left| (\delta^{\text{sys}} \text{Re} \tilde{h}_Y(z_L), \delta^{\text{sys}} \text{Im} \tilde{h}_Y(z_L)) \cdot (\sin(P^z y z_L), \cos(P^z y z_L)) \right| \\ &< \frac{1}{\pi |y|} \sqrt{\left(\delta^{\text{sys}} \text{Re} \tilde{h}_Y(z_L)\right)^2 + \left(\delta^{\text{sys}} \text{Im} \tilde{h}_Y(z_L)\right)^2} ,\end{aligned}\quad (3.87)$$

where the first line is written in the form of vector product. Since the statistical and systematic uncer-

tainties are irrelevant to each other, the total uncertainty is estimated as,

$$\begin{aligned} \delta \tilde{f}_Y(y, P^z)_{P^z \rightarrow \infty} &= \sqrt{\left(\delta^{\text{stat}} \tilde{f}_Y(y, P^z)_{P^z \rightarrow \infty}\right)^2 + \left(\delta^{\text{sys}} \tilde{f}_Y(y, P^z)_{P^z \rightarrow \infty}\right)^2} \\ &< \frac{1}{\pi|y|} \sqrt{\left(\delta^{\text{stat}} \text{Re} \tilde{h}_Y(z_L)\right)^2 + \left(\delta^{\text{stat}} \text{Im} \tilde{h}_Y(z_L)\right)^2 + \left(\delta^{\text{sys}} \text{Re} \tilde{h}_Y(z_L)\right)^2 + \left(\delta^{\text{sys}} \text{Im} \tilde{h}_Y(z_L)\right)^2}. \end{aligned} \quad (3.88)$$

In the above derivation, we utilize the upper bounds that $\sin^2(P^z y z_L), \cos^2(P^z y z_L) \leq 1$. If one keeps those triangular functions, one obtains the oscillation period in the error, $\Delta y = \pi/(P^z z_L)$.

Then, the above logic is generalized to all three types of terms in Eq. (3.44) or other quasi correlators,

$$\tilde{h}_{\text{asym}}(z) = \tilde{h}_X(z) + \tilde{h}_Y(z) + \tilde{h}_Z(z), \quad (3.89)$$

where

$$\tilde{h}_X(z) = \sum_{X,l=0} \frac{e^{-iP^z z} e^{-\Lambda_X |z|}}{|z|^{\frac{3}{2}(N_X-1)+l}} A_{X,l} e^{i \text{sign}(z) \phi_{X,l}}, \quad (3.90)$$

$$\tilde{h}_Z(z) = \sum_{Z,l=0} \frac{e^{iP^z z} e^{-\Lambda_Z |z|}}{|z|^{\frac{3}{2}(N_Z-1)+l}} A_{Z,l} e^{i \text{sign}(z) \phi_{Z,l}}, \quad (3.91)$$

whose FT in the large P^z limit is,

$$\begin{aligned}
\tilde{f}_{\text{asym}}(y, P^z)_{P^z \rightarrow \infty} &= P^z \int_{|z| > z_L} \frac{dz}{2\pi} e^{iyP^z z} \tilde{h}_{\text{asym}}(z) \Big|_{P^z \rightarrow \infty} \\
&= - \frac{\text{Re} [\tilde{h}_X(z_L)] \sin(P^z y z_L) + \text{Im} [\tilde{h}_X(z_L)] \cos(P^z y z_L)}{\pi(y-1)} \\
&\quad - \frac{\text{Re} [\tilde{h}_Y(z_L)] \sin(P^z y z_L) + \text{Im} [\tilde{h}_Y(z_L)] \cos(P^z y z_L)}{\pi y} \\
&\quad - \frac{\text{Re} [\tilde{h}_Z(z_L)] \sin(P^z y z_L) + \text{Im} [\tilde{h}_Z(z_L)] \cos(P^z y z_L)}{\pi(y+1)}, \tag{3.92}
\end{aligned}$$

which mainly contributes to different end-point regions in momentum space. To estimate the error of $\tilde{f}_{\text{asym}}$, one can first evaluate the contribution from each term individually following similar logics as Eqs. (3.86) and (3.87),

$$\delta^{\text{stat}} \tilde{f}_X(y, P^z)_{P^z \rightarrow \infty} < \frac{1}{\pi|y-1|} \sqrt{\left(\delta^{\text{stat}} \text{Re} \tilde{h}_X(z_L)\right)^2 + \left(\delta^{\text{stat}} \text{Im} \tilde{h}_X(z_L)\right)^2}, \tag{3.93}$$

$$\delta^{\text{stat}} \tilde{f}_Z(y, P^z)_{P^z \rightarrow \infty} < \frac{1}{\pi|y+1|} \sqrt{\left(\delta^{\text{stat}} \text{Re} \tilde{h}_Z(z_L)\right)^2 + \left(\delta^{\text{stat}} \text{Im} \tilde{h}_Z(z_L)\right)^2}, \tag{3.94}$$

$$\delta^{\text{sys}} \tilde{f}_X(y, P^z)_{P^z \rightarrow \infty} < \frac{1}{\pi|y-1|} \sqrt{\left(\delta^{\text{sys}} \text{Re} \tilde{h}_X(z_L)\right)^2 + \left(\delta^{\text{sys}} \text{Im} \tilde{h}_X(z_L)\right)^2}, \tag{3.95}$$

$$\delta^{\text{sys}} \tilde{f}_Z(y, P^z)_{P^z \rightarrow \infty} < \frac{1}{\pi|y+1|} \sqrt{\left(\delta^{\text{sys}} \text{Re} \tilde{h}_Z(z_L)\right)^2 + \left(\delta^{\text{sys}} \text{Im} \tilde{h}_Z(z_L)\right)^2}, \tag{3.96}$$

which share similar mathematical structures as Eqs. (3.86) and (3.87) except for the overall y -dependent

factors. Then, the most conservative error bounds for total statistical and systematic uncertainties are estimated as,

$$\delta^{\text{stat}} \tilde{f}_{\text{asym}}(y, P^z)_{P^z \rightarrow \infty} < \delta^{\text{stat}} \tilde{f}_X(y, P^z)_{P^z \rightarrow \infty} + \delta^{\text{stat}} \tilde{f}_Y(y, P^z)_{P^z \rightarrow \infty} + \delta^{\text{stat}} \tilde{f}_Z(y, P^z)_{P^z \rightarrow \infty} , \quad (3.97)$$

$$\delta^{\text{sys}} \tilde{f}_{\text{asym}}(y, P^z)_{P^z \rightarrow \infty} < \delta^{\text{sys}} \tilde{f}_X(y, P^z)_{P^z \rightarrow \infty} + \delta^{\text{sys}} \tilde{f}_Y(y, P^z)_{P^z \rightarrow \infty} + \delta^{\text{sys}} \tilde{f}_Z(y, P^z)_{P^z \rightarrow \infty} , \quad (3.98)$$

and the total uncertainty is,

$$\delta \tilde{f}_{\text{asym}}(y, P^z)_{P^z \rightarrow \infty} < \sqrt{\left(\delta^{\text{stat}} \tilde{f}_{\text{asym}}(y, P^z)_{P^z \rightarrow \infty} \right)^2 + \left(\delta^{\text{sys}} \tilde{f}_{\text{asym}}(y, P^z)_{P^z \rightarrow \infty} \right)^2} , \quad (3.99)$$

which treats statistical and systematic uncertainties as independent error budgets.

Therefore, the FT precision from asymptotic analysis is completely under control in the moderate y range, which depends on the error of the asymptotic form at z_L , that is further decomposed into the statistical uncertainty from the fit with lattice data, and systematic uncertainty from the omitted higher order terms in the large distance expansion. As will be shown in most numerical tests with current lattice data using $z_L \sim 0.8 - 1$ fm, the leading asymptotics mainly comes from $\tilde{h}_Y(z_L)$ ⁸, whose statistical uncertainty is

$$\delta^{\text{stat}} \text{Re} \tilde{h}_Y(z_L) \sim \delta^{\text{stat}} \text{Im} \tilde{h}_Y(z_L) \sim 0.05 , \quad (3.100)$$

⁸This is the case for most quasi-correlators except for pion DA where both X and Y terms contribute the similar amounts to the leading asymptotics. However, the uncertainty in pion DA at moderate y is still a few percent.

and the systematic uncertainties from the omitted higher order terms scale as,

$$\begin{aligned} \delta^{\text{sys}} \text{Re} \tilde{h}_X(z_L) &\sim \delta^{\text{sys}} \text{Im} \tilde{h}_X(z_L) \sim \delta^{\text{sys}} \text{Re} \tilde{h}_Y(z_L) \sim \delta^{\text{sys}} \text{Im} \tilde{h}_Y(z_L) \\ &\sim \delta^{\text{sys}} \text{Re} \tilde{h}_Z(z_L) \sim \delta^{\text{sys}} \text{Im} \tilde{h}_Z(z_L) \sim 0.01 . \end{aligned} \quad (3.101)$$

The error budget can be estimated using Eq. (3.99),

$$\delta \tilde{f}_{\text{asym}}(y, P^z)_{P^z \rightarrow \infty} < \sqrt{\frac{0.05^2 + 0.05^2}{(\pi y)^2} + \left(\frac{\sqrt{2} \times 0.01}{\pi |y - 1|} + \frac{\sqrt{2} \times 0.01}{\pi |y|} + \frac{\sqrt{2} \times 0.01}{\pi |y + 1|} \right)^2}, \quad (3.102)$$

which gives $\delta \tilde{f}_{\text{asym}}(y = 0.5, P^z)_{P^z \rightarrow \infty} < 0.05$, which is just a few percent compared to most quasi-PDFs at $O(1)$ for $y = 0.5$. This precision can be improved in the near future with recently proposed techniques [44, 218, 279].

3.4.1.3 FT precision in the end point regions

The error estimate in Eq. (3.99) does not apply to the end-point regions, such as $y \rightarrow -1, 0$, or 1 , because the large momentum limit $P^z \rightarrow \infty$ does not commute with the end-point limits $y \rightarrow -1, 0$, or 1 . In this subsection, we derive the FT error bounds for the end-point regions, which are slightly larger than the moderate y range but still under control.

Keep only the leading exponential decay for each term for simplicity,

$$\tilde{h}_{\text{asym}}(z) = \tilde{h}_X(z) + \tilde{h}_Y(z) + \tilde{h}_Z(z), \quad (3.103)$$

where

$$\tilde{h}_X(z) = \sum_{l=0} \frac{e^{-iP^z z} e^{-\Lambda_X |z|}}{|z|^{\frac{3}{2}(N_X-1)+l}} A_{X,l} e^{i \text{sign}(z) \phi_{X,l}} , \quad (3.104)$$

$$\tilde{h}_Y(z) = \sum_l \frac{e^{-\Lambda_Y |z|} A_{Y,l} e^{i \text{sign}(z) \phi_{Y,l}}}{|z|^{\frac{3}{2}(N_Y-1)+l}} , \quad (3.105)$$

$$\tilde{h}_Z(z) = \sum_{l=0} \frac{e^{iP^z z} e^{-\Lambda_Z |z|}}{|z|^{\frac{3}{2}(N_Z-1)+l}} A_{Z,l} e^{i \text{sign}(z) \phi_{Z,l}} . \quad (3.106)$$

Its contribution to Fourier transformation is,

$$\tilde{f}_{\text{asym}}(y, P^z) = P^z \int_{|z| > z_L} \frac{dz}{2\pi} e^{iyP^z z} \tilde{h}_{\text{asym}}(z) . \quad (3.107)$$

First take the end-point limits $y \rightarrow -1$ or 0 or 1 , and then perform the large z_L and P^z expansions,

$$\tilde{f}_{\text{asym}}(1, P^z) = \frac{\text{Re} \left[e^{iz_L P^z} \tilde{h}_X(z_L) \right]}{\pi} \frac{P^z}{\Lambda_X} \left(1 + O \left(\frac{1}{z_L}, \frac{1}{P^z} \right) \right) , \quad (3.108)$$

$$\tilde{f}_{\text{asym}}(0, P^z) = \frac{\text{Re} \left[\tilde{h}_Y(z_L) \right]}{\pi} \frac{P^z}{\Lambda_Y} \left(1 + O \left(\frac{1}{z_L}, \frac{1}{P^z} \right) \right) , \quad (3.109)$$

$$\tilde{f}_{\text{asym}}(-1, P^z) = \frac{\text{Re} \left[e^{-iz_L P^z} \tilde{h}_Z(z_L) \right]}{\pi} \frac{P^z}{\Lambda_Z} \left(1 + O \left(\frac{1}{z_L}, \frac{1}{P^z} \right) \right) . \quad (3.110)$$

Therefore, the asymptotic uncertainties at the end-point regions are⁹,

$$\begin{aligned} \delta \tilde{f}_{\text{asym}}(1, P^z) \lesssim & \frac{P^z}{\Lambda_X \pi} \left[\left(\delta^{\text{sys}} \text{Re} \left[e^{iz_L P^z} \tilde{h}_X(z_L) \right] \right)^2 + \left(\delta^{\text{stat}} \text{Re} \left[e^{iz_L P^z} \tilde{h}_X(z_L) \right] \right)^2 \right. \\ & \left. + \left(\text{Re} \left[e^{iz_L P^z} \tilde{h}_X(z_L) \right] \frac{\delta \Lambda_X}{\Lambda_X} \right)^2 \right]^{1/2}, \end{aligned} \quad (3.111)$$

$$\delta \tilde{f}_{\text{asym}}(0, P^z) \lesssim \frac{P^z}{\Lambda_Y \pi} \sqrt{\left(\delta^{\text{sys}} \text{Re} \tilde{h}_Y(z_L) \right)^2 + \left(\delta^{\text{stat}} \text{Re} \tilde{h}_Y(z_L) \right)^2 + \left(\text{Re} \left[\tilde{h}_Y(z_L) \right] \frac{\delta \Lambda_Y}{\Lambda_Y} \right)^2}, \quad (3.112)$$

$$\begin{aligned} \delta \tilde{f}_{\text{asym}}(-1, P^z) \lesssim & \frac{P^z}{\Lambda_Z \pi} \left[\left(\delta^{\text{sys}} \text{Re} \left[e^{-iz_L P^z} \tilde{h}_Z(z_L) \right] \right)^2 + \left(\delta^{\text{stat}} \text{Re} \left[e^{-iz_L P^z} \tilde{h}_Z(z_L) \right] \right)^2 \right. \\ & \left. + \left(\text{Re} \left[e^{-iz_L P^z} \tilde{h}_Z(z_L) \right] \frac{\delta \Lambda_Z}{\Lambda_Z} \right)^2 \right]^{1/2}. \end{aligned} \quad (3.113)$$

Compared to the errors in the moderate y range in Eq. (3.99), the errors in the end-point regions are enhanced by $O(P^z/2\Lambda)$ but still finite and under control. At current lattice status for $P^z \sim 2 \text{ GeV}$ and $\Lambda \sim 0.5 \text{ GeV}$, this enhancement factor is ~ 2 .¹⁰

3.4.2 Applications to realistic lattice data

In this subsection, we apply the asymptotic analysis discussed in Sec. 3.4.1.1 to realistic lattice data to demonstrate controlled precision using numerical methods. In all the tested cases, the uncertainty from asymptotic analysis is just a few percent in the moderate y range.

⁹The symbol “ $\lesssim$ ” means the power corrections in the large z_L and P^z expansions are omitted in the error estimate.

¹⁰In most cases tested later, the major uncertainty arises from the Y term, corresponding to the endpoint uncertainty near $y \sim 0$. In the matching process, the propagation of the $y \sim 0$ uncertainty to the moderate x range is negligible according to the discussions in Ref. [133], which means the moderate x -range targeted in LaMET analysis is almost not contaminated by the slightly larger uncertainty near $y \sim 0$.

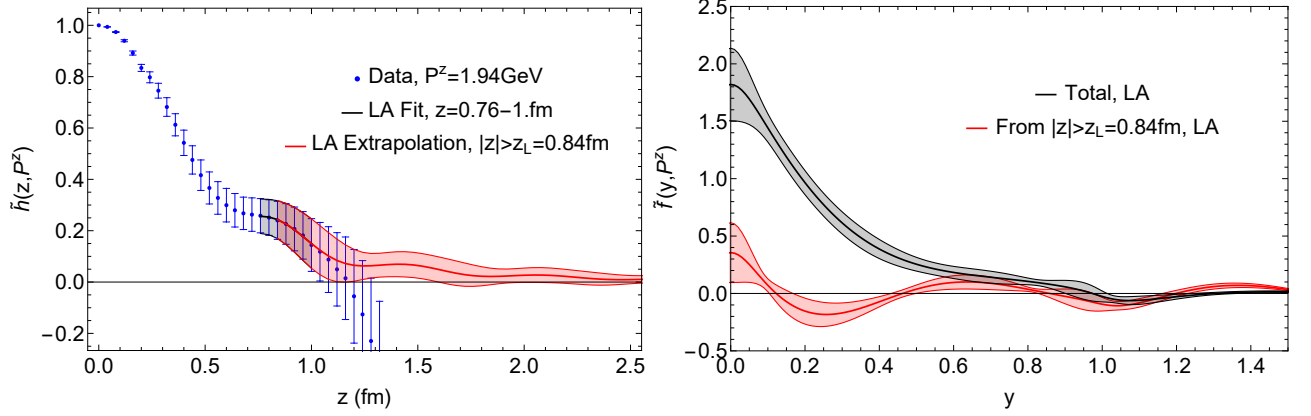


Figure 3.7: The leading asymptotic (LA) analysis for pion valence PDF. The left panel shows the renormalized matrix element in coordinate space, where the blue points are renormalized lattice data, the black band covers the fit range, and the red band denotes the extrapolated result. The right panel displays the quasi-PDF in momentum space obtained from FT, where the red band only includes the contribution from the extrapolated form, and the black band denotes the total result. The bands only include the statistical uncertainty.

3.4.2.1 Pion valence PDF

We conduct numerical studies of the large distance extrapolation and Fourier transformation on lattice QCD matrix elements for the valence quark distribution of a pion at lattice spacing $a = 0.04$ fm and momentum $P^z = 1.94$ GeV. These matrix elements have been calculated by the BNL/ANL collaboration and analyzed in the papers [15–17, 22, 105, 115, 117, 232, 397]. The renormalization of the lattice data is performed using the hybrid scheme [287] with the PV prescription following the method in Ref. [232], shown as blue data points in the left panel of Fig. 3.7. The signal-to-noise ratio becomes worse for larger distances, which forbids the direct Fourier transformation of the data. To control the FT precision, we perform the large distance asymptotic analysis following Sec. 3.4.1.1.

According to the discussions in Sec. 3.2.4, the leading asymptotics (LA) is,

$$\tilde{h}^{\text{LA}}(z, P^z) = [A_2 + 2A_1 \cos(\phi - zP^z)] e^{-\Lambda|z|}. \quad (3.114)$$

We fit it with lattice data. As discussed in Sec. 3.2.4, the mass gap is estimated as $\Lambda \sim 0.5 \text{ GeV}$, corresponding to the binding energy of a heavy-light meson under the PV prescription. We choose the fit range as $z = 0.76 - 1 \text{ fm}$, where the lower bound is roughly $2/\Lambda$, making it suitable for LA analysis. The mass gap Λ is treated as a fit parameter, loosely constrained as $\Lambda \in [0.2, 0.8] \text{ GeV}$ to ensure the stability of the fit. The quantities A_2 , A_1 , and ϕ are taken as free parameters in the fit. The fitted result is shown as the black band in the left panel of Fig. 3.7, which agrees well with lattice data, with $\chi^2/\text{dof} = 1.46$. The fitted values are $\Lambda = 0.48(25) \text{ GeV}$, $\tilde{h}_Y^{\text{LA}}(z_L, P^z) = 0.20(09)$ and $\tilde{h}_X^{\text{LA}}(z_L, P^z) = \tilde{h}_Z^{\text{LA}}(z_L, P^z) = 0.023(17)$.

We then perform the FT, with the results shown in the right panel of Fig. 3.7. The black curve represents the total result given in Eq. (4.10), while the red curve shows the contribution arising solely from the region $|z| > z_L$ using the fitted LA. In the moderate y range, the contribution from the extrapolated form takes a small amount of the total result; for example, -0.07 compared to 0.39 at $y = 0.4$, and 0.04 compared to 0.25 at $y = 0.5$.

The red curve in the left panel of Fig. 3.9 represents the momentum-space statistical uncertainty propagated from the LA¹¹, which is the uncertainty of the red band in the right panel of Fig. 3.7. This uncertainty is slightly larger near $y \sim 0$, but well under control in the moderate y range (e.g., $\delta^{\text{stat}} \tilde{f}_{\text{asym}}(y = 0.5) = 0.036$). The curve exhibits local minima that are spaced by $\Delta y \sim 0.36$.

The above error estimates from numerical methods are consistent with the theoretical discussions

¹¹Since the LA parameters are fitted from lattice data, the statistical uncertainty from the data is propagated to the LA, which is then transmitted to the momentum-space quasi-PDF after FT.

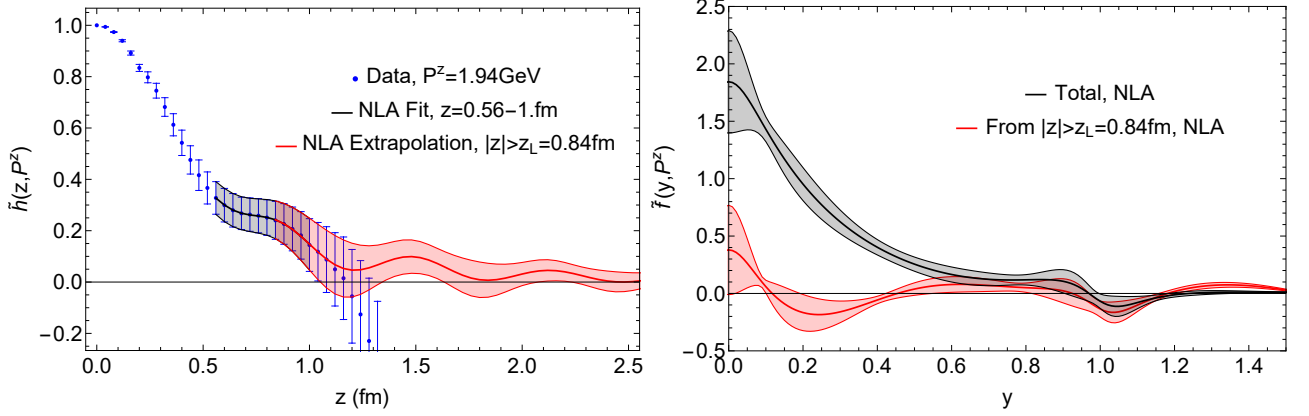


Figure 3.8: Similar to Fig. 3.7 except for next-to-leading asymptotic (NLA) analysis.

in Secs. 3.4.1.2 and 3.4.1.3. The analytical upper error bound derived in Eq. (3.97) is¹²,

$$\begin{aligned} \delta^{\text{stat}} \tilde{f}_{\text{asym}}(y, P^z) &\lesssim \frac{\delta^{\text{stat}} \tilde{h}_Y^{\text{LA}}(z_L, P^z)}{\pi|y|} + \frac{\delta^{\text{stat}} \tilde{h}_X^{\text{LA}}(z_L, P^z)}{\pi|y-1|} + \frac{\delta^{\text{stat}} \tilde{h}_Z^{\text{LA}}(z_L, P^z)}{\pi|y+1|} \\ &= \frac{0.09}{\pi|y|} + \frac{0.017}{\pi|y-1|} + \frac{0.017}{\pi|y+1|}, \end{aligned} \quad (3.115)$$

which gives $\delta^{\text{stat}} \tilde{f}_{\text{asym}}(y = 0.5) \lesssim 0.072$ compatible with the numerical calculation. The theoretical derivation predicts an increased error as $y \rightarrow 0$, which is consistent with the trend in the numerical result displayed as the red curve in the left panel of Fig. 3.9. The oscillation period from theoretical derivation in Eq. (3.85) is $\Delta y = \pi/(P^z z_L) \sim 0.38$, close to the numerical result.

To estimate the systematic uncertainty from the omitted higher order terms in the truncated large distance expansion, we fit lattice data with the following NLA form discussed in Sec. 3.2.4,

$$\tilde{h}^{\text{NLA}}(z, P^z) = \left[A_2 + \frac{A'_2}{|z|} + 2A_1 \cos(\phi - zP^z) + \frac{2A'_1 \cos(\phi' - zP^z)}{|z|} \right] e^{-\Lambda|z|}. \quad (3.116)$$

The fit range is chosen as $[0.56, 1]$ fm, where the lower bound is slightly smaller than the LA case but

¹²The symbol “ $\lesssim$ ” means the omitted power corrections in the large P^z expansion could slightly influence the error estimate (e.g. $\sim 10\%$ of the total error in the moderate y range for the tested pion valence PDF data). Due to their small impacts, they have not been discussed in detail for simplicity.

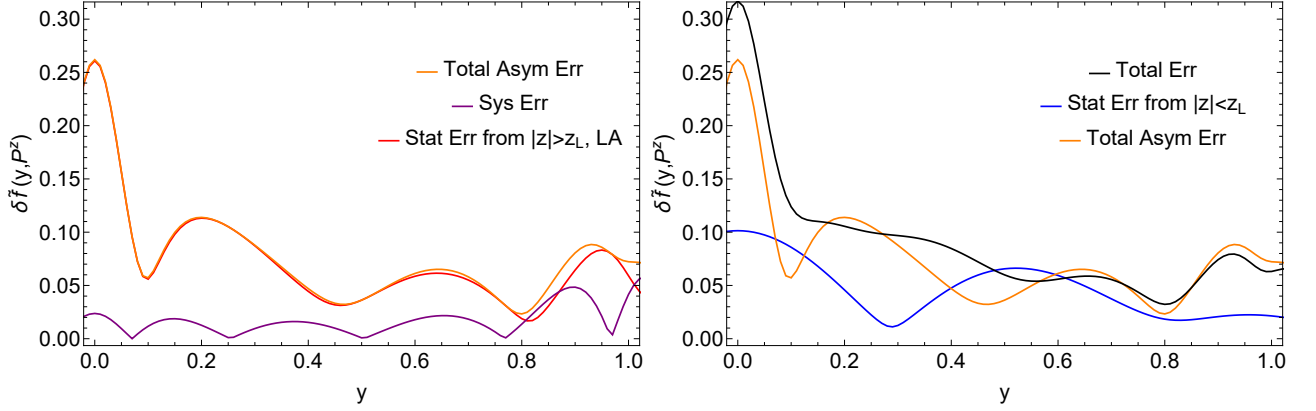


Figure 3.9: Error budgets of FT for pion valence quasi-PDF calculated using numerical methods. The left panel displays the uncertainties from the large distance asymptotic analysis, corresponding to $|z| > z_L$. The red curve originates from the statistical uncertainty of LA, which comes from the fit to lattice data. The purple curve represents the systematic uncertainty from the omitted higher-order terms in the truncated large distance expansion, estimated as the difference between LA and NLA results. The right panel contains all the sources of FT uncertainties, where the orange curve is the total uncertainty from asymptotic analysis, and the blue curve denotes the statistical uncertainty propagated from the data in the region $|z| < z_L$. The black curve is the total FT uncertainty.

larger than $\sim 1/\Lambda$. The constraints $|A'_2|/z_L < |A_2|$ and $|A'_1|/z_L < |A_1|$ are set during the fit because the NLA correction should be smaller than LA. As shown in the left panel of Fig. 3.8, the fitted results agree well with data. The right panel of Fig. 3.8 displays the quasi-PDF in momentum space. The difference between the LA and NLA results in momentum space is considered as the systematic uncertainty, shown as the purple curve in the left panel of Fig. 3.9, where the value at $y = 0.4$ is just 0.015. That means the omitted higher-order terms are negligible in our calculations.

On the other hand, the upper bound of systematic uncertainty can be estimated using the theoretically-derived error bound in Eq. (3.98). The omitted higher-order terms in LA can be estimated using the fit

parameters of NLA,

$$\begin{aligned}\delta^{\text{sys}}\text{Re}\tilde{h}_X(z_L) &= \delta^{\text{sys}}\text{Re}\tilde{h}_Z(z_L) = \frac{|A'_1 \cos[\phi' - z_L P^z]|}{z_L} e^{-\Lambda z_L} = 0.04(5) , \\ \delta^{\text{sys}}\text{Re}\tilde{h}_Y(z_L) &= \frac{|A'_2|}{z_L} e^{-\Lambda z_L} = 0.05(16) .\end{aligned}\tag{3.117}$$

Plug the above values into Eq. (3.98),

$$\delta^{\text{sys}}\tilde{f}_{\text{asym}}(y, P^z) \lesssim \frac{0.05}{\pi|y|} + \frac{0.04}{\pi|y-1|} + \frac{0.04}{\pi|y+1|} ,\tag{3.118}$$

which gives $\delta^{\text{sys}}\tilde{f}_{\text{asym}}(y = 0.4, P^z) \lesssim 0.07$, which is a very loose bound compared to the error estimate using numerical methods.

We compare various origins of FT uncertainties in Fig. 3.9, calculated using numerical methods. The left panel shows the errors from asymptotic analysis, which are well under control in the moderate y range. The systematic uncertainty is much smaller compared to the statistical one. The right panel compares asymptotic-analysis uncertainty from $|z| > z_L$ and statistical uncertainty from $|z| < z_L$. Near $y \sim 0$, asymptotic uncertainty makes a dominant contribution to the total error. In the moderate y range, the two error budgets are similarly small in our test, and the total error is well controlled.

3.4.2.2 Proton quark unpolarized PDF

The proton quark unpolarized quasi-PDF matrix elements are generated by Lattice Parton Collaboration (LPC) and have been used to study the ‘‘EMC effect’’ of a deuteronlike dibaryon system [50]. We choose the isovector combination with $P^z = 1.85$ GeV generated on C32P29 ensemble [398, 399]. The data are renormalized under hybrid scheme [287] and converted to the ‘‘PV-prescription’’ for mass

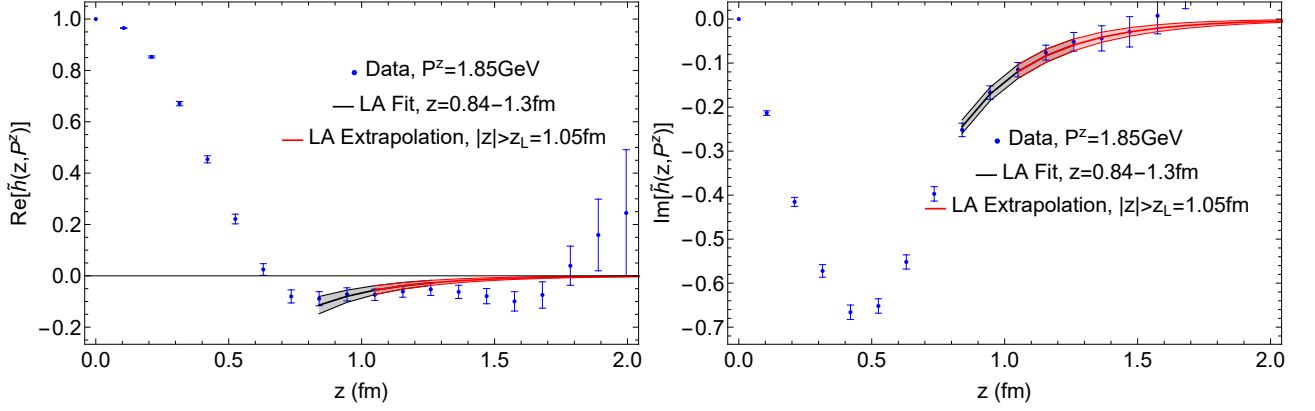


Figure 3.10: The leading asymptotic (LA) analysis for unpolarized nucleon quasi-PDF matrix element. The left panel shows the real part of the renormalized matrix element in coordinate space, where the blue points are renormalized lattice data, the black band covers the fit range, and the red band denotes the extrapolated result. The right panel displays the imaginary part.

renormalon regularization [232]. Using these data, we demonstrate the controlled precision of FT following the asymptotic analysis in Sec. 3.4.1.1.

The LA and NLA fit formulas for the isovector quark quasi-PDF of an unpolarized nucleon are discussed in Sec. 3.2.4,

$$\tilde{h}^{\text{LA}}(z, P^z) = A e^{i\phi \text{sign}(z)} e^{-\Lambda|z|}, \quad (3.119)$$

$$\tilde{h}^{\text{NLA}}(z, P^z) = \left[A e^{i\phi \text{sign}(z)} + \frac{A' e^{i\phi' \text{sign}(z)}}{|z|} \right] e^{-\Lambda|z|}. \quad (3.120)$$

The fit parameters A , ϕ , Λ , A' and ϕ' are real numbers. Compared to the Y terms in the pion valence PDF case, they contain a phase dependent on the sign of z .

Figs. 3.10 and 3.11 display the fits with lattice data, where the mass gap Λ is loosely bounded within [0.01, 0.99] GeV to improve the stability, and the constraint $|A'| < |A|z_L/2$ is set in the NLA

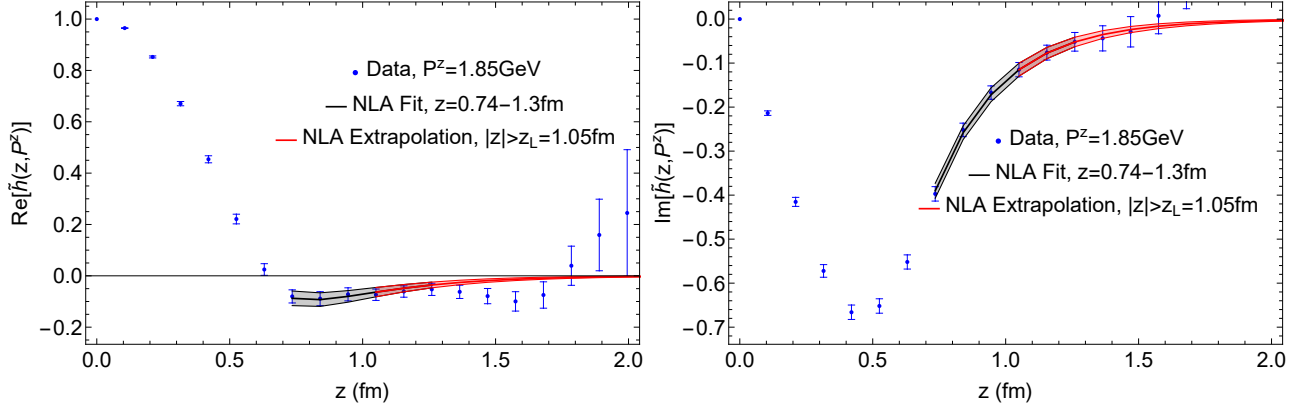


Figure 3.11: Similar to Fig. 3.10 except for the next-to-leading asymptotic (NLA) analysis.

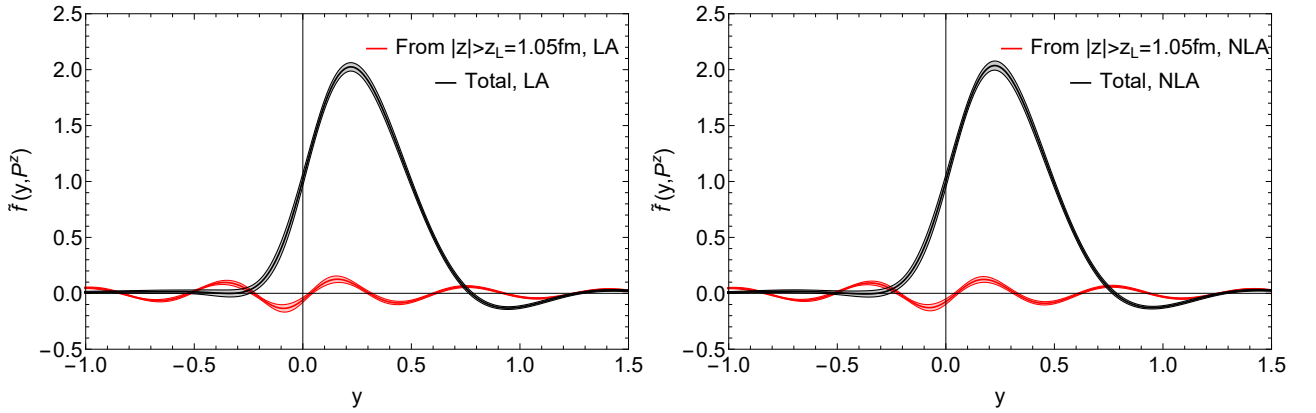


Figure 3.12: The unpolarized nucleon quasi-PDF in momentum space obtained from FT, where the black curve denotes the total result in Eq. (4.10), and the red curve only includes the contribution from the asymptotic form. The bands only contain the statistical uncertainty. The left panel comes from LA analysis in Fig. 3.10 and the right panel is for NLA analysis in Fig. 3.11.

case required by the asymptotic expansion. Both LA and NLA forms agree well with data.

Fig. 3.12 shows the unpolarized nucleon quasi-PDF in momentum space obtained from FT. The contributions from asymptotic analysis only take a small amount of the total results (e.g. -0.072 compared to 0.985 at $y = 0.5$).

We compare various sources of FT uncertainties in Fig. 3.13. The left panel displays the uncertainties from asymptotic analysis, which is slightly larger near $y \sim 0$ (e.g. 0.026 at $y = 0$) and well under control in the moderate y range (e.g. 0.014 at $y = 0.5$). The error size is much smaller than

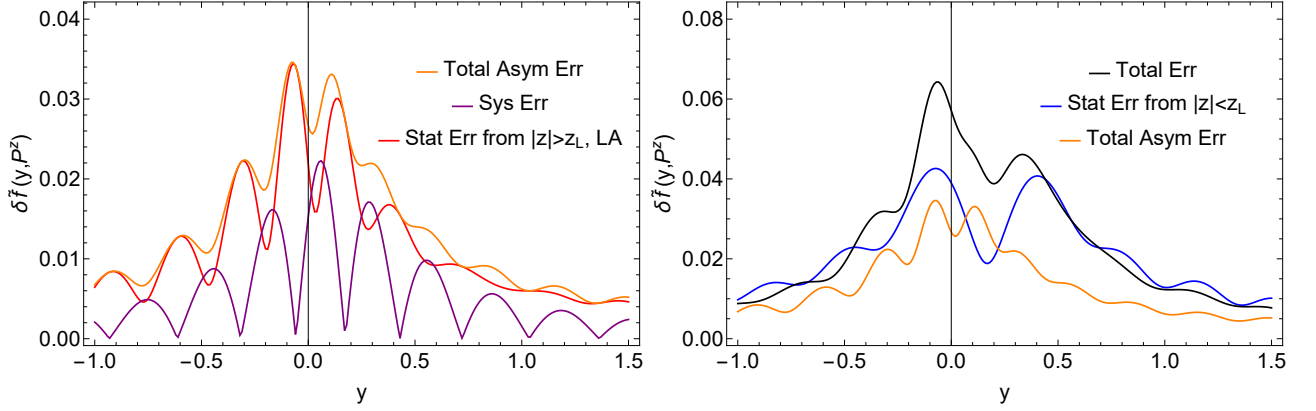


Figure 3.13: Error budgets of FT for unpolarized proton quasi-PDF calculated using numerical methods. The left panel displays the uncertainties from the large distance asymptotic analysis, corresponding to $|z| > z_L$. The red curve originates from the statistical uncertainty of LA, which comes from the fit to lattice data. The purple curve represents the systematic uncertainty from the omitted higher-order terms in the truncated large distance expansion, estimated as the difference between LA and NLA results. The right panel contains all the sources of FT uncertainties, where the orange curve is the total uncertainty from asymptotic analysis, and the blue curve denotes the statistical uncertainty propagated from the data in the region $|z| < z_L$. The black curve is the total FT uncertainty.

the pion valence PDF case because the data here are more precise at large distance, providing smaller statistical error in the asymptotic analysis. The right panel indicates that at most y the asymptotic error from $|z| > z_L$ is smaller than the statistical error from $|z| < z_L$, though both of them are just a few percent compared to the center values of PDF.

3.4.2.3 Proton quark transversity PDF

The proton transversity quasi-PDF matrix elements are calculated by Lattice Parton Collaboration (LPC) and presented in Ref. [12] with a careful study of continuum, physical pion mass and infinite momentum extrapolations. We take $P^z = 1.82$ GeV generated on CLS H102 ensemble [318] as an example to demonstrate the large distance asymptotic analysis following Sec. 3.4.1.1.

The asymptotic expansions of proton quark transversity PDF have been derived in Sec. 3.3.1,

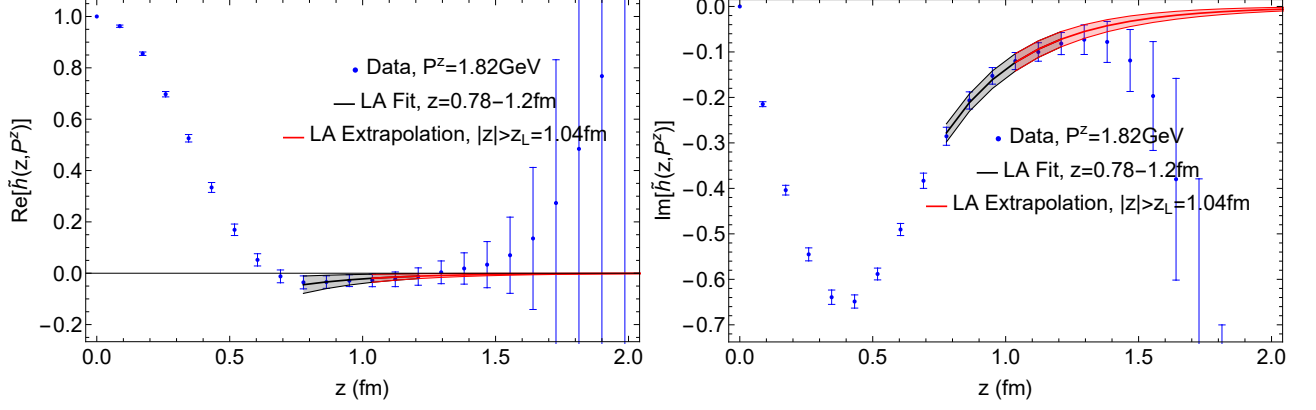


Figure 3.14: Similar to Fig. 3.10 except for nucleon transversity quasi-PDF matrix element.

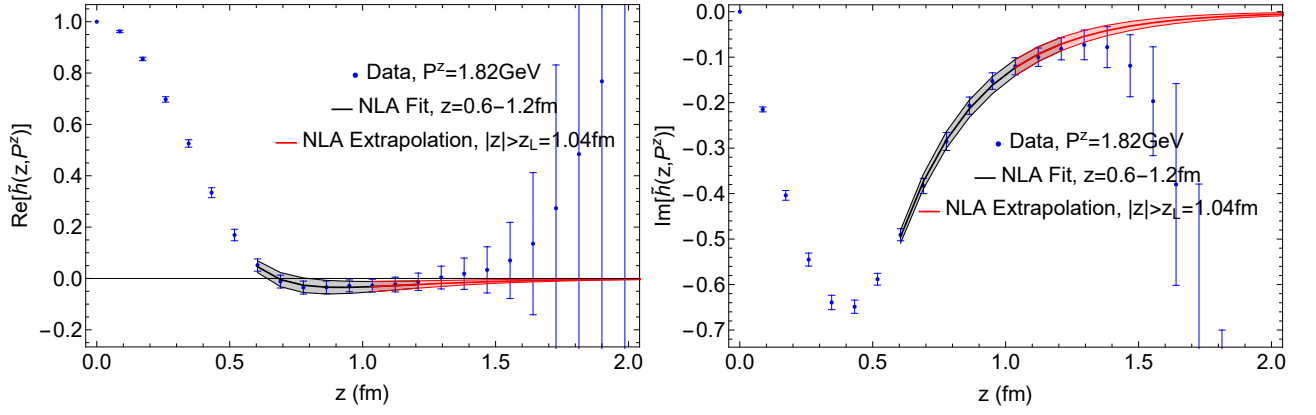


Figure 3.15: Similar to Fig. 3.14 except for the next-to-leading asymptotic (NLA) analysis.

where the LA and NLA fit formulas are,

$$\tilde{h}^{\text{LA}}(z, P^z) = A e^{i\phi \text{sign}(z)} e^{-\Lambda|z|}, \quad (3.121)$$

$$\tilde{h}^{\text{NLA}}(z, P^z) = \left[A e^{i\phi \text{sign}(z)} + \frac{A' e^{i\phi' \text{sign}(z)}}{|z|} \right] e^{-\Lambda|z|}. \quad (3.122)$$

Similar to the unpolarized proton PDF case, the LA does not contain the polynomial- z enhancement because the quantum numbers of the operators forbid the forward singularity.

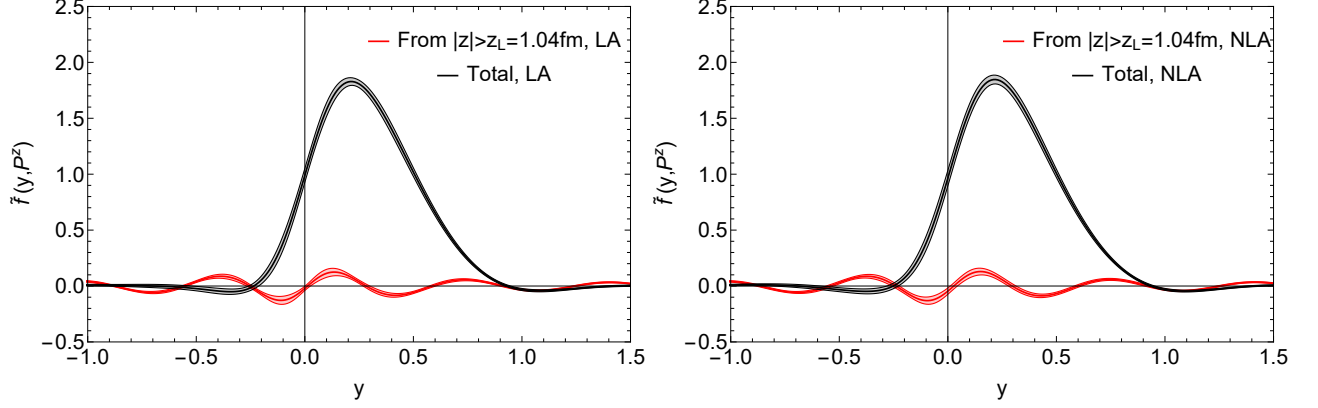


Figure 3.16: Similar to Fig. 3.12 except for the nucleon transversity quasi-PDF.

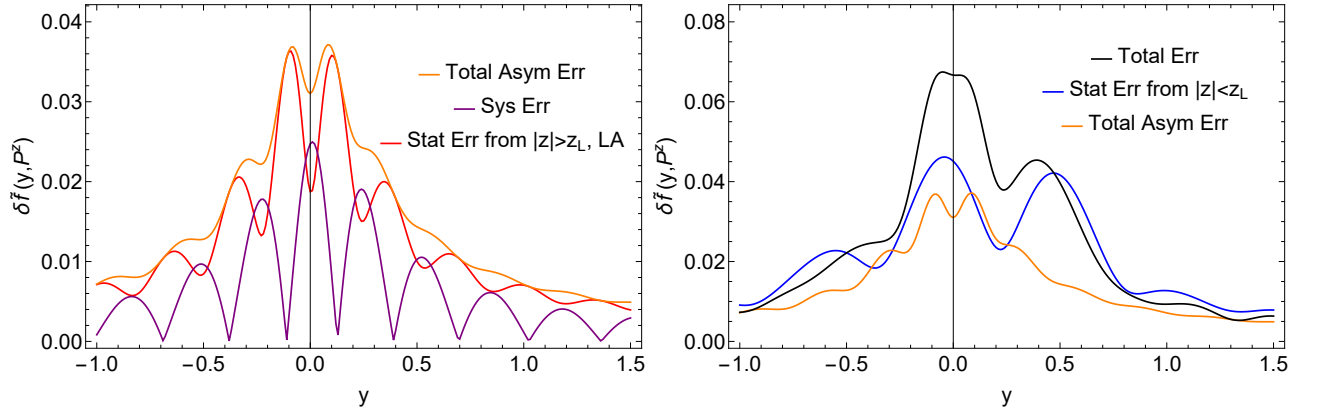


Figure 3.17: Similar to Fig. 3.13 except for nucleon transversity quasi-PDF.

Following the procedure and criteria in Sec. 3.4.1.1, we fit the asymptotic forms with lattice data shown in Figs. 3.14 and 3.15. The mass gap Λ is loosely bounded within $[0.01, 0.99]$ GeV to improve the stability, and the constraint $|A'|/z_L < |A|/2$ is set in the NLA case. The fitted results agree well with lattice data. Particularly, the NLA fit can even describe the non-monotonic behavior of lattice data.

Fig. 3.16 shows the momentum space quasi-PDF obtained through FT. Again, the large distance asymptotic form only contributes a small percent to the total result.

Fig. 3.17 includes various FT uncertainties. The asymptotic analysis has a controlled precision with a few percent uncertainty compared to quasi PDF at $O(1)$ (e.g. 0.015 compared to 1.0 at $y = 0.5$).

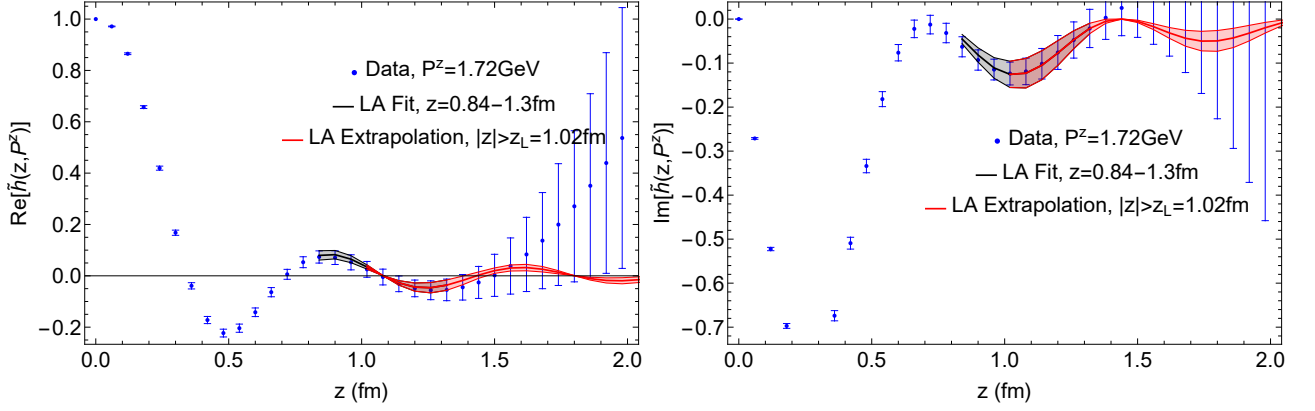


Figure 3.18: Similar to Fig. 3.10 except for pion quasi-DA matrix element.

3.4.2.4 Pion DA

The pion quasi-DA matrix elements are generated by LPC and studied in Ref. [23] with hybrid renormalization, continuum extrapolation and infinite momentum limit. We choose $P^z = 1.72$ GeV on a09m130 ensemble [325, 326] to study the asymptotic analysis in Sec. 3.4.1.1.

Sec. 3.3.2 shows the large distance asymptotic forms for the meson quasi-DA matrix elements. Constrained by the isospin symmetry of pion quasi-DA, the LA and NLA asymptotic forms are,

$$\tilde{h}^{\text{LA}}(z, P^z) = [Ae^{i\phi \text{sign}(z)}e^{-izP^z} + Ae^{-i\phi \text{sign}(z)}]e^{-\Lambda|z|}, \quad (3.123)$$

$$\begin{aligned} \tilde{h}^{\text{NLA}}(z, P^z) = & [(Ae^{i\phi \text{sign}(z)}e^{-izP^z} + Ae^{-i\phi \text{sign}(z)}) \\ & + (A'e^{i\phi' \text{sign}(z)}e^{-izP^z} + A'e^{-i\phi' \text{sign}(z)}) \frac{1}{|z|}] e^{-\Lambda|z|}. \end{aligned} \quad (3.124)$$

These fit formulas exhibit stronger oscillations than the nucleon quasi-PDF cases.

In Figs. 3.18 and 3.19, we fit the asymptotic forms with lattice data. The mass gap Λ is loosely bounded within [0.2, 0.8] GeV to improve the stability, and the constraint $|A'|/z_L < |A|/2$ is set in

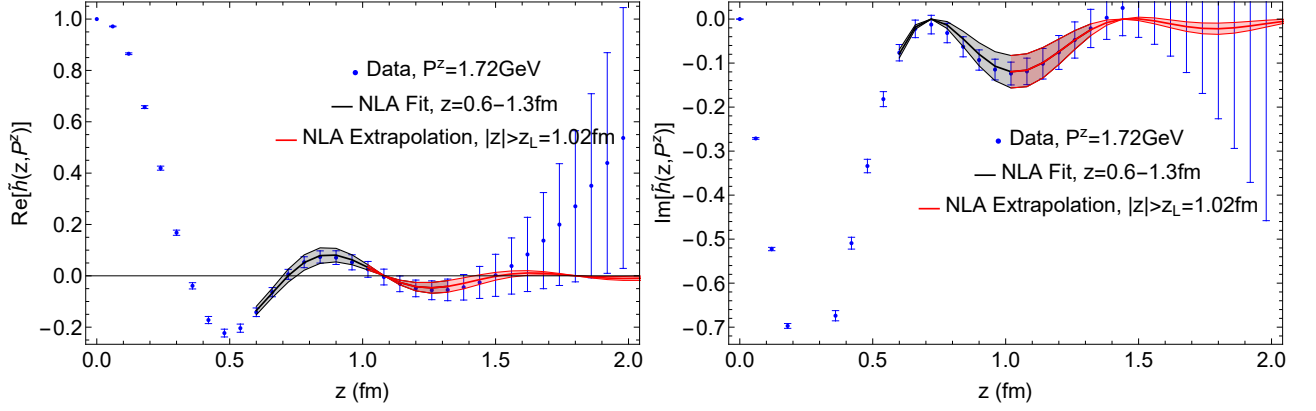


Figure 3.19: Similar to Fig. 3.18 except for the next-to-leading asymptotic (NLA) analysis.

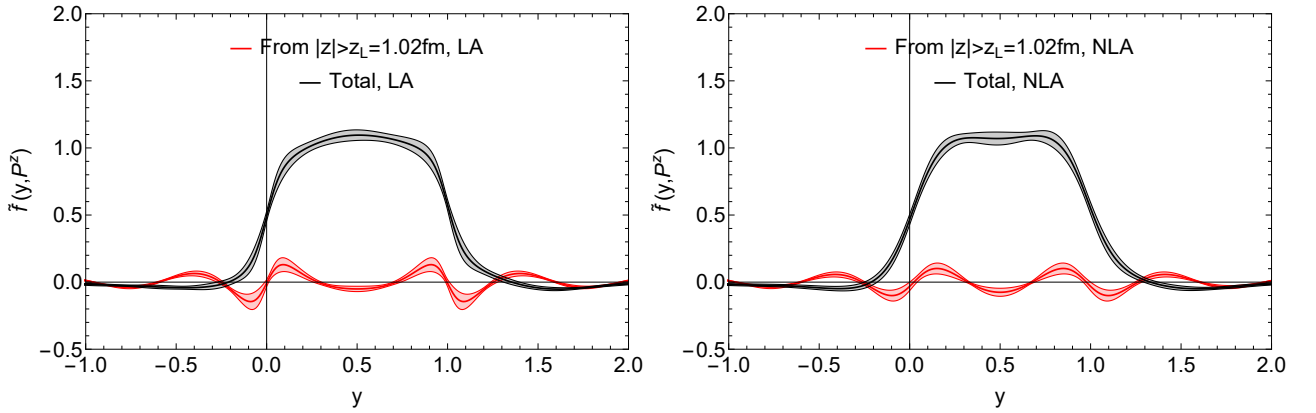


Figure 3.20: Similar to Fig. 3.12 except for the pion quasi-DA.

the NLA case. Both LA and NLA forms agree well with lattice data, especially for the oscillatory behavior.

Fig. 3.20 displays the quasi-DA in momentum space obtained from FT. The contribution from large distance asymptotic analysis is negligible compared to the total result.

Fig. 3.21 shows various contributions to FT uncertainties. The total asymptotic error is well controlled in the moderate y range (e.g. 0.03 at $y = 0.5$ compared to the center value 1.06).

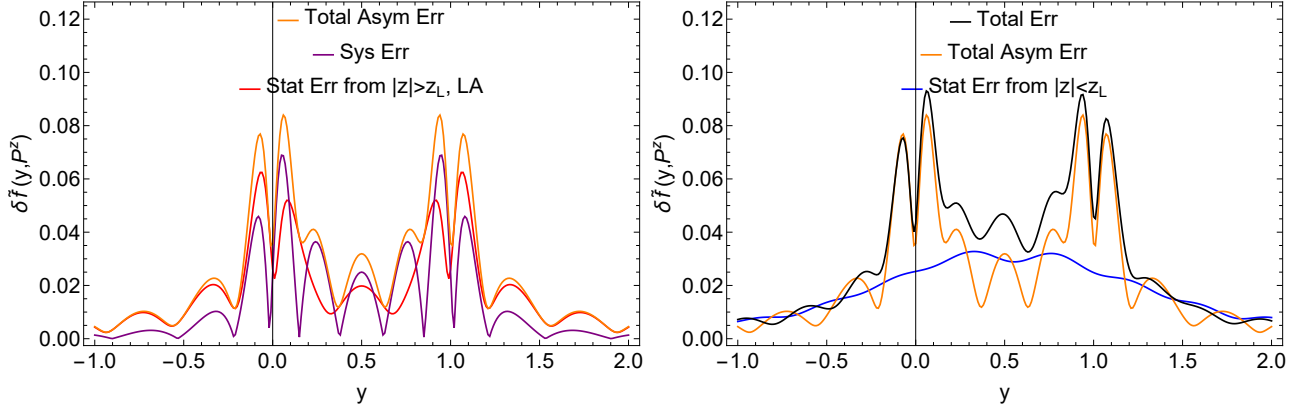


Figure 3.21: Similar to Fig. 3.13 except for pion quasi-DA.

3.5 Conclusion and discussion

In this paper, we use HQET reduction, dispersive analysis, and Lorentz symmetry to derive the large distance asymptotic forms of quasi-correlators, where many aspects are confirmed by two-loop calculations in ϕ^3 theory. We demonstrate the controlled precision of asymptotic analysis, which is a few percent with current lattice data and is expected to improve in the near future. Our results are valuable for precision calculations under LaMET.

A natural question is whether the parameters appearing in the asymptotic forms can be calculated directly rather than determined by fitting. It is possible to calculate the mass gaps in Eq. (3.44) from lattice QCD. The physical masses in Eq. (3.44) can be directly extracted from two-point functions of heavy quark systems on the lattice. By subtracting off the heavy quark pole mass defined under the same renormalon regularization scheme as the linear divergence, one obtains the binding energies that can be directly used in the extrapolated forms. However, it is challenging to calculate the coefficients $a_{X,l}$, $a_{Y,l}$, and $a_{Z,l}$ in Eq. (3.43) for a spacelike x . Although they have matrix element definitions that depend on the momenta k_i of inserted states, the spacelike correlator requires the analytical continuation of k_i to a spatial direction with imaginary numbers, which is difficult to perform with numerical

values. Since the FT uncertainties in the moderate y range mostly depend on the error of the magnitude at the truncation point z_L , fixing the binding energies but not the coefficients does not improve the precision there.

Chapter 4: Unpolarized $u - d$ proton PDF

This chapter presents the LaMET calculation of the unpolarized $u - d$ proton PDF, an unpublished result of the author. The formal definition of the targeted object is [57],

$$f(x, \mu) \equiv \int_{-\infty}^{+\infty} \frac{d\xi^-}{4\pi} e^{-ixP^+\xi^-} \langle N(P) | [\bar{u}(\xi^-)\gamma^+U(\xi^-, 0)u(0) - \bar{d}(\xi^-)\gamma^+U(\xi^-, 0)d(0)] | N(P) \rangle . \quad (4.1)$$

The quark and anti-quark fields u and $\bar{u}$ (or d and $\bar{d}$) are separated along the light cone by the distance ξ^- . They are connected by a straight-line gauge link $U(\xi^-, 0)$. The Dirac structure γ^+ is for the unpolarized projection. The external state $|N(P)\rangle$ is an unpolarized proton, and $f(x, \mu)$ is independent of the momentum P . The quantity x is the momentum fraction of the parton to the proton. The variable μ is the scale for the $\overline{\text{MS}}$ scheme.

4.1 Motivation

This section discusses the motivation for studying the unpolarized $u - d$ proton PDF, a natural benchmark for LaMET.

The unpolarized $u - d$ proton PDF can be extracted from experimental data through global fits, such as ATLASpdf21 [59], NNPDF4.0 [60], MSHT20 [61], CT18 [9], ABMP16 [62], HERAPDF2.0 [63], and JR14 [64], and Fig. 4.1 shows the results from two global fit groups. Numerous experimental datasets, including DIS and DY processes, can be used to impose a joint constraint on this quantity; see Fig. 4.2 for the datasets adopted by NNPDF. Furthermore, the flavor-isovector channel ($u - d$) during DGLAP evolution does not mix with the gluon PDF, which is usually noisier than the quark PDF. As a result, it has been determined in high precision, with $\sim 2\%$ relative uncertainties

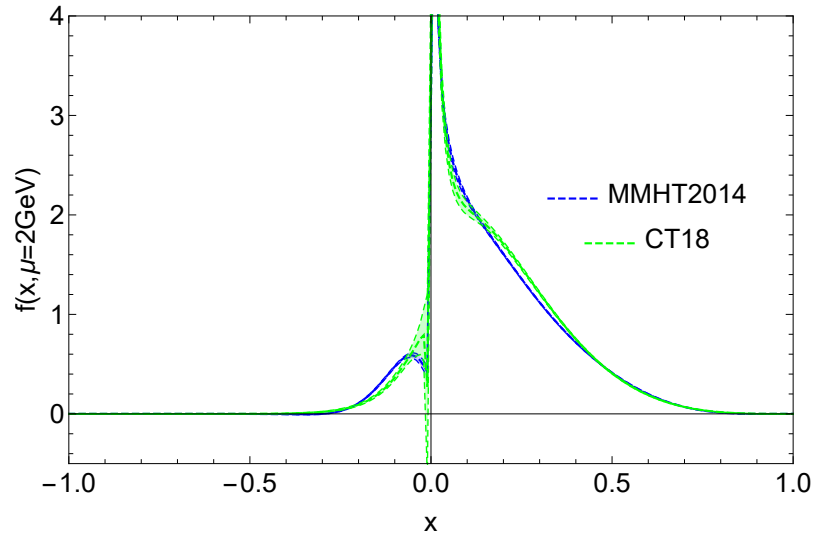


Figure 4.1: The unpolarized $u - d$ proton PDF extracted from experimental data through global fit by MMHT2014 [45] and CT18 [9]. The uncertainties are shown as the bands.

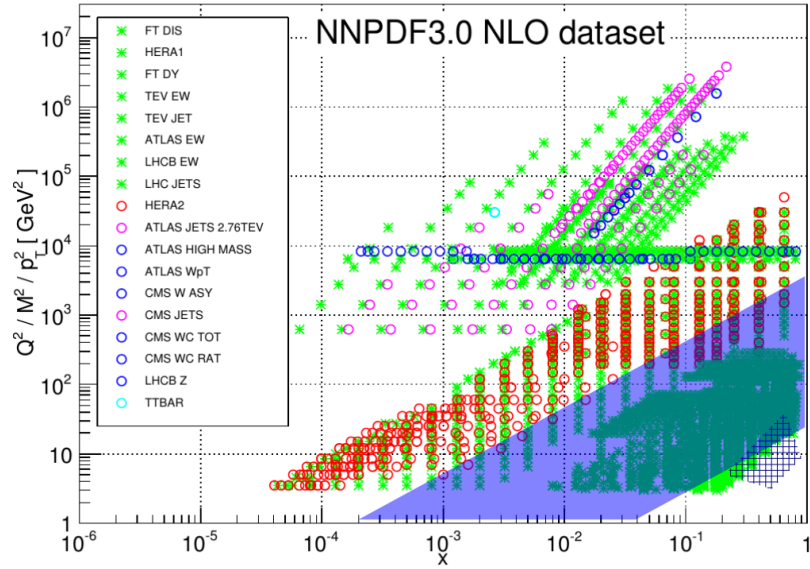


Figure 4.2: The NNPDF dataset from Ref. [46].

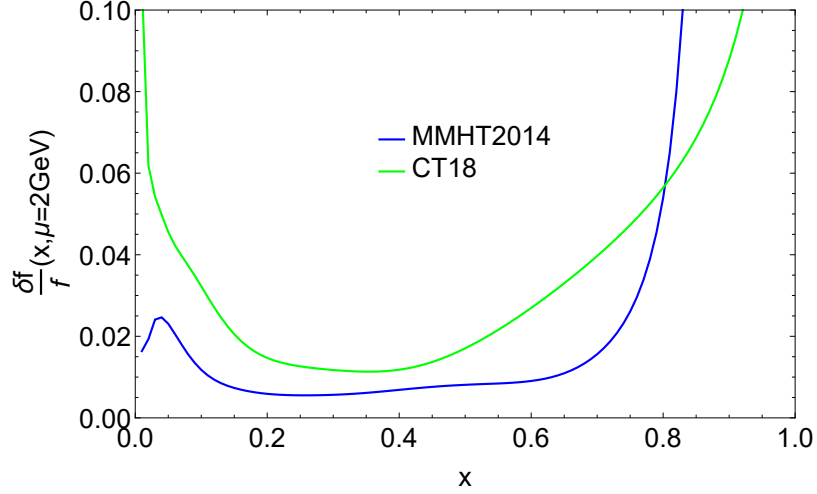


Figure 4.3: The relative uncertainties of unpolarized $u - d$ proton PDF from MMHT2014 [45] and CT18 [9].

in the moderate x range, as shown in Fig. 4.3.

On the other hand, the moderate x -dependence of the unpolarized $u - d$ proton PDF can be calculated from first principles under the LaMET framework, using recently developed high-precision techniques. From the lattice QCD side, the $(u - d)$ combination does not have quark-line disconnected diagrams, thereby avoiding all-to-all propagators and making it easier to simulate.

For the aforementioned reasons, the unpolarized $u - d$ proton PDF serves as a natural benchmark for validating and calibrating the LaMET analysis procedure. The comparison between LaMET and global fit for this quantity will demonstrate LaMET's precision-control capability.

4.2 LaMET analysis procedure

This section summarizes the standard LaMET analysis procedure using the unpolarized $u - d$ proton PDF as an example, but generalizable to other parton-physics calculations.

The first step is to generate lattice QCD data for unpolarized $u - d$ quasi-PDF matrix elements

in a nucleon,

$$\tilde{h}^B(z, P^z, a) = \langle N(P) | [\bar{u}(z)U(z, 0)\gamma^t u(0) - \bar{d}(z)U(z, 0)\gamma^t d(0)] | N(P) \rangle . \quad (4.2)$$

The quark and anti-quark fields are now separated along the spatial direction by the distance z , connected by the straight line Wilson link $U(z, 0)$. The state $|N(P)\rangle$ is an unpolarized proton, with momentum $P = \left(\sqrt{m_N^2 + P_z^2}, 0, 0, P^z\right)$. We need both $P^z \gg m_N$ for large momentum expansion and $P^z = 0$ for renormalization in the follow-up analysis. The UV divergences of this matrix element, including linear and logarithmic divergences [280–282, 285, 400], are regularized by the finite lattice spacing a .

The second step is to perform the hybrid renormalization [287],

$$\tilde{h}^R(z, P^z) = \frac{\tilde{h}^B(z, P^z, a)}{\tilde{h}^B(z, 0, a)} \theta[z_s - |z|] + \frac{\tilde{h}^B(z, P^z, a)}{Z_R(z, a, \mu)} \frac{Z_R(z_s, a, \mu)}{\tilde{h}^B(z_s, 0, a)} \theta[|z| - z_s] . \quad (4.3)$$

The quantity $\tilde{h}^B(z, 0, a)$ is the zero momentum matrix element generated on lattice. The factor $Z_R(z, a, \mu)$, converting lattice data to $\overline{\text{MS}}$ scheme, can be extracted using self renormalization [231]. It is parametrized as a small a asymptotic expansion inspired by perturbation theory,

$$Z_R(z, a, \mu) = \exp \left[\frac{kz}{a \ln[a\Lambda_{\text{QCD}}]} + m_0 z + \frac{3C_F}{b_0} \ln \left[\frac{\ln[1/(a\Lambda_{\text{QCD}})]}{\ln[\mu/\Lambda_{\overline{\text{MS}}}] } \right] + \ln \left[1 + \frac{d}{\ln(a\Lambda_{\text{QCD}})} \right] + f(z)a \right] . \quad (4.4)$$

The parameters k , Λ_{QCD} , and $f(z)$ are extracted from $\tilde{h}^B(z, 0, a)$ across multiple lattice spacings. The renormalon-scheme-dependent constant m_0 and the renormalization-scheme-dependent constant d are determined by fitting the renormalized lattice zero momentum matrix element $\tilde{h}^B(z, 0, a)/Z_R(z, a, \mu)$

with the NNLO+RGR+LRR [232] $\overline{\text{MS}}$ perturbative zero momentum matrix element $C_0(z^2\mu^2)$ at short distance,

$$\frac{\tilde{h}^B(z, 0, a)}{Z_R(z, a, \mu)} = C_0(z^2\mu^2) \quad \text{for } a < z \ll 1/\Lambda_{\text{QCD}}. \quad (4.5)$$

In Eq. (4.3), the hybrid cutoff z_s is also chosen within the window of $a < z_s \ll 1/\Lambda_{\text{QCD}}$, so the z_s dependence can be described by continuum perturbation theory. The motivation for hybrid renormalization is to implement an almost minimal UV subtraction scheme, except for mitigating the short-distance discrepancies between lattice and continuum theories. For $|z| > z_s$, it is the same as the $\overline{\text{MS}}$ scheme up to an overall constant¹, which purely subtracts off the UV divergences without introducing additional IR effects. For $|z| < z_s$, the hybrid renormalization is the same as the ratio method, which eliminates the UV divergences as well as part of the discretization effects.

The third step calibrates the vector charge $\langle N(P) | [\bar{u}\gamma^t u - \bar{d}\gamma^t d] | N(P) \rangle$ as well as the first moment $\langle x \rangle$,

$$\tilde{h}(z, P^z) = \frac{\tilde{h}^R(z, P^z)}{\tilde{h}^R(0, P^z)} e^{izP^z\Gamma}. \quad (4.6)$$

Here, the parameter Γ is determined by fitting the imaginary part of $\tilde{h}(z, P^z)$ at $|z| < z_s$ with the short distance OPE [79, 80, 103],

$$-izP^z \langle x \rangle(\mu) \frac{C_1(z^2\mu^2)}{C_0(z^2\mu^2)} + \dots \quad (4.7)$$

where the Wilson coefficients $C_\#(z^2\mu^2)$ [103] are chosen to be at NNLO+RGR+LRR accuracy [232],

¹The factor $Z_R(z_s, a, \mu)/\tilde{h}^B(z_s, 0, a)$ is introduced to make the curve continuous at $|z| = z_s$.

and the first moment is taken to be the same as CT18 [9], $\langle x \rangle(\mu = 2 \text{ GeV}) = 0.1562(24)$. The purpose of this step is to mitigate lattice artifacts arising from excited-state contamination, unphysical pion mass, or discretization effects.²

The fourth step implements large-distance asymptotic analysis and the Fourier transform, following Ref. [233]. The Fourier transform formally requires lattice data over an infinite z -range, including the large distance region (e.g. $> 1 \text{ fm}$). In principle, due to confinement, the correlator at large distances should decay exponentially, and the long tail region contributes only a small amount to the Fourier transform. In practice, lattice data at large distances suffer from exponentially deteriorating signal-to-noise ratios and cannot be used directly. To resolve this issue, Ref. [233] derives the large-distance asymptotic forms for various quasi correlators based on HQET reduction, dispersive analysis, Lorentz symmetry, and heavy flavor spectra. As for the unpolarized $u - d$ quasi-PDF case, the leading asymptotics (LA) and next-to-leading asymptotics (NLA) in the $z \rightarrow \infty$ expansion take the forms,

$$\tilde{h}^{\text{LA}}(z, P^z) = A e^{i\phi \text{sign}(z)} e^{-\Lambda|z|}, \quad (4.8)$$

$$\tilde{h}^{\text{NLA}}(z, P^z) = \left[A e^{i\phi \text{sign}(z)} + \frac{A' e^{i\phi' \text{sign}(z)}}{|z|} \right] e^{-\Lambda|z|}, \quad (4.9)$$

where the parameters A , ϕ , Λ , A' and ϕ' are real numbers. In particular, the mass gap Λ is the binding energy of a vector heavy-light meson, which is estimated as $\sim 400 - 600 \text{ MeV}$. We fit the parameters in the asymptotic form $\tilde{h}_{\text{asym}}(z, P^z)$ ³ with lattice data $\tilde{h}(z, P^z)$ at the distance larger than $\sim 1/\Lambda$ (e.g.

²These lattice artifacts could exist in the entire z -range, causing the deviation of the first moment from the physical value. One piece of evidence appears in the short-distance imaginary part, dominated by the first moment. We aim to calibrate the first moment, and a natural approach is to introduce an overall phase factor that affects the entire z -range and is determined using the short-distance OPE with the CT18 first moment.

³This notation is for $\tilde{h}^{\text{LA}}(z, P^z)$ or $\tilde{h}^{\text{NLA}}(z, P^z)$.

$0.7 \sim 1$ fm). Then, we perform the Fourier transform using lattice data $\tilde{h}(z, P^z)$ at $|z| < z_L$ and asymptotic form $\tilde{h}_{\text{asym}}(z, P^z)$ at $|z| > z_L$ to obtain the quasi-PDF,

$$\tilde{f}(y, P^z) = P^z \int_{-\infty}^{+\infty} \frac{dz}{2\pi} e^{izP^zy} \left[\tilde{h}(z, P^z) \theta(z_L - |z|) + \tilde{h}_{\text{asym}}(z, P^z) \theta(|z| - z_L) \right], \quad (4.10)$$

where the switching point z_L can be chosen around $0.7 \sim 1$ fm. This step converts the coordinate-space result to momentum space to facilitate the large-momentum expansion and perturbative matching in the next step.

Finally, we perform the perturbative matching on the quasi-PDF to obtain the light-cone PDF defined in Eq. (4.1),

$$f(x, \mu) = \int_{-\infty}^{+\infty} \frac{dy}{|y|} C\left(\frac{x}{y}, \frac{\mu}{yP^z}\right) \tilde{f}(y, P^z) + O\left(\frac{\Lambda_{\text{QCD}}^2}{x^2 P_z^2}, \frac{\Lambda_{\text{QCD}}^2}{(1-x)^2 P_z^2}\right), \quad (4.11)$$

where the hybrid scheme matching kernel $C(x/y, \mu/yP^z)$ at the NNLO+RGR+LRR+TR accuracy can be found in Ref. [17]. The renormalon regularization scheme of the matching kernel is consistent with the constant m_0 in Eq. (4.4) so that we can achieve linear power accuracy [232]. This step addresses the UV difference between light-cone and quasi-PDFs.

4.3 Lattice QCD datasets

This section applies the LaMET analysis procedure described above to realistic lattice QCD datasets in the world.

The lattice QCD matrix elements for unpolarized $u - d$ proton PDF, as defined in Eq. (4.2), have been generated in many previous works, including ETMC18 [96], ETMC19 [104], ETMC20 [48],

Works	MSU20 [47]	ETMC20 [48]	LPC18 [49]	CLQCD24 [50]
a (fm)	0.12, 0.09, 0.06	0.0820	0.0854	0.105
m_π (MeV)	310, 230, 138	373	356	293
P^z (GeV)	2.2	1.89	2.27	1.85

Table 4.1: Selected lattice QCD datasets of unpolarized $u - d$ proton quasi PDF matrix elements. Their information, including lattice spacing a , pion mass m_π , and hadron momentum P^z , is listed in the table.

LPC18 [49], CLQCD24 [50], JLab20 [401], BNL/MSU [83], MSU20 [47], and ANL/BNL22 [119].

Considering the large momentum expansion, statistical precision, and data accessibility, four datasets [47–50] are selected for our analysis in this section, as shown in Tab. 4.1.

4.3.1 MSU20

In this subsection, we re-analyze the data from MSU20 [47] following the latest LaMET analysis procedure in Sec. 4.2. The continuum and physical pion mass extrapolations can be performed using this dataset, thereby yielding the PDF at the physical limits.

We start by implementing the first and second steps in Sec. 4.2. The data can be digitized from Fig. 4 in Ref. [47]. There are multiple strategies for the physical pion mass and continuum extrapolations, among which we choose the red curve (a^1, M_π^2) because the leading discretization effect should be $O(a)$ due to the mixed Fermion action effect, and the pion mass dependence should be $O(M_\pi^2)$ according to Ref. [402]. The results in Fig. 4 are the renormalized matrix elements in the RI/MOM scheme, denoted as $\tilde{h}^{\text{RM}}(z, P^z)$, and we convert them to the hybrid scheme,

$$\tilde{h}^R(z, P^z) = \frac{\tilde{h}^{\text{RM}}(z, P^z) Z^{\text{RM-inp}}(z, \mu)}{\tilde{h}^{\text{B-inp}}(z, 0, \mu)} \theta[z_s - |z|] + \frac{\tilde{h}^{\text{RM}}(z, P^z) Z^{\text{RM-inp}}(z, \mu)}{\tilde{h}^{\text{B-inp}}(z_s, 0, \mu)} \theta[|z| - z_s] , \quad (4.12)$$

where $Z^{\text{RM-inp}}(z, \mu)$ and $\tilde{h}^{\text{B-inp}}(z, 0, \mu)$ denote the intrinsic IR physics in the RI/MOM factor $Z^{\text{RM}}(z, a)$

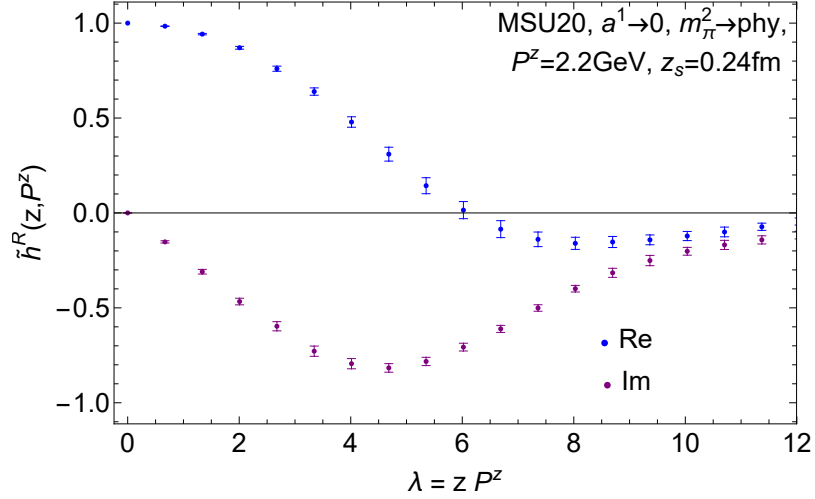


Figure 4.4: The hybrid renormalized matrix element for MSU20 dataset [47]. The blue and purple points are the real and imaginary parts, respectively. The continuum and physical pion mass extrapolations have already been done under $O(a^1, M_\pi^2)$.

and the zero momentum matrix element $\tilde{h}^B(z, 0, a)$, respectively, after the UV renormalization factor $Z_R(z, a, \mu)$ has been factorized out,

$$\begin{aligned} Z^{\text{RM}}(z, a) &= Z_R(z, a, \mu) Z^{\text{RM-inp}}(z, \mu), \\ \tilde{h}^B(z, 0, a) &= Z_R(z, a, \mu) \tilde{h}^{\text{B-inp}}(z, 0, \mu). \end{aligned} \quad (4.13)$$

Here, the RI/MOM factor $Z^{\text{RM}}(z, a)$ is from Ref. [231], and $Z_R(z, a, \mu)$ is parameterized as in Eq. (4.4) and extracted using the self renormalization method [231] with m_0 determined using the NNLO + RGR + LRR [232] Wilson coefficients. The zero momentum matrix element $\tilde{h}^B(z, 0, a)$ is the same as the clover Nucleon hyp1 case in Fig. 23 of Ref. [231], and its Z_R is extracted similarly. As a result, Eq. (4.12) is equivalent to Eq. (4.3) and it is displayed in Fig. 4.4.

Then, we implement the third step, as displayed in Fig. 4.5. The left panel depicts the determination of the phase rotation using the CT18 first moment. The right panel presents the results after the calibration of the first moment for the lattice data.

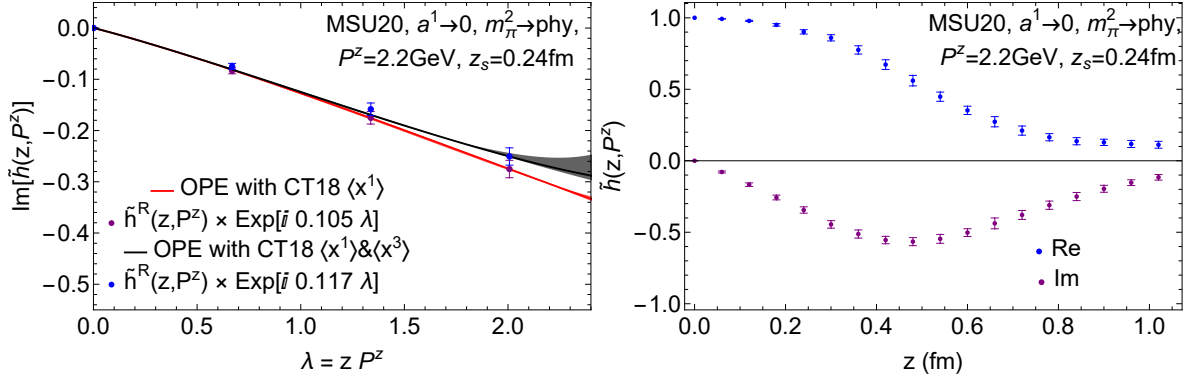


Figure 4.5: The calibration of the first moment $\langle x \rangle$. In the left panel, we determine the parameter Γ for the phase rotation by matching the imaginary part of lattice result $\tilde{h}(z, P^z) = \tilde{h}^R(z, P^z)/\tilde{h}^R(0, P^z) \times e^{izP^z\Gamma}$ (blue and purple points) to the short-distance OPE constructed using CT18 moments (black and red curves) with NNLO+RGR+LRR Wilson coefficients. The red curve only considers the first moment, while the black curve includes both first and third moments. They lead to slightly different Γ that will be considered as a systematic error. The right panel displays the results after the calibration of the first moment, with $\Gamma = (0.125 + 0.11)/2 = 0.1175$.

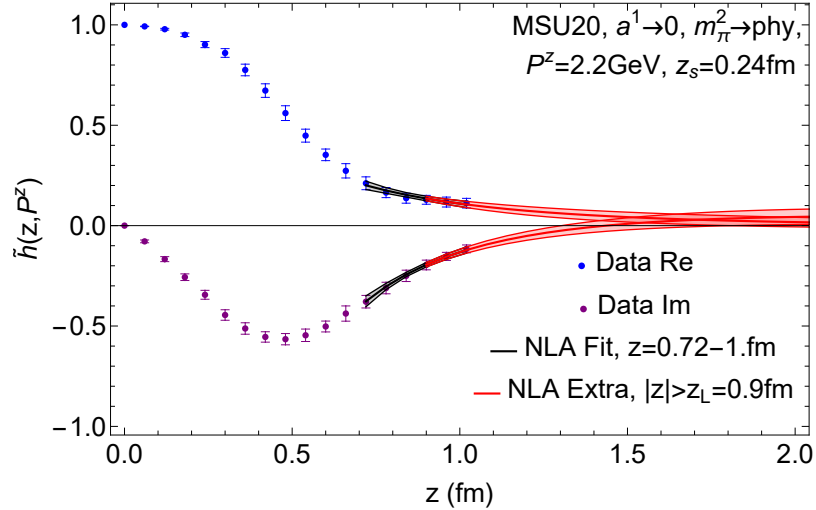


Figure 4.6: The large distance asymptotic analysis with NLA. The blue points are calibrated lattice data. The black curve covers the fit range, and the red curve is the extrapolated result.

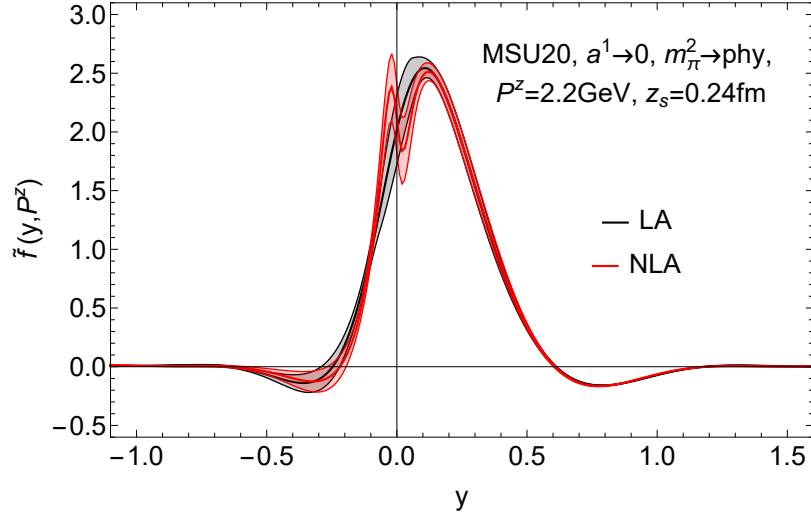


Figure 4.7: The momentum space quasi-PDFs obtained through the Fourier transform. The red and black curves correspond to the LA and NLA extrapolations, respectively. Their difference will be considered as one of the error budgets.

We apply the fourth step. The fit and extrapolation of $\tilde{h}(z, P^z)$ with NLA are shown in Fig. 4.6.

The momentum space quasi-PDFs obtained through the Fourier transform are depicted in Fig. 4.7.

The last step is to perform the NNLO+RGR+LRR+TR perturbative matching [17]. The final result is shown as the red band in Fig. 4.8, which is consistent with the global fit PDFs within $\sim 1\sigma$.

The error budgets are displayed in Fig. 4.9.

The above analysis considers the continuum and physical pion mass limits. For comparison, we also depict the results obtained from each individual ensemble in Fig. 4.10, which slightly differ in their inclination from the $O(a, m_\pi^2)$ extrapolated lattice result and the global fit PDFs. However, the smaller lattice spacing tends to approach the global-fit PDF. Therefore, it is important to account for the physical limits in precision calculations under LaMET.

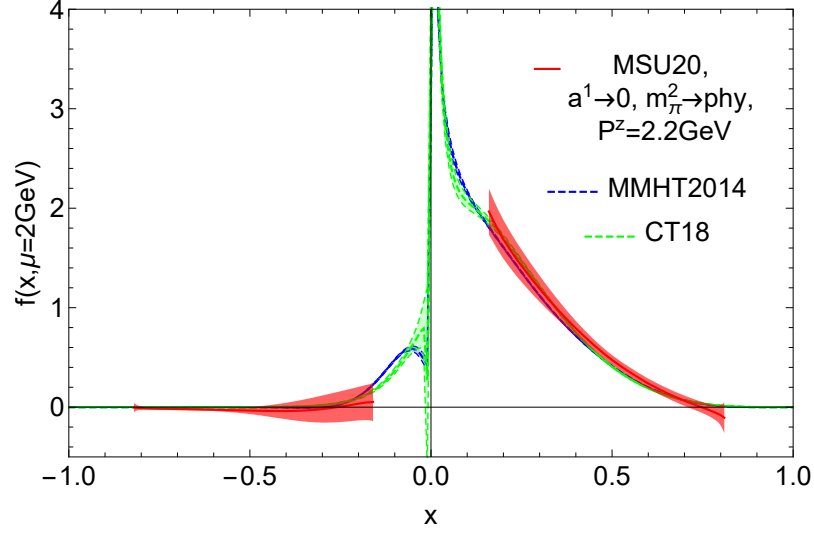


Figure 4.8: The light cone PDFs for unpolarized $u - d$ in a proton. The red band is calculated based on the latest LaMET analysis procedure in Sec. 4.2 using MSU20 data extrapolated to the continuum and physical pion mass limits [47]. The band covers several uncertainties that are detailed in Fig. 4.9. The blue and green bands are extracted from experimental data through global fit by MMHT2014 [45] and CT18 [9].

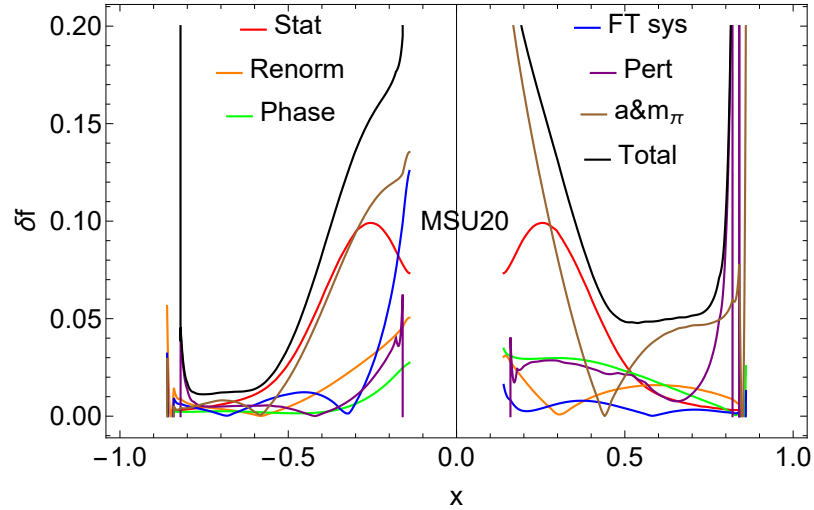


Figure 4.9: The error budgets of the red band in Fig. 4.8. The red curve is the statistical uncertainty digitized from Fig. 5 in Ref. [47]. The orange curve arises from the uncertainty of m_0 in the renormalization. The green curve is the uncertainty from the phase rotation parameter Γ as mentioned in Fig. 4.5. The blue curve is the systematic uncertainty of the Fourier transform estimated as the difference between LA and NLA as discussed in Fig. 4.7. The purple curve denotes the uncertainty of perturbative matching evaluated through the variations of physical scales in the RG evolution studied in Ref. [17]. The brown curve is the systematics from the continuum and physical pion mass extrapolations, estimated using the deviation among various ansätze in Fig. 4 of Ref. [47]. The black curve is the total uncertainty obtained by combining the individual uncertainties uncorrelatedly.

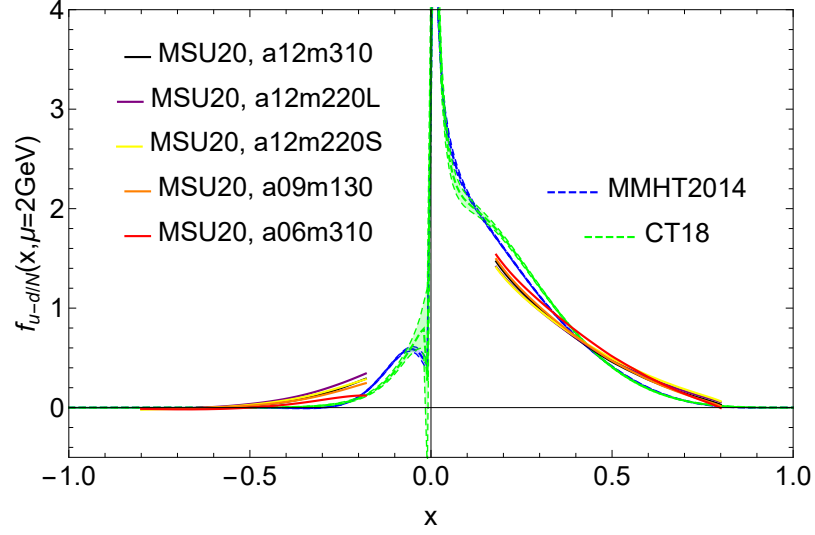


Figure 4.10: Same as Fig. 4.8 except for each individual ensemble. The uncertainty bands of lattice PDFs are not depicted for illustrative purposes.

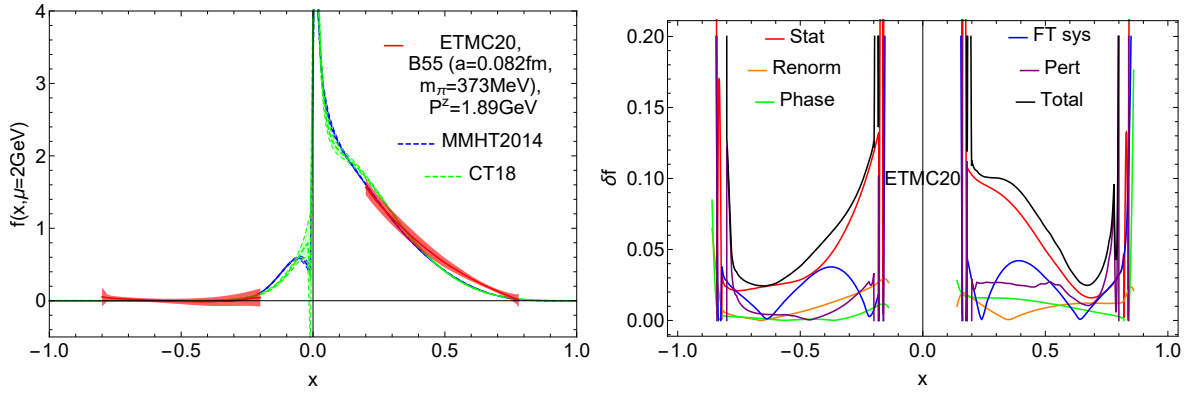


Figure 4.11: Left panel: the light cone PDF for unpolarized $u - d$ in a proton calculated based on LaMET analysis in Sec. 4.2 using ETMC20 data [48] on B55 ensemble, shown as the red band, compared with the ones extracted from experimental data through global fit, such as MMHT2014 (blue) [45] and CT18 (green) [9]. Right panel: error budgets of the PDF calculated under LaMET, and the definitions of the curves can be found in Fig. 4.9.

4.3.2 ETMC20

In this subsection, we re-analyze the data from ETMC20 [48] following the latest LaMET analysis procedure in Sec. 4.2. We use the ETMC B55 ensemble, with lattice spacing $a = 0.0820$ fm, an unphysical pion mass $m_\pi = 373$ MeV, and a proton momentum $P^z = 1.89$ GeV. Since we have access to the bootstrap samples of the large momentum and zero momentum matrix elements, the statistical uncertainties can be estimated, and the hybrid renormalization can be directly performed following Sec. 4.2. The final results and error budgets are depicted in Fig. 4.11. The shape is similar to that of the global fit PDFs, and the modest deviation may arise from discretization effects or unphysical pion mass.

4.3.3 LPC18

We re-analyze the data from LPC18 [49] following the latest LaMET analysis procedure in Sec. 4.2. The LPC18 is calculated on the CLS H102 ensemble with lattice spacing $a = 0.085$ fm, unphysical pion mass $m_\pi = 356$ MeV, and proton momentum $P^z = 2.3$ GeV. We have access to the bootstrap samples of the large momentum and zero momentum matrix elements. The final results are shown in Fig. 4.12.

4.3.4 CLQCD24

The lattice data are generated by the Lattice Parton Collaboration (LPC) and have been used to study the “EMC effect” of a deuteronlike dibaryon system [50]. We choose the data with proton momentum $P^z = 1.8$ GeV generated on the C32P29 ensemble [398,399] with $a = 0.105$ fm and $m_\pi = 292.9$ MeV. We have access to the bootstrap samples of the large momentum and zero momentum

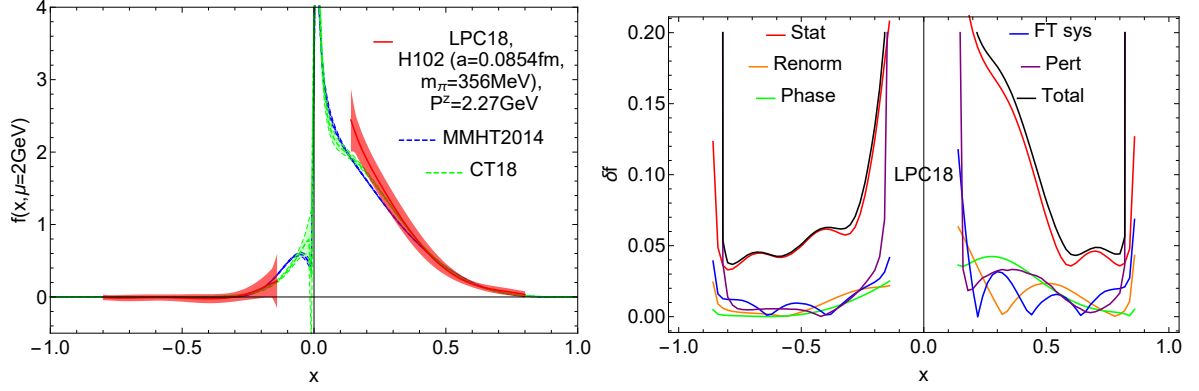


Figure 4.12: Similar to Fig. 4.11 except for LPC18 data [49].

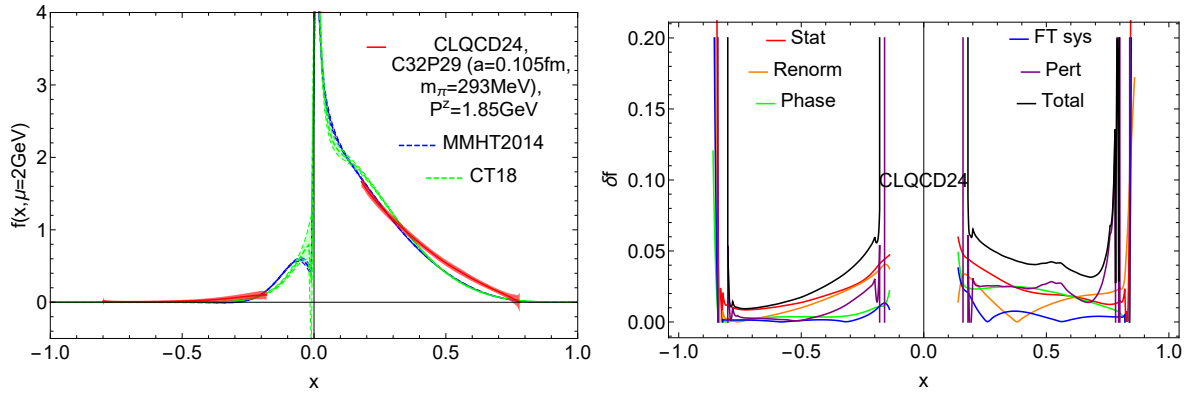


Figure 4.13: Similar to Fig. 4.12 except for CLQCD24 data [50].

matrix elements. The final results and error budgets are displayed in Fig. 4.13.

4.4 Summary and outlook

Among all datasets analyzed under LaMET, the final results exhibit shapes similar to the global-fit PDFs. Moreover, the MSU20 dataset, extrapolated to the physical pion mass and continuum limit, is consistent with the global fit PDFs within $\sim 1\sigma$. These results justify the LaMET analysis procedure and demonstrate LaMET's precision-control capability.

The analysis can be further improved in the future if the following data become available,

- MSU20 [47]: bootstrap samples for each ensemble in Fig. 4 and RI/MOM factors in Fig. 2;
- ETMC20 [48]: different pion masses for physical pion mass extrapolation; several momenta so one can interpolate to the same momentum for continuum extrapolation;
- LPC18 [49]: several pion masses and lattice spacings for physical extrapolations;
- CLQCD24 [50]: several pion masses and lattice spacings for physical extrapolations.

The remaining theoretical questions of LaMET precision control correspond to discretization at $z \sim a$ and excited-state control. The former affects hybrid renormalization, involving the short-distance ratio and the m_0 parameter. It may be analyzed under lattice perturbation theory [93, 283, 333, 334] and hypercubic symmetry [335, 336]. The renormalization of the derivative operators [70, 72] could also inspire this study. The latter influences lattice data generation and may be studied by adopting the methods in Refs. [290–294, 296].

Chapter 5: Conclusion

Until now, the large momentum effective theory (LaMET) appears to be the only method to directly calculate the x -dependence of parton physics in the region $\Lambda_{\text{QCD}}/2P^z \lesssim x \lesssim 1 - \Lambda_{\text{QCD}}/2P^z$. Other approaches, such as local quark bilinear operators with covariant derivatives [65–78] and the short-distance factorization of nonlocal operators [15, 79–83], provide complementary global information, such as several low-order moments of parton distributions.

Since the proposal of LaMET in 2013, substantial progress has been made in the first-principles calculation of parton physics. The theoretical foundation, including the justification of the matching relation and the development of the renormalization framework, is now well established. Precision calculation techniques, such as lattice interpolators for boosted hadrons, resummation of logarithms, linear renormalon and power corrections, and large-distance asymptotic analysis, have been extensively developed. Approximately one hundred state-of-the-art lattice QCD calculations have been carried out by major collaborations, including the Argonne National Laboratory/Brookhaven National Laboratory (ANL/BNL) Collaboration, the China Lattice QCD (CLQCD) Collaboration, the Extended Twisted Mass Collaboration (ETMC), the Lattice Parton Collaboration (LPC), the Massachusetts Institute of Technology (MIT), and Michigan State University (MSU).

Significant research opportunities still remain. Further progress requires improved control of lattice systematics, particularly excited-state contamination in nucleon matrix elements. It is also important to extend LaMET to a broader range of parton observables, including gluon PDFs, heavy-meson DAs, baryon DAs, and DPDs, many of which are still in an early stage of development. In addition, alternative quasi-correlator constructions, including Coulomb-gauge formulations and correlators involving heavy-light or light-light currents, continue to provide new directions for theoretical exploration.

LaMET calculations may also influence experimental or phenomenological studies. Heavy-flavor decay processes measured at LHCb [403], which probe charge conjugation-parity (CP) violation and the Cabibbo-Kobayashi-Maskawa (CKM) matrix, require non-perturbative standard model inputs such as heavy-meson distribution amplitudes. These quantities are now being calculated within LaMET combined with HQET. Three-dimensional partonic distributions, including TMDs and GPDs, encode essential information about hadron structures, including the origin of mass, spin, and confinement, and constitute major scientific goals at Jefferson Lab [404] and the future Electron-Ion Collider (EIC) in the United States [258, 259] and China [405]. These observables have also been extensively explored within the LaMET framework, with steadily improving precision as theoretical and numerical techniques advance. In addition, the generalized parton distributions through universal moment parameterization (GUMP) program [406–409] combines experimental and lattice inputs in global analysis of GPDs, where LaMET calculations provide important constraints in kinematic regions that remain difficult to access experimentally.

Appendix A: Supporting evidence for large distance asymptotic analysis

This Appendix provides the supporting evidence for the large distance asymptotic analysis discussed in Chapter 3.

A.1 Matching between dynamical heavy quark and gauge link

This section aims to justify the coordinate space HQET reduction in Eqs. (3.14) and (3.18), through the expansion by region analysis, one-loop perturbative calculations, and complex analysis.

A.1.1 Expansion by region analysis in the timelike separation

This subsection justifies Eqs. (3.14) and (3.18) for the timelike case using the expansion-by-region analysis.

First, let's recall the HQET reduction for an on-shell external heavy quark $p = (m_Q, 0, 0, 0)$ and define the velocity vector $v \equiv (1, 0, 0, 0)$. Consider multi-gluon insertions on the quark line with

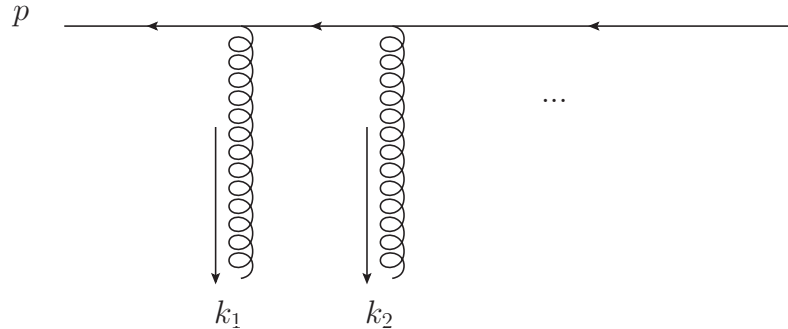


Figure A.1: An on-shell external heavy quark inserted by multiple gluons.

momenta $k_1, k_2, \dots$, as shown in Fig. A.1,

$$I = \bar{u}(p) \gamma^{\mu_1} t^{a_1} \frac{i(\not{p} - \not{k}_1 + m_Q)}{(p + k_1)^2 - m_Q^2} \gamma^{\mu_2} t^{a_2} \frac{i(\not{p} + \not{k}_1 + \not{k}_2 + m_Q)}{(p + k_1 + k_2)^2 - m_Q^2} \dots \quad (\text{A.1})$$

In the heavy quark limit $m_Q \rightarrow +\infty$, the gluon momenta are counted as $k_i \sim \lambda m_Q (1, 1, 1, 1)$ where $\lambda \ll 1$. Therefore, the quark propagators can be reduced as,

$$\begin{aligned} I &\approx \bar{u}(p) \gamma^{\mu_1} t^{a_1} \frac{i(\not{p} + 1)}{2 \left(p^0 + k_1^0 - \frac{(\vec{p} + \vec{k}_1)^2}{2m_Q} - m_Q \right)} \gamma^{\mu_2} t^{a_2} \frac{i(\not{p} + 1)}{2 \left(p^0 + k_1^0 + k_2^0 - \frac{(\vec{p} + \vec{k}_1 + \vec{k}_2)^2}{2m_Q} - m_Q \right)} \dots \\ &\approx \bar{u}(p) v^{\mu_1} t^{a_1} \frac{i(\not{p} + 1)}{2v \cdot k_1} v^{\mu_2} t^{a_2} \frac{i(\not{p} + 1)}{2v \cdot k_2} \dots \\ &= \bar{u}(p) \frac{i v^{\mu_1} t^{a_1}}{v \cdot k_1} \frac{i v^{\mu_2} t^{a_2}}{v \cdot k_2} \dots \end{aligned} \quad (\text{A.2})$$

In the first line, the non-relativistic expansion is performed in the denominator. For a heavy quark at rest, the “kinematic energy terms” $(\vec{p} + \vec{k}_1)^2/(2m_Q)$ can be ignored, which leads to the second line. The projection of Dirac gamma matrices has also been applied here $\gamma^{\mu_1} = \not{p} v^{\mu_1} + \dots$, where “...” does not contribute to the leading term. The final result is eikonalized as a gauge link.

For a coordinate-space heavy quark propagator $Q(t)\bar{Q}(0)$ of interest in Eqs. (3.14) and (3.18), the HQET reduction becomes slightly different in the way that the heavy quark momentum p has quantum fluctuations around the timelike direction and the “kinematic energy terms” should be kept in the power counting. To understand this, we first consider the tree-level propagator under non-relativistic

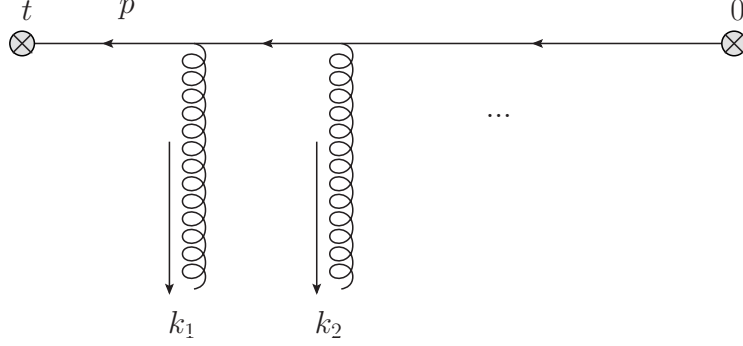


Figure A.2: A coordinate-space heavy quark propagator $Q(t)\bar{Q}(0)$ inserted by multiple gluons.

expansion,

$$\begin{aligned}
 I'_{\text{tree}} &= \int \frac{d^4 p}{(2\pi)^4} e^{-ip^0 t} \frac{i(\not{p} + m_Q)}{p^2 - m_Q^2} \\
 &\approx \int \frac{d^4 p}{(2\pi)^4} e^{-ip^0 t} \frac{1}{\left(p^0 - m_Q - \frac{\vec{p}^2}{2m_Q} + i0^+\right)} \frac{i(\not{p} + 1)}{2}
 \end{aligned} \tag{A.3}$$

In this case, the kinematic term $\vec{p}^2/(2m_Q)$ can not be ignored otherwise the integral becomes divergent.

Pick up the residue pole of p^0 and integrate out $\vec{p}$,

$$I'_{\text{tree}} \approx \int \frac{d^3 \vec{p}}{(2\pi)^3} e^{-im_Q t} e^{-i\frac{\vec{p}^2}{2m_Q} t} \frac{\not{p} + 1}{2} = \frac{m_Q^3 e^{-im_Q t}}{2\sqrt{2}\pi^{3/2} (im_Q t)^{3/2}} \frac{\not{p} + 1}{2}. \tag{A.4}$$

In this calculation, the momentum mode $p \sim (m_Q, \sqrt{2m_Q/t}, \sqrt{2m_Q/t}, \sqrt{2m_Q/t})$ plays a crucial role regarding the width of the Gaussian factor $e^{-i\frac{\vec{p}^2}{2m_Q} t}$. The overall polynomial dependence $m_Q^3/(m_Q t)^{3/2}$ is related to the size of quantum fluctuations along the spatial directions.

The next step is to check if HQET reduction holds for the momentum mode $p \sim (m_Q, \sqrt{2m_Q/t}, \sqrt{2m_Q/t}, \sqrt{2m_Q/t})$. Here p is almost on-shell, and the off-shellness is treated as higher orders in non-relativistic expansion, $p^0 - m_Q - \frac{\vec{p}^2}{2m_Q} \sim O(\vec{p}^4/m_Q^3)$. And the power counting for the gluon momenta

will be $k_i \sim \lambda m_Q(1, 1, 1, 1)$, where $\vec{p}^4/m_Q^4 \ll \lambda \ll 1$. Start with a coordinate space heavy quark propagator with multi-gluon insertions in Fig. A.2,

$$I' = \int \frac{d^4 p}{(2\pi)^4} e^{-ip^0 t} \frac{i(\not{p} + m_Q)}{p^2 - m_Q^2} \gamma^{\mu_1} t^{a_1} \frac{i(\not{p} - \not{k}_1 + m_Q)}{(p + k_1)^2 - m_Q^2} \gamma^{\mu_2} t^{a_2} \frac{i(\not{p} + \not{k}_1 + \not{k}_2 + m_Q)}{(p + k_1 + k_2)^2 - m_Q^2} \dots \quad (\text{A.5})$$

Perform the non-relativistic expansion up to the kinematic energy term,

$$I' \approx \int \frac{d^4 p}{(2\pi)^4} e^{-ip^0 t} \frac{i(\not{p} + 1)}{2 \left(p^0 - m_Q - \frac{\vec{p}^2}{2m_Q} + i0^+ \right)} \times v^{\mu_1} t^{a_1} \frac{i(\not{p} + 1)}{2 \left(p^0 + k_1^0 - \frac{(\vec{p} + \vec{k}_1)^2}{2m_Q} - m_Q \right)} v^{\mu_2} t^{a_2} \frac{i(\not{p} + 1)}{2 \left(p^0 + k_1^0 + k_2^0 - \frac{(\vec{p} + \vec{k}_1 + \vec{k}_2)^2}{2m_Q} - m_Q \right)} \dots \quad (\text{A.6})$$

The key point here is the kinematic terms $(\vec{p} + \vec{k}_1)^2/(2m_Q)$, $(\vec{p} + \vec{k}_1 + \vec{k}_2)^2/(2m_Q)$ cannot be ignored anymore. They can be reduced by power counting,

$$k_1^0 - \frac{(\vec{p} + \vec{k}_1)^2}{2m_Q} = k_1^0 - \frac{\vec{p}^2}{2m_Q} - \frac{\vec{k}_1^2}{2m_Q} - \frac{\vec{p} \cdot \vec{k}_1}{m_Q} \approx k_1^0 - \frac{\vec{p}^2}{2m_Q} + O(\lambda^2 m_Q) + O\left(\frac{\lambda m_Q}{\sqrt{t m_Q}}\right), \quad (\text{A.7})$$

where the $\vec{k}_1^2/(2m_Q) \sim \lambda^2 m_Q$ and $\vec{p} \cdot \vec{k}_1/m_Q \sim \lambda m_Q/\sqrt{t m_Q}$ are ignored since they are more suppressed than $k_1^0 \sim \lambda m_Q$. The kinematic term $\vec{p}^2/(2m_Q) \sim 1/t$ is kept since it can be similarly large as $k_1^0 \sim \lambda m_Q$ as we do not require the hierarchy between $1/(m_Q t)$ and λ . Therefore, the integral becomes,

$$I' \approx \int \frac{d^4 p}{(2\pi)^4} e^{-ip^0 t} \frac{i(\not{p} + 1)}{2 \left(p^0 - m_Q - \frac{\vec{p}^2}{2m_Q} + i0^+ \right)} \times v^{\mu_1} t^{a_1} \frac{i(\not{p} + 1)}{2 \left(p^0 + k_1^0 - \frac{\vec{p}^2}{2m_Q} - m_Q \right)} v^{\mu_2} t^{a_2} \frac{i(\not{p} + 1)}{2 \left(p^0 + k_1^0 + k_2^0 - \frac{\vec{p}^2}{2m_Q} - m_Q \right)} \dots \quad (\text{A.8})$$

By redefining $p'^0 = p^0 - \frac{\vec{p}^2}{2m_Q} - m_Q \sim O(\vec{p}^4/m_Q^3)$, one obtains,

$$\begin{aligned}
I' &\approx \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{dp'^0}{2\pi} e^{-ip'^0 t} e^{-im_Q t} e^{-i\frac{\vec{p}^2}{2m_Q} t} \frac{i(\not{p} + 1)}{2(p'^0 + i0^+)} v^{\mu_1} t^{a_1} \frac{i(\not{p} + 1)}{2(p'^0 + k_1^0)} v^{\mu_2} t^{a_2} \frac{i(\not{p} + 1)}{2(p'^0 + k_1^0 + k_2^0)} \cdots \\
&\approx \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{dp'^0}{2\pi} e^{-im_Q t} e^{-i\frac{\vec{p}^2}{2m_Q} t} \frac{i(\not{p} + 1)}{2(p'^0 + i0^+)} v^{\mu_1} t^{a_1} \frac{i(\not{p} + 1)}{2k_1^0} v^{\mu_2} t^{a_2} \frac{i(\not{p} + 1)}{2(k_1^0 + k_2^0)} \cdots \\
&= \frac{m_Q^3 e^{-im_Q t}}{2\sqrt{2}\pi^{3/2} (im_Q t)^{3/2}} \frac{\not{p} + 1}{2} \frac{iv^{\mu_1} t^{a_1}}{v \cdot k_1} \frac{iv^{\mu_2} t^{a_2}}{v \cdot (k_1 + k_2)} \cdots
\end{aligned} \tag{A.9}$$

where the dynamical, Dirac and color structures are fully factorized. The color structure is the same as a gauge link along the timelike direction v . The above region analysis justifies Eqs. (3.14) and (3.18) for the timelike case in the main text.

A.1.2 One-loop verification for the timelike case

This subsection targets at justifying Eq. (3.18) in the timelike separation $x = (t, 0, 0, 0)$ up to one-loop in a massless on-shell quark external state with $P = (P^z, 0, 0, P^z)$. In order to regulate the UV and IR divergences, we implement the dimensional regularization for the loop momentum of gluon, while the momentum of heavy quark propagator is fixed at 4D for simplicity. m_Q is treated as a pole mass in our calculation and the other UV divergences are renormalized to $\overline{\text{MS}}$ scheme.

The quantities are expanded in terms of QCD coupling $\alpha(\mu)$,

$$M_{\Gamma_1\Gamma_2}(x, P, \mu) = M_{\Gamma_1\Gamma_2}^{(0)}(x, P, \mu) + \frac{\alpha(\mu)C_F}{2\pi} M_{\Gamma_1\Gamma_2}^{(1)}(x, P, \mu) + \dots \tag{A.10}$$

$$\tilde{h}_{\Gamma_1\Gamma_2}(x, P, \mu) = \tilde{h}_{\Gamma_1\Gamma_2}^{(0)}(x, P, \mu) + \frac{\alpha(\mu)C_F}{2\pi} \tilde{h}_{\Gamma_1\Gamma_2}^{(1)}(x, P, \mu) + \dots \tag{A.11}$$

$$H_{\Gamma_1\Gamma_2,\hat{x}}\left(\frac{m_Q^2}{\mu^2}\right) = H_{\Gamma_1\Gamma_2,\hat{x}}^{(0)}\left(\frac{m_Q^2}{\mu^2}\right) + \frac{\alpha(\mu)C_F}{2\pi}H_{\Gamma_1\Gamma_2,\hat{x}}^{(1)}\left(\frac{m_Q^2}{\mu^2}\right) + \dots \quad (\text{A.12})$$

The tree level results are,

$$M_{\Gamma_1\Gamma_2}^{(0)}(x, P, \mu) = \frac{m_Q^3 e^{-im_Q t}}{2\sqrt{2}\pi^{3/2} (im_Q t)^{3/2}} e^{itPt} \bar{u}(P) \Gamma_1 \frac{\gamma^t + 1}{2} \Gamma_2 u(P) \left[1 + O\left(\frac{1}{m_Q t}\right) \right], \quad (\text{A.13})$$

$$\tilde{h}_{\Gamma_1\Gamma_2}^{(0)}(x, P, \mu) = e^{itPt} \bar{u}(P) \Gamma_1 \frac{\gamma^t + 1}{2} \Gamma_2 u(P). \quad (\text{A.14})$$

Therefore, the hard kernel at tree-level is obtained by comparing the above results with Eq. (3.18),

$$H_{\Gamma_1\Gamma_2,\hat{x}}^{(0)}\left(\frac{m_Q^2}{\mu^2}\right) = 1, \quad (\text{A.15})$$

which does not have P or x dependences as expected.

The one-loop diagrams are calculated under Feynman gauge and decomposed based on diagram topology,

$$\begin{aligned} M_{\Gamma_1\Gamma_2}^{(1)}(x, P, \mu) &= [M_{\Gamma_1\Gamma_2}^{\text{vertex}}(x, P, \mu) + M_{\Gamma_1\Gamma_2}^{\text{sail}}(x, P, \mu) \\ &+ M_{\Gamma_1\Gamma_2}^{\text{t.p.}}(x, P, \mu) + M_{\Gamma_1\Gamma_2}^{\text{w.fn.}}(x, P, \mu)] M_{\Gamma_1\Gamma_2}^{(0)}(x, P, \mu), \end{aligned} \quad (\text{A.16})$$

$$\begin{aligned} \tilde{h}_{\Gamma_1\Gamma_2}^{(1)}(x, P, \mu) &= [\tilde{h}_{\Gamma_1\Gamma_2}^{\text{vertex}}(x, P, \mu) + \tilde{h}_{\Gamma_1\Gamma_2}^{\text{sail}}(x, P, \mu) \\ &+ \tilde{h}_{\Gamma_1\Gamma_2}^{\text{t.p.}}(x, P, \mu) + \tilde{h}_{\Gamma_1\Gamma_2}^{\text{w.fn.}}(x, P, \mu)] \tilde{h}_{\Gamma_1\Gamma_2}^{(0)}(x, P, \mu), \end{aligned} \quad (\text{A.17})$$

where the names of Feynman diagrams are the same as Ref. [103]. The overall tree-level factors are factorized out.

First, it is obvious that

$$M_{\Gamma_1\Gamma_2}^{\text{vertex}}(x, P, \mu) = \tilde{h}_{\Gamma_1\Gamma_2}^{\text{vertex}}(x, P, \mu) + O\left(\frac{1}{m_Q t}\right), \quad (\text{A.18})$$

$$M_{\Gamma_1\Gamma_2}^{\text{w.fn.}}(x, P, \mu) = \tilde{h}_{\Gamma_1\Gamma_2}^{\text{w.fn.}}(x, P, \mu) + O\left(\frac{1}{m_Q t}\right), \quad (\text{A.19})$$

because the gluon and heavy quark momentum integrals can be studied independently in those diagrams. One can integrate out the heavy quark momentum and perform the large m_Q expansion, the remaining gluon momentum integral is the same as the corresponding contribution from $\tilde{h}_{\Gamma_1\Gamma_2}^{(1)}(x, P, \mu)$.

The perturbative results from the heavy quark self energy are,

$$M_{\Gamma_1\Gamma_2}^{\text{t.p.}}(x, P, \mu) = \frac{3}{2} \ln \frac{m_Q^2}{\mu^2} + \ln \frac{-t^2 \mu^2}{4e^{-2\gamma_E}} + O\left(\frac{1}{m_Q t}\right), \quad (\text{A.20})$$

$$\tilde{h}_{\Gamma_1\Gamma_2}^{\text{t.p.}}(x, P, \mu) = \ln \frac{-t^2 \mu^2}{4e^{-2\gamma_E}} + 2, \quad (\text{A.21})$$

where $M_{\Gamma_1\Gamma_2}^{\text{t.p.}}(x, P, \mu)$ is calculated using the cut diagram techniques and non-relativistic expansion of heavy quark momentum similar as Eq. (A.55).

The perturbative results from the sail diagrams are,

$$\begin{aligned}\tilde{h}_{\Gamma_1\Gamma_2}^{\text{sail}}(x, P, \mu) = & 2 \left(\frac{1}{\epsilon_{\text{IR}}} - \ln \frac{4P_z^2}{\mu^2} + 2 \ln(-i\lambda e^{\gamma_E}) + 1 \right) \\ & \times \left(\ln(-i\lambda e^{\gamma_E}) + \frac{i}{\lambda} - \frac{ie^{i\lambda}}{\lambda} + \Gamma(0, -i\lambda) - 1 \right) \\ & - \ln \frac{4P_z^2}{\mu^2} + 2 \ln(-i\lambda e^{\gamma_E}) + 4i\lambda {}_3F_3(1, 1, 1; 2, 2, 2; i\lambda)\end{aligned}\quad (\text{A.22})$$

$$M_{\Gamma_1\Gamma_2}^{\text{sail}}(x, P, \mu) = \tilde{h}_{\Gamma_1\Gamma_2}^{\text{sail}}(x, P, \mu) + V_{\Gamma_1, \hat{x}} \left(\frac{m_Q^2}{\mu^2} \right) + V_{\Gamma_2, \hat{x}} \left(\frac{m_Q^2}{\mu^2} \right) + O \left(\frac{1}{m_Q t} \right), \quad (\text{A.23})$$

where dimensionless distance $\lambda \equiv -P^t t$. The correction V is,

$$V_{\Gamma_1, \hat{x}} \left(\frac{m_Q^2}{\mu^2} \right) = \begin{cases} -\frac{3}{2} \ln \frac{m_Q^2}{\mu^2} + 2, & \Gamma_1 = I \\ \frac{i}{4\sqrt{-x^2}} \text{Tr}[\not{x}\gamma^\mu] - 1, & \Gamma_1 = \gamma^\mu \end{cases}, \quad (\text{A.24})$$

which has been calculated for $\Gamma_1, \Gamma_2 \in \{I, \gamma^t, \gamma^x, \gamma^y, \gamma^z\}$ as examples.

By comparing the above expressions, one validates the matching relation Eq. (3.18) in the time-like separation up to one-loop. The hard kernel can be extracted,

$$H_{\Gamma_1\Gamma_2, \hat{x}}^{(1)} \left(\frac{m_Q^2}{\mu^2} \right) = \frac{3}{2} \ln \frac{m_Q^2}{\mu^2} - 2 + V_{\Gamma_1, \hat{x}} \left(\frac{m_Q^2}{\mu^2} \right) + V_{\Gamma_2, \hat{x}} \left(\frac{m_Q^2}{\mu^2} \right), \quad (\text{A.25})$$

which only depends on the hard scale m_Q and irrelevant to other physical scales.

A.1.3 Gauge link reduction at spacelike separations

This subsection generalizes Eq. (3.18) to the spacelike case using the methods of complex analysis. We refer the readers to our original paper [233].

A.2 Large distance behavior for correlator in ϕ^3 theory

In this section, we study the large-distance asymptotic behavior of the “quasi-PDF matrix element” in ϕ^3 theory, using perturbative calculations up to two-loop order. These results provide supporting evidence for multiple features of dispersive analysis in Secs. 3.2.2 and 3.2.3, such as the physical origins of mass gaps, the impacts of form factor connectedness on the phases, power counting for polynomial dependences, cancellation and impact of forward singularity, and Lorentz symmetry to connect timelike and spacelike correlators. It also demonstrates the controlled precision of the large distance expansion truncated to a certain order, supporting the discussions in Sec. 3.4.1.1.

A.2.1 Definitions and notations

The Lagrangian for ϕ^3 theory,

$$\mathcal{L} = \frac{1}{2}(\partial^\mu \phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{g}{3!}\phi^3 + \frac{1}{2}\delta_Z(\partial_\mu \phi)^2 - \frac{1}{2}\delta_m\phi^2 - \frac{\delta_g}{3!}\phi^3, \quad (\text{A.26})$$

where δ_Z , δ_m and δ_g are renormalization counterterms. δ_m is defined for m to be a pole mass, and δ_Z, δ_g are in $\overline{\text{MS}}$ scheme.

To mimic the quasi-PDF matrix element in QCD, the following correlator is constructed,

$$M \left(\sqrt{-x^2}, \frac{iP \cdot x}{\sqrt{-x^2}} \right) \equiv Z_\phi^{-1} \langle P, \text{out} | \phi(x) \phi(0) | P, \text{in} \rangle_c, \quad (\text{A.27})$$

where the $|P\rangle$ is a large momentum single-particle state, with $P = (\sqrt{P_z^2 + m^2}, 0, 0, P^z)$ and $P^z \gg m$. They are defined as “in” and “out” states respectively so that the amplitude is a forward amplitude. $\phi(x)\phi(0)$ are scalar fields separated by x , which can be either timelike or spacelike. The disconnected diagrams are subtracted off as indicated by the lower script “c” defined in Eq. (3.19), and the wave-function diagram $Z_\phi = \langle P | \phi | \Omega \rangle \langle \Omega | \phi | P \rangle$ is eliminated out. The vacuum bubbles are not considered in the calculations.

The large distance ($x^2 \rightarrow \infty$ with fixed $P \cdot x / \sqrt{-x^2}$) asymptotic forms can be obtained using form factor representation¹,

$$M \left(\sqrt{-x^2}, \frac{iP \cdot x}{\sqrt{-x^2}} \right) = Z_\phi^{-1} \sum_n \int d\Gamma_n(k) e^{i(P-k) \cdot x} \langle P, \text{in} | \phi | n(k), \text{out} \rangle \langle n(k), \text{out} | \phi | P, \text{in} \rangle_c, \quad (\text{A.28})$$

where the intermediate particles are chosen as “out” states. The external state on the left is switched from “out” state to “in” state, which causes no difference for a stable particle but makes the discussions more convenient [57]. $\Gamma_n(k)$ is the multi-particle phase space and k is the total momentum as defined in or around Eq. (3.24). The form factor representation can be classified based on the connectedness

¹For the timelike case, we choose $t > 0$ to do the dispersive analysis.

of form factors,

$$\begin{aligned}
M\left(\sqrt{-x^2}, \frac{iP \cdot x}{\sqrt{-x^2}}\right) &= M_X\left(\sqrt{-x^2}, \frac{iP \cdot x}{\sqrt{-x^2}}\right) \\
&+ M_Y\left(\sqrt{-x^2}, \frac{iP \cdot x}{\sqrt{-x^2}}\right) + M_Z\left(\sqrt{-x^2}, \frac{iP \cdot x}{\sqrt{-x^2}}\right),
\end{aligned} \tag{A.29}$$

which follows the same convention as Eq. (3.27).

A.2.2 Perturbative results up to two-loop order

In this subsection, we present perturbative results on the large-distance asymptotic behavior up to two-loop order, which provide supporting evidence for Lorentz symmetry in connecting timelike and spacelike correlators, for the methods to determine the physical origins of mass gaps, for the relation between form factor connectedness and phases, and for power counting of polynomial dependences.

The perturbative results for a timelike separation $x = (t, 0, 0, 0)$ in the $t \rightarrow +\infty$ limit with fixed large P^t up to two-loop order,

$$\begin{aligned}
M(it, P^t)_{t \rightarrow +\infty} &= \left[M^{(0)}(it, P^t) + \frac{g^2}{(2\pi)^2} M^{(1)}(it, P^t) + \frac{g^4}{(2\pi)^4} M^{(2)}(it, P^t) \right] \\
&\times \left[1 + O\left(\frac{1}{imt}\right) \right] + O\left(e^{-\frac{\sqrt{3}}{2}|P^z t|}\right) + O(g^6),
\end{aligned} \tag{A.30}$$

where the tree-level result is,

$$M^{(0)}(it, P^t) = e^{itP^t} + e^{-itP^t}. \tag{A.31}$$

The one-loop result is (colors distinguish the contributions from different form factor representations:

red for M_X , blue for M_Y and orange for M_Z),

$$\begin{aligned}
M^{(1)}(it, P^t) &= \frac{e^{itP^t} e^{-imt}}{(imt)^{3/2}} \frac{\sqrt{\pi/2}}{(m - 2P^t)^2} \\
&+ \frac{e^{-imt}}{(imt)^{1/2}} \frac{\sqrt{\pi/2}(4P_t^2 + m^2)}{2m^2(4P_t^2 - m^2)} + \frac{e^{-itP^t} e^{-imt}}{(imt)^{3/2}} \frac{\sqrt{\pi/2}}{(m + 2P^t)^2}, \tag{A.32}
\end{aligned}$$

which is evaluated in both covariant perturbation theory and form factor representation, and the results are consistent between the two methods. The two-loop result is calculated using the form factor representation under stationary phase analysis,

$$\begin{aligned}
M^{(2)}(it, P^t) &= \frac{e^{itP^t} e^{-imt}}{(imt)^{3/2}} \frac{2\sqrt{2}\pi^{5/2}}{(m - 2P^t)^2} [2I_{\text{tr}}(2m(m - P^t)) - 2I_{\text{se}}(2m(m - P^t)) - I_{\text{se}}(m^2)] \\
&+ \frac{e^{itP^t} e^{-i2mt}}{(imt)^3} \frac{(7m - 2P^t)^2 \pi}{576m^2 (m - 2P^t)^2 (m - P^t)^2} \\
&+ \frac{e^{-imt}}{(imt)^{1/2}} \sqrt{2}\pi^{5/2} C(mP^t) + \frac{e^{-i2mt}}{(imt)^2} \frac{(4P_t^2 + m^2) \pi}{36m^4 (4P_t^2 - m^2)} \\
&+ \frac{e^{-itP^t} e^{-imt}}{(imt)^{3/2}} \frac{2\sqrt{2}\pi^{5/2}}{(m + 2P^t)^2} [2I_{\text{tr}}(2m(m + P^t)) - 2I_{\text{se}}(2m(m + P^t)) - I_{\text{se}}(m^2)] \\
&+ \frac{e^{-itP^t} e^{-i2mt}}{(imt)^3} \frac{(7m + 2P^t)^2 \pi}{576m^2 (m + 2P^t)^2 (m + P^t)^2}, \tag{A.33}
\end{aligned}$$

where the coefficient $C(k \cdot P)$ is,

$$\begin{aligned}
C(k \cdot P) = & -I_{\text{cbox}}(2m^2 + 2k \cdot P) - I_{\text{cbox}}(2m^2 - 2k \cdot P) \\
& - I_{\text{box}}(2m^2 + 2k \cdot P) - I_{\text{box}}(2m^2 - 2k \cdot P) \\
& - \frac{2I_{\text{tr}}(2m^2 + 2k \cdot P)}{m^2 + 2k \cdot P} - \frac{2I_{\text{tr}}(2m^2 - 2k \cdot P)}{m^2 - 2k \cdot P} - \frac{2I_{\text{tr0}}(m^2)}{m^2} \\
& + \frac{I_{\text{se}}(2m^2 + 2k \cdot P)}{m^2 + 2k \cdot P} + \frac{I_{\text{se}}(2m^2 - 2k \cdot P)}{m^2 - 2k \cdot P} + \frac{4m^2 I_{\text{se}}(m^2)}{m^4 - 4(k \cdot P)^2} - \frac{I_{\text{se}}(0) + 2I_{\text{se}}(m^2)}{m^2},
\end{aligned} \tag{A.34}$$

where the master integrals are,

$$\begin{aligned}
I_{\text{cbox}}(s = (P + k)^2) = & i \int \frac{d^4 q}{(2\pi)^4} \frac{1}{q^2 - m^2} \frac{1}{(q - P)^2 - m^2} \\
& \times \frac{1}{(q - k)^2 - m^2} \frac{1}{(q - k - P)^2 - m^2} \Big|_{k^2=m^2, P^2=m^2},
\end{aligned} \tag{A.35}$$

$$\begin{aligned}
I_{\text{box}}(s = (P + k)^2) = & i \int \frac{d^4 q}{(2\pi)^4} \frac{1}{(P - q + k)^2 - m^2} \left(\frac{1}{(P - q)^2 - m^2} \right)^2 \frac{1}{q^2 - m^2} \Big|_{k^2=m^2, P^2=m^2},
\end{aligned} \tag{A.36}$$

$$\begin{aligned}
I_{\text{tr}}(t = (P - k)^2) = & i \int \frac{d^4 q}{(2\pi)^4} \frac{1}{(q - k)^2 - m^2} \frac{1}{q^2 - m^2} \frac{1}{(q - P)^2 - m^2} \Big|_{k^2=m^2, P^2=m^2},
\end{aligned} \tag{A.37}$$

$$I_{\text{tr}0}(k^2) = \int \frac{d^4 q}{(2\pi)^4} \frac{i}{(k-q)^2 - m^2} \left(\frac{i}{q^2 - m^2} \right)^2, \quad (\text{A.38})$$

$$I_{\text{se}}(k^2) = \frac{i}{k^2 - m^2} \frac{1}{2} \left(\frac{\mu^2 e^{\gamma_E}}{4\pi} \right)^\epsilon \int \frac{d^d q}{(2\pi)^d} \frac{i}{(k-q)^2 - m^2} \frac{i}{q^2 - m^2} - \frac{k^2 \delta_Z^{(1)} - \delta_m^{(1)}}{k^2 - m^2}, \quad (\text{A.39})$$

where the renormalization contour terms at one-loop is,

$$\delta_Z^{(1)} = 0, \quad \delta_m^{(1)} = \frac{1}{32\pi^2} \left(-\frac{1}{\epsilon_{\text{UV}}} + \ln \frac{m^2}{\mu^2} - 2 + \frac{\pi}{\sqrt{3}} \right). \quad (\text{A.40})$$

The three-point and four-point scalar integrals I_{tr} , I_{box} , and I_{cbox} have been studied in Refs. [410–413]. In Eq. (A.30), for each type of exponential behavior, we only keep the leading polynomial dependence and the omitted higher order terms are denoted as $O(1/(imt))$. The exponential decaying terms regarding the single-particle on-shell production in the form factors are ignored because they are at $O\left(e^{-\frac{\sqrt{3}}{2}|P^z t|}\right)$ which are much more suppressed compared to other terms for a large momentum external state. All the propagators in the master integrals are Feynman propagators with $+i\epsilon$ prescription.

The perturbative calculations have also been performed for a spacelike separation $x = (0, 0, 0, z)$ in the $z \rightarrow +\infty$ limit with fixed large P^z ,

$$\begin{aligned} M(z, -iP^z)_{z \rightarrow +\infty} &= \left[M^{(0)}(z, -iP^z) + \frac{g^2}{(2\pi)^2} M^{(1)}(z, -iP^z) + \frac{g^4}{(2\pi)^4} M^{(2)}(z, -iP^z) \right] \\ &\times \left[1 + O\left(\frac{1}{mz}\right) \right] + O\left(e^{-\frac{\sqrt{3}}{2}\sqrt{P_z^2 + m^2}|z|}\right) + O(g^6), \end{aligned} \quad (\text{A.41})$$

where the one-loop result is,

$$\begin{aligned}
M^{(1)}(z, -iP^z) &= \frac{e^{-iP^z z - mz}}{(mz)^{3/2}} \frac{\sqrt{\pi/2}}{(m + 2iP^z)^2} \\
&+ \frac{e^{-mz}}{\sqrt{mz}} \frac{\sqrt{\pi/2}(-m^2 + 4P_z^2)}{2m^2(m^2 + 4P_z^2)} + \frac{e^{iP^z z} e^{-mz}}{(mz)^{3/2}} \frac{\sqrt{\pi/2}}{(m - 2iP^z)^2},
\end{aligned} \tag{A.42}$$

and the two-loop result is,

$$\begin{aligned}
M^{(2)}(z, -iP^z) &= \frac{e^{-izP^z} e^{-mz}}{(zm)^{3/2}} \frac{2\sqrt{2}\pi^{5/2}}{(m + 2iP^z)^2} (2I_{\text{tr}}(2m(m + iP^z)) - 2I_{\text{se}}(2m(m + iP^z)) - I_{\text{se}}(m^2)) \\
&+ \frac{e^{-izP^z} e^{-2mz}}{(zm)^3} \frac{\pi(7m + 2iP^z)^2}{576m^2(m^2 + 3imP^z - 2P_z^2)^2} \\
&+ \frac{e^{-mz}}{\sqrt{zm}} \sqrt{2}\pi^{5/2} C(-imP^z) + \frac{e^{-2mz}}{(zm)^2} \frac{\pi(-m^2 + 4P_z^2)}{36m^4(m^2 + 4P_z^2)} \\
&+ \frac{e^{izP^z} e^{-mz}}{(zm)^{3/2}} \frac{2\sqrt{2}\pi^{5/2}}{(m - 2iP^z)^2} (2I_{\text{tr}}(2m(m - iP^z)) - 2I_{\text{se}}(2m(m - iP^z)) - I_{\text{se}}(m^2)) \\
&+ \frac{e^{izP^z} e^{-2mz}}{(zm)^3} \frac{\pi(7m - 2iP^z)^2}{576m^2(m^2 - 3imP^z - 2P_z^2)^2}.
\end{aligned} \tag{A.43}$$

Please check the definitions for I_{tr} in Eq. (A.37), I_{se} in Eq. (A.39), and C in Eq. (A.34).

The timelike and spacelike results in Eqs. (A.32), (A.33), (A.42) and (A.43) are consistent with the following Lorentz invariant forms,

$$\begin{aligned}
M^{(1)}\left(\sqrt{-x^2}, \mathcal{P}_x \equiv \frac{ix \cdot P}{\sqrt{-x^2}}\right) &= \frac{e^{ix \cdot P} e^{-m\sqrt{-x^2}}}{(m\sqrt{-x^2})^{3/2}} \frac{\sqrt{\pi/2}}{(m - 2\mathcal{P}_x)^2} \\
&+ \frac{e^{-m\sqrt{-x^2}}}{(m\sqrt{-x^2})^{1/2}} \frac{\sqrt{\pi/2}(4\mathcal{P}_x^2 + m^2)}{2m^2(4\mathcal{P}_x^2 - m^2)} + \frac{e^{-ix \cdot P} e^{-m\sqrt{-x^2}}}{(m\sqrt{-x^2})^{3/2}} \frac{\sqrt{\pi/2}}{(m + 2\mathcal{P}_x)^2},
\end{aligned} \tag{A.44}$$

$$\begin{aligned}
M^{(2)} \left(\sqrt{-x^2}, \mathcal{P}_x \equiv \frac{ix \cdot P}{\sqrt{-x^2}} \right) &= \frac{e^{ix \cdot P} e^{-m\sqrt{-x^2}}}{(m\sqrt{-x^2})^{3/2}} \frac{2\sqrt{2}\pi^{5/2}}{(m-2\mathcal{P}_x)^2} [2I_{\text{tr}}(2m(m-\mathcal{P}_x)) - 2I_{\text{se}}(2m(m-\mathcal{P}_x)) - I_{\text{se}}(m^2)] \\
&+ \frac{e^{ix \cdot P} e^{-2m\sqrt{-x^2}}}{(m\sqrt{-x^2})^3} \frac{(7m-2\mathcal{P}_x)^2 \pi}{576m^2 (m-2\mathcal{P}_x)^2 (m-\mathcal{P}_x)^2} \\
&+ \frac{e^{-m\sqrt{-x^2}}}{(m\sqrt{-x^2})^{1/2}} \sqrt{2}\pi^{5/2} C(m\mathcal{P}_x) + \frac{e^{-2m\sqrt{-x^2}}}{(m\sqrt{-x^2})^2} \frac{(4\mathcal{P}_x^2 + m^2) \pi}{36m^4 (4\mathcal{P}_x^2 - m^2)} \\
&+ \frac{e^{-ix \cdot P} e^{-m\sqrt{-x^2}}}{(m\sqrt{-x^2})^{3/2}} \frac{2\sqrt{2}\pi^{5/2}}{(m+2\mathcal{P}_x)^2} [2I_{\text{tr}}(2m(m+\mathcal{P}_x)) - 2I_{\text{se}}(2m(m+\mathcal{P}_x)) - I_{\text{se}}(m^2)] \\
&+ \frac{e^{-ix \cdot P} e^{-2m\sqrt{-x^2}}}{(m\sqrt{-x^2})^3} \frac{(7m+2\mathcal{P}_x)^2 \pi}{576m^2 (m+2\mathcal{P}_x)^2 (m+\mathcal{P}_x)^2}, \tag{A.45}
\end{aligned}$$

which justifies the usage of Lorentz symmetry in Eq. (3.44) in the main text. Also see the discussions below Eq. (3.15) on the prescription of $\sqrt{-x^2}$.

In the above results, the mass gaps in the exponential factors are m and $2m$, which are the threshold masses of one-particle and two-particle intermediate states, respectively. They are covered by the spectra of the quantum numbers $\phi|P\rangle$ for M_X , $\phi|\Omega\rangle$ for M_Y , and $\phi|\bar{P}\rangle$ for M_Z , that appear at one loop and two loop. Therefore, these results substantiate the methods to identify the physical origins of mass gaps in Eqs. (3.30), (3.39), (3.32), and (3.42).

Consider the overall phase factors associated with the external momentum P . The factor $\sim e^{ix \cdot P}$ appears in the fully connected form factor representation M_X . If one form factor is connected and the other one is not, as in M_Y , there is no overall phase factor involving the external momentum. The factor $\sim e^{-ix \cdot P}$ corresponds to M_Z , where both form factors are disconnected. Consequently, these perturbative calculations provide supporting evidence for the relation between form factor connectedness (M_X, M_Y, M_Z) and phases ($e^{ix \cdot P}$, 1 or $e^{-ix \cdot P}$) in Eqs. (3.30), (3.39), (3.32), and (3.42).

Regarding the polynomial dependences on $\sqrt{-x^2}$, the asymptotic behaviors of M_X and M_Z , in-

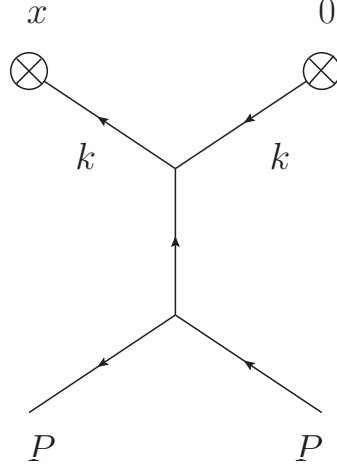


Figure A.3: The one-loop example I_t in covariant perturbation theory for “quasi-PDF matrix element” in ϕ^3 theory.

cluding one-particle and two-particle insertions, agree with the naive power counting $k_i \sim (m_i, \sqrt{m_i/t}, \sqrt{m_i/t}, \sqrt{m_i/t})$ explained below Eq. (3.30). In the case of M_Y here, the polynomial dependence is linearly enhanced as a consequence of the forward singularity, while this enhancement does not appear in Eq. (3.39) in the main text because it is forbidden by the spin quantum numbers as explained around Eq. (3.38). Further discussions on the forward singularity will be provided in the next subsection.

A.2.3 Cancellation and impact of forward singularity

In this subsection, we explicitly show how the forward singularity is canceled in a representative one-loop diagram, thereby substantiating the discussion surrounding Eq. (3.36). We further interpret the resulting enhanced polynomial dependence as the remnant of the forward singularity, providing additional support for Eq. (3.37).

Consider the “ t -channel” diagram at one-loop displayed in Fig. A.3. In covariant perturbation

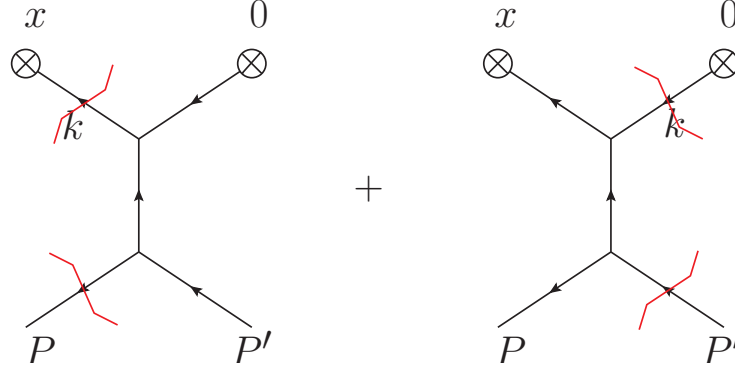


Figure A.4: The form factor representation of Fig. A.3, which belongs to M_Y in Eq. (A.29), where one form factor is connected and the other one is disconnected.

theory, the corresponding integral is

$$I_t(\sqrt{-x^2}) = -g^2 \int \frac{d^4 k}{(2\pi)^4} e^{-ik \cdot x} \left(\frac{i}{k^2 - m^2 + i\epsilon} \right)^2 \frac{i}{-m^2 + i\epsilon} = \frac{g^2 K_0(m\sqrt{-x^2})}{m^2 (8\pi^2)}, \quad (\text{A.46})$$

which can be evaluated directly in a Lorentz-invariant manner using Schwinger parameters. The final result is finite and can be written in terms of the modified Bessel function of the second kind, K_0 .

However, its form factor representation contains a potential singularity originating from the quadratic term $[i/(k^2 - m^2 + i\epsilon)]^2$ due to the symmetric structure of the Feynman diagram. When one attempts to derive the form-factor representation by cutting one of the propagators using the Cutkosky cutting rule, the other one becomes singular. Such singularity has also been studied in Ref. [57].

To regulate this singularity, one introduces off-forwardness, which takes the external momenta to be different,

$$\langle P, \text{out} | \phi(x) \phi(0) | P, \text{in} \rangle_c \rightarrow \langle P, \text{out} | \phi(x) \phi(0) | P', \text{in} \rangle_c. \quad (\text{A.47})$$

Then, the form factor representation, in Fig. A.4, can be obtained by directly applying Feynman rules

of form factors,

$$\begin{aligned}
I_t(\sqrt{-x^2}) &= \frac{g^2}{(P' - P)^2 - m^2} \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{e^{-ik \cdot x}}{2E_k} \frac{1}{(E_{P'} - E_P - E_k)^2 - E_{k+P-P'}^2 + i\epsilon} \\
&+ \frac{g^2 e^{i(P-P') \cdot x}}{(P' - P)^2 - m^2} \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{e^{-ik \cdot x}}{2E_k} \frac{1}{(E_P - E_{P'} - E_k)^2 - E_{k+P'-P}^2 - i\epsilon}, \quad (\text{A.48})
\end{aligned}$$

where $E_k \equiv \sqrt{\vec{k}^2 + m^2}$. The above expression contains two terms, and each of them becomes singular in the forward limit $P' \rightarrow P$, which is referred to as forward singularity in our terminology. However, we will demonstrate the cancellation of this forward singularity between the two terms. Also note that the signs of the $i\epsilon$ prescription originate from the time-ordering regarding the “in” and “out” states in Eq. (A.28), which are essential for verifying this cancellation.

We begin by examining the timelike case $x = (t, 0, 0, 0)$. After changing integral variables and recombining terms, one obtains

$$\begin{aligned}
I_t(it) &= \frac{g^2}{(P' - P)^2 - m^2} \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{e^{-iE_k t}}{2E_k} \left[\frac{e^{-i(E_{k+P'-P} - E_k)t} - e^{i(E_P - E_{P'})t}}{2E_{k+P'-P}(E_P - E_{P'} + E_{k+P'-P} - E_k + i\epsilon)} \right. \\
&\left. + \frac{1}{2E_{P'-P-k}(E_{P'} - E_P - E_k - E_{P'-P-k})} + \frac{e^{i(E_P - E_{P'})t}}{2E_{P-P'-k}(E_P - E_{P'} - E_k - E_{P-P'-k})} \right], \quad (\text{A.49})
\end{aligned}$$

where, in the first term in the square brackets, the forward singularity is canceled between the numerator and the denominator. The forward limit $P' \rightarrow P$ can then be smoothly taken,

$$I_t(it) = \frac{g^2}{-m^2} \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{e^{-iE_k t}}{2E_k} \left[\frac{-it}{2E_k} - \frac{1}{2E_k^2} \right], \quad (\text{A.50})$$

where the factor $-it$ originates from the expansion of the phase factor $e^{-i(E_{k+P'-P} - E_k)t} - e^{i(E_P - E_{P'})t}$, which cancels the singularity from the denominator and enhances the polynomial dependence com-

pared to naive power counting.

The cancellation of forward singularity can also be studied in a spacelike case $x = (0, 0, 0, z)$, where the expression can be simplified by redefining the integral variables,

$$\begin{aligned}
I_t(z) &= \frac{g^2}{(P' - P)^2 - m^2} \int \frac{d^3 \vec{k}}{(2\pi)^3} \frac{e^{-iPz} e^{ikz}}{2E_{k-P} 2E_{k-P'}} \\
&\times \left[\frac{1}{E_{P'} - E_P - E_{k-P} - E_{k-P'}} - \frac{1}{E_{P'} - E_P - E_{k-P} + E_{k-P'} - i\epsilon} \right. \\
&\quad \left. + \frac{1}{E_P - E_{P'} - E_{k-P'} - E_{k-P}} - \frac{1}{E_P - E_{P'} - E_{k-P'} + E_{k-P} + i\epsilon} \right], \tag{A.51}
\end{aligned}$$

in which the forward singularity is exactly canceled between the second and fourth terms in the square bracket. One can then take the forward limit $P' \rightarrow P$,

$$I_t(z) = \frac{g^2}{-m^2} \int \frac{d^3 \vec{k}}{(2\pi)^3} \frac{e^{ikz}}{2E_k} \left[\frac{-1}{2E_k^2} \right], \tag{A.52}$$

where the energy denominator exhibits a more singular behavior at $|\vec{k}| = \pm im$ compared to the M_X and M_Z cases, which causes the enhancement of the polynomial dependence, as we will show in detail as the last example in the next subsection.

A.2.4 Calculation methods under form factor representation

In this subsection, we present our methods for calculating the large distance asymptotic forms in Eqs. (A.30) and (A.41), using selected one-loop diagrams as examples. This subsection is not just a collection of technical details but also a pedagogical journey deepening our understanding of the origins of asymptotic forms.

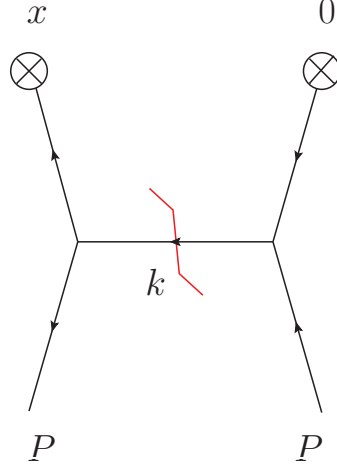


Figure A.5: The one-loop example $I_{s,1}$ belonging to M_X in Eq. (A.29), where both form factors are connected.

We first consider a one-loop example associated with M_X in Eq. (A.29),

$$I_{s,1} \left(\sqrt{-x^2}, \frac{iP \cdot x}{\sqrt{-x^2}} \right) = g^2 \int \frac{d^3 \vec{k}}{(2\pi)^3} \frac{e^{i(P-k) \cdot x}}{2E_k} \left[\frac{1}{(E_k - E_P)^2 - E_{k-P}^2} \right]^2, \quad (\text{A.53})$$

with the corresponding Feynman diagram displayed in Fig. A.5. The quantity E_k is defined below Eq. (A.48).

For a timelike separation $x = (t, 0, 0, 0)$, the integral becomes,

$$I_{s,1} (it, P^t) = \frac{g^2 e^{iP^t t}}{(2\pi)^2} \int \frac{d\Omega}{4\pi} \int_0^{+\infty} dk \, 2k^2 \frac{e^{-iE_k t}}{2E_k} \left[\frac{1}{(E_k - E_P)^2 - E_{k-P}^2} \right]^2. \quad (\text{A.54})$$

In the large time limit, the leading asymptotics can be obtained using the stationary phase analysis,

$$\begin{aligned} I_{s,1} (it, P^t)_{t \rightarrow +\infty} &= \frac{g^2 e^{iP^t t}}{(2\pi)^2} \int \frac{d\Omega}{4\pi} \int_0^{+\infty} dk \, 2k^2 \frac{e^{-imt - i\frac{k^2}{2m}t}}{2m} \left[\frac{1}{(m - P^t)^2 - P_t^2} \right]^2 \\ &\times [1 + O(kP^z \cos(\theta)) + O(k^2)], \end{aligned} \quad (\text{A.55})$$

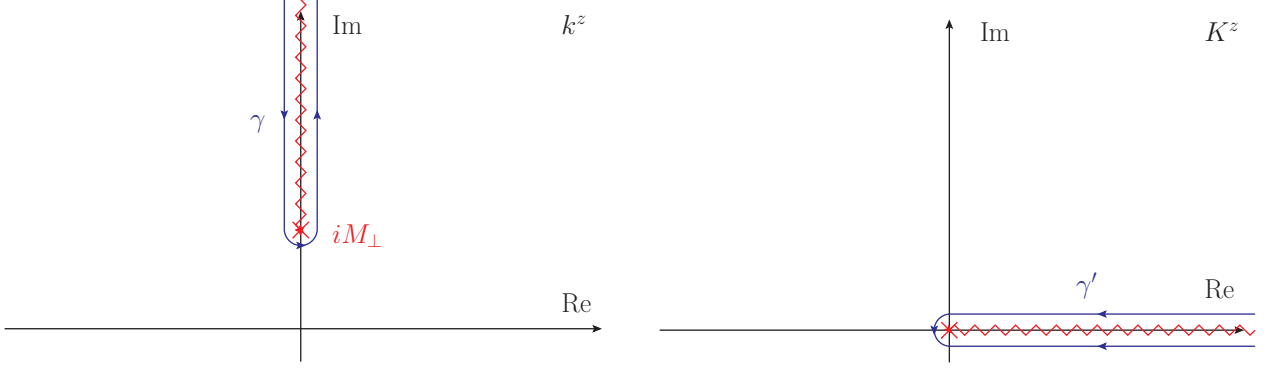


Figure A.6: Integral contour in the complex plane of k^z or K^z for evaluating the large distance asymptotic form.

where, in the exponent, we have retained the non-relativistic expansion up to the kinetic energy term $k^2/(2m)$. Here θ denotes the angle between $\vec{k}$ and $\vec{P}$. Before doing this integral explicitly, we anticipate an overall exponential factor of the form $\sim e^{iP^t t} e^{-imt}$, related to the energy m at the stationary point. Spatial fluctuations around this stationary point scale as $\sim (\sqrt{m/t}, \sqrt{m/t}, \sqrt{m/t})$, which determines the polynomial dependence on t ; by simple power counting in k , the leading contribution scales as $1/t^{3/2}$. The remaining k -dependence in the integrand is expanded order by order, which corresponds to various higher-order polynomial contributions in the large t expansion. Integrating out k and Ω gives,

$$I_{s,1}(it, P^t)_{t \rightarrow +\infty} = \frac{g^2}{(2\pi)^2} \frac{e^{iP^t t} e^{-imt}}{(imt)^{3/2}} \frac{\sqrt{\pi/2}}{(m - 2P^t)^2} \left[1 + O\left(\frac{1}{imt}\right) \right], \quad (\text{A.56})$$

where the leading exponential and polynomial dependences agree with our expectations. The subleading term is suppressed by $O(1/(imt))$ instead of $O(1/\sqrt{imt})$ because the $O(kP^z \cos \theta)$ term vanishes after we integrate out the solid angle, a consequence of rotational symmetry.

For a spacelike separation $x = (0, 0, 0, z)$, Eq. (A.53) becomes,

$$I_{s,1}(z, -iP^z) = \frac{g^2 e^{-iP^z z}}{(2\pi)^2} \int_m^{+\infty} dM_\perp \int_{-\infty}^{+\infty} dk^z M_\perp e^{ik^z z} f_{s,1}(k^z, M_\perp), \quad (\text{A.57})$$

where

$$f_{s,1}(k^z, M_\perp) \equiv \frac{1}{2E_k} \left[\frac{1}{(E_k - E_P)^2 - E_{k-P}^2} \right]^2, \quad (\text{A.58})$$

and $M_\perp \equiv \sqrt{m^2 + k_x^2 + k_y^2}$ represents the effective mass associated with the transverse momentum. To evaluate the results for $z \rightarrow +\infty$, we deform the k^z integral contour upward in the complex plane around the non-holomorphic structures of the integrand. The energy denominator $1/E_k$ has a branch cut starting at $k^z = iM_\perp$ and extending upward along the imaginary axis, which leads to the exponential behavior $\sim e^{-iP^z z} e^{-mz}$. In addition, the factor $[1/(E_k - E_P)^2 - E_{k-P}^2]^2$ contains a second order residue pole located at $k^z = \frac{P^z}{2} + i\sqrt{\frac{M_\perp^2}{m^2} - \frac{1}{4}}P^t$, yielding the exponential factor $\sim e^{-i\frac{P^z}{2}z - \frac{\sqrt{3}}{2}P^t z}$. For $P^z \gg m$, the latter one is highly suppressed compared to the first one. Therefore, we focus on the first one,

$$I_{s,1}(z, -iP^z)_{z \rightarrow +\infty} = \frac{g^2 e^{-iP^z z}}{(2\pi)^2} \int_m^{+\infty} dM_\perp \int_\gamma dk^z M_\perp e^{ik^z z} f_{s,1}(k^z, M_\perp) + O\left(e^{-\frac{\sqrt{3}}{2}P^t z}\right), \quad (\text{A.59})$$

where γ is a contour around the branch cut of the energy denominator $1/\sqrt{k_z^2 + M_\perp^2}$ as depicted in the left panel of Fig. A.6. We now perform the change of variables $k^z \rightarrow i\left(\frac{K^z}{z} + M_\perp\right)$ followed by

$$M_{\perp} \rightarrow \frac{M_{\perp}}{z} + m,$$

$$\begin{aligned} I_{s,1}(z, -iP^z)_{z \rightarrow +\infty} &= \frac{g^2 e^{-iP^z z} e^{-mz}}{(2\pi)^2} \int_0^{+\infty} dM_{\perp} \int_{\gamma'} dK^z \frac{i}{z^2} \left(\frac{M_{\perp}}{z} + m \right) \\ &\times e^{-K^z - M_{\perp}} f_{s,1} \left(i \left(\frac{K^z}{z} + \frac{M_{\perp}}{z} + m \right), \frac{M_{\perp}}{z} + m \right) + O \left(e^{-\frac{\sqrt{3}}{2} P^t z} \right), \end{aligned} \quad (\text{A.60})$$

where γ' now denotes a contour along the real axis shown in the right panel of Fig. A.6. The exponential behavior $\sim e^{-iP^z z} e^{-mz}$ has already been factorized out and is consistent with our expectation from the start of the branch cut, $\vec{k} \sim (0, 0, im)$. To calculate the leading polynomial z dependence, we perform the large z expansion for the integrand,

$$\begin{aligned} I_{s,1}(z, -iP^z)_{z \rightarrow +\infty} &= \frac{g^2 e^{-iP^z z} e^{-mz}}{(2\pi)^2} \int_0^{+\infty} dM_{\perp} \int_{\gamma'} dK^z e^{-K^z - M_{\perp}} \\ &\times \frac{im}{z^2} \frac{\sqrt{z}}{2\sqrt{2}\sqrt{-K^z m}} \frac{1}{(m^2 + 2imP^z)^2} \left[1 + O \left(\frac{1}{mz} \right) \right] + O \left(e^{-\frac{\sqrt{3}}{2} P^t z} \right) \\ &= \frac{g^2 e^{-iP^z z} e^{-mz}}{(2\pi)^2} \int_0^{+\infty} dM_{\perp} \int_0^{+\infty} dK^z e^{-K^z - M_{\perp}} \\ &\times \frac{im}{z^2} \frac{\sqrt{z}}{\sqrt{2i}\sqrt{K^z m}} \frac{1}{(m^2 + 2imP^z)^2} \left[1 + O \left(\frac{1}{mz} \right) \right] + O \left(e^{-\frac{\sqrt{3}}{2} P^t z} \right) \\ &= \frac{g^2}{(2\pi)^2} \frac{e^{-iP^z z} e^{-mz}}{(mz)^{3/2}} \frac{\sqrt{\pi/2}}{(m + 2iP^z)^2} \left[1 + O \left(\frac{1}{mz} \right) \right] + O \left(e^{-\frac{\sqrt{3}}{2} P^t z} \right), \end{aligned} \quad (\text{A.61})$$

where the 3D phase space provides $1/z^2$ from spatial fluctuations $\sim (\sqrt{m/z}, \sqrt{m/z}, 1/z)$ and the energy denominator $1/E_k$ yields $\sqrt{z/m}$ due to its singular structure at $k^z \sim iM_{\perp}$ or $K^z \sim 0$.

Although the mathematical origins of exponential and polynomial behaviors seem quite different between the timelike and spacelike cases, the final results for the leading asymptotics can be cast in a

Lorentz invariant format,

$$I_{s,1} \left(\sqrt{-x^2}, \mathcal{P}_x \equiv \frac{ix \cdot P}{\sqrt{-x^2}} \right)_{\sqrt{-x^2} \rightarrow \infty} = \frac{g^2}{(2\pi)^2} \frac{e^{iP \cdot x} e^{-m\sqrt{-x^2}}}{(m\sqrt{-x^2})^{3/2}} \frac{\sqrt{\pi/2}}{(m - 2\mathcal{P}_x)^2} \left[1 + O \left(\frac{1}{m\sqrt{-x^2}} \right) \right]. \quad (\text{A.62})$$

Moreover, even the power counting in the two cases can be related by a complex Lorentz transformation [377]. In the timelike result in Eq. (A.56), the exponential and polynomial t dependences can be reproduced by power counting $k \sim (m + C/it, \sqrt{m/it}, \sqrt{m/it}, \sqrt{m/it})$ for the integrand and $dk \sim (C/it, \sqrt{m/it}, \sqrt{m/it}, \sqrt{m/it})$ for the phase space $\int d^3\vec{k}$ in Eq. (A.53). We also impose $k^2 - m^2 \sim 1/t^2$, implying that the momentum is nearly on-shell. The $O(1/t)$ in the time component does not affect the leading asymptotics. On the other hand, the exponential and polynomial z dependences in the spacelike result in Eq. (A.61) follows from power counting $k \sim (i\sqrt{m/z}, \sqrt{m/z}, \sqrt{m/z}, im + iC/z)$ and $dk \sim (i\sqrt{m/z}, \sqrt{m/z}, \sqrt{m/z}, iC/z)$ in Eq. (A.53). In this case, the $O(1/z)$ term does contribute to the leading asymptotics. The momentum modes in the spacelike case can be obtained from the timelike case via the following complex Lorentz transformation²,

$$\Lambda(\zeta) = \begin{pmatrix} \cosh(\zeta) & 0 & 0 & \sinh(\zeta) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh(\zeta) & 0 & 0 & \cosh(\zeta) \end{pmatrix}, \quad (\text{A.63})$$

with $\zeta = i\pi/2$, which implies $\cosh(\zeta) = 0$ and $\sinh(\zeta) = i$.

To gain further insight into how the forward singularity influences the polynomial dependence,

²This is to relate the Lorentz vector structures of the momentum modes. The Lorentz scalars it and z are linked through the invariant $\sqrt{-x^2}$.

we present the calculations for I_t defined in Eq. (A.46), associated with M_Y in Eq. (A.29). The corresponding form factor representation in the timelike region is given in Eq. (A.50). Its large time asymptotic behavior can be calculated using stationary phase analysis,

$$\begin{aligned} I_t(it)_{t \rightarrow +\infty} &= \frac{g^2}{-m^2(2\pi)^2} \int \frac{d\Omega}{4\pi} \int_0^{+\infty} dk \, 2k^2 \frac{e^{-imt - i\frac{k^2}{2m}t}}{2m} \frac{(-it)}{2m} \left[1 + O\left(\frac{1}{it}\right) + O(k^2) \right] \\ &= \frac{g^2}{(2\pi)^2} \frac{e^{-imt}}{\sqrt{imt}} \frac{\sqrt{\pi}}{2\sqrt{2}m^2} \left[1 + O\left(\frac{1}{imt}\right) \right], \end{aligned} \quad (\text{A.64})$$

where the naive power counting of phase space provides $1/t^{3/2}$, and the remnant of forward singularity gives t . As a result, the polynomial t dependence becomes $1/t^{1/2}$.

The expression for the spacelike case has been obtained in Eq. (A.52),

$$\begin{aligned} I_t(z) &= \frac{g^2}{(2\pi)^2 m^2} \int_m^{+\infty} dM_\perp \int_{-\infty}^{+\infty} dk^z M_\perp e^{ik^z z} \frac{1}{4 \left(\sqrt{k_z^2 + M_\perp^2} \right)^3} \\ &= \frac{g^2}{(2\pi)^2 m^2} \int_m^{+\infty} dM_\perp \int_\gamma dk^z M_\perp e^{ik^z z} \frac{1}{4 \left(\sqrt{k_z^2 + M_\perp^2} \right)^3}, \end{aligned} \quad (\text{A.65})$$

where the contour γ is depicted in the left panel of Fig. A.6. Change the integral variable $k^z \rightarrow i \left(\frac{K^z}{z} + M_\perp \right)$ followed by $M_\perp \rightarrow \frac{M_\perp}{z} + m$, and perform the large z expansion,

$$I_t(z)_{z \rightarrow +\infty} = \frac{g^2 e^{-mz}}{(2\pi)^2 m^2} \int_0^{+\infty} dM_\perp \int_{\gamma'} dK^z e^{-K^z - M_\perp} \frac{im}{z^2} \frac{z^{3/2}}{(-K^z)^{3/2} 8\sqrt{2}m^{3/2}} \left[1 + O\left(\frac{1}{mz}\right) \right], \quad (\text{A.66})$$

where γ' denotes a contour illustrated in the right panel of Fig. A.6. The phase space produces $1/z^2$, and the singularity from $1/E_k^3$ generates $z^{3/2}$, which leads to an overall scaling of $1/z^{1/2}$.

To complete the calculation, we apply the derivative trick,

$$\frac{\partial}{\partial K^z} \frac{2}{(-K^z)^{1/2}} = \frac{1}{(-K^z)^{3/2}}, \quad (\text{A.67})$$

together with integration by parts,

$$\begin{aligned} I_t(z)_{z \rightarrow +\infty} &= \frac{g^2 e^{-mz}}{(2\pi)^2 m^2} \int_0^{+\infty} dM_\perp \int_{\gamma'} dK^z e^{-K^z - M_\perp} \frac{im}{z^2} \frac{2z^{3/2}}{(-K^z)^{1/2} 8\sqrt{2}m^{3/2}} \left[1 + O\left(\frac{1}{mz}\right) \right] \\ &= \frac{g^2 e^{-mz}}{(2\pi)^2 m^2} \int_0^{+\infty} dM_\perp \int_0^{+\infty} dK^z e^{-K^z - M_\perp} \frac{im}{z^2} \frac{4z^{3/2}}{i(K^z)^{1/2} 8\sqrt{2}m^{3/2}} \left[1 + O\left(\frac{1}{mz}\right) \right] \\ &= \frac{g^2}{(2\pi)^2} \frac{e^{-mz}}{\sqrt{zm}} \frac{\sqrt{\pi}}{2\sqrt{2}m^2} \left[1 + O\left(\frac{1}{mz}\right) \right]. \end{aligned} \quad (\text{A.68})$$

Although the mechanism for the enhanced polynomial dependence originating from the forward singularity seems quite different between the timelike and spacelike cases, the final results, including the polynomial dependences, are connected by Lorentz symmetry.

A.2.5 Forward singularity under split mass regularization

This subsection studies the forward singularity under split mass regularization, which is a different method compared to the off-forward regulator adopted in Eq. (3.35), Appendices A.2.3 and A.2.4. It provides a complementary perspective on the forward singularity and further justifies its cancellation and impact. We refer the readers to our original paper [233].

A.2.6 Non-convergent expansion and controlled precision

In the previous subsections, we have performed the large- z expansion when evaluating the integrals. This naturally leads to the question: Does this large- z expansion converge? The answer is No. The large z asymptotic expansion is divergent because the point $z = \infty$ is non-analytic. Nonetheless, the expansion is Borel summable, which means it yields an unambiguous result if regularized in the Borel plane. Furthermore, truncating the series at a relatively low order already provides an accurate approximation of the original function at large distances; For instance, truncation at the minimum point leads to an exponentially small error.

We demonstrate the above idea in the simplest case I_t , whose full result has been evaluated in Eq. (A.46),

$$I_t(z) = \frac{g^2 K_0(m|z|)}{8\pi^2 m^2}, \quad (\text{A.69})$$

where K_0 is a modified second kind Bessel function. For convenience, we introduce the notation $Z \equiv m|z|$. The large Z asymptotic series of $K_0(Z)$ has been worked out by Hankel [414, 415],

$$K_0(Z) = \sqrt{\frac{\pi}{2Z}} e^{-Z} \left[1 + \sum_{l=0}^{\infty} (-1)^{l+1} \frac{((2l+1)!!)^2}{(l+1)!(8Z)^{l+1}} \right]. \quad (\text{A.70})$$

In the large l limit, the coefficient behaves as

$$(-1)^{l+1} \frac{((2l+1)!!)^2}{(l+1)! 8^{l+1}} \xrightarrow{l \rightarrow +\infty} (-1)^{l+1} \frac{l!}{2^{l+1} \pi}, \quad (\text{A.71})$$

where the factorial growth $l!$ indicates that the series is divergent.

Since the series is alternating, it is Borel summable, which means there is no ambiguity if regularized in the Borel plane. Applying the Borel transform gives

$$\begin{aligned} K_0(Z) &= \sqrt{\frac{\pi}{2Z}} e^{-Z} \left[1 + \int_C dt e^{-tZ} \sum_{l=0}^{\infty} \frac{t^l}{l!} (-1)^{l+1} \frac{((2l+1)!!)^2}{(l+1)! 8^{l+1}} \right] \\ &= \sqrt{\frac{\pi}{2Z}} e^{-Z} \left[1 - \frac{1}{8} \int_C dt e^{-tZ} {}_2F_1 \left(\frac{3}{2}, \frac{3}{2}; 2; -\frac{t}{2} \right) \right], \end{aligned} \quad (\text{A.72})$$

where C is a contour in the complex t -plane from 0 to $+\infty$. The function ${}_2F_1 \left(\frac{3}{2}, \frac{3}{2}; 2; -\frac{t}{2} \right)$ is a hypergeometric function whose only branch cut lies along the negative real axis for $t < -2$. Consequently, different choices of contours around the positive axis of t do not introduce any ambiguity.

One can also perform a truncation near the minimum point,

$$\frac{\partial}{\partial l_{\min}} \frac{l_{\min}!}{2^{l_{\min}+1} \pi Z^{l_{\min}+1}} = 0 \quad \Rightarrow \quad l_{\min} \sim 2Z, \quad (\text{A.73})$$

where Stirling's approximation, $n! \sim \sqrt{2\pi n} (n/e)^n$, has been employed. The truncation error is then of the order of the minimal term,

$$\sim \frac{l_{\min}!}{2^{l_{\min}+1} \pi Z^{l_{\min}+1}} \sim \frac{e^{-2Z}}{\sqrt{\pi} \sqrt{Z}}, \quad (\text{A.74})$$

which is exponentially suppressed in the large Z limit.

If the series is truncated at $l < l_{\min}$, the truncation error can be approximated by a higher-order polynomial dependence.

As shown in Fig. A.7, the full result is well described by large distance asymptotic expansions for $z \gtrsim 1/m$. At $z = 1/m$, LA only deviates from the full result by 9.5%, NLA by 4.2%, and NNLA

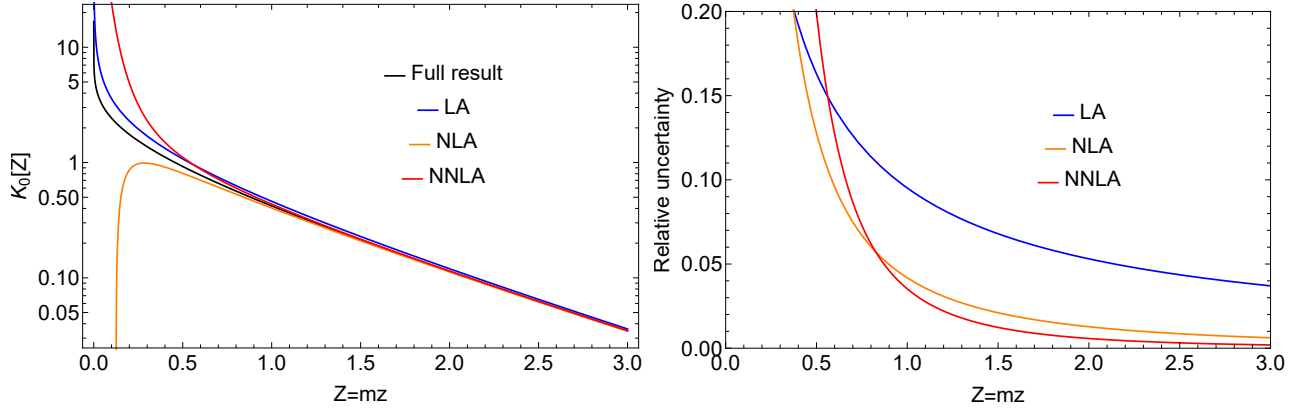


Figure A.7: Various truncation orders of the large distance asymptotic forms for $K_0[Z]$, compared with the full result. The left panel displays LA (blue), NLA (orange), and NNLA (red), which correspond to truncations up to $l = -1$, 0 , and 1 in Eq. (A.70), respectively. The black curve denotes the full result. The right panel displays the relative uncertainty of each order, $|\text{trunc. res.} - \text{full res.}|/\text{full res.}$.

by 3.5%. At $z = 2/m$, LA by 5.3%, NLA by 1.3%, and NNLA by 0.6%.

Notice that if z becomes an imaginary time or real time with $1 \pm i0$ prescriptions, then the threshold-type terms with phases $e^{\pm izP^z}$ become exponentially enhanced or suppressed, while the “small- x ” type terms discussed here are exponentially suppressed and acquire phase ambiguities. These phase ambiguities cancel with the non Borel-summabilities of the leading threshold-type terms, due to poles and branch cuts along the integration paths, which are absent in case of space-like correlators.

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