Scaling Symmetric Positive Definite Matrices to Prescribed Row Sums

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Abstract

We show that any symmetric positive definite matrix can be symmetrically scaled by a positive diagonal matrix, or by a diagonal matrix with arbitrary signs, to have arbitrary positive rows sums. The scaling can be constructed by solving an ordinary differential equation.

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1 Introduction

We consider the following problem: Given a matrix $W \in R^{n\times n}$ and an n-vector $u > 0$, find a diagonal matrix X so that the scaled matrix XWX has row-sums equal to the elements of u. In other words, given W and u, solve the nonlinear equation

$$XWXe = u\,,$$

where e is an n-vector with all entries equal to one.

Scaling problems have been a topic of intense investigation. Brualdi [1] gave necessary and sufficient conditions for the existence of such a diagonal scaling when W is symmetric with nonnegative elements. Other authors have considered scalings of nonsymmetric matrices, allowing different diagonal matrices on the left and the right; see, for example, [2–4] and the references therein. The inverse problem of finding matrices of given sign patterns with given row and column sums has also been investigated, for example, in [5,6].

In this paper, we prove that if W is symmetric and positive definite, then a solution X exists. In fact, there are 2^n solutions, one for each sign pattern for X.

2 A Constructive Existence Proof

Suppose we are given a symmetric positive definite matrix $W \in R^{n\times n}$ and an n-vector $u > 0$. We want to show that there exists a positive diagonal matrix X so that the scaled matrix XWX has row-sums equal to the elements of u.

We will prove this result by considering the matrix

$$V(t) = (1-t)I + tW\,.$$

Then $V(0) = I$, and $V(1) = W$. The notation $\|.\|$ will denote the 2-norm for vectors and matrices.

We will study the mapping

$$H(t, x) = X(t)V(t)X(t)e - u = 0\,,$$

where $X(t)$ is a positive diagonal matrix with entries x_i. For $t = 0$, we have a unique positive solution $x(0) = \hat{x}$ with $\hat{x}_i = \sqrt{u_i}$.

If we can find a positive solution vector x for $t = 1$, then the solution to our scaling problem is the corresponding matrix $X(1)$.

Differentiating our mapping, we obtain

$$\partial_x H(t, x)x'(t) + \partial_t H(t, x) = 0\,, \quad x(0) = \hat{x}\,, \tag{1}$$

where

$$\begin{aligned} \partial_x H(t, x) &= X(t)V(t) + \mathrm{diag}(\mathrm{V(t)X(t)e})\,, \\ \partial_t H(t, x) &= X(t)(W - I)X(t)e\,. \end{aligned}$$

The matrix $V(t)$ is positive definite on some interval $(-\sigma, \tau)$ where $\sigma > 0$ and $\tau > 1$. Let $2\epsilon = \min(\sigma, \tau - 1)$. Then $V(t)$ is uniformly positive definite on the interval $(-\epsilon, 1+\epsilon)$, with eigenvalues $(1-t)+t$ times the eigenvalues of W, and we define the bounds on its eigenvalues to be $\lambda_{min} > 0$ and $\lambda_{max} < \infty$.

The proof of our theorem relies on three lemmas, one establishing the boundedness of $X(t)$, one showing Lipschitz continuity of $f(t,x) = \partial_x H(t,x)^{-1}\partial_t H(t,x)$, and one rather standard result concerning existence of solutions to initial value problems.

Lemma 1: There exist scalars $\xi_\ell > 0$ and $\xi_u < \infty$, independent of t, such that if $x(t) > 0$ satisfies (1) for some value of $t \in [-\epsilon, 1+\epsilon]$, then

$$\xi_\ell \leq \min_i x_i(t) \leq \max_i x_i(t) \leq \xi_u \,.$$

Proof: The matrix $X(t)$ satisfies $X(t)V(t)X(t)e - u = 0$, so

$$e^T X(t)V(t)X(t)e = e^T u > 0 \,.$$

Since $Xe = x$, we know that

$$e^T u \geq \lambda_{min}(V)\|x\|^2 \geq \lambda_{min}\, x_i^2 \,, i = 1, \ldots, n,$$

so

$$x_i^2 \leq \frac{e^T u}{\lambda_{min}} \equiv \xi_u^2.$$

This means that the elements of $X(t)$ are uniformly bounded above for $t \in [-\epsilon, 1+\epsilon]$.

Now since $x_i(t)(V(t)X(t)e)_i = u_i$, $i = 1, \ldots, n$, we have

$$x_i(t) = \frac{u_i}{(V(t)X(t)e)_i} \geq \frac{u_i}{\|V(t)\|\|X(t)\|\sqrt{n}} \,,$$

so we can define

$$\xi_\ell = \frac{\min_i u_i}{\lambda_{max}\xi_u\sqrt{n}} \,.$$

[]

Lemma 2: Let

$$\Omega = \{(t,x) : -\epsilon < t < 1+\epsilon, \frac{1}{2}\xi_\ell e < x(t) < 2\xi_u e, V(t)x(t) > 0, \} .$$

For a fixed value of t, the function $f(t,x)$ is Lipschitz continuous on Ω, where f is defined by

$$\partial_x H(t,x) f(x) = -\partial_t H(t,x) . \tag{2}$$

Proof: The matrix $\partial_x H(t,x)X(t)$ is symmetric and positive definite on Ω, so the inverse of $\partial_x H(t,x)$ must exist, and it is a continuous function of x and t. The right-hand side $-X(t)(W-I)X(t)e$ is continuous on Ω, Therefore, $f(x)$ is continuous.

Now, for a fixed $t \in [-\epsilon, 1+\epsilon]$, we show that $f(t,x)$ satisfies a Lipschitz condition in x.

Let (t,x) and $(t,\hat{x})$ be two points in Ω. Let $Y = X(W-I)Xe$ and $Z = XVX + \text{diag(XVx)}$, and define $\hat{Y}$ and $\hat{Z}$ by substituting $\hat{X}$ for X in these expressions. Then we have these bounds:

$$\begin{aligned} \|\hat{X}\|\,,\ \|X\| &\le \xi_u\,, \\ \|\hat{Y}\|\,,\ \|Y\| &\le \sqrt{n}\|W-I\|\xi_u^2\,, \\ \|\hat{Z}^{-1}\|\,,\ \|Z^{-1}\| &\le \frac{1}{\xi_\ell^2 \lambda_{min}}\,. \end{aligned}$$

We compute

$$\begin{aligned} \|f(t,\hat{x}) - f(t,x)\| &= \|\hat{X}\hat{Z}^{-1}\hat{Y} - XZ^{-1}Y \\ &= \|(\hat{X}-X)\hat{Z}^{-1}\hat{Y} + X\hat{Z}^{-1}(\hat{Y}-Y) + X(\hat{Z}^{-1}-Z^{-1})Y\| \\ &\le \|(\hat{X}-X)\|\,\|\hat{Z}^{-1}\|\,\|\hat{Y}\| + \|X\|\,\|\hat{Z}^{-1}\|\,\|\hat{Y}-Y\| \\ &\quad + \|X\|\,\|\hat{Z}^{-1}-Z^{-1}\|\,\|Y\|\,. \end{aligned}$$

We already have bounds on many of these norms, so to conclude that f is Lipschitz continuous, it suffices to bound $\|\hat{Y}-Y\|$ and $\|\hat{Z}^{-1}-Z^{-1}\|$ in terms of $\|\hat{X}-X\|$, since $\|\hat{X}-X\| \le \|\hat{x}-x\|$.

We compute the Y bound by noting that

$$\hat{Y} - Y = \hat{X}(W-I)\hat{X}e - X(W-I)Xe = (\hat{X}-X)(W-I)\hat{X}e + X(W-I)(\hat{X}-X)e\,,$$

so

$$\|\hat{Y} - Y\| \leq 2\|W - I\|\xi_u\sqrt{n}\|\hat{X} - X\| \,.$$

Now we bound the Z term. Let $D = \text{diag(XVx)}$, and similarly for $\hat{D}$, and note that

$$\begin{aligned}\hat{Z}^{-1} - Z^{-1} &= (\hat{X}V\hat{X} + \hat{D})^{-1} - (XVX + D)^{-1} \\ &= (\hat{X}V\hat{X} + \hat{D})^{-1}[-\hat{X}V(\hat{X} - X) + (\hat{X} - X)VX - \hat{D} + D](XVX + D)^{-1}\end{aligned}$$

The norms of the first and last factors are bounded, so we just need to bound the norm of the middle expression:

$$\| - \hat{X}V(\hat{X} - X) + (\hat{X} - X)VX - \hat{D} + D\| \leq 2\xi_u\lambda_{max}\|\hat{X} - X\| + \|\hat{D} - D\| \,.$$

Focusing on the last term gives

$$\begin{aligned}(\hat{D} - D)_i &= \hat{x}_i \sum_j w_{ij}\hat{x}_j - x_i \sum_j w_{ij}x_j \\ &= (\hat{x}_i - x_i)\sum_j w_{ij}\hat{x}_j + x_i \sum_j w_{ij}(\hat{x}_j - x_j)\end{aligned}$$

so

$$|(\hat{D} - D)_i| \leq \lambda_{max}\xi_u|\hat{x}_i - x_i| + \xi_u\lambda_{max}\|\hat{x}_j - x_j\|$$

and thus we have a bound on every term in terms of $\|\hat{x} - x\|$, yielding a conclusion of Lipschitz continuity for f. []

Lemma 3: Let Ω be a bounded domain in R^{n+1} with $(0, x_0) \in \Omega$. If f is continuous in Ω and locally satisfies a Lipschitz condition in the x variables, then there exists a solution of the initial value problem

$$x'(t) = f(t, x) \ , \quad x(0) = x_0$$

that can be uniquely extended arbitrarily close to the boundary of Ω.

Proof: See, for example, Hurewicz [7, Theorem 11]. []

Now we use our three lemmas to prove that the scaling matrix exists.

Theorem: Given a symmetric positive definite matrix $W \in R^{n\times n}$ and an n-vector $u > 0$, there exists a positive diagonal matrix X so that the scaled matrix XWX has row-sums equal to the elements of u.

Proof: To construct our scaling X, we use Lemma 3 to show that (1) has a solution at $t = 1$.

It is clear that $(0, x_0) \in \Omega$, and Lemma 2 assures us that the function f defined by (2) is Lipschitz continuous on Ω. Thus, the assumptions of Lemma 3 are satisfied, so a solution to (1) can be extended to the boundary of Ω.

Now, consider any solution point $(t, x(t))$ for $t \in [-\epsilon, 1+\epsilon]$ with $x > 0$. By Lemma 1, $\xi_\ell e \le x \le \xi_u e$, and thus, since $XV(t)x = u > 0$, we must have

$$V(t)x \ge \frac{1}{\xi_u}u > 0\,.$$

Therefore, any solution point $(t, x(t))$ with $t \in [-\epsilon, 1+\epsilon]$ has x bounded away from the constraints

$$\frac{1}{2}\xi_\ell e < x(t) < 2\xi_u e, \;\; V(t)x(t) > 0$$

that define Ω. Therefore, we must be able to extend the solution from $t = 0$ to the boundary $t = 1 + \epsilon$, and thus the solution exists for $t = 1$. []

By replacing $V(t)$ by the positive definite matrix $EV(t)E$, where E is a diagonal matrix with entries ± 1, we can see that there are actually 2^n scaling matrices, one for each quadrant, that give the prescribed row sums. For $t = 1$, the equation $XVXe = u$ is a polynomial system of degree 2^n, so this accounts for all possible solutions.

Corollary: The equation $XWXe = u$, with W symmetric positive definite and X a diagonal matrix, has 2^n solutions, one per quadrant, so we can scale the matrix W by a diagonal matrix with arbitrary signs, so that it has prescribed row sums.

3 Conclusions and Remarks

We have presented an existence proof showing that any symmetric positive definite matrix can be scaled by a positive diagonal matrix, or by a diagonal matrix with arbitrary signs, to have arbitrary positive row sums.

The proof is constructive in that it leads to algorithms for computing such a scaling: apply an ordinary differential equation solver to (1). This is one particular homotopy method applied to the solution of the nonlinear equation $XWXe - u = 0$; other methods for solution of nonlinear equations could also be applied.

If the matrix is not positive definite, then the homotopy breaks down at values t for which $(1-t)I + tW$ is singular.

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