ABSTRACT

Title of Dissertation: NECESSARY AND SUFFICIENT CONDITIONS FOR THE TRANSFER OF KINETIC ENERGY AT ASYMPTOTICALLY LARGE REYNOLDS NUMBERS

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At high Reynolds numbers, an incompressible fluid will become turbulent — a phenomenon where the fluid is sensitive to external noise, develops chaotic time dynamics, and can develop eddy-type structures which break apart into smaller and smaller versions of themselves to dissipate energy as heat. In the field of mathematical physics, turbulent flows are typically modeled using stochastic partial differential equations to model the apparent randomness of the turbulent flow. Moreover, from a physics perspective, this method accounts for external noise on the system, such as the vibrations of the table holding the cup of coffee. Parameterizing these solutions by the viscosity (or the inverse of the Reynolds number) we can then study the behavior of the flow in the inviscid limit — or as the viscosity decreases toward 0.

One striking feature of three-dimensional turbulence is the presence of anomalous dissipation, or that the mean rate of energy dissipation is bounded below by a positive number in the inviscid limit. This is thought to be due to the convective acceleration acting in part like a

dissipation mechanism instead of a pure transport mechanism at asymptotically large Reynold's numbers (i.e. in the inviscid limit). Moreover the amount of anomalous dissipation dictates how fast the various eddy structures in the flow can break apart into smaller and smaller versions of themselves known as an energy cascade. In 1941, Kolmogorov predicted the rate of the energy cascade to be $\frac{4}{5}\varepsilon\ell$ where ℓ is the size of the eddy structure and ε is the amount of anomalous dissipation. Kolmogorov's work has been experimentally verified and simulated in numerous studies, but has faced serious mathematical obstacles in its analysis. In the first part of this dissertation we focus on finding necessary and sufficient conditions for Kolmogorov's flux laws on the movement of kinetic energy. We complete this over a torus in both two and three dimensions and discusses both the physical and mathematical differences encountered due to the dimension.

In the second part of this dissertation we examine the anomalous dissipation assumption itself. Here we consider the case of a bounded domain, subject to the Navier slip condition and show that the existence of (global) anomalous dissipation — anomalous dissipation over the entire domain — can be caused in a linear problem through lack of control over the tangential component of the velocity at the boundary.

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by

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Table of Contents

List of Figures

Chapter 1: Introduction

Despite the ubiquitousness of fluid turbulence in our everyday lives, rigorous mathematical analysis of such phenomena is minimal. Examples of fluid turbulence encountered daily range from the airflow around our car or bike, to mixing milk and coffee in the morning, to the weather itself. While all of these examples are distinct from one another, the underlying statistics about the flow tend to be the same: sensitive to external noise, far away from the boundary the flow is isotropic, and the fluid tends to develop eddy-type structures which break apart into smaller and smaller eddies [32]. Here eddies mean fluid structures that hold kinetic energy and deviate from the general flow field. Examples of such structures include vortices (the most common type of eddy structure), plumes, and Rossby waves. When these eddies break down into smaller and smaller versions of themselves, the kinetic energy stored within the structure also moves across length scales until it can be dissipated as heat and the eddy itself vanishes from the flow. This movement of energy across length scales is known as an *energy cascade*.

The focus of this dissertation is to study sufficient and necessary conditions for this breakdown of eddy structures within an incompressible fluid over a domain $D \subset \mathbb{R}^d$ for $d = 2, 3$. To do this, we consider the stochastically forced incompressible Navier Stokes Equations:

$$\begin{cases} du^\nu + u^\nu \cdot \nabla u^\nu \, dt = (\nu \Delta u^\nu - \nabla p^\nu + f)\, dt + g \, dW_t \quad x \in D, \ t > 0, \\ \nabla \cdot u^\nu = 0 \\ u^\nu(0) = u_0 \end{cases} \tag{1.1}$$

where $\nu > 0$ is the kinematic viscosity, u^ν is the viscous velocity field, p^ν is the associated pressure which enforces the conservation of mass on the fluid, $u_0 \in [L^2(D)]^3$ is a divergence free vector field and represents the initial flow velocity, and f, g are $[L^2((0,T) \times D)]^3$ vector fields which are divergence free and model the external forces on the system. We let W_t be a one-dimensional Wiener Process supported on a canonical filtered probability space denoted as $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$. Then we model the external noise — which is defined in the Ito sense i.e. gdW_t — as a white-in-time, colored-in-space Gaussian process which satisfies the coloring condition

$$\mathbf{E} \int_0^T \|g\|^2_{L^2(D)} < \infty$$

for some $T > 0$. Additionally we will assume that u^ν satisfies some appropriate boundary conditions depending on the choice of domain D.

Equation (1.1) models the macroscopic motion of the fluid with constant density and should be understood as the conservation of momentum (i.e. Newton's second law) and the conservation of mass for a fluid over the domain D. However, we are introducing a random white-in-time, colored-in-space stochastic forcing to the system to represent noise from the external environment such as vibrations in the ground slightly affecting our coffee as we mix in the milk. Due to this noise, solutions to (1.1) are stochastic processes with an underlying stochastic basis, meaning that individual realizations of the flow may be different but the statistics of the flow are the same. This, along with the fact that (1.1) has a positive Lyapunov exponent [7] and is both Galilean and rotationally invariant, in the absence of boundaries, makes (1.1) a good representative model for a turbulent fluid. See [9, 12, 36, 51, 61] and the references within for examples.

While there is no rigorous definition for when a flow is turbulent, all of the features of a turbulent flow (i.e. the sensitivity to external noise, break down of eddy structures, chaotic time dynamics, statistically spatially homogeneous far from the boundaries) tend to appear when the Reynolds number [32]

$$Re = \frac{UL}{\nu} \gg 2500.$$

Here U is the mean velocity of the flow, and L is the length scale of the flow which were implicitly normalized when formulating (1.1). Thus in our problem the Reynolds number becomes $Re = \nu^{-1}$ and we study the inviscid limit problem $\nu \to 0$ in order to extract information about the transfer of energy and the breakdown of eddy structures for turbulent flows at asymptotically large Reynolds numbers.

1.1 Anomalous Dissipation

While solutions to (1.1) exist in a weak sense (see Chapter 2) they need not be unique. In the deterministic case, this is known to be true in the absence of forcing over the torus $\mathbb{T}^3$ [14]. However, the underlying statistics of the flow should remain the same. This means that the quantities such as the average velocity of the flow or the average energy injected via the noise will be the same across the various realizations of the flow. Surprisingly, experimental evidence suggests that on average the amount of kinetic energy dissipated by the viscosity does *not* vanish as $\nu \to 0$. Specifically, this means that there exists $\varepsilon_0 > 0$ such that if T is the maximum time of existence of the flow, then

$$\limsup_{\nu \to 0} \nu \mathbf{E} \int_0^T \|\nabla u^\nu\|_{L^2(D)}^2 \, dt \geq \varepsilon_0 > 0. \tag{1.2}$$

See for example [32, 62, 74, 79] and the references within. This phenomenon is known as (global) *anomalous dissipation* and is thought to be due to the convective term, $u^\nu \cdot \nabla u^\nu$, accelerating the rate of dissipation of kinetic energy[9, 59] and acting as a dissipation mechanism for sufficiently rough velocity fields [25, 26]. While this is the usual notion of what is meant by anomalous dissipation, we mention that this is a quantity which is global in space as this will be an important distinction in the case of bounded domains. Here (1.2) represents a particular rate of blow up for the H^1 norm for u^ν of $O(\nu^{-1})$. Heuristically this comes about similarly to the formulation of the Weierstrass function: the dissipation only smooths out the smallest length scales, so as $\nu \to 0$ more small scale oscillations are included until the limiting solution does not have a weak derivative in $L^2(D)$ anymore. More physically, this can be understood as the convective term stretching each of the vortex tubes within the flow, thereby increasing the total vorticity (tendency of fluid to swirl and form vortex eddys) within the flow and making the "texture of the motion even finer"[59].

It should be noted that while (1.2) is usually taken to be the definition of anomalous dissipation within a fluid, for stochastic flows this definition can be accomplished by statistically stationary solutions to the stochastic heat equation (i.e. the law of u^ν is invariant under time translations) [9]. This is an issue as the stochastic heat equation does not possess a nonlinear term so there is nothing "anomalous" occurring. As such several other notions of "weak anomalous dissipation" have been proposed which try to account for the participation of the nonlinearity [9, 26, 60]. We will explore this more in Chapters 3 and 4. Furthermore, even in the deterministic case where such issues are not present, there are still no analytical examples of flows which exhibit anomalous dissipation in the sense of (1.2). However there has been recent progress in the case of passive scalar flows over a 2D torus [4] and strong 2.5 dimensional solutions (when u^ν is a 3 dimensional vector field but only depends on 2 spatial input variables) over $\mathbb{T}^3$ whose inviscid limit blows up in finite time [13].

1.1.1 In the Context of Bounded Domains

In a bounded domain, there is an additional mechanism by which the total vorticity within the flow may increase: through the shear stress at the wall [54]. Due to the friction the fluid experiences along the wall, thin regions known as boundary layers begin to develop next to the wall in which dissipation plays the dominant role in the dynamics [65] and as such while both eddies and vorticity are generated at the wall, only a shear flow persists next to the wall. It is not until the flow is further from the wall that the vortical structures can mix together, transport energy, and possibly separate from the wall entirely [63, 75]. These regions of dissipation are thought to be as a boundary correction as the bulk flow wishes to freely slip past the boundary while at some non-zero viscosity, the viscous solution is experiencing friction at the boundary hindering its ability to slip past. For instance the viscous fluid may stick to the boundary which is known as a no-slip condition. As a result, this potential mismatch in boundary conditions between (1.1) and its inviscid limit may introduce an uncontrollable amount of vorticity into the flow in order to satisfy both conditions. When the viscous fluid is allowed to slip along the boundary, the difference between the behavior of the viscous and inviscid fluids is less distinct leading to less restrictions on the behavior of fluid near the boundary. However, we will show in Chapter 4 that it is still possible to introduce an infinite amount of vorticity from the wall in this case as well. The mismatch of boundary conditions, particularly in the no-slip case, has been the subject of much research regarding the issue of boundary layers, see [41, 50, 69, 70] and the references within.

1.1.2 Onsager's conjecture

While we have discussed the physical implications of anomalous dissipation, from a mathematical standpoint, this is an issue of the invariant regularity of the system in the inviscid limit. This was first conjectured by Onsager in 1949, [59]: if $f, g \equiv 0$ and a solution to the Euler equations (i.e. (1.1) with $\nu = 0$) is in the class $[C(0, T, C^s(D))]^3$

- for some $s > \frac{1}{3}$, then u^0 conserves kinetic energy;
- for some $s \leq \frac{1}{3}$, then u^0 dissipates kinetic energy.

This is particularly astonishing as the Euler system formally defines a Hamiltonian system, meaning it should always conserve its kinetic energy. When $D = \mathbb{T}^3$ is the three-dimensional torus, both directions of Onsager's conjecture have been shown to be true in the deterministic case and even extended to the slightly larger space of L^3 Besov spaces $[C(0, T, B^s_{3,\infty}(D))]^3$ [19, 25, 28, 37, 58]. When $s \leq \frac{1}{3}$ the dissipative examples were constructed using "convex integration" techniques which in the majority of cases can be obtained as the inviscid limit of *mild* solutions to Navier Stokes equations [14]. Therefore, in the context of anomalous dissipation, Onsager's conjecture can be restated that if the family of viscous solutions $\{u^\nu\}_{\nu>0}$ is precompact in $[C(0,T,C^s(D))]^3$ then (1.2) is satisfied if and only if $s \leq \frac{1}{3}$. While this seems easier now that examples of dissipative Euler flows have been constructed, explicitly identifying a family of viscous solutions which approximate the dissipative Euler solution are incredibly difficult to do. Hence, despite this open path to constructing families of solutions which exhibit anomalous dissipation, no such work has been completed to the authors knowledge.

Nevertheless, in the presence of a boundary, some results are already known. For instance, Kato [41] showed that the absence of anomalous dissipation for C^1 solutions with a no-slip boundary condition is equivalent to

$$\lim_{\nu\to 0} \nu \int_0^T \|\nabla u^\nu\|^2_{L^2(C_\nu)} = 0$$

where $C_\nu = \{y \in D \mid \operatorname{dist}(y, \partial D) < \nu\}$ is a strip of width ν next to the boundary. Meanwhile, Kato's work has been improved to larger classes of functions [17, 24, 76], but a similar theory to Kato's seems to be missing for flows with boundary conditions other than the no-slip condition. Regardless, one can quickly surmise that control over how vorticity is injected into the system from the wall should be a sufficient condition for the lack of anomalous dissipation. In Chapter 4 we confirm this by constructing solutions to the Stokes problem which are not uniformly controlled at the boundary and exhibit anomalous dissipation.

1.2 The Energy Cascade in Three Dimensions

As previously discussed anomalous dissipation occurs when the convective term stretches the vortex tubes within the flow and increases the total vorticity making the texture of the fluid even finer. It is within these infinitesimally small scale structures that energy is being removed from the system as heat. However, typically energy is injected into the system at large length scales (i.e. on the size of the system), so in order for the system to reach a stationary distribution the kinetic energy must somehow move between the two length scales where it is injected and dissipated as heat. In 1922, Richardson [66] conjectured that there was a cascade of kinetic energy from the large to small length scales by breaking apart the eddies into smaller and smaller eddies. In order to experimentally measure the fluctuation in the velocity, Taylor considered the difference in velocity at two different points: $\delta_h u^\nu(x) = u^\nu(x+h) - u^\nu(x)$ for all $h \in \mathbb{R}^3$, and averaged multiples of these differences together [32, 77]. These combinations are referred to as structure functions. Later, Kolmogorov provided a heuristic argument for the rate at which energy is transferred to the small scales for statistically stationary, isotropic flows in a series of papers in 1941 now known as K41 theory [43–45].

To do this, Kolmogorov made three assumptions about the statistics of the flow:

1. the flow is homogeneous and isotropic away from the boundary i.e. $\delta_h u^\nu$ is statistically invariant under translations and rotations away from the boundary (in the case of a bounded domain);

2. the flow has mono-fractal scaling, i.e. the system is self similar with a single fractal scaling exponent $s \in \mathbb{R}$ such that $\delta_{\zeta\ell} u^\nu = \zeta^s \delta_\ell u^\nu$ for all $\zeta > 0$ and the increments $\zeta, \zeta\ell \ll \ell_{inj}$ where ℓ_{inj} is the scale of energy injection [71];

3. there is an finite non-zero mean rate of dissipation ε per unit mass independent of ν, i.e. u^ν satisfies the anomalous dissipation condition.

Using these three assumptions on top of assuming that the flow is statistically stationary, Kolmogorov argued that there is an inertial range $\ell_\nu \ll \ell \ll \ell_{inj}$ such that

$$S^{\|}_{vel,p}(\ell) := \mathbf{E}(\delta_{\ell n} u^\nu \cdot n)^p \sim C_p \varepsilon^{p/3} \ell^{p/3}.$$

Here C_p is a fixed diffusion constant depending on the dimension and the power of p (and independent of ν), $\mathbf{E}$ is the expected value with respect to the underlying probability measure $\mathbb{P}$ and $S^{\|}_{vel,p}$ is the pth order universal longitudinal velocity structure function. Landau famously objected to Kolmogorov's second assumption of mono-fractal scaling due to intermittency effects — rare events and non-uniformity in the roughness of the velocity field [32]. Indeed, the mono-fractal scaling assumptions has been found to be generally false due to such intermittency effects [9, 23, 32, 39]. However such errors can be corrected for by including an intermittency corrector $\tau(p)$ to the exponent on the right hand side of the law. See [39] and the references within for more details regarding intermittency. Nevertheless, experimental results show that the $p = 3$ case correlates exceptionally close to Kolmogorov's prediction to the point that it is considered the only exact law in the physics literature [23]. This is because the $p = 3$ case doesn't rely on the statistical assumptions of isotropy and mono-fractal scaling, but can be derived directly from the Navier Stokes equations via the von Karman-Howarth-Monin (KHM) relations (see Section 3.2 for more details).

When $p = 3$, Kolmogorov's prediction is known as the $\frac{4}{5}$-law due to its diffusion coefficient:

$$\mathbf{E}\Big(\delta_h u^\nu \cdot \frac{h}{|h|}\Big)^3 \sim -\frac{4}{5}\varepsilon|h| \qquad \ell_\nu \ll |h| \ll \ell_{inj} \tag{1.3}$$

where the dissipation scale is given by $\ell_\nu = (\nu^3/\varepsilon)^{1/4}$ is the scale at which dissipation effects begin to dominate the flow and is known as the Kolmogorov scale.

The negative sign on the flux indicates that energy is moving from large scales to small scales and one can see that the rate of the flux decreases proportional to the scale. By this we mean that if we pick a point x_0 in the domain D and construct balls with radii $\ell_\nu \ll \ell \ll \ell_{inj}$ around the point x_0, then the flux of energy across the various balls moves linearly from the largest ball to the smallest at a rate proportional to the radius of the ball. However, this energy transfer is only within the scale space (i.e. with respect to ℓ) and not the physical space (i.e. with respect to x_0) due the homogenity of the flow. This means that due to the homogeneity of the flow, there is no physical movement of energy in space, but the amount of energy within the balls decreases linearly with the size of the radius.

1.3 The Dual Cascade in Two Dimensions

The Richardson-Kolmogorov theory of an energy cascade has also been extended to 2D flows by Batchelor [6], Kraichnan [46], Leith [48], and Fjørtoft [30] independent of each other. They each heuristically showed that while energy does still move across length scales, it no longer goes from large to small length scales. The culprit for this change in the energy cascade is the lack of vortex stretching in two-dimensions. Due to the lack of vortex stretching the vorticity is no longer deformed by the convection term and is only transported through the flow, leading to a second conserved quantity: the mean enstrophy. Hence, as long as the initial velocity has finite mean enstrophy

$$\mathbf{E}\|\omega^\nu(0)\|^2_{L^2(D)} < \infty \quad \omega^\nu(t) = \partial_1 u_2^\nu(t) - \partial_2 u_1^\nu(t)$$

then the 2D dynamics are largely different from the 3D dynamics, in that there is a cascade of enstrophy from large to small scales and an inverse cascade of kinetic energy from small to large scales.

More specifically, let us consider the corresponding vorticity equation to (1.1) in two dimensions:

$$d\omega^\nu + (u^\nu \cdot \nabla)\omega^\nu \, dt = (\nu\Delta\omega^\nu + \mathrm{curl} f) \, dt + \mathrm{curl} g \, dW_t \tag{1.4}$$

$$u^\nu = \nabla^\perp(-\Delta)^{-1}\omega^\nu.$$

Here $\nabla^\perp := (-\partial_2, \partial_1)$ and we call the second relation $u^\nu = \nabla^\perp(-\Delta)^{-1}\omega^\nu$ the Biot-Savart law. As there is no vortex stretching, the vorticity equation is an advection-diffusion equation which is fairly well behaved. First, one can construct mild solutions through use of a semi-group operator (see Section 2.1.3 for the definition and further details). Second, the total enstrophy is conserved for all time, in the absence of forcing due to a simple energy argument. This results in a second invariant for the system which *should* be conserved in the invisicid limit in some sense.

Hence, while extending Kolmogorov's K41 theory to 2D, Kraichnan [46], Batchelor [6], Leith [48], and Fjortoft [30] independently observed two different inertial zones, one for each conserved quantity. It was observed that now energy moves toward large scales in an inverse cascade, while enstrophy moves to small scales via a (direct) cascade. Such situations are frequently called dual cascades and are generally known to exist in systems with two conserved quantities [8]. Specifically the direct cascade flux laws for the velocity (and the vorticity as predicted by Eyink [29]) can be informally stated as follows: there exists a dissipation scale ℓ_ν such that for

$h \in \mathbb{R}^2$ within the inertial range $\ell_\nu \ll |h| \ll \ell_{inj}$

$$
\begin{aligned}
\mathbf{E}\Big(|\delta_h \omega^\nu|^2 \delta_h u^\nu \cdot \frac{h}{|h|}\Big) &\sim -2\eta |h| \\
\mathbf{E}\Big(|\delta_h u^\nu|^2 \delta_{\ell n} u^\nu \cdot \frac{h}{|h|}\Big) &\sim \frac{1}{4}\eta |h|^3 \\
\mathbf{E}\Big(\delta_h u^\nu \cdot \frac{h}{|h|}\Big)^3\rangle &\sim \frac{1}{8}\eta |h|^3.
\end{aligned}
$$

Moreover, there is a *second* inertial range $\ell_{inj} \ll |h| \ll \tilde{\ell}_\nu \le \lambda$ (where λ is the size of the system) for the inverse cascade such that

$$
\begin{aligned}
\mathbf{E}\Big(|\delta_h u^\nu|^2 \delta_h u^\nu \cdot \frac{h}{|h|}\Big) &\sim 2\varepsilon |h| \\
\mathbf{E}\Big(\delta_h u^\nu \cdot \frac{h}{|h|}\Big)^3 &\sim \frac{3}{2}\varepsilon |h|
\end{aligned}
$$

where

$$
\limsup_{\nu \to 0} \nu \mathbf{E} \int_0^T \|\omega^\nu\|_{L^2}^2 \, dt =: \varepsilon \tag{1.5}
$$

$$
\limsup_{\nu \to 0} \nu \mathbf{E} \int_0^T \|\nabla \omega^\nu\|_{L^2}^2 \, dt =: \eta. \tag{1.6}
$$

A simple heuristic argument for the existence of two separate inertial ranges with different flux coefficients can be seen from the two associated balance laws for the system (assuming

$f \equiv 0$, the absence of boundaries, and the flow is statistically stationary):

$$\nu \mathbf{E} \int_0^T \|\nabla \omega^\nu\|_{L^2}^2 \, dt = \frac{1}{2} \mathbf{E} \int_0^T \|\text{curl} g\|_{L^2}^2 = \eta < \infty;$$
$$\nu \mathbf{E} \int_0^T \|\omega^\nu\|_{L^2}^2 \, dt = \frac{1}{2} \mathbf{E} \int_0^T \|g\|_{L^2}^2 = \varepsilon < \infty.$$

The first comes from Ito's lemma and computing the energy balance for stationary solutions to the vorticity equation (1.4), while the second is the energy balance for stationary solutions to the Navier Stokes equations (1.1) after we applied the Biot-Savart Law. Since neither of the right hand sides depend on the viscosity, it can be seen that at least one wave-number of ω^ν must become singular. Moreover, if there are two singular wave-numbers then the flux of energy to these scales must occur at different rates as one is moving like $|k|^2 \mathbf{E} |\widehat{\omega^\nu}(k)|^2$ and the other is moving like $\mathbf{E} |\widehat{\omega^\nu}(k)|^2$.

While the existence of both cascades simultaneously had been conjectured for a while, only recently in [12] did researchers experimentally observe such a phenomena in soap films. This is because the inverse cascade is somewhat unstable due to the difference in the amount of energy at large scales and how much energy the force pushes towards this end of the spectrum. Moreover, as we show in Corollary 3.12.1 if the enstrophy does not blow up fast enough at small wave-numbers then there is no inverse cascade. For these reasons most works regarding inverse cascades include a friction force to dampen the contribution of the large wave-number terms as well as to collect the energy at small wave-numbers; or a hyper/hypo-viscosity term to heighten the effects of the viscous dissipation. See [61, 68] and the references within for more information. Within the mathematics literature far fewer two dimensional theoretical analysis works have been produced compared to the three dimensional case. However, we note that a major inspiration for this project was the work by Bedrossian et. al. [8] where they showed sufficient conditions for the existence of a dual cascade to the Navier Stoke's equations with a frictional term of the form: $\alpha(-\Delta)^{-2\gamma}u^{\nu}$.

1.4 Outline of Thesis

In Chapter 2 we cover the definition and existence of martingale solutions to (1.1). We also outline how one uses the stochastic basis to construct the noise in the system as well as why we specifically restrict ourselves to additive noise throughout this work. Then in Chapter 3 we present necessary and sufficient conditions on the velocity profile for a Kolmogorv type energy cascade to occur. This project is completed over a torus in both two and three dimensions, and shows that Kolmogorov's flux laws measures how much kinetic energy "escapes to infinity" in Fourier space in the inviscid limit. Finally in Chapter 4 we construct a solution to the linear Stokes problem which blows up at the boundary in the inviscid limit and exhibits (global) anomalous dissipation in the sense of (1.2).

1.5 Notation

Let X be any Banach space containing scalar fields $h : \mathbb{R}^d \to \mathbb{R}$ subject to some requirement. We will denote the vector space equivalent as $[X]^d = \{h \mid h_i \in X,\ i \in \{1, 2, \ldots, d\}\}$ for $d = 2, 3$. For $x \in \mathbb{R}^d$ and $R > 0$ we write $B(x, R) \subset \mathbb{R}^d$ for the ball centered at x with a radius of R. Let $C_c^\infty(D)$ be the space of continuously differentiable functions compactly supported within D. And when D has a boundary, we will use $\mathcal{H}^{d-1}$ to express the surface measure of ∂D.

For $s \in (0,1)$ and define the following spaces

$$\begin{aligned}
[L^2_\sigma(D)]^d &= \{f \in [L^2(D)]^d \mid \nabla \cdot f = 0\} \\
[H^1_{\sigma,\tau}(D)]^d &= \{f \in [H^1(D)]^d \mid \nabla \cdot f = 0, \quad n^{\partial D} \cdot \nabla f_\tau + \alpha f_\tau = 0|_{\partial D}\} \\
\dot{H}^s(D) &= \{f \in L^2(D) \mid \|f\|^2_{\dot{H}^s} := \int_D \int_D \frac{|f(x) - f(y)|^2}{|x-y|^{3+2s}} \, dxdy < \infty\}.
\end{aligned}$$

Each space will be equipped with its induced (semi-) norm. Without loss of generality, the (semi-) norm over X and $[X]^3$ will be denoted by $\|\cdot\|_X$ for simplicity. Here $[H^1_{\sigma,\tau}(D)]^d$ will be only used in the case that D has a boundary subject to the Navier-slip boundary condition, and f_τ is the tangential component of f to the boundary and $n^{\partial D} \cdot \nabla f_\tau + \alpha f_\tau = 0|_{\partial D}$ holds in the sense of distributions.

Let S be the space of Schwartz functions, then for $h \in S$ the Fourier transform of h is given by

$$\widehat{h}(\xi) = \int_{\mathbb{R}^3} e^{-ix\cdot\xi} h(x) \, dx.$$

In the case when $D = \mathbb{T}^d_\lambda = \mathbb{R}^d \setminus 2\pi\lambda\mathbb{Z}^d$, we denote the averaged L^2-norm in space as $\|f\|_\lambda := \left(\fint_{\mathbb{T}^d_\lambda} |f(x)|^2 \, dx \right)^{1/2}$. Furthermore, we will use the following Fourier analysis conventions:

$$f(x) = \frac{1}{\lambda^d} \sum_{\xi \in (\frac{2\pi}{\lambda}\mathbb{Z})^d} \widehat{f}(\xi) e^{ix\cdot\xi}.$$

Throughout this work we will use component free tensor notation. For example, given two vectors a and b we will denote the tensor product $(a \otimes b)_{ij} = a_i b_j$. Also for convenience we will use the Einstein summation convention: $a_j b_j = \sum_{j=1}^d a_j b_j$. Moreover, for two given rank-two tensors A and B we define the Frobenius product and norm as $A : B = A_{ij}B_{ij} = \sum_i \sum_j A_{ij} B_{ij}$

and $|A| = \sqrt{A : A}$.

Finally, we will commonly use C to denote a constant independent of ν, u^ν, but may depend on $s, D, \partial D$ or T. Furthermore, without loss of generality, we will use C repeatedly across inequalities even if the constant of the coefficient increases/decreases.

Chapter 2: Martingale Solutions

In order for us to study the movement of kinetic energy, we need a rigorous definition of what a solution to (1.1) is, whether such a solution exists, and its underlying regularity. In this thesis we will only concern ourselves with solutions which satisfy (1.1) in the sense of distributions, known as Leray-Hopf solutions in the deterministic literature. However, due to the presence of the stochastic forcing, solutions to (1.1) will in fact consist of a stochastic basis as well as a solution process u^ν. When the associated stochastic basis is generated along with the solution process (i.e. the basis is unknown apriori) such a solution is known as a probabilistically weak solution. Martingale solutions to (1.1), are the probabilistically weak-weak solution processes to (1.1) which satisfy the associated martingale problem. In this chapter we will cover the existence of such solutions in both bounded and periodic domains as well as the existence of stationary solutions.

2.1 Existence and Regularity of Martingale Solutions

To be more specific, we define a martingale solution as:

Definition 2.1. *A stochastic basis* $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,T]}, \{\{\beta_j(t)\}_{t\in[0,T]}\}_{j\in\mathbb{N}})$ *along with a complete right-continuous filtration and an* $\mathcal{F}_t$*-progressively measurable stochastic process* $u^\nu : \Omega \times [0,T] \to [L^2_\sigma(D)]^d$ *is called a martingale solution to* (1.1) *on* $[0,T]$ *in* d*-dimensions if*

- *the trajectories of u^ν (i.e. the sample paths $u^\nu(\cdot,t)$ for $t\in[0,T]$) belong $\mathbb{P}$-a.s. to*

$$[C(0,T;\dot{H}^\zeta(D))]^d \cap [L^\infty(0,T;L^2_\sigma(D))]^d \cap [L^2(0,T;H^1_\sigma(D))]^d, \quad \textit{for some } \zeta<0;$$

- *for all $t\in[0,T]$, u^ν solves (1.1) in the sense of distributions $\mathbb{P}$-a.s.*

Throughout this thesis, a process u^ν will be called a martingale solution without reference to its stochastic basis unless there is a risk of confusion.

The existence of martingale solutions to (1.1) were first shown to exist in $\mathbb{R}^3$ [10] through an analogous argument to Leray's construction of deterministic weak solutions [49]. Later, Flandoli and Gaterak [31] constructed martingale solutions over a bounded regular domain subject to the no-slip condition (i.e. homogeneous Dirichlet boundary conditions $u^\nu|_{\partial D}=0$). Their approach consisted of a Faedo-Galerkin type argument where the solution (and the associated noise) is constructed from the eigenfunctions of the Stokes operator. It is this approach that we adapt here to both periodic and Navier-slip boundary conditions. It should be noted that this argument is very similar to the deterministic case (see [16, 57, 81] and the reference within) except we also require a stochastic basis on which to construct the solution.

2.1.1 The Stokes Operator

Define $A=\mathrm{Pr}(-\Delta)$ to be the Stokes operator where $\mathrm{Pr}:[L^2(D)]^d\to[L^2_\sigma(D)]^d$ is the Leray projection and $\Delta:[H^2(D)]^d\cap[H^1(D)]^d\to[L^2(D)]^d$ is the Laplacian subject to the appropriate boundary conditions. When A is subject to Dirchlet, Periodic, or the Navier-slip conditions with slip length $\alpha\geq 0$, then A is known to be a positive, self-adjoint, unbounded linear operator, with a compact resolvent. Hence, by the Spectral Theorem the eigenvalues of A are strictly positive and non-decreasing (i.e. $0 < \lambda_1 \leq \lambda_2 \leq \dots$) and their associated eigenfunctions $\{q_j\}_{j\in\mathbb{N}}$ form an orthonormal basis for $[L^2_\sigma(D)]^d$ [3, 16].

Therefore, the noise can be represented in $[L^\infty(0,T,L^2(D))]^d$ $\mathbb{P}$-a.s. as

$$gdW_t = \sum_{j=1}^{\infty} \langle g, q_j\rangle \; q_j d\beta_j(t)$$

with the cross-variation $\int_0^T [gdW_t, gdW_t] = \int_0^T \|g\|^2_{L^2(D)} < \infty$. Here $\langle\cdot,\cdot\rangle$ is the $[L^2(D)]^d$ inner product, and $\{\beta_j(t)\}_{j\in\mathbb{N}}$ is a family of i.i.d. 1-dimensional standard Brownian motions. For more explicit information regarding the construction of the noise, see [21]. Similarly, we consider a finite dimensional approximation to the solution u^ν constructed as

$$u^\nu_N = \sum_{j=1}^{N} \langle u^\nu, q_j\rangle \; q_j. \tag{2.1}$$

Then we apply (2.1) to (1.1) and find uniform energy estimates on u^ν_N with respect to N using Ito's lemma. However, these uniform energy bounds are in $L^2(\Omega)$ and therefore do not have to hold $\mathbb{P}$-a.s. to u^ν as $N \to \infty$. In order to remedy this issue, we employ the Skorohod Embedding theorem to represent a subsequence of u^ν_N on a new stochastic basis where u^ν_N and its duplicate share the same law, but now convergence occurs $\mathbb{P}$-a.s.. This step of representing u^ν_N on a new stochastic basis, is why the stochastic basis of the full solution u^ν is unknown apriori is probabilistically weak. The full details of this approach can be found in [31] for the case of no-slip boundary conditions.

Let A_D, A_P, and A_{NS} be the Stokes operator subject to homogeneous Dirichlet boundary conditions (i.e. the no-slip condition), periodic boundary conditions, and Navier-slip boundary conditions respectively. Then using integration by parts, we can define for all $h \in H^1(D)$

$$\|A_D^{1/2} h\|_{L^2(D)}^2 := \|\nabla h\|_{L^2(D)}^2$$
$$\|A_P^{1/2} h\|_{L^2(D)}^2 := \|\nabla h\|_{L^2(D)}^2$$
$$\|A_{NS}^{1/2} h\|_{L^2(D)}^2 := \|\nabla h\|_{L^2(D)}^2 + \|\sqrt{\alpha}\gamma(h)\|_{L^2(\partial D)}^2$$

where $\gamma : H^1(D) \to L^2(\partial D)$ is the trace operator.

Theorem 2.2. *Consider the following two kinds of boundary conditions:*

- *When $D = \mathbb{T}^d$ (i.e. u^ν is subject to periodic boundary conditions) there exists a martingale solution u^ν to (1.1) and*

$$\sup_{\nu>0} \mathbf{E}\Big(\sup_{t\in[0,T]} \|u^\nu\|_{L^2(D)}^2 \Big) < \infty$$
$$\sup_{\nu>0} \nu \mathbf{E} \int_0^T \|A_P^{1/2} u^\nu\|_{L^2(D)}^2 < \infty.$$

- *Suppose $D \subset \mathbb{R}^d$ is bounded and regular. Furthermore assume u^ν is subject to the Navier-slip boundary condition then there exists a martingale solution u^ν to (1.1) and*

$$\sup_{\nu>0} \mathbf{E}\Big(\sup_{t\in[0,T]} \|u^\nu\|_{L^2(D)}^2 \Big) < \infty$$
$$\sup_{\nu>0} \nu \mathbf{E} \int_0^T \|A_{NS}^{1/2} u^\nu\|_{L^2(D)}^2 < \infty.$$

Proof. The proof of existence is exactly the same as in [31] using the definition of $\|A_{NS}^{1/2} \cdot \|_{L^2(D)}^2$ or $\|A_P^{1/2} \cdot \|_{L^2(D)}$ instead of $\|A_D^{1/2} \cdot \|_{L^2(D)}^2$ for the viscous dissipation term. The proof was briefly sketched above. The uniform bounds follow from Ito's formula applied to the martingale solutions to (1.1). Explicitly it means that u^ν satisfy an "on average energy inequality": for all $t > 0$

$$
\begin{aligned}
\frac{1}{2}\mathbf{E}\|u^\nu(t)\|^2_{L^2(D)} + \nu\mathbf{E}\int_0^t \|A_\cdot^{1/2}u^\nu\|^2_{L^2(D)} \\
\leq \frac{1}{2}\mathbf{E}\|u_0\|^2_{L^2(D)} + \mathbf{E}\int_0^t\int_D f\cdot u^\nu\, dxdt + \frac{1}{2}\mathbf{E}\int_0^t \|g\|^2_{L^2(D)}.
\end{aligned}
\tag{2.2}
$$

Here $A_\cdot$ is either A_P or A_{NS} depending on the choice of boundary conditions. Then by Young's inequality and the fact that u_0, f, and g are all independent of ν the uniform bounds (with respect to ν) follow. □

Remark 2.3. *Flandoli and Gatarek's [31] method was proven to work for an even more general class of noise, $G(u^\nu)dW_t$ where $G : L^2_\sigma(D) \to L_2(\Omega, L^2_\sigma(D))$ is a continuous mapping and grows at most linearly with respect to u^ν. This means that G is allowed to takes values in the space of Hilbert-Schmidt operators from Ω to $L^2_\sigma(D)$ as long as there exists constants $\lambda_0, \rho > 0$ such that*

$$
\|G(u^\nu)\|^2_{L_2(\Omega, L^2_\sigma(D))} \leq \lambda_0\|u^\nu\|^2_{L^2(D)} + \rho.
$$

As such Theorem 2.2 similarly holds for multiplicative noise, whenever λ_0 and ρ are independent of ν with λ_0 sufficiently small.

Even though Theorem 2.2 can be stated for multiplicative noise as in Remark 2.3, we purposely choose to restrict ourself to the case of additive noise for one important reason: the existence of statistically stationary solutions.

2.1.2 Stationary Martingale Solutions

In a general dynamical system, fixed point solutions, or solutions which do not change over time make up an important class of solutions. Within the case of martingale solutions, these fixed point type solutions are known as "statistically stationary" solutions as the law of the trajectories is independent of time translations while the trajectory itself is not. This allows for the construction of global in time solutions, and are defined as

Definition 2.4. *A d-dimensional stochastic basis $(\Omega, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P}, \{\{\beta_j(t)\}_{t\in[0,\infty)}\}_{j\in\mathbb{N}})$ along with a complete right-continuous filtration and an $\mathcal{F}_t$-progressively measurable stochastic process $u^\nu : \Omega \times [0,\infty) \to [L^2_\sigma(D)]^d$ is called a statistically stationary martingale solution to (1.1) on $[0,\infty)$ if*

- *u^ν is a martingale solution in the sense of Definition 2.1;*
- *the path of $u(\cdot + s) = u(\cdot)$ in law on $[C([0,\infty); L^2(D))]^d$ for all $s > 0$.*

Again, we will simply call a process u^ν a statistically stationary martingale solution and omit reference to the associated stochastic basis.

The existence of statistically stationary martingale solutions to the Navier Stokes equations was first shown in [31], provided the linear growth rate of the noise was bounded by twice the smallest eigenfunction of the Stokes operator A. In the general case of multiplicative noise $G(u^\nu)dW_t$ as commented on in Remark 2.3, this means that as long as

$$\lambda_0 < 2\lambda_1$$

there exists a statistically stationary martingale solution to (1.1). Importantly, when the noise is additive it is always guaranteed that one can construct a statistically stationary martingale solution.

2.1.3 Mild Solutions in 2D

Due to the lack of vortex stretching in two-dimensions, there is a stark difference in the existence and regularity of solutions which we will comment on in this section. First and foremost, in two-dimensions the vorticity is a passive scalar within the flow, meaning that the vorticity equation (1.4) is an advection-diffusion equation and is fairly well-behaved. The improved behavior of the solutions means that the martingale solutions can be upgraded from weak solutions to mild solutions which are evolving under a semi-group operator [8]. For convenience we only consider domains without boundary such as tori $\mathbb{T}^2_\lambda$ where $\lambda > 0$, then the associated semi-group over the torus is formed from the Stokes operator A_P.

Definition 2.5. *Given a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$, a mild solution (ω^ν_t) to (1.4) is an $\mathcal{F}_t$ adapted process $\omega^\nu : [0,T]\times\Omega \to L^2(\mathbb{T}^2_\lambda)$ satisfying*

$$\begin{aligned} \omega^\nu(t) = e^{-\nu A_P t}\omega_0 - \int_0^t e^{-\nu(t-s)A_P}(u^\nu(s)\cdot\nabla\omega^\nu(s))\,ds + \int_0^t e^{-\nu(t-s)A_P}\mathrm{curl}\ f\ ds \\ + \int_0^t e^{-\nu(t-s)A_P}\mathrm{curl}\ g\ dW_s \\ u^\nu(t) = \nabla^\perp(-\Delta)^{-1}\omega^\nu(t). \end{aligned} \tag{2.3}$$

where Δ is the Laplacian operator subject to periodic boundary conditions on $\mathbb{T}^2_\lambda$ and $\mathrm{curl} h = \partial_{x_1}h_2 - \partial_{x_2}h_1$ for all $h \in H^1(\mathbb{T}^2_\lambda)$ is the two-dimensional curl operator.

Similarly,

Definition 2.6. *A complete filtered probability space* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$*, along with a process* (ω^ν) *is called a statistically stationary mild solution to* (1.4) *provided*

- ω^ν *is a mild solution in the sense of Definition 2.5;*
- *the path of* $\omega^\nu(\cdot) = \omega^\nu(\cdot + s)$ *in law on* $C([0,\infty); L^2(\mathbb{T}^2_\lambda))$ *for all* $s > 0$*.*

Again, we will simply call a process ω^ν a (statistically stationary) mild solution and omit reference to the associated stochastic basis unless there is a risk of confusion.

In order to guarantee the existence of stationary measures we make the following compactness assumption on the forcing:

Assumption 2.7. *There exists a constant* $C > 0$ *such that*

$$\sup_{\lambda\in(1,\infty)} \frac{1}{|\mathbb{T}^2_\lambda|}\mathbf{E}\|\nabla^3 g\|^2_{L^2(\mathbb{T}^2_\lambda)} \leq C.$$

This is a purely mathematical assumption inspired by the work of Bedrossian et. al. [8] which ensures the existence of stationary measures for the 2D vorticity equation due to the following well known result. See for instance [47] for details.

Proposition 2.8. *Suppose that* ε *and* η *(from* (1.5)*) are both independent of* $\lambda \geq 1$ *and that Assumption 2.7 holds. Then for all* $\nu > 0$ *and* $\lambda \geq 1$*, the vorticity equation* (1.4) *admits a global in time,* $\mathbb{P}$*-a.s. unique, mild solution* ω^ν *with initial data* ω^ν_0*. Moreover,* ω^ν *defines a Feller Markov process which has at least one stationary probability measure* μ *supported on* $W^{3,2}(\mathbb{T}^2_\lambda)$*. That is, a measure satisfying the following: for all bounded measurable functionals* $\phi : L^2(\mathbb{T}^2_\lambda) \to \mathbb{R}$ *and*

$t \geq 0:$

$$\int_{L^2(\mathbb{T}^2_\lambda)} \mathbf{E}\phi(\omega^\nu(t))\mu(d\omega_0^\nu) = \int_{L^2(\mathbb{T}^2_\lambda)} \phi \, d\mu.$$

Then we have the following corollary which we reproduce from [8]

Corollary 2.8.1. *Under the same assumptions as Proposition 2.8 there exists a statistically stationary mild solution to* (1.4) *in the sense of Definition 2.6.*

Proof. By Proposition 2.8 there exists a stationary probability measure μ on $W^{3,2}(\mathbb{T}^2_\lambda)$. Choose ω_0^ν to be distributed according to μ and define $\omega^\nu(t)$ as in (2.3). As μ is stationary, so is the law of ω^ν, hence ω^ν is a statistically stationary mild solution to (1.4). $\square$

Chapter 3: Necessary and Sufficient Conditions for Kolmogorov's Flux Laws on $\mathbb{T}^2$ and $\mathbb{T}^3$

The contents of this chapter have been published in [26].

In this chapter, we characterize the necessary and sufficient conditions for the formulation of Kolmogorov's turbulent flux laws in 2 and 3 dimensions. For simplicity, we consider only toroidal domains and such that the external forcing is drift free (i.e. $D = \mathbb{T}^d_\lambda = \mathbb{R}^d \setminus 2\pi\lambda\mathbb{Z}^d$ for $d = 2, 3$ and $\lambda > 0$, and $f \equiv 0$ in equation (1.1)). The reason for this is that the presence of a boundary introduces anisotropy into the flow and makes the KHM equations (see Section 3.2) we use to analyze the flow of energy across length scales no longer hold. Furthermore by setting $f \equiv 0$, we can balance the amount of energy dissipation exactly with the amount of energy injected by the noise a useful fact to carry out several computations in the proof of Kolmogorov cascade laws. Lastly, in order to guarantee the existence of statistically stationary solutions we will consider $g \in C^\infty(\mathbb{T}^d_\lambda)$ which is divergence free and independent of time, such that Assumption 2.7 holds when $d = 2$.

3.1 Turbulence Flux Laws

As outlined in Chapter 1, numerous experimental and numerical simulations support the existence of an energy cascade in turbulent fluids along the lines of Kolmogorov's predictions. However, an important distinction the requires attention is which statistical assumptions about the flow are truly required and which are byproducts of the very physics governing the motion of the fluid itself.

Inspired by the ideas in [8, 9, 25, 60], in this work we characterize the presence of a direct energy cascade in terms of (possible) finite time singularities that arise due to the invariant regularity of the flow and the amount of the energy which moves out to infinite wave-numbers. This is summed up in the anomalous dissipation condition (1.2). However, it was shown in [9] that in the stochastic regime, such a condition can be exhibit by statistically stationary solutions to the heat equation, where nothing "anomalous" is occurring. As such we consider what we will refer to as *weak anomalous dissipation*:

Definition 3.1. *We say that a sequence $\{u^\nu\}_{\nu \geq 0}$ of stationary martingale solutions satisfies weak anomalous dissipation if either*

$$\liminf_{\nu \to 0} \mathbf{E} D(u^\nu) > 0 \tag{3.1}$$

where $D(u^\nu)$ is a positive Radon measure defined in 3.15, or there exists $N_\nu \to \infty$ such that

$$\liminf_{\nu \to 0} \nu \sum_{|k| \geq N_\nu} \mathbf{E} \fint_0^T |\widehat{\nabla u^\nu}(k)|^2 > 0. \tag{3.2}$$

Physically this would mean that in the inviscid limit, infinitely many length scales have been introduced to the system in order to let the energy "escape to the molecular level" where it can be dissipated as heat by the particles as they collide with one another, or that the flow velocity is kinked in such a why that energy must be expended/dissipated in order for the flow to remain statistically stationary.

Remark 3.2. *Previously, the authors in* [9] *and* [60] *defined weak anomalous dissipation in terms of the Taylor micro-scale*

$$\lim_{\nu \to 0} \nu \mathbf{E} \|u^\nu(t)\|_\lambda^2 = 0 \quad \forall\, t \geq 0.$$

Such a condition is sufficient to guarantee the existence of a sequence N_ν such that (3.2) *holds. For example, if $N_\nu^{-2} = o(\nu \mathbf{E} \|u^\nu\|_{L^2}^2)$ then*

$$\limsup_{\nu \to 0} \nu \sum_{|k| \leq N_\nu} \mathbf{E} \fint_0^T |\widehat{\nabla u^\nu}(k)|^2 \leq \limsup_{\nu \to 0} N_\nu^2 \nu \sum_{|k| \leq N_\nu} \mathbf{E} |\widehat{u^\nu}(k)|^2 \leq \limsup_{\nu \to 0} N_\nu^2 \nu \mathbf{E} \|u^\nu\|_{L^2}^2 = 0.$$

Therefore

$$\liminf_{\nu \to 0} \nu \sum_{|k| \geq N_\nu} \mathbf{E} \fint_0^T |\widehat{\nabla u^\nu}(k)|^2 = \varepsilon - \limsup_{\nu \to 0} \nu \sum_{|k| \leq N_\nu} \mathbf{E} \fint_0^T |\widehat{\nabla u^\nu}(k)|^2 = \varepsilon > 0.$$

where $\varepsilon = \frac{1}{2} \mathbf{E} \fint_0^T \|g\|_\lambda^2 dt$ is the average energy injected by the noise.

As discussed previously, due to the lack of vortex-stretching in two-dimensions, the movement of kinetic energy is different in two and three dimensions, as such we discuss the direct cascade separately for both dimensions.

Theorem 3.3 (3D Direct Cascade Characterization)**.** *Suppose $\{u^\nu\}_{\nu > 0}$ is a sequence of statistically stationary solutions to the statistically forced Navier Stokes equations* (1.1) *on $\mathbb{T}_\lambda^3$. There* *exists* $N_\nu \geq 1$ *such that* $\lim\limits_{\nu\to 0} N_\nu = \infty$ *and*

$$\liminf_{\nu\to 0}\Big(\nu \sum_{|k|\geq N_\nu} \mathbf{E}\fint_0^T |\widehat{\nabla u}(k)|^2 + \mathbf{E}D(u^\nu)\Big) = \varepsilon^*$$

if and only if there exists $\ell_\nu \in (0,1)$ *such that* $\lim\limits_{\nu\to 0} \ell_\nu = 0$ *and*

$$\lim_{\ell_I\to 0}\limsup_{\nu\to 0}\sup_{\ell\in[\ell_\nu,\ell_I]}\Big|\frac{1}{\ell}\mathbf{E}\fint_{S^2}\fint_0^T\fint_{\mathbb{T}^3_\lambda}|\delta_{\ell n}u^\nu|^2(\delta_{\ell n}u^\nu\cdot n)\,dxdtdS(n) + \frac{4}{3}\varepsilon^*\Big| = 0; \tag{3.3}$$

$$\lim_{\ell_I\to 0}\limsup_{\nu\to 0}\sup_{\ell\in[\ell_\nu,\ell_I]}\Big|\frac{1}{\ell}\mathbf{E}\fint_{S^2}\fint_0^T\fint_{\mathbb{T}^3_\lambda}(\delta_{\ell n}u^\nu\cdot n)^3\,dxdtdS(n) + \frac{4}{5}\varepsilon^*\Big| = 0. \tag{3.4}$$

A similar characterization in 2D also holds, but now in terms of the dissipation of enstrophy instead of the dissipation of kinetic energy.

Theorem 3.4 (2D Direct Cascade Characterization)**.** *Let* $\{u^\nu\}_{\nu>0}$ *be a sequence of statistically stationary solutions to the stochastically forced Navier Stokes equations* (1.1) *and the 2D vorticity equation* (1.4) *on* $\mathbb{T}^2_\lambda$*. There exists* $N_\nu \geq 1$ *such that* $\lim\limits_{\nu\to 0} N_\nu = \infty$ *and*

$$\liminf_{\nu\to 0}\nu\sum_{|k|\geq N_\nu}\mathbf{E}|\widehat{\nabla\omega^\nu}(k)|^2 = \eta^*$$

if and only if there exists $\ell_\nu \in (0,1)$ *such that* $\lim\limits_{\nu\to 0} \ell_\nu = 0$ *and*

$$\lim_{\ell_I\to 0}\limsup_{\nu\to 0}\sup_{\ell\in[\ell_\nu,\ell_I]}\Big|\frac{1}{\ell^3}\mathbf{E}\fint_{S^1}\fint_0^T\fint_{\mathbb{T}^2_\lambda}|\delta_{\ell n}\omega^\nu|^2(\delta_{\ell n}u^\nu\cdot n)\,dxdtdS(n) + 2\eta^*\Big| = 0; \tag{3.5}$$

$$\lim_{\ell_I\to 0}\limsup_{\nu\to 0}\sup_{\ell\in[\ell_\nu,\ell_I]}\Big|\frac{1}{\ell^3}\mathbf{E}\fint_{S^1}\fint_0^T\fint_{\mathbb{T}^2_\lambda}|\delta_{\ell n}u^\nu|^2(\delta_{\ell n}u^\nu\cdot n)\,dxdtdS(n) - \frac{1}{4}\eta^*\Big| = 0; \tag{3.6}$$

$$\lim_{\ell_I\to 0}\limsup_{\nu\to 0}\sup_{\ell\in[\ell_\nu,\ell_I]}\Big|\frac{1}{\ell^3}\mathbf{E}\fint_{S^1}\fint_0^T\fint_{\mathbb{T}^2_\lambda}(\delta_{\ell n}u^\nu\cdot n)^3\,dxdtdS(n) - \frac{1}{8}\eta^*\Big| = 0. \tag{3.7}$$

Here the limit $\ell_I \to 0$ should *not* be confused with injection scale of energy moving to small scales. Instead ℓ_I is acting as a dummy variable signifying when the scales go to 0. This is similar to the limits shown in [8, 9, 32, 60].

Remark 3.5. *In both 2D and 3D, the size of the torus λ is insignificant. This is because the direct cascade limit localizes the structure function, i.e. only nearby points contribute to the energy dynamics. For this reason, one can also extend these ideas to domains with boundaries by localizing each of the quantities. See* [60] *for details.*

Remark 3.6. *This characterization of a direct cascade is in agreement with the weak anomalous dissipation results from* [9]*, under the additional assumption that $\mathbf{E}D(u^\nu) \equiv 0$ in 3D. By this we mean, that if the energy blows up at most like $1/\nu$, then a sufficient estimate for N_ν is given by:*

$$N_\nu^2 = o((\nu \mathbf{E}\|u^\nu\|_\lambda^2)^{-1}).$$

See Remark 3.2 for details.

In contrast to the direct cascade, we characterize an inverse cascade by how much energy moves toward the wave-numbers near 0 due to an imbalance in the strength of the noise across the entire domain. This would mean that at the largest length scales in the system some parts of the domain receive more energy from the noise than other areas leading to some of the fluid's kinetic energy moving up in length scales to balance out the system. In this way, the existence of an inverse cascade is nearly identical in both two and three dimensions. However, an inverse cascade typically requires some kind of dampening process to erase the contribution of the high Fourier modes and emphasize the low order modes, see [8, 11, 18, 64] and the references within for various examples. In this chapter, we accomplish this by taking the period of the domain $\lambda \to \infty$. Physically this would be equivalent to filtering out all of the non-constant modes from the spectral solution.

Theorem 3.7 (Inverse Cascade Characterization)**.** *Suppose that $\lambda = \lambda(\nu) < \infty$ is a continuous monotone increasing function such that $\lim_{\nu \to 0} \lambda = \infty$. Let $\{u^\nu\}_{\nu>0}$ be a sequence of statistically stationary solutions to (1.1) with divergence free forcing g. There exists a decreasing sequence M_ν satisfying $\lim_{\nu\to 0} M_\nu = 0$ such that*

$$\liminf_{\nu \to 0} \nu \sum_{|k| \leq M_\nu} \mathbf{E} \fint_0^T |\widehat{\nabla u^\nu}(k)|^2 = \varepsilon^* + \frac{1}{2}\mathbf{E}|\widehat{g}(0)|^2$$

if and only if there exists an increasing sequence $\tilde{\ell}_\nu \in (1, \lambda)$ satisfying $\lim_{\nu \to 0} \tilde{\ell}_\nu = \infty$ such that:

$$\lim_{\ell_I \to \infty} \limsup_{\nu \to 0} \sup_{\ell \in [\ell_I, \tilde{\ell}_\nu]} \left| \frac{1}{\ell} \mathbf{E} \fint_{S^{d-1}} \fint_0^T \fint_{\mathbb{T}^d_\lambda} |\delta_{\ell n} u^\nu|^2 (\delta_{\ell n} u^\nu \cdot n) \, dx dt dS(n) - \gamma_d \varepsilon^* \right| = 0 \tag{3.8}$$

$$\lim_{\ell_I \to \infty} \limsup_{\nu \to 0} \sup_{\ell \in [\ell_I, \tilde{\ell}_\nu]} \left| \frac{1}{\ell} \mathbf{E} \fint_{S^{d-1}} \fint_0^T \fint_{\mathbb{T}^d_\lambda} (\delta_{\ell n} u^\nu \cdot n)^3 \, dx dt dS(n) - \kappa_d \varepsilon^* \right| = 0 \tag{3.9}$$

where the coefficients are given by

$$\gamma_d = \begin{cases} 2 & d = 2 \\ 4/3 & d = 3 \end{cases} \qquad \kappa_d = \begin{cases} 3/2 & d = 2 \\ 4/5 & d = 3 \end{cases}$$

Remark 3.8. *Notice that the difference in coefficients γ_d and κ_d when $d = 2, 3$ are due to the difference in dimension but the proof is otherwise exactly the same.*

Remark 3.9. *Due to the Biot-Savart law, in 2D, the equivalent condition for the inverse cascade* *can be reformulated as*

$$\liminf_{\nu\to 0} \nu \sum_{|k|\le M_\nu} \mathbf{E}|\widehat{\omega^\nu}(k)|^2 = \varepsilon^* + \frac{1}{2}\mathbf{E}|\widehat{g}(0)|^2.$$

Remark 3.10. *Notice that the flux laws for the direct cascades are defined in the energy/enstrophy dissipated by the viscosity at the infinite wave-numbers and any potential finite time local singularities. Whereas, as an artifact of the proof to Theorem 3.7 an inverse cascade is a difference between the average force and the energy dissipation at the largest scales.*

Remark 3.11. *If one knows the spectrum of u^ν through a scaling argument or an ad-hoc assumption, then one can quickly find the sequences N_ν and M_ν such that either an inverse or direct cascade can occur.*

Corollary 3.11.1. *Suppose that for some $\alpha \in (0,1)$ the family of viscous flows $\{u^\nu\}_{\nu>0}$ takes values in $L^\infty(0,T,C^\alpha(\mathbb{T}^d))$ uniformly in ν. Then the smallest possible cutoff sequence is given by $N_\nu = o(\nu^{\frac{1}{2\alpha-2}})$. In particular if $\alpha = \frac{1}{3}$, then $\mathbf{E}D(u^\nu) \equiv 0$ and $N_\nu = o(\nu^{-3/4})$ which is the inverse of the Kolmogorov microscale.*

Proof. For any $0 < s < \alpha$

$$\begin{aligned}\mathbf{E}\fint_0^T \|u^\nu\|^2_{\dot{H}^s(\mathbb{T}^d_\lambda)} &= C\mathbf{E}\fint_0^T \int_{\mathbb{T}^d_\lambda}\int_{\mathbb{T}^d_\lambda} \frac{|u^\nu(x)-u^\nu(y)|^2}{|x-y|^{d+2s}}\,dxdydt \\ &\le C\fint_0^T \int_{\mathbb{T}^d}\int_{\mathbb{T}^d} \frac{1}{|x-y|^{d+2(s-\alpha)}}\,dxdydt < \infty.\end{aligned}$$

Hence if we use the Fourier transform definition of $\dot{H}^s$ instead we get that

$$\nu \sum_{|k|<N_\nu} \mathbf{E} \fint_0^T |\widehat{\nabla u^\nu}(k)|^2 = \nu \sum_{|k|<N_\nu} |k|^{2-2s}|k|^{2s}\mathbf{E} \fint_0^T |\widehat{u^\nu}(k)|^2 \leq \nu N_\nu^{2-2s}\mathbf{E} \fint_0^T \|u^\nu\|^2_{\dot{H}^s(\mathbb{T}^d)}.$$

Choosing $N_\nu = o(\nu^{\frac{1}{2s-2}})$, the infimum of which being $o(\nu^{\frac{1}{2\alpha-2}})$, results in

$$\liminf_{\nu\to 0} \nu \sum_{|k|\geq N_\nu} \mathbf{E} \fint_0^T |\widehat{\nabla u^\nu}(k)|^2 = \varepsilon - \limsup_{\nu\to 0} \nu \sum_{|k|<N_\nu} \mathbf{E} \fint_0^T |\widehat{\nabla u^\nu}(k)|^2 = \varepsilon > 0.$$

This implies there is a direct cascade of kinetic energy in the sense of Theorem 3.3. A similar argument can be made for the direct cascade of enstrophy as in Theorem 3.4.

Furthermore, if $\alpha = \frac{1}{3}$ then

$$|\mathbf{E}D(u^\nu)| \leq \lim_{\ell\to 0} C\mathbf{E} \int_0^T \int_{\mathbb{T}^d_\lambda} |\delta_{\ell n} u^\nu|^3 \, dn \leq C \lim_{\ell\to 0} \ell = 0$$

and $N_\nu = o(\nu^{\frac{1}{2s-2}})$ for all $0 < s < \frac{1}{3}$ the infimum of which is $o(\nu^{-3/4})$. □

Remark 3.12. *At first glance one may consider the equivalence conditions for Theorems 3.3, 3.4, and 3.7 may seem difficult to prove. However, by Remark 3.2, it is enough to know that the expected value of the kinetic energy does not grow faster than ν^{-1}. In the physics and engineering literature, it is common to assume that the the kinetic energy remains finite independent of the viscosity ν making it feasible to show the conditions for a cascade are satisfied.*

Furthermore, better estimates on the cutoff sequences N_ν and M_ν can be achieved based on the regularity of the viscous flow fields. For instance in Corollary 3.11.1 we showed that if u^ν are uniformly (in ν) $C^{\frac{1}{3}}$-valued random fields then one can find $N_\nu \sim \nu^{-3/4}$ which is the inverse *of Kolmogorov microscale, while the roughness measure*

$$\liminf_{\nu>0} \mathbf{E}D(u^\nu) = 0.$$

Regardless of how one estimates the cutoff sequences, the equivalent conditions for each of the main Theorems describe how small scale structures develop by making sure enough energy can "escape to infinity" in spectral space or the solutions are sufficiently rough (in a fractal-like manner) so that the convection term $D(u^\nu)$ dissipates the energy instead. Therefore, one needs only to relate the regularity of the solution to its Fourier spectrum such as through Besov spaces as seen in Remark 3.2 and Corollary 3.11.1.

Not only does the work here provide necessary and sufficient conditions for direct and indirect cascades as noted by Kolmogorov[43], Batchelor [6], Kraichnan [46], and Fjortoft[30], but it also characterizes split cascades (were energy moves in both directions), isolated cascades (movement in only one direction) without frictional forces in both 2D and 3D, and condensates where energy becomes trapped within an eddy structure at a finite non-zero length scale. For example flows which exhibit split cascades have been observed in experiments in thin film turbulence [34, 55] and soap films [68]. Furthermore, using the Biot-Savart law, one can reformulate the dual cascade problem (where there is a direct cascade of enstrophy and an inverse cascade of energy simultaneously) of 2D turbulence, as a the velocity having a split cascade [46].

Corollary 3.12.1. *Suppose $\{u^\nu\}_{\nu>0}$ is a sequence of statistically stationary solutions to* (1.1) *on $\mathbb{T}^3_\lambda$, then*

$$\lim_{\nu\to 0} \nu \mathbf{E}\|u^\nu(t)\|^2_{L^2(\mathbb{T}^3_\lambda)} = 0 \quad \forall\, t$$

is a sufficient condition for an isolated direct cascade in the sense of Theorem 3.3

This follows directly from Theorem 3.3 and Remark 3.2.

Corollary 3.12.2. *Suppose* $\{u^\nu\}_{\nu>0}$ *is a sequence of smooth statistically stationary solutions to* (1.1) *on* $\mathbb{T}^3_\lambda$*, and* $|\widehat{g}(0)| > 0$*. Assume there exists a wave-number* $c \in (0,\infty)$ *such that*

$$\lim_{\nu\to 0} \nu \sum_{|k|\geq c} \mathbf{E}|\widehat{u^\nu}(k)|^2 = 0 \quad \textit{and} \quad \lim_{\nu\to 0} \nu \sum_{|k|\leq c} \mathbf{E}|\widehat{\nabla u^\nu}(k)|^2 < \varepsilon.$$

If there is no flux of energy to large scales in the sense of (3.8) *and* (3.9)*, then there is a split cascade (i.e. there exist both sequences* N_ν *and* M_ν*).*

Proof. By assumption the zero energy flux of the structure functions at large scales, it follows from Theorem 3.7 that there exists $M_\nu \to 0$ such that

$$\liminf_{\nu\to 0} \nu \sum_{|k|\leq M_\nu} \mathbf{E} \fint_0^T |\widehat{\nabla u^\nu}(k)|^2 = \frac{1}{2}\mathbf{E}|\widehat{g}(0)|^2 > 0.$$

Now choose $N_\nu \to \infty$ such that $\lim_{\nu\to\infty} \nu|N_\nu|^2 \sum_{|k|\geq c} \mathbf{E}|\widehat{u^\nu}(k)|^2 = 0$. Then

$$\liminf_{\nu\to 0} \nu \sum_{|k|\geq N_\nu} \mathbf{E} \fint_0^T |\widehat{\nabla u^\nu}(k)|^2 = \varepsilon - \limsup_{\nu\to 0} \nu \sum_{|k|\leq c} \mathbf{E} \fint_0^T |\widehat{\nabla u^\nu}(k)|^2 = \varepsilon^* > 0.$$

□

Finally in the case of 2D flows

Corollary 3.12.3. *Suppose* $\{u^\nu\}_{\nu>0}$ *is a sequence of statistically stationary solutions to both the Navier Stokes equations* (1.1) *and the vorticity equation* (1.4) *on* $\mathbb{T}^2_\lambda$ *with mean-zero forcing (i.e.*

$\widehat{g}(0) = 0$). Assume there exists a wave-number $c \in (0, \infty)$ such that

$$\lim_{\nu \to 0} \nu \sum_{|k| \geq c} \mathbf{E}|\widehat{\omega^\nu}(k)|^2 = 0 \quad \textit{and} \; \lim_{\nu \to 0} \nu \sum_{|k| \leq c} \mathbf{E}|\widehat{\nabla\omega^\nu}(k)|^2 = 0.$$

Then there is a dual cascade in the sense of Theorems 3.4 and 3.7.

Proof. Pick $N_\nu \to \infty$ such that $N_\nu^2 \nu \sum_{c \leq |k| \leq N_\nu} \mathbf{E} \fint_0^T |\widehat{\omega^\nu}(k)|^2 \to 0$ as $\nu \to 0$. Then

$$\nu \sum_{|k| \leq N_\nu} \mathbf{E} \fint_0^T |\widehat{\nabla\omega^\nu}(k)|^2 \leq \nu \sum_{|k| \leq c} \mathbf{E} \fint_0^T |\widehat{\nabla\omega^\nu}(k)|^2 + \nu \sum_{c \leq |k| \leq N_\nu} \mathbf{E} \fint_0^T N_\nu^2 |\widehat{\omega^\nu}(k)|^2 \to 0 \quad \text{as } \nu \to 0.$$

Hence by the enstrophy balance (3.17)

$$\lim_{\nu \to 0} \nu \sum_{|k| \geq N_\nu} \mathbf{E}|\widehat{\nabla\omega^\nu}(k)|^2 = \eta > 0.$$

Similarly, for the inverse cascade we pick $M_\nu \to 0$ such that for M_ν sufficiently small it holds that $\nu \sum_{M_\nu \leq |k| \leq c} \frac{|k|^2}{M_\nu^2} \mathbf{E} \fint_0^T |\widehat{\omega^\nu}(k)|^2 \to 0$ as $\nu \to 0$. Then

$$\nu \sum_{|k| \geq M_\nu} \mathbf{E} \fint_0^T |\widehat{\omega^\nu}(k)|^2 \leq \nu \sum_{|k| \geq c} \mathbf{E} \fint_0^T |\widehat{\omega^\nu}(k)|^2 + \nu \sum_{M_\nu \leq |k| \leq c} \frac{|k|^2}{M_\nu^2} \mathbf{E} \fint_0^T |\widehat{\omega^\nu}(k)|^2 \to 0 \quad \text{as } \nu \to 0.$$

Thus by the energy balance (3.14) (after applying the Biot-Savart law) it follows that

$$\lim_{\nu \to 0} \nu \sum_{|k| \leq M_\nu} \mathbf{E}|\widehat{\omega^\nu}(k)|^2 = \varepsilon > 0.$$

□

3.1.1 Outline of the Chapter

The remainder of the chapter focuses on proving each of the three main Theorems listed above. First in section 3.2 we discuss the von Karman-Howarth-Monin relations, which we use to derive both an energy balance as well as an enstrophy balance (in the two-dimensional case). Furthermore, we show that the isotropric tensors used to construct the structure functions has a nice power series representation in spectral space. Then in Section 3.3 we prove Kolmogorov's $\frac{4}{3}$ and $\frac{4}{5}$ flux laws, while in Section 3.4 we prove the Batchelor-Kraichnan flux laws hold true in two-dimensions. Finally in Section 3.5 we find necessary and sufficient conditions for the inverse cascade laws in both two and three dimensions.

3.2 The von Karman-Howarth-Monin Relations

The main tool we use to analyze the existence of Kolmogorov's flux laws are the von Karman-Howarth-Monin (KHM) relations. These relations connect the various structure functions to two point correlations of the velocity or vorticity dissipation through the Navier Stokes equations. More simply stated, they are the weak formulation of (1.1) applied to a mollified version of the flow velocity. These kinds of energy relations were first derived for classical solutions to the deterministic Navier Stokes equations by Karman and Howarth for the 4/5th law (1.3) in [22]. Later, they were generalized (for strong solutions) by Monin in [53]. It was shown in [32] that the KHM relations eradicated the need of Kolmogorov's mono-fractal scaling assumption, creating a consistant theory of turbulence with experimental results. In the stochastic regime, Bedrossian et.al. [8, 9] generalized them for stationary martingale solutions on $\mathbb{T}^d$ for $d = 2, 3$.

To clarify how the KHM relations are found, let us briefly take $\ell \in (0, \lambda)$ and $\phi \in$

$[C_c^\infty(\mathbb{T}_\lambda^d)]^d$ such that $\nabla \cdot \phi = 0$. Now we consider the weak formulation of (1.1) on $D = \mathbb{T}_\lambda^d$:

$$\begin{aligned}\int_{\mathbb{T}_\lambda^d} u^\nu(t)\cdot\phi(t) + \nu\int_0^t\int_{\mathbb{T}_\lambda^d}\nabla u^\nu : \nabla\phi - \int_0^t\int_{\mathbb{T}_\lambda^d} u^\nu\cdot(u^\nu\cdot\nabla)\phi \\ = \int_{\mathbb{T}_\lambda^d} u_0\cdot\phi(0) + \int_0^t\int_{\mathbb{T}_\lambda^d}\phi\cdot g dW_s + \int_0^t\int_{\mathbb{T}_\lambda^d} u^\nu\cdot d\phi + \int_0^t\int_{\mathbb{T}_\lambda^d}[gdW_t, d\phi]\end{aligned} \tag{3.10}$$

where $[\cdot,\cdot]$ is the cross variation of the random processes gW_t and ϕ. To derive the KHM relationships, we choose ϕ to be a mollified version of u^ν of length ℓ: $\phi = \mathbf{a}\psi_\ell * u^\nu$ where ψ_ℓ is a standard mollifier of size ℓ (scalar valued) and $\mathbf{a}$ is an isotropic 2-tensor. In particular, we focus on the cases when $\mathbf{a} = \mathbf{I}$ (identity tensor) or $\mathbf{a} = n\otimes n$ for $n \in S^{d-1}$. For a closer look at how this is applied see the proof of Theorem 3.14.

Let T_y be the translation operator $T_y h(\cdot) = h(\cdot + y)$ for all $y \in \mathbb{R}^d$, then for any martingale solution to the Navier Stokes equations (1.1) and the vorticity equation (1.4) (when $d = 2$) we define the following vorticity and velocity structure functions

$$\begin{aligned}S_{vor}(\ell) &= \mathbf{E}\fint_{S^{d-1}}\fint_0^T\fint_{\mathbb{T}_\lambda^d}|\delta_{\ell n}\omega^\nu|^2\delta_{\ell n}u^\nu\cdot n\; dxdtdS(n)\\ S_{vel}(\ell) &= \mathbf{E}\fint_{S^{d-1}}\fint_0^T\fint_{\mathbb{T}_\lambda^d}|\delta_{\ell n}u^\nu|^2\delta_{\ell n}u^\nu\cdot n\; dxdtdS(n)\\ S_{vel}^{\|}(\ell) &= \mathbf{E}\fint_{S^{d-1}}\fint_0^T\fint_{\mathbb{T}_\lambda^d}(\delta_{\ell n}u^\nu\cdot n)^3\; dxdtdS(n)\end{aligned}$$

where $dS(n)$ be the surface measure of the unit sphere S^{d-1}. We also define the the following two-point correlations:

$$
\begin{aligned}
\Gamma_{vor}(\ell) &= \mathbf{E} \fint_0^T \fint_{S^{d-1}} \fint_{\mathbb{T}^d_\lambda} \omega^\nu \cdot T_{\ell n} \omega^\nu \; dx dS(n) dt \\
\Gamma_{vel}(\ell) &= \mathbf{E} \fint_0^T \fint_{S^{d-1}} \fint_{\mathbb{T}^d_\lambda} u^\nu \cdot T_{\ell n} u^\nu \; dx dS(n) dt \\
\Gamma^{||}_{vel}(\ell) &= \mathbf{E} \fint_0^T \fint_{S^{d-1}} \fint_{\mathbb{T}^d_\lambda} (n \otimes n) : u^\nu \otimes T_{\ell n} u^\nu \; dx dS(n) dt \\
a_{vor}(\ell) &= \frac{1}{2} \mathbf{E} \fint_{S^{d-1}} \fint_{\mathbb{T}^d_\lambda} \mathrm{curl} g : T_{\ell n} \mathrm{curl} g \; dx dS(n) \\
a_{vel}(\ell) &= \frac{1}{2} \mathbf{E} \fint_{S^{d-1}} \fint_{\mathbb{T}^d_\lambda} g \cdot T_{\ell n} g \; dx dS(n) \\
a^{||}_{vel}(\ell) &= \frac{1}{2} \mathbf{E} \fint_{S^{d-1}} \fint_{\mathbb{T}^d_\lambda} (n \otimes n) : g \otimes T_{\ell n} g \; dx dS(n).
\end{aligned}
$$

Proposition 3.13. *Suppose u^ν is a solution to (1.1) in the sense of Definition 2.4 on $\mathbb{T}^d_\lambda$ for $d = 2, 3$. Then*

$$
S_{vel}(\ell) = -4\nu \Gamma'_{vel}(\ell) - \frac{4}{\ell^{d-1}} \int_0^\ell r^{d-1} a_{vel}(r) \, dr; \tag{3.11}
$$

$$
S^{||}_{vel}(\ell) = -4\nu (\Gamma^{||}_{vel})'(\ell) + \frac{2}{\ell^{d+1}} \int_0^\ell r^d S_{vel}(r) \, dr - \frac{4}{\ell^{d+1}} \int_0^\ell r^{d+1} a^{||}_{vel}(r) \, dr. \tag{3.12}
$$

When $d = 2$, if ω^ν is a solution to (1.4) in the sense of Definition 2.6 then there is the additional KHM relation:

$$
S_{vor}(\ell) = -4\nu \Gamma'_{vor}(\ell) - \frac{4}{\ell} \int_0^\ell r a_{vor}(r) \, dr. \tag{3.13}
$$

Moreover, for three-dimensional flows u^ν, the correlations $\Gamma_{vel}, \Gamma^{||}_{vel} \in C^2(\mathbb{R})$ while for two-dimensional flows the correlations $\Gamma_{vor} \in C^3(\mathbb{R})$ and $\Gamma_{vel}, \Gamma^{||}_{vel} \in C^4(\mathbb{R})$.

Proof. See [9] for details about the derivation and regularity of the identities in 3D. In the two-dimensional case, the derivation of the structure functions is similar to the 3D case, but there is an increase in regularity due to boundedness of the average enstrophy. See [8] for more details. □

3.2.1 Balance Laws for the Stochastic Navier Stokes Equations

In Proposition 3.13, we used the fact that u^ν is statistically stationary to eliminate any initial and final kinetic energy correlation terms. This is because when u^ν is statistically stationary, then on average the total kinetic energy is conserved for all time:

$$\frac{1}{2}\mathbf{E}\|u^\nu(t)\|_\lambda^2 = \frac{1}{2}\mathbf{E}\|u^\nu(s)\|_\lambda^2 \quad \forall\, t, s \in [0, \infty).$$

Moreover, this remains true for the autocorrelation for the kinetic energy. As such if we take $\ell \to 0$ then (3.11) will provide us with the mean viscous kinetic energy balance for the system. This idea was first heuristically considered in the work on Onsager [59] and then again more rigorously by Duchon and Robert in [25] for deterministic flows. As such this result has been around for awhile in the deterministic setting and is relatively well-known [23], but the author could find no reference to it within the stochastic literature. Therefore, the proof of this result will be included here.

Theorem 3.14. *Suppose u^ν is a statistically stationary martingale solution to* (1.1) *in the sense of Definition 2.4. Then the following energy equality holds:*

$$\nu\mathbf{E}\fint_0^T \|\nabla u^\nu(s)\|_\lambda^2\, ds + \mathbf{E}\fint_{\mathbb{T}_\lambda^d} D(u^\nu)(dx) = \frac{1}{2}\mathbf{E}\fint_0^T \|g\|_\lambda^2 = \varepsilon \tag{3.14}$$

where

$$D(u^\nu)(x) := \lim_{\ell\to 0} \frac{1}{4} \fint_0^T \fint_{\mathbb{T}^d} \nabla\phi_\ell(y)\cdot \delta_y u^\nu(x)|\delta_y u^\nu(x)|^2 \, dydt \tag{3.15}$$

is a non-negative Radon measure and ϕ_ℓ is a standard mollifier of size ℓ. Moreover, $D(u^\nu)$ is independent of the choice of mollifier and $D(u^\nu) \equiv 0$ when $d = 2$ (i.e. 2D flow).

Proof. This is a simple application of the ideas from [25] to the case of martingale solutions. Let $\phi \in C_c^\infty(B(0,1))$ such that $\phi \geq 0$ and $\int_{B(0,1)} \phi \, dx = 1$. For $\ell \ll 1$ we define $\phi_\ell(x) = \ell^{-d}\phi(x/\ell)$ and mollify

$$u_\ell^\nu = \phi_\ell * u^\nu, \quad (u_i^\nu u_j^\nu)_\ell = \phi_\ell * (u_i^\nu u_j^\nu), \quad p_\ell^\nu = \phi_\ell^\nu * p, \quad (gdW_t)_\ell = \phi_\ell * gdW_t.$$

Implicit here is that we are using the isotropic tensor $\mathbf{a} = \mathbf{I}$ in our mollification. Each of these mollified variables now resides in $C_c^\infty(\mathbb{T}_\lambda^d)$, and as u^ν solves the Navier Stokes equations (1.1) it follows that:

$$du_\ell^\nu + (\nabla\cdot(u^\nu \otimes u^\nu)_\ell + \nabla p_\ell^\nu)dt = \nu\Delta u_\ell^\nu \, dt + (gdW_t)_\ell$$

$$\nabla\cdot u_\ell^\nu = 0.$$

Now fix t, we apply Ito's lemma, integration by parts, and the independence of the Brownian motions to get

$$d\int_{\mathbb{T}^d}(u^\nu\cdot u^\nu_\ell)\;dx=\int_{\mathbb{T}^d}u^\nu_\ell\cdot du^\nu\;dx+\int_{\mathbb{T}^d}u^\nu\cdot du^\nu_\ell\;dx+\int_{\mathbb{T}^d}d[u^\nu,u^\nu_\ell]\;dx$$
$$=-2\nu\int_{\mathbb{T}^d}\nabla u^\nu:\nabla u^\nu_\ell\;dxdt+\int_{\mathbb{T}^d}\Big((u^\nu\otimes u^\nu):\nabla u^\nu_\ell-u^\nu\cdot\nabla\cdot(u^\nu\otimes u^\nu)_\ell\Big)\;dxdt$$
$$+\int_{\mathbb{T}^d}g\cdot g_\ell\;dxdt+\int_{\mathbb{T}^d}(u^\nu_\ell\cdot gdW_t+u^\nu\cdot(gdW_t)_\ell)\;dx.$$

Define

$$A_\ell(s):=\int_{\mathbb{T}^d}u^\nu(s)\cdot\nabla\cdot(u^\nu(s)\otimes u^\nu(s))_\ell\;dx-\int_{\mathbb{T}^d}(u^\nu(s)\otimes u^\nu(s)):\nabla u^\nu_\ell(s)\;dx.$$

As $g\in L^2(\mathbb{T}^d_\lambda)$, the noise terms are each martingales with mean 0. Thus when we take the expected value of both sides and write everything in integral form we get

$$\mathbf{E}\int_{\mathbb{T}^d}u^\nu(T)\cdot u^\nu_\ell(T)\;dx=\mathbf{E}\int_{\mathbb{T}^d}u^\nu(0)\cdot u^\nu_\ell(0)\;dx-2\nu\mathbf{E}\int_0^T\int_{\mathbb{T}^d}\nabla u^\nu(s):\nabla u^\nu_\ell(t)\;dxdt$$
$$-\mathbf{E}\int_0^T A_\ell(t)\;dt+\mathbf{E}\int_0^T\int_{\mathbb{T}^d}g\cdot g_\ell\;dxdt.$$

Next, since $u^\nu\in[L^2(\Omega\times[0,T],H^1(\mathbb{T}^d_\lambda))\cap L^2(\Omega,L^\infty(0,T,L^2(\mathbb{T}^d_\lambda)))]^d$, the map $t\mapsto\|u^\nu(t)\|_{L^2}$

is bounded over the interval $[0,T]$ and due to the continuity of translations in L^2 it follows that

$$\begin{aligned}
&\Big|\mathbf{E}\int_{\mathbb{T}^d} u^\nu(T)\cdot u^\nu_\ell(T)\,dx - \mathbf{E}\int_{\mathbb{T}^d} u^\nu(T)\cdot u^\nu(T)\,dx\Big| \\
&\qquad = \Big|\mathbf{E}\int_{\mathbb{T}^d}\phi_\ell(y)\int_{\mathbb{T}^d}\big(u^\nu(T,x+y)-u^\nu(T,x)\big)\cdot u^\nu(T,x)\,dxdy\Big| \\
&\qquad \le C\mathbf{E}\int_{\mathrm{supp}\ \phi_\ell}\|u^\nu(T)\|_{L^2}\Big(\int_{\mathbb{T}^d}|u^\nu(T,x+y)-u^\nu(T,x)|^2\,dx\Big)^{1/2}dy \\
&\qquad \to 0 \\
&\Big|\mathbf{E}\int_{\mathbb{T}^d} u^\nu(0)\cdot u^\nu_\ell(0)\,dx - \mathbf{E}\int_{\mathbb{T}^d} u^\nu(0)\cdot u^\nu(0)\,dx\Big| \\
&\qquad = \Big|\mathbf{E}\int_{\mathbb{T}^d}\phi_\ell(y)\int_{\mathbb{T}^d}\big(u^\nu(0,x+y)-u^\nu(0,x)\big)\cdot u^\nu(0,x)\,dxdy\Big| \\
&\qquad \le C\mathbf{E}\int_{\mathrm{supp}\ \phi_\ell}\|u^\nu(0)\|_{L^2}\Big(\int_{\mathbb{T}^d}|u^\nu(0,x+y)-u^\nu(0,x)|^2\,dx\Big)^{1/2}dy \\
&\qquad \to 0
\end{aligned}$$

as $\ell\to 0$. Similar analysis shows that $\mathbf{E}\int_0^T\int_{\mathbb{T}^d_\lambda} g\cdot g_\ell\,dxdt \to \mathbf{E}\int_0^T\|g\|^2_{L^2}\,dt$ and $\mathbf{E}\int_0^T\int_{\mathbb{T}^d}\nabla u^\nu : \nabla u^\nu_\ell\,dxdt \to \mathbf{E}\int_0^T\|\nabla u^\nu(t)\|^2_{L^2}\,dt$ as $\ell\to 0$. Hence by the stationarity of the law of u^ν:

$$\lim_{\ell\to 0}\mathbf{E}\fint_0^T A_\ell\,dt = -2\nu\mathbf{E}\fint_0^T\|\nabla u^\nu\|^2_{L^2} + \mathbf{E}\fint_0^T\|g\|^2_{L^2}. \tag{3.16}$$

Next, akin to [25] we consider the function:

$$D_\ell(u^\nu)(x) := \frac{1}{4}\fint_0^T\int_{\mathbb{T}^d}\nabla\phi_\ell(y)\cdot\delta_y u^\nu|\delta_y u^\nu|^2\,dydt.$$

Let $T_y h(\cdot) = h(\cdot + y)$ for all $y\in\mathbb{R}^d$. Applying the incompressibility condition, Fubini's theorem, and integration by parts we get:

$$
\begin{aligned}
\int_{\mathbb{T}^d}\int_{\mathbb{T}^d} & \nabla\phi_\ell(y)\cdot\delta_y u^\nu |\delta_y u^\nu|^2 \, dydx \\
&= \int\int \partial_{y_j}\phi_\ell(y)\Big(T_y u_j^\nu T_y u_k^\nu T_y u_k^\nu - 2T_y u_j^\nu T_y u_k^\nu u_k^\nu \\
&\quad + T_y u_j^\nu u_k^\nu u_k^\nu - u_j^\nu T_y u_k^\nu T_y u_k^\nu + 2u_j^\nu u_k^\nu T_y u_k^\nu - u_j^\nu u_k^\nu u_k^\nu\Big) \, dydx \\
&= \int \partial_{y_j}\phi_\ell(y)\Big(\int u_j^\nu u_k^\nu u_k^\nu \, dx - 2\int T_y u_j^\nu T_y u_k^\nu u_k^\nu \, dx + \int T_y u_j^\nu u_k^\nu u_k^\nu \, dx \\
&\quad - \int T_{-y} u_j^\nu u_k^\nu u_k^\nu \, dx + 2\int u_j^\nu u_k^\nu T_y u_k^\nu \, dx - \int u_j^\nu u_k^\nu u_k^\nu \, dx\Big) \, dy \\
&= -2\int \partial_{y_j}\phi_\ell(y)\Big(\int T_y u_j^\nu T_y u_k^\nu u_k^\nu \, dx - \int u_j^\nu u_k^\nu T_y u_k^\nu \, dx\Big) \, dy \\
&= -2\int u_j^\nu(z) u_k^\nu(z)\partial_{z_j}\int \phi_\ell(z-x) u_k^\nu(x) \, dxdz \\
&\quad + 2\int u_k^\nu(z)\partial_{z_j}\int \phi_\ell(z-x) u_j^\nu(x) u_k^\nu(x) \, dxdz \\
&= 2\int u_k^\nu \partial_j (u_j^\nu u_k^\nu)_\ell \, dz - 2\int u_j^\nu u_k^\nu \partial_j (u_k^\nu)_\ell \, dz \\
&= 2A_\ell.
\end{aligned}
$$

Define

$$
\fint_{\mathbb{T}^d_\lambda} D(u^\nu)(dx) := \lim_{\ell\to 0}\fint_{\mathbb{T}^d_\lambda} D_\ell(u^\nu)(dx) = \frac{1}{2|\mathbb{T}^d_\lambda|}\lim_{\ell\to 0}\fint_0^T A_\ell \, dt.
$$

Then after combining $D(u^\nu)$ with Equation (3.16) the desired energy balance appears:

$$
\nu\mathbf{E}\fint_0^T \|\nabla u^\nu\|_\lambda^2 + \mathbf{E}\fint_{\mathbb{T}^d_\lambda} D(u^\nu)(dx) = \frac{1}{2}\mathbf{E}\fint_0^T \|g\|_\lambda^2.
$$

It follows directly from [25] that $D(u^\nu)$ is at most non-negative in 3 dimensions and is identically zero in 2 dimensions using the same proof so we omit the details. $\square$

Duchon and Robert [25] showed that in two-dimensions $D(u^\nu) \equiv 0$ which leads to a pure energy balance. We show here that for a fixed $\nu > 0$ the same argument is true for the enstrophy. Therefore, when (u^ν, ω^ν) are statistically stationary solutions to (1.1) and (1.4) respectively, there are two conserved invariants leading to the presence of a dual-cascade.

Corollary 3.14.1. *Let ω^ν be a statistically stationary mild solution to* (1.4)*, then*

$$\nu \mathbf{E} \fint_0^T \|\nabla \omega^\nu(s)\|_\lambda^2 \, ds = \frac{1}{2} \mathbf{E} \fint_0^T \|\text{curl} g\|_\lambda^2 = \eta. \tag{3.17}$$

Proof. Similar to the velocity case let $\phi \in C_c^\infty(B(0,1))$ such that $\phi \geq 0$ and $\int_{B(0,1)} \phi \, dx = 1$. For $\ell \ll 1$ we define $\phi_\ell(x) = \ell^{-2} \phi(x/\ell)$ and mollify

$$u_\ell^\nu = \phi_\ell * \omega^\nu, \quad (u^\nu \omega^\nu)_\ell = \phi_\ell * (u^\nu \omega^\nu), \quad (\text{curl } g dW_t)_\ell = \phi_\ell * \text{curl } g dW_t.$$

We also define

$$D_\ell(u^\nu, \omega^\nu)(x) := \frac{1}{4} \fint_0^T \int_{\mathbb{T}^d} \nabla \phi_\ell(y) \cdot \delta_y u^\nu |\delta_y \omega^\nu|^2 \, dy dt.$$

Then similar to the proof of (3.14) we apply Ito's lemma, integration by parts and the continuity of translation in L^p spaces to get

$$\nu \mathbf{E} \fint_0^T \|\nabla \omega^\nu\|_\lambda^2 + \lim_{\ell \to 0} \mathbf{E} \fint_{\mathbb{T}_\lambda^2} D(u^\nu, \omega^\nu)(dx) = \frac{1}{2} \mathbf{E} \fint_0^T \|\text{curl } g\|_\lambda^2 = \eta.$$

Note that for any norm $\|\cdot\|$ we have that $\|\delta_y z\| \leq 2\|z\|$ and in particular when it is the L^2 norm we can write $\|\delta_y z\|_{L^2} \leq |y| \|z\|_{H^1}$. Moreover, due to the interpolation inequality $\|z\|_{L^3} \leq$

$C\|z\|_{L^2}^{2/3}\|z\|_{H^1}^{1/3}$ it follows that

$$\begin{aligned}
\int_{\mathbb{T}_\lambda^2} \Big| D_\ell(u^\nu, \omega^\nu)(dx) \Big| &\leq \frac{1}{4} \fint_0^T \int_{B(0,1)} \int_{\mathbb{T}^d} |\nabla\phi(\xi)||\delta_{\ell\xi} u^\nu|\delta_{\ell\xi}\omega^\nu|^2 \, dx d\xi dt \\
&\leq C\ell^{-1} \fint_0^T \sup_{|\xi|\leq 1} \|\delta_{\ell\xi} u^\nu\|_{L^3(\mathbb{T}_\lambda^2)} \|\delta_{\ell\xi}\omega^\nu\|_{L^3(\mathbb{T}_\lambda^2)}^2 \\
&\leq C\ell^{-1} \fint_0^T \sup_{|\xi|\leq 1} \|\delta_{\ell\xi} u^\nu\|_{L^2(\mathbb{T}_\lambda^2)}^{2/3} \|\delta_{\ell\xi} u^\nu\|_{H^1(\mathbb{T}_\lambda^2)}^{1/3} \|\delta_{\ell\xi}\omega^\nu\|_{L^2(\mathbb{T}_\lambda^2)}^{4/3} \|\delta_{\ell\xi}\omega^\nu\|_{H^1(\mathbb{T}_\lambda^2)}^{2/3} \\
&\leq C \fint_0^T \ell \|u^\nu\|_{H^1(\mathbb{T}_\lambda^2)} \|\omega^\nu\|_{H^1(\mathbb{T}_\lambda^2)}^2 \\
&\leq C\ell \|\omega^\nu\|_{L^\infty(0,T,L^2(\mathbb{T}_\lambda^2))} \|\omega^\nu\|_{L^2(0,T,H^1(\mathbb{T}_\lambda^2))}
\end{aligned}$$

which goes to 0 as $\ell \to 0$. Note that in the last line we have used the fact that $\|u^\nu\|_{H^1(\mathbb{T}_\lambda^2)} = \|\omega^\nu\|_{L^2(\mathbb{T}_\lambda^2)}$ by integration by parts as well as Holder's inequality along with the fact that $\omega^\nu \in L^\infty(0,T,L^2(\mathbb{T}_\lambda^2)) \cap L^2(0,T,H^1(\mathbb{T}_\lambda^2)$ $\mathbb{P}$-a.s. □

3.2.2 Expression for Isotropic Tensors

In order to prove Kolmogorov's flux laws hold, we will take the KHM equations and consider two subsequent limits: $\nu \to 0$ and $\ell \to 0$. Here $\nu \to 0$ analyzes the affect of the singular limit (i.e. the disappearance of viscous dissipation) on the auto-correlations, while $\ell \to 0$ analyzes the inherent structure of the isotropic tensor used in the auto-correlation. In this section, we construct power-series representations for the structure functions to make applying the limits $\ell \to 0$ and $\nu \to 0$ easier in the future.

Lemma 3.15. *Let $k \in \mathbb{N}$ and $\{i_j\}_{j=1}^{2k} \subset \{1, 2, \ldots, d\}$. We define $\mathcal{S}_k$ to be the space of all pairwise combinations of the elements $\{i_j\}$ and let $n \in S^{d-1}$ for $d = 2, 3$ and $dS(n)$ be the surface measure of the unit sphere S^{d-1}. If $\sigma(p)$ is the pth particular set of pairings of $\{i_j\}$ (i.e.*

$\sigma(p) \in \mathcal{S}_k$, *with* $1 \leq p \leq |\mathcal{S}_k|$*), then the following identity holds:*

$$\fint_{S^{d-1}} \prod_{j=1}^{2k} n_{i_j} \; dS(n) = \beta_d(k) \sum_{j=1}^{|\mathcal{S}_k|} \prod_{(a,b)\in\sigma(j)} \delta_{a,b} \tag{3.18}$$

where

$$\beta_d(k) = \begin{cases} \frac{1}{2^k k!} & d = 2 \\ \\ \frac{2^k k!}{(2k+1)!} & d = 3 \end{cases}$$

and $\delta_{ij} = \begin{cases} 1 & i = j \\ \\ 0 & \textit{otherwise} \end{cases}$ *is the Kronecker delta tensor.*

Proof. Due to the odd symmetry of each of the components n_i over the unit sphere S^{d-1}, the isotropic tensor

$$\fint_{S^{d-1}} \prod_{j=1}^{k} n_{i_j} \; dS(n) > 0$$

if and only if there is an even number of each of the components (i.e. k is an even number). From [42] it is known that products of Kronecker delta tensors of all possible pairings of the elements $\{i_j\}$ form a spanning set for the space of isotropic tensors.

Claim: Moreover, as the ordering of the $\{i_j\}$ are interchangeable in $\fint_{S^{d-1}} \prod_{j=1}^{2k} n_{i_j} \; dS(n)$, all possible Kronecker delta pairings for the same set $\{i_j\}$ must occur with the same factor.

Proof. For simplicity, we will show this in the case for $k = 2$ but the result can be repeated for any $k \geq 2$ by the same argument (just the notation is messier). Let $\{i_j\} = \{\alpha, \beta, \gamma, \varepsilon\}$ where $\alpha, \beta, \gamma, \varepsilon \in \{1, \ldots d\}$. As

$$I = \int_{S^{d-1}} n_\alpha n_\beta n_\gamma n_\varepsilon \; dS(n)$$

is an isotropic tensor, we can express it by a linear combination of product of Kronecker deltas [42]. So

$$I = a_1 \delta_{\alpha\beta}\delta_{\gamma\varepsilon} + a_2 \delta_{\alpha\gamma}\delta_{\beta\varepsilon} + a_3 \delta_{\alpha\varepsilon}\delta_{\beta\gamma}.$$

Now swap β and γ. As $n_\beta n_\gamma = n_\gamma n_\beta$ (as these are real functions thus commutative) the value of I remains the same. However, the expansion of Kronecker delta products become

$$I = a_1 \delta_{\alpha\gamma}\delta_{\beta\varepsilon} + a_2 \delta_{\alpha\beta}\delta_{\gamma\varepsilon} + a_3 \delta_{\alpha\varepsilon}\delta_{\beta\gamma}.$$

Comparing this with the previous expansion we see that $a_1 = a_2$. Repeat this process with α and β to get that $a_3 = a_2$. Hence all of the possible Kronecker delta pairings for the set $\{i_j\}$ have the same coefficient factor. $\square$

Now we return to proof of Lemma 3.15. Then

$$\fint_{S^{d-1}} \prod_{j=1}^{2k} n_{i_j} \, dS(n) = \beta_d(k) \sum_{j=1}^{|\mathcal{S}_k|} \prod_{(a,b)\in\sigma(j)} \delta_{a,b}$$

where $\mathcal{S}_k$ is the collection of all pairwise combinations of $\{i_j\}$ and $\sigma(p) \in \mathcal{S}_k$ is the pth collection of pairings. For example when $k = 2$ we have

$$\mathcal{S}_2 = \Big\{ \underbrace{\{(i_1,i_2),(i_3,i_4)\}}_{\sigma(1)}, \underbrace{\{(i_1,i_3),(i_2,i_4)\}}_{\sigma(2)}, \underbrace{\{(i_1,i_4),(i_2,i_3)\}}_{\sigma(3)} \Big\}.$$

It is a well known combinatorial result that the number of pairings is $|\mathcal{S}_k| = \frac{(2k)!}{2^k k!}$.

Now all that is left is to compute the constant $\beta_d(k)$. As all possible pairings have the same factor we consider the case with $i_1 = i_2 = \cdots = i_{2k}$. As none of the terms on the left hand side vanish, we get

$$|\mathcal{S}_k|\beta_d(k) = \fint_{S^{d-1}} n_{i_1}^{2k}\, dS(n).$$

The right hand side depends on the dimension so we consider each case separately. When $d = 2$ we use the reduction formula to deduce:

$$\fint_{S^1} (n_{i_1})^{2k}\, dS(n) = \frac{1}{2\pi}\int_0^{2\pi} \cos^{2k}\theta\, d\theta = \frac{(2k-1)!!}{(2k)!!} = \frac{(2k-1)!}{(2k-2)!!(2k)!!} = \frac{2(2k-1)!}{4^k(k-1)!k!}$$

where we have used the identity $(2k)!! = \prod_{j=1}^{k} 2j = 2^k k!$. Thus

$$\beta_2(k) = \frac{2(2k-1)!}{4^k(k-1)!k!}\frac{1}{|\mathcal{S}_k|} = \frac{1}{2^k k!}.$$

Whereas when $d = 3$:

$$\fint_{S^2} (n_{i_1})^{2k}\, dS(n) = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \cos^{2k}\phi \sin\phi\, d\phi d\theta = \frac{1}{2k+1}$$

so

$$\beta_3(k) = \frac{1}{2k+1}\frac{1}{|\mathcal{S}_k|} = \frac{2^k k!}{(2k+1)!}.$$

□

Two direct consequences of Lemma 3.15 are the following lemmas:

Lemma 3.16. *Let $r \in \mathbb{R}$, $p \in \mathbb{N} \cup \{0\}$, and $k \in \mathbb{R}^d$. Then*

$$\Re\Big(\fint_{S^{d-1}} (n\cdot k)^{2p} e^{-ir(n\cdot k)}\, dS(n)\Big) = |k|^{2p}\sum_{m=0}^{\infty}\beta_d(m+p)\frac{(-1)^m(r|k|)^{2m}(2m+2p)!}{2^{m+p}(m+p)!(2m)!}$$

where $\Re(z)$ is the real part of a complex number z.

Proof. We use the power series expansion of e^x and the fact that it is an entire function to consider the integration term-wise:

$$\begin{aligned}\Re\Big(\fint_{S^{d-1}} (n\cdot k)^{2p} e^{-ir(n\cdot k)}\, dS(n)\Big) &= \Re\Big(\sum_{m=0}^{\infty} \frac{(-ir)^m}{m!} \fint_{S^{d-1}} (n\cdot k)^{m+2p}\, dS(n)\Big) \\ &= \sum_{m=0}^{\infty} \frac{(-1)^m r^{2m}}{(2m)!} \fint_{S^{d-1}} \prod_{j=1}^{2(m+p)} k_{i_j} n_{i_j}\, dS(n)\end{aligned}$$

where $k_{i_j} n_{i_j}$ is the jth copy of the inner product $k_i n_i = k\cdot n$. Then it follows from Lemma 3.15 that

$$\begin{aligned}\fint_{S^{d-1}} \prod_{j=1}^{2(m+p)} k_{i_j} n_{i_j}\, dS(n) &= \prod_{j=1}^{2(m+p)} k_{i_j} \fint_{S^{d-1}} \prod_{j=1}^{2(m+p)} n_{i_j}\, dS(n) \\ &= \beta_d(m+p) \prod_{j=1}^{2(m+p)} k_{i_j} \sum_{q=1}^{|\mathcal{S}_{m+p}|} \prod_{(a,b)\in\sigma(q)} \delta_{ab} \\ &= \beta_d(m+p) \sum_{q=1}^{|\mathcal{S}_{m+p}|} |k|^{2m+2p} \\ &= \beta_d(m+p)|\mathcal{S}_{m+p}||k|^{2m+2p} \\ &= \beta_d(m+p)\frac{(2m+2p)!}{2^{m+p}(m+p)!}|k|^{2m+2p}.\end{aligned}$$

Now we substitute this back into the sum to conclude the proof. $\square$

Lemma 3.17. *Let u be a divergence free random vector field of dimension d and let $p \in \mathbb{N}\cup\{0\}$*

and $r \in \mathbb{R}$*, then for* $a, b \in \{1, \dots, d\}$

$$\Re\Big(\fint_{S^{d-1}} n_a n_b (n \cdot k)^{2p} e^{-irn\cdot k} \, dS(n) \mathbf{E} \widehat{u}_a(k) \widehat{u}_b(k) \Big) \\ = |k|^{2p} \sum_{m=0}^{\infty} \beta_d(m+p+1) \frac{(-1)^m (r|k|)^{2m} (2m+2p)!}{2^{m+p}(m+p)!(2m)!} \mathbf{E}|\widehat{u}(k)|^2$$

where $\Re(z)$ *is the real part of a complex number* z*.*

Proof. Similar to the proof of Lemma 3.16, we use the power series expansion of e^x and the fact that it is an entire function to consider the integration term-wise:

$$\Re\Big(\fint_{S^{d-1}} n_a n_b (n \cdot k)^{2p} e^{-irn\cdot k} \, dS(n) \mathbf{E} \widehat{u}_a(k) \widehat{u}_b(k) \Big) \\ = \Re\Big(\sum_{m=0}^{\infty} \frac{(-ir)^m}{m!} \fint_{S^{d-1}} n_a n_b (n \cdot k)^{m+2p} \, dS(n) \mathbf{E} \widehat{u}_a(k) \widehat{u}_b(k) \Big) \\ = \sum_{m=0}^{\infty} \frac{(-1)^m r^{2m}}{(2m)!} \fint_{S^{d-1}} n_a n_b \prod_{j=1}^{2(m+p)} k_{i_j} n_{i_j} \, dS(n) \mathbf{E} \widehat{u}_a(k) \widehat{u}_b(k).$$

Note that on the Fourier side, the divergence free condition can be written as $k_j \widehat{u_j} = 0$. Now, as there are $2n+2p+2$ many n_j when we apply Lemma 3.15 we get

$$
\begin{aligned}
&\fint_{S^{d-1}} n_a n_b \prod_{j=1}^{2(m+p)} k_{i_j} n_{i_j}\, dS(n)\mathbf{E}\widehat{u}_a(k)\widehat{u}_b(k) \\
&\qquad = \prod_{j=1}^{2(m+p)} k_{i_j} \fint_{S^{d-1}} n_a n_b \prod_{j=1}^{2(m+p)} n_{i_j}\, dS(n)\mathbf{E}\widehat{u}_a(k)\widehat{u}_b(k) \\
&\qquad = \beta_d(m+p+1) \prod_{j=1}^{2(m+p)} k_{i_j} \Big(\sum_{q=1}^{|\mathcal{S}_{m+p}|} \delta_{ab} \prod_{(a',b')\in\sigma(q)} \delta_{a'b'} \mathbf{E}\widehat{u}_a(k)\widehat{u}_b(k) \\
&\qquad\quad + \sum_{q=1}^{|\mathcal{S}_{m+p}|} \sum_{(a',b')\in\sigma(q)} (\delta_{ab'}\delta_{a'b} + \delta_{a'a}\delta_{b'b}) \prod_{\substack{(c',d')\in\sigma(q)\\(c',d')\neq(a',b')}} \delta_{c'd'} \mathbf{E}\widehat{u}_a(k)\widehat{u}_b(k) \Big) \\
&\qquad = \beta_d(m+p+1) \sum_{q=1}^{|\mathcal{S}_{m+p}|} |k|^{2m+2p} \mathbf{E}|\widehat{u}(k)|^2 \\
&\qquad\quad + 2\beta_d(m+p+1) \sum_{q=1}^{|\mathcal{S}_{m+p}|} |k|^{2m+2p-2} \mathbf{E} k_a \widehat{u}_a(k) k_b \widehat{u}_b(k) \\
&\qquad = \beta_d(m+p+1) |\mathcal{S}_{m+p}| |k|^{2m+2p} \mathbf{E}|\widehat{u}(k)|^2 \\
&\qquad = \beta_d(m+p+1) \frac{(2m+2p)!}{2^{m+p}(m+p)!} |k|^{2m+2p} \mathbf{E}|\widehat{u}(k)|^2.
\end{aligned}
$$

Substituting this back into the sum concludes the proof. $\square$

3.2.3 Filtration Limits

Now that we have expressions for the isotropic tensors in spectral space, we want to filter out the high frequency modes or low frequency modes in order to characterize the existence of a direct or inverse cascade. We begin with characterizing the direct cascade by filtering out the low frequency terms leaving only the energy which can "escape to infinity".

Lemma 3.18 (Small Scale Limit). *Let $c : \mathbb{R}_+ \times \mathbb{Z}^d \to \mathbb{R}$ be a bounded function such that*

$\lim\limits_{\ell|k|\to 0} c(\ell,k) = 0$ *and* $\lim\limits_{\ell|k|\to\infty} c(\ell,k) = L$ *with* $L \neq 0$. *Let* f^ν *be a sequence of* $[L^2(\Omega \times \mathbb{T}^d_\lambda)]^d$ *random variables such that*

$$\limsup_{\nu\to 0} \mathbf{E}\|f^\nu\|^2_\lambda < \infty.$$

Then there exists $\ell_\nu \to 0$ *such that*

$$\lim_{\ell_I\to 0} \liminf_{\nu\to 0} \inf_{\ell\in(\ell_\nu,\ell_I)} \sum_{k\in(\frac{2\pi}{\lambda}\mathbb{Z})^3} c(\ell,k)\mathbf{E}|\widehat{f^\nu}(k)|^2 = L\delta$$

if and only if there exists $N_\nu \to \infty$ *such that*

$$\liminf_{\nu\to 0} \sum_{|k|\geq N_\nu} \mathbf{E}|\widehat{f^\nu}(k)|^2 = \delta.$$

Proof. ($\Rightarrow$) Suppose there exists $\ell_\nu \to 0$ such that

$$\lim_{\ell_I\to 0} \liminf_{\nu\to 0} \inf_{\ell\in(\ell_\nu,\ell_I)} \sum_{k\in(\frac{2\pi}{\lambda}\mathbb{Z})^d} c(\ell,k)\mathbf{E}|\widehat{f^\nu}(k)|^2 = L\delta.$$

Pick $N_\nu \to \infty$ such that $\lim\limits_{\nu\to 0} \ell_\nu N_\nu = \infty$ and consider the sums

$$\sum_{k\in(\frac{2\pi}{\lambda}\mathbb{Z})^d} c(\ell,k)\mathbf{E}|\widehat{f^\nu}(k)|^2 = \sum_{|k|<N_\nu} c(\ell,k)\mathbf{E}|\widehat{f^\nu}(k)|^2 + \sum_{|k|\geq N_\nu} c(\ell,k)\mathbf{E}|\widehat{f^\nu}(k)|^2 =: I_1 + I_2.$$

First consider I_2 and observe

$$\inf_{|k|\geq N_\nu} c(\ell,k) \sum_{|k|\geq N_\nu} \mathbf{E}|\widehat{f^\nu}(k)|^2 \leq I_2 \leq \sup_{|k|\geq N_\nu} c(\ell,k) \sum_{|k|\geq N_\nu} \mathbf{E}|\widehat{f^\nu}(k)|^2.$$

Apply the inf in ℓ to each side of the inequality:

$$\begin{aligned}\inf_{\ell|k|\geq\ell_\nu N_\nu} c(\ell,k)\sum_{|k|\geq N_\nu}\mathbf{E}|\widehat{f^\nu}(k)|^2 &\leq \inf_{\ell\in(\ell_\nu,\ell_I)} I_2\\ &\leq \inf_{\ell|k|\geq\ell_\nu N_\nu} c(\ell,k)\sum_{|k|\geq N_\nu}\mathbf{E}|\widehat{f^\nu}(k)|^2\\ &\leq \sup_{\ell|k|\geq\ell_\nu N_\nu} c(\ell,k)\sum_{|k|\geq N_\nu}\mathbf{E}|\widehat{f^\nu}(k)|^2.\end{aligned}$$

Next we take the limits in ν and ℓ_I. Note that by our choice of N_ν and the fact that $\lim\limits_{\ell|k|\to\infty} c(\ell,k) = L$ it follows that

$$\lim_{\ell_I\to 0}\liminf_{\nu\to 0}\inf_{\ell|k|\geq\ell_\nu N_\nu} c(\ell,k)\sum_{|k|\geq N_\nu}\mathbf{E}|\widehat{f^\nu}(k)|^2 = L\liminf_{\nu\to 0}\sum_{|k|\geq N_\nu}\mathbf{E}|\widehat{f^\nu}(k)|^2$$

$$\lim_{\ell_I\to 0}\liminf_{\nu\to 0}\sup_{\ell|k|\geq\ell_\nu N_\nu} c(\ell,k)\sum_{|k|\geq N_\nu}\mathbf{E}|\widehat{f^\nu}(k)|^2 = L\liminf_{\nu\to 0}\sum_{|k|\geq N_\nu}\mathbf{E}|\widehat{f^\nu}(k)|^2.$$

Hence by the Squeeze theorem

$$\lim_{\ell_I\to 0}\liminf_{\nu\to 0}\inf_{\ell|k|\geq\ell_\nu N_\nu} I_2 = L\liminf_{\nu\to 0}\sum_{|k|\geq N_\nu}\mathbf{E}|\widehat{f^\nu}(k)|^2.$$

Next we claim that

$$\lim_{\ell_I\to 0}\liminf_{\nu\to 0}\inf_{\ell\in(\ell_\nu,\ell_I)}|I_1| = 0.$$

For the sake of contradiction suppose that

$$\lim_{\ell_I\to 0}\liminf_{\nu\to 0}\inf_{\ell\in(\ell_\nu,\ell_I)}|I_1| > 0.$$

Then there exists $|k| < N_\nu$ and $\varepsilon_0 > 0$ such that for all $\nu > 0, \ell_I > 0$ there exists $\nu^* < \nu$ and $\ell_I^* < \ell_I$ such that

$$\varepsilon_0 < \inf_{\ell\in(0,\ell_I^*)} |c(\ell,k)|\mathbf{E}|\widehat{f^{\nu^*}}(k)|^2 \leq \inf_{\ell\in(0,\ell_I^*)} |c(\ell,k)|\mathbf{E}\|f^{\nu^*}\|_\lambda^2 \leq C \sup_{\ell\in 0,\ell_I)} |c(\ell,k)|.$$

However, as $\lim\limits_{\ell|k|\to 0} c(\ell,k) = 0$, we can find ℓ_I small enough such that

$$\sup_{\ell\in(0,\ell_I)} |c(\ell,k)| < \varepsilon_0/C$$

a contradiction. Thus $|I_1| \to 0$ as $\ell_I \to 0$ and $\nu \to 0$. Putting it all together now we get

$$\liminf_{\nu\to 0} \sum_{|k|\geq N_\nu} \mathbf{E}|\widehat{f^\nu}(k)|^2 = \frac{1}{L} \lim_{\ell_I\to 0} \liminf_{\nu\to 0} \inf_{\ell\in(\ell_\nu,\ell_I)} \sum_{k\in(\frac{2\pi}{\lambda}\mathbb{Z})^3} c(\ell,k)\mathbf{E}|\widehat{f^\nu}(k)|^2 = \delta.$$

($\Leftarrow$) Now we assume that there exists $N_\nu \to \infty$ such that

$$\liminf_{\nu\to 0} \sum_{|k|\geq N_\nu} \mathbf{E}|\widehat{f^\nu}(k)|^2 = \delta.$$

Pick $\ell_\nu \to 0$ such that $\lim\limits_{\nu\to 0} \ell_\nu N_\nu = \infty$. Repeat the steps above to conclude that

$$\lim_{\ell_I\to 0} \liminf_{\nu\to 0} \inf_{\ell\in(\ell_\nu,\ell_I)} I_1 = 0$$

$$\lim_{\ell_I\to 0} \liminf_{\nu\to 0} \inf_{\ell\in(\ell_\nu,\ell_I)} I_2 = L\delta.$$

□

Notice that the small scale limit filters out everything except what happens to the random variables at infinite wave-numbers. As such we will sometimes refer to Lemma 3.18 as the high-frequency filtration limit. Similarly we can characterize the large scale limit as filtering out everything except what happens at low frequencies.

Lemma 3.19 (Large Scale Limit)**.** *Let $\lambda = \lambda(\nu) < \infty$ be a continuous monotone increasing function such that $\lim_{\nu\to 0}\lambda = \infty$. Let $c : \mathbb{R}_+ \times \mathbb{Z}^d \to \mathbb{R}$ be a bounded continuous function such that $\lim\limits_{\ell|k|\to 0} c(\ell,k) = 0$ and $\lim\limits_{\ell|k|\to\infty} c(\ell,k) = L$ with $L \neq 0$. Let f^ν is a sequence of $[L^2(\Omega\times\mathbb{T}^d_\lambda)]^d$ random variables such that*

$$\limsup_{\nu\to 0} \mathbf{E}\|f^\nu\|_\lambda^2 = \Delta < \infty.$$

Then there exists $\tilde{\ell}_\nu \leq \lambda$ and $\tilde{\ell}_\nu \to \infty$ such that

$$\lim_{\ell_I\to\infty}\limsup_{\nu\to 0}\sup_{\ell\in(\ell_I,\tilde{\ell}_\nu)}\sum_{k\in(\frac{2\pi}{\lambda}\mathbb{Z})^d} c(\ell,k)\mathbf{E}|\widehat{f^\nu}(k)|^2 = L(\Delta-\delta)$$

if and only if there exists $M_\nu \to 0$ such that

$$\liminf_{\nu\to 0}\sum_{|k|\leq M_\nu}\mathbf{E}|\widehat{f^\nu}(k)|^2 = \delta.$$

Proof. The proof is very similar to the proof of Lemma 3.18.

($\Rightarrow$) Suppose there exists $\tilde{\ell}_\nu \leq \lambda$ such that $\lim\limits_{\nu\to 0}\tilde{\ell}_\nu = \infty$ and

$$\lim_{\ell_I\to\infty}\limsup_{\nu\to 0}\sup_{\ell\in(\ell_I,\tilde{\ell}_\nu)}\sum_{k\in(\frac{2\pi}{\lambda}\mathbb{Z})^d} c(\ell,k)\mathbf{E}|\widehat{f^\nu}(k)|^2 = L(\Delta-\delta).$$

Pick $M_\nu \to 0$ such that $M_\nu = O(\tilde{\ell}_\nu^{-1})$. By assumption $c(\ell, 0) = 0$ for all ℓ. Hence

$$\begin{aligned}
\sum_{k \in (\frac{2\pi}{\lambda}\mathbb{Z})^d} c(\ell, k)\mathbf{E}|\widehat{f^\nu}(k)|^2 &= \sum_{|k|>0} c(\ell, k)\mathbf{E}|\widehat{f^\nu}(k)|^2 \\
&= \sum_{\frac{2\pi}{\lambda} \le |k| \le M_\nu} c(\ell, k)\mathbf{E}|\widehat{f^\nu}(k)|^2 + \sum_{|k| \ge M_\nu} c(\ell, k)\mathbf{E}|\widehat{f^\nu}(k)|^2 \\
&=: I_1 + I_2.
\end{aligned}$$

As the sequence $c(\ell, k)$ is bounded uniformly in ℓ and k and from our choice of M_ν it follows that

$$\lim_{\ell_I \to \infty} \limsup_{\nu \to 0} \sup_{\ell \in (\ell_I, \tilde{\ell}_\nu)} |I_1| \le C \limsup_{\nu \to 0} \sum_{\frac{2\pi}{\lambda} \le |k| \le M_\nu} \mathbf{E}|\widehat{f^\nu}(k)|^2 \le C \limsup_{\nu \to 0} \sum_{0 < |k| \le M_\nu} \mathbf{E}|\widehat{f^\nu}(k)|^2 = 0.$$

Next consider the term I_2. Then

$$\inf_{|k| \ge M_\nu} c(\ell, k) \sum_{|k| \ge M_\nu} \mathbf{E}|\widehat{f^\nu}(k)|^2 \le I_2 \le \sup_{|k| \ge M_\nu} c(\ell, k) \sum_{|k| \ge M_\nu} \mathbf{E}|\widehat{f^\nu}(k)|^2.$$

Apply the sup in ℓ to each side of the inequality:

$$\begin{aligned}
\inf_{\ell|k| \ge \ell_I M_\nu} c(\ell, k) \sum_{|k| \ge M_\nu} \mathbf{E}|\widehat{f^\nu}(k)|^2 &\le \sup_{\ell \ge \ell_I} \inf_{|k| > M_\nu} c(\ell, k) \sum_{|k| \ge M_\nu} \mathbf{E}|\widehat{f^\nu}(k)|^2 \\
&\le \sup_{\ell \in (\ell_I, \tilde{\ell}_I)} I_2 \\
&\le \sup_{\ell|k| > \ell_I M_\nu} c(\ell, k) \sum_{|k| \ge M_\nu} \mathbf{E}|\widehat{f^\nu}(k)|^2.
\end{aligned}$$

As $k \in \left(\frac{2\pi}{\lambda}\mathbb{Z}\right)^d$, we can represent k as $k = \frac{(2\pi)^d z}{\lambda^d}$ for $z \in \mathbb{Z}^d$. By our choice of M_ν and the fact that $\tilde{\ell}_\nu \leq \lambda$ along with $d > 1$ we have

$$\ell|k| \geq \ell_I M_\nu = \ell|z| \geq \ell_I M_\nu \frac{\lambda^d}{(2\pi)^d} = \ell_I \tilde{\ell}_\nu M_\nu \frac{\lambda^d}{\tilde{\ell}_\nu (2\pi)^d} \to \infty$$

as $\nu \to 0$. Therefore, using the fact that $\lim\limits_{\ell|k| \to \infty} c(\ell, k) = L$ we get:

$$\lim_{\ell_I \to \infty} \limsup_{\nu \to 0} \inf_{\ell|k| \geq \ell_I M_\nu} c(\ell, k) \sum_{|k| \geq M_\nu} \mathbf{E}|\widehat{f^\nu}(k)|^2 = L \limsup_{\nu \to 0} \sum_{|k| \geq M_\nu} \mathbf{E}|\widehat{f^\nu}(k)|^2$$

$$\lim_{\ell_I \to \infty} \limsup_{\nu \to 0} \sup_{\ell|k| \geq \ell_I M_\nu} c(\ell, k) \sum_{|k| \geq M_\nu^*} \mathbf{E}|\widehat{f^\nu}(k)|^2 = L \limsup_{\nu \to 0} \sum_{|k| \geq M_\nu} \mathbf{E}|\widehat{f^\nu}(k)|^2.$$

Hence by the Squeeze theorem

$$\lim_{\ell_I \to \infty} \limsup_{\nu \to 0} \sup_{\ell \in (\ell_I, \tilde{\ell}_\nu)} I_2 = L \limsup_{\nu \to 0} \sum_{|k| \geq M_\nu} \mathbf{E}|\widehat{f^\nu}(k)|^2.$$

Putting it all together results in

$$\begin{aligned}
\liminf_{\nu \to 0} \sum_{|k| \leq M_\nu} \mathbf{E}|\widehat{f^\nu}(k)|^2 &= \Delta - \limsup_{\nu \to 0} \sum_{|k| \geq M_\nu} \mathbf{E}|\widehat{f^\nu}(k)|^2 \\
&= \Delta - \frac{1}{L} \lim_{\ell_I \to \infty} \limsup_{\nu \to 0} \sup_{\ell \in (\ell_I, \tilde{\ell}_\nu)} \sum_{k \in (\frac{2\pi}{\lambda}\mathbb{Z})^d} c(\ell, k) \mathbf{E}|\widehat{f^\nu}(k)|^2 \\
&= \delta.
\end{aligned}$$

($\Leftarrow$) Now assume that there exists $M_\nu \to 0$ such that

$$\delta = \liminf_{\nu \to 0} \sum_{|k| \leq M_\nu} \mathbf{E}|\widehat{f^\nu}(k)|^2 = \Delta - \limsup_{\nu \to 0} \sum_{|k| \geq M_\nu} \mathbf{E}|\widehat{f^\nu}(k)|^2.$$

Simplifying, we have $\limsup_{\nu\to 0} \sum_{|k|\geq M_\nu} \mathbf{E}|\widehat{f^\nu}(k)|^2 = \Delta - \delta$.

Choose $\tilde{\ell}_\nu = O(M_\nu^{-1})$ and repeat the steps above to conclude that

$$\lim_{\ell_I\to 0} \limsup_{\nu\to 0} \sup_{\ell\in(\ell_\nu,\ell_I)} I_1 = 0$$

$$\lim_{\ell_I\to 0} \limsup_{\nu\to 0} \sup_{\ell\in(\ell_\nu,\ell_I)} I_2 = L(\Delta-\delta).$$

□

3.3 The 3D Direct Cascade

In this section we establish the characterization of the direct cascade flux laws in 3D.

Theorem 3.20 (3D Direct Cascade Characterization)**.** *Suppose $\{u^\nu\}_{\nu>0}$ is a sequence of statistically stationary solutions. There exists $N_\nu \geq 1$ such that $\lim_{\nu\to 0} N_\nu = \infty$ and*

$$\liminf_{\nu\to 0} \left(\nu \sum_{|k|\geq N_\nu} \mathbf{E}|\widehat{\nabla u}(k)|^2 + \mathbf{E}D(u^\nu)\right) = \varepsilon^*$$

if and only if there exists $\ell_\nu \in (0,1)$ such that $\lim_{\nu\to 0} \ell_\nu = 0$ and

$$\lim_{\ell_I\to 0} \limsup_{\nu\to 0} \sup_{\ell\in[\ell_\nu,\ell_I]} \left|\frac{S_{vel}(\ell)}{\ell^3} + \frac{4}{3}\varepsilon^*\right| = 0; \tag{3.3}$$

$$\lim_{\ell_I\to 0} \limsup_{\nu\to 0} \sup_{\ell\in[\ell_\nu,\ell_I]} \left|\frac{S_{vel}^{||}(\ell)}{\ell} + \frac{4}{5}\varepsilon^*\right| = 0. \tag{3.4}$$

3.3.1 Proof of (3.3)

First, recall the KHM equation for S_{vel} in three dimensions:

$$\frac{S_{vel}}{\ell} = -\frac{4\nu}{\ell}\Gamma'_{vel}(\ell) - \frac{4}{\ell^3}\int_0^\ell r^2 a_{vel}(r)\, dr.$$

Step 1: Beginning with the forcing term, we show that

$$\frac{-4}{\ell^3}\int_0^\ell r^2 a_{vel}(r)\, dr = -\frac{4}{3}\varepsilon + o_{\ell\to 0}(1).$$

Note that

$$a_{vel}(0) = \frac{1}{2}\mathbf{E}\fint_0^T \fint_{\mathbb{T}^3_\lambda} |g(x)|^2\, dxdt =: \varepsilon.$$

Furthermore, as g is smooth, we Taylor expand around $r = 0$ using the Peano formulation of the error (as we do not need a rigorous analysis of the error in this case).

$$\begin{aligned}
\frac{-4}{\ell^3}\int_0^\ell r^2 a_{vel}(r)\, dr &= \frac{-2}{\ell^3}\int_0^\ell r^2 \mathbf{E}\fint_0^T \fint_{S^2}\fint_{\mathbb{T}^3_\lambda} g\cdot T_{rn}g\; dxdS(n)dtdr \\
&= \frac{-2}{\ell^3}\int_0^\ell r^2 \mathbf{E}\fint_0^T \fint_{S^2}\fint_{\mathbb{T}^3_\lambda} |g|^2\; dxdS(n)dt + r^2 h_0(r)dr \\
&= \frac{-2}{3}\fint_0^T \fint_{\mathbb{T}^3_\lambda} \mathbf{E}|g|^2\, dx + \frac{-2}{\ell^3}\int_0^\ell r^2 h_0(r)\, dr \\
&= \frac{-4}{3}\varepsilon + \frac{-2}{\ell^3}\int_0^\ell r^2 h_0(r)\, dr
\end{aligned}$$

where $h_0(r)$ is a function such that $\lim_{r\to 0} h_0(r) = 0$. The error disappears as $\ell \to 0$ or in other words $\frac{-2}{\ell^3}\int_0^\ell r^2 h_0(r)\, dr = o_{\ell\to 0}(1)$.

Step 2: Next we show that

$$\frac{-4\nu}{\ell}\Gamma'_{vel}(\ell) = \frac{4}{3}\varepsilon - \frac{4}{3}\mathbf{E}D(u^\nu) - \sum_{k\in(\frac{2\pi}{\lambda}\mathbb{Z})^3} c(k,\ell)\mathbf{E}\fint_0^T |\widehat{\nabla u^\nu}(k)|^2$$

where $\lim_{\ell|k|\to 0} c(\ell,k) = 0$ and $\lim_{\ell|k|\to\infty} c(\ell,k) = \frac{4}{3}$.

From Proposition 3.13 it is known that $\Gamma_{vel} \in C^2(\mathbb{R})$, so if we Taylor expand Γ'_{vel} around 0 and use the Lagrange formulation of the error to get:

$$\frac{-4\nu}{\ell}\Gamma'_{vel}(\ell) = \frac{-4\nu}{\ell}\Gamma'_{vel}(0) + \frac{-4\nu}{\ell}\int_0^\ell \Gamma''_{vel}(r)\, dr.$$

By integration by parts

$$\Gamma'_{vel}(0) = -\mathbf{E}\fint_{S^2}\fint_{\mathbb{T}^2_\lambda} n_i u^\nu_m \partial_i u^\nu_m \, dx dS(n) = 0.$$

Next, consider the error term, which we will note is real valued. We use Plancheral's theorem to rewrite the L^2 inner product in x into a Fourier series. Then we apply Fubini's theorem and

Lemma 3.16 when $p = 1$ to get

$$
\begin{aligned}
\frac{-4\nu}{\ell}\int_0^\ell \Gamma''_{vel}(r)\, dr &= \frac{4\nu}{\ell}\int_0^\ell \mathbf{E} \fint_0^T \fint_{S^2} \fint_{\mathbb{T}^3_\lambda} n_i n_j \partial_i u^\nu_m \partial_j T_{rn} u^\nu_m \; dx dS(n) dt dr \\
&= \frac{-4\nu}{\ell}\sum_{k\in\mathbb{Z}^3}\int_0^\ell \fint_{S^2} (n\cdot k)^2 e^{-irn\cdot k}\mathbf{E}\fint_0^T |\widehat{u^\nu}(k)|^2 \; dt dS(n) dr \\
&= \frac{-4\nu}{\ell}\sum_{k\in\mathbb{Z}^3}\int_0^\ell |k|^2 \sum_{m=0}^\infty \frac{(-1)^m (r|k|)^{2m}}{(2m)!(2m+3)}\; dr \mathbf{E}\fint_0^T |\widehat{u^\nu}(k)|^2 \; dt \\
&= 4\nu \sum_{k\in(\frac{2\pi}{\lambda}\mathbb{Z})^3}\sum_{m=0}^\infty \frac{(-1)^m(\ell|k|)^{2m}}{(2m+1)!(2m+3)}\mathbf{E}\fint_0^T |\widehat{\nabla u^\nu}(k)|^2 \\
&= \frac{4\nu}{3}\sum_{k\in(\frac{2\pi}{\lambda}\mathbb{Z})^3} \mathbf{E}\fint_0^T |\widehat{\nabla u^\nu}(k)|^2 \; dt \\
&\quad - 4\nu \sum_{k\in(\frac{2\pi}{\lambda}\mathbb{Z})^3}\sum_{m=1}^\infty \frac{(-1)^{m+1}(\ell|k|)^{2m}}{(2m+1)!(2m+3)}\mathbf{E}\fint_0^T |\widehat{\nabla u^\nu}(k)|^2 \; dt \\
&= \frac{4}{3}\varepsilon - \frac{4}{3}\mathbf{E}D(u^\nu) - \sum_{k\in(\frac{2\pi}{\lambda}\mathbb{Z})^3} c(\ell,k)\mathbf{E}\fint_0^T |\widehat{\nabla u^\nu}(k)|^2 \; dt.
\end{aligned}
$$

Notice that when applying Lemma 3.16 we substituted in for the value of $\beta_3(m+1)$ and simplified the coefficient in the power series in m. Also the ultimate equality comes from the energy balance (3.14) to write the first term as $\frac{4}{3}\varepsilon$.

From the definition of $c(\ell, k)$ it is readily seen that $\lim_{\ell|k|\to 0} c(\ell, k) = 0$.

$$
\begin{aligned}
\Big|\frac{4}{3} - c(\ell, k)\Big| &= \Big|4\sum_{m=0}^{\infty} \frac{(-1)^m(\ell|k|)^{2m}}{(2m+1)!(2m+3)}\Big| \\
&= \Big|\frac{4}{(\ell|k|)^3}\sum_{m=0}^{\infty} \frac{(-1)^m(\ell|k|)^{2m+3}}{(2m+1)!(2m+3)}\Big| \\
&= \Big|\frac{4}{(\ell|k|)^3}\sum_{m=0}^{\infty} \int_0^{\ell|k|} \frac{(-1)^m t^{2m+2}}{(2m+1)!}\Big| \\
&= \Big|\frac{4}{(\ell|k|)^3}\int_0^{\ell|k|} t\sin t \, dt\Big| \\
&\leq \frac{4}{(\ell|k|)^3}\int_0^{\ell|k|} t \, dt \\
&= \frac{2}{\ell|k|}
\end{aligned}
$$

so $\lim_{\ell|k|\to\infty} c(\ell, k) = \frac{4}{3}$.

Step 3: It follows from Lemma 3.18 that there exists $\ell_\nu \to 0$ such that

$$
\begin{aligned}
\lim_{\ell_I\to 0}\limsup_{\nu\to 0}\sup_{\ell\in[\ell_\nu,\ell_I]} \frac{-4\nu}{\ell}\Gamma'(\ell) &= \frac{4}{3}\varepsilon - \frac{4}{3}\liminf_{\nu\to 0}\mathbf{E}D(u^\nu) \\
&\quad - \lim_{\ell_I\to 0}\liminf_{\nu\to 0}\inf_{\ell\in[\ell_\nu,\ell_I]} \nu \sum_{k\in(\frac{2\pi}{\lambda}\mathbb{Z})^3} c(\ell, k)\mathbf{E}|\widehat{\nabla u^\nu}(k)|^2 \\
&= \frac{4\varepsilon}{3} - \frac{4\varepsilon^*}{3}
\end{aligned}
$$

if and only if there exists $N_\nu \to \infty$ such that

$$
\liminf_{\nu\to 0}\mathbf{E}D(u^\nu) + \liminf_{\nu\to 0}\nu\sum_{|k|\geq N_\nu}\mathbf{E}|\widehat{\nabla u^\nu}(k)|^2 = \varepsilon^*.
$$

Finally, combine Steps 1-3 with the KHM equation to derive (3.3).

3.3.2 Proof of (3.4)

Recall the KHM equation for $S^{\|}_{vel}$ on $\mathbb{T}^3_\lambda$:

$$\frac{S^{\|}_{vel}(\ell)}{\ell} = -\frac{4\nu}{\ell}(\Gamma^{\|}_{vel})'(\ell) + \frac{2}{\ell^5}\int_0^\ell r^3 S_{vel}(r)\, dr - \frac{4}{\ell^5}\int_0^\ell r^4 a^{\|}_{vel}(r)\, dr.$$

Step 1: Beginning with the forcing term, we show that

$$\frac{-4}{\ell^5}\int_0^\ell r^4 a^{\|}_{vel}(r)\, dr = -\frac{-4}{15}\varepsilon + o_{\ell\to 0}(1).$$

Note that

$$a^{\|}_{vel}(0) = \frac{1}{3}\mathbf{E}\fint_0^T \fint_{\mathbb{T}^3_\lambda} |g(x)|^2\, dxdt = \frac{\varepsilon}{3}.$$

Furthermore, as g is smooth, we Taylor expand around $r = 0$ using the Peano formulation of the error (as we do not need a rigorous analysis of the error in this case).

$$\begin{aligned}\frac{-4}{\ell^5}\int_0^\ell r^4 a^{\|}_{vel}(r)\, dr &= \frac{-4}{\ell^5}\int_0^\ell r^4 a^{\|}_{vel}(0) + r^4 h_0(r)\, dr \\ &= \frac{-4}{15}\varepsilon - \frac{4}{\ell^5}\int_0^\ell r^4 h_0(r)\, dr\end{aligned}$$

where $h_0(r)$ is a function such that $\lim_{r\to 0} h_0(r) = 0$. Hence by L'hopital's Theorem, the error disappears as $\ell \to 0$ uniformly with respect to ν or in other words $\frac{-4}{\ell^5}\int_0^\ell r^2 h_0(r)\, dr = o_{\ell\to 0}(1)$.

Step 2: Next we show that

$$\frac{-4\nu}{\ell}(\Gamma^{\|}_{vel})'(\ell) = \frac{4}{15}\varepsilon - \sum_{k\in(\frac{2\pi}{\lambda}\mathbb{Z})^3} c(\ell,k)\mathbf{E}|\widehat{\nabla u^\nu}(k)|^2$$

where the coefficients satisfy $\lim_{\ell|k|\to 0} c(\ell,k) = 0$ and $\lim_{\ell|k|\to\infty} c(\ell,k) = \frac{4}{15}$.

From Proposition 3.13, we know $\Gamma^{\|}_{vel} \in C^2(\mathbb{R})$. We Taylor expand this about $r = 0$ and use the Lagrange formulation of the error to get:

$$\frac{-4\nu}{\ell}(\Gamma^{\|}_{vel})'(\ell) = \frac{-4\nu}{\ell}(\Gamma^{\|}_{vel})'(0) - \frac{4\nu}{\ell}\int_0^{\ell}(\Gamma^{\|}_{vel})''(r)\,dr.$$

By integration by parts

$$\frac{-4\nu}{\ell}\Gamma'_{\|}(0) = \frac{4\nu}{\ell}\mathbf{E}\fint_{S^2}\fint_{\mathbb{T}^2_\lambda} n_i n_j n_k \partial_k u_i^\nu u_j^\nu \, dx dS(n) = 0.$$

Next, consider the error term, which we will note is real valued. We use Plancheral's theorem to rewrite the L^2 inner product in x into a Fourier series. Then we apply Fubini's theorem and

Lemma 3.17 when $p = 1$ to get

$$
\begin{aligned}
\frac{-4\nu}{\ell}\int_0^\ell \Gamma_\parallel''(r)\,dr &= \frac{4\nu}{\ell}\int_0^\ell \mathbf{E}\fint_0^T \fint_{S^2}\fint_{\mathbb{T}^3_\lambda} n_i n_j n_p n_q \partial_i u_p^\nu \partial_j T_{rn} u_q^\nu \,dxdS(n)dr \\
&= \frac{-4\nu}{\ell}\sum_{k\in(\frac{2\pi}{\lambda}\mathbb{Z})^3}\int_0^\ell \fint_{S^2} n_p n_q (n\cdot k)^2 e^{-irn\cdot k}\mathbf{E}\fint_0^T \widehat{u_p^\nu}(k)\widehat{u_q^\nu}(k)\,dS(n)dr \\
&= \frac{4\nu}{\ell}\sum_{k\in(\frac{2\pi}{\lambda}\mathbb{Z})^3}\int_0^\ell \sum_{m=0}^\infty \frac{(-1)^m (r|k|)^{2m}}{(2m)!(2m+3)(2m+5)}\,dr\mathbf{E}\fint_0^T |\widehat{\nabla u^\nu}(k)|^2 \\
&= 4\nu \sum_{k\in(\frac{2\pi}{\lambda}\mathbb{Z})^3}\sum_{m=0}^\infty \frac{(-1)^m(\ell|k|)^{2m}}{(2m+1)!(2m+3)(2m+5)}\mathbf{E}\fint_0^T |\widehat{\nabla u^\nu}(k)|^2 \\
&= \frac{4\nu}{15}\sum_{k\in(\frac{2\pi}{\lambda}\mathbb{Z})^3}\mathbf{E}\fint_0^T |\widehat{\nabla u^\nu}(k)|^2 \\
&\quad - 4\nu \sum_{k\in(\frac{2\pi}{\lambda}\mathbb{Z})^3}\sum_{m=1}^\infty \frac{(-1)^{m+1}(\ell|k|)^{2m}}{(2m+1)!(2m+3)(2m+5)}\mathbf{E}\fint_0^T |\widehat{\nabla u^\nu}(k)|^2 \\
&= \frac{4\varepsilon}{15} - \frac{4}{15}\mathbf{E}D(u^\nu) - \sum_{k\in(\frac{2\pi}{\lambda}\mathbb{Z})^3} c(\ell,k)\mathbf{E}\fint_0^T |\widehat{\nabla u^\nu}(k)|^2.
\end{aligned}
$$

Note that when we applied Lemma 3.17, we substituted in the definition of $\beta_3(m+2)$ and then simplified the coefficient. Then in the final equality we have applied the energy balance (3.14).

From the definition of $c(\ell, k)$ it is readily seen that $\lim_{\ell|k|\to 0} c(\ell, k) = 0$. Also

$$\begin{aligned}
\left|\frac{4}{15} - c(\ell, k)\right| &= \left|4\sum_{m=0}^{\infty} \frac{(-1)^m(\ell|k|)^{2m}}{(2m+1)!(2m+3)(2m+5)}\right| \\
&= \left|\frac{4}{(\ell|k|)^5}\sum_{m=0}^{\infty} \frac{(-1)^m(\ell|k|)^{2m+5}}{(2m+1)!(2m+3)(2m+5)}\right| \\
&= \left|\frac{4}{(\ell|k|)^5}\int_0^{\ell|k|}\sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!(2m+3)}t^{2m+4}\,dt\right| \\
&= \left|\frac{4}{(\ell|k|)^5}\int_0^{\ell|k|} t\int_0^t\sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!}s^{2m+2}\,dsdt\right| \\
&= \left|\frac{4}{(\ell|k|)^5}\int_0^{\ell|k|} t\int_0^t s\sin(s)\,dsdt\right| \\
&\leq \frac{C}{(\ell|k|)^5}\int_0^{\ell|k|} t\int_0^t s\,dsdt \\
&\leq \frac{C}{\ell|k|}
\end{aligned}$$

so $\lim_{\ell|k|\to\infty} c(\ell, k) = \frac{4}{15}$.

Step 3: Apply Lemma 3.18 to conclude that there exists $\ell_\nu \to 0$ such that

$$\begin{aligned}
\lim_{\ell_I\to 0}\limsup_{\nu\to 0}\sup_{\ell\in(\ell_\nu,\ell_I)} \frac{-4\nu}{\ell}\Gamma_{\|}'(\ell) &= \frac{4\varepsilon}{15} - \lim_{\ell_I\to 0}\liminf_{\nu\to 0}\inf_{\ell\in(\ell_\nu,\ell_I)}\sum_{k\in(\frac{2\pi}{\lambda}\mathbb{Z})^3} c(\ell, k)\mathbf{E}|\widehat{\nabla u^\nu}(k)|^2 \\
&= \frac{4}{15}\varepsilon - \frac{4}{15}\varepsilon^*
\end{aligned}$$

if and only if there exists $N_\nu \to \infty$ such that

$$\liminf_{\nu\to 0}\mathbf{E}D(u^\nu) + \liminf_{\nu\to 0}\nu\sum_{|k|\geq N_\nu}\mathbf{E}|\widehat{\nabla u^\nu}(k)|^2 = \varepsilon^*.$$

Step 4: Next we use Equation (3.3), to conclude that the if and only if condition in step 3, also implies

$$\frac{2}{\ell^5}\int_0^\ell r^3 S_{vel}(r)\,dr = \frac{2}{\ell^5}\int_0^\ell r^4 \frac{S_{vel}(r)}{r}\,dr = \frac{-8}{15}\varepsilon^* + o_{\ell\to 0}(1).$$

Finally combine Steps 1-4 with the KHM equation to derive (3.4).

3.4 The 2D Direct Cascade

In this section, we characterize the existence of a direct cascade in 2D. Note that the proof is almost identical to the 3D case, but as some of the details are not trivial we fully show the derivation in 2D.

Theorem 3.21 (2D Direct Cascade Characterization)**.** *Suppose* $\{(u^\nu, \omega^\nu)\}_{\nu>0}$ *is a sequence of statistically stationary solutions satisfying both the Navier Stokes equations* (1.1) *and the vorticity equation* (1.4)*. Then there exists* $N_\nu \geq 1$ *such that* $\lim_{\nu\to 0} N_\nu = \infty$ *and*

$$\liminf_{\nu\to 0} \nu \sum_{|k|\geq N_\nu} \mathbf{E}|\widehat{\nabla\omega^\nu}(k)|^2 = \eta^*$$

if and only if there exists $\ell_\nu \in (0,1)$ *such that* $\lim_{\nu\to 0} \ell_\nu = 0$ *and*

$$\lim_{\ell_I\to 0}\limsup_{\nu\to 0}\sup_{\ell\in[\ell_\nu,\ell_I]}\left|\frac{S_{vor}(\ell)}{\ell} + 2\eta^*\right| = 0; \tag{3.5}$$

$$\lim_{\ell_I\to 0}\limsup_{\nu\to 0}\sup_{\ell\in[\ell_\nu,\ell_I]}\left|\frac{S_{vel}(\ell)}{\ell^3} - \frac{1}{4}\eta^*\right| = 0; \tag{3.6}$$

$$\lim_{\ell_I\to 0}\limsup_{\nu\to 0}\sup_{\ell\in[\ell_\nu,\ell_I]}\left|\frac{S^{||}_{vel}(\ell)}{\ell^3} - \frac{1}{8}\eta^*\right| = 0. \tag{3.7}$$

3.4.1 Proof of (3.5)

First, recall the KHM equation for S_{vor} :

$$\frac{S_{vor}(\ell)}{\ell} = -\frac{4\nu}{\ell}\Gamma'_{vor}(\ell) - \frac{4}{\ell^2}\int_0^\ell r a_{vor}(r)\, dr.$$

Step 1: We begin with the forcing term and show that

$$\frac{-4}{\ell^2}\int_0^\ell r a_{vor}(r)\, dr = -2\eta + o_{\ell\to 0}(1).$$

Note that

$$a_{vor}(0) = \frac{1}{2}\mathbf{E}\fint_0^T \fint_{\mathbb{T}^2_\lambda} |\mathrm{curl} g|^2\, dxdt =: \eta.$$

Furthermore, as $\mathrm{curl} g$ is smooth we Taylor expand around $r = 0$ and use the Peano formulation of the error:

$$\begin{aligned}\frac{-4}{\ell^2}\int_0^\ell r a_{vor}(r)\, dr &= \frac{-4}{\ell^2}\int_0^\ell r a_{vor}(0)\, dr + \frac{-4}{\ell^2}\int_0^\ell r^2 h_1(r)\, dr \\ &= -2\eta + \frac{-4}{\ell^2}\int_0^\ell r^2 h_1(r)\, dr\end{aligned}$$

where $\lim_{r\to 0} h_1(r) = 0$. Note, this implies that $\frac{1}{\ell^2}\int_0^\ell r^2 h_1(r)\, dr = o_{\ell\to 0}(1)$.

Step 2: Next we show that

$$\frac{-4\nu}{\ell}\Gamma'_{vor}(\ell) = 2\eta - \nu \sum_{k\in(\frac{2\pi}{\lambda}\mathbb{Z})^2} c(\ell,k)\mathbf{E}|\widehat{\nabla\omega^\nu}(k)|^2$$

where $\lim_{\ell|k|\to 0} c(\ell,k)=0$ and $\lim_{\ell|k|\to\infty} c(\ell,k)=2$.

From Proposition 3.13 we know that $\Gamma_{vor}\in C^3(\mathbb{R})$, we Taylor expand Γ' about 0 using the Lagrange formulation of the error:

$$\frac{-4\nu}{\ell}\Gamma'_{vor}(\ell)=\frac{-4\nu}{\ell}\Gamma'_{vor}(0)+\frac{-4\nu}{\ell}\int_0^\ell \Gamma''_{vor}(r)\,dr.$$

Due to the fact that $⨍_{S^1} n\,dS(n)=0$ it follows from Fubini that

$$\Gamma'_{vor}(0)=\mathbf{E}⨍_0^T ⨍_{S^1}⨍_{\mathbb{T}^2_\lambda} n_i\partial_i\omega^\nu\omega^\nu\,dxdS(n)\,dt=0.$$

Consider the error term, which we will note is real valued. We apply Fourier transformation to the integral in x, Fubini, along with Lemma 3.16 when $p=1$:

$$\begin{aligned}
\frac{-4\nu}{\ell}\int_0^\ell \Gamma''_{vor}(r)\,dr &= \frac{4\nu}{\ell}\int_0^\ell \mathbf{E}⨍_0^T ⨍_{S^1}⨍_{\mathbb{T}^2_\lambda} n_in_j\partial_i\omega^\nu T_{rn}\partial_j\omega^\nu\,dxdS(n)dtdr\\
&=\frac{-4\nu}{\ell}\sum_{k\in(\frac{2\pi}{\lambda}\mathbb{Z})^2}\int_0^\ell \mathbf{E}⨍_0^T⨍_{S^1}(n\cdot k)^2e^{-irn\cdot k}\mathbf{E}|\widehat{\omega^\nu}(k)|^2\,dS(n)dtdr\\
&=-\frac{4\nu}{\ell}\sum_{k\in(\frac{2\pi}{\lambda}\mathbb{Z})^2}\int_0^\ell |k|^2\sum_{m=0}^\infty\frac{(-1)^m(r|k|)^{2m}(2m+2)!}{4^{m+1}(2m)!(m+1)!(m+1)!}\,dr⨍_0^T\mathbf{E}|\widehat{\omega^\nu}(k)|^2dt\\
&=2\nu\sum_{k\in(\frac{2\pi}{\lambda}\mathbb{Z})^2}\sum_{m=0}^\infty\frac{(-1)^m(\ell|k|)^{2m}}{4^m(m+1)!m!}⨍_0^T\mathbf{E}|\widehat{\nabla\omega^\nu}(k)|^2dt\\
&=2\nu\sum_{k\in(\frac{2\pi}{\lambda}\mathbb{Z})^2}\mathbf{E}⨍_0^T|\widehat{\nabla\omega^\nu}(k)|^2\,dt\\
&\quad-2\nu\sum_{k\in(\frac{2\pi}{\lambda}\mathbb{Z})^2}\sum_{m=1}^\infty\frac{(-1)^{m+1}(\ell|k|)^{2m}}{4^m(m+1)!m!}\mathbf{E}⨍_0^T|\widehat{\nabla\omega^\nu}(k)|^2dt\\
&=2\eta-\nu\sum_{k\in(\frac{2\pi}{\lambda}\mathbb{Z})^2}c(\ell,k)\mathbf{E}⨍_0^T|\widehat{\nabla\omega^\nu}(k)|^2dt.
\end{aligned}$$

Note that the application of Lemma 3.16 we use the definition of $\beta_2(m+1)$ and simplified the coefficient. Then in the ultimate equality we have used the enstrophy balance (3.17) to get the 2η for the first component. It is clear from the definition of $c(\ell, k)$ that $\lim_{\ell|k|\to 0} c(\ell, k) = 0$. Moreover, using the power series representation of $J_1(x)$[1] (note J_1 is the first order Bessel function of the first kind) one can write

$$|2 - c(\ell, k)| = \Big|2\sum_{m=0}^{\infty} \frac{(-1)^m(\ell|k|)^{2m}}{4^m(m+1)!m!}\Big| = \Big|4\frac{J_1(\ell|k|)}{\ell|k|}\Big|.$$

Since $J_1(x)/x \to 0$ as $x \to \infty$, this implies that $\lim_{\ell|k|\to\infty} c(\ell, k) = 2$.

Step 3: Next we apply Lemma 3.18 to conclude that there exists $\ell_\nu \to 0$ such that

$$\begin{aligned}\lim_{\ell_I\to 0}\limsup_{\nu\to 0}\sup_{\ell\in[\ell_\nu,\ell_I]} \frac{-4\nu}{\ell}\Gamma'_{vor}(\ell) &= 2\eta - \lim_{\ell_I\to 0}\liminf_{\nu\to 0}\inf_{\ell\in[\ell_\nu,\ell_I]} \nu \sum_{k\in(\frac{2\pi}{\lambda}\mathbb{Z})^2} c(\ell, k)\mathbf{E} ⨏_0^T |\widehat{\nabla\omega^\nu}(k)|^2\, dt \\ &= 2\eta - 2\eta^*\end{aligned}$$

if and only if there exists $N_\nu \to \infty$ such that

$$\liminf_{\nu\to 0}\nu\sum_{|k|\geq N_\nu}\mathbf{E} ⨏_0^T |\widehat{\nabla\omega^\nu}(k)|^2\, dt = \eta^*.$$

Finally, combine Steps 1-3 with the KHM equation to derive (3.5).

3.4.2 Proof of (3.6)

First, recall the KHM equation for S_{vel} :

$$\frac{S_{vel}}{\ell^3} = -\frac{4\nu}{\ell^3}\Gamma'_{vel}(\ell) - \frac{4}{\ell^4}\int_0^\ell r a_{vel}(r)\, dr.$$

Step 1: Once again, we begin with the forcing term and show that

$$\frac{-4}{\ell^4}\int_0^\ell r a_{vel}(r)\, dr = \frac{-2\varepsilon}{\ell^2} + \frac{\eta}{4} + o_{\ell\to 0}(1).$$

Recall that a_{vel} is defined by

$$a_{vel}(r) = \frac{1}{2}\mathbf{E}\fint_0^T \fint_S \fint_{\mathbb{T}^2_\lambda} g \cdot T_{rn} g \; dx dS(n) dt.$$

As g is smooth we Taylor expand them around $r = 0$ and use the Peano formulation of the error (this is because we don't require an in-depth analysis of the error term for the forcing term). Moreover, as $\fint_S n_i \; dS(n) = 0$ it follows from Fubini that

$$\frac{1}{2}\mathbf{E}\fint_0^T \fint_S \fint_{\mathbb{T}^2_\lambda} g \cdot n_i \partial_i g \; dx dS(n) = 0.$$

Then by integration by parts we obtain

$$\begin{aligned}
\mathbf{E}\fint_0^T \fint_{\mathbb{T}^2_\lambda} g \cdot T_{rn} g \; dx &= \mathbf{E}\fint_0^T \fint_{\mathbb{T}^2_\lambda} |g|^2 + \frac{r^2}{2} n_i n_m g \cdot \partial_i \partial_m g + r^2 h_2(r) \; dx dt \\
&= \mathbf{E}\fint_0^T \fint_{\mathbb{T}^2_\lambda} |g|^2 - \frac{r^2}{2} n_i n_m \partial_i g \cdot \partial_m g \; dx + r^2 h_2(r) \; dx dt
\end{aligned}$$

where $\lim_{r\to 0} h_2(r) = 0$. Since $⨍_S n_i n_m = \frac{1}{2}\delta_{im}$ we use the definition of η and ε to conclude that

$$\begin{aligned}
\frac{-4}{\ell^4}\int_0^\ell r a_{vel}(r)\,dr &= \frac{-2}{\ell^4}\int_0^\ell r\mathbf{E}⨍_0^T ⨍_{\mathbb{T}^2_\lambda} |g|^2\,dxdr + \frac{1}{2\ell^4}\int_0^\ell r^3\mathbf{E}⨍_0^T ⨍_{\mathbb{T}^2_\lambda} |\nabla g|^2\,dxdr \\
&\quad + \frac{-4}{\ell^4}\int_0^\ell r^3 h_2(r)\,dx \\
&= \frac{-4}{\ell^4}\int_0^\ell \varepsilon r\,dr + \frac{1}{\ell^4}\int_0^\ell r^3\eta\,dr + \frac{-4}{\ell^4}\int_0^\ell r^3 h_2(r)\,dr \\
&= \frac{-2\varepsilon}{\ell^2} + \frac{\eta}{4} + \frac{-4}{\ell^4}\int_0^\ell r^3 h_2(r)\,dr.
\end{aligned}$$

Finally, note that since $\lim_{r\to 0} h_2(r) = 0$ the term $\frac{1}{\ell^4}\int_0^\ell r^3 h_2(r)\,dr = o_{\ell\to 0}(1)$.

Step 2: Next we show that

$$\frac{-4\nu}{\ell^3}\Gamma'_{vel}(\ell) = \frac{2\varepsilon}{\ell^2} - \frac{\eta}{4} - \nu \sum_{k\in(\frac{2\pi}{\lambda}\mathbb{Z})^2} c(\ell,k)\mathbf{E}|\widehat{\nabla\omega^\nu}(k)|^2$$

where $\lim_{\ell|k|\to 0} c(\ell,k) = 0$ and $\lim_{\ell|k|\to\infty} c(\ell,k) = \dfrac{-1}{4}$ as $\ell|k|\to\infty$.

From Proposition 3.13 we know that $\Gamma_{vel}\in C^4(\mathbb{R})$, thus we Taylor expand about 0 while using the Lagrange formulation of the error:

$$\frac{-4\nu}{\ell^3}\Gamma'_{vel}(\ell) = \frac{-4\nu}{\ell^3}\Gamma'_{vel}(0) + \frac{-4\nu}{\ell^2}\Gamma''_{vel}(0) + \frac{-2\nu}{\ell}\Gamma'''_{vel}(0) + \frac{-2\nu}{\ell^3}\int_0^\ell (\ell-s)^2\Gamma''''_{vel}(s)\,ds.$$

From Lemma 3.15 we have $⨍_{S^1} n\,dS(n) = 0$, $⨍_{S^1} n_i n_j\,dS(n) = \frac{1}{2}\delta_{ij}$, and $⨍_{S^1} n_i n_j n_k\,dS(n) = 0$, hence by Fubini

$$
\begin{aligned}
\Gamma'_{vel}(0) &= -\mathbf{E} \fint_S \fint_{\mathbb{T}^2_\lambda} n_i \partial_i u^\nu_m u^\nu_m \; dx dS(n) = 0; \\
\Gamma''_{vel}(0) &= -\mathbf{E} \fint_S \fint_{\mathbb{T}^2_\lambda} n_i n_j \partial_i u^\nu_m \partial_j u^\nu_m \; dx dS(n) = -\frac{1}{2}\mathbf{E}\|\nabla u^\nu\|^2_\lambda; \\
\Gamma'''_{vel}(0) &= \mathbf{E} \fint_S \fint_{\mathbb{T}^2_\lambda} n_i n_j n_k \partial_i \partial_j u^\nu_m \partial_k u^\nu_m \; dx dS(n) = 0.
\end{aligned}
$$

Considering the error term, which we note is real valued. We use Fourier analysis to convert the integral in x to a Fourier series in k (wave-number), and apply Fubini along with Lemma 3.16 when $p = 2$:

$$
\begin{aligned}
&\int_0^\ell \frac{-2\nu(\ell - r)^2}{\ell^3} \Gamma''''_{vel}(r) \; dr \\
&\qquad = \int_0^\ell \frac{-2\nu(\ell - r)^2}{\ell^3} \mathbf{E} \fint_S \fint_{\mathbb{T}^2_\lambda} n_i n_j n_m n_p \partial_m \partial_i u^\nu_q \partial_p \partial_j T_{rn} u^\nu_q \; dx dS(n) dr \\
&\qquad = \sum_{k \in (\frac{2\pi}{\lambda}\mathbb{Z})^2} \int_0^\ell \frac{-2\nu(\ell - r)^2}{\ell^3} \fint_S (n \cdot k)^4 e^{-irn\cdot k} \mathbf{E}|\widehat{u^\nu}(k)|^2 \; dS(n) dr \\
&\qquad = \sum_{k \in (\frac{2\pi}{\lambda}\mathbb{Z})^2} \int_0^\ell \frac{-2\nu(\ell - r)^2}{\ell^3} \frac{2|k|^4}{4^2} \sum_{m=0}^\infty \frac{(-1)^m (2m+3)!(r|k|)^{2m}}{4^m (2m)!(m+2)!(m+1)!} \mathbf{E}|\widehat{u^\nu}(k)|^2 \\
&\qquad = \frac{-\nu}{4} \sum_{k \in (\frac{2\pi}{\lambda}\mathbb{Z})^2} \sum_{m=0}^\infty \frac{(-1)^m (2m+3)!|k|^{2m}}{4^m (2m)!(m+2)!(m+1)!} \int_0^\ell \frac{(\ell - r)^2 r^{2m}}{\ell^3} \; dr \mathbf{E}|\widehat{\nabla\omega^\nu}(k)|^2 \\
&\qquad = \frac{-\nu}{2} \sum_{k \in (\frac{2\pi}{\lambda}\mathbb{Z})^2} \sum_{m=0}^\infty \frac{(-1)^m (\ell|k|)^{2m}}{4^m (m+2)!(m+1)!} \mathbf{E}|\widehat{\nabla\omega^\nu}(k)|^2 \\
&\qquad = \frac{-\nu}{4} \sum_{k \in (\frac{2\pi}{\lambda}\mathbb{Z})^2} \mathbf{E}|\widehat{\nabla\omega^\nu}(k)|^2 - \frac{\nu}{2} \sum_{k \in (\frac{2\pi}{\lambda}\mathbb{Z})^2} \sum_{m=1}^\infty \frac{(-1)^m (\ell|k|)^{2m}}{4^m (m+2)!(m+1)!} \mathbf{E}|\widehat{\nabla\omega^\nu}(k)|^2 \\
&\qquad =: \frac{-\eta}{4} - \nu \sum_{k \in (\frac{2\pi}{\lambda}\mathbb{Z})^2} c(\ell, k) \mathbf{E}|\widehat{\nabla\omega^\nu}(k)|^2.
\end{aligned}
$$

Note that when applying Lemma 3.16 we use the definition of $\beta_2(m+2)$ and simplified the coefficient. Furthermore, we can relate $\widehat{u^\nu}$ and $\widehat{\omega^\nu}$ via the Biot-Savart law:

$$|k|^4|\widehat{u^\nu}(k)|^2 = |k|^4\frac{-|k|^2}{|k|^4}|\widehat{\omega^\nu}(k)|^2 = -|k|^2|\widehat{\omega^\nu}(k)|^2 = |\widehat{\nabla\omega^\nu}(k)|^2.$$

Then in the ultimate equality we have used the enstrophy balance (3.17) to get the $\eta/4$ for the first component. It is clear from the definition of $c(\ell,k)$ that $\lim\limits_{\ell|k|\to 0} c(\ell,k) = 0$. Moreover,

$$\begin{aligned}
|\frac{-1}{4} - c(\ell,k)| &= \Big|\frac{-1}{2}\sum_{m=0}^{\infty}\frac{(-1)^m(\ell|k|)^{2m}}{4^m(m+2)!(m+1)!}\Big| = \Big|\frac{1}{(\ell|k|)^2}\sum_{m=0}\frac{(-1)^m(\ell|k|)^{2m+2}}{4^m(m+2)!m!(2m+2)}\Big| \\
&= \Big|\frac{1}{(\ell|k|)^2}\int_0^{\ell|k|}\sum_{m=0}\frac{(-1)^m t^{2m+3}}{4^m(m+2)!m!}\,dt\Big| = \Big|\frac{4}{(\ell|k|)^2}\int_0^{\ell|k|} tJ_2(t)\,dt\Big| \\
&\leq \frac{4}{(\ell|k|)^2}\int_0^{\ell|k|} t|J_2(t)|\,dt \leq \frac{C}{(\ell|k|)^2}\int_0^{\ell|k|} t^{1/2}\,dt \\
&= \frac{C}{(\ell|k|)^{-1/2}}
\end{aligned}$$

where we have used the power series representation of the Bessel function $J_2(x)$ and that there exists $C > 0$ such that $|J_2(x)| \leq C|x|^{-1/2}$ for all x [1]. Hence $\lim\limits_{\ell|k|\to\infty} c(\ell,k) = \frac{-1}{4}$.

Step 3: Next we apply Lemma 3.18 to conclude that there exists $\ell_\nu \to 0$ such that

$$\begin{aligned}
&\lim_{\ell_I\to 0}\limsup_{\nu\to 0}\sup_{\ell\in[\ell_\nu,\ell_I]}\int_0^{\ell}\frac{-2\nu(\ell-s)^2}{\ell^3}\Gamma''''(s)\,ds \\
&\qquad = \frac{-\eta}{4} - \lim_{\ell_I\to 0}\liminf_{\nu\to 0}\inf_{\ell\in[\ell_\nu,\ell_I]}\nu\sum_{k\in(\frac{2\pi}{\lambda}\mathbb{Z})^2} c(\ell,k)\mathbf{E}|\widehat{\nabla\omega}(k)|^2 \\
&\qquad = \frac{-\eta}{4} + \frac{\eta^*}{4}
\end{aligned}$$

if and only if there exists $N_\nu \to \infty$ such that

$$\liminf_{\nu \to 0} \nu \sum_{|k| \geq N_\nu} \mathbf{E}|\widehat{\nabla \omega}(k)|^2 = \eta^*.$$

Finally, combine Steps 1-3 with the KHM equation to derive (3.6).

3.4.3 Proof of (3.7)

Recall the KHM equation for $S^{||}_{vel}$:

$$\frac{S^{||}_{vel}(\ell)}{\ell^3} = -\frac{4\nu}{\ell^3}(\Gamma^{||}_{vel})'(\ell) + \frac{2}{\ell^6}\int_0^\ell r^2 S_{vel}(r)\, dr - \frac{4}{\ell^6}\int_0^\ell r^3 a^{||}_{vel}(r)\, dr.$$

Step 1: Once again, we begin with the energy injection term and show that

$$\frac{-4}{\ell^6}\int_0^\ell r^3 a^{||}_{vel}(r)\, dr = \frac{-\varepsilon}{2\ell^2} + \frac{\eta}{24} + o_{\ell \to 0}(1).$$

This follows exactly the same as in [9] but we will include it for the sake of completeness. Recall that

$$a^{||}_{vel}(r) = \frac{1}{2}\mathbf{E}\fint_0^T \fint_{S^1}\fint_{\mathbb{T}^2_\lambda} (n \cdot g)(n \cdot T_{rn} g)\, dx dS(n) dt.$$

As g is smooth, we Taylor expand about $r = 0$ and use the Peano formulation of the error. After integration by parts we have

$$\mathbf{E}\fint_0^T \fint_{\mathbb{T}^2_\lambda} (n \cdot g)(n \cdot T_{rn} g)\, dx dt = \mathbf{E}\fint_0^T \fint_{\mathbb{T}^2_\lambda} (n \cdot g)^2 - \frac{r^2}{2} n_i n_k n_p n_q \partial_k g_i \partial_p g_q\, dx dt + r^3 h_2(r)$$

where $\lim_{r\to 0} h_2(r) = 0$. Note that this implies $\frac{1}{\ell^6}\int_0^\ell r^6 h_2(r)\ dr = o_{\ell\to 0}(1)$. From Lemma 3.15 we have the identities $\fint_S n_i n_j\ dS(n) = \frac{1}{2}\delta_{ij}$ and

$$\fint_S n_i n_k n_p n_q\ dS(n) = \frac{1}{8}(\delta_{ik}\delta_{pq} + \delta_{ip}\delta_{kq} + \delta_{iq}\delta_{pk}).$$

Since the vector field g is divergence free it follows that

$$\frac{2}{\ell^6}\mathbf{E}\fint_0^T \int_0^\ell r^3 \fint_S \fint_{\mathbb{T}^2_\lambda} (n\cdot g)^2\ dxdS(n)drdt = \frac{1}{\ell^6}\int_0^\ell r^3\varepsilon = \frac{\varepsilon}{2\ell^2}$$

$$-\frac{2}{\ell^6}\mathbf{E}\fint_0^T \int_0^\ell r^5 \fint_S \fint_{\mathbb{T}^2_\lambda} n_i n_k n_p n_q \partial_k g_i \partial_p g_q\ dxdS(n)drdt = \frac{-1}{4\ell^6}\mathbf{E}\fint_0^T \int_0^\ell r^5 \fint_{\mathbb{T}^2_\lambda} |\nabla g|^2\ dxdrdt = \frac{-\eta}{24}.$$

Then putting everything together we get

$$\frac{-4}{\ell^6}\int_0^\ell r^3 a^{\|}_{vel}(r)\ dr = \frac{-2}{\ell^6}\int_0^\ell r^3 \mathbf{E}\fint_0^T \fint_S \fint_{\mathbb{T}^2_\lambda} (n\cdot g)(n\cdot T_{rn} g)\ dxdS(n)dtdr = -\frac{\varepsilon}{2\ell^2} + \frac{\eta}{24} + o_{\ell\to 0}(1).$$

Step 2: Next we show that

$$\frac{-4\nu}{\ell^3}(\Gamma^{\|}_{vel})'(\ell) = \frac{\varepsilon}{2\ell^2} - \frac{\eta}{24} + \sum_{k\in(\frac{2\pi}{\lambda}\mathbb{Z})^2} c(\ell,k)\mathbf{E}|\widehat{\nabla\omega}(k)|^2$$

where the coefficients satisfy $\lim_{\ell|k|\to 0} c(\ell,k) = 0$ and $\lim_{\ell|k|\to\infty} c(\ell,k) = \dfrac{1}{24}$.

From Proposition 3.13, $\Gamma_{\|} \in C^4(\mathbb{R})$ so we Taylor expand it about $\ell = 0$ with the Lagrange formulation of the error to get

$$\frac{-4\nu}{\ell^3}(\Gamma^{\|}_{vel})'(\ell) = \frac{-4\nu}{\ell^3}(\Gamma^{\|}_{vel})'(0) - \frac{4\nu}{\ell^2}(\Gamma^{\|}_{vel})''(0) - \frac{2\nu}{\ell}(\Gamma^{\|}_{vel})'''(0) - \frac{2\nu}{\ell^3}\int_0^\ell (\ell-r)^2(\Gamma^{\|}_{vel})''''(r)\ dr.$$

We know from Lemma 3.15 that

$$\fint_{S^1} n_i n_j n_k \; dS(n) = \fint_{S^1} n_i n_j n_k n_p n_q \; dS(n) = 0$$
$$\fint_{S^1} n_i n_j n_k n_m \; dS(n) = \frac{1}{8}(\delta_{ij}\delta_{km} + \delta_{ik}\delta_{jm} + \delta_{im}\delta_{jk}).$$

Now we apply Fubini's theorem and the energy balance (3.14) to see

$$\frac{-4\nu}{\ell^3}(\Gamma^{\|}_{vel})'(0) = \frac{4\nu}{\ell^3}\mathbf{E}\fint_{S^1}\fint_{\mathbb{T}^2_\lambda} n_i n_j n_k \partial_k u_i^\nu u_j^\nu \; dxdS(n) = 0$$
$$\frac{-4\nu}{\ell^2}(\Gamma^{\|}_{vel})''(0) = \frac{4\nu}{\ell^2}\mathbf{E}\fint_{S^1}\fint_{\mathbb{T}^2_\lambda} n_i n_j n_k n_p \partial_k u_i^\nu \partial_p u_j^\nu \; dxdS(n) = \frac{\nu}{2\ell^2}\mathbf{E}\|\nabla u^\nu\|^2_\lambda = \frac{\varepsilon}{2\ell^2}$$
$$\frac{-2\nu}{\ell}(\Gamma^{\|}_{vel})'''(0) = \frac{2\nu}{\ell}\mathbf{E}\fint_{S^1}\fint_{\mathbb{T}^2_\lambda} n_i n_j n_k n_p n_q \partial_k \partial_p u_i^\nu \partial_q u_j^\nu \; dxdS(n) = 0.$$

Next, we consider the error term, which we will note is real valued. Using Fourier Analysis, Fubini's theorem, and Lemma 3.17 when $p = 2$ it follows that

$$\begin{aligned}
&\frac{2\nu}{\ell^3}\int_0^\ell (\ell - r)^2 (\Gamma^{\|}_{vel})''''(r) \; dr \\
&\qquad = \sum_{k \in (\frac{2\pi}{\lambda}\mathbb{Z})^3} \frac{2\nu}{\ell^3}\int_0^\ell (\ell - r)^2 \fint_S n_i n_j (n \cdot k)^4 e^{-ir(n\cdot k)} \mathbf{E}\widehat{u_i}(k)\widehat{u_j}(k) \; dS(n)dr \\
&\qquad = \sum_{k \in (\frac{2\pi}{\lambda}\mathbb{Z})^2} \frac{2\nu}{\ell^3}\int_0^\ell (\ell - r)^2 \frac{2|k|^2}{4^2}\sum_{m=0}^\infty \frac{(-1)^m (2m+3)!(r|k|)^{2m}}{4^m (m+1)!(m+3)!(2m)!}\mathbf{E}|\widehat{\omega}(k)|^2 \\
&\qquad = \frac{\nu}{4}\sum_{k \in (\frac{2\pi}{\lambda}\mathbb{Z})^2}\sum_{m=0}^\infty \frac{(-1)^m (\ell|k|)^{2m}}{4^m (m+1)!(m+3)!}\mathbf{E}|\widehat{\nabla\omega}(k)|^2 \\
&\qquad = \frac{\nu}{24}\mathbf{E}|\widehat{\nabla\omega}(k)|^2 - \frac{1}{4}\nu \sum_{k \in (\frac{2\pi}{\lambda}\mathbb{Z})^2}\sum_{m=1}^\infty \frac{(-1)^{m+1} (\ell|k|)^{2m}}{4^m (m+1)!(m+3)!}\mathbf{E}|\widehat{\nabla\omega}(k)|^2 \\
&\qquad =: \frac{\eta}{24} - \sum_{k \in (\frac{2\pi}{\lambda}\mathbb{Z})^2} c(\ell, k)\mathbf{E}|\widehat{\nabla\omega}(k)|^2.
\end{aligned}$$

Note that we applied the definition of $\beta_2(m+3)$ and simplified the coefficient when applying Lemma 3.17. Also by construction $\lim_{\ell|k|\to 0} c(\ell,k) = 0$. Moreover,

$$\begin{aligned}
\Big|\frac{1}{24} - c(\ell,k)\Big| &= \Big|\frac{1}{4}\sum_{m=0}^{\infty}\frac{(-1)^m(\ell|k|)^{2m}}{4^m(m+1)!(m+3)!}\Big| = \Big|\frac{1}{2(\ell|k|)^2}\sum_{m=0}\frac{(-1)^m(\ell|k|)^{2m+2}}{4^m m!(m+3)!(2m+2)}\Big| \\
&= \Big|\frac{1}{2(\ell|k|)^2}\int_0^{\ell|k|}\sum_{m=0}\frac{(-1)^m t^{2m+3}}{4^m m!(m+3)!}\,dt\Big| = \Big|\frac{4}{(\ell|k|)^2}\int_0^{\ell|k|} J_3(t)\,dt\Big| \\
&\leq \frac{4}{(\ell|k|)^2}\int_0^{\ell|k|} |J_3(t)|\,dt \leq \frac{C}{\ell|k|}
\end{aligned}$$

where we have used the power series representation of the Bessel function $J_3(x)$ and the fact that there exists $C > 0$ such that $|J_3(x)| \leq C$ uniformly for all x[1]. Hence $\lim_{\ell|k|\to\infty} c(\ell,k) = \frac{1}{24}$.

Step 3: Apply Lemma 3.18 to conclude that there exists $\ell_\nu \to 0$ such that

$$\frac{-4\nu}{\ell^3}(\Gamma^{||}_{vel})'(\ell) = \frac{\varepsilon}{2\ell^2} - \frac{\eta}{24} - \frac{\eta^*}{24} + o_{\ell\to 0}(1)$$

if and only if there exists $N_\nu \to \infty$ such that

$$\liminf_{\nu\to 0}\nu\sum_{|k|\geq N_\nu}\mathbf{E}|\widehat{\nabla\omega^\nu}(k)|^2 = \eta^*.$$

Step 4: Next we use Equation (3.6) which was proved in Section 3.4.2, to conclude that the if and only if established in step 3, also equivalently shows that

$$\frac{2}{\ell^6}\int_0^\ell r^2 S_{vel}(r)\,dr = \frac{2}{\ell^6}\int_0^\ell r^5\frac{S_{vel}(r)}{r^3}\,dr = \frac{\eta^*}{12} + o_{\ell\to 0}(1).$$

Finally combine Steps 1-4 with the KHM equation to derive (3.7).

3.5 The Inverse Cascade

In this section, we characterize the existence of an inverse cascade in both 2D and 3D.

Theorem 3.22 (Characterization of the Inverse Cascade)**.** *Suppose that $\lambda = \lambda(\nu) < \infty$ is a continuous monotone increasing function such that $\lim_{\nu\to 0} \lambda = \infty$. Let $\{u^\nu\}_{\nu>0}$ be a sequence of statistically stationary solutions to the Navier Stokes equations (1.1) on $\mathbb{T}^d_\lambda$ with divergence free stochastic forcing g. Then there exists a decreasing sequence M_ν such that $\lim_{\nu\to 0} M_\nu = 0$ and*

$$\liminf_{\nu\to 0} \nu \sum_{|k|\le M_\nu} \mathbf{E}|\widehat{\nabla u^\nu}(k)|^2 = \varepsilon^* + \frac{1}{2}\mathbf{E}|\widehat{g}(0)|^2$$

if and only if there exists $\tilde{\ell}_\nu \in (1,\lambda)$ such that $\lim_{\nu\to 0} \tilde{\ell}_\nu = \infty$ and

$$\lim_{\ell_I\to\infty} \limsup_{\nu\to 0} \sup_{\ell\in[\ell_I,\tilde{\ell}_\nu]} \left|\frac{S_{vel}(\ell)}{\ell} - 4\beta_d(1)\varepsilon^*\right| = 0; \tag{3.19}$$

$$\lim_{\ell_I\to\infty} \limsup_{\nu\to 0} \sup_{\ell\in[\ell_I,\tilde{\ell}_\nu]} \left|\frac{S^{||}_{vel}(\ell)}{\ell} - \big(4\beta_d(2) + \frac{8\beta_d(1)}{d+2}\big)\varepsilon^*\right| = 0. \tag{3.20}$$

Explicitly the coefficients can be computed as

$$4\beta_d(1) = \begin{cases} 2 & d=2 \\ 4/3 & d=3 \end{cases} \qquad 4\beta_d(2) + \frac{8\beta_d(1)}{d+2} = \begin{cases} 3/2 & d=2 \\ 4/5 & d=3. \end{cases}$$

The proof of Theorem 3.7 is split into two different subsections, one for each of the flux laws. While the details are slightly different for each one, similar to the direct cascade case the mechanics are similar for both flux laws.

3.5.1 Proof of (3.19)

First, recall the KHM equation for S_{vel}:

$$\frac{S_{vel}(\ell)}{\ell} = -\frac{4\nu}{\ell}\Gamma'_{vel}(\ell) - \frac{4}{\ell^d}\int_0^\ell r^{d-1}a_{vel}(r)\,dr.$$

Step 1: Beginning with the source term $a_{vel}(r)$, we apply Fourier analysis, Fubini and Lemma 3.16 with $p = 0$ to derive:

$$\begin{aligned}
\frac{4}{\ell^d}\int_0^\ell r^{d-1}a_{vel}(r)\,dr &= \frac{2}{\ell^d}\sum_{k\in(\frac{2\pi}{\lambda}\mathbb{Z})^d}\int_0^\ell \fint_{S^{d-1}} r^{d-1}e^{-irn\cdot k}\,dS(n)dr\mathbf{E}|\widehat{g}(k)|^2 \\
&= \frac{2}{\ell^d}\sum_{k\in(\frac{2\pi}{\lambda}\mathbb{Z})^d}\int_0^\ell r^{d-1}\sum_{m=0}^{\infty}\beta_d(m)\frac{(-1)^m(r|k|)^{2m}}{2^m m!}\,dr\mathbf{E}|\widehat{g}(k)|^2 \\
&= 2\sum_{k\in(\frac{2\pi}{\lambda}\mathbb{Z})^d}\sum_{m=0}^{\infty}\beta_d(m)\frac{(-1)^m(\ell|k|)^{2m}}{2^m(2m+d)m!}\mathbf{E}|\widehat{g}(k)|^2 \\
&:= \frac{2\beta_d(0)}{d}\mathbf{E}\|g\|_{L^2}^2 + \sum_{k\in(\frac{2\pi}{\lambda}\mathbb{Z})^d} c(\ell,k)\mathbf{E}|\widehat{g}(k)|^2.
\end{aligned}$$

Recall that when $d = 2$, $\beta_2(m) = \dfrac{1}{2^m m!}$ so

$$\Big|\frac{2\beta_2(0)}{2} + c(\ell,k)\Big| = \Big|\sum_{m=0}^{\infty}\frac{(-1)^m(\ell|k|)^{2m}}{4^m(m+1)!m!}\Big| = \Big|\frac{J_1(\ell|k|)}{\ell|k|}\Big| \to 0$$

as $\ell|k| \to \infty$. Here we have used the power series representation of the Bessel function of the first kind and that $|J_1(x)| < 1$ uniformly for all $x \in \mathbb{R}$ [1]. Alternatively when $d = 3$, recall that

$\beta_3(m) = \frac{2^m m!}{(2m+1)!}$ so

$$\begin{aligned}
\Big|\frac{2\beta_3(0)}{3} + c(\ell,k)\Big| &= \Big|\sum_{m=0}^{\infty} \frac{(-1)^m(\ell|k|)^{2m}}{(2m+3)(2m+1)!}\Big| \\
&= \Big|\frac{2}{(\ell|k|)^3}\int_0^{\ell|k|} \sum_{m=0}^{\infty} \frac{(-1)^m r^{2m+2}}{(2m+1)!}\,dr\Big| \\
&= \Big|\frac{2}{(\ell|k|)^3}\int_0^{\ell|k|} r\sin(r)\,dr\Big| \\
&\leq \frac{1}{\ell|k|} \to 0
\end{aligned}$$

as $\ell|k| \to \infty$. Hence in both cases $c(\ell,k) \to -\frac{2\beta_d(0)}{d}$. By Lemma 3.19 it follows that

$$\begin{aligned}
\lim_{\ell_I\to\infty}\limsup_{\nu\to 0}\sup_{\ell\in(\ell_I,\tilde{\ell_\nu})} \frac{4}{\ell^d}\int_0^{\ell} r^{d-1}a_{vel}(r)\,dr &= \frac{2\beta_d(0)}{d}\|g\|_{L^2}^2 - \frac{2\beta_d(0)}{d}(\|g\|_{L^2}^2 - |\widehat{g}(0)|^2) \\
&= \frac{2\beta_d(0)}{d}|\widehat{g}(0)|^2.
\end{aligned}$$

Step 2: Next, we use Fourier analysis to translate the energy correlation term Γ_{vel} as

$$\frac{-4\nu}{\ell}\Gamma_{vel}'(\ell) = 4\beta_d(1)\varepsilon + \nu\sum_{k\in(\frac{2\pi}{\lambda}\mathbb{Z})^d} c(\ell,k)\mathbf{E}\fint_0^T |\widehat{\nabla u^\nu}(k)|^2$$

where $\lim_{\ell|k|\to 0} c(\ell,k) = 0$ and $\lim_{\ell|k|\to\infty} c(\ell,k) = -4\beta_d(1)$.

By Proposition 3.13 we know $\Gamma_{vel} \in C^2(\mathbb{R})$. Thus we Taylor expand Γ_{vel}' about 0 and use the Lagrange formulation of the error:

$$\frac{-4\nu}{\ell}\Gamma_{vel}'(\ell) = \frac{-4\nu}{\ell}\Gamma_{vel}'(0) + \frac{-4\nu}{\ell}\int_0^{\ell}\Gamma_{vel}''(r)\,dr.$$

From the proof of Lemma 3.15 we know that $\fint_{S^{d-1}} n\ dS(n) = 0$, so after applying Fubini's theorem we have

$$\Gamma'_{vel}(0) = -\mathbf{E} \fint_{S^{d-1}} \fint_0^T \fint_{\mathbb{T}^d_\lambda} n_i \partial_i u^\nu_m u^\nu_m \, dxdtdS(n) = 0.$$

Next, we consider the error term, which is real valued. We use Plancheral's theorem to rewrite the integral in x in terms of the Fourier series and apply both Fubini and Lemma 3.16 when $p = 1$ to get:

$$\begin{aligned}
\frac{-4\nu}{\ell} \int_0^\ell \Gamma''_{vel}(r)\, dr &= \frac{-4\nu}{\ell} \sum_{k \in (\frac{2\pi}{\lambda}\mathbb{Z})^d} \int_0^\ell \mathbf{E} \fint_{S^{d-1}} (n \cdot k)^2 e^{-irn\cdot k} \mathbf{E} \fint_0^T |\widehat{u^\nu}(k)|^2 \, dtdS(n)dr \\
&= -\frac{4\nu}{\ell} \sum_{k \in (\frac{2\pi}{\lambda}\mathbb{Z})^d} \int_0^\ell |k|^2 \sum_{m=0}^\infty \beta_d(m+1) \frac{(-1)^m (r|k|)^{2m}(2m+2)!}{2^{m+1}(2m)!(m+1)!} \, dr \mathbf{E} \fint_0^T |\widehat{u^\nu}(k)|^2 \\
&= 4\nu \sum_{k \in (\frac{2\pi}{\lambda}\mathbb{Z})^d} \sum_{m=0}^\infty \beta_d(m+1) \frac{(-1)^m (\ell|k|)^{2m}}{2^m m!} \mathbf{E} \fint_0^T |\widehat{\nabla u^\nu}(k)|^2 \\
&= 4\beta_d(1)\nu \sum_{k \in (\frac{2\pi}{\lambda}\mathbb{Z})^d} \mathbf{E} \fint_0^T |\widehat{\nabla u^\nu}(k)|^2 \\
&\quad + 4\nu \sum_{k \in (\frac{2\pi}{\lambda}\mathbb{Z})^d} \sum_{m=1}^\infty \beta_d(m+1) \frac{(-1)^m (\ell|k|)^{2m}}{2^m m!} \mathbf{E} \fint_0^T |\widehat{\nabla u^\nu}(k)|^2 \\
&=: 4\beta_d(1)\varepsilon - 4\beta_d(1)\mathbf{E}D(u^\nu) + \nu \sum_{k \in (\frac{2\pi}{\lambda}\mathbb{Z})^d} c(\ell,k) \mathbf{E} |\widehat{\nabla u^\nu}(k)|^2.
\end{aligned}$$

Note that the final equality follows from the energy balance (3.14).

It is clear from the definition of $c(\ell,k)$ that $\lim_{\ell|k|\to 0} c(\ell,k) = 0$. Moreover, using the same argument as in step 1 we can show

$$|4\beta_d(1) + c(\ell,k)| = \left| 4 \sum_{m=0}^\infty \beta_d(m+1) \frac{(-1)^m (\ell|k|)^{2m}}{2^m m!} \right| \le \frac{C}{\ell|k|}.$$

Therefore

$$\lim_{\ell|k|\to\infty} c(\ell,k) = -4\beta_d(1).$$

Step 3: Apply Lemma 3.19 to conclude that there exists $\tilde{\ell}_\nu \to \infty$ such that

$$\begin{aligned}
&\lim_{\ell_I\to\infty}\limsup_{\nu\to 0}\sup_{\ell\in[\ell_I,\tilde{\ell}_\nu]} \frac{-4\nu}{\ell}\Gamma'(\ell) \\
&\qquad = 4\beta_d(1)(\varepsilon - \mathbf{E}D(u^\nu)) + \lim_{\ell_I\to\infty}\limsup_{\nu\to 0}\sup_{\ell\in[\ell_I,\tilde{\ell}_\nu]} \nu \sum_{k\in(\frac{2\pi}{\lambda}\mathbb{Z})^3} c(\ell,k)\mathbf{E}|\widehat{\nabla u}(k)|^2 \\
&\qquad = 4\beta_d(1)(\varepsilon - \mathbf{E}D(u^\nu)) - 4\beta_d(1)\Big(\varepsilon - \mathbf{E}D(u^\nu) - \varepsilon^* - \frac{\beta_d(0)}{2d\beta_d(1)}|\widehat{g}(0)|^2\Big) \\
&\qquad = 4\beta_d(1)\varepsilon^* + \frac{2\beta_d(0)}{d}|\widehat{g}(0)|^2.
\end{aligned}$$

if and only if there exists $M_\nu \to 0$ such that

$$\liminf_{\nu\to 0}\nu\sum_{|k|\le M_\nu}\mathbf{E}|\widehat{\nabla u^\nu}(k)|^2 = \varepsilon^* + \frac{\beta_d(0)}{2d\beta_d(1)}|\widehat{g}(0)|^2 = \varepsilon^* + \frac{1}{2}|\widehat{g}(0)|^2.$$

Finally, combine Steps 1-3 with the KHM equation to derive (3.19).

3.5.2 Proof of (3.20)

Recall the KHM equation for $S^{\|}_{vel}$:

$$\frac{S^{\|}_{vel}(\ell)}{\ell} = -\frac{4\nu}{\ell}(\Gamma^{\|}_{vel})'(\ell) + \frac{2}{\ell^{d+2}}\int_0^\ell r^d S_{vel}(r)\, dr - \frac{4}{\ell^{d+2}}\int_0^\ell r^{d+1} a^{\|}_{vel}(r)\, dr.$$

Step 1: Beginning with the source term $a^{\parallel}_{vel}(r)$, we apply Fourier analysis, Fubini and Lemma 3.17 with $p = 0$ to derive:

$$\begin{aligned}
\frac{4}{\ell^{d+2}}\int_0^\ell r^{d+1}a^{\parallel}_{vel}(r)\,dr &= \frac{2}{\ell^{d+2}}\sum_{k\in(\frac{2\pi}{\lambda}\mathbb{Z})^d}\int_0^\ell \fint_{\S^{d-1}} r^{d+1}n_a n_b e^{-irn\cdot k}\,dS(n)dr\widehat{g}_a(k)\widehat{g}_b(k)\\
&= \frac{2}{\ell^{d+2}}\sum_{k\in(\frac{2\pi}{\lambda}\mathbb{Z})^d}\sum_{m=0}^{\infty}\int_0^\ell \beta_d(m+1)r^{d+1}\frac{(-1)^m(r|k|)^{2m}}{2^m m!}\,dr|\widehat{g}(k)|^2\\
&= 2\sum_{k\in(\frac{2\pi}{\lambda}\mathbb{Z})^d}\sum_{m=0}^{\infty}\beta_d(m+1)\frac{(-1)^m(\ell|k|)^{2m}}{2^m m!(2m+d+2)}|\widehat{g}(k)|^2\\
&= \frac{2\beta_d(1)}{d+2}\|g\|^2_{L^2} + 2\sum_{k\in(\frac{2\pi}{\lambda}\mathbb{Z})^d}\sum_{m=0}^{\infty}\beta_d(m+1)\frac{(-1)^m(\ell|k|)^{2m}}{2^m m!(2m+d+2)}|\widehat{g}(k)|^2\\
&:= \frac{2\beta_d(1)}{d+2}\|g\|^2_{L^2} + \sum_{k\in(\frac{2\pi}{\lambda}\mathbb{Z})^d} c(\ell,k)|\widehat{g}(k)|^2.
\end{aligned}$$

Recall that when $d = 2$, $\beta_2(m) = \dfrac{1}{2^m m!}$ so

$$\Big|\frac{2\beta_2(1)}{4} + c(\ell,k)\Big| = \Big|\frac{1}{2}\sum_{m=0}^{\infty}\frac{(-1)^m(\ell|k|)^{2m}}{4^m m!(m+2)!}\Big| = \Big|\frac{2J_2(\ell|k|)}{(\ell|k|)^2}\Big| \to 0 \quad \text{as } \ell|k| \to \infty.$$

Here we have used the power series representation of the Bessel function of the first kind and that $|J_2(x)| < 1$ uniformly for all $x \in \mathbb{R}$ [1]. Alternatively when $d = 3$, recall that $\beta_3(m) = \frac{2^m m!}{(2m+1)!}$

so

$$\begin{aligned}
\Big|\frac{2\beta_3(1)}{5}+c(\ell,k)\Big| &= \Big|2\sum_{m=0}^{\infty}\frac{(-1)^m(\ell|k|)^{2m}}{(2m+5)(2m+3)(2m+1)!}\Big| \\
&= \Big|\frac{2}{(\ell|k|)^5}\int_0^{\ell|k|}\sum_{m=0}^{\infty}\frac{(-1)^m r^{2m+4}}{(2m+3)(2m+1)!}\;dr\Big| \\
&= \Big|\frac{2}{(\ell|k|)^5}\int_0^{\ell|k|} r\int_0^{r}\sum_{m=0}^{\infty}\frac{(-1)^m q^{2m+2}}{(2m+1)!}\;dqdr\Big| \\
&= \Big|\frac{2}{(\ell|k|)^5}\int_0^{\ell|k|} r\int_0^{r} q\sin(q)\;dqdr\Big| \\
&\le \frac{C}{(\ell|k|)} \to 0 \quad \text{ as } \ell|k|\to\infty.
\end{aligned}$$

Hence in both cases $\lim_{\ell|k|\to\infty} c(\ell,k) = -\frac{2\beta_d(1)}{d+2}$. Moreover, by definition the coefficients $c(\ell,k)\to 0$ as $\ell|k|\to 0$. By Lemma 3.19 it follows that

$$\begin{aligned}
\lim_{\ell_I\to\infty}\limsup_{\nu\to 0}\sup_{\ell\in(\ell_I,\tilde{\ell_\nu})}\frac{4}{\ell^d}\int_0^{\ell} r^{d-1}a_{vel}(r)\;dr &= \frac{2\beta_d(1)}{d+2}\|g\|_{L^2}^2 - \frac{2\beta_d(1)}{d+2}(\|g\|_{L^2}^2-|\widehat{g}(0)|^2) \\
&= \frac{2\beta_d(1)}{d+2}|\widehat{g}(0)|^2.
\end{aligned}$$

Step 2: Next we show that

$$\frac{-4\nu}{\ell}(\Gamma_{vel}^{||})'(\ell) = 4\beta_d(2)\varepsilon - \sum_{k\in(\frac{2\pi}{\lambda}\mathbb{Z})^d} c(\ell,k)\mathbf{E}|\widehat{\omega}(k)|^2$$

where the coefficients satisfy $\lim\limits_{\ell|k|\to 0} c(\ell,k)=0$ and $\lim\limits_{\ell|k|\to\infty} c(\ell,k) = -4\beta_d(2)$ (Note these coefficients $c(\ell,k)$ may be different from the ones found in Step 1).

From Proposition 3.13, $\Gamma_{vel}^{||}\in C^2(\mathbb{R})$. Thus we Taylor expand $(\Gamma_{vel}^{||})'$ about $\ell=0$ and use the Lagrange formulation of the error to get

$$\frac{-4\nu}{\ell}(\Gamma^{\|}_{vel})'(\ell) = \frac{-4\nu}{\ell}(\Gamma^{\|}_{vel})'(0) - \frac{4\nu}{\ell}\int_0^\ell (\Gamma^{\|}_{vel})''(r)\, dr.$$

As $\fint_{S^{d-1}} n_i n_j n_k \, dS(n) = 0$ it follows from Fubini's theorem that

$$\frac{-4\nu}{\ell}(\Gamma^{\|}_{vel})'(0) = \frac{4\nu}{\ell}\mathbf{E}\fint_{S^{d-1}}\fint_{\mathbb{T}^d_\lambda} n_i n_j n_k \partial_k u_i u_j \, dx dS(n) = 0.$$

Next, consider the error term, which we will note is real valued. Using Fourier Analysis, Fubini's Theorem, and Lemma 3.17 when $p = 1$ it follows that

$$\begin{aligned}
\frac{-4\nu}{\ell}\int_0^\ell (\Gamma^{\|}_{vel})''(r)\, dr &= \sum_{k\in(\frac{2\pi}{\lambda}\mathbb{Z})^d} \frac{4\nu}{\ell}\int_0^\ell \fint_{S^{d-1}} n_i n_j (n\cdot k)^2 e^{-ir(n\cdot k)}\mathbf{E}\fint_0^T \widehat{u_i}(k)\widehat{u_j}(k)\, dt dS(n) dr \\
&= \sum_{k\in(\frac{2\pi}{\lambda}\mathbb{Z})^d} \frac{4\nu}{\ell}\int_0^\ell \sum_{m=0}^\infty \beta_d(m+2)\frac{(-1)^m(2m+2)!(r|k|)^{2m}}{2^{m+1}(m+1)!(2m)!}\, dr\mathbf{E}\fint_0^T |\widehat{\nabla u^\nu}(k)|^2 \\
&= 4\nu \sum_{k\in(\frac{2\pi}{\lambda}\mathbb{Z})^d}\sum_{m=0}^\infty \beta_d(m+2)\frac{(-1)^m(\ell|k|)^{2m}}{2^m m!}\mathbf{E}\fint_0^T |\widehat{\nabla u^\nu}(k)|^2 \\
&= 4\nu\beta_d(2)\mathbf{E}\fint_0^T \|\nabla u^\nu\|^2 \\
&\quad + 4\nu \sum_{k\in(\frac{2\pi}{\lambda}\mathbb{Z})^d}\sum_{m=1}^\infty \beta_d(m+2)\frac{(-1)^m(\ell|k|)^{2m}}{2^m m!}\mathbf{E}\fint_0^T |\widehat{\nabla u^\nu}(k)|^2 \\
&=: 4\beta_d(2)\varepsilon - 4\beta_d(2)\mathbf{E}D(u^\nu) + \sum_{k\in(\frac{2\pi}{\lambda}\mathbb{Z})^d} c(\ell,k)\mathbf{E}\fint_0^T |\widehat{\nabla u^\nu}(k)|^2.
\end{aligned}$$

Note that by construction $\lim_{\ell|k|\to 0} c(\ell,k) = 0$. Moreover, by a similar argument to that done in step 1, one can show that $\lim_{\ell|k|\to\infty} c(\ell,k) = -4\beta_d(2)$.

Step 3: Apply Lemma 3.19 to the result from step 2 to conclude that there exists $\tilde{\ell}_\nu \to \infty$

such that

$$
\begin{aligned}
\lim_{\ell_I\to\infty}\limsup_{\nu\to 0}\sup_{\ell\in(\ell_I,\tilde{\ell}_\nu)}\frac{-4\nu}{\ell}(\Gamma^{\|}_{vel})'(\ell) &= 4\beta_d(2)\varepsilon - 4\beta_d(2)\liminf_{\nu\to 0}\mathbf{E}D(u^\nu) \\
&\quad + \lim_{\ell_I\to\infty}\limsup_{\nu\to 0}\sup_{\ell\in(\ell_I,\tilde{\ell}_\nu)}\sum_{k\in(\frac{2\pi}{\lambda}\mathbb{Z})^d} c(\ell,k)\mathbf{E}|\widehat{\nabla u^\nu}(k)|^2 \\
&= 4\beta_d(2)\varepsilon - 4\beta_d(2)\liminf_{\nu\to 0}\mathbf{E}D(u^\nu) \\
&\quad - 4\beta_d(2)\big(\varepsilon - \liminf_{\nu\to 0}\mathbf{E}D(u^\nu) - \varepsilon^* - \frac{\beta_d(0)}{2d\beta_d(1)}|\widehat{g}(0)|^2\big) \\
&= 4\beta_d(2)\varepsilon^* + \frac{2\beta_d(2)\beta_d(0)}{d\beta_d(1)}|\widehat{g}(0)|^2
\end{aligned}
$$

if and only if there exists $M_\nu \to 0$ such that

$$
\liminf_{\nu\to 0}\nu\sum_{|k|\le M_\nu}\mathbf{E}|\widehat{\nabla u^\nu}|^2 = \varepsilon^* + \frac{\beta_d(0)}{2d\beta_d(1)}|\widehat{g}(0)|^2 = \varepsilon^* + \frac{1}{2}|\widehat{g}(0)|^2.
$$

Step 4: Next we use Equation (3.19), which was proved in Section 3.5.1, to conclude that the if and only if established in Step 3, also equivalently shows that

$$
\frac{2}{\ell^{d+2}}\int_0^\ell r^d S_{vel}(r)\,dr = \frac{2}{\ell^{d+2}}\int_0^\ell r^{d+1}\frac{S_{vel}(r)}{r}\,dr = \frac{8\beta_d(1)}{d+2}\varepsilon^* + o_{\ell\to 0}(1).
$$

Step 5: For both $d = 2, 3$, one can check that

$$
\frac{2\beta_d(2)\beta_d(0)}{d\beta_d(1)} - \frac{2\beta_d(1)}{d+2} = 0.
$$

Now we combine Steps 1-4 with the KHM equation to derive (3.20).

Chapter 4: On the Existence of Anomalous Dissipation for the Stokes equation over Bounded Domains

In this chapter we consider the problem of defining a flow field which exhibits anomalous dissipation over a bounded domain. We recall from Chapter 1 that anomalous dissipation is a non-vanishing amount of mean viscous dissipation which remains in the system as $\nu \to 0$ (or equivalently as the Reynolds number $Re \to \infty$). This is thought to be due to the nonlinear convective acceleration term $u^\nu \cdot \nabla u^\nu$ elongating vortex tubules within the flow which in turn increases the total vorticity (and small scale oscillations) within the flow. However as this is a rather abstract quantity, the question becomes how to measure it. Recall that (in the absence of boundaries) the Navier Stokes equations (1.1) satisfy the energy inequality

$$\frac{1}{2}\mathbf{E}\|u^\nu(t)\|_{L^2(D)}^2 + \nu\mathbf{E}\int_0^t \|\nabla u^\nu\|_{L^2(D)}^2 \leq \frac{1}{2}\mathbf{E}\|u^\nu(0)\|_{L^2(D)}^2 + \mathbf{E}\int_0^t\int_D f \cdot u^\nu + \frac{1}{2}\mathbf{E}\int_0^t \|g\|_{L^2(D)}^2.$$

Formally, if we assumed that rather than an inequality the above was a perfect energy balance (i.e. an equality instead) and we took the limit $\nu \to 0$ then we should achieve the energy balance for the Euler system. Hence the usual way to define anomalous dissipation is to check that the viscous dissipation is non-vanishing: there exists $\varepsilon_0 > 0$ such that

$$\limsup_{\nu \to 0} \nu \mathbf{E} \int_0^t \|\nabla u^\nu\|^2_{L^2(D)} = \varepsilon_0 > 0. \tag{1.2}$$

This definition of (global) anomalous dissipation [32] can be readily extended to a local type definition over compact subsets $K \subset D$ and is easily computable within a numerical simulation meaning it is used extensively within the literature. An interesting mathematical question, is how much regularity does u^ν need to retain in its inviscid limit for (1.2) to hold. Onsager conjectured that if $u^\nu \in [C(0, T, C^\gamma(D))]^3$ with $\gamma \leq \frac{1}{3}$ then u^ν exhibits anomalous dissipation. Onsager's conjecture is supported by the fact that there exist Euler flows over unbounded domains with Holder regularity $\gamma \leq \frac{1}{3}$ which are dissipative [37] and even in the larger space of Besov regular solutions [58]. Moreover, recently Armstrong and Vicol [4] were able to construct a highly oscillatory flow field by gluing together shear flows together at infinitely many length scales such that a passive scalar subject to this flow field exhibits anomalous dissipation. Based on this approach, Brue and de Lellis [13] were able to construct 2.5 dimensional solutions (when u^ν is a 3 dimensional vector field but only depends on 2 spatial input variables) to the Navier Stokes equations which exhibit (1.2) by setting the z-velocity to be a passive scalar type quantity. Despite this great progress, the case of a fully three-dimensional analytical velocity field to the Navier Stokes equations which exhibits (1.2) remains open to the authors knowledge.

In this work, we wish to address how the presence of a boundary affects the existence of anomalous dissipation. Typically, the presence of a boundary makes most classical techniques harder to use, for instance Calderon-Zygmund estimates for the pressure fail to hold due to the boundary, and in the inviscid limit a boundary layer will form [40, 65, 69]. In the case of a no-slip boundary condition, Kato was able to show that for a highly regular domain and $[C^1((0,T)\times D)]^3$ solutions then there is *no* anomalous dissipation if and only if the amount of vorticity within a thin region next to the boundary vanishes [40]. Kato's proof uses a boundary correction type approach and there have been many variants of Kato's criterion for the lack of anomalous dissipation. In particular, we refer to the result by Drivas and Nguyen [24] which uses a mollifier type approach and breaks the flow into the bulk and boundary components to show the regularity assumptions required on each part. In that work the authors were able to show that if the bulk flow is uniformly $[L^3(0,T,B^{\sigma}_{3,c_0}(D)]^3$ where $\sigma \in [\frac{1}{3},1]$ and the boundary flow is $[L^\infty((0,T)\times D)]^3$ then there is no anomalous dissipation. However, the authors in [24] work with an unforced flow which would almost never appear in nature. Motivated in part by their work, we work with the Navier-slip condition instead of the no-slip condition and we will study flows which are only $L^2((0,T)\cap H^s(D))$ near the boundary with $s\in(0,\frac{1}{2})$. This is substantially less control over how the flows behave at the boundary and we will show that we get a non-zero limit in (1.2) which is the opposite result from [24]. Moreover our example should be able to be extended to show an non-vanishing limit over any compactly supported subset in the domain by taking a properly weighted sum of external forces which concentrate energy on concentric shells with radii from a countable dense set in $\mathbb{R}_+$ which shows that one sufficient condition for (1.2) is to concentrate energy over sets of Lebesgue measure 0 within the domain. Furthermore, contrary to most comments within articles for unbounded domains, our example uses a fixed external forcing which is not highly irregular (it is in fact C^∞ within the interior of the domain) but rather just concentrates energy at the boundary. This is highly analogous to how Stokes 2nd problem which is not "anomalous" which shows that (1.2) is not a good way of measuring the contribution of the nonlinear term to the acceleration of viscous dissipation in the inviscid limit.

Results of this kind are known already in the context of stationary solutions to the stochastically forced Navier Stokes equations over unbounded domains. For instance, Bedrossian et.al. [9] showed that (1.2) is not a good definition for anomalous dissipation in the case of statistically stationary solutions, because stationary solutions to the stochastic heat equation subject to a zero drift, white-in-time colored-in-space noise (which is independent of ν) will satisfy the non-vanishing viscous dissipation assumption due to its energy balance:

$$\begin{cases} dw = \nu\Delta w dt + g dW_t \\ w(0) = w_0 \end{cases} \Rightarrow \nu\mathbf{E}\int_0^t \|\nabla w\|_{L^2(D)}^2 = \frac{1}{2}\mathbf{E}\int_0^t \|g\|_{L^2(D)}^2 \neq 0.$$

As such in the case of stochastic flows, authors have tried finding alternative definitions for anomalous dissipation which are directly related to how the convection term contributes to the energy balance. See [9, 23, 26, 60] and Chapter (3). However to the authors knowledge, the example we outline here is the first such example in the for non-statistically stationary solutions. In particular, we show that in the presence of a boundary, we can build non-statistically stationary solutions to the linear Stokes equation which still satisfy (1.2). This analysis also works for the linear heat equation, but we will work with the Stokes operator as it is closely tied to the Navier Stokes equation which is the motivating system for this problem. Heuristically, one can argue that our approach still increases the total vorticity within the flow — without any vortex stretching — as the boundary is a natural source for generating vorticity within the flow.

4.1 The Physical Implications of Including a Boundary

In this work, we take the domain $D = B(0,R) \subset \mathbb{R}^3$ where $R \in (0,\infty)$ is a fixed value. We will further assume that this boundary is impermeable to the fluid, but the total shear stress induced on the fluid is proportional to the momentum of the fluid at the wall, that is to say z^ν satisfies the Stokes equation with the Navier-slip conditions:

$$
\begin{cases}
dz^\nu = \big(\nu\Delta z^\nu + \nabla p^\nu + f\big)dt + g dW_t & x \in D,\ t > 0 \\
\nabla \cdot z^\nu = 0 & x \in D,\ t > 0 \\
z^\nu \cdot n^{\partial D} = 0 & x \in \partial D,\ t > 0 \\
n^{\partial D} \cdot \nabla z^\nu_\tau + \alpha z^\nu_\tau = 0 & x \in \partial D,\ t > 0 \\
z^\nu(0) = z_0 & x \in D.
\end{cases}
\tag{4.1}
$$

Here $\alpha \in L^\infty(\partial D)$ is called the slip length, $n^{\partial D}$ is the outward unit normal vector to the boundary ∂D, and $z^\nu_\tau|_{\partial D} = z^\nu|_{\partial D} - (n^{\partial D} \cdot z^\nu)n^{\partial D}|_{\partial D}$ is the trace of z^ν in the tangential direction along the boundary.

The slip-type boundary condition proposed here was first considered by Navier [56]. Here $\alpha \geq 0$ is a "constant" of proportionality that balances the friction the fluid experiences along the boundary with the acceleration due to the pressure gradient. Alternatively, one can consider the slip condition as a growth rate for the tangential component of the vorticity generated at the wall (after accounting for the curvature of the wall as well) [16]. In particular, this friction-type boundary condition has been derived from the kinetic theory of homogenization in [5, 20] and has been justified as the effective boundary condition for flows over rough boundary [33, 52]. Furthermore, experimental evidence suggests that at sufficiently large Reynolds numbers or in domains with curvature the no-slip boundary condition fails to capture important information about the flow [27, 38, 82]. As such, the study of (4.1) with Navier Slip boundary condition has become more prevalent within the literature, see [16, 57, 81] and the references within for a sample. Moreover, as $\alpha \to \infty$ one can recover the non-slip condition and as $\alpha \to 0$ we recover the full slip (also known as the zero-flux) boundary condition; marking the Navier-slip condition as a generalization of the other types of physical boundary conditions typically encountered in experiments [16]. In this work, the Navier-slip condition is used not only to account for physical relevance but also as a measure to ensure the kinetic energy at the boundary is non-zero. As we will show in the following sections, lacking control over the kinetic energy at the boundary is a sufficient condition to show the existence of anomalous dissipation over the entire domain even if there is no convective term to mix length scales together.

4.1.1 Outline for the Chapter

First in Section 4.2.1 we elabortate on how the Stokes problem and the Navier Stokes problem are related, and physical intuition for the problem. Then in Section 4.2 we construct a family of non-statistically stationary solutions to the Stokes problems which satisfy (1.2). Finally, in Section 4.3 we numerically simulation our example using finite differences to illustrate how the family of constructed solutions from the previous section behave in both the stochastic and deterministic setting.

4.2 Existence of (Global) Anomalous Dissipation

In order to construct our family of martingale solutions $\{z^\nu\}_{\nu>0}$ which satisfy both the linear Stokes problem as well as (1.2) we will concentrate the external forcing near the boundary. This has the affect of speeding up the fluid velocity at the wall causing larger shearing affects (due to the slip boundary condition) which then moves into the interior as small scale oscillations. Before getting into the specific details to ensure the family of solutions exhibit anomalous dissipation, we first recall a few results about the linearized Stoke problem.

4.2.1 The Linearized Stokes problem

Consider a solution process z^ν to the equation

$$\begin{cases} dz^\nu = \big(-\nu\,\mathrm{Pr}(-\Delta_{NS})z^\nu + f\big)dt + g dW_t & x \in D,\ t>0 \\ \nabla \cdot z^\nu = 0 & x \in D,\ t>0 \\ n^{\partial D} \cdot z^\nu = 0 & x \in \partial D,\ t>0 \\ n^{\partial D} \cdot \nabla z^\nu_\tau + \alpha z^\nu_\tau = 0 & x \in \partial D,\ t>0 \\ z^\nu(0) = 0 & x \in D \end{cases}$$

which can be recast using the Stokes operator A_{NS} as a general linear parabolic problem:

$$\begin{cases} dz^\nu = \big(-\nu A_{NS} z^\nu + f\big)dt + g dW_t & x \in D,\ t>0 \\ z^\nu(0) = 0 & x \in D. \end{cases} \tag{4.2}$$

It is known that $-A_{NS}$ generates an analytic semigroup provided D is at least $C^{1,1}$[78]. As such define:

$$z^\nu(t) = \int_0^t e^{-\nu(t-s)A_{NS}} f \, ds + \int_0^t e^{-\nu(t-s)A_{NS}} g \, dW_s.$$

Then z^ν is the unique weak solution to (4.2) [21], and by both Ito's formula and Young's inequality z^ν satisfies the following energy inequality

$$\begin{aligned}\frac{1}{2}\mathbf{E}\big(\sup_{t\in[0,T]} \|z^\nu(t)\|^2_{L^2(D)}\big) + 2\nu\mathbf{E}\int_0^T \|A^{1/2}_{NS} z^\nu\|^2_{L^2(D)} \\ \leq 2\mathbf{E}\int_0^T \|f\|^2_{L^2(D)} + \mathbf{E}\int_0^T \|g\|^2_{L^2(D)}.\end{aligned} \tag{4.3}$$

Thus the trajectories of z^ν are in $[L^\infty(0,T,L^2_\sigma(D))]^3 \cap [L^2(0,T,H^1_{\sigma,\tau}(D))]^3$ $\mathbb{P}$-a.s. when $\nu > 0$. It is possible to show higher classes of regularity by appealing to the analyticity of the semigroup operator, but such results are unneeded here.

4.2.2 The Nonlinear Problem

It is worth noting that solutions to the full Navier Stokes problem (1.1) can be formed using the Stokes problem as a base: For each fixed $\omega \in \Omega$ we consider the trajectory of $v^\nu(\omega) =$

$u^\nu(\omega) - z^\nu(\omega)$ which will be the weak solution to random differential equation:

$$\begin{cases} dv^\nu = \big(-\nu \, \mathrm{Pr}(-\Delta_{NS}) v^\nu - \mathrm{Pr}((v^\nu + z^\nu \cdot \nabla (v^\nu + z^\nu)\big) dt & x \in D, \ t > 0 \\ \nabla \cdot v^\nu = 0 & x \in D, \ t > 0 \\ n^{\partial D} \cdot v^\nu = 0 & x \in \partial D, \ t > 0 \\ n^{\partial D} \cdot \nabla v^\nu_\tau + \alpha v^\nu_\tau = 0 & x \in \partial D, \ t > 0 \\ v^\nu(0) = u_0 & x \in D. \end{cases} \tag{4.4}$$

We remark that (4.4) is entirely deterministic (for fixed ω) and v^ν only inherits its probabilistic behavior from the linear solution z^ν through the convection term. Furthermore, construction of weak solutions to (4.4) are straight forward to establish by using the eigenfunctions of the Stokes operator A_{NS} similar to the method used in [31] and the full solution $u^\nu = z^\nu + v^\nu$ will be a martingale solution to (1.1).

Remark 4.1. *Here the convection term $u^\nu \cdot \nabla u^\nu$ shows the mixing of the (dissipative) linear problem across different length scales within the interior of the domain.*

Remark 4.2. *The property that v^ν is a suitable weak solution to* (4.4) *so that $u^\nu = v^\nu + z^\nu$ is a weak solution to* (1.1) *seems to depend on the solution z^ν to the linear problem. However this is not true. See Theorem 2.8 in* [67] *for the proof of such a result. As such when $g \equiv 0$ there is no difference between suitable weak solutions in the classical deterministic sense of Caffarelli, Kohn and Nirenberg* [15] *and the stochastic case.*

Remark 4.3. *In the next subsection we choose the external forcing f, g which are concentrated* *at the boundary to ensure that*

$$0 < \varepsilon_0 \leq \lim_{\nu \to 0} \nu \mathbf{E} \int_0^t \|\nabla z^\nu\|_{L^2(D)}^2.$$

This is done through the use of the trace theorem to bound the viscous dissipation below by the behavior of the flow at the wall. Hence if v^ν is compactly supported within the bulk/interior of the domain then a similar argument shows that u^ν will also exhibit anomalous dissipation. We note that while we were unable to show that v^ν is compactly supported within the interior in order to show the existence of solutions to the Navier Stokes problem which satisfy (1.1)*, this kind of assumption is well supported by experimental evidence* [63] *and the references within.*

4.2.3 Construction of a Stokes Family Exhibiting Global Anomalous Dissipation

Now we will choose specific choices for f, g so that the family of (viscous) martingale solutions $\{z^\nu\}$ to the Stokes problem (4.2) satisfy (1.2) by blowing up at the boundary. While it may be possible to generalize the arguments used here to arbitrary $C^{1,1}$ or even Lipschitz domains, for clarity we will restrict ourselves to the case when $D = B(0, R) \subset \mathbb{R}^3$ to make the computation along the boundary easier to work with. Furthermore, we expect a similar behavior to be true for the full Navier Stokes solution $\{u^\nu\}$, however our approach is not immediately amendable to this question due to possible contributions v^ν may make at the boundary. See 4.3.

Before getting into all of the details, let us naively sketch out our approach.

Remark 4.4. *Let $0 < \varepsilon \ll 1$. Since $D \in C^1$ there exists a trace operator $\gamma : H^{\frac{1}{2}+\varepsilon}(D) \to$*

$L^2(\partial D)$ [72]. *Moreover* $H^{\frac{1}{2}+\varepsilon}(D)$ *is an interpolation space for* $L^2(D)$ *and* $H^1(D)$*. Thus by Holder's inequality*

$$\begin{aligned}\nu^{\frac{1}{2}+\varepsilon}\mathbf{E}\int_0^T \|z^\nu\|^2_{L^2(\partial D)} &\leq \nu^{\frac{1}{2}+\varepsilon}C\mathbf{E}\int_0^T \|z^\nu\|^2_{H^{\frac{1}{2}+\varepsilon}(D)} \\ &\leq C\Big(\mathbf{E}\int_0^T \|z^\nu\|^2_{L^2(D)}\Big)^{1/2-\varepsilon}\Big(\nu\mathbf{E}\int_0^T \|z^\nu\|^2_{H^1(D)}\Big)^{1/2+\varepsilon}.\end{aligned}$$

Recall that when $f, g \in [L^2(D)]^3$ *then the kinetic energy within the interior of the domain is uniformly bounded meaning that if* z^ν *blows up at the boundary then its* H^1 *norm also blows up at the same rate. Note that there is no contradiction in requiring* z^ν *to have a uniform bound on its bulk kinetic energy (i.e. kinetic energy over the interior) and requiring the kinetic energy at the boundary to blow up since the boundary* ∂D *is a set of 3-dimensional Lebesgue measure 0.*

While this naive approach seems great at first, its impossible to construct a solution z^ν which blows up faster than $\nu^{\frac{1}{2}}$ at the boundary.

Remark 4.5. *Suppose* $a \in (\frac{1}{2}, 1)$*. Now select* $\varepsilon = \frac{2a-1}{4} \in (0, \frac{1}{4})$*. Then the linear trace operator* $\gamma : H^{\frac{1}{2}+\varepsilon}(D) \to L^2(\partial D)$ *is uniformly bounded and by the same argument as in Remark 4.4*

$$\nu^a\mathbf{E}\int_0^T \|z^\nu\|^2_{L^2(\partial D)}\,dt \leq C\nu^{a-\varepsilon-\frac{1}{2}}(\mathbf{E}\int_0^T \|z^\nu\|^2_{L^2(D)})^{\frac{1}{2}-\varepsilon}(\nu\mathbf{E}\int_0^T \|z^\nu\|^2_{H^1(D)})^{\frac{1}{2}+\varepsilon} \leq C\nu^{a-\varepsilon-\frac{1}{2}}.$$

Here the last inequality comes from the uniform bounds on the mean kinetic energy and viscous dissipation of z^ν *from* (4.3)*. Hence the right hand side vanishes as* $\nu \to 0$*. As such our naive approach to the problem which we outlined in Remark 4.4 needs to be adjusted slightly.*

Since the Stokes problem 4.2.1 is linear, to examine how z^ν behaves at the wall it is enough to study how the noise/ external forcing interacts with the wall to introduce vorticity into the fluid.

We now explicitly construct an example of external forcing which is in $[L^2_\sigma(D)]^3$ and blows up along the boundary when convolved with a Gaussian kernel.

Proposition 4.6. *Let $\delta \in (0,1)$, $R > 0$, $D = B(0,R)$, and $\{\mathbf{e}_j\}_{j=1}^3$ be the standard basis in $\mathbb{R}^3$. Let $\operatorname{dist}(x,A) = \inf_{y\in A} \|x-y\|_{\ell^2}$ be the Euclidean distance to set the A. Define*

$$g(x) := \frac{1}{\operatorname{dist}(x,\partial D)^{\delta/2}} \frac{-\sqrt{x_1^2+x_2^2}\mathrm{e}_1 + x_3\mathrm{e}_2}{|x|} \quad x \in D.$$

Then $g \in [L^2(0,T,L^2_\sigma(D))]^3$. Furthermore, for all $c,b,w > 0$ there exists a constant $C_\delta = C(c,b,\delta) > 0$ such that

$$\liminf_{\nu\to 0} \nu^{\delta/2}\mathbf{E}\int_0^T \int_{\partial D} \Big| \int_0^t \int_D \frac{c}{(\nu(t-s))^{3/2}} e^{-b\frac{|x-y|^2}{\nu(t-s)}} e^{-w\nu(t-s)} g(y)dydW_s\Big|^2 \mathcal{H}^2(dx)dt$$
$$\geq C_\delta T^{2-\delta/2} > 0.$$

Proof. Notice that for $x \in B(0,R)$ then $\operatorname{dist}(x,\partial D) = R - |x|$ and $\widehat{\phi}(x) = \frac{-\sqrt{x_1^2+x_2^2}\mathrm{e}_1+x_3\mathrm{e}_2}{|x|}$ is the azimuthal unit vector. As such we can rewrite g is spherical coordinates as

$$g(x) = \frac{1}{(R-r)^\delta}\widehat{\phi}$$

where $|x| = r$. Then using spherical coordinates and the change of variables $R - r \mapsto r$ provides

$$\begin{aligned}\int_0^T \int_D |g|^2 \, dxdt &= \int_0^T \int_0^R \int_{S^2} \frac{r^2}{(R-r)^\delta} \, dS(n)drdt \\ &\leq 4\pi T \int_0^R \frac{(R-r)^2}{r^\delta} dr < \infty\end{aligned}$$

and by applying the definition of divergence in spherical coordinates we can show that g is divergence free:

$$\nabla \cdot g = \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\Big(\frac{1}{(R-r)^{\delta/2}}\Big) = 0.$$

Let $\chi_D(x)$ be the indicator function for the set D and let $x \in \partial D$. Consider the change of variables $\xi = \frac{x-y}{\sqrt{\nu t}}$. Note the Triangle inequality implies that for all $h \in \mathbb{R}^3$

$$|x-h| \leq |x| + |h| = R + |h| \Rightarrow |g(x-h)|^2 \geq \frac{1}{|h|^\delta}.$$

Thus by applying Ito's isometry and our lower bound on $g(x-h)$ we obtain

$$\begin{aligned}
S_\nu(t) &:= \nu^{\delta/2}\mathbf{E}\Big|\int_0^t\int_D \frac{c}{(\nu(t-s))^{3/2}}e^{-b\frac{|x-y|^2}{\nu(t-s)}}e^{w\nu(t-s)}g(y)dydW_s\Big|^2 \\
&= c^2\nu^{\delta/2}\mathbf{E}\Big|\int_0^t\int_{\mathbb{R}^3} e^{-b|\xi|^2}e^{-w\nu(t-s)}g(x-\xi\sqrt{\nu(t-s)})\chi_D(x-\xi\sqrt{\nu(t-s)})\ d\xi dW_s\Big|^2 \\
&= c^2\nu^{\delta/2}\int_0^t\int_{\mathbb{R}^3} |e^{-b|\xi|^2}e^{-w\nu(t-s)}g(x-\xi\sqrt{\nu(t-s)})|^2\chi_D(x-\xi\sqrt{\nu(t-s)})\ d\xi ds \\
&\geq c^2\nu^{\delta/2}\int_0^t\int_{\mathbb{R}^3} \frac{e^{-2b|\xi|^2}e^{-2w\nu(t-s)}}{|\xi|^\delta(\nu(t-s))^{\delta/2}}\chi_D(x-\xi\sqrt{\nu(t-s)})\ d\xi ds.
\end{aligned}$$

Since $D = B(0,R)$ is convex, the radial lines $x - \xi\sqrt{\nu(t-s)}$ remain inside D for ν sufficiently small whenever the angle between the vectors x and ξ is between $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ radians. In other words if $\langle\cdot,\cdot\rangle_2$ is the ℓ^2 inner product in $\mathbb{R}^3$ (i.e. the standard dot product in $\mathbb{R}^3$) then $x - \xi\sqrt{\nu(t-s)} \in D$ for some ν sufficiently small whenever $\langle x, \xi\rangle_2 < 0$. As such, the indicator functions $\chi_D(x-\xi\sqrt{\nu(t-s)}) \to \chi_{\langle\xi,x\rangle_2<0}$ in the sense of distributions as $\nu \to 0$ and for any fixed $x \in \partial D$, in the inviscid limit, the indicator function is supported over exactly half of the unit sphere (in ξ) relative to x. Hence by applying Fatou's Lemma and writing the ξ integral in spherical coordinates, we obtain

$$
\begin{aligned}
&\liminf_{\nu\to 0}\int_0^T\int_{\partial D} S_\nu(t)\ \mathcal{H}^2(dx)dt\\
&\qquad\geq \int_0^T\int_{\partial D}\liminf_{\nu\to 0} S_\nu(t)\ \mathcal{H}^2(dx)dt\\
&\qquad= c^2\int_0^T\int_0^t (t-s)^{-\delta/2}dsdt\int_{S^2}\int_{\{\xi\in\mathbb{R}^3|\langle\xi,x\rangle_2<0\}}\frac{e^{-2b|\xi|^2}}{|\xi|^\delta}\ d\xi dS(x)\\
&\qquad= \frac{c^2T^{2-\delta/2}}{(1-\delta/2)(2-\delta/2)}\int_{S^2}\int_{\{y\in S^2|\langle y,x\rangle_2<0\}}\int_0^\infty e^{-2br^2}r^{2-\delta}\ drdS(y)dS(x)\\
&\qquad= \frac{2\pi^2c^2T^{2-\delta/2}}{(2b)^{(3-\delta)/2}(1-\delta/2)(2-\delta/2)}\Gamma(3-\delta)\\
&\qquad= C_\delta T^{2-\delta/2}>0.
\end{aligned}
$$

□

Proposition 4.7. *For $\delta\in(0,1)$, let $p\delta<1$ and $s\in(0,1)$ such that*

$$\delta-2s+3-\frac{6}{p}>0 \tag{4.5}$$

then $g\in[L^2(0,T,H^s(D))]^3$.

Proof. By the definition of $\widehat{\phi}$, the Triangle inequality, and Young's inequality

$$
\begin{aligned}
|\widehat{\phi}(x)-\widehat{\phi}(y)|^2 &= \Big(\frac{\sqrt{x_1^2+x_2^2}}{|x|}-\frac{\sqrt{y_1^2+y_2^2}}{|y|}\Big)^2+\Big(\frac{x_3}{|x|}-\frac{y_3}{|y|}\Big)^2\\
&\leq \frac{2|y|^2|x-y|^2+2|y|^2|x-y|^2}{|x|^2|y|^2}\\
&= \frac{4|x-y|^2}{|x|^2}
\end{aligned}
$$

Then using the Triangle (and Reverse Triangle) inequality, we have that for all $x, y \in D$

$$\begin{aligned}
|g(x) - g(y)| &= \Big| \frac{\widehat{\phi}(x)}{(R - |x|)^{\delta/2}} - \frac{\widehat{\phi}(y)}{(R - |y|)^{\delta/2}} \Big| \\
&= \Big| \frac{1}{(R - |x|)^{\delta/2}} - \frac{1}{(R - |y|)^{\delta/2}} \Big| + \frac{|\widehat{\phi}(x) - \widehat{\phi}(y)|}{(R - |y|)^{\delta/2}} \\
&\leq \frac{|x - y|^{\delta/2}}{(R - |x|)^{\delta/2}(R - |y|)^{\delta/2}} + \frac{2|x - y|}{|x|(R - |y|)^{\delta/2}}.
\end{aligned}$$

Moreover, for $x, y \in D$ the max difference $|x - y| \leq 2R$ and we can extend g to $\mathbb{R}^3$ as

$$\tilde{g}(x) = \begin{cases} \frac{1}{(R - |x|)^{\delta/2}} & |x| < R \\ \\ 0 & |x| > R \end{cases}.$$

Putting this all together allows us to bound the H^s semi-norm by

$$\begin{aligned}
\int_D \int_D \frac{|g(x) - g(y)|^2}{|x - y|^{3+2s}} dx dy &\leq \int_D \int_D \frac{|x - y|^{\delta}}{|x - y|^{3+2s}(R - |x|)^{\delta}(R - |y|)^{\delta}} \, dy dx \\
&\quad + \int_D \int_D \frac{4}{(R - |y|)^{\delta}|x|^2|x - y|^{1+2s}} \, dx dy \\
&\leq \int_D \frac{1}{(R - |x|)^{\delta}} \int_{|x-y| \leq R} \frac{|x - y|^{\delta}}{|x - y|^{3+2s}} \tilde{g}(y)^2 \, dy dx \\
&\quad + \int_D \frac{1}{(R - |x|)^{\delta}} \int_{R \leq |x-y| \leq 2R} \frac{|x - y|^{\delta}}{|x - y|^{3+2s}} \tilde{g}(y)^2 \, dy dx \\
&\quad + \int_D \int_D \frac{4}{(R - |y|)^{\delta}|x|^2|x - y|^{1+2s}} \, dx dy \\
&= I_1 + I_2 + I_3.
\end{aligned}$$

As $\tilde{g} \in L^2(\mathbb{R}^3)$ it follows quickly that for any $s > 0$ and $\delta \in (0, 1)$

$$I_2 \leq \int_D \frac{1}{(R - |x|)^\delta} \int_{R \leq |x-y| \leq 2R} \frac{(2R)^\delta}{R^{3+2s}} \tilde{g}(y)^2 \, dy dx \leq 2^\delta R^{\delta - 2s - 3} \|g\|^2_{L^2(D)} < \infty.$$

Next to study I_1 we use a technique from [35] to write the inner integral as a convolution which is averaged over a ball of radius R. Let $p > 1$ be chosen such that $p\delta < 1$ and let $q \geq 1$ be its Holder conjugate, i.e. $\frac{1}{p} + \frac{1}{q} = 1$. Then by Young's Convolution inequality for all $r > 0$

$$\|r^{-3} \chi_{B(0,r)} * \tilde{g}^2\|_{L^q(\mathbb{R}^3)} \leq r^{-3} \|\chi_{B(0,r)}\|_{L^w(\mathbb{R}^3)} \|\tilde{g}^2\|_{L^p(\mathbb{R}^3)} = r^{\frac{3-3w}{w}} (\frac{4\pi}{3})^{1/w} \|g^2\|_{L^p(D)}.$$

Here

$$1 + \frac{1}{q} = \frac{1}{p} + \frac{1}{w} = 1 - \frac{1}{q} + \frac{1}{w} \Rightarrow w = \frac{q}{2}$$

and when $p\delta < 1$

$$\|g^2\|^p_{L^p(D)} = \int_D |g|^{2p} \, dx = \int_{S^2} \int_0^R \frac{r^2}{(R - r)^{p\delta}} dr dS(n) \leq 4\pi R^2 \int_0^R u^{-p\delta} du < \infty.$$

Thus by Holder's inequality

$$\begin{aligned}
I_1 &= \int_D \frac{1}{(R - |x|)^\delta} \sum_{n=0}^{\infty} \int_{R2^{-n-1} \leq |x-y| \leq R2^{-n}} \frac{|x - y|^\delta}{|x - y|^{3+2s}} \tilde{g}^2(y) \, dy dx \\
&\leq \int_D \frac{1}{(R - |x|)^\delta} \sum_{n=0}^{\infty} \frac{(R2^{-n})^\delta}{(R2^{-n-1})^{3+2s}} \int_{|x-y| \leq R2^{-n}} \tilde{g}^2(y) \, dy dx \\
&\leq 8 \sum_{n=0}^{\infty} (R2^{-n})^{\delta - 2s} (R2^{-n})^{-3} \int_D g^2(x) \chi_{B(0,R2^{-n})} * \tilde{g}^2(x) \, dx \\
&\leq C \|g^2\|^2_{L^p(D)} \sum_{n=0}^{\infty} (R2^{-n})^{\delta - 2s + \frac{6-3q}{q}}.
\end{aligned}$$

To ensure that this geometric series converges we choose δ, s, p, q such that $p\delta < 1$ and

$$\delta - 2s + \frac{6-3q}{q} = \delta - 2s + \frac{3p-6}{p} > 0.$$

Lastly, we consider the I_3 integral due to differences in the azimuthal angle. It follows by Young's convolution inequality that provided $t \geq 1$ satisfies

$$\frac{1}{p} + \frac{3}{4} + \frac{1}{t} = 2 \quad \text{and} \quad t(1+2s) < 3$$

then

$$\begin{aligned} I_3 &= \int_D \int_D \frac{1}{(R-|y|)^\delta} \frac{1}{|x-y|^{1+2s}} \frac{1}{|x|^2}\, dxdy \\ &\leq \Big(\int_D \frac{1}{(R-|y|)^{p\delta}} dy \Big)^{1/p} \Big(\int_D \frac{1}{|y|^{8/3}} dy \Big)^{3/4} \Big(\int_D \frac{1}{|y|^{t(1+2s)}}\, dy)^{1/t} < \infty. \end{aligned}$$

Importantly we note that it is always possible to choose t such that this is true. We see that by substituting in our relationship for t into 4.5 results in

$$0 < \delta - 2s + 3 - \frac{6}{p} = \delta - 2s - 9 + 6(2 - \frac{1}{p}) = \delta - 2s - \frac{9}{2} + \frac{6}{t}$$

and thus

$$t(1+2s) < t(\delta - \frac{7}{2}) + 6 < 3 \Rightarrow t > \max\{1, \frac{3}{\frac{7}{2} - \delta}\}.$$

The supremum (in δ) of which is $\frac{6}{5}$, so for every value of δ we can choose t sufficiently small so that $t(1+2s) < 3$ which implies I_3 is finite. $\square$

Remark 4.8. *Let* $0 < \varepsilon \ll \frac{1}{2}$ *and define* $s = \frac{1-\delta+2\varepsilon}{2-\delta}$ *then* (4.5) *is satisfied for all* $\delta < \frac{11-\sqrt{41}}{10} \approx 0.45969$.

To see why, note that by substituting in for s *we make a common dominator for the fractions to get*

$$\delta - \frac{2(1-\delta+2\varepsilon)}{2-\delta} + 3 - \frac{6}{p} = \frac{p(\delta-\delta^2+4-4\varepsilon)+6\delta-12}{p(2-\delta)} > 0$$
$$\Rightarrow p > \frac{12-6\delta}{4+\delta-\delta^2-4\varepsilon}.$$

Also, it is assumed that $p\delta < 1$ *so*

$$\frac{12-6\delta}{4+\delta-\delta^2-4\varepsilon} < p < \frac{1}{\delta} \Rightarrow 0 < 4(1-\varepsilon) - 11\delta + 5\delta^2.$$

Which is satisfied for $\delta < \frac{11-\sqrt{121-80(1-\varepsilon)}}{10} < \frac{11-\sqrt{41}}{10} \approx 0.45969$

Now that we have a possible candidate for our noise coloring, let us show that z^ν inherits this behavior at the wall. To avoid confusion, we keep g according to Proposition 4.6 but for simplicity we take $f \equiv 0$.

Theorem 4.9. *Let* g *be as in Proposition 4.6 and assume* $f \equiv 0$. *Then there exists a kernel* $K_t \in [L^\infty(\overline{D} \times \overline{D})]^{3\times 3}$ *for all* $t > 0$ *such that*

$$z^\nu(x,t) = \int_0^t \int_D K_{\nu(t-s)}(x,y) g(y)\, dy dW_s$$

is the unique weak solution to the linear Stokes problem (4.2.1) *and*

$$\liminf_{\nu\to 0} \nu^{\delta/2}\mathbf{E}\int_0^T \|z^\nu\|^2_{L^2(\partial D)}\, dt > 0.$$

Proof. Recall that $B(0,R)$ is of class C^1 and $\alpha \in L^\infty(\partial D)$. Moreover the Stokes operator $A_{NS} = \mathbb{P}(-\Delta)$ is associated with the sesquilinear form $a_{\alpha,\nu} : H^1_{\sigma,\tau}(D) \times H^1_{\sigma,\tau}(D) \to \mathbb{R}$

$$a_{\alpha,\nu}(z,\phi) = \nu\int_{\partial D} \alpha z\cdot\phi\, \mathcal{H}^2(dx) + \nu\int_D \nabla z : \nabla\phi\, dx - \int_D \phi\cdot g dW_t(x).$$

Thus the semigroup generated by A_{NS}: $e^{-tA_{NS}}$, has a kernel $K_t \in [L^\infty(\overline{D}\times\overline{D})]^3$ for all $t > 0$ [78]. As such the solution z^ν can be represented for all $x \in \overline{D}$ and $t > 0$ as

$$z^\nu(x,t) = \int_0^t e^{-\nu(t-s)A_{NS}} g\, dW_s = \int_0^t\int_D K_{\nu(t-s)}(x,y)g(y)\, dW_s(y).$$

Moreover, as A_{NS} is a self-adjoint operator, K_t also satisfies the lower Gaussian bound [78]: for all $x, y \in \overline{D}$ and $t > 0$ there exists $b, c, w > 0$ such that

$$K_t(x,y) \geq ct^{-3/2}e^{-b\frac{|x-y|^2}{t}}e^{-wt} =: H_t(x,y).$$

Then as $g \geq 0$ and $K_t, H_t \geq 0$

$$\mathbf{E}\Big|\int_0^t\int_D K_{\nu(t-s)}(x,y)g(y)dydW_s\Big|^2 \geq \mathbf{E}\Big|\int_0^t\int_D H_{\nu(t-s)}(x,y)g(y)dydW_s\Big|^2.$$

Therefore, by Proposition 4.6

$$\begin{aligned}\liminf_{\nu\to 0}\nu^{\delta/2}\mathbf{E}\int_0^T\|z^\nu\|^2_{L^2(\partial D)} &= \liminf_{\nu\to 0}\nu^{\delta/2}\mathbf{E}\int_0^T\int_{\partial D}\Big|\int_0^t\int_D K_{\nu(t-s)}(x,y)g(y)\ dydW_s\Big|^2 d\mathcal{H}^2(x)dt\\ &\geq \liminf_{\nu\to 0}\nu^{\delta/2}\mathbf{E}\int_0^T\int_{\partial D}\Big|\int_0^t\int_D H_{\nu(t-s)}(x,y)g(y)\ dydW_s\Big|^2 d\mathcal{H}^2(x)dt\\ &\geq C_\delta T^{2-\delta/2}>0.\end{aligned}$$

□

Remark 4.10. *While we have shown that the kinetic energy at the wall can be made to blow up like $\nu^{\delta/2}$ we cannot apply the same arguement as in Remark 4.4 since its blow up rate is too slow for the interpolation between $L^2(D)$ and $H^1(D)$. Instead we amend this approach to instead interpolate between H^s and H^1 for an appropriate choice of s.*

Proposition 4.11. *Let z^ν_δ be the solution to the Stokes problem under the same assumptions as in Theorem 4.9. Let $\delta<\frac{11-\sqrt{41}}{10}$ and $0<\varepsilon\ll\frac{\delta}{4}$ then $\gamma=\frac{1-\delta+2\varepsilon}{2-\delta}\in(\frac{1-\delta}{2-\delta},\frac{1}{2})$ and*

$$\sup_{\nu\in(0,1)}\mathbf{E}\int_0^T\|z^\nu_\delta\|_{H^\gamma(D)}<\infty.$$

Proof. By Ito's Isometry

$$\mathbf{E}\int_0^T\|z^\nu_\delta\|^2_{H^\gamma(D)}=\mathbf{E}\int_0^T\int_0^t\|e^{-\nu(t-s)A_{NS}}g\|^2_{H^\gamma(D)}\ dsdt.$$

Then since $B(0,R)$ is C^∞, it follows that the Stokes semigroup $e^{-\nu tA_{NS}}$ is C_0 [78] and thus there exists constants $\omega_0\geq 0$ and $M\geq 1$ such that $\|e^{-\nu tA_{NS}}\|\leq Me^{\nu\omega_0 t}$. As $T<\infty$ it follows from

Remark 4.8 that

$$\sup_{\nu\in(0,1)} \mathbf{E}\int_0^T \|z_\delta^\nu\|^2_{H^\gamma(D)} \leq \sup_{\nu\in(0,1)} M\mathbf{E}\int_0^T\int_0^t e^{\omega_0(t-s)}\|g\|^2_{H^\gamma(D)}\, dsdt < \infty.$$

□

Theorem 4.12. *Under the same assumpations as Theorem 4.9 with $\delta < \frac{11-\sqrt{41}}{10}$, if z_δ^ν is the martingale solution to (4.2), then*

$$\limsup_{\nu\to 0} \nu\mathbf{E}\int_0^T \|\nabla z_\delta^\nu\|^2_{L^2(D)} > 0.$$

Proof. **Step 1:** Let $0 < \varepsilon \ll \frac{\delta}{2}$. Define $s_\delta = \frac{1-\delta+2\varepsilon}{2-\delta}$. By Proposition 4.11

$$\sup_{\nu\in(0,1)} \mathbf{E}\int_0^T \|z_\delta^\nu\|^2_{H^{s_\delta}(D)} < \infty.$$

Step 2: As $D = B(0,R)$ is C^1, there exists a bounded linear trace operator $\mathrm{tr} : [H^{1/2+\varepsilon}(D)]^3 \to [L^2(\partial D)]^3$ [2, 73]. Moreover, the Sobolev space $H^{1/2+\varepsilon}(D)$ is an interpolation space for $H^{s_\delta}(D)$ and $H^1(D)$ with interpolation exponent $\theta \in (0,1)$ given by [2, 80]

$$\frac{1}{2} + \varepsilon = (1-\theta)s_\delta + \theta \Rightarrow \theta = \frac{\delta}{2}.$$

Therefore, after applying Holder's inequality with conjugates $p = \frac{2}{\delta}$ and $q = \frac{2}{2-\delta}$ we get

$$\begin{aligned}
\mathbf{E}\int_0^T \|z_\delta^\nu\|^2_{L^2(\partial D)} &\leq C\mathbf{E}\int_0^T \|z_\delta^\nu\|^2_{H^{1/2+\varepsilon}(D)} \\
&\leq C\Big(\mathbf{E}\int_0^T \|z_\delta^\nu\|^2_{H^{s_\delta}(D)}\Big)^{\frac{2-\delta}{2}}\Big(\mathbf{E}\int_0^T \|z_\delta^\nu\|^2_{H^1(D)}\Big)^{\frac{\delta}{2}}.
\end{aligned} \tag{4.6}$$

Step 3: It then follows from Theorem 4.9 and (4.6) that

$$\begin{aligned}
0 &< \limsup_{\nu\to 0} \nu^{\frac{\delta}{2}}\mathbf{E}\int_0^T \|z_\delta^\nu\|^2_{L^2(\partial D)}\, dt \\
&\leq \limsup_{\nu\to 0} C\Big(\mathbf{E}\int_0^T \|z_\delta^\nu\|^2_{H^{s_\delta}(D)}\Big)^{\frac{2-\delta}{2}}\Big(\nu\mathbf{E}\int_0^T \|z_\delta^\nu\|^2_{H^1(D)}\Big)^{\frac{\delta}{2}} \\
&\leq C\limsup_{\nu\to 0}\Big(\nu\mathbf{E}\int_0^T \|\nabla z_\delta^\nu\|^2_{L^2(D)}\, dt\Big)^{\frac{\delta}{2}}.
\end{aligned}$$

Here the last inequality is due to the uniform bound on the kinetic energy in (4.3) so

$$\begin{aligned}
\limsup_{\nu\to 0}\nu\mathbf{E}\int_0^T \|z_\delta\|^2_{H^1(D)} &= \limsup_{\nu\to 0}\nu\mathbf{E}\int_0^T \|z_\delta\|^2_{L^2(D)} + \limsup_{\nu\to 0}\nu\mathbf{E}\int_0^T \|\nabla z_\delta\|^2_{L^2(D)} \\
&= \limsup_{\nu\to 0}\nu\mathbf{E}\int_0^T \|\nabla z_\delta\|^2_{L^2(D)}.
\end{aligned}$$

□

Remark 4.13. *Here we were able to construct a linear problem with a non-vanishing viscous dissipation term by concentrating the kinetic energy at the wall. This shows that even for non-statistically stationary solutions* (1.2) *is not a good measure of how the nonlinearity contributes to the increase in small scale oscillations and accelerates viscous dissipation within a turbulent flow. In part, this is because* (1.2) *measures the total enstrophy in the system and not how the convective acceleration speeds up dissipation within the interior* [16]. *As such if we choose our* *forcing appropriately we can create a sufficient amount of vorticity at the boundary so that the average total enstrophy in the system blows up at its max possible rate of* $O(\nu^{-1})$.

Remark 4.14. *Previously Bedrossian et.al.* [9] *showed that global anomalous dissipation could be achieved by statistically stationary solutions to the heat equation where nothing "anomalous" is actually occurring. This pointed to the fact that* (1.2) *is not a good definition for saying a fluid exhibits anomalous dissipation in the context of stochastic flows. Similarly, we have constructed an example of a solution to the linear Stokes problem which satisfies* (1.2) *but is not statistically stationary which is a complementary result. However, our approach can also be extended to the case of deterministic solutions by choosing* f *to blow up at the boundary and setting* $g \equiv 0$*. Hence* (1.2) *is not a good definition for saying a fluid exhibits anomalous dissipation if one only requires the external forcing to be* L^2.

Remark 4.15. *In order to account for the issue of statistically stationary solutions Bedrossian et.al.* [9] *suggested the concept of a solution satisfying what they call as weak anomalous dissipation where*

$$\lim_{\nu \to 0} \nu \mathbf{E} \|u^\nu(t)\|^2_{L^2(D)} = 0 \quad \forall t \in [0, T].$$

However by the uniform bounds on the kinetic energy we see that both u^ν *and* z^ν *satisfy this definition as well meaning this does not capture the role of the nonlinearity to accelerate the amount of dissipation that occurs within the interior of the flow. Instead we posit that anomalous dissipation should be measured as a local phenomena instead of a global one.*

One possible local approach would be that instead of using (1.2)*, for a non-statistically* *stationary fluid to truly exhibit anomalous dissipation it should hold that*

$$\limsup_{\nu\to 0} \nu \mathbf{E} \int_0^T \|\nabla u^\nu\|^2_{L^2_{loc}(D)} > 0.$$

This does not work in the case of statistically stationary solutions due to the heat equation (again) satisfying this bound without anything anomalous occurring. As such we expect that one can similarly construct non-statistically stationary solutions using the techniques here which also satisfy this requirement possibly by using g like we defined here but it concentrates energy on concentric shells with radii from a countable dense set subset of $(0, \infty)$.

Another (local) approach, is to just measure how much the nonlinearity contributes to the dissipation of energy using the regularity measure of Duchon and Robert[25]*:*

$$\limsup_{\nu\to 0} \mathbf{E} \int_0^T \int_U D(u^\nu)(dx)dt > 0$$

where

$$D(u^\nu)(x) = \lim_{\ell\to 0} \frac{1}{4} \int_D \nabla\phi_\ell(y) \cdot \delta_y u^\nu(x) |\delta_y u^\nu(x)|^2 \, dy \quad \forall\, x \in D.$$

Here ϕ_ℓ is a standard mollifier of size ℓ and $\delta_y u^\nu(x) = u^\nu(x+y) - u^\nu(x)$. See Theorem 3.14 for how $D(u^\nu)$ acts as a measure of the dissipation due to the convective term in the energy balance over a torus.

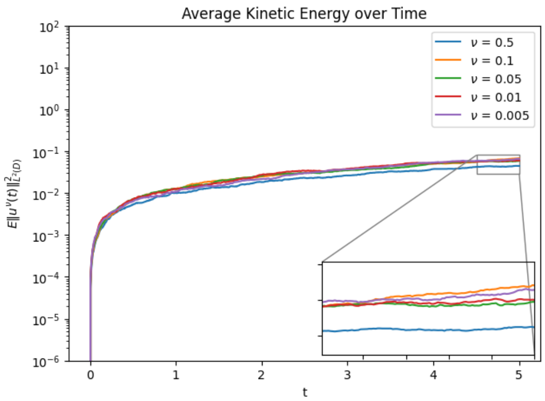
Remark 4.16. *By construction the noise $g dW_t$ is concentrated at the boundary. This is strictly different from Kolmogorov's theory of turbulence which assumes that the external forcing on the system is confined to only the largest length scales of the problem (i.e. within the interior of the domain). Nevertheless one can still analyze the structure functions over the interior such as in*

[60] *to examine the impact of the nonlinearity on the energy dynamics.*

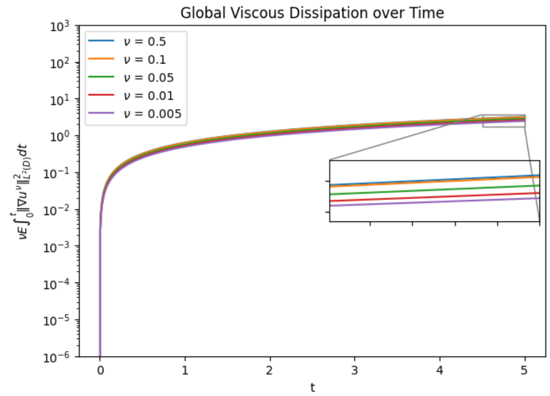
Remark 4.17. *While we have shown that the existence of (global) anomalous dissipation for a system subject to an arbitrary choice of $[L^2(D)]^3$ forcing and initial conditions, is not well defined as a nonlinear phenomena, one could alternatively redefine the problem to be subject to $[L^2(D)]^3$ forcing which is bounded over the closure of D, like in Kolmogorov's theory of turbulence. In this way the forcing will only exist over the largest length scales of the problem and (possibly) cannot be concentrated on sets of Lebesgue measure 0. In this case, the nonlinearity will (possibly) be the driving source of vorticity generation and once again allow* (1.2) *to be a good definition of anomalous dissipation. This problem remains open, however we note that vorticity will still be generated at the wall so one will need to account for how this affects the existence of global anomalous dissipation.*

4.3 Simulation Examples

In order to confirm the results from Section 4.2 we will simulate the Linear Stokes problem over a semi-infinite plate and inside a sphere and measure the amount of viscous dissipation as $\nu \to 0$. Throughout this section we implement a finite difference approach while evolving in time using the Euler-Maruyama method (a forward Euler approach in the context of deterministic flows) for simplicity. In order to ensure that the dissipation is fully resolved we take the spatial step size on the order of the Kolmogorov length scale $dy = \nu^{3/4}$ and temporal step size of $dt = 0.005$ (smaller than the Kolmogorov time scale for all values of ν we consider here and is compatible with the CFL condition). Finally for all stochastic simulations we average the results over 250 different realizations of the flow.



(a) Average Kinetic Energy



(b) Average Dissipation

Figure 4.1: The Average Kinetic Energy and Viscous Dissipation above an infinite plate

First we consider the numerically simplest case: an incompressible fluid above an infinite stationary plate contained within the plane $y = 0$. We will also assume that the flow is symmetric with respect to both the x and z axes and decays to 0 as $y \to \infty$. We take our coloring to be $g(x, y, z) = [0, 0, \frac{1}{y^{\delta/2}}]$ with the initial condition of $z_0 \equiv 0$. For now we will fix $\delta = 0.75$. Later we will check to see how the various quantities of interest change with δ. For the actual simulation space we will take the domain to have a height of $y_{max} = 10$, and we fix the slip length $\alpha = 0.0005$.

Due to the symmetry assumption in this setting we can reduce the full 3D problem to a 1D problem which is perpendicular to the plate greatly simplifying the numerical complexity of the problem. In Figure 4.1, we show that both the mean global kinetic energy and global viscous dissipation remain roughly constant as $\nu \to 0$. This agrees with our earlier analysis that along the boundary the kinetic energy is blowing up as $\nu \to 0$, so even through the energy within the interior of the flow is uniformly bounded with respect to ν, the viscous dissipation does not vanish as vorticity continues to be created from the singularity in the noise at the boundary. See how the mean kinetic energy at the boundary remains roughly constant in Figure 4.2.

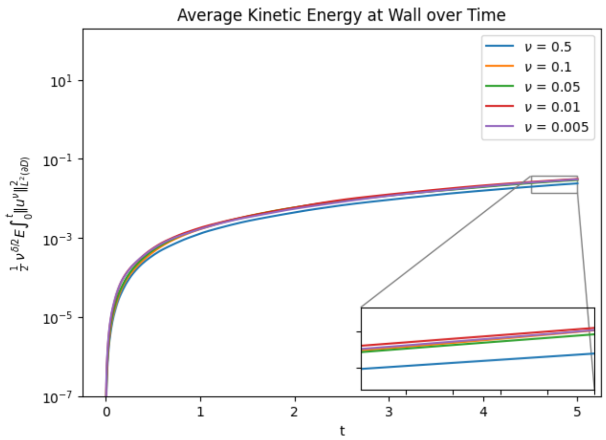
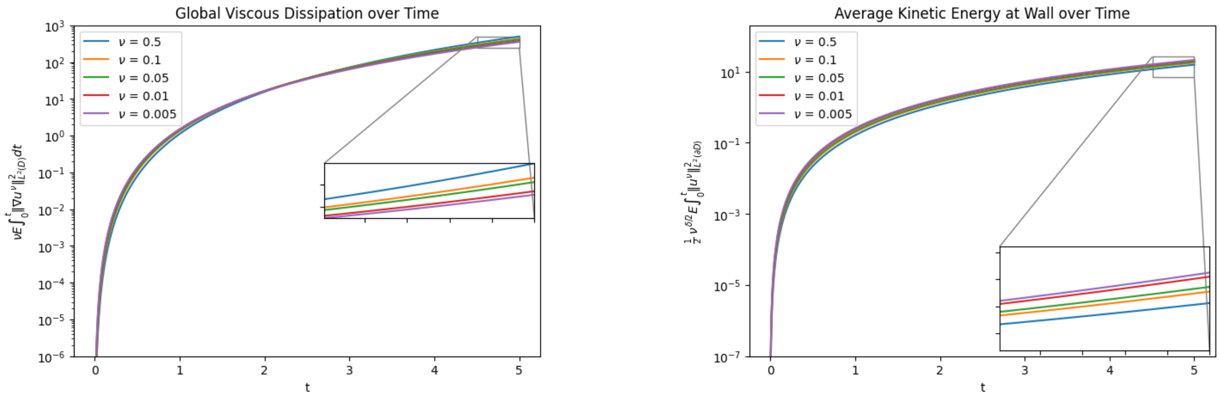


Figure 4.2: The Average Kinetic Energy at the Plate

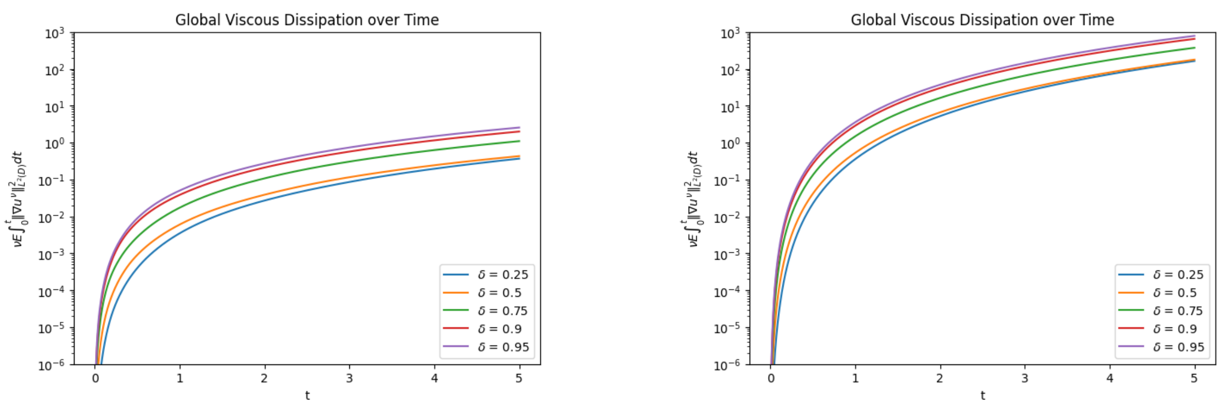
We note that a similar problem to what we have analyzed so far is Stokes' 1st problem where we take a no-slip condition at the boundary and allow the plate to oscillate with fixed speed U. This is analogous to the problem we have analyzed as the forcing is concentrated at the boundary and a Brownian motion is normally distributed about 0 so in our case the noise gdW_t may flip signs with every time step. However we note that the creation of anomalous dissipation and the blow up of the kinetic energy are not due to the oscillations but instead due to the singularity of g at the boundary. Indeed, if we instead look at the deterministic system with external forcing still given as $f(x,y,z) = [0, 0, \frac{1}{y^{\delta/2}}]$ we still see the existence of global anomalous dissipation and a blow up in the kinetic energy at the boundary. See Figure 4.3. Moreover, looking at the various quantities for the deterministic and stochastic settings, we see that the oscillations at the boundary actually reduce how much energy can accumulate at the boundary and how much vorticity can be introduced into the bulk flow. Most likely this is because in the deterministic setting, the forcing pushes the flow in only one direction, while in the stochastic case the forcing is able to change directions — meaning that in the deterministic setting the fluid velocity at the boundary increases much faster due to a mono-directional forcing while the oscillatory-like



(a) Global Viscous Dissipation

(b) Kinetic Energy at the Wall

Figure 4.3: The Total (Global) Viscous Dissipation above an infinite plate in a Deterministic System and Energy at the Wall



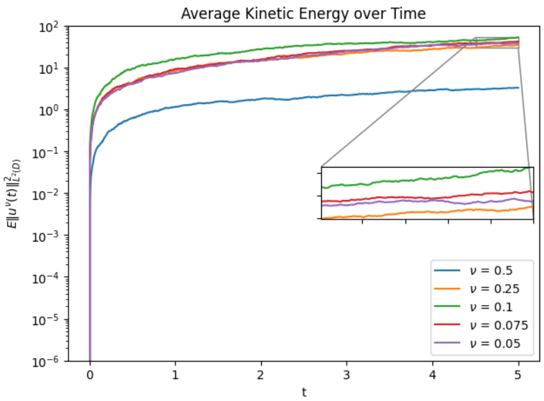
(a) Average Dissipation in Stochastic System

(b) Dissipation in Deterministic System

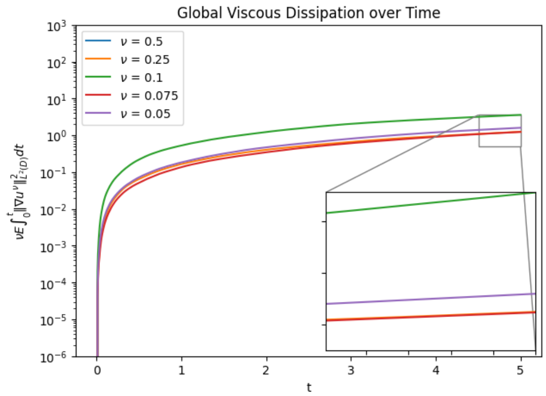
Figure 4.4: The Average Viscous Dissipation above an infinite plate in a Stochastic and a Deterministic System as $\delta \to 1^-$

forcing from the stochastic case limits how fast the velocity at the boundary can grow.

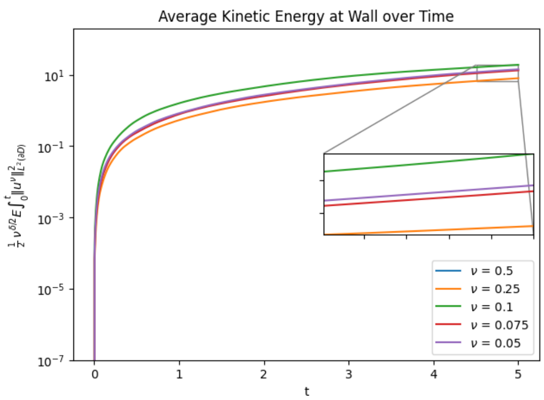
So far we have fixed δ and seen how the quantities of interest change with ν. Moreover the choice of δ used is outside of the values we used in Theorem 4.12, however this is inconsequential as can be seen in Figure 4.4 both the stochastic and deterministic systems behave the same for each value of δ with only the total amount of viscous dissipation over time increases with the strength of the singularity at the wall. This suggests that Theorem 4.12 should be able to be extended to all $\delta \in (0, 1)$ but the validity of which remains open.



(a) Average Kinetic Energy



(b) Average Viscous Dissipation



(c) Average Kinetic Energy at the Wall

Figure 4.5: The Average Viscous Dissipation and Kinetic Energy over a Sphere of Radius 5

Next we consider the same (Stokes) problem but over a sphere of radius $R = 5$ instead of in a flat semi-infinite domain. We do this first to confirm the results from Theorem 4.12 as well as to check if the curvature / symmetry assumptions we used in the previous simulation are impacting the results. Note that in this setting the coloring on the noise is given by $g(r, \theta, \phi) = [0, 0, \frac{1}{(5-r)^{\delta/2}}]$.

Once again we use a finite difference approach to compute the solution z^ν but now we take the mesh sizing in the r, θ, ϕ directions to all be on the size of the Kolmogorov scale $\nu^{3/4}$. Since this mesh sizing is in all 3 dimensions, the cost of each time step has greatly increased compared to the halfplane case where the symmetry assumptions reduced the problem to 1 dimension.

Moreover we impost the (implicit) assumption used throughout that at the origin z^ν is bounded. In practice this was done by saying that at the origin z^ν is equal to the average of z^ν over the smallest shell containing the origin.

In order to compensate for the extra computational difficulties, in this case we restrict ν to a smaller range of values: $\nu = [0.5, 0.25, 0.1, 0.075, 0.05]$. Nevertheless, we still observe that once again the solution to the linear Stokes problem z^ν exhibits a blow up in the kinetic energy at the wall as well as satisfies the global anomalous dissipation assumption. See Figure 4.5. However when comparing the spherical case in Figure 4.5 with the infinite plate case in 4.1 and 4.2 we see that the curvature of the domain causes the kinetic energy at the boundary to blow up much faster. Moreover, it causes the total amount of viscous dissipation and kinetic energy to increase. Most likely this is because in the bounded domain, any oscillations that do not dissipate out near the origin are eventually reflected back at the wall and the curvature makes the newly created oscillations which are moving into the interior to collide with one another leading to the waves amplifying one another.

4.4 Future Directions

This work has shown that (1.2) is not a good definition for how nonlinear affects contribute to anomalous dissipation even in the non-statistically stationary/ deterministic setting when the external forcing is taken to be only L^2. But it remains open whether requiring the external forcing to be in L^∞ could fix this issue. Another area of interest is to consider the converse: what are sufficient conditions such that solutions to the Navier Stokes equations u^ν behave well-enough such that

$$\limsup_{\nu \to 0} \nu \mathbf{E} \int_0^t \|\nabla u^\nu\|_{L^2(D)}^2 = 0.$$

One possible approach is to examine the techniques used in [24] for the no-slip boundary condition to the slip boundary condition where certain quantities may be easier to work with. Another interesting approach is that Bourgain, Brezis, and Mironescu showed that for Lipschitz domains D and $h \in [H^1(D)]^3$

$$\lim_{s \to 1^-} (1-s) \int_D \int_D \frac{|h(x) - h(y)|^2}{|x-y|^{3+2s}} \, dxdy = \frac{2\pi}{3} \|\nabla h\|_{L^2(D)}^2$$

Hence if one can find a condition so that this point-wise limit in the H^s semi-norms is uniform, then using interpolation theory

$$\limsup_{\nu \to 0^+} \nu \mathbf{E} \int_0^T \|\nabla u^\nu\|_{L^2(D)}^2 = \lim_{s \to 1^-} \limsup_{\nu \to 0^+} (1-s) \nu \mathbf{E} \int_0^T \|u^\nu\|_{\dot{H}^s(D)}^2 = 0.$$

Bibliography

[1] Milton Abramowitz, Irene A Stegun, and Robert H Romer, *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, American Association of Physics Teachers, 1988.

[2] Robert A Adams and John JF Fournier, *Sobolev spaces*, Elsevier, 2003.

[3] Chérif Amrouche, Miguel Escobedo, and Amrita Ghosh, *Semigroup theory for the stokes operator with navier boundary condition on lp spaces*, Springer International Publishing, Cham, 2021.

[4] Scott Armstrong and Vlad Vicol, *Anomalous diffusion by fractal homogenization*, arXiv preprint arXiv:2305.05048 (2023).

[5] Claude Bardos, François Golse, and Yoshio Sone, *Half-space problems for the boltzmann equation: a survey*, Journal of statistical physics **124** (2006), no. 2, 275–300.

[6] George K Batchelor, *Computation of the energy spectrum in homogeneous two-dimensional turbulence*, The Physics of Fluids **12** (1969), no. 12, II–233.

[7] Jacob Bedrossian, Alex Blumenthal, and Sam Punshon-Smith, *Lagrangian chaos and scalar advection in stochastic fluid mechanics*, Journal of the European Mathematical Society **24** (2022), no. 6, 1893–1990.

[8] Jacob Bedrossian, Michele Coti Zelati, Sam Punshon-Smith, and Franziska Weber, *Sufficient conditions for dual cascade flux laws in the stochastic 2d navier–stokes equations*, Archive for Rational Mechanics and Analysis **237** (2020), no. 1, 103–145.

[9] Jacob Bedrossian, Michele Coti Zelati, Samuel Punshon-Smith, and Franziska Weber, *A sufficient condition for the kolmogorov 4/5 law for stationary martingale solutions to the 3d navier–stokes equations*, Communications in Mathematical Physics **367** (2019), no. 3, 1045–1075.

[10] Alain Bensoussan and Roger Temam, *Equations stochastiques du type navier-stokes*, Journal of Functional Analysis **13** (1973), no. 2, 195–222.

[11] Luca Biferale, Stefano Musacchio, and Federico Toschi, *Inverse energy cascade in three-dimensional isotropic turbulence*, Physical review letters **108** (2012), no. 16, 164501.

[12] Guido Boffetta and S Musacchio, *Evidence for the double cascade scenario in two-dimensional turbulence*, Physical Review E **82** (2010), no. 1, 016307.

[13] Elia Bruè and Camillo De Lellis, *Anomalous dissipation for the forced 3d navier–stokes equations*, Communications in Mathematical Physics **400** (2023), no. 3, 1507–1533.

[14] Tristan Buckmaster and Vlad Vicol, *Nonuniqueness of weak solutions to the navier-stokes equation*, Annals of Mathematics **189** (2019), no. 1, 101–144.

[15] Luis Caffarelli, Robert Kohn, and Louis Nirenberg, *Partial regularity of suitable weak solutions of the navier-stokes equations*, Communications on pure and applied mathematics **35** (1982), no. 6, 771–831.

[16] Gui-Qiang Chen and Zhongmin Qian, *A study of the navier-stokes equations with the kinematic and navier boundary conditions*, Indiana University mathematics journal (2010), 721–760.

[17] Robin Ming Chen, Zhilei Liang, and Dehua Wang, *A kato-type criterion for vanishing viscosity near onsager's critical regularity*, Archive for Rational Mechanics and Analysis **246** (2022), no. 2, 535–559.

[18] Shiyi Chen, Robert E Ecke, Gregory L Eyink, Michael Rivera, Minping Wan, and Zuoli Xiao, *Physical mechanism of the two-dimensional inverse energy cascade*, Physical review letters **96** (2006), no. 8, 084502.

[19] Peter Constantin, Weinan E, and Edriss S Titi, *Onsager's conjecture on the energy conservation for solutions of euler's equation*, Communications in Mathematical Physics (1994).

[20] François Coron, François Golse, and Catherine Sulem, *A classification of well-posed kinetic layer problems*, Communications on Pure and Applied Mathematics **41** (1988), no. 4, 409–435.

[21] Giuseppe Da Prato and Jerzy Zabczyk, *Stochastic equations in infinite dimensions*, Vol. 152, Cambridge university press, 2014.

[22] Theodore De Karman and Leslie Howarth, *On the statistical theory of isotropic turbulence*, Proceedings of the Royal Society of London. Series A-Mathematical and Physical Sciences **164** (1938), no. 917, 192–215.

[23] Theodore D Drivas, *Self-regularization in turbulence from the kolmogorov 4/5-law and alignment*, Philosophical Transactions of the Royal Society A **380** (2022), no. 2226, 20210033.

[24] Theodore D Drivas and Huy Q Nguyen, *Onsager's conjecture and anomalous dissipation on domains with boundary*, SIAM Journal on Mathematical Analysis **50** (2018), no. 5, 4785–4811.

[25] Jean Duchon and Raoul Robert, *Inertial energy dissipation for weak solutions of incompressible euler and navier-stokes equations*, Nonlinearity **13** (2000), no. 1, 249.

[26] Ethan Dudley, *Necessary and sufficient conditions for kolmogorov's flux laws on* $\mathbb{T}^2$ *and* $\mathbb{T}^3$, Nonlinearity **37** (2024), no. 9.

[27] Dietrich Einzel, Peter Panzer, and Mario Liu, *Boundary condition for fluid flow: curved or rough surfaces*, Physical review letters **64** (1990), no. 19, 2269.

[28] Gregory L Eyink, *Energy dissipation without viscosity in ideal hydrodynamics i. fourier analysis and local energy transfer*, Physica D: Nonlinear Phenomena **78** (1994), no. 3-4, 222–240.

[29] ———, *Exact results on stationary turbulence in 2d: consequences of vorticity conservation*, Physica D: Nonlinear Phenomena **91** (1996), no. 1-2, 97–142.

[30] Ragnar Fjørtoft, *On the changes in the spectral distribution of kinetic energy for twodimensional, nondivergent flow*, Tellus **5** (1953), no. 3, 225–230.

[31] Franco Flandoli and Dariusz Gatarek, *Martingale and stationary solutions for stochastic navier-stokes equations*, Probability Theory and Related Fields **102** (1995), no. 3, 367–391.

[32] Uriel Frisch and Andreĭ Nikolaevich Kolmogorov, *Turbulence: the legacy of an kolmogorov*, Cambridge university press, 1995.

[33] David Gérard-Varet and Nader Masmoudi, *Relevance of the slip condition for fluid flows near an irregular boundary*, Communications in Mathematical Physics **295** (2010), no. 1, 99–137.

[34] W.I. Goldburg, M.A. Rutgers, and X. l. Wu, *Experiments on turbulence in soap films*, Physica A: Statistical Mechanics and its Applications **239** (1997), no. 1, 340–349.

[35] Lars Inge Hedberg, *On certain convolution inequalities*, Proceedings of the American Mathematical Society **36** (1972), no. 2, 505–510.

[36] Atsushi Inoue and Tadahisa Funaki, *On a new derivation of the navier-stokes equation*, Communications in Mathematical Physics **65** (1979), 83–90.

[37] Philip Isett, *A proof of onsager's conjecture*, Annals of Mathematics **188** (2018), no. 3, 871–963.

[38] Willi Jäger and Andro Mikelić, *Couette flows over a rough boundary and drag reduction*, Communications in Mathematical Physics **232** (2003), no. 3, 429–455.

[39] Javier Jiménez, *Intermittency in turbulence*, Proc. 15th "aha huliko" a winter workshop, extreme events, 2007, pp. 81–90.

[40] Tosio Kato, *Nonstationary flows of viscous and ideal fluids in* R^3, Journal of functional Analysis **9** (1972), no. 3, 296–305.

[41] ______, *Remarks on zero viscosity limit for nonstationary navier-stokes flows with boundary*, Seminar on nonlinear partial differential equations, 1984, pp. 85–98.

[42] Elliot A Kearsley and Jeffrey T Fong, *Linearly independent sets of isotropic cartesian tensors of ranks up to eight*, Journal of Research of the National Bureau of Standards (1975), 49–58.

[43] A.N. Kolmogorov, *Dissipation of energy in the locally isotropic turbulence*, Dokl. akad. nauk. sssr, 1941, pp. 19–21.

[44] ______, *The local structure of turbulence in incompressible viscous fluid for very large reynolds numbers*, C. R. (Doklady) Academy of Science URSS (N.S.) **30** (1941), 301–305.

[45] ______, *On degeneration of isotropic turbulence in an incompressible viscous liquid*, Dokl. akad. nauk. sssr, 1941, pp. 538–540.

[46] Robert H Kraichnan, *Inertial ranges in two-dimensional turbulence*, The Physics of Fluids **10** (1967), no. 7, 1417–1423.

[47] Sergei Kuksin and Armen Shirikyan, *Mathematics of two-dimensional turbulence*, Vol. 194, Cambridge University Press, 2012.

[48] Cecil E Leith, *Diffusion approximation for two-dimensional turbulence*, The Physics of Fluids **11** (1968), no. 3, 671–672.

[49] Jean Leray, *Sur le mouvement d'un liquide visqueux emplissant l'espace*, Acta mathematica **63** (1934), 193–248.

[50] Yasunori Maekawa, *On the inviscid limit problem of the vorticity equations for viscous incompressible flows in the half-plane*, Communications on Pure and Applied Mathematics **67** (2014), no. 7, 1045–1128.

[51] Jonathan Christopher Mattingly, *The stochastic navier-stokes equation: Energy estimates and phase space contraction*, Princeton University, 1998.

[52] Andro Mikelic and Willi Jäger, *On the interface boundary condition of beavers, joseph, and saffman*, SIAM Journal on Applied Mathematics **60** (2000), no. 4, 1111–1127.

[53] A.S. Monin and A. M. Yaglom, *Statistical fluid mechanics, volume ii: mechanics of turbulence*, Vol. 2, Courier Corporation, 2013.

[54] Bruce R Morton, *The generation and decay of vorticity*, Geophysical & Astrophysical Fluid Dynamics **28** (1984), no. 3-4, 277–308.

[55] Stefano Musacchio and Guido Boffetta, *Split energy cascade in turbulent thin fluid layers*, Physics of Fluids **29** (2017), no. 11, 111106.

[56] Claude Navier, *Mémoire sur les lois du mouvement des fluides*, Vol. 1, Mémoires de l'Académie Royale des Sciences de l'Institut de France, 1816.

[57] Jiří Neustupa and Patrick Penel, *On regularity of a weak solution to the navier–stokes equations with the generalized navier slip boundary conditions*, Advances in Mathematical Physics **2018** (2018), no. 1, 4617020.

[58] Matthew Novack and Vlad Vicol, *An intermittent onsager theorem*, Inventiones mathematicae **233** (2023), no. 1, 223–323.

[59] Lars Onsager, *Statistical hydrodynamics*, Il Nuovo Cimento (1943-1954) **6** (1949), no. Suppl 2, 279–287.

[60] Stavros Papathanasiou, *Sufficient conditions for local scaling laws for stationary martingale solutions to the 3d navier–stokes equations*, Nonlinearity **34** (2021), no. 5, 2937.

[61] Jérôme Paret and Patrick Tabeling, *Experimental observation of the two-dimensional inverse energy cascade*, Physical review letters **79** (1997), no. 21, 4162.

[62] Bruce R Pearson, P-Å Krogstad, and Willem van de Water, *Measurements of the turbulent energy dissipation rate*, Physics of fluids **14** (2002), no. 3, 1288–1290.

[63] Sergio Pirozzoli, Matteo Bernardini, and Francesco Grasso, *On the dynamical relevance of coherent vortical structures in turbulent boundary layers*, Journal of fluid mechanics **648** (2010), 325–349.

[64] Franck Plunian, Andrei Teimurazov, Rodion Stepanov, and Mahendra Kumar Verma, *Inverse cascade of energy in helical turbulence*, Journal of Fluid Mechanics **895** (2020), A13.

[65] Ludwig Prandtl, *Motion of fluids with very little viscosity*, 1928.

[66] Lewis F Richardson, *Weather prediction by numerical process*, University Press, 1922.

[67] Marco Romito, *Existence of martingale and stationary suitable weak solutions for a stochastic navier–stokes system*, Stochastics: An International Journal of Probability and Stochastics Processes **82** (2010), no. 3, 327–337.

[68] Maarten A Rutgers, *Forced 2d turbulence: experimental evidence of simultaneous inverse energy and forward enstrophy cascades*, Physical review letters **81** (1998), no. 11, 2244.

[69] Marco Sammartino and Russel E Caflisch, *Zero viscosity limit for analytic solutions, of the navier-stokes equation on a half-space.ℙ I. existence for euler and prandtl equations*, Communications in mathematical physics **192** (1998), no. 2, 433–461.

[70] ______, *Zero viscosity limit for analytic solutions of the navier-stokes equation on a half-space.ℙ II. construction of the navier-stokes solution*, Communications in mathematical physics **192** (1998), no. 2, 463–491.

[71] DA Schaffner and Michael R Brown, *Multifractal and monofractal scaling in a laboratory magnetohydrodynamic turbulence experiment*, The Astrophysical Journal **811** (2015), no. 1, 61.

[72] Cornelia Schneider, *Trace operators in besov and triebel–lizorkin spaces*, Zeitschrift für Analysis und ihre Anwendungen **29** (2010), no. 3, 275–302.

[73] ______, *Traces of besov and triebel-lizorkin spaces on domains*, Mathematische Nachrichten **284** (2011), no. 5-6, 572–586.

[74] Katepalli R Sreenivasan, *On the scaling of the turbulence energy dissipation rate*, The Physics of fluids **27** (1984), no. 5, 1048–1051.

[75] Joseph Sternberg, *A theory for the viscous sublayer of a turbulent flow*, Journal of Fluid Mechanics **13** (1962), no. 2, 241–271.

[76] Franck Sueur, *On the inviscid limit for the compressible navier–stokes system in an impermeable bounded domain*, Journal of Mathematical Fluid Mechanics **16** (2014), no. 1, 163–178.

[77] Geoffrey Ingram Taylor, *Statistical theory of turbulence-ii*, Proceedings of the Royal Society of London. Series A-Mathematical and Physical Sciences **151** (1935), no. 873, 444–454.

[78] AFM Ter Elst and MF Wong, *Hölder kernel estimates for robin operators and dirichlet-to-neumann operators*, Journal of Evolution Equations **20** (2020), no. 3, 1195–1225.

[79] Hatem Touil, Jean-Pierre Bertoglio, and Liang Shao, *The decay of turbulence in a bounded domain*, Journal of Turbulence **3** (2002), no. 1, 049.

[80] Hans Triebel, *Spaces of distributions of besov type on euclidean n-space. duality, interpolation*, Arkiv för Matematik **11** (1973), no. 1, 13–64.

[81] Yuelong Xiao and Zhouping Xin, *On the vanishing viscosity limit for the 3d navier-stokes equations with a slip boundary condition*, Communications on pure and applied mathematics **60** (2007), no. 7, 1027–1055.

[82] Yingxi Zhu and Steve Granick, *Limits of the hydrodynamic no-slip boundary condition*, Physical review letters **88** (2002), no. 10, 106102.