TECHNICAL RESEARCH REPORT

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Supported by the National Science Foundation Engineering Research Center Program (NSFD CD 8803012), Industry and the University

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by E. Zafiriou and H-W. Chiou

TR 92-108

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Abstract

The presence of constraints in the on-line optimization problem solved by Model Predictive Control algorithms results in a nonlinear control system, even if the plant and model dynamics are linear. This is the case both for physical constraints, like saturation constraints, as well for performance or safety constraints on outputs or other variables of the process. Performance constraints can usually be softened by allowing violation if necessary. This is advisable, as hard constraints can lead to stability problems. The determination of the necessary degree of softening is usually a trial-and-error matter. This paper utilizes a theoretical framework that allows to relate hard as well as soft constraints to closed-loop stability. The problem of determining the appropriate degree of softening is addressed by treating the parameters (weights) affecting the amount of softening as one-sided real-valued uncertainty and solving a robust stability problem.

1 Approach

Model Predictive Control (MPC) algorithms solve on-line a constrained optimization problem at each sampling time. Standard formulations include hard constraints on inputs and outputs of the process. The optimization is carried out over a future horizon assuming no feedback is used during this future period. However, only the first in a sequence of "optimal" future inputs is implemented and the problem is solved again at the next sampling point with new information from feedback used to modify it. This fact often results in closed-loop behavior quite different from the one predicted by the optimal solution of the on-line optimization, which assumes uninterrupted open-loop implementation. In particular, we have shown that output constraints can result in instability for processes with model error or discrete models with unstable zeros (Zafiriou and Marchal, 1991). The framework that has been developed has allowed us to show that constrained MPC algorithms with linear models give rise to piece-wise linear control systems (Zafiriou, 1990). This framework can be used

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for tuning the MPC parameters to guarantee robust stability by simultaneous design of a set of linear feedback controllers each of which corresponds to a constraint combination. However, the use of hard constraints restricts the degrees of freedom available to the designer. When a particular combination of constraints is shown to result in instability, often the only course of action is to remove them from the on-line optimization. This is usually the case with output constraints placed on the first few points of the future constraint window of the on-line optimization. Removal is equivalent to infinite softening of these constraints. This "binary" choice between hard and infinitely softened constraints can be shown to be unsatisfactory in many cases, where the resulting closed-loop performance often is no better than that of the unconstrained algorithm. An alternative is to allow those constraints that are not physically hard to be softened by a "finite" amount. Such is usually the case for output constraints. This can be accomplished by allowing the constraints to be violated by an ϵ amount, while new terms of the form $W^2\epsilon^2$ are added to the standard quadratic objective function (Ricker *et al*, 1989). Removal of a constraint is equivalent to using W=0, while the constraint becomes hard as W becomes infinite. The tuning question is that of determining the appropriate values for the various W weights. The framework that was developed for hard constraints has been extended to the case of mixed hard and soft constraints (Zafiriou, 1991). However, the effort required to tune all parameters via a simultaneous design is too great. A different approach has been followed by starting with the tuning of the unconstrained algorithm first, so that it is robustly stable. This is a task that is relatively simple to accomplish via linear robust control techniques. The next step is to concentrate on the tuning of the Ws. It can be shown that the stability problem can be formulated as one where the Ws are treated as "uncertain" real-valued parameters with nominal value W=0. The task is to find the largest value for which the system remains stable. For the case of one output, relatively simple conditions can be derived for both nominal and robust stability of the constrained MPC algorithm. For the general case, recent developments in the area of robust stability conditions for one-sided real-valued parameter uncertainty can be used to carry out the necessary computations.

2 Closed-loop Stability

An state space model is used to describe the process:

$$\begin{aligned} x(k+1) &= \Phi x(k) + \Theta u(k) \\ y(k) &= Cx(k) + d(k) \end{aligned}$$

where $x(k)$ is the state vector; $u(k)$ and $y(k)$ are the input and output vectors of the model respectively; $d(k)$ is the disturbance effect at the output at k; Φ, Θ, C are the coefficient matrices. The plant is assumed to be open-loop stable. Other types of models can also be used, e.g., step response models (Garcia and Morshedi, 1986).

At sampling point k, the following optimization is carried out on-line:

$$\min_{\Delta u(k),\ldots,\Delta u(k+M-1)} \sum_{l=1}^{P} [e(k+l)^T \Gamma^2 e(k+l) + \Delta u(k+l-1)^T D^2 \Delta u(k+l-1)] \tag{1}$$

The minimization of the objective function is carried out over the values of $\Delta u(k)$, $\Delta u(k+1)$,..., $\Delta u(k+M-1)$, where M is a specified parameter. The minimization is subject to possible hard constraints on the inputs u, their rate of change Δu, the outputs y and other process variables usually referred to as associated variables. The details on the formulation of the optimization problem can be found in Prett and Garcia (1988). After the problem is solved on-line at k, only the optimal value for the first input $\Delta u(k)$ is implemented and the problem is solved again at $k+1$.

The optimization problem can be rewritten as the standard quadratic programming problem:

$$\min_v J(v) = \frac{1}{2} v^T G v + g^T v \tag{2}$$

subject to

$$A^T v \geq b,$$

where $v = [u(k), \ldots, u(k+M-1)]^T$; the matrices G, A and vectors g, b are functions of the MPC tuning parameters (P, M, Γ, D); the vectors g, b are also linear functions of $x(k)$, $d(k)$, $u(k-1)$. Then, the optimal solution v^* corresponding to an active constraint situation is computed from the following equation (Fletcher, 1987):

$$\begin{bmatrix} G & -\hat{A} \\ -\hat{A}^T & 0 \end{bmatrix} \begin{bmatrix} v^* \\ \lambda^* \end{bmatrix} = - \begin{bmatrix} g \\ \hat{b} \end{bmatrix} \tag{3}$$

where $\hat{A}^T$, $\hat{b}$ consist of the rows of A^T, b that correspond to the constraints that are active at the optimum and λ^* is the vector of the Lagrange multiplier corresponding to the active constraint situation. The coefficient matrix is referred to as Lagrangian matrix and is symmetric but not positive definite. If the inverse exists, then (Fletcher, 1987):

$$\begin{bmatrix} G & -\hat{A} \\ -\hat{A}^T & 0 \end{bmatrix}^{-1} = \begin{bmatrix} H & -T \\ -T^T & \mathcal{U} \end{bmatrix} \tag{4}$$

From (3),(4) we obtain $u(k)$ in a state feedback form corresponding to a particular set J_i of constraints that are active at the optimum of the optimization solved at k:

$$\begin{aligned} u(k) &= [\, I \; 0 \ldots 0 \,](-Hg + T\hat{b}) \\ &= \varphi_1 x(k) + \varphi_2 d(k) + \varphi_3 u(k-1) + \varphi_4 \end{aligned}$$

when J_i is the empty set (unconstrained), $H = G^{-1}$:

$$\begin{aligned} u(k) &= -[\, I \; 0 \ldots 0 \,] G^{-1} g \\ &= \bar{\varphi}_1 x(k) + \bar{\varphi}_2 d(k) + \bar{\varphi}_3 u(k-1) \end{aligned}$$

$\bar{\varphi}_1$, $\bar{\varphi}_2$, $\bar{\varphi}_3, \varphi_1$, φ_2, φ_3, φ_4 are functions of the MPC tuning parameters, state space model coefficient matrices, and J_i. I is the identity matrix.

For linear model dynamics, Zafiriou (1990) showed that the *constrained* MPC is *piece-wise linear*, meaning that the dynamics of MPC for a certain constraint set J_i active, are those of a discrete linear controller. This linear controller, denoted $c_{J_i}(z)$, depends explicitly only on J_i; it depends only implicitly on the past and current values of the plant inputs and outputs.

These values together with external inputs (setpoints, disturbances) determine the J_i that corresponds to a sampling point. However, if at different sampling points the Quadratic Program (QP) solution results in the same J_i, the MPC dynamics at those points are those of the *same* linear controller. For the case with the same number of inputs and outputs (for other cases, c_{J_i} also can be derived from the above control laws), c_{J_i} as computed from the above control laws, is given by:

$$c_{J_i}(z) = \tilde{P}^*(z)^{-1}(\beta^{-1} - I) \tag{5}$$

where $\tilde{P}^*(z)$ is the discrete model of the process $= C(zI - \Phi)^{-1}\Theta$, and

$$\beta = C[zI - (\Phi + \Theta(I - z^{-1}\varphi_3)^{-1}\varphi_1)]^{-1}\Theta(I - z^{-1}\varphi_3)^{-1}\varphi_2 + I$$

A necessary condition for the closed-loop operator mapping the states of the system (plant + controller) from one sampling point to the next, is that each of these linear controllers yields a closed-loop stable system. Note that the contraction property implies closed-loop stability. For more details and discussion the reader is referred to Zafiriou (1990).

In this paper we consider the case of output constraints only. These are defined over a future prediction horizon:

$$y_L \leq y(k+l) \leq y_U, \quad w_b \leq l \leq w_e \tag{6}$$

where y_L, y_U are the lower and upper limits respectively. In Zafiriou and Marchal (1991) the expressions for the c_{J_i} are given for special cases of combinations of points in the horizon, at which the hard constraints may become active at the optimum of the on-line optimization. It is also shown that for many important cases, the corresponding c_{J_i} result in an unstable closed-loop system, regardless of the values of the tuning parameters of the objective function. In such cases the only option is to soften the constraints by allowing violation by an amount ϵ. In the formulation here, the same violation variable $\epsilon \geq 0$ is used for all the points in the constraint window. Hence the output constraints are softened to be:

$$y_L - \epsilon \leq y(k+l) \leq y_U + \epsilon, \quad w_b \leq l \leq w_e \tag{7}$$

The term $W^2\epsilon^2$ is added to the objective function, where W is the weight that determines the extent of softening. For $W = \infty$ we get hard constraints. $W = 0$ corresponds to completely removing the constraints. For a nonzero finite W, and when the on-line QP results in a nonzero ϵ, then at the optimum for at least one of the points in the constraint window, say for $N_a \in [w_b, w_e]$, we will have $y(k+N_a) = y_U + \epsilon$ or $y(k+N_a) = y_L - \epsilon$. Otherwise a smaller ϵ would reduce the objective function, while still satisfying the constraints. This point is the one for which satisfaction of the constraint presents the greatest difficulty. The objective function that allows for output softening can be written as:

$$\min_{v,\epsilon} J(v,\epsilon) = \frac{1}{2}v^T(k)Gv(k) + g^T v(k) + \frac{1}{2}\epsilon^T W^T W \epsilon \tag{8}$$

where ϵ is the violation vector and W is the diagonal weight matrix. The output hard constraints are described by (7).

Rewrite the objective function as:

$$\min_{\bar{v}} J(\bar{v}) = \frac{1}{2}\bar{v}^T(k)\begin{bmatrix} G & 0 \\ 0 & W^TW \end{bmatrix}\bar{v}(k) + [g^T \quad 0]\bar{v}(k) \tag{9}$$

where $\bar{v}^T(k) = [v(k) \quad \epsilon]$.

Then, by following the same procedure as in the hard constraint case, we can set up the stability analysis method by setting the diagonal entries of W (≥ 0) as one-sided uncertainties in the c_{J_i} . The nominal value is $W = 0$ (unconstrained). The control law for the case with active output constraint can be written as:

$$u(k) = [I \quad 0 \ldots 0](-\bar{H}g + \bar{T}\hat{b}) \tag{10}$$

where

$$\begin{aligned} \bar{H} &= G^{-1} + G^{-1}\hat{A}[-(W^TW)^{-1} - \hat{A}^TG^{-1}\hat{A}]^{-1}\hat{A}^TG^{-1} \\ \bar{T} &= -G^{-1}\hat{A}[-(W^TW)^{-1} - \hat{A}^TG^{-1}\hat{A}]^{-1} \end{aligned}$$

From this control law, we can construct the c_{J_i} as that in the hard contraint case (5) but with the diagonal entries of W as one-sided uncertainties. The design task consists of two steps. First design a stable unconstrained MPC ($W = 0$) with linear control theory. Then use robust control theory to find the largest W for which stability is maintained.

3 Example

We consider a Multi-Effect Evaporator modeled by Ricker *et al*(1989):

$$\tilde{P}(s) = \frac{2.69(-6s+1)e^{-1.5s}}{(20s+1)(5s+1)}$$

For a sampling time of 3 minutes,the discrete model is:

$$\tilde{P}^*(z) = \frac{-0.174(z^2 - 1.0837z - 0.88585)z^{-1}}{z^2 - 1.4095z + 0.47237}$$

We select the following tuning parameters:

$$M = 1, \; P = 30, \; \Gamma = I, \; D = 0$$

These result in a stable unconstrained controller. Constraints are set on predictive outputs $y(k+1)$ and $y(k+2)$. The controller that corresponds to the case that these two constraints are active with softening weights is:

$$c_{J_i} = \frac{(p_1/q_1) + (p_2/q_1)W_1^2 + (p_3/q_1)W_2^2}{1 + (q_2/q_1)W_1^2 + (q_3/q_1)W_2^2}$$

where W_1, W_2 are the softening weights for the hard constraints of $y(k+1)$, $y(k+2)$ respectively. And,

$$
\begin{aligned}
p_1 &= p_3 p_2 (877.63z^5 - 2188.28z^4 + 978.12z^3 + 646.19z^2 - 367.15z) \\
p_2 &= -2.89z^5 + 7.19z^4 - 3.22z^3 - 2.12z^2 + 1.2z \\
p_3 &= -3.83z^5 + 9.54z^4 - 4.26z^3 - 2.82z^2 + 1.6z \\
q_1 &= q_3 q_2 (1978.86z^5 - 4728z^4 + 1786.47z^3 + 1351.73z^2 - 508.71z + 119.80) \\
q_2 &= 0.5z^5 - 1.59z^4 + 0.79z^3 + 1.26z^2 - 0.57z - 0.39 \\
q_3 &= 0.88z^5 - 3.47z^4 + 2.97z^3 + 1.69z^2 - 1.56z - 0.52
\end{aligned}
$$

We treat the weights W_1, W_2 as uncertainties in the control system which are real-valued numbers and greater than zero. We search for their upper bound such that the control system remains stable.(The nominal value is $W_1 = W_2 = 0$, which corresponds to a closed-loop stable system.) The control block diagram is shown in figure A and it can be rearranged as shown in the figure B. The Δ_c is the uncertainty matrix which is:

$$
\begin{bmatrix}
W_1^2 & 0 & 0 & 0 \\
0 & W_2^2 & 0 & 0 \\
0 & 0 & W_1^2 & 0 \\
0 & 0 & 0 & W_2^2
\end{bmatrix}
$$

And, M_u is:

$$
\begin{bmatrix}
-\tilde{P}^*(p_2/q_1)\psi & -\tilde{P}^*(p_2/q_1)\psi & \tilde{P}^*(p_2/q_1)\psi(q_2/q_1) & \tilde{P}^*(p_2/q_1)\psi(q_3/q_1) \\
-\tilde{P}^*(p_3/q_1)\psi & -\tilde{P}^*(p_3/q_1)\psi & \tilde{P}^*(p_3/q_1)\psi(q_2/q_1) & \tilde{P}^*(p_3/q_1)\psi(q_3/q_1) \\
\psi & \psi & -\psi(q_2/q_1) & -\psi(q_3/q_1) \\
\psi & \psi & -\psi(q_2/q_1) & -\psi(q_3/q_1)
\end{bmatrix}
$$

where $\psi = (1 + (p_1/q_1)\tilde{P}^*)^{-1}$.

We compute the structured singular value (Doyle, 1982) corresponding to the block diagram in figure B, and we can get the maximum value over the frequencies 0 to π/T (T is the sampling time) from figure C. The maximum value is equal to 0.0077. The computation of the structured singular value is made according to Lee and Tits (1992). The maximum softening weight can be computed from this value, and it is equal to 11.396.

4 Conclusions

This paper provides a method for obtaining the weights used in softening output constraints of MPC algorithms. The technique results in the largest weight (hardest constraint) that will not cause any stability problems. The task can be carried out as a second design step following the design of the unconstrained MPC algorithm. Thus, although the control system is nonlinear because of the constraint, the methods that have been developed in the literature for designing unconstrained controllers can still be used in the first step.

The method is based on the idea of handling the weights as "uncertain" parameters. Finding their largest value can be thought of as a robust stability problem, in which the

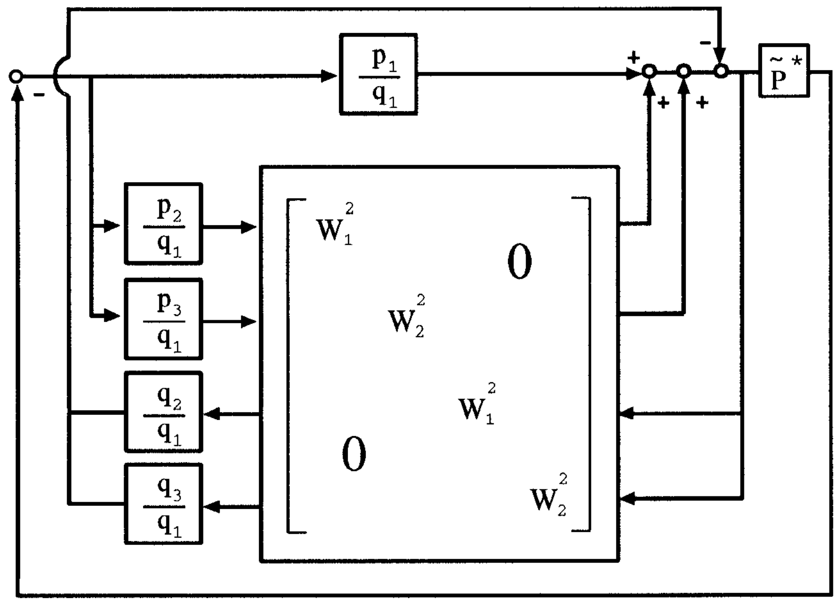

Figure A: The control block diagram

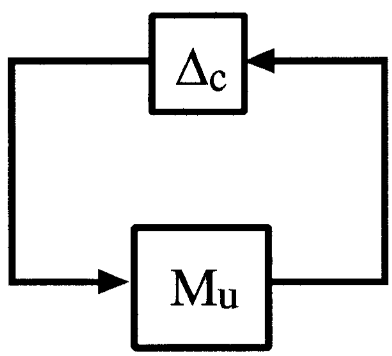

Figure B: The rearranged control block diagram

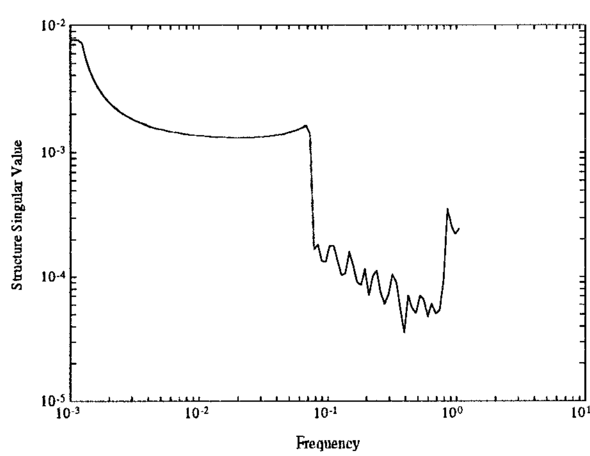

Figure C: The structured singular value for the example uncertain parameters are real-valued with their nominal value at one end of the uncertainty interval.

Acknowledgements

This project is supported by the National Science Foundation (PYI grant CTS-9057292), the Systems Research Center, and an unrestricted grant from Shell Development. The authors are grateful to Dr. Li Lee and Prof. André Tits for making the software for computing the structured singular value available.

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