Path Curves and Plant Buds: An Introduction to the Work of Lawrence Edwards

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Contents

To discover in the world of nature the geometrical forms of our own thinking can be one of our most exciting experiences. A child delights in the hexagonal symmetry of a snow flake, and Kepler and thousands after him have joyed in the beauty of the laws of planetary motion. These experiences stir us, for they reveal that behind material nature there is a creative world in which we can participate through our thinking.

Such experiences are even more moving when they come from the world of living forms. The work of Lawrence Edwards (1913 – 2004) on plant buds offers the finest example known to me. In over four-fifths of the species he has examined, the bud profiles are fit extremely closely by a family of curves known as path curves, for they are the paths taken by points under repeated application of a projective transformation of space.

Edwards's own description [1] of the mathematics of these curves flows beautifully, but has proven perplexing to readers not well acquainted with these matters. I have therefore undertaken to provide an introduction to his work in terms of mathematics which is widely known. In the first section, I explain the construction of the bud-form curves. This section uses only plane geometry and suffices to understand the computations actually made by Edwards. It does not explain what these curves have to do with projective geometry. That is the business of section 2, which uses coordinate geometry and vectors and matrices for expressing linear equations. It also makes use of the idea of characteristic roots and vectors of a matrix. Still, this section does not give us algebraic equations for the path curves. Section 3 handles these matters, but it is necessary to use a bit more mathematics, namely linear differential equations and the elementary properties of complex numbers. With these formulae in hand, we turn, in the final section, to the statistical fitting of the path curves. Here I use data kindly provided

* The theoretical part of this paper was worked out in about 1978. The computations were done when I was at the International Institute for Applied Systems Analysis (IIASA) in Laxenburg, Austria in 1978-79, and the paper first appeared in the Professional Papers series of that institute in 1979. That version is now (2013) on the website of the Institute. In preparing it for entry into the digital repository at the University of Maryland (DRUM) it has been reviewed and some clarifications introduced and typographical corrections made. I was first introduced to the subject by Martin Levin and owe my interest in it to his enthusiasm for it. Martin McCrea suggested the use of homogeneous coordinates. My greatest debt, of course, is to Lawrence Edwards, who has painstakingly written answers to many questions and has shared the data he has accumulated over years of work. Calculations were done on the IIASA computer.

by Edwards and fit path curves by least squares. For many species, the average absolute percentage error is less than two percent.

The first section should be intelligible to any interested reader; and the last section is intelligible without reading the intermediate sections if one will accept the formula derived for the path curves.

1. Construction of a Plant-Bud Path Curve

We begin with the construction of the bud-form curve. On a given line, *a*, pick points O, A and B, as shown in Figure 1. Our first task is to find the point C on the line such that

$$OC/OB = OB/OA = \lambda_1.$$

That is to say, we are looking for the point C that makes the distance from O grow by the same percentage between B and C as it did between A and B. Here is the construction that makes that happen.

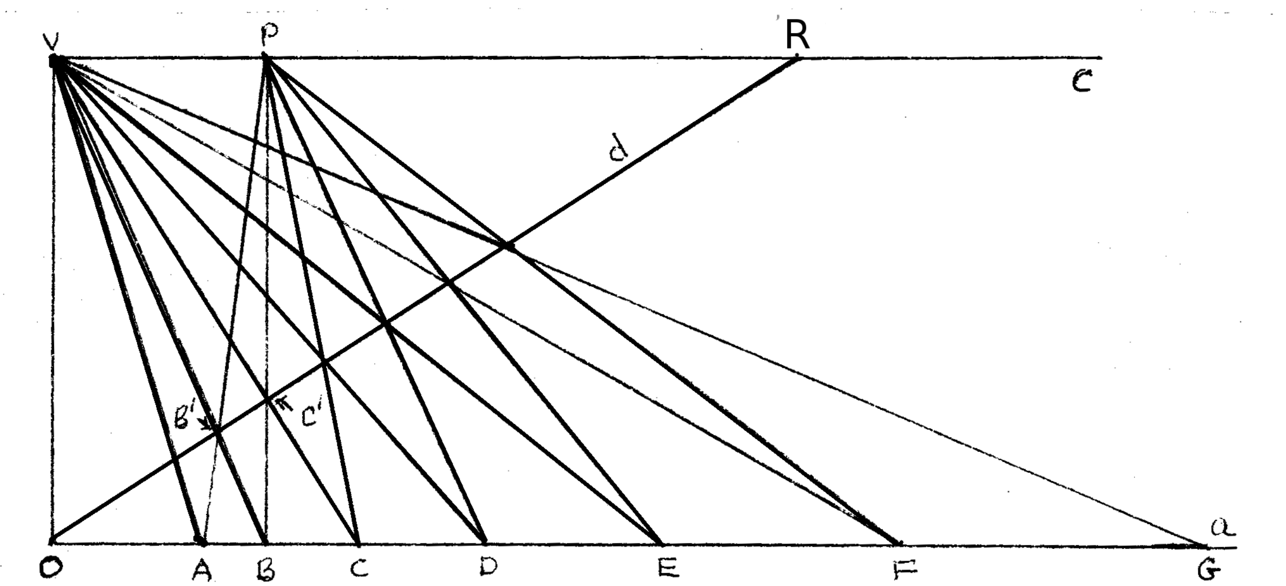


Figure 1. An Expanding Series of Points

Draw a line *c* parallel to *a* and choose V on c so that the line from V to O will be perpendicular to *c*. Draw a line *d* passing through O and not identical with *a*. Mark its intersection with *c* by R. Draw VB and mark its intersection with *d* by B′. Draw the line AB′ and mark its intersection with *c* by P. Draw PB and mark its intersection with *d* by C′. Draw the line connecting V and C′ and mark its intersection with *a* by C. This C is the desired point.

Proof: The triangle OAB′ is similar to PRB' and OBC′ is similar PRC'. Therefore,

(a) OA/AB′ = PR/PB′

and

(b) OB/B′ = PR/PC′.

Furthermore ABB′ is similar to VPB′ and BCC′ is similar to VPC′, so

(c) AB/AB′ = VP/PB′

and

(d) BC/BC′ = VP/PC′.

Adding (a) and (c) gives

(e) (OA+AB)/AB′ = (VP+PR)/PB′.

But OA+AB = OB and VP+PR = VR, so

(f) OB/AB′ = VR/PB′.

Dividing each side of (f) by the corresponding side of (a) gives

(g) OB/OA = VR/PR

By similar reasoning, (b) and (d) imply

(h) OC/OB = VR/PR.

But the right sides of (g) and (h) are the same, so their left sides must be equal. Thus

OC/OB = OB/OA

as was to be demonstrated. (The proof did not use the fact that VO was perpendicular to *c,* but we will draw the figure that way in what follows.)

As shown in Figure 1, an expanding sequence of points A, B, C, D, E, etc. can be constructed on the line *a* in the same way. We say that λ_1 is the *multiplier* of this sequence. Similarly, in Figure 2, we construct a contracting series of points A″, B″, C″, D″, … along the line *c*. Let λ_2 = V″B″/VA″ be the multiplier on *c*.

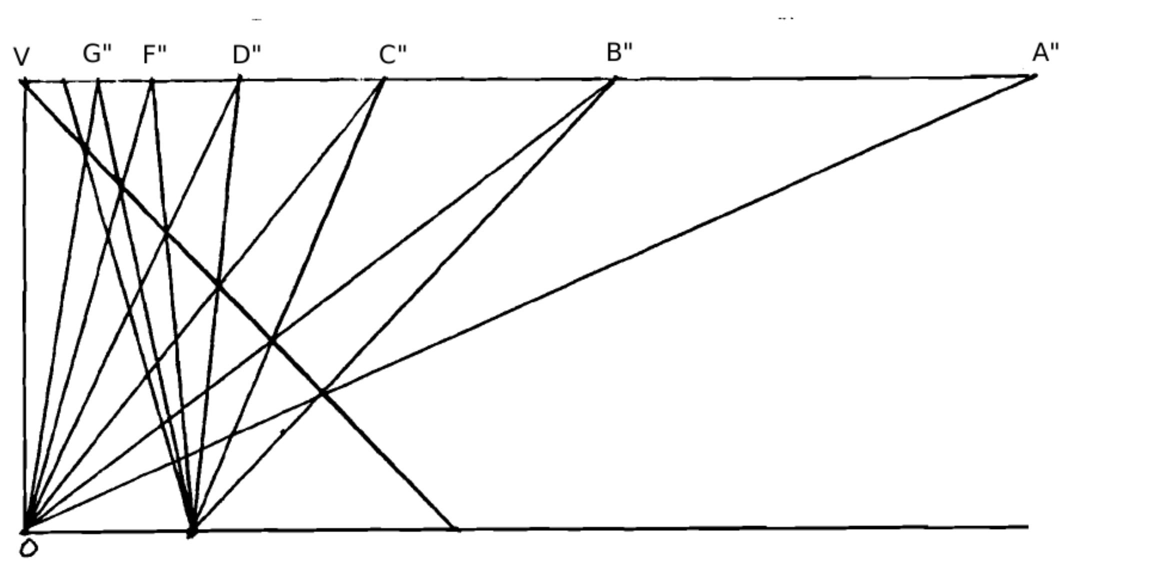


Figure 2. The Contracting Sequence of Points.

Now in Figure 3, we combine Figure 1 and Figure 2, but to avoid confusion we show only the lines passing through V and O.

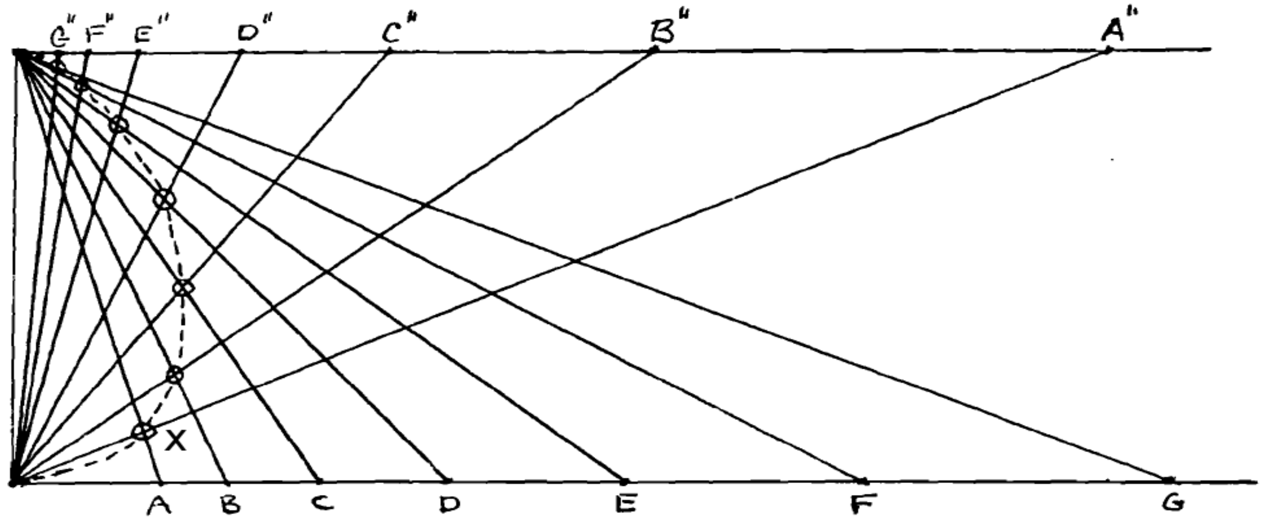


Figure 3. The Path Curve Emerges from the Expanding and Contracting Sequences

Where the line to A meets the line to A' mark the point x. Think now of x stepping along from line to line on the expanding sequence of lines and, simultaneously, on the contracting sequence. Its "footprints" will fall on the circled points of Figure 3.

Now suppose that, instead of having multiplier λ_1 on *a* and multiplier λ_2 on c, we had multipliers of $\lambda_1^{1/2}$ on *a* and $\lambda_2^{1/2}$ on *c*. Then two steps of this "walk" are equivalent to one of the original. All of the "footprints" of x on the first walk remain footprints on the second, but the second has an extra print between each pair of the first walk. If we took a walk with $\lambda_1^{1/3}$ on *a* and $\lambda_2^{1/3}$ on *c*, then x would make two footprints between each pair of the original ones. With $\lambda_1^{2/3}$ and $\lambda_2^{2/3}$, x would have every other one of these footprints. Clearly x is traversing the same "path" on all of the walks; only its step-length differs. For all of the step-lengths, the ratio

$$\lambda = \log\lambda_2^{\alpha}/\log\lambda_1^{\alpha} = \log\lambda_2/\log\lambda_1$$

remains the same and characterizes the path itself.

It is these *path curves* which Edwards has shown to give the profile of plant buds. For a particular species, he collects numerous buds at the point just before opening. Then, using tweezers and a magnifying glass, he carefully removes the outer petals and reveals the form of the inner inflorescence. If a tiny petal budges, the specimen is lost. He then photographs the bud and enlarges it to be four inches high. At half-inch intervals along the vertical axis of the enlarged bud he measures the diameter. These measurements on at least seven buds of the species are averaged and plotted as in Figure 4. These measurements for 150 species and varieties are given in Appendix 2. They are radii in inches of the 4-inch high buds, starting from the top.

Edwards then takes two points on the profile, say T and E in Figure 4, draws lines from O and V through them, and computes the multipliers λ_1 and λ_2 on *a* and *c* respectively, and then calculates

$\lambda = \log \lambda_2 / \log \lambda_1$.

If the profile is a perfect path curve, each pair of points gives the same value of λ. (Of course, it is not necessary actually to draw the figure; the value of λ for a pair of points can be easily calculated directly from the measurements of the diameters without introducing. any drafting error.)

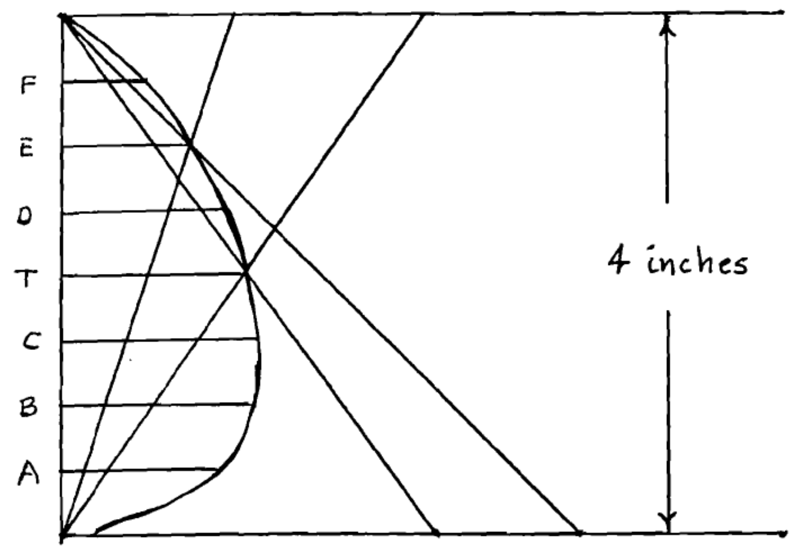


Figure 4. Plot of bud measurements after averaging

For ease of computation, Edwards takes the midpoint, marked T, in conjunction with each of the other points. For fifty-five species and varieties, Edwards reports in [2] the average absolute percentage deviations of the resulting six λ's from their mean. He also indicates that deviations of ten percent or less mean extremely close fit to a path curve. Thirty of the species have average deviations of less than 10 percent; twenty of them have average deviations between 10 and 20 percent, four, between 20 and 30 percent, and only one over 30 percent.

We shall present in section 4 the results of fitting the path curves to Edwards's data in another way, minimizing the sum of the squared percentage errors between observed and "theoretical" values of the bud diameters. We present there also the average absolute error in the fit, which makes it easy even for the inexperienced to appreciate the extraordinary closeness of fit.

Although we have constructed the particular bud-form path curve, we have not seen its connection with projective transformations, nor have we developed the algebraic formula necessary for statistical fitting to the diameters, nor have we seen how to generalize from two-dimensional figures to path curves in three dimensions. The next two sections concern these matters.

2 Projective Transformations and Homogeneous Coordinates

Projective geometry deals with the properties of figures which are preserved under projective transformation. Figure 5 shows a typical projective transformation of the line *a* into itself.

The transformation is determined by a second line, d, and two points, p and q, not on *a* or *d*. The transformation of the point x is then found as follows. Draw the line determined by p and x; where it intersects *d*, mark the point x'. Draw the line determined by x' and q and mark x″ where this line intersects *a*. This x″ is the image of x under the transformation. Any point x is transformed into a unique x″, and any x″ comes from a unique x.

Our familiar Figure 1 from the first section also shows an example of a projective transformation of the line *a*, with point A being transformed into B; B. into C; C into D and so on. Figure 6 shows yet another special case of a projective transformation of a line. This one gives the solution of the the problem of representing in proper perspective on the plane a row of equally spaced telephone poles of equal height. It uses the fact that any three parallel lines in space will all meet at one point on the plane (or appear as parallel).

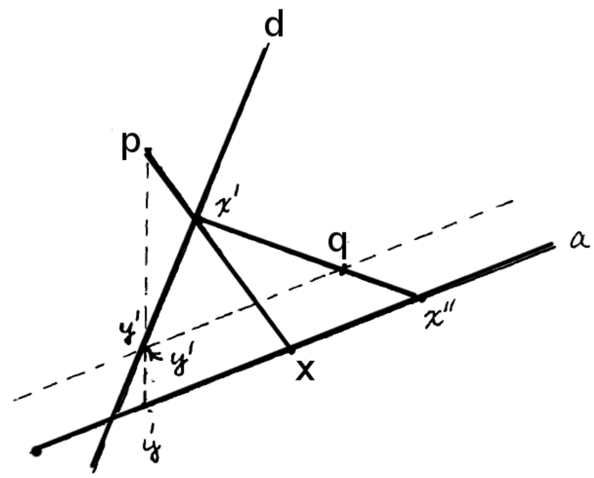


Figure 5. Projective transformation of a line into itself

Let us return now to Figure 5 and notice that if we happen to start from the particular point marked y, we will find that the line from y' to q is parallel to *a*. We say, however, that the two parallel lines determine the point at infinity or intersect at the point at infinity on *a*, which is, of course, also the point at infinity on the line through y' and q. Consequently, we see that parallel lines must all have the same point at infinity.

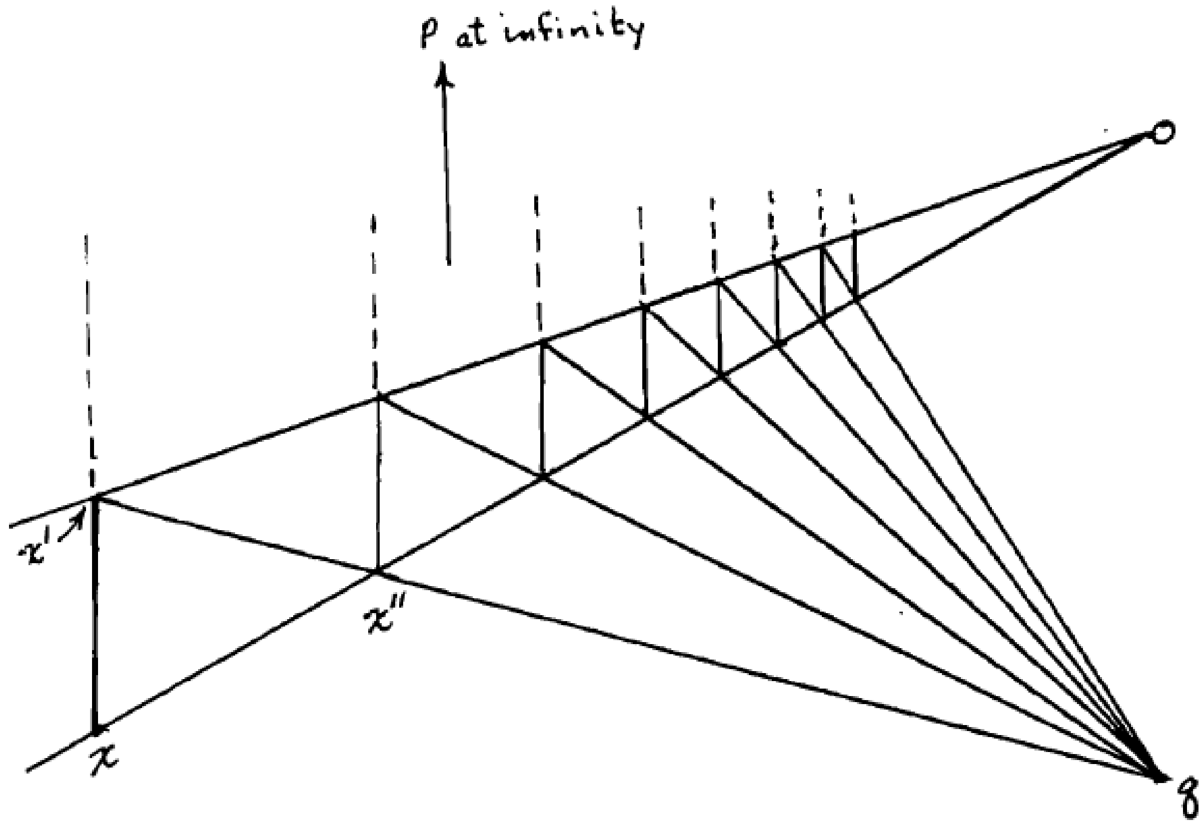


Figure 6. Special case of projective transformation when p is at infinity.

If we use ordinary, Cartesian coordinates, this point can only be written as (∞, ∞). But the point at infinity on the line *d* can also be written only as (∞, ∞). This notation gives the unfortunate impression that *a* and *d* intersect at infinity, when, in fact, they intersect at a finite point. Consequently, projective geometry requires a coordinate system which can distinguish between the different points at infinity. This distinction is achieved by adding one more coordinate and agreeing that all multiples of the same vector represent the same point. This manner of representation is called *homogeneous coordinates*. Thus the vectors (4, 1, 1), (8, 2, 2) and (1, .25, .25) all denote the same point in homogeneous coordinates. We shall write $x \approx y$ to mean that the vectors x and y are proportional and thus denote, in homogeneous coordinates, the same point.

For plotting. one picks a normalization. a row vector h, and plots the first two coordinates of the vector x/hx. The most common choice for h is h = (0,0,1), so that one plots $(x_1/x_3, x_2/x_3)$. Any other choice of a non-zero h is equally valid, though of course the homogeneous representation of a point depends upon which h is used. For example, the Cartesian coordinates (1,1) may represent (1,1,1) if h = (0,0,1), but (1,1,-1) if h = (1,1,1). Only the vector (0,0,0) never arises as the homogeneous representation of a point.

If we think of the homogeneous coordinates of a figure as the Cartesian coordinates of a three-dimensional figure, then the normalization amounts to a projection through the origin onto the plane hx = 1. Then, in plotting only the first two coordinates, we are, in effect, looking at this planar figure from infinitely far out on the x 3 axis. More formally, we are projecting onto the plane

$$0x_1 + 0x_2 + 1x_3 = 1$$

from the point $(0, 0, \infty)$.

Thus, plotting from homogeneous coordinates is formally equivalent to projection of a figure in three-dimensional space onto a plane.

In what follows, we shall denote the column vectors for the homogeneous coordinates of points with letters from p to z; row vectors for equations we denote with letters from a to h and scalars we denote with k, m, and n.

In homogeneous coordinates, the points on the line connecting x and y may be written as mx + ny for any values of m and n. In the plane, the equation of a line may be written ax = 0, where x gives the homogeneous coordinates of points on the line. With h = (0,0,1), this equation corresponds to the equation

$$a_1x_1 + a_2x_2 = 1$$

in Cartesian coordinates; we just set $x_3 = 1$ and $a_3 = -1$. Similarly, in three-dimensional space, the equation of any plane can be written as ax = 0, where x is a four-element vector giving the homogeneous coordinates of points on the plane.

Let us now consider the transformation, not of a line, as in Figure 5, but of an entire plane by a projective transformation. That is, let us start with a point *x* in plane A, project it through point *p* into point *x′* in a plane B, and then project *x′* through a point *q* back into plane A at *x″*. (The points *p* and *q* must not lie in A or B.) We shall show that this transformation, *which is quite non-linear in Cartesian coordinates, is linear in homogeneous coordinates* and may be represented by u″ = Cu, where u and u″ are homogeneous coordinates of the original and transformed (or image) points, respectively, and C is a square matrix. It will prove convenient to take as plane A the "horizontal" plane with $x_3 = 0$ in its Cartesian coordinates.

The line from x through p is given by the points $mx + np$, where x and p are 4-element column vectors, homogeneous coordinates of points in three-dimensional space, and m and n are any real numbers. The requirement that the point x' lie in B we may write as $bx' = 0$, so

$$bx' = b(mx + np) = 0$$

hence, once we pick an m, n must satisfy

$$n = - mbx/bp$$

and

$$x' = mx + np = mx - m(bx/bp)p$$

for all m, so we may as well take this $m = 1$. The line from x' through q is therefore all points of the form $m(x -(bx/bp)p) + nq$ for all m and n. (These are a new m and n for the second projection; for the first projection we have already fixed $m = 1$ and $n = -bx/bp$.) The requirement that $x_3'' = 0$ – i.e., the requirement that x lie in the plane A – implies that x'' is given by the m and n that satisfy

$$-m(bx/bp)p_3 + nq_3 = 0$$

since we chose A as the plane with $x_3 = 0$. Therefore, for x''

$$n = m(bx/bp)k$$

where $k = p_3/q_3$ so that

$$x'' = m(x - (bx/bp)p + k(bx/bp)q).$$

Since this equation is valid for all m, we may pick $m = (bp)$ and write

$$x'' = (bp)x - (bx)p + k(bx)q.$$

Since (bx) is a scalar, $(bx)p = p(bx) = [pb]x$, where [ ] marks a square matrix.
Likewise, $(bx)q = q(bx) = [qb]x$, so

$$x'' = ((bp)I - [pb] + k[qb])x$$

where I is the identity matrix. If we now denote the entire matrix on the right of this equation by B, the equation becomes just

$$x'' = Bx.$$

Furthermore, because both $x_3 = 0$ and $x''_3 = 0$, we can strike out the third row and column of B to get a 3-by-3 matrix C such that

$$u'' = Cu \qquad (1\ )$$

for the three-element vectors u'' and u derived by striking out the third coordinate of x'' and x. Thus, u and u'' are just the homogeneous coordinates of points in the plane A.

We have therefore shown that *a projective transformation of a plane into itself can be represented by a linear transformation of its homogeneous coordinates.* Though we have conducted this proof in three dimensions. it immediately generalizes to n dimensions by just replacing "three" or "3" by "n". The matrix C of this transformation will be non-singular, for no point transforms into the non-point (0,0,0),

What does a projective transformation of the plane look like geometrically? From matrix theory, we know that that a 3-by-3 matrix will have three characteristic vectors, v_1, v_2 and v_3, with corresponding characteristic values m_1, m_2 and m_3. For each of these

$$Cv_i = m_i v_i.$$

But $m_i v_i$ and v_i are the homogeneous coordinates of the same point. Therefore these characteristic vectors are fixed points of the transformation. In this section, we shall assume that all three characteristic values are real and distinct. (In the next section we shall treat also complex characteristic values.) With this assumption, we can plot the three fixed points as in Figure 7.

Now any point on the line *a* determined by v_1 and v_2 is transformed into a point on this line, for

$$C(n_1 v_1 + n_2 v_2) = n_1 m_1 v_1 + n_2 m_2 v_2$$

We express this fact by saying that the line *a* is invariant under the transformation. The lines *b* and *c* determined by v_1 and v_3 and by v_2 and v_3 are likewise invariant. These three invariant lines have been drawn on Figure 7.

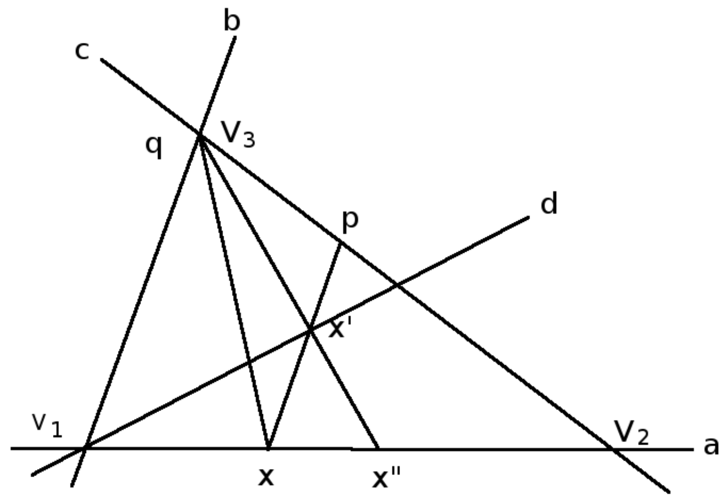


Figure 7. Fixed points of the transformation

What can we say about the transformation of the line *a* into itself? Well, it is precisely a projective transformation of the type described by Figure 5 with the point v_3 of Figure 7 playing the role of the point q in Figure 5. We have therefore labeled the point both v_3 and q, but as a vector we will define q numerically as $v_3 - v_2$. We may take *d* to be any line through v_1 not containing v_2 or v_3. We then choose the point p on *c* such that

$$1 - (dv_2/dp) = m_2/m_1 \tag{2}$$

or

$$dp = dv_2 /(1 - m_2/m_1)$$

Note that p is not an arbitrary point but depends on characteristic values and vectors of the matrix. That is only to be expected. We are not trying to show that *any* projective transformation of the line *a* into itself is equivalent to that of a *given* matrix, but only that, given the arbitrary line *d*, there is *one* such transformation and it depends on the characteristic values of the matrix and a characteristic vector, as well as, of course, the line *d*.

To check that this geometrical construction will give the same transformation as does the matrix C, let us pick an arbitrary point

$$x \approx n_1v_1 + n_2v_2$$

on *a* and find its image, x″, in both ways. The matrix transformation gives immediately

$$x'' \approx C(n_1v_1 + n_2v_2) = n_1m_1v_1 + n_2m_2v_2 \tag{3}$$

The transformation done in Figure 5 style gives first

$$x' \approx x - (dx/dp)p \tag{4}$$

Clearly this x′ is on the line determined by x and p and $dx' = 0$ so it is on the line *d* also. Next,

$$x'' \approx x' - (ax'/ aq)q. \tag{5}$$

It is clearly on the line determined by x′ and q; and $ax'' = 0$, so it is on the line *a* also. We need to show that this x″ is the same as that found in equation (3). Since $ax = 0$, premultiplying (4) by the vector a gives

$$ax' = -(dx/dp)(ap), \tag{6}$$

and substituting from (4) and (6) into (5) gives

$$x'' \approx x - (dx/dp)p + (dx/dp)(ap/aq)q = x - (dx/dp)(p - (ap/aq)q) \tag{7}$$

Now p satisfies the equation

$$p = v2 + (ap/aq)q \tag{8}$$

To see this, note that ap can be thought of as the distance from the line *a* to the line through p parallel to *a,* while aq is the distance from *a* to the line parallel to *a* through q. To get to the point p from v_2 along the vector q, we have to go the fraction (ap/aq) of the total length of q. That is what is expressed as equation (8). We can rewrite (8) as

$$v_2 = p - (ap/aq)q \tag{8}$$

and substitute this expression into (7) to get

$$x'' \approx x - (dx/dp)v_2 \tag{9}$$

Now substituting the expression for x in terms of v_1 and v_2

$$x'' \approx n_1v_1 + n_2v_2 - ((n_1dv_1 + n_2dv_2/dp)v_2$$

But $dv_1 = 0$, so

$$x'' \approx n_1v_1 + n_2(1 - dv_2/dp)v_2$$

or, by using (2),

$$x'' \approx n_1v_1 + n_2(m_2/m_1)v_2 \tag{10}$$

Multiplying through by m_1 gives

$$x'' \approx n_1m_1v_1 + n_2m_2v_2 \tag{11}$$

which is exactly the expression for x″ which we obtained in (3) from the matrix transformation. (Remember, the ≈ means the vectors on either side are proportional; one side of the ≈ can be multiplied by a scalar without multiplying the other side by the same scalar.)

Thus, the projective transformation of the plane given by

$$x'' = Cx$$

induces a projective transformation into itself of each of the lines connecting its three fixed points.

Edwards refers to the ratio m_2/m_1 as the multiplier of the transformation along *a*, and denotes it by λ_1. The multipliers $\lambda_2 = m_3/m_2$ and $\lambda_3 = m_1/m_3$ on the two other invariant lines are similarly defined, and the identity

$$\lambda_1 \lambda_2 \lambda_3 = 1$$

becomes obvious.

If we know the transformation of two of the invariant lines, we can easily determine geometrically the transformation of the entire plane. In Figure 8, for example, let us suppose that we know the transformation induced by C on the lines *a* and *c*. Whither will the point x, not on either line, be transformed?

From v_3, project x onto *a* at point y. This y will be transformed to some other point on *a*, say y″, and v_3 is transformed into itself. Now the transformation induced by the matrix takes lines into lines:

$$C(k_1u_1 + k_2u_2) = k_1(Cu_1) + k_2(Cu_2)$$

for any k's and u's. Therefore, the line through v_3 and y will be transformed into the (dashed) line through v_3 and y″. Likewise, the line through v_1 through x to z on the line *c* will be transformed into the (dashed) line through v_1 and some z″ on *c*. Since x″, the image of x under the transformation, must lie on both the dashed lines, its location is fully determined.

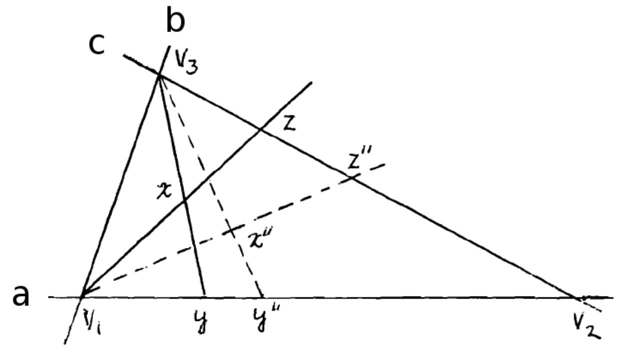


Figure 8. Transformation of the plane induced by that on lines a and c.

We obtain our old friend figure 3, the cornerstone of Edwards's work, if we happen to have

$$v_1 = (0, 0, 1)^T,\ v_2 = (1, 0, 0)^T,\ \text{and}\ v_3 = (0, 1, 1)^T$$

where the superscript T denotes transposition. Then v_2 is the point at infinity on the horizontal axis, so the "triangle" appears exactly as the lines *a*, *b*, and *c* of Figure 3. If we rewrite formula (4) taking $n_1 = 1$ and $n_2 = n$, so that

$$x'' \approx v_1 + n\lambda_1 v_2,$$

then we see that the point on *a* at distance n from v1, is transformed into the point at $n\lambda_1$; and this point, into the one at distance $n\lambda_1^2$, and so on, exactly as in Figure 3. Similarly, the distance of a point on *c* from v_3 is changing by the multiplier λ_2, just as in Figure 3. Moreover, the projective transformation of the two lines *a* and *c* induces a projective transformation on the entire plane and the successive "footprints" of the point x are, in fact, where it is moved by successive applications of a projective transformation of the plane into itself.

3 Continuous Path Curves

We now want to think of taking walks with shorter and shorter steps taken faster and faster. We therefore introduce a continuous parameter t and consider the sequence of projective transformations described by

$$x(t + \Delta t) = (I + A\Delta t)\, x(t) \quad t = 0, \Delta t, 2\Delta t, 3\Delta t \ldots .$$

Here Δt is a fixed, finite change in t; $I + A\Delta t$ is the matrix C of equation (1) and x(t) and $x(t + \Delta t)$ correspond to u and u″ of equation (1). They are n+1 dimensional vectors giving the homogeneous coordinates of points in n-dimensional space relative to some normalization vector, h. If we fix A and take Δt smaller and smaller, we do not, in fact, always walk along the same "path" as described at the end of section 1. Instead, the route changes slightly with each shorter Δt, but these routes converge to a limiting, continuous path. On each walk we have

$$(x(t+\Delta t) - x(t))/\Delta t = Ax(t)$$

and as $\Delta t \to 0$, we obtain the differential equations

$$\dot{x}(t) = Ax(t) \tag{12}$$

where the dot over x denotes the derivative with respect to t.

This is a system of n+1 linear, homogeneous differential equations with constant coefficients. Its solution is well-known, and we need only review it here. We can then easily show that it is, indeed, the path followed by infinitely many finite-step walks, and, in fact, that all such walks take such a path.

The general solution to the system (12) is

$$x(t) = \sum_{i=1}^{n+1} k_i v_i e^{m_i t} \tag{13}$$

where the k's are constants depending on initial conditions, and the v's and m's are characteristic vectors and values of the matrix A, as we shall explain. If x(t) is a solution of (5), kx(t) is also a solution, as is readily checked. Likewise, the sum of two solutions is a solution. Consequently, to investigate whether (13) is a solution of (12), we need only know the conditions for ve^{mt} to be a solution. For this, we must have

$$\dot{x} = mve^{mt} = Ave^{mt} \quad \text{for all t.}$$

Therefore, we must have

mv = Av

or

(A – mI)v = 0.

If this last equation is to have a solution other than v = 0, the matrix (A - mI) must be singular, so the determinant $|A\text{-}mI|$ must be zero. The expansion of the determinant produces a polynomial of degree n+1 in m, which will have n+1 roots. These roots are the m's of (13). We shall assume that they are distinct, and this assumption is sufficient to guarantee that the v's are linearly independent. Let V be the matrix of the v's. Then $V^{-1}AV = M,$ where M is a diagonal matrix having the m's, the characteristic values of A, down the diagonal.

Because Edwards always worked with finite step walks while we are moving into continuous curves, we need to be careful about the relation of the two. We therefore ask, first, Are there finite-step walks which move along the curve given by (13)? Indeed there are. First, pick any Δt and calculate by (13) the points x(0), x(Δt), x(2Δt), x(3Δt), etc. These are all points on the continuous path. Now, to repeat the question more precisely, Can we find a matrix C which will “walk” x(0) with exactly these points as footprints? Then answer is Yes, but before we display that C we need to be clearer about the footprints. Let

$$e^{m_i \Delta t} = f_i \tag{14}$$

then from (13) we see that

$$x(j\Delta t) = \sum_{j=1}^{n+1} k_i v_i f_i^j = VKF^j \quad \text{for j = 0, 1, 2, 3, … .} \tag{15}$$

where K is the diagonal matrix of the k_i and F is the diagonal matrix of the f_i. These successive points for j = 0, 1, 2, … are the "footprints" on the path.

Now we need to find the matrix C which will, starting from the first of these, transform it into the second, and transform the second to the third, and so on. We define a matrix C by

$$C = VFV^{-1}$$

where F is the diagonal matrix of the f_i. We now need to show that it will produce the sequence of "footprints" given by (15). Let us apply it to $x(j\Delta T)$.

$$Cx(j\Delta t) = VFV^{-1}VKF^{j} = VFKF^{j} = VKFF^{j} = VKF^{j+1} = x((j+1)\Delta t)$$

so that repeated application of the transformation C "walks" the point x precisely along path given by the differential equation. (The third of the equalities in the above equation used $FK = KF$, which is true because both matrices are diagonal.)

Conversely, given C, we can define F by $F = V^{-1}CV$, where V is the matrix of characteristic vectors of C, and then use (14) to define the m_i for a given Δt. The given C will then walk x in finite steps, each step falling on the continuous path determined by (13) with these v's and m's.

Let us now look at a few special cases. First with n = 2, (and n+1 = 3 for the homogeneous coordinates) suppose that we have

$$V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

and $k_2 = k_3 = 1$ and $k_1 = k$. If we then write out the formula for each component of x(t) by (13) we get

$$x_{1t} = ke^{m_1 t}, \quad x_{2t} = e^{m_2 t}, \quad x_{3t} = e^{m_2 t} + e^{m_3 t}.$$

We now normalize with the vector h = (0,0,1), which is to say, we divide x_1 and x_2 by x_3 and call the first component of the normalized vector r(t), and the second h(t):

$$r(t) = ke^{m_1 t}/(e^{m_2 t} + e^{m_3 t})$$

$$h(t) = e^{m_2 t}/(e^{m_2 t} + e^{m_3 t}) \ .$$

Now divide numerator and denominator of the expressions for r(t) and h(t) by $e^{m_3 t}$

$$r(t) = ke^{(m_1 - m_3)t}/(e^{(m_2 - m_3)t} + e^{0})$$

$$h(t) = e^{(m_2 - m_3)t}/(e^{(m_2 - m_3)t} + e^{0}) \ .$$

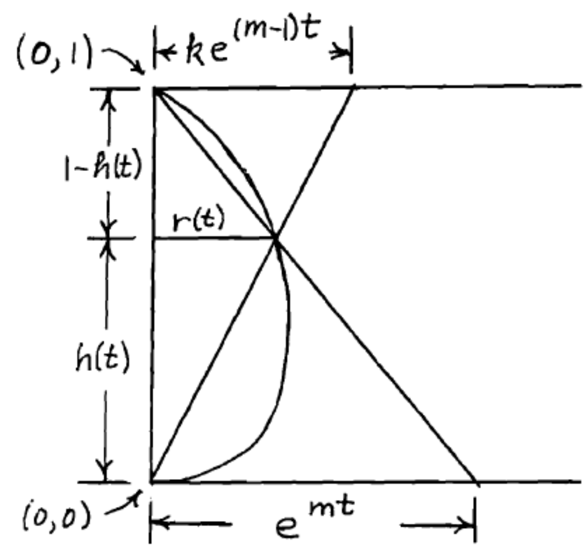
Next, choose the units of t so that $m_2 - m_3 = 1$, and define $m = m_1 - m_3$. We then finally obtain

$$r(t) = ke^{mt}/(e^{t} + 1) \tag{16}$$

$$h(t) = e^{t}/(e^{t} + 1) \tag{17}$$

These are the parametric equations of the bud forms, with r as the radius of the bud at height h. When t = 0, h = 1/2 and r = k, so k is the radius of the bud at mid height.

Figure 9 shows such a curve for the case 0 < m < 1. Its connection with Figure 3 becomes apparent if we draw a line from (0, 0) through (r,h) to intersect the horizontal line h = 1 at the point whose distance from the vertical axis is $ke^{(m-1)t}$. Clearly this distance is contracting exponentially, just as in Figure 3. Likewise, the line from (0,1) through (r,h) intersects the horizontal axis at ke^{mt}, This distance from the origin is expanding exponentially, just as in Figure 3. Therefore, (16) and (17) do indeed give the continuous form of the path on which x was walking in Figure 3. With step length of Δt, the multipliers on the top and bottom lines are $e^{(m-1)\Delta t}$ and $ke^{m\Delta t}$, respectively. Now λ is the ratio of the logarithms of these multipliers, so $\lambda = (m - 1)/m$.



For statistical fitting, it is convenient to divide (17) into (16) to get

$$r(t)/h(t) = ke^{(m-1)t} \tag{18}$$

or, in logarithmic form

$$\log r(t) - \log h(t) = \log k + (m-1)t. \tag{19}$$

We also solve (17) for t in terms of h, thus

$$t = \log [h(t)/(1 - h(t)]. \tag{20}$$

Now given observations on r at various values of h. we compute by (20) the values of t corresponding to these h's and then fit (19) by least squares. Note that the r, h, and t in (19) are known, and we seek log k and m-1 to give the closest fit to (19). Results are given in the next section.

We need to comment briefly on the case of complex roots in (6). If they occur, they occur in conjugate pairs, and it is easy to see that the corresponding v's are also complex conjugates. Furthermore, since the initial point is real, the corresponding k's must also be conjugate. We then use the definition

$$e^{(a+ib)t} = e^{at}(\cos(bt) + i\sin(bt))$$

Clearly, the exponential functions of conjugates are conjugate, so the vectors on the right of (13) occur in conjugate pairs and their imaginary components cancel in the summation. Suppose then that we have the following V matrix

$$V = \begin{pmatrix} 1 & 1 & 0 \\ -i & i & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

For simplicity, let us suppose that all the k's of (13) are 1, and that $m_1 = a + ib$, $m_2 = a - ib$, and m3 = m. Then (13) gives

$$x_1(t) = 2e^{at}\cos(bt)$$

$$x_2(t) = 2e^{at}\sin(bt)$$

$$x_3(t) = e^{mt}$$

If we now normalize with h = (0,0,1), we get a logarithmic spiral. Thus this beautiful curve, also found in nature, is a path curve.

Finally, we may go to three-dimensional space – described by 4-element homogeneous coordinates – and consider the equations with

$$V = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -i & i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The first three components will be as in the previous example, and the fourth will be another exponential function. Normalizing with h =(0.0,1.1) gives a family of three-dimensional curves that wind around a bud shape, as described in Edwards's second article [3]. Here we see how the three-dimensional form of the bud whispers of the fourth dimension.

Many plants look much alike in their first two leaves before they expand in infinite variety of shapes in their leaves and stems. Then in the bud they again contract, again touch back to the archetypal plant and let a higher dimension breathe through them before they open into the glory of the blossom. In the bud. the plant meditates; in the blossom, it works in this world.

4. How the Curves Fit the Buds

To fit a path curve to a bud profile by least squares, we find the values of log k and m- 1 to minimize

$$S = \sum_{i=1}^{7} [\log r_i - \log h_i - (\log(k) + (m-1)t_i)]^2$$

where the r_i are the observed values of the radius of the bud at the chosen heights h_i and the t_i are defined by (20) as

$t_i = \log[h_i/(1 - h_i)]$, i = 1, …, 7.

In the terms usually used in regression analysis, the dependent variable is $\log r_i - \log h_i$ and the independent variable is t_i ; the constant term found by the regression is log k, and the regression coefficient on the one independent variable is (m-1). If the heights at which the radii are measured are the same for all buds, then the independent variable is the same for all buds, making the calculations fairly easily done even by hand.

A slight complication is added by uncertainty about where the bottom of the bud is. The bud is always attached to the plant, so it cannot come down to a zero radius at h = O. Therefore, there is some uncertainty about where the bottom of the bud is. On each photograph, Edwards extrapolates freehand the path-curve-like part of the profile downwards to a point at the bottom. Consequently. this lower point is somewhat uncertain. If, for example, the bud was estimated to be 8 mm long but was in fact 9 mm long, then the 7 radii, measured at 1 mm intervals from the top would not have been at the heights

$h_i = 1 - i/8$ i = 1, 2, …, 7

on the standardized bud of height 1, but at the heights

$h_i = 1 - i/9$ i = 1, 2, …, 7.

More generally,

$$h_i = 1 - \frac{i}{b} \quad \text{i = 1, 2, …, 7.}$$

where b is the true length of the bud in units in which it was estimated to have been 8. I follow Edwards's practice of varying b and picking the b that gives the best fit. The variation in length, however, has been limited to five percent on either side of the “best guess” length estimated by Edwards by eye.

The results of fitting are shown in the accompanying tables. Edwards has collected bud profiles not only where he lives, in Scotland, but also on his travels in New Zealand and Australia. He gave me the data grouped by country. and I have left it in this grouping.

In the tables, the first column gives the value of m – 1, the parameter actually estimated by regression; the second column gives the value of λ derived from m - 1. The third column. labeled “ro" gives the value of k in the regression. (The notation ro recalls the fact that it is the “theoretlcal” radius at h =1/2, which corresponds to t = 0.) The column labeled “|er|" gives the average absolute error for the fit. The units are in hundredths of an inch on a bud shape four inches high. The column labeled “emax" gives the maximum absolute error, also in hundredths of an inch on a four in bud. The “mape” column give the mean absolute percentage error. The "rho" column gives the autocorrelation of errors along the curve. (Values of rho close to zero are good; they indicate that relatively large errors in the same direction do not tend to occur next to one another along the bud height. Thus, low values of rho indicate that the errors are "noise" in the observations rather than evidence of systematic deviation from path-curve forms.) Finally, the column labeled “length" gives the optimal-fit, theoretical length as a percent of the best-guess, free-hand estimate of the length of the bud.

Buds have not been excluded after taking measurements because they gave a poor fit. On the other hand, some plants have non-symmetric buds which are obviously not path curves, and Edwards has not gone to the trouble to measure them to prove the obvious. The outer budcase of the rhododendron, a beautiful, big bud visible all winter, was measured but was not a good path curve and is not given here. The inner inflorescence, however, is a good path curve. It was actually this plant from which Edwards learned to measure the inner shape, not the outer case.

In New Zealand and Australia, Edwards was often unable to identify the plant whose buds he collected. The observation is, however, no less relevant for the main point, namely, that plants produce path curves in their buds. The accuracy with which these plants, quite unrelated to those of Edwards's home country, produce path curves shows that this capacity is little connected with family lines, but comes directly from the nature of the plant kingdom.

Of the 150 species or varieties in the tables, 125 have an emax of less than 4 hundredths of an inch. That is, for these 125, the maximum deviation is less than one percent of the height of the bud. No bud had a maximum deviation of as much as two percent of the height. The mean absolute percentage error was under two percent for 109 of the buds; for only two of them did it slightly exceed four percent. The buds are higher than wide, so the errors are a larger percent of the radius than of the height.

Edwards has asked his classes to draw buds or to draw ovals flat at one end and pointed at the other. Only three or four percent of the results were path curves in the range of accuracy with which plants produce them. I have tried drawing bud-form path curves with only slightly better results. I suggest that you try a few freehand curves. You can then take two points on the curve, and – using the method of Figure 3 – check other points. Whether or not you prove better at it than I am, you may share my amazement that plants all over the world are out there producing path curves by the billion. And if you share my amazement, maybe you will share my joy.

Appendix 1: Results of Fitting a Path Curve to Bud Profiles

Scotland

bud	m - 1	lambda	ro	|er|	emax	mape	rho	length
aubresia	0.436	1.29	1.11	0.91	1.54	0.91	0.24	96.
blackberry	0.481	1.08	1.83	1.03	3.66	0.72	-0.31	102.
bluebell single bud	0.478	1.09	0.72	2.01	3.45	3.39	0.35	95.
bluebell inflor.	0.445	1.25	0.72	1.03	1.66	1.73	0.55	95.
buttercup i	0.309	2.24	1.80	1.03	2.97	0.66	-0.11	105.
buttercup 1i	0.364	1.75	2.27	2.06	4.01	1.12	0.03	101.
campanula	0.446	1.24	0.89	0.71	1.98	0.93	0.30	95.
campion, pink	0.260	2.85	0.96	0.88	2.05	1.18	0.24	98.
campion, white	0.271	2.69	0.99	2.68	4.06	3.26	0.52	95.
celandine	0.416	1.41	1.52	1.89	4.21	1.59	0.04	99.
cherry, ornamental	0.411	1.44	1.23	0.75	1.41	0.77	-0.02	99.
cherry, wild	0.470	1.13	1.70	1.15	3.81	0.77	-0.24	96.
chickweed i	0.353	1.83	0.62	0.70	1.58	1.31	-0.44	95.
chickweed ii	0.337	1.97	0.83	0.27	0.72	0.38	-0.09	103.
clematis	0.187	4.34	0.68	0.99	1.70	1.83	0.31	104.
columbine	0.148	5.75	0.68	1.61	2.91	3.16	0.24	104.
comfrey	0.542	0.85	0.62	0.49	0.73	0.89	0.05	913 .
convolvulus	0.253	2.94	0.47	0.60	1.30	1.53	0.21	103.
cornflower	0.327	2.05	1.30	0.48	1.22	0.41	-0.14	103.
creeping jenny	0.300	2.34	0.94	0.62	1.23	0.84	-0.01	102.
crow'sfoot	0.460	1.18	1.53	3.08	4.85	2.30	0.52	95.
currant, flowering I	0.308	2.25	1.08	2.48	4.96	2.57	0.23	105.
currant, flowering ii	0.319	2.14	1.06	2.24	4.68	2.46	0.16	105.
currant, flowering iii	0.377	1.65	1.03	1.40	2.65	1.58	0.53	105.
daffodil	0.523	0.91	0.50	1.14	1.85	2.73	0.40	95.
elm leafbud	0.357	1.80	0.85	0.66	2.33	0.86	-0.42	102.
forsythea	0.343	1.92	0.67	1.18	2.82	2.05	0.21	102.
fuschia	0.257	2.88	0.53	1.53	3.49	3.73	0.30	100.
garden briar rose	0.195	4.13	0.76	1.53	3.57	2.42	0.36	104.
garlic	0.194	4.16	0.80	2.27	3.93	3.18	0.58	105.
geranium	0.396	1.53	1.04	0.87	1.84	0.96	-0.10	103.
grape hyacinth inflo.	0.374	1.67	0.73	1.76	3.11	2.97	0.23	96.
hawthorn	0.504	0.98	1.82	1.01	2.63	0.68	-0.40	101.
honesty	0.468	1.14	0.71	0.70	1.66	1.10	-0.45	99.
hypericum	0.335	1.99	1.21	1.27	3.23	1.25	0.30	100.
ivy leafbud	0.428	1.33	1.15	0.65	1.40	0.64	-0.19	101.
jasmine, winter	0.325	2.07	0.84	0.67	1.30	0.92	0.41	100.
kale (flower bud)	0.441	1.21	1.18	0.77	1.92	0.78	0.30	99.
knapweed	0.453	1.21	1.54	0.88	2.64	0.66	0.02	101.
1arch (male flower bud)	0.455	1.20	1.60	0.89	2.06	0.69	-0.27	98.
1ime leafbud	0.357	1.80	0.75	0.54	1.51	0.89	0.24	101.
mahonia	0.485	1.06	0.85	0.85	1.45	1.17	0.23	95.

Scotland

bud	m – 1	lambda	ro	|er|	emax	mape	rho	length
narcissus	0.349	1.86	0.53	0.68	1.09	1.64	0.43	95.
phlox, dwarf	0.410	1.44	0.74	1.59	4.63	2.46	-0.08	102.
potentilla	0.447	1. 24	1.75	1.10	2.29	0.72	0.24	98.
oak (leafbud)	0.281	2.56	0.84	0.96	2.25	1.52	0.27	105.
poplar (leaf bud)	0.263	2.80	0.64	0.50	1.21	0.98	0.19	105.
poppy	0.277	2.62	1.00	1.44	2.61	1.59	0.31	105.
poppy (another variety)	0.346	1.89	0.96	0.92	2.29	1.14	0.04	105.
primrose	0.369	1.71	1.01	2.61	5.87	3.11	0.28	100.
pussy willow (catkin)	0.434	1.31	0.53	0.57	1.05	1. 21	0.48	95.
red may	0.511	0.96	1.90	1.01	3.29	0.57	-0.11	99.
rhododendron	0.344	1.91	1.24	0.85	2.71	0.77	-0.18	105.
sage	0.212	3.71	1.10	2.91	3.91	3.55	0.23	105.
sea thrift	0.290	2.45	1.16	1.01	2.06	1.11	-0.14	100.
sibirica	0.447	1.24	0.81	0.56	1.74	0.92	-0.15	98.
snowdrop	0.402	1.49	0.60	1.43	2.23	3.07	0.27	95.
speed well	0.299	2.35	0.93	0.80	2.09	0.97	0.09	105.
star of bethlehem	0.380	1.63	0.69	0.53	1.02	0.92	0.47	95.
stitchwort	0.371	1.70	0.76	1.12	2.04	1.86	0.07	99.
strawberry, wild	0.386	1.59	1.25	0.88	1.54	0.77	-0.11	104.
summer snow	0.450	1.22	0.87	0.59	1.21	0.82	0.01	98.
summer snow ii	0.422	1.37	0.84	0.67	1.79	0.94	-0.43	96.
sycamore flower bud	0.450	1.22	0.92	0.81	1.34	1.04	0.32	95.
sycamore leafbud	0.404	1.47	0.91	0.47	1.19	0.59	0.12	95.
syringa	0.322	2.11	0.90	2.74	5.20	3.55	0.54	105.
veronica	0.386	1.59	0.80	0.94	2.05	1.31	-0.02	101.
water lily	0.312	2.20	0.83	0.73	1.40	1.03	-0.18	103.
whortleberry	0.343	1.92	1.37	3.67	5.93	3.35	0.25	95.
wild iris	0.421	1.38	0.48	0.97	1.97	2.31	0.34	97.
wood sorrel	0.349	1.86	0.88	0.68	1.45	0.94	-0.09	101.
wood sorrel (dif.year)	0.345	1.90	0.69	0.45	0.88	0.77	0.35	99.

Australia

bud	m – 1	lambda	ro	|er|	emax	mape	rho	length
boromia	0.385	1.59	1.82	0.92	1.73	0.58	0·39	101.
camelia	0.324	2.09	1.42	1.68	4.16	1.39	0.14	104.
Christmas bells ?	0.435	1.30	1.08	0.54	1.57	0.56	-0.41	99.
clematis, bush	0.366	1.73	0.92	1.23	2.65	1.52	0.18	105.
epocris	0.198	4.04	0.68	0.59	1.37	1.00	-0.05	103.
eriostemon i	0.389	1.57	1.00	1.15	2.56	1.39	0.10	101.
eriostemon ii	0.392	1.55	1.30	2.17	3.70	1.93	0.51	95.
eriostemon iii	0.347	1.88	0.99	1.58	2.95	2.12	0.09	95.
eucalyptus i	0.404	1.48	1.34	1.83	2.84	1.69	0.21	95.
eucalyptus ii	0.137	6.31	1.14	1.58	3.39	1.68	0.39	105.
gardenia	0.340	1.95	0.99	0.89	1.70	1.23	0.13	100.
hibertia stricta	0.244	3.09	1.08	0.88	2.36	1.10	-0.27	103.
hibiscus i	0.343	1.91	0.80	0.44	1.17	0.61	0.05	100.
hibiscus ii	0.440	1.27	1.57	0.64	1.86	0.48	0.03	99.
hibiscus iii	0.292	2.43	0.78	1.09	3.17	1.56	-0.06	95.
mallow	0.265	2.78	0.70	1.0 1	1.82	1.99	0.12	99.
wild vine	0.129	6.78	0.81	2.17	4.14	2.81	0.18	105.
unidentified i	0.299	2.35	0.90	2.13	2.87	3.21	0.25	95.
unidentified ii	0.360	1.77	1.62	0.57	1.35	0.39	0.10	99.
unidentified iii	0.387	1.58	0.65	0.73	2.17	1.26	0.21	95.
unidentified iv	0.410	1.44	0.98	0.75	1.47	0.94	-0.07	98.
unidentified v	0.148	5.74	0.87	3.46	6.46	4.25	0.18	105.
uniden tified vi	0.377	1.65	0.85	0.90	1.50	1.33	0.09	102.
unidentified vii	0.376	1.66	0.85	1.11	2.88	1.42	0.01	95.
unidentified viii	0.494	1.03	0.85	1.80	4.04	2.50	-0.06	95.
unidentified ix	0.395	1.53	0.84	0.61	0.99	0.88	0.28	98.
unidentified x	0.377	1.65	0.98	0.71	1.43	0.80	-0.22	99.
unidentified xi	0.353	1.84	0.98	0.61	1.04	0.67	0.04	103.
unidentified xii	0.373	1.68	1.01	1.13	3.06	1.23	-0.03	96.
unidentified xiii	0.559	0.79	1.55	3.52	6.15	2.57	0.34	101.
unidentified xiv	0.251	2.99	0.46	0.27	0.54	0.81	-0.08	97.

New Zealand

bud	m – 1	lambda	ro	|er|	emax	mape	rho	length
apple	0.463	1.16	1.32	0.50	1.09	0.44	-0.22	98.
berberis	0.575	0.74	1.53	1.69	3.40	1.28	-0.06	98.
cabbage (flower plant)	0.438	1.28	1.40	0.80	1.39	0.70	-0.18	101.
cabbage tree	0.609	0.64	1.02	1.09	2.04	1.18	0.16	95.
california fuschia	0.248	3.04	1.06	0.72	1.84	0.70	-0.08	104.
camelia	0.346	1.89	1.50	1.95	4.45	1.56	0.16	105.
Chinese lantern	0.401	1.49	1.47	0.60	1.58	0.50	-0.25	100.
clematis, bush	0.413	1.42	1.29	1.55	2.55	1.50	0.18	104.
cress	0.477	1.10	1.23	1.83	3.43	1.75	0.24	97.
daphne	0.385	1.60	0.86	0.57	0.89	0.77	-0.16	100.

New Zealand (continued)

bud	m – 1	lambda	ro	|er|	emax	mape	rho	length
diosma	0.268	2.73	0.77	2.66	5.68	4 .11	0.47	105.
flax	0.210	3.76	0.76	2.00	4.17	3.21	0.13	105.
fremontia	0.190	4.26	1.18	2.12	5.15	1.81	0.33	105.
geranium	0.384	1.60	0.90	0.87	3.07	1.13	-0.28	105.
ginger	0.519	0.93	0.94	1.83	3.20	2.25	0.61	95.
holly	0.620	0.61	1.53	0.72	1.51	0.51	0.32	99.
horopito	0.327	2.06	2.48	2.77	5.11	1.25	0.28	102.
indian hawthorn, pink	0.264	2.79	1.28	1.07	2.34	1.16	-0.09	100.
indian hawthorn, white	0.274	2.65	1.14	2.00	3.25	1.98	0.33	105.
iris, new zealand	0.479	1.09	1.12	0.41	0.94	0.44	-0.06	96.
jasmine, minature	0.294	2.40	1.09	1.21	2.50	1.30	-0.35	95.
japonica	0.323	2.10	1.35	1.32	2.76	1.15	0.35	99.
karaka	0.397	1.52	1.55	0.62	1.11	0.44	-0.04	97.
keria	0.240	3.17	0.81	0.40	0.86	0.64	-0.08	101.
lady's smock	0.399	1.51	0.69	1.05	2.33	1.94	0.04	97.
lemon	0.553	0.81	1.50	1.79	2.72	1.39	0.16	99.
magnolia	0.315	2.18	0.65	1.03	2.61	1.92	0.33	99.
magnolia stellata	0.314	2.18	0.63	2.23	4.45	4.32	0.37	103.
magnolia, port wine	0.333	2.00	1.09	0.38	0.62	0.39	-0.36	101.
malus	0.453	1.21	1.18	0.58	1.61	0.57	0.11	98.
maori privet	0.279	2.58	1.71	3.47	7.48	2.35	0.38	104.
ngaio	0.378	1.64	0.94	1.41	2.47	1.68	-0.07	99.
oxalis	0.384	1.61	0.50	0.42	0.55	0.97	-0.12	99.
peach	0.396	1.52	1.45	1.42	5.28	1.06	-0.15	102.
periwinkle	0.245	3.07	0.87	2.91	4.84	4.38	0.43	95.
pimpernel, scarlet	0.313	2.20	0.92	2.09	3.37	2.88	0.23	96.
poroporo	0.517	0.93	1.13	0.50	1.32	0.50	0.37	95.
quince	0.251	2.98	1.04	1.63	3.11	1.85	0.47	105.
rose, wild	0.196	4.10	0.91	2.24	3.48	2.95	0.20	105.
tawari	0.318	2.14	1.75	0.97	2.31	0.68	-0.20	103.
vibernum	0.425	1.35	1.74	1.65	2.86	1.05	0.04	102.
wiegela	0.544	0.84	0.89	0.81	1.57	1.04	0.28	97.
unidentified i	0.354	1.83	1.70	2.13	5.13	1.44	0.18	103.
unidentified ii	0.606	0.65	0.93	1.33	2.76	1.64	0.29	104.
unidentified iii	0.195	4.13	0.55	1.54	3.14	3.55	0.30	102.
unidentified iv te mata?	0.336	1.97	0.59	0.38	0.89	0.71	-0.42	100.
unidentified v	0.412	1.43	1.92	1.80	3.25	1.09	0.35	97.

Appendix 2. Data on Plant Buds

Scotland Data.

	F	E	D	T	C	B	A
auresia	0.530	0.880	1.043	1.120	1.110	1.945	0.910
blackberry	1.180	1.570	1.760	1.830	1.780	1.550	1.200
bluebell single bud	0.470	0.580	0.654	0.700	0.590	0.650	0.590
bluebell inflor.	0.430	0.570	0.550	0.700	0.715	0.590	0.600
buttercup i	0.840	1.300	1.650	1.830	1.940	1.870	1.570
buttercup ii	1.180	1.670	2.040	2.250	2.360	2.310	1.910
campanula	0.520	0.710	0.820	0.880	0.870	0.840	0.740
campion, pink	0.400	0.620	0.795	0.940	1.040	1.100	1.030
campion, white	0.430	0.640	0.790	0.940	1.040	1.130	1.110
celandine	0.875	1.150	1.375	1.520	1.555	1.460	1.210
cherry, ornamental	0.670	0.975	1.145	1.220	1.245	1.185	0.985
cherry, wild	1.040	1.405	1.615	1.710	1.690	1.535	1.330
chickweed i	0.305	0.440	0.540	0.630	0.640	0.650	0.590
chickweed ii	0.410	0.610	0.750	0.840	0.880	0.860	0.710
clematis	0.260	0.420	0.560	0.680	0.790	0.950	0.800
columbine	0.250	0.395	0.540	0.690	0.820	0.910	0.880
comfrey	0.435	0.565	0.620	0.620	0.600	0.540	0.440
convolvulus	0.200	0.310	0.400	0.460	0.520	0.540	0.480
cornflower	0.630	0.950	1.160	1.320	1.385	1.360	1.135
creeping jenny	0.433	0.650	0.825	0.955	1.024	1.010	0.890
crowsfoot	0.930	1.260	1.400	1.470	1.480	1.460	1.250
currant, flowering i	0.515	0.755	0.944	1.100	1.210	1.170	0.910
currant, flowering ii	0.524	0.760	0.915	1.100	1.170	1.120	0.890
currant, flowering iii	0.545	0.790	0.970	1.065	1.090	0.990	0.745
daffodil	0.355	0.430	0.480	0.485	0.475	0.440	0.380
elm (1eaf bud)	0.430	0.640	0.770	0.855	0.890	0.830	0.725
forsythea	0.340	0.490	0.580	0.656	0.700	0.710	0.575
fuschia	0.235	0.340	0.420	0.525	0.600	0.620	0.565
garden briar rose	0.300	0.480	0.620	0.760	0.880	0.965	0.885
garlic	0.290	0.500	0.710	0.860	0.945	0.958	0.885
geranium	0.574	0.795	0.970	1.064	1.064	0.980	0.780
grape hyacinth inflores	0.390	0.520	0.630	0.7Q0	0.760	0.770	0.660
hawthorn	1.230	1.560	1.770	1.820	1.760	1.540	1.160
honesty	0.440	0.590	0.690	0.715	0.715	0.630	0.520
hypericum	0.590	0.870	1.040	1.190	1.270	1.280	1.100
ivy leafbud	0.670	0.920	1.080	1.170	1.160	1.060	0.860
jasmine, winter	0.400	0.600	0.730	0.825	0.890	0.890	0.780
kale (flower bud)	0.705	0.950	1.090	1.180	1.190	1.110	0.900
knapweed	0.940	1.260	1.450	1.540	1.550	1.390	1.080
larch (male flower bud)	0.945	1.325	1.505	1.590	1.595	1.460	1.210
lime leafbud	0.385	0.555	0.665	0.755	0.790	0.765	0.545
mahonia	0.540	0.700	0.790	0.835	0.840	0.790	0.645
narcissus	0.260	0.370	0.450	0.510	0.545	0.550	0.510
phlox, dwarf	0.425	0.580	0.670	0.730	0.750	0.745	0.540

Scotland cont.

	F	E	D	T	C	B	A
potentilla	1.020	1.420	1.650	1.765	1.730	1.620	1.340
oak leafbud	0.385	0.575	0.720	0.R50	0.920	0.920	0.780
poplar (leaf bud)	0.280	0.435	0.550	0.650	0.715	0.710	0.620
poppy	0.440	0.675	0.890	1.040	1.120	1.095	0.915
poppy (another variety)	0.475	0.725	0.885	0.980	1.005	0.935	0.765
primrose	0.565	0.755	0.880	1.020	1.075	1.085	0.885
pussy willow (catkin)	0.305	0.425	0.490	0.520	0.530	0.510	0.455
red may	1.280	1.650	1.840	1.935	1.840	1.630	1.270
rhododendron	0.620	0.930	1.140	1.260	1.300	1.215	1.000
sage	0.400	0.760	0.990	1.170	1.260	1.280	1.190
sea thrift	0.520	0.780	1.007	1.180	1.264	1.260	1.160
sibirica	0.490	0.640	0.756	0.810	0.310	0.760	0.625
snowdrop	0.344	0.440	0.525	0.590	0.610	0.600	0.550
speedwell	0.423	0.665	0.840	0.953	1.000	0.965	0.830
star of Bethlehem	0.355	0.510	0.605	0.675	0.700	0.695	0.635
stitchwort	0.403	0.550	0.670	0.770	0.805	0.765	0.660
strawberry, wild	0.675	0.980	1.140	1.245	1.290	1.200	0.925
summer snow	0.510	0.720	0.825	0.865	0.860	0.810	0.655
summer snow ii	0.470	0.640	0.790	0.840	0.850	0.810	0.700
sycamore flower bud	0.545	0.730	0.840	0.900	0.910	0.880	0.745
sycamore 1eafbud	0.485	0.695	0.835	0.910	0.930	0.890	0.800
syringa	0.410	0.660	0.860	0.970	0.980	0.900	0.740
veronica	0.425	0.630	0.745	0.785	0.815	0.795	0.650
water lily	0.390	0.580	0.740	0.850	0.890	0.860	0.750
whort1eberry	0.610	1.010	1.253	1.373	1.400	1.370	1.325
wild iris	0.260	0.380	0.460	0.500	0.486	0.445	0.397
wood sorrel	0.445	0.640	0.790	0.890	0.940	0.895	0.770
wood sorrel (dif. year)	0.330	0.500	0.620	0.690	0.715	0.700	0.625

Australia

	F	E	D	T	C	B	A
boromia	0.960	1.395	1.685	1.840	1.870	1.760	1.470
camelia	0.670	1.045	1.305	1.460	1.515	1.430	1.230
Christmas bells ?	0.630	0.860	1.010	1.075	1.100	1.010	0.830
clematis, bush	0.475	0.700	0.870	0.950	0.950	0.870	0.700
epocris	0.255	0.425	0.590	0.700	0.730	0.815	0.790
eriostemon i	0.550	0.760	0.905	0.995	1.025	1.000	0.800
eriostemon ii	0.695	0.970	1.150	1.245	1.300	1.320	1.180
eriostemon iii	0.490	0.575	0.840	0.965	1.030	1.055	0.940
eucalyptus i	0.740	1.000	1.195	1.310	1.360	1.355	1.180
eucalyptus ii	0.375	0.695	0.980	1.200	1.370	1.465	1.495
gardenia	0.495	0.750	0.870	0.995	1.050	1.030	0.905
hibertia stricta	0.450	0.695	0.935	1.105	1.210	1.230	1.130
hibiscus i	0.390	0.575	0.715	0.810	0.840	0.815	0.715
hibiscus ii	0.915	1.270	1.490	1.570	1.570	1.450	1.200
hibiscus iii	0.330	0.520	0.695	0.760	0.815	0.850	0.825
mallow	0.305	0.450	0.585	0.700	0.780	0.800	0.745
wild vine	0.275	0.460	0.660	0.850	1.015	1.095	1.050
unidentified i	0.415	0.575	0.735	0.855	0.955	0.990	0.950
unidentified ii	0.810	1.195	1.465	1.625	1.685	1.630	1.435
unidentified iii	0.340	0.485	0.585	0.620	0.660	0.660	0.590
unidentified iv	0.530	0.775	0.905	0.985	0.995	0,935	0.310
unidentified v	0.315	0.500	0.705	0.910	1.105	1.180	1.070
unidentified vi	0.460	0.635	0.770	0.815	0.835	0.840	0.685
unidentified vii	0.435	0.635	0.750	0.825	0.875	0.895	0.775
unidentified viii	0.535	0.740	0.825	0.855	0.825	0.735	0.570
unijentified ix	0.455	0.635	0.755	0.830	0.855	0.830	0.710
unidentified x	0.510	0.730	0.895	0.990	1.020	0.965	0.845
unidentified xi	0.495	0.730	0.900	1.000	1.020	0.975	0.815
unidentified xii	0.515	0.750	0.885	0.990	1.045	1.055	0.905
unidentified xiii	1.105	1.450	1.605	1.600	1.430	1.190	0.880
unidentified xiv	0.185	0.290	0.380	0.455	0.500	0.520	0.505

New Zealand

apple	0.810	1.080	1.250	1.330	1.310	1.205	0.990
berberis	1.140	1.465	1.545	1.545	1.425	1.230	0.940
cabbage (flower plant)	0.815	1.150	1.325	1.395	1.400	1.275	1.020
cabbage tree	0.845	0.970	1.030	1.030	0.970	0.850	0.645
california fuschia	0.440	0.710	0.930	1.090	1.190	1.190	1.090
camelia	0.740	1.140	1.410	1.540	1.580	1.460	1.200
chinese lantern	0.810	1.130	1.360	1.480	1.510	1.420	1.180
clematis, bush	0.710	1.055	1.230	1.305	1.290	1.170	0.900
cress	0.745	1.040	1.195	1.250	1.210	1.085	0.920
daphne	0.460	0.650	0.785	0.870	0.895	0.840	0.715
diosma	0.310	0.540	0.700	0.840	0.865	0.820	0.710
flax	0.310	0.470	0.625	0.770	0.880	0.945	0.820
fremontia	0.445	0.730	1.010	1.250	1.425	1.455	1.335
geranium	0.490	0.700	0.840	0.915	0.920	0.825	0.665

	F	E	D	T	C	B	A
ginger	0.650	0.830	0.890	0.910	0.900	0.845	0.705
holly	1.260	1.510	1.585	1.535	1.390	1.170	0.830
horopito	1.170	1.790	2.270	2.540	2.610	2.530	2.225
indian hawthorn, pink	0.545	0.830	1.085	1.290	1.405	1.430	1.340
indian hawthorn, white	0.495	0.780	1.045	1.200	1.270	1.230	1.070
iris, new zealand	0.705	0.925	1.060	1.115	1.110	1.030	0.850
jasmine, minature	0.470	0.710	0.930	1.075	1.160	1.150	1.140
japonica	0.620	0.965	1.220	1.365	1.420	1.410	1.285
karaka	0.825	1.175	1.420	1.550	1.580	1.520	1.330
keria	0.330	0.525	0.690	0.815	0.903	0.945	0.880
lady's smock	0.355	0.550	0.640	0.685	0.590	0.670	0.590
lemon	1.065	1.395	1.510	1.515	1.415	1.210	0.925
magnolia	0.290	0.465	0.600	0.660	0.690	0.680	0.630
magnolia stellata	0.320	0.440	0.520	0.615	0.695	0.695	0.570
magnolia,port wine	0.520	0.790	0.965	1.095	1.150	1.120	0.975
malus	0.700	0.965	1.125	1.175	1.170	1.085	0.900
maori privet	0.730	1.200	1.590	1.795	1.850	1.820	1.635
ngaio	0.480	0.720	0.865	0.915	0.940	0.950	0.800
oxalis	0.265	0.375	0.450	0.505	0.520	0.490	0.425
peach	0.795	1.135	1.345	1.400	1.485	1.405	1.118
periwinkle	0.370	0.530	0.670	0.815	0.940	1.020	1.020
pimpernel, scarlet	0.435	0.610	0.765	0.890	0.990	1.022	0.920
poroporo	0.760	0.985	1.095	1.125	1.090	1.010	0.830
quince	0.425	0.710	0.945	1.085	1.175	1.160	1.025
rose, wild	0.360	0.560	0.745	0.930	1.080	1.140	1.020
tawari	0.840	1.250	1.570	1.760	1.870	1.860	1.560
vibernum	1.010	1.420	1.610	1.720	1.760	1.640	1.260
wiegela	0.625	0.810	0.895	0.900	0.850	0.750	0.595
unidentified i	0.850	1.255	1.600	1.745	1.780	1.675	1.400
unidentified ii	0.750	0.920	0.980	0.935	0.820	0.660	0.420
unidentified iii	0.220	0.335	0.440	0.545	0.640	0.705	0.555
unidentified iv te mata?	0.285	0.435	0.525	0.590	0.535	0.615	0.540
unidentified v	1.070	1.480	1.730	1.875	1.960	1.890	1.600

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There are also several websites devoted to the work of Lawrence Edwards, path curves, and projective geometry