PH.D. THESIS

Large Deviations for Partial Sum Processes on Infinite Time Intervals with Applications to Single-Server Queues and Effective Bandwidths

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Abstract

Title of Dissertation: Large Deviations for Partial Sum Processes on Infinite Time Intervals with Applications to Single-Server Queues and Effective Bandwidths

Lionel Banege, Doctor of Philosophy, 1996

Dissertation directed by: Professor Armand M. Makowski
Department of Electrical Engineering

In this Dissertation, we establish large deviations results for partial sum processes on infinite time intervals, and apply them to the characterization of the large deviations behavior of the stationary and transient output processes of a single-server queue with time-varying capacities. We first show that the extension of a partial sum process on the infinite time interval $[0, \infty)$ satisfies the Large Deviations Principle (LDP) in the function space $D[0, \infty)$, provided the partial sum process itself satisfies the LDP in the space $D[0, 1]$. Furthermore, for a stationary random sequence whose associated partial sum process satisfies the LDP in $D[0, 1]$, we establish the LDP jointly for a partial sum process based on the entire past and future of the sequence, a result especially useful in queueing theory. Through a functional approach at the sample path level, the Contraction Principle then enables us to derive the sample path LDP for processes of interest in the study of single-server queues, from that of the inputs. Finally, using our results, we refine the newly introduced notion of effective bandwidths.

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Dedication

To my parents

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Chapter 1

Introduction and Background on Large Deviations

In this Dissertation, we establish large deviations results for a special class of processes, and apply them to characterize the large deviations of the output of a single-server queue with time-varying capacities.

This introductory chapter is organized as follows. In the first section, we introduce and motivate our work. In Section 1.2, we present the approach taken here and briefly review the difficulties encountered. The outline of the Dissertation is presented in Section 1.3, while the main contributions are outlined in Section 1.4. Finally, the chapter is completed with a brief review of the principal large deviations definitions and techniques used in the Dissertation.

1.1 Introduction

The characterization of the output processes of general single-server queues has been a long standing and interesting problem in queueing theory [19]. Their interest lies in the fact that such processes may themselves be inputs to other queueing systems. The recent renewed interest in large deviations and in particular in their applications to queueing theory has created new opportunities to revisit this problem from a large deviations stand point.

Indeed, the characterization of the large deviations of the output process of a single-server queue has numerous applications. The first one lies in the recently introduced notion of effective bandwidth, which we now briefly discuss.

1.1.1 Effective bandwidths

A great deal of attention has recently been devoted to the development of the notion of effective bandwidth as a means to help resolve bandwidth allocation issues in high speed networks. In this approach, the network is viewed as a collection of interconnected nodes, each of them modeled as a discrete-time queue. Let $\{a_{t+1},\ t=0,1,\ldots\}$ denote a traffic stream into any one of the queues, with the usual understanding that a_{t+1} cells arrive into the node during time slot $[t,t+1)$. To define the effective bandwidth of this arrival stream we offer it to a fictitious single server queue with constant release rate c cells/slot. If the fictitious server is equipped with an infinite buffer, the buffer content sequence $\{x_t,\ t=0,1,\ldots\}$ evolves according to

$$x_0 = x; \quad x_{t+1} = [x_t + a_{t+1} - c]^+, \quad t = 0,1,\ldots . \tag{1.1}$$

If the arrival process $\{a_{t+1},\ t = 0,1,\ldots\}$ is stationary and ergodic, and if $\mathbf{E}[a_1] < c$, then steady state is eventually reached in the sense that $x_t \Longrightarrow_t x_\infty$ for some $\mathbb{R}_+$-valued rv x_∞. Several authors [16, 20, 22, 28, 35, 42] have shown that under reasonably mild assumptions, the following buffer asymptotics

$$\lim_{b\to\infty} \frac{1}{b} \ln \mathbf{P}[x_\infty > b] = -\theta^\star$$

hold for some positive constant $\theta^\star$, i.e.,

$$\mathbf{P}[x_\infty > b] \sim \exp(-b\theta^\star), \quad b \to \infty . \tag{1.2}$$

The constant $\theta^\star$ is an increasing function $\theta^\star(c)$ of the release rate c, and is determined by the statistics of the arrival process $\{a_{t+1},\ t=0,1,\ldots\}$. Building upon this result, it was further shown [22, 42] that the performance of the system could be held above a certain level if and only if a linear constraint on the statistics of the input were satisfied, namely

$$\lim_{b\to\infty} \frac{1}{b} \ln \mathbf{P}[x_\infty > b] \le -\delta \iff \alpha(\delta) \le c, \quad \delta > 0 \tag{1.3}$$

where the **effective bandwidth** $\alpha(\delta)$ is a function of the statistics of the process $\{a_{t+1}.\ t=0,1,\ldots\}$. In view of (1.2), we can interpret $\alpha(\delta)$ as the smallest release rate that supports the QoS level characterized by

$$\mathbf{P}[x_\infty > b] \sim \exp(-b\delta), \quad b \to \infty .$$

What renders this notion particularly useful for bandwidth allocation, is its separability property with respect to independent sources. In other words, for

K independent sources, it can be shown [22], that under some mild additional assumptions, (1.3) becomes

$$\lim_{b\to\infty} \frac{1}{b} \ln \mathbf{P}[x_\infty > b] \le -\delta \iff \sum_{k=1}^{K} \alpha_k(\delta) \le c, \quad \delta > 0$$

where for each $k = 1, \ldots, K$, the effective bandwidth $\alpha_k(\delta)$ is a function of the statistics of the k^{th} source only. The use of effective bandwidth for bandwidth allocation has been discussed in [37].

In the literature, the most general situation in which the effective bandwidth can be derived is that of a multiplexer fed by a stationary and ergodic traffic stream whose sample mean sequence, among other assumptions, satisfies the Large Deviations Principle (LDP) in $\mathbb{R}$ with good rate function given as the Legendre-Fenchel transform of the associated logarithmic moment generating function Λ. The effective bandwidth $\alpha(\delta)$ then becomes a simple function of Λ, namely $\alpha(\delta) = \dfrac{\Lambda(\delta)}{\delta}$.

It is plain however that for this notion to be operationally useful, one needs to understand how the effective bandwidth of a given stream is altered as the stream traverses the network, through its interaction for network resources with other streams at various nodes. The simplest possible situation that addresses this issue is obtained when considering the output process from a single node network, in which case the issue really lies in the characterization of the large deviations of the output process. The motivation for considering time-varying capacities comes from the possible application of the effective bandwidth to bandwidth allocation on satellite channels or impaired wireless channels.

1.1.2 Convergence of the departures from an infinite series of queues

Secondly, in the context of tandem queues, the large deviations characterization of the output process could yield new results, or confirm earlier results on the convergence of the departures from series of queues. Indeed, ever since the results of Vere-Jones [65], there has been considerable interest in the study of infinite series of queues [1, 4, 5, 34, 49]. Lately, it has (finally) been proved [4] that the departures from an infinite series of independent exponential servers with rate unity fed by a stationary and ergodic source with rate $\alpha < 1$, weakly converges to a Poisson process. By considering the queueing system as an operator on point processes, the convergence of the departures of an infinite series of identical queues can then be rephrased as the existence of a fixed point for this input-output operator [4, 5, 65] .

It is plain that a similar functional approach can be applied to the rate functions governing the large deviations of the input and output processes of a queue. In this context, it is then interesting to investigate the following issues:

i) Are the existing results on the existence of a fixed point for the input-output mapping confirmed for the rate function operator?

ii) In the general case, does the rate function operator admit a fixed point? What about the particular situation when the service distribution is predetermined, e.g., exponential, or when the input distribution is given?

iii) What is the rate function, or the effective bandwidth of the departures from an infinite series of queues?

Although the same issues applied to the processes themselves would be fairly complex, one would expect their counterparts for the rate functions being simpler as the functional now operates on rate functions, rather than on processes.

1.1.3 Results obtained

We point out that the problem of characterizing the large deviations of the output process is a difficult one. Indeed, because one is often interested in the steady-state behavior of the system, the output process of interest is in fact the stationary one. Unfortunately, unlike the transient output process which has finite memory over the inputs to the system, the stationary output process has infinite memory, and is characterized as a function of the entire past of the system. This renders its large deviations analysis much more complicated than that of the transient output process.

Because the output process is a function of the inputs to the system over a certain period of time, a sample path approach, where the processes are embedded into continuous-time ones, appears promising, as it embodies the statistics of the process over an entire time interval.

However, for the the stationary output, the approach is further complicated by the fact that the sample path approach must be performed on an infinite time interval, whereas compact time intervals are enough for the transient case.

In this Dissertation, we have obtained the following results. First, we have achieved a major step towards the characterization of the large deviations of the output process of a general single-server queue with **time-varying** capacities, in that we have established the existence of the sample path Large Deviations Principle for both the stationary and transient output processes, from that of the inputs. That had not been previously done in the literature. An interesting question is then whether the rate functions associated with these two LDP coincide. Indeed, if they were to coincide, then many problems would be greatly simplified, as the transient output process is much more easily handled. Although we have obtained expressions for both rate functions, the question of knowing whether or not they are equal, or of finding conditions under which they coincide remains open.

Secondly, we have been able, trough a sample path approach, to refine earlier results on the effective bandwidth. In particular, once some remaining technical problems are solved, this approach will easily yield the effective bandwidth of a stream which traverses an intree network.

Finally, on a more theoretical front, we have established that for a particular class of functional processes, namely partial sum processes, the LDP for the process defined on compact time intervals implies the LDP for its extension on infinite time intervals. We have also obtained a companion result for stationary sequences, as in that case the LDP for the process defined on a compact time interval implies the LDP **jointly** for a process defined on the **entire** time axis. In particular, when the rate function is of integral form, we have shown that in terms of the sample path large deviations, the sequence behaves as if the past were independent of the future. Those theoretical results are foreseen to have many implications in queueing theory, and we here only illustrate their use in the context of large deviations for single-server queues.

In the next section, we present in some details the sample path approach we have taken in the Dissertation.

1.2 The sample path approach

To help understand what motivated our sample path approach, we develop a representation of the output process of a single-server queue in terms of the basic inputs random variables of the model.

1.2.1 The output process of a single-server queue

We consider a discrete-time **G/G/1** queue with arrival and capacity sequences $\{a_{t+1},\ t=0,1,\ldots\}$ and $\{c_{t+1},\ t=0,1,\ldots\}$, where a_{t+1} (resp. c_{t+1}) denotes the number of arrivals (resp. capacity) in the time interval $[t,t+1)$, $t=0,1,\ldots$. The queue length sequence $\{q_t,\ t=0,1,\ldots\}$ is then generated through the Lindley recursion

$$q_0=q; \quad q_{t+1}=[q_t+a_{t+1}-c_{t+1}]^+, \quad t=0,1,\ldots, \tag{1.4}$$

where the initial condition q is some $\mathbb{R}_+$-valued random variable.

The output process $\{b_{t+1},\ t = 0, 1, \ldots\}$, where b_{t+1} represents the number of departures in the time interval $[t, t+1)$, is given by

$$b_{t+1} = a_{t+1} - (q_{t+1} - q_t), \quad t = 0, 1, \ldots \ . \tag{1.5}$$

If we use the notation

$$X(t_1, t_2) = \begin{cases} \sum_{t=t_1}^{t_2} x_t & \text{if} \quad t_1 \le t_2 \\ 0 & \text{otherwise} \end{cases}$$

for any random sequence $\{x_{t+1},\ y = 0, 1, \ldots\}$, then it is easily checked that

$$B(1,t) = C(1,t) + \min\left\{0,\ q + \min_{s=1,\ldots,t}(A(1,s) - C(1,s))\right\}, \quad t = 0, 1, \ldots \ .$$

Defining the random variables $\{m(1,t),\ t = 0, 1, \ldots\}$ by

$$m(1,t) \equiv \min_{s=1,\ldots,t}(A(1,s) - C(1,s)), \quad t = 1, 2, \ldots,$$

we note that

$$B(1,t) = F(C(1,t), q + m(1,t)), \quad t = 1, 2, \ldots \tag{1.6}$$

where the mapping $F : \mathbb{R}^2 \to \mathbb{R}$ is given by

$$F(x,y) = x + \min(0,y), \quad x,\ y \in \mathbb{R}.$$

It is clear from the relation (1.6) that the statistics of the transient output process ($q = 0$) up to time t are a function of those of the inputs on the **same** time interval $[0, t]$.

On the other hand, the stationary output process is obtained from (1.6) with $q = q_{st}$ and

$$q_{st} = \left[\max_{t=0,1,\ldots}(A(-t,0) - C(-t,0))\right]^+,$$

so that its statistics up to time t can be expressed as a function of those of the inputs on the **entire past** and the future up to t.

Although (1.6) suggests that an approach at the sample path level might be more promising, with all discrete-time processes of interest being embedded into continuous-time ones, our last comment already suggests that establishing the LDP for the stationary output process is inherently much more difficult.

The difficulty is to find conditions on the input process which are strong enough to yield the result, and at the same time weak enough to ensure that they fully propagate to the output process. In that sense, the combination of the sample path LDP with the Contraction Principle seems to be a natural choice.

Let $D[0,1]$ denote the space of right-continuous functions on $[0,1]$ with left-hand limits, endowed with the uniform norm, and for any sequence $\{x_t,\ t = 0,1,\ldots\}$, define the random element $X_n(\cdot)$ in $D[0,1]$ by setting

$$X_n(t) \equiv \begin{cases} \frac{1}{n}\sum_{i=1}^{\lfloor nt \rfloor} x_i & \text{if } \lfloor nt \rfloor \geq 1 \\ 0 & \text{otherwise} \end{cases}, \quad t \in [0,1], \quad n = 1,2,\ldots \quad (1.7)$$

Following [23], we refer to $X_n(\cdot) \equiv \{X_n(t),\ t \in [0,1]\}$ as the partial sum process associated with $\{x_t,\ t = 0,1,\ldots\}$.

1.2.2 Formulation of the problem

Using (1.6), we write

$$\frac{B(1,\lfloor nt \rfloor)}{n} = F\left(\frac{C(1,\lfloor nt \rfloor)}{n}, \frac{q}{n} + \frac{m(1,\lfloor nt \rfloor)}{n}\right), \quad t \in [0,1], \quad (1.8)$$

where

$$\begin{aligned} \frac{m(1,\lfloor nt \rfloor)}{n} &= \min_{s=1,\ldots,\lfloor nt \rfloor} \left(n^{-1}(A(1,s) - C(1,s))\right) \\ &= \inf_{0 \leq s \leq t} \left(n^{-1}(A(1,s) - C(1,s))\right), \quad t \in [0,1]. \end{aligned}$$

Therefore, defining the families of random processes $\{q_n(\cdot),\ n = 1,2,\ldots\}$ and $\{m_n(\cdot),\ n = 1,2,\ldots\}$ by

$$q_n(t) \equiv \frac{q}{n} \quad \text{and} \quad m_n(t) \equiv \inf_{0 \leq s \leq t} (A_n(s) - C_n(s)), \quad t \in [0,1],$$

we can rewrite (1.8) as

$$B_n(\cdot) = \tilde{F}(C_n(\cdot), q_n(\cdot) + m_n(\cdot)), \quad n = 1,2,\ldots, \quad (1.9)$$

where the (continuous) mapping $\tilde{F}: D[0,1]^2 \to D[0,1]$ is defined by

$$\tilde{F}(\psi_1, \psi_2) \equiv \psi_1 + \min\{0, \psi_2\}, \quad \psi_1, \psi_2 \in D[0,1].$$

In principle, the sample path LDP for the transient output process $(q = 0)$ as well as for its stationary version $(q = q_{st})$ can now be derived from the joint

LDP for the family of partial sum processes $\{(C_n(\cdot), m_n(\cdot), q_n(\cdot)),\ n = 1, 2, \ldots\}$ through a simple application of the Contraction Principle.

The use of the sample path LDP together with the Contraction Principle was first introduced in [16] in the context of effective bandwidths. It is shown there under the assumption that the arrivals sequence satisfies the sample path LDP and is bounded, that the **transient** output process of a single server queues with **constant** capacity satisfies also a sample path LDP. In that situation, $q = 0$, and it is readily seen from (1.9) that the partial sum process associated with the transient output is a simple function of the processes $A_n(\cdot)$ and $C_n(\cdot)$. Furthermore, under the assumption of bounded arrivals and constant capacity, $B_n(\cdot)$ can then be expressed as a function of a process exponentially equivalent to $A_n(\cdot)$, so that the result follows easily by applying the Contraction Principle (in one dimension), upon noting that the reflection mapping is continuous in $D[0, 1]$.

Although this natural sample path approach seems promising, in the general setup of **G/G/1** queues, it rises several difficult technical problems.

1.2.3 Technical difficulties encountered

When applying the sample path approach to the **stationary** output process of a single-server queue with **time-varying** capacities three problems immediately arise: First, we need to establish the LDP **jointly** for $C_n(\cdot)$, $m_n(\cdot)$ and $q_n(\cdot)$ in the product space, in order to obtain that for $B_n^{st}(\cdot)$ via the Contraction Principle and (1.9). Secondly, the LDP for the inputs is required on the **entire past**, and thirdly, we must somehow show that the LDP is preserved when going to the limit in order to get that of $q_n(\cdot)$.

The underlying problem in establishing the LDP in the product space and then applying the Contraction Principle is that unless the space is separable, the triplet of random elements may not anymore be a random element in the product space. In particular, its distribution law may not be defined on the Borel σ-field of the product space, which then prevents us to use most of the large deviations techniques, as it is often required that the probability measures be either Borel (e.g. in the framework of [25]), or defined on a σ-field which contains the Borel σ-field [24, p. 5].

This forces us to use the separable Skorohod topology, as the space $D[0, 1]$ endowed with the uniform topology is **not** separable. Consequently, many functionals loose their continuity properties, thus generating further technical difficulties in applying the Contraction Principle.

To establish the LDP jointly for $\{(C_n(\cdot), m_n(\cdot), q_n(\cdot)),\ n = 1,2,\ldots\}$, we observe that $q_n(\cdot)$ can be expressed as

$$q_n(t) = \sup_{s\geq 0}\left(A_n^-(s) - C_n^-(s)\right), \quad t \in [0,1] \tag{1.10}$$

where for a stationary sequence $\{x_t,\ t = 0, \pm 1, \pm 2, \ldots\}$, we have defined

$$X_n^-(t) = \begin{cases} \sum_{i=1-\lceil nt\rceil}^{0} x_i & \text{if } 1 \leq \lfloor nt \rfloor \\ 0 & \text{otherwise} \end{cases}, \quad t \in [0,\infty), \quad n = 1,2,\ldots .$$

Thus, the desired LDP could in principle be derived through the Contraction Principle if we knew that the family $\{(A_n(\cdot), C_n(\cdot), A_n^-(\cdot), C_n^-(\cdot)),\ n = 1,2,\ldots\}$ satisfies the LDP in the product space $D[0,1]^2 \times D_l[0,\infty)^2$, where $D_l[0,\infty)$ is the space of left continuous functions $f : [0,\infty) \to \mathbb{R}$ with right-hand limits, endowed with the extended Skorohod topology.

The problem really consists in establishing, for a stationary sequence $\{x_t,\ t = 0, \pm 1, \pm 2, \ldots\}$, the LDP jointly for $\{(X_n(\cdot), X_n^-(\cdot)),\ n = 1,2,\ldots\}$ from that of $\{X_n(\cdot),\ n = 1,2,\ldots\}$. Using again the Contraction Principle and the stationarity of the sequence, we first derive the LDP for the restrictions of $\{(X_n(\cdot), X_n^-(\cdot)),\ n = 1,2,\ldots\}$ on $D[0,K] \times D_l[0,K]$, for $K = 1,2,\ldots$. Then, by restricting, we obtain the LDP in the spaces $D[0,1] \times D_l[0,T]$, for T irrational. From there, the natural way to establish the LDP in the space $D[0,1] \times D_l[0,\infty)$ is to use the Dawson-Gärtner Theorem, and identify the space $D[0,1] \times D_l[0,\infty)$ with the projective limit of the spaces $(D[0,1] \times D_l[0,T])_{T>0}$. In our case, this approach is further complicated by the fact that the restriction mappings are **not** continuous in the Skorohod topology. Nevertheless, we are able to overcome this difficulty by considering the LDP in a particular subspace of $D[0,1] \times D_l[0,\infty)$, and projecting on suitably chosen subspaces of $D[0,1] \times D_l[0,T]$ where the restrictions are continuous in the relative Skorohod topology.

Finally, once the LDP is obtained jointly on the entire past and the future, we need to translate it to the LDP for the supremum in (1.10). This we achieve by first considering the LDP for the supremum functional, and then by downsizing the LDP on a particular subspace where the Contraction Principle can be applied for the passage to the limit.

We point out that everything turns out to work well because the Contraction Principle can be extended to mappings which are only Borel-measurable and continuous on the effective domain, and that typically, i) the effective domain of the rate functions associated with the LDP of partial sum processes is contained in the space of absolutely continuous functions, and ii), on that space the induced Skorohod topology coincide with the uniform topology.

Thus, the simple fact of considering the stationary output processes of single-server queue with time-varying capacities really generates major technical difficulties; this may help explain why people have focused mostly on the transient output process.

1.2.4 Literature survey

The problem of deriving the effective bandwidth for the output process of a single-server queue has recently been addressed by several authors [7, 16, 18, 29, 51, 20]. The difference between those treatments lies in the model chosen, e.g., discrete-time vs. continuous-time single-server queues, constant vs. time-varying capacities, and stationary vs. transient output processes. In what appears to be the first paper on this issue, de Veciana et al. [20] study the large deviations behavior of the stationary output process of a discrete-time **G/D/1** queue. As pointed out in [7], the result of [20] holds under certain technical assumptions on the arrival process which are developed in [23]; however, the authors in [20] do not show that these technical assumptions hold for the departure process, thereby precluding their result to be applied inductively to networks of queues.

In [16], Chang derives closure properties (sum, reduction, composition and reflection mapping) for sample path large deviations. The model considered there is still a discrete-time single-server queue with deterministic capacity, but the **transient** output process (output process resulting from a queue with empty initial buffer content) is considered. The author assumes that the arrival process is stationary, ergodic, **bounded** (a very restricted assumption) and adapted to a filtration. The existence and differentiability of a limiting log moment generating function are also assumed. Under some technical assumptions, it is shown that the transient output process satisfies a sample path LDP and that the buffer asymptotics (1.2) hold for the queue fed by the transient output process. In [18], the effective bandwidth of the **stationary** output process of a discrete-time queue with bounded arrivals and time-varying capacity is heuristically derived. However, the argument are not made at the sample path level and, as pointed out by the authors, cannot be used inductively. An important point made in [18] is that, except when the queue has constant capacity, the effective bandwidth of the stationary and transient output processes are a priori different.

In [7] Bertsimas, Paschalidis and Tsitsiklis study the large deviations behavior of networks of **G/G/1** queues and establish, under some technical assumptions, that the stationary output process of a continuous-time single server queue with i.i.d. service times satisfies the Large Deviation Principle (LDP). They also show that their assumptions are satisfied by the output process, a fact which allows them to propagate the result.

We became aware only recently of a revised version of [51] in which the author establishes the sample path LDP for the stationary output process of a **G/G/1** queue from that of the input. The assumptions required are joint sample path LDP for arrivals and service processes and bounded exponential moments. The author uses a sample path approach not on the partial sum processes of the sequences of interest, but rather on their continuous polygonal approximations. By doing so, some of the technicalities arising from the use of the Skorohod topology disappear. However, the author implicitly uses the incorrect fact that the space of measurable functions equipped with the uniform topology is separable. Furthermore, the author uses without any justification the fact that the LDP for each of the sequences $\{q_n(\cdot),\ n = 1, 2, \ldots\}$ and $\{(C_n(\cdot), m_n(\cdot)),\ n = 1, 2, \ldots\}$ implies the joint LDP in the product space. Although this is true for i.i.d. sequences, we do not know of any results which would yield the joint LDP when the arrivals and capacity sequences are only stationary. Our results were obtained independently.

Finally, at this point, we do not know of any result in the literature which establishes the LDP for partial sum processes in $D[0, \infty)$ from that in $D[0, 1]$.

1.3 Outline of the Dissertation

The dissertation is organized in a linear way, as follows. Chapters 2 and 3 consist of the theoretical results on the LDP for the extension of partial sum processes on infinite time intervals. The finite time interval case is treated in Chapter 2, where we introduce the Skorohod topology on the spaces $D[0, T]$ and $D_l[0, T]$, and establish some results on the LDP for the extension of partial sum processes on these spaces. The infinite time interval case is then presented in Chapter 3. There, we begin by introducing the extended Skorohod topology on the space $D[0, \infty)$ and then formally transpose it to the space $D_l[0, \infty)$. Topological properties needed later, in particular Borel-measurability and continuity results for some functional are discussed. The general results on the existence of the LDP jointly for the partial sum process associated with the entire past and future of a stationary sequence is finally presented.

Building upon Chapter 3, we then establish in Chapter 4 the LDP for some functionals of particular interest in the study of Lindley processes or single-server queues. Those results are then applied to the characterization of the large deviations behavior of Lindley processes and discrete-time single-server queues in Chapters 5 and 6, respectively. In particular, the derivation of the sample path LDP for the output process of the single-server queue is presented in Section 6.3. Finally, those large deviations results are in turn applied to analyzing the effective bandwidth at the sample path level in Chapter 7, where the applicability to queues in series is also investigated.

A few words on the numbering used in the Dissertation. Within a chapter, Theorems, Propositions and Lemmas are all numbered sequentially in order of appearance.

1.4 Contributions

The contributions of this Dissertation are twofold.

First in the context of large deviations for partial sum processes, we were able to establish two useful results. The first one states that the LDP in $D[0, \infty)$ for the extension of a partial sum process on $[0, \infty)$ is a consequence of that of the partial sum process in $D[0, 1]$. In addition, when the associated rate function is of the integral form, then so is the new rate function. The second one, for stationary random sequences, is very interesting. It basically states that under some mild continuity assumptions which are satisfied in all known situations where the LDP holds, the LDP jointly for a partial sum process based on the **entire past and future** of the random sequence follows from that of the partial sum process in $D[0, 1]$. It also shows that, in terms of large deviations, the past and future behave as if they were independent.

The other contributions relate to the large deviations behavior of the **G/G/1** queues, and the notion of effective bandwidths. In this context, the most important result is the propagation of the sample path LDP from the input to a single-server queue with **time-varying** capacities to its output process, under some very mild assumptions on the associated rate function. In particular, it does not require that neither the arrivals and capacity processes, nor the associated exponential moments be bounded, so that our assumptions are much weaker than those in [16, 51].

We also show that the steady-state queue length satisfies the LDP in the function space $D[0, 1]$, under the assumption that either the input and capacity process jointly satisfy a sample path LDP, or the queue length satisfies asymptotics of a particular form. Finally, our sample path approach yields a sample path definition of the effective bandwidth which greatly simplify the assumptions required in the traditional derivation.

1.5 Background on large deviations

In this section, we review some of the large deviations techniques that we use in the course of this dissertation. All the material presented here can be found in [24].

1.5.1 Definitions and related properties

We use the definitions and conventions of [24]. For a topological space $(\mathcal{X}, \tau_{\mathcal{X}})$, $\mathcal{B}_{\mathcal{X}}$ will always denote its Borel σ-field, i.e., the smallest σ-field containing all the open sets.

Let $(\mathcal{X}, \tau)$ be a topological space.

Definition 1.1 *A rate function I is a lower semi-continuous mapping $I : \mathcal{X} \to [0, \infty]$. A rate function I is said to be a good rate function if it is level-compact, i.e., all its level sets $\Psi_I(\alpha) \equiv \{x : I(x) \leq \alpha\}$ are compact subsets of $\mathcal{X}$. The effective domain $\mathcal{D}_I$ of I is the set of points in $\mathcal{X}$ of finite rate, namely $\mathcal{D}_I \equiv \{x \in \mathcal{X} : I(x) < \infty\}$.*

Let $\{\mu_n,\ n = 1, 2, \ldots\}$ be a collection of probability measures defined on some σ-field $\mathcal{B}$ of $\mathcal{X}$, which **contains** the Borel σ-field $\mathcal{B}_{\mathcal{X}}$.

Definition 1.2 *The family $\{\mu_n,\ n = 1, 2, \ldots\}$ satisfies the Large Deviations Principle (LDP) in $(\mathcal{X}, \tau)$ with a rate function $I : \mathcal{X} \to [0, \infty]$ if*

$$\liminf_{n\to\infty} \frac{1}{n} \ln \mu_n(\Gamma) \ \geq \ -\inf_{x\in\Gamma^\circ} I(x) \tag{1.11}$$

and

$$\limsup_{n\to\infty} \frac{1}{n} \ln \mu_n(\Gamma) \ \leq \ -\inf_{x\in\overline{\Gamma}} I(x) \tag{1.12}$$

for every Γ in $\mathcal{B}_{\mathcal{X}}$.

We stress that we require in the definition above that the bounds (1.11) and (1.12) hold for all **Borel sets** Γ. This definition differs from that given in [24, p. 5] as there the bounds are required to hold only on the σ-field $\mathcal{B}$ on which the probability measures are defined. The motivation for imposing the bounds in the LDP to hold on the Borel σ-field comes from the fact that most of the definitions and properties in [24] require the inclusion $\mathcal{B}_{\mathcal{X}} \subseteq \mathcal{B}$, to hold. To avoid any technicalities, we will always consider Borel probability measures, in which case they are defined on the Borel σ-field , and Definition 1.2 becomes equivalent to the definition of the LDP given in [24, p. 5].

We list below some easy yet important properties of rate functions which we use throughout the discussion.

Lemma 1.3 1. *Let $I : \mathcal{X} \to [0, \infty]$ be a rate function. Then I is Borel-measurable. Moreover, I achieves its minimum on any non-empty compact subset of $\mathcal{X}$.*

2. *Let $I : \mathcal{X} \to [0, \infty]$ be a good rate function. Then I achieves its minimum on any closed subset of $\mathcal{X}$ which has a non-empty intersection with its effective domain $\mathcal{D}_I$.*

3. *Assume the collection $\{\mu_n,\ n = 1, 2, \ldots\}$ to satisfy the LDP with rate function $I : \mathcal{X} \to [0, \infty]$. Then, $\inf_{x \in \mathcal{X}} I(x) = 0$. Furthermore, if the rate function I is good, then the infimum is achieved, i.e., there exists c in $\mathcal{X}$ such that*

$$I(c) \;=\; \inf_{x \in \mathcal{X}} I(x) \;=\; 0.$$

4. *A family of probability measures $\{\mu_n,\ n = 1, 2, \ldots\}$ on a Hausdorff regular topological space can have at most one rate function associated with its LDP.*

Proof: **(1)**: Borel-measurability of I follows easily from the closeness property of lower semi-continuous functions [12, Prop. I, p. 361].

That I achieves its minimum on any non-empty compact subset of $\mathcal{X}$ is proved in [12, Theorem 3, p. 361].

(2): Let F be a closed subset of $\mathcal{X}$ such that $F \cap \mathcal{D}_I \neq \emptyset$, and let x_0 be in $F \cap \mathcal{D}_I$. Clearly,

$$\inf_{x \in F} I(x) \;=\; \inf \{ I(x) : x \in F \cap \Psi_I(I(x_0)) \}$$

whence, $F \cap \Psi_I(I(x_0))$ being compact [12, Prop. III, p. 86] and non empty, I achieves its minimum by assertion **(1)**.

(3): We follow [25, Exercise 2.1.14, p. 44]. The LDP lower and upper bounds (with $\Gamma = \mathcal{X}$) yield $\inf_{x \in \mathcal{X}} I(x) = 0$ and the first part of the assertion is proved.

Hence there exists x_0 in $\mathcal{X}$ such that $I(x_0) < \infty$, and it is now plain that

$$\inf_{x \in \mathcal{X}} I(x) \;=\; \inf\{ I(x) : x \in \mathcal{X},\ I(x) \leq I(x_0) \},$$

so that, when I is a good rate function, it follows from assertion **(1)** that the infimum is achieved.

(4): This result is given as Lemma 4.1.4 in [24, p. 103]. ■

So far, the only structure we had on the space $\mathcal{X}$ was its topology. However, additional constraints on the topology are required in order to avoid embarrassing situations such as the non-uniqueness of the rate function associated with a LDP. Therefore, we will always assume that the topological spaces considered are Hausdorff and regular, which we refer to as simply regular. Since a metric space is regular, this will be the case in all our applications of large deviations techniques, as we always consider metrizable topological spaces.

For now on, $\{\mu_n,\ n = 1, 2, \ldots\}$ is a collection of probability measure defined on the Borel σ-field $\mathcal{B}_{\mathcal{X}}$ of a Hausdorff regular topological space $(\mathcal{X}, \tau)$.

There exists a weaker formulation of the LDP, which in some instances is the first step in establishing the full LDP.

Definition 1.4 *The family of probability measures $\{\mu_n,\ n = 1, 2, \ldots\}$ satisfies the weak LDP with a rate function I if the lower bound (1.11) holds for all open sets and the upper bound (1.12) holds for all compact sets.*

As shown in [24, p. 7], there exist families of probability measures that satisfy the weak LDP with a good rate function, but which do not satisfy the full LDP. Nevertheless, in some cases the weak LDP implies the full LDP. To discuss this point, we need a definition.

Definition 1.5 *The family of probability measures $\{\mu_n,\ n = 1, 2, \ldots\}$ is exponentially tight if for every $\alpha < \infty$, there exists a compact set $K_\alpha \subset \mathcal{X}$ such that*

$$\limsup_{n\to\infty} \frac{1}{n} \ln \mu_n(K_\alpha^c) \ < \ -\alpha. \tag{1.13}$$

Lemma 1.6 ([24, Lemma 1.2.18 p. 8]) *Let $\{\mu_n,\ n = 1, 2, \ldots\}$ be an exponentially tight family. If it satisfies the weak LDP with rate function I, then it satisfies the full LDP and I is a good rate function.*

Conversely, in some cases, the full LDP implies exponential tightness.

Proposition 1.7 **1.** *Assume that $\mathcal{X}$ is locally compact. If the family $\{\mu_n,\ n = 1, 2, \ldots\}$ satisfies the LDP with good rate function I, then it is exponentially tight.*

2. *Assume that $\mathcal{X}$ is a Polish space and that the family $\{\mu_n,\ n = 1, 2, \ldots\}$ satisfies the LDP upper bound with good rate function I. Then, the sequence of measures $\{\mu_n,\ n = 1, 2, \ldots\}$ is exponentially tight.*

Proof: The proposition is given in [24] as Exercise 1.2.19 p. 9 and 4.1.10 p. 105. A proof can be found in Appendix A.1. ■

As we discuss large deviations at the sample path level, we will be led to consider random elements in a topological space $(\mathcal{X}, \tau)$. A random element ξ on some probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and taking on values in a topological space $(\mathcal{X}, \tau)$ is defined as a measurable mapping $\xi : (\Omega, \mathcal{F}) \to (\mathcal{X}, \mathcal{B}_{\mathcal{X}})$, where $\mathcal{B}_{\mathcal{X}}$ is the Borel σ-field on the topological space $(\mathcal{X}, \tau)$. The distribution law of the random element ξ is then the probability measure $\mu : \mathcal{B}_{\mathcal{X}} \to [0, 1]$ defined by

$$\mu(B) \equiv \mathbf{P}[\xi \in B], \quad B \in \mathcal{B}_{\mathcal{X}}.$$

Given a collection of random elements $\{\xi_n,\ n = 1, 2, \ldots\}$ in the topological space $(\mathcal{X}, \tau)$, we define the LDP for $\{\xi_n,\ n = 1, 2, \ldots\}$ as the LDP for the collection of probability measure $\{\mu_n,\ n = 1, 2, \ldots\}$, where for each $n = 1, 2, \ldots$, μ_n is the distribution law of ξ_n. In this setup, it should be clear that for each $n = 1, 2, \ldots$, μ_n is indeed a probability measure defined on the Borel σ-field $\mathcal{B}_{\mathcal{X}}$ of $(\mathcal{X}, \tau)$, as was required earlier.

Similarly, the family of random elements $\{\xi_n,\ n = 1, 2, \ldots\}$ is said to be exponentially tight if the family of distribution laws $\{\mu_n,\ n = 1, 2, \ldots\}$ is exponentially tight.

The sample path approach we take to study the large deviations behavior of single-server queues in series requires establishing the LDP for n-uples of random elements. Thus, we need to place ourselves in a framework where an n-uple of random elements is still a random element (in the Cartesian product space) and its distribution law is a Borel probability measure on the product topological space. More precisely, for each $n = 1, 2, \ldots$, let X_n be a random element in the topological space $(\mathcal{X}_n, \tau_n)$, i.e., $X_n : (\Omega, \mathcal{F}) \to (\mathcal{X}_n, \mathcal{B}_{\mathcal{X}_n})$ is a measurable mapping. We can then construct the mapping $X : \Omega \to \prod_n \mathcal{X}_n$ by setting

$$X(\omega) = (X_1(\omega), X_2(\omega), \ldots), \quad \omega \in \Omega.$$

If $\mathcal{B}$ is the tensor product of the σ-fields $\mathcal{B}_{\mathcal{X}_n}$, i.e.,

$$\mathcal{B} \equiv \mathcal{B}_{\mathcal{X}_1} \otimes \mathcal{B}_{\mathcal{X}_2} \otimes \ldots,$$

then it is well known [27, p. 55] that X is $\mathcal{F}/\mathcal{B}$-measurable. However, in order for X to be a random element in the topological product space $\prod_n \mathcal{X}_n$ (as previously defined), it is necessary that X be a measurable mapping from $(\Omega, \mathcal{F})$ into $(\prod_n \mathcal{X}_n, \mathcal{B}_{\prod \mathcal{X}_n})$, where $\mathcal{B}_{\prod \mathcal{X}_n}$ is the Borel σ-field of the product space $\prod_n \mathcal{X}_n$ endowed with the product topology.

Although the inclusion

$$\otimes_{i=1}^{\infty} \mathcal{B}_{\mathcal{X}_i} \subseteq \mathcal{B}_{\prod \mathcal{X}_i}$$

always holds for any collection of topological spaces $\{\mathcal{X}_i,\ i = 1, 2, \ldots\}$ [53, p. 6], the converse inclusion

$$\mathcal{B}_{\prod \mathcal{X}_i} \subseteq \otimes_{i=1}^{\infty} \mathcal{B}_{\mathcal{X}_i}$$

requires additional properties on the topologies to hold. In particular it holds for separable Hausdorff spaces [53, Theorem 1.10 p. 6].

Therefore, in this dissertation, we shall consider only **separable** Hausdorff spaces, so that n-uples of random elements are themselves random elements. In addition, this automatically ensures that the distribution law of the n-uple of random elements be defined on the Borel σ-field of the product space, as required in our treatment of large deviations. We note that this separability requirement does not appear in the general abstract framework of [24], to the contrary of that in [25, p. 35] which consists of Borel probability measures on separable metric spaces.

In some cases, the joint LDP in product spaces is a consequence of the LDP in each of the component spaces.

Proposition 1.8 *Let $\{X_n,\ n = 1, 2, \ldots\}$ and $\{Y_n,\ n = 1, 2, \ldots\}$ be two mutually independent families of random elements in the separable regular topological spaces $\mathcal{X}$ and $\mathcal{Y}$, respectively. Assume the families $\{X_n,\ n = 1, 2, \ldots\}$ and $\{Y_n,\ n = 1, 2, \ldots\}$ each to satisfy the LDP respectively in $\mathcal{X}$ and $\mathcal{Y}$, with good rate functions I_X and I_Y, and to be exponentially tight.*

Then the family $\{(X_n, Y_n),\ n = 1, 2, \ldots\}$ satisfies the LDP in $\mathcal{X} \times \mathcal{Y}$ (endowed with the product topology) with good rate function $I_{X,Y}$ given by

$$I_{X,Y}(x, y) = I_X(x) + I_Y(y), \quad x \in \mathcal{X}, \quad y \in \mathcal{Y} .$$

Proof: This result is given as Exercise 4.2.7 of [24, p. 113], and its proof can be found in Appendix A.2. ■

Because $\mathcal{X}$ and $\mathcal{Y}$ need not be Polish, the assumption on exponential tightness is required, as it does not automatically follow from Proposition 1.7. However, if $\mathcal{X}$ and $\mathcal{Y}$ are Polish spaces, Proposition 1.8 simplifies to the following Corollary.

Corollary 1.9 *Let $\{X_n,\ n = 1, 2, \ldots\}$ and $\{Y_n,\ n = 1, 2, \ldots\}$ be two mutually independent families of random elements in the Polish spaces $\mathcal{X}$ and $\mathcal{Y}$, respectively. Assume the families $\{X_n,\ n = 1, 2, \ldots\}$ and $\{Y_n,\ n = 1, 2, \ldots\}$ each to satisfy the LDP respectively in $\mathcal{X}$ and $\mathcal{Y}$, with good rate functions I_X and I_Y.*

Then the family $\{(X_n, Y_n),\ n = 1, 2, \ldots\}$ satisfies the LDP in $\mathcal{X} \times \mathcal{Y}$ (endowed with the product topology) with good rate function $I_{X,Y}$ given by

$$I_{X,Y}(x, y) \ = \ I_X(x) + I_Y(y), \quad x \in \mathcal{X}, \quad y \in \mathcal{Y}.$$

Some of the known cases where a LDP indeed holds, including the theorems of Cramér and Gärtner-Ellis , are reviewed in Chapter 7, where the LDP is used to derive the effective bandwidth for certain classes of queueing systems.

1.5.2 General principles

We review here some of the most important properties of the LDP that we use in the forthcoming chapters. All the properties stated in this section are borrowed from [24].

The next result is our main tool in establishing LDPs. For the sake of completeness, because the statement is only proved for the case f continuous in [24, Theorem 4.2.1 p. 110], a proof is presented in Appendix A.3.

Theorem 1.10 (Contraction Principle) *Let $\mathcal{X}$ and $\mathcal{Y}$ be two regular topological spaces and let $\{\mu_n,\ n = 1, 2, \ldots\}$ be a family of probability measures on the Borel σ-field $\mathcal{B}_X$ of X. Assume the family $\{\mu_n,\ n = 1, 2, \ldots\}$ satisfies the LDP in $\mathcal{X}$ with good rate function $I : \mathcal{X} \to [0, \infty]$, and let the Borel-measurable mapping $f : \mathcal{X} \to \mathcal{Y}$ be continuous on the effective domain $\mathcal{D}_I$ of I.*

Then the family $\{\mu_n \circ f^{-1},\ n = 1, 2, \ldots\}$ of probability measures on $\mathcal{B}_\mathcal{Y}$ satisfies the LDP in $\mathcal{Y}$ with good rate function I' given by

$$I'(y) \ \equiv \ \inf_{x \in \mathcal{X}} \{I(x) : y = f(x)\}, \quad y \in \mathcal{Y}. \tag{1.14}$$

In particular, if f is continuous on the whole space $\mathcal{X}$, then the family $\{\mu_n \circ f^{-1},\ n = 1, 2, \ldots\}$ satisfies the LDP in $\mathcal{Y}$ with good rate function I' as above.

We complete these general principles with two useful results which yield the LDP in one space from that in another space.

First, as demonstrated by the following lemma, the LDP can either be lifted in a bigger space or down-sized to a smaller space. Because this Lemma will be needed in proving a crucial step in Chapter 4, we present its proof (based on that sketched in [24]) in Appendix A.4.

Recall that for any subset $\mathcal{E}$ of $\mathcal{X}$, the Borel σ-field $\mathcal{B}_{\mathcal{E}}$ of $\mathcal{E}$ is the trace on $\mathcal{E}$ of the Borel σ-field of $\mathcal{X}$, i.e., $\mathcal{B}_{\mathcal{E}} = \mathcal{B}_{\mathcal{X}} \cap \mathcal{E}$ [53, Theorem 1.9 p. 5].

Lemma 1.11 **([24, Lemma 4.1.5 p. 104])** *Consider a Borel-measurable subset $\mathcal{E}$ of the regular topological space $\mathcal{X}$, endowed with the topology induced by $\mathcal{X}$. Let $\{\mu_n,\ n = 1,2,\ldots\}$ be a family of Borel probability measure on $\mathcal{X}$, satisfying $\mu_n(\mathcal{E}) = 1$ for all $n = 1,2,\ldots$.*

1. *If the family $\{\mu_n,\ n = 1,2,\ldots\}$ satisfies the LDP in $\mathcal{X}$ with rate function I, then $\mathcal{D}_I \subset \mathrm{cl}_{\mathcal{X}}(\mathcal{E})$.*

2. *If $\mathcal{E}$ is a closed subset of $\mathcal{X}$ and the restriction of the family $\{\mu_n,\ n = 1,2,\ldots\}$ to the Borel σ-field $\mathcal{B}_{\mathcal{E}}$ satisfies the LDP in $\mathcal{E}$ with rate function (resp. good rate function) I, then the family $\{\mu_n,\ n = 1,2,\ldots\}$ satisfies the LDP in $\mathcal{X}$ with rate function (resp. good rate function) I' given by*

$$I'(x) = \begin{cases} I(x), & x \in \mathcal{E} \\ \infty, & x \notin \mathcal{E}. \end{cases} \tag{1.15}$$

3. *If the family $\{\mu_n,\ n = 1,2,\ldots\}$ satisfies the LDP in $\mathcal{X}$ with rate function (resp. good rate function) I such that $\mathcal{D}_I \subset \mathcal{E}$, then its restriction to the Borel σ-field $\mathcal{B}_{\mathcal{E}}$ satisfies the LDP in $\mathcal{E}$ with rate function (resp. good rate function) the restriction of I to $\mathcal{E}$.*

Finally, for comparable topologies, the LDP in a finer topology yields the LDP in a coarser one, and conversely in presence of exponential tightness or if the topologies coincide when relativized to the effective domain.

Lemma 1.12 *Let $\mathcal{X}$ be a set and let τ_1, τ_2 be two regular topologies on $\mathcal{X}$, such that τ_1 is finer than τ_2, i.e., $\tau_2 \subset \tau_1$. Let $\{\mu_n,\ n = 1,2,\ldots\}$ be a family of probability measure defined on the Borel σ-field $\mathcal{B}_{\tau_1}$ of τ_1.*

1. *If the family $\{\mu_n,\ n = 1,2,\ldots\}$ satisfies the LDP in $(\mathcal{X},\tau_1)$ with good rate function $I : \mathcal{X} \to [0,\infty]$, then its restriction to the Borel σ-field $\mathcal{B}_{\tau_2}$ satisfies the LDP in the coarser topology τ_2 with the same (good) rate function.*

2. *If the family* $\{\mu_n,\ n = 1, 2, \ldots\}$ *is exponentially tight in* τ_1 *and its restriction to the Borel* σ*-field* $\mathcal{B}_{\tau_2}$ *satisfies the LDP in* $(\mathcal{X}, \tau_2)$ *with rate function* $I : \mathcal{X} \to [0, \infty]$, *then it satisfies the LDP in the finer topology* τ_1 *with the same rate function.*

3. *If the restriction of the family* $\{\mu_n,\ n = 1, 2, \ldots\}$ *to the Borel* σ*-field* $\mathcal{B}_{\tau_2}$ *satisfies the LDP in* $(\mathcal{X}, \tau_2)$ *with rate function (resp. good rate function)* $I : \mathcal{X} \to [0, \infty]$, *and the two topologies coincide when relativized to the effective domain of* I, *then the family* $\{\mu_n,\ n = 1, 2, \ldots\}$ *satisfies the LDP in the finer topology* τ_1 *with the same rate function (resp. good rate function).*

We point out that in Assertion (3), it is required that the family of probability measure $\{\mu_n,\ n = 1, 2, \ldots\}$ be defined on the Borel σ-field $\mathcal{B}_{\tau_1}$ of the **finer** topology.

Proof: The first assertion follows easily from the Contraction Principle by taking the natural embedding from $(\mathcal{X}, \tau_1)$ onto $(\mathcal{X}, \tau_2)$ as the (continuous) contraction mapping, while the second one is given as Corollary 4.2.6 in [24, p. 113].

Next, the LDP bounds in the last assertion follow from the LDP in $(\mathcal{X}, \tau_2)$ and the inequality

$$\mu_n\left(\mathrm{int}_{\tau_2}(\Gamma)\right) \leq \mu_n(\Gamma) \leq \mu_n\left(\mathrm{cl}_{\tau_2}(\Gamma)\right), \quad n = 1, 2, \ldots, \quad \Gamma \in \mathcal{B}_{\tau_1},$$

once we note the elementary facts

$$\begin{aligned} \mathcal{D}_I \cap \mathrm{int}_{\tau_1}(\Gamma) &= \mathcal{D}_I \cap \bigcup\{O : O \in \tau_1,\ O \subset \Gamma\} \\ &= \bigcup\{O \cap \mathcal{D}_I : O \in \tau_1,\ O \subset \Gamma\} \\ &= \bigcup\{O \cap \mathcal{D}_I : O \in \tau_2,\ O \subset \Gamma\} \\ &= \mathcal{D}_I \cap \mathrm{int}_{\tau_2}(\Gamma), \quad \Gamma \subset \mathcal{X}, \end{aligned}$$

$$\begin{aligned} \mathcal{D}_I \cap \mathrm{cl}_{\tau_1}(\Gamma) &= \mathcal{D}_I \cap \bigcap\{O^c : O \in \tau_1,\ \Gamma \subset O\} \\ &= \bigcap\{O^c \cap \mathcal{D}_I : O \in \tau_1,\ \Gamma \subset O\} \\ &= \bigcap\{O^c \cap \mathcal{D}_I : O \in \tau_2,\ \Gamma \subset O\} \\ &= \mathcal{D}_I \cap \mathrm{cl}_{\tau_2}(\Gamma), \quad \Gamma \subset \mathcal{X}. \end{aligned}$$

For $\alpha \geq 0$, $\Psi_I(\alpha)$ is closed in τ_2, thus closed in the finer topology τ_1, so that I is a rate function in $(\mathcal{X}, \tau_1)$. If I is good in $(\mathcal{X}, \tau_2)$, then $\Psi_I(\alpha)$ is compact in $(\mathcal{X}, \tau_2)$, hence in $(\mathcal{D}_I, \tau_2)$ by Lemma A.3. As the topologies τ_1 and τ_2 coincide on $\mathcal{D}_I$, $\Psi_I(\alpha)$ is also compact in (D_I, τ_1), hence by Lemma A.3 in $(\mathcal{X}, \tau_1)$, so that I is good in $(\mathcal{X}, \tau_1)$. ■

With these large deviations techniques now on hand, we are ready to study LDPs for a certain class of random elements in a function space, namely the partial sum processes.

Chapter 2

LDP for Partial Sum Processes in $D[a,b]$ and $D_l[a,b]$

In this chapter we derive preliminary results on the Large Deviations Principle (LDP) for partial sum processes in the spaces $D[a,b]$ and $D_l[a,b]$. We begin by reviewing the Skorohod topology on these spaces as well as some topological properties. Next, we derive under suitable assumptions the LDP for partial sum processes in $D[0,T]$ for arbitrary $T > 0$, and then jointly for a particular partial sum process in the product space $D[0,T] \times D_l[0,T]$. General formulas for the associated rate functions are given, and a closed form expression valid in most situations is derived.

The results obtained in this chapter are of independent interest, especially the LDP obtained for the particular partial sum process in the product space $D[0,T] \times D_l[0,T]$, but most importantly, they will be used in Chapter 3 to establish the LDP in $D[0,\infty)$ and $D_l[0,\infty)$ for the extension of the partial sum processes on $[0,\infty)$.

2.1 The spaces $D[a,b]$ and $D_l[a,b]$

For $p = 1,2,\ldots$, let $D[a,b]^p$ (resp. $D_l[a,b]^p$) denote the space of functions $f : [a,b] \to \mathbb{R}^p$ which are right continuous (resp. left continuous) with left-hand limits (resp. right-hand limits).

Two topologies are traditionally used in the space $D[a,b]$, the uniform topology τ_∞, induced by the uniform metric, and the Skorohod topology. The uniform metric is defined by

$$d_\infty(x,y) \equiv \sup_{t\in[a,b]} |x(t) - y(t)|, \quad x,y \in D[a,b] \;\; (\text{or } D_l[a,b]). \qquad (2.1)$$

It is well known [8, p. 150] that $(D[a,b], \tau_\infty)$ is a complete metric space which is **not** separable. In view of the comments made in Section 1.5.1, the uniform topology is not well adapted to using on n-uples of random elements the large deviations techniques discussed in Section 1.5. This calls for the use of the (separable) Skorohod topology.

The Skorohod topology was originally introduced on the space $D[0,1]$ in the context of weak convergence [59], and has been since then extensively studied [6, 8, 31, 33, 39, 43, 44, 53, 55, 61, 66]. Interestingly enough, the transposition of the properties of the Skorohod topology on the space $D_l[a,b]$ does not seem to have attracted much attention.

Because we eventually study partial sum processes on both spaces $D[0,T]$ and $D_l[0,T]$, we begin by reviewing the main properties of the Skorohod topology on $D[a,b]$ and then formally transpose them to the space $D_l[a,b]$.

2.1.1 The Skorohod topology

In his seminal paper [59], Skorohod defined several types of convergence in the functional space $D[0,1]$, including J_1-convergence:

Definition 2.1 ([59]) *The sequence $\{x_n,\ n=1,2,\ldots\}$ in $D[0,1]$ is called J_1-convergent to x if there exists a sequence of continuous, strictly increasing and one-to-one mappings $\lambda_n : [0,1] \to [0,1]$ such that*

$$\lim_{n\to\infty} \sup_{t\in[0,1]} |x_n(t) - x\circ\lambda_n(t)| = \lim_{n\to\infty} \sup_{t\in[0,1]} |\lambda_n(t) - t| = 0. \tag{2.2}$$

Because λ_n is a bijection with λ_n^{-1} continuous as well as strictly increasing, the definition of J_1-convergence is actually equivalent to the existence of a sequence of continuous, strictly increasing and one-to-one mappings $\mu_n : [0,1] \to [0,1]$ such that

$$\lim_{n\to\infty} \sup_{t\in[0,1]} |x_n \circ \mu_n(t) - x(t)| = \lim_{n\to\infty} \sup_{t\in[0,1]} |\mu_n(t) - t| = 0. \tag{2.3}$$

Kolmogorov then proved the existence of a complete metric on $D[0,1]$ which induces Skorohod's J_1-convergence [43]. The topology induced by this metric is called the Skorohod topology.

Later, Billingsley [8] introduced two metrics on $D[0,1]$ (denoted by d and d_0 in his book) which both induce the Skorohod topology. Although both yield separability of $D[0,1]$, only one (d_0) makes $D[0,1]$ into a complete metric space. We denote here Billinsgley's d metric by d_1.

To define the metrics d_1 and d_0, we introduce Λ_{ab} as the set of continuous one-to-one functions λ from $[a,b]$ onto itself which are strictly increasing. Note that if λ belongs to Λ_{ab}, then we necessarily have $\lambda(a)=a$ and $\lambda(b)=b$.

For x,y in $D[a,b]$, we set

$$\begin{aligned} d_1(x,y) &\equiv \inf_{\lambda\in\Lambda_{ab}}\left(\sup_{t\in[a,b]}|x(t)-y\circ\lambda(t)| \;\vee\; \sup_{t\in[a,b]}|\lambda(t)-t|\right) \\ &= \inf\left\{\varepsilon>0 : \exists\lambda\in\Lambda_{ab} \text{ s.t. } \begin{array}{l} \sup_{t\in[a,b]}|x(t)-y\circ\lambda(t)|\le\varepsilon \\ \sup_{t\in[a,b]}|\lambda(t)-t|\le\varepsilon \end{array}\right\} \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} d_0(x,y) &\equiv \inf_{\lambda\in\Lambda_{ab}}\left(\sup_{t\in[a,b]}|x(t)-y\circ\lambda(t)| \;\vee\; \sup_{s\neq t}\left|\ln\frac{\lambda(s)-\lambda(t)}{s-t}\right|\right) \\ &= \inf\left\{\varepsilon>0 : \exists\lambda\in\Lambda_{ab} \text{ s.t. } \begin{array}{l} \sup_{t\in[a,b]}|x(t)-y\circ\lambda(t)|\le\varepsilon \\ \sup_{s\neq t}\left|\ln\frac{\lambda(s)-\lambda(t)}{s-t}\right|\le\varepsilon \end{array}\right\}. \end{aligned} \tag{2.5}$$

When proving topological properties for functionals on $D[a,b]$ we shall use either one of the two metrics d_1 and d_0 as they both induce the same topology.

The following elementary result will be useful later.

Lemma 2.2 *For $\alpha\neq0$ and β in $\mathbb{R}$, let $c\equiv\min\{\alpha a+\beta\,,\,\alpha b+\beta\}$ and $d\equiv\max\{\alpha a+\beta\,,\,\alpha b+\beta\}$. Define the mapping $f:[c,d]\to[a,b]$ by*

$$f(t)=\frac{1}{\alpha}(t-\beta),\quad t\in[c,d].$$

Then, for each x,y in $D[a,b]$ (resp. in $D_l[a,b]$), we have

$$\begin{aligned} &\left(\sup_{t\in[c,d]}|x\circ f(t)-y\circ f\circ\lambda(t)|\right)\vee\left(\sup_{t\in[c,d]}|\lambda(t)-t|\right) \\ &\quad=\left(\sup_{t\in[a,b]}|x(t)-y\circ f\circ\lambda\circ f^{-1}(t)|\right)\vee\left(|\alpha|\sup_{t\in[a,b]}|f\circ\lambda\circ f^{-1}(t)-t|\right), \end{aligned} \tag{2.6}$$

for all λ in Λ_{cd}, and

$$\begin{aligned} &\inf_{\lambda'\in\Lambda_{ab}}\left(\sup_{t\in[a,b]}|x(t)-y\circ\lambda'(t)|\vee\sup_{t\neq s\in[a,b]}\left|\ln\frac{\lambda'(t)-\lambda'(s)}{t-s}\right|\right) \\ &\quad=\inf_{\lambda\in\Lambda_{cd}}\left(\sup_{t\in[c,d]}|x\circ f(t)-y\circ f\circ\lambda(t)|\vee\sup_{t\neq s\in[c,d]}\left|\ln\frac{\lambda(t)-\lambda(s)}{t-s}\right|\right). \end{aligned} \tag{2.7}$$

Proof: Let x, y in $D[a,b]$ or $D_l[a,b]$. First note that for any λ in Λ_{cd},

$$x \circ f(t) - y \circ f \circ \lambda(t) = x(f(t)) - y \circ f \circ \lambda\left(f^{-1}(f(t))\right), \quad t \in [c,d],$$

whence, f being a bijection,

$$\sup_{t\in[c,d]} |x \circ f(t) - y \circ f \circ \lambda(t)| = \sup_{t\in[a,b]} |x(t) - y \circ f \circ \lambda \circ f^{-1}(t)|. \tag{2.8}$$

On the other hand, for each λ in Λ_{cd}, we also have

$$\begin{aligned} \sup_{t\in[c,d]} |\lambda(t) - t| &= \sup_{t\in[a,b]} |\lambda(f^{-1}(t)) - f^{-1}(t)| \\ &= \sup_{t\in[a,b]} |\alpha| \left|f\left(\lambda(f^{-1}(t)\right) - f\left(f^{-1}(t)\right)\right| \\ &= \sup_{t\in[a,b]} |\alpha||f \circ \lambda \circ f^{-1}(t) - t|, \end{aligned} \tag{2.9}$$

as we see, from the definition of f, that

$$f(t) - f(s) = \frac{1}{\alpha}(t-s), \quad s,t \in [c,d].$$

Equality (2.6) then becomes an easy consequence of (2.8) and (2.9).

Similarly, for all λ in Λ_{cd} we have

$$\begin{aligned} \sup_{s\neq t\in[c,d]} \left|\ln \frac{\lambda(t)-\lambda(s)}{t-s}\right| &= \sup_{s\neq t\in[a,b]} \left|\ln \frac{\lambda \circ f^{-1}(t) - \lambda \circ f^{-1}(s)}{f^{-1}(t) - f^{-1}(s)}\right| \\ &= \sup_{s\neq t\in[a,b]} \left|\ln \frac{f(\lambda \circ f^{-1}(t)) - f(\lambda \circ f^{-1}(s))}{f(f^{-1}(t)) - f(f^{-1}(s))}\right| \\ &= \sup_{s\neq t\in[a,b]} \left|\ln \frac{f \circ \lambda \circ f^{-1}(t) - f \circ \lambda \circ f^{-1}(s)}{t-s}\right|. \end{aligned} \tag{2.10}$$

Finally, f and f^{-1} are either both strictly increasing ($\alpha > 0$), or both strictly decreasing ($\alpha < 0$). Hence, it is easily checked that $f \circ \lambda \circ f^{-1}$ spans Λ_{ab} as λ spans Λ_{cd}, and the desired equality (2.7) follows from (2.8) and (2.10). ■

We now show how to define the Skorohod topology on $D_l[a,b]$. To this end, consider the mapping $\varphi_{ab} : [a,b] \to [a,b]$ defined by

$$\varphi_{ab}(t) = a + b - t, \quad t \in [a,b],$$

and define the mapping $\Phi_{ab} : D_l[a,b] \to D[a,b]$ by

$$\Phi_{ab}(x) \equiv x \circ \varphi_{ab}, \quad x \in D_l[a,b]. \tag{2.11}$$

From these definitions, it is plain that φ_{ab} and Φ_{ab} are both bijections with $\varphi_{ab}^{-1} = \varphi_{ab}$ and $\Phi_{ab}^{-1}(x) = x \circ \varphi_{ab}$ for all x in $D[a,b]$.

If we define the mapping $d_0^l : D_l[a,b] \times D_l[a,b] \to \mathbb{R}_+$ by setting, identically to the definition of d_0 on $D[a,b]$,

$$d_0^l(x,y) \equiv \inf_{\lambda \in \Lambda_{ab}} \left(\sup_{t\in[a,b]} |x(t) - y \circ \lambda(t)| \vee \sup_{s \neq t} |\ln \frac{\lambda(s) - \lambda(t)}{s-t}| \right), \qquad (2.12)$$

for x, y in $D_l[a,b]$, then Lemma 2.2 (with $\alpha = -1$ and $\beta = a+b$) yields

$$\begin{aligned} d_0^l(x,y) &= \inf_{\lambda' \in \Lambda_{ab}} \left(\sup_{t\in[a,b]} |x(t) - y \circ \lambda'(t)| \vee \sup_{s \neq t} |\ln \frac{\lambda'(t) - \lambda'(s)}{t-s}| \right) \\ &= \inf_{\lambda \in \Lambda_{ab}} \left(\sup_{t\in[a,b]} |x \circ \varphi_{ab}(t) - y \circ \varphi_{ab} \circ \lambda(t)| \vee \sup_{s \neq t} |\ln \frac{\lambda(t) - \lambda(s)}{t-s}| \right) \\ &= d_0(\Phi_{ab}(x), \Phi_{ab}(y)), \quad x, y \in D_l[a,b]. \end{aligned} \qquad (2.13)$$

Therefore, Φ_{ab} being one-to-one, it follows that d_0^l is a metric on $D_l[a,b]$; in fact, $(D_l[a,b], d_0^l)$ and $(D[a,b], d_0)$ are isometric. All topological and metric properties of $(D[a,b], d_0)$ then translate to $(D_l[a,b], d_0^l)$, and in particular, $(D_l[a,b], d_0^l)$ is a Polish space.

On the other hand, applying Definition 2.1 to $D_l[a,b]$ defines J_1-convergence on that space which, as shown below, is equivalent to d_0^l-convergence.

Lemma 2.3 *Let $\{x_n,\ n = 1,2,\ldots\}$ be a sequence in $D_l[a,b]$, and let x in $D_l[a,b]$. The sequence $\{x_n,\ n = 1,2,\ldots\}$ J_1-converges to x in the sense of Definition 2.1 if and only if*

$$\lim_{n\to\infty} d_0^l(x_n, x) = 0.$$

Proof: By (2.13), $\lim_{n\to\infty} d_0^l(x_n,x) = 0$ if and only if $\lim_{n\to\infty} d_0(\Phi_{ab}(x_n), \Phi_{ab}(x)) = 0$.

Therefore, d_0-convergence being equivalent to J_1-convergence on $D[a,b]$, we have $\lim_{n\to\infty} d_0^l(x_n,x) = 0$ if and only if there exists a sequence $\{\lambda_n,\ n = 1,2,\ldots\}$ in Λ_{ab} such that

$$\lim_{n\to\infty} \sup_{t\in[a,b]} |\Phi_{ab}(x_n)(t) - \Phi_{ab}(x) \circ \lambda_n(t)| = \lim_{n\to\infty} \sup_{t\in[a,b]} |\lambda_n(t) - t| = 0,$$

or equivalently,

$$\lim_{n\to\infty} \sup_{t\in[a,b]} |x_n \circ \varphi_{ab}(t) - x \circ \varphi_{ab} \circ \lambda_n(t)| = \lim_{n\to\infty} \sup_{t\in[a,b]} |\lambda_n(t) - t| = 0.$$

By Lemma 2.2 this last statement is equivalent to

$$\lim_{n\to\infty} \sup_{t\in[a,b]} |x_n(t) - x \circ \lambda_n'(t)| \;=\; \lim_{n\to\infty} \sup_{t\in[a,b]} |\lambda_n'(t) - t| \;=\; 0,$$

where for each $n = 1, 2, \ldots$, the mapping $\lambda_n' \equiv \varphi_{ab} \circ \lambda_n \circ \varphi_{ab}^{-1}$ belongs to Λ_{ab}, and the desired result follows easily. ■

In short, there exists a metric d_0^l on $D_l[a,b]$ (whose definition is identical to that of d_0 on $D[a,b]$) which makes $D_l[a,b]$ into a Polish space, and d_0^l-convergence is equivalent to J_1-convergence in $D_l[a,b]$. We shall refer to the topology induced by the metric d_0^l as the Skorohod topology on $D_l[a,b]$ and in the sequel we shall use either one of the two characterizations of Skorohod convergence in this space.

Noteworthy is the fact that the mapping Φ_{ab} also renders $(D_l[a,b], d_\infty)$ isometric to $(D[a,b], d_\infty)$ since

$$d_\infty(x,y) \;=\; d_\infty(\Phi_{ab}(x), \Phi_{ab}(y)), \quad x, y \in D_l[a,b].$$

We would like to point out that the existence of the isometry Φ_{ab} does not guarantee that the topological properties of a given functional on $D[a,b]$ hold true for its equivalent functional on $D_l[a,b]$ (i.e., the functional defined identically on $D_l[a,b]$). In particular, continuity of a mapping on $D[a,b]$ in the Skorohod topology does not automatically imply continuity of its equivalent mapping in $D_l[a,b]$ endowed with the Skorohod topology.

We shall abuse the notation somewhat by writing d_0 for d_0^l, and τ_∞, τ_0 for the uniform and Skorohod topologies on $D_l[a,b]$ induced respectively by the metrics d_0^l and d_∞.

The topologies and metrics discussed so far are defined for the spaces $D[a,b]$ and $D_l[a,b]$. On the spaces $D[a,b]^p$ (resp. $D_l[a,b]^p$) of right continuous (resp. left continuous) with left-hand (resp. right-hand) limits mappings taking on values in $\mathbb{R}^p$, we shall use the Skorohod metrics d_0 and d_1 (resp. d_0^l) and the uniform metric d_∞ defined similarly to the one-dimensional case, but with the absolute value on $\mathbb{R}$ in the definitions (2.5), (2.4) (resp. (2.12)) and (2.1), replaced by the norm $|\cdot|_p$ on $\mathbb{R}^p$ defined by

$$|x-y|_p \;=\; \max_{i=1,\ldots,p} |x_i - y_i|$$

for $x = (x_1, \ldots, x_p)$ and $y = (y_1, \ldots, y_p)$ in $\mathbb{R}^p$.

It should be clear to the reader that for each $p = 1, 2, \ldots$, the mapping Φ_{ab} considered as a mapping from $D_l[a,b]^p$ into $D[a,b]^p$ is still an isometry, and that all the properties obtained on $D[a,b]$ or $D_l[a,b]$ carry over to $D[a,b]^p$ and $D_l[a,b]^p$. For the sake of simplicity, we do not make any distinction in the notation for the metrics or topologies on $D[a,b]^p$ and on $D_l[a,b]^p$, from those on $D[a,b]$ and $D_l[a,b]$.

A prefix U will refer to the uniform topology and a prefix S to the Skorohod topology (e.g., U-continuous and S-continuous).

Let $C[a,b]^p$ denote the space of $\mathbb{R}^p$-valued continuous function on $[a,b]$ and let $AC_0[a,b]^p$ (resp. $AC[a,b]^p$) denote the space of functions $f : [a,b] \to \mathbb{R}^p$ which are absolutely continuous with $f(a) = 0$ (resp. absolutely continuous). Throughout, we use the following properties [8] (and, through the isometry Φ_{ab}, their counterparts in $D_l[a,b]^p$) without further reference:

- $(D[a,b]^p, d_0)$ is a Polish space.
- $(D[a,b]^p, d_\infty)$ is complete but not separable.
- For all x, y in $D[a,b]^p$, $d_0(x,y) \le d_\infty(x,y)$, so that the uniform topology is finer that the Skorohod topology ($\tau_0 \subset \tau_\infty$).
- On $C[a,b]^p$, d_0 and d_∞ coincide.

Finally, as for the Cartesian product spaces $(D[a,b])^p$ and $(D_l[a,b])^p$, we shall use the product metrics d^p_∞ and d^p_0 (which induce the product topology) defined on $(D[a,b])^p$ (or $(D_l[a,b])^p$) by

$$d^p_\infty(x,y) = \max_{i=1,\dots,p} d_\infty(x_i, y_i) \tag{2.14}$$

and

$$d^p_0(x,y) = \max_{i=1,\dots,p} d_0(x_i, y_i) \tag{2.15}$$

for $x = (x_1, \dots, x_p)$ and $y = (y_1, \dots, y_p)$ in $(D[a,b])^p$ (or in $(D_l[a,b])^p$).

Although the product space $(D[a,b])^p$ can be identified to the multi-dimensional space $D[a,b]^p$, their (Skorohod) topologies do not coincide. We shall come back to this later in Section 3.2.3.

2.1.2 Topological properties on $D[a,b]$ and $D_l[a,b]$

We complete this review of the Skorohod topology with a characterization of the Borel σ-fields on $(D[a,b]^p, \tau_0)$ and $(D_l[a,b]^p, \tau_0)$, and with results on the Borel-measurability of functionals on $D[a,b]^p$ and $D_l[a,b]^p$ that are needed later. The results obtained here are often counter-intuitive as many of these functionals are continuous in the uniform topology but not in the Skorohod topology. Unless stated otherwise, measurability is to be understood with respect to the Skorohod Borel σ-fields .

Lemma 2.4 *Let t_0 in $[a,b]$ and define the projection mapping $\pi_{t_0} : D[a,b]^p \to \mathbb{R}^p$ by*

$$\pi_{t_0}(x)(t) = x(t_0), \quad t \in [a,b], \quad x \in D[a,b]^p.$$

Then the mappings π_a and π_b are S-continuous, while for $a < t_0 < b$, the Borel-measurable mapping π_{t_0} is S-continuous at x if and only if t_0 is a continuity point of x.

The same properties hold true on $D_l[a,b]^p$ for the projection mapping $\pi^l_{t_0} : D_l[a,b]^p \to \mathbb{R}^p$.

Proof: A proof of Borel-measurability of π_{t_0} can be found in Theorem 1 of [6, p. 170] together with the S-continuity results .

The same properties are easily checked to hold true for $\pi^l_{t_0}$, upon noting that for each t_0 in $[a,b]$ and x in $D_l[a,b]^p$,

$$\begin{aligned} \pi^l_{t_0}(x) &= x(t_0) \\ &= x \circ \varphi_{ab}(\varphi^{-1}_{ab}(t_0)) \\ &= \pi_{\varphi^{-1}_{ab}(t_0)}(x \circ \varphi_{ab}), \end{aligned} \tag{2.16}$$

and that continuity of x at t_0 is equivalent to continuity of $\Phi_{ab}(x) = x \circ \varphi_{ab}$ at $\varphi^{-1}_{ab}(t_0)$. ■

The following Proposition is crucial in circumventing all measurability issues.

Proposition 2.5 *The Borel σ-fields $\mathcal{B}_{\tau_0}$ and $\mathcal{B}^l_{\tau_0}$ on $(D[a,b]^p, \tau_0)$ and $(D_l[a,b]^p, \tau_0)$ coincide respectively with the projection σ-fields $\mathcal{P}_{ab}$ and $\mathcal{P}^l_{ab}$ defined as the smallest σ-fields on which the natural projections $\{\pi_t,\ a \le t \le b\}$ and $\{\pi^l_t,\ a \le t \le b\}$ are measurable.*

Proof: A proof of the result on the space $D[a,b]^p$ can be found in [53, Theorem 7.1 p. 249].

The result on $D_l[a,b]^p$ can be established from that on $D[a,b]^p$ as follows: The mapping Φ_{ab} being an isometry, it is plain that

$$\mathcal{B}^l_{\tau_0} = \Phi^{-1}_{ab}(\mathcal{B}_{\tau_0}) = \Phi^{-1}_{ab}(\mathcal{P}_{ab}),$$

and the proof will be complete once it is shown that $\mathcal{P}^l_{ab} = \Phi^{-1}_{ab}(\mathcal{P}_{ab})$.

Because

$$\pi^l_t = \pi_{\varphi^{-1}_{ab}(t)} \circ \Phi_{ab} \quad \text{and} \quad \pi_t = \pi^l_{\varphi_{ab}(t)} \circ \Phi^{-1}_{ab}, \quad t \in [a,b],$$

we see for all A in $\mathcal{B}_{\mathbf{R}}$ that

$$(\pi^l_t)^{-1}(A) = \Phi^{-1}_{ab}\left(\pi^{-1}_{\varphi^{-1}_{ab}(t)}(A)\right) \in \Phi^{-1}_{ab}(\mathcal{P}_{ab}), \quad t \in [a,b],$$

as well as

$$\pi^{-1}_t(A) \in \Phi_{ab}(\mathcal{P}^l_{ab}).$$

In short, the inclusions

$$\mathcal{P}^l_{ab} \subset \Phi^{-1}_{ab}(\mathcal{P}_{ab}) \quad \text{and} \quad \mathcal{P}_{ab} \subset \Phi_{ab}(\mathcal{P}^l_{ab})$$

hold, and the mapping Φ_{ab} being a bijection, the equality $\mathcal{P}^l_{ab} = \Phi^{-1}_{ab}(\mathcal{P}_{ab})$ follows trivially. ■

The next result on Borel-measurability and continuity of the restriction mappings needs the following preliminary Lemma, whose proof can be found in Appendix A.5.

Lemma 2.6 *Let $\{x_n,\ n = 1,2,\ldots\}$ be a sequence which is S-converging to x in $D[a,b]^p$, and let c in (a,b) be a continuity point of x. Then the restrictions of $\{x_n,\ n = 1,2,\ldots\}$ in $D[a,c]^p$ (resp. $D[c,b]^p$) S-converge to the restriction of x in $D[a,c]^p$ (resp. $D[c,b]^p$).*

Lemma 2.7 *Let $[c,d] \subseteq [a,b]$, and define the restriction $r_{cd} : D[a,b]^p \to D[c,d]^p$ by*

$$r_{cd}(x)(t) = x(t), \quad t \in [c,d], \quad x \in D[a,b]^p.$$

Then the mapping r_{cd} is Borel-measurable, and is S-continuous at x in $D[a,b]^p$ if and only if c and d are continuity points of x or endpoints of $[a,b]$.

The same properties hold true on $D_l[a,b]^p$ for the restriction $r^l_{cd} : D_l[a,b]^p \to D_l[c,d]^p$.

Proof: Borel-measurability of r_{cd} is shown as Lemma 2.3 in [66].

To show the S-continuity of r_{cd}, let x in $D[a,b]^p$, continuous at c if $c > a$ and at d if $d < b$, and consider a sequence $\{x_n,\ n = 1,2,\ldots\}$ in $D[a,b]^p$ which is S-converging to x.

By applying Lemma 2.6 twice, first to $D[c,b]^p$ and then to $D[c,d]^p$, it is readily seen that the restrictions of x_n on $[c,d]$ S-converge to the restriction of x on $[c,d]$. In short, $r_{cd}(x_n)$ S-converges to $r_{cd}(x)$ in $D[c,d]^p$, and r_{cd} is thus S-continuous at x.

We now establish the "only if" part of the assertion of the Lemma by proving its contrapositive. To this end, let x be in $D[a,b]^p$ such that x has a discontinuity at c and let $\pi_c^{ab} : D[a,b]^p \to \mathbb{R}^p$ and $\pi_c^{cd} : D[c,d]^p \to \mathbb{R}^p$ denote the natural projections at c. Suppose now that r_{cd} were S-continuous at x. Because c is an endpoint for $[c,d]$, π_c^{cd} is S-continuous on $D[c,d]^p$ (Lemma 2.4), so in particular at $r_{cd}(x)$. The equality $\pi_c^{ab} = \pi_c^{cd} \circ r_{cd}$ then yields S-continuity of π_c^{ab} at x, a contradiction with Lemma 2.4 since c is a discontinuity point of x. A similar argument holds if d is a discontinuity point, and we conclude that r_{cd} is not S-continuous at x whenever c or d are discontinuity point of x.

In order to establish the measurability and S-continuity properties of r_{cd}^l, we observe that for any x in $D_l[a,b]^p$ the relation

$$r_{cd}^l(x) = \Phi_{\varphi_{ab}^{-1}(d)\varphi_{ab}^{-1}(c)} \circ r_{\varphi_{ab}^{-1}(d)\varphi_{ab}^{-1}(c)} \circ \Phi_{ab}(x)$$

holds. The properties of r_{cd}^l then follows from that of $r_{\varphi_{ab}^{-1}(d)\varphi_{ab}^{-1}(c)}$ and S-continuity of Φ_{ab} and $\Phi_{\varphi_{ab}^{-1}(d)\varphi_{ab}^{-1}(c)}$, upon noting that c, d are continuity points of x or endpoints of $[a,b]$ if and only if $\varphi_{ab}^{-1}(d), \varphi_{ab}^{-1}(c)$ are continuity points of $\Phi_{ab}(x)$ or endpoints of $[a,b]$. ■

Lemma 2.8 *The addition mapping $S : D[a,b]^p \times D[a,b]^p \to D[a,b]^p$ (resp. $S^l : D_l[a,b]^p \times D_l[a,b]^p \to D_l[a,b]^p$) is Borel-measurable, and S-continuous at those points (x,y) which do not have common discontinuity points.*

Proof: A proof of the results for the addition on $D[a,b]^p \times D[a,b]^p$ can be found in [66, Theorem 4.1].

The same result is easily checked to hold on $D_l[a,b]^p \times D_l[a,b]^p$, once we observe that

$$x + y = \Phi_{ab}^{-1}\left(\Phi_{ab}(x) + \Phi_{ab}(y)\right), \quad x, y \in D_l[a,b]^p,$$

or equivalently,

$$S^l(x,y) = \Phi_{ab}^{-1}\left(S\left(\Phi_{ab}(x), \Phi_{ab}(y)\right)\right), \quad x, y \in D_l[a,b]^p$$

and that x and y have no common discontinuity points if and only if $\Phi_{ab}(x)$ and $\Phi_{ab}(y)$ have no common discontinuity points. ■

Lemma 2.9 *For $\alpha > 0$, β in $\mathbb{R}$ and $\gamma \neq 0$, define the mapping $F : D[a,b]^p \to D[\alpha a+\beta, \alpha b+\beta]^p$ (resp. $D_l[a,b]^p \to D_l[\alpha a+\beta, \alpha b+\beta]^p$) by*

$$F(x)(t) \;=\; \gamma\, x\left(\frac{1}{\alpha}(t-\beta)\right), \quad t \in [\alpha a+\beta, \alpha b+\beta], \quad x \in D[a,b]^p \;\; (\textit{resp. } D_l[a,b]^p).$$

Then F is d_0-Lipschitz (resp. d_0^l-Lipschitz), hence S-continuous on $D[a,b]^p$ (resp. $D_l[a,b]^p$).

Proof: Fix $\alpha > 0$, β in $\mathbb{R}$ and $\gamma \neq 0$, and define the mapping $f : [\alpha a+\beta\,,\, \alpha b+\beta] \to [a,b]$ by $f(t) = \frac{1}{\alpha}(t-\beta)$ for all t in $[\alpha a+\beta, \alpha b+\beta]$. Then, for all x in $D[a,b]^p$ (resp. $D_l[a,b]^p$), $F(x) = \gamma\, x \circ f$, and from Lemma 2.2 we get

$$\begin{aligned} d_0(F(x), F(y)) &= d_0(\gamma\, x \circ f, \gamma\, y \circ f) \\ &= d_0(\gamma\, x, \gamma\, y) \\ &\leq \max\{1, |\gamma|\}\, d_0(x,y), \quad x, y \in D[a,b]^p \;\; (\text{resp. } D_l[a,b]^p). \end{aligned}$$

■

In what follows the infimum is to be taken component-wise.

Lemma 2.10 *The infimum mapping $m : D[a,b]^p \to D[a,b]^p$ (resp. $D_l[a,b]^p \to D_l[a,b]^p$) defined by*

$$m(x)(t) = \inf_{a \leq s \leq t} x(s), \quad t \in [a,b], \quad x \in D[a,b]^p \;\; (\textit{resp. } D_l[a,b]^p),$$

is d_0-Lipschitz, hence S-continuous.

Proof: A proof given in the context of the space $D[a,b]^p$ can be found in [66, Theorem 6], and is readily seen to carry over to the space $D_l[a,b]^p$. ■

Finally the following continuity result will be needed in Chapter 6.

Lemma 2.11 *The mapping $H : D[a,b]^p \to D[a,b]^p$ defined by $H(x) = \min\{0, x\}$ for x in $D[a,b]^p$ is S-continuous.*

Proof: Let $\{x_n,\ n = 1,2,\ldots\}$ be a sequence in $D[a,b]^p$ which S-converges to x. Then there exists a sequence $\{\lambda_n,\ n = 1,2,\ldots\}$ in Λ_{ab} such that

$$\lim_{n\to\infty} d_\infty(x_n, x \circ \lambda_n) \;=\; \lim_{n\to\infty} d_\infty(\lambda_n, e) \;=\; 0 \tag{2.17}$$

where e denotes the identity mapping from $[a,b]$ onto itself.

The mapping H being U-continuous, in fact d_∞-Lipschitz, the previous equalities yield

$$\lim_{n\to\infty} d_\infty(H(x_n), H(x\circ\lambda_n)) \;=\; \lim_{n\to\infty} d_\infty(\lambda_n, e) \;=\; 0, \tag{2.18}$$

and the desired result follows upon noting that $H(x\circ\lambda_n) = H(x)\circ\lambda_n$ for each $n = 1, 2, \ldots$. ■

We conclude this section with a few properties of the Skorohod topology induced on a particular subspace; this will be very useful in Chapter 3 in overcoming the technical difficulties which arise from the fact that the restriction mapping is not S-continuous.

Let $\mathbf{Q}$ denote the set of rational numbers in $\mathbb{R}$. We define the subspaces $D^{Q}[a,b]^p$ and $D_l^{Q}[a,b]^p$ of $D[a,b]^p$ and $D_l[a,b]^p$, respectively, of functions which are continuous at each **irrational** t in $[a,b]$, i.e., which are continuous on $[a,b]\cap \mathbf{Q}^c$. As shown below, $D^{Q}[a,b]^p$ and $D_l^{Q}[a,b]^p$ endowed with the metric induced respectively by d_0 and d_0^l are still Polish spaces.

Lemma 2.12 *The sets $D^{Q}[a,b]^p$ and $D_l^{Q}[a,b]^p$ are closed under the metric d_0 and d_0^l, respectively, so that $(D^{Q}[a,b]^p, d_0)$ and $(D_l^{Q}[a,b]^p, d_0^l)$ are Polish spaces.*

Proof: We prove the statement for $D^{Q}[a,b]^p$ only, as the proof can be easily modified to accommodate $D_l^{Q}[a,b]^p$. The argument below is based on one used by Lindvall [45] to show that the subspace of $D[0,1]$ of functions continuous at $t=1$ is a Polish space.

Let $\{x_n,\ n=1,2,\ldots\}$ be a sequence in $D^{Q}[a,b]^p$ which is S-converging to the element x of $D[a,b]^p$. We need to show that x is continuous at each irrational t in $[a,b]$. In view of the comments made after the definition of J_1-convergence, there exists a sequence of mappings $\{\lambda_n,\ n=1,2,\ldots\}$ in Λ_{ab} such that

$$\lim_{n\to\infty} d_\infty(x_n\circ\lambda_n, x) \;=\; \lim_{n\to\infty} d_\infty(\lambda_n, e) \;=\; 0 \tag{2.19}$$

where e denotes the identity mapping from $[a,b]$ onto itself.

For each $n = 1, 2, \ldots$, let y_n denote $x_n\circ\lambda_n$. Then, for t in $[a,b]\cap\mathbf{Q}^c$, we observe that

$$\begin{aligned}|x(t)-x(t^-)| &= \limsup_{s\to t^-} |x(t)-x(s)| \\ &\le \limsup_{s\to t^-} \left(|x(t)-y_n(t)| + |y_n(t)-y_n(s)| + |y_n(s)-x(s)|\right)\end{aligned}$$

$$\begin{aligned}&\leq\ 2\,d_\infty(y_n,x)\ +\ \limsup_{s\to t^-}|y_n(t)-y_n(s)|\\&=\ 2\,d_\infty(y_n,x)\end{aligned}\tag{2.20}$$

where the last step follows from the fact that for each $n=1,2,\ldots,\ y_n=x_n\circ\lambda_n$ is continuous at t.

Upon letting $n\to\infty$ in (2.20), we get from (2.19) that

$$x(t)=x(t^-),\quad t\in[a,b]\cap\mathbf{Q}^c,$$

so that x belongs to $D^Q[a,b]^p$.

The space $(D^Q[a,b]^p,d_0)$ is then complete and separable as a closed subspace of a complete and separable metric space. In other words, $(D^Q[a,b]^p,d_0)$ is a Polish space. ■

In the sequel, unless otherwise mentioned, the topologies considered on $D[a,b]^p$ and $D_l[a,b]^p$, as well as on any of their subspaces are the Skorohod topologies induced by the metric d_0, and are therefore metrizable.

2.2 LDP for partial sum processes

Partial sum processes appear in many applications of applied probability. Most of the work on large deviations for partial sum processes has focused on obtaining sufficient conditions for a LDP to hold [9, 23, 48, 54, 58, 64], [24, Section 5.1 p. 152]. The results are typically obtained on the space $(D[0,1]^p,\tau_\infty)$. Although it is often noted that the method used to obtain the desired LDP would yield the LDP in $(D[0,T]^p,\tau_\infty)$ for arbitrary $T>0$, to the best of our knowledge there are no result in the literature on how one is obtained from the other. In this section, we present results showing that the LDP for partial sum processes in $D[0,T]^p$ for any $T>0$ is a consequence of that in $D[0,1]^p$. We actually go further, as we show that the LDP in $D[0,1]^p$ implies a LDP jointly in $D[0,T]\times D_l[0,T]$ for any $T>0$, for a partial sum process associated with the past and future of a stationary random sequence.

2.2.1 Partial sum processes

We assume all random variables to be defined on a common, possibly enlarged, probability space $(\Omega,\mathcal{F},\mathbf{P})$. For a family of $\mathbb{R}^p$-valued ($p=1,2,\ldots$) random variables $\{x_n,\ n=1,2,\ldots\}$, we define the random variable $X(n_1,n_2)$ by

$$X(n_1,n_2) \equiv \begin{cases} \sum_{i=n_1}^{n_2} x_i & \text{if } n_1 \le n_2 \\ 0 & \text{otherwise} \end{cases}, \quad n_1,n_2 = 0,1,\ldots .$$

For each $T>0$, we can then construct a family $\{X_n^T(\cdot),\ n=1,2,\ldots\}$ of mappings $X_n^T(\cdot):\Omega\to D[0,T]^p$ by setting for each $n=1,2,\ldots$

$$\begin{aligned} X_n^T(t)(\omega) &\equiv \frac{1}{n}X(1,\lfloor nt\rfloor)(\omega) \\ &= \begin{cases} \frac{1}{n}\sum_{i=1}^{\lfloor nt\rfloor} x_i(\omega) & \text{if } \lfloor nt\rfloor \ge 1 \\ 0 & \text{otherwise} \end{cases}, \quad t\in[0,T], \quad \omega\in\Omega. \end{aligned}$$

As shown below, the mapping $X_n^T(\cdot)$ is a random element in $(D[0,T]^p,\tau_\infty)$, as well as in $(D[0,T]^p,\tau_0)$.

Lemma 2.13 *For each $T>0$ and $n=1,2,\ldots$, the mapping $X_n^T(\cdot):\Omega\to D[0,T]^p$ is $\mathcal{F}/\mathcal{B}_{\tau_\infty}$ and $\mathcal{F}/\mathcal{B}_{\tau_0}$-measurable, where $\mathcal{B}_{\tau_\infty}$ and $\mathcal{B}_{\tau_0}$ are the Borel σ-fields of $D[0,T^p]$ with respect to the uniform and Skorohod topologies, respectively.*

Proof: Fix $T>0$ and $n=1,2,\ldots$. We first write

$$X_n^T(t)(\omega) = \frac{1}{n}\sum_{i=1}^{\lfloor nt\rfloor} x_i(\omega) = \frac{1}{n}\sum_{i=1}^{\infty} x_i(\omega)\,\mathbf{1}\{i\le\lfloor nt\rfloor\}, \quad t\in[0,T], \quad \omega\in\Omega. \tag{2.21}$$

Clearly, for each $i=1,2,\ldots$, the (constant) mapping $\omega\to\mathbf{1}\{i\le\lfloor n\cdot\rfloor\}$ is $\mathcal{F}/\mathcal{B}_{\tau_\infty}$-measurable.

On the other hand, if $\bar{B}_\delta(\varphi)$ denotes the closed ball of center φ and radius δ in $(D[0,1],\tau_\infty)$, then we have

$$\begin{aligned} x_i^{-1}(\bar{B}_\delta(\varphi)) &= \{\omega\in\Omega : \sup_{t\in[0,T]} |\varphi(t)-x_i(\omega)|_t \le \delta\} \\ &= \{\omega\in\Omega : \varphi(t)-\delta\le x_i(\omega)\le\varphi(t)+\delta,\ t\in[0,T]\} \\ &= \{\Omega\in\Omega : \sup_{t\in[0,T]}\varphi(t)-\delta\le x_i(\omega)\le \inf_{T\in[0,T]}\varphi(t)+\delta\} \end{aligned}$$

and it follows from $\mathcal{F}/\mathcal{B}_{\mathbb{R}}$-measurability of x_i that $x_i^{-1}(\bar{B}_\delta(\varphi))\subset\mathcal{F}$. As this inclusion holds for all $\delta>0$ and all φ in $D[0,T]$, we immediately conclude that for each $i = 1, 2, \ldots$, x_i is $\mathcal{F}/\mathcal{B}_{\tau_\infty}$-measurable, and using (2.21), we find that $X_n^T(\cdot)$ is $\mathcal{F}/\mathcal{B}_{\tau_\infty}$-measurable, as the countable sum of products of measurable mappings.

Finally, the uniform topology τ_∞ being finer that the Skorohod topology τ_0, $\mathcal{F}/\mathcal{B}_{\tau_0}$-measurability of $X_n^T(\cdot)$ follows by considering the (continuous) natural embedding from $(D[0,T]^p, \tau_\infty)$ onto $(D[0,T]^p, \tau_0)$. ■

Following [23, 24], we refer to $\{X_n^T(\cdot),\ n = 1, 2, \ldots\}$ as the family of partial sum processes on $D[0,T]^p$ associated with $\{x_n,\ n = 1, 2, \ldots\}$. For each $n = 1, 2, \ldots$, we write $X_n(\cdot)$ for $X_n^1(\cdot)$.

For a bi-infinite $\mathbb{R}^p$-valued random sequence $\{x_n,\ n = 0, \pm 1, \pm 2, \ldots\}$, we define similarly for each $T > 0$ a family of mappings $X_n^{T,-}(\cdot) : \Omega \to D_l[0,T]^p$, by setting for each $n = 1, 2, \ldots$

$$X_n^{T,-}(t)(\omega) \equiv \begin{cases} \dfrac{1}{n} \displaystyle\sum_{i=1-\lceil nt \rceil}^{0} x_i(\omega) & \text{if } \lceil nt \rceil \geq 1 \\ 0 & \text{otherwise} \end{cases}, \quad t \in [0,T], \quad \omega \in \Omega.$$

We emphasize that for each ω in Ω, the mapping $X_n^{T,-}(\cdot)(\omega)$ is **left continuous with right-hand limits**, and thus belongs to $D_l[0,T]^p$ and **not** to $D[0,T]^p$.

By slightly modifying the arguments given in the proof of Lemma 2.13, $\mathcal{F}/\mathcal{B}^l_{\tau_\infty}$ and $\mathcal{F}/\mathcal{B}^l_{\tau_0}$-measurability of $X_n^{T,-}(\cdot)$ is easily established. We refer to $X_n^{T,-}(\cdot)$ as the **negative partial sum process** on $D_l[0,T]^p$ associated with $\{x_n,\ n = 0, \pm 1, \pm 2, \ldots\}$.

It is plain from the definition of $X_n^T(\cdot)$ and $X_n^{T,-}(\cdot)$ that for each $n = 1, 2, \ldots$, $T > 0$ and ω in Ω, the only possible discontinuities occur for t **rational**. In fact, as shown below, for each $n = 1, 2, \ldots$ and $T > 0$, the partial sum processes $X_n^T(\cdot)$ and $X_n^{T,-}(\cdot)$ are random elements in $D^Q[0,T]^p$ and $D_l^Q[0,T]^p$, respectively.

Lemma 2.14 *For each $T > 0$ and $n = 1, 2, \ldots$, $X_n^T(\cdot) : \Omega \to D^Q[0,T]^p$ and $X_n^{T,-}(\cdot) : \Omega \to D_l^Q[0,T]^p$ are random elements in $(D^Q[0,T]^p, \tau_0)$ and $(D_l^Q[0,T]^p, \tau_0)$, respectively.*

Proof: Fix $T > 0$ and $n = 1, 2, \ldots$. By Theorem 1.9 in [53, p. 5] the Borel σ-field on $D^Q[0,T]^p$ is the trace on $D^Q[0,T]^p$ of the Borel σ-field on $D[0,T]^p$, i.e,

$$\mathcal{B}_{D^Q[0,T]^p} = \left\{ B \cap D^Q[0,T]^p : B \in \mathcal{B}_{D[0,T]^p} \right\}.$$

Therefore, for any Borel set B in $D^Q[0,T]^p$, we have $B = D^Q[0,T]^p \cap A$ for some Borel set A in $D[0,T]^p$, and by Lemma 2.13 it follows that

$$\begin{aligned} X_n^T(\cdot)^{-1}(B) &= X_n^T(\cdot)^{-1}(D^Q[0,T]^p) \cap X_n^T(\cdot)^{-1}(A) \\ &= \mathcal{F} \cap X_n^T(\cdot)^{-1}(A) \in \mathcal{F}, \end{aligned}$$

or equivalently, that $X_n(\cdot)$ is $\mathcal{F}/\mathcal{B}_{D^Q[0,T]^p}$-measurable. ∎

The motivation for introducing this negative partial sum process will become clear in Chapter 5 as we study the asymptotics of Lindley processes, and in Chapter 6 where the steady state behavior of a **G/G/1** queue is discussed. Indeed, the steady-state characteristics of a queueing system are often expressed as a function of the entire past of the input processes. Thus any large deviations analysis at the sample path level of such systems will require the LDP for the extension on $[0,\infty)$ of the partial sum process associated with the inputs to the system. The LDP for $X_n^T(\cdot)$ or $X_n^{T,-}(\cdot)$, beyond its independent interest, is a first step towards the LDP in $D[0,\infty)$ or $D_l[0,\infty)$.

The LDP for the family of partial sum processes $X_n(\cdot)$ on $D[0,1]^p$ is often referred to as the sample path LDP for the sequence $\{x_n,\ n = 0, \pm 1, \pm 2, \ldots\}$. Noteworthy is the fact that the sample path LDP yields the LDP for the sample mean sequence [35].

Lemma 2.15 *Assume that $\{X_n(\cdot),\ n = 1,2,\ldots\}$ satisfy the LDP in $(D[0,1]^p, \tau_0)$ (or for that matter in $(D[0,1]^p, \tau_\infty)$) with good rate function $I_X : D[0,1]^p \to [0,\infty]$. For each $n = 1,2,\ldots$, define the sample mean S_n by*

$$S_n \equiv \frac{1}{n}\sum_{i=1}^{n} x_i .$$

Then $\{S_n,\ n = 1,2,\ldots\}$ satisfies the LDP in $\mathbb{R}^p$ with good rate function $r : \mathbb{R}^p \to [0,\infty]$ given by

$$r(x) \equiv \inf\{I_X(\psi) : x = \psi(1),\ \psi \in D[0,1]^p\}, \quad x \in \mathbb{R}^p . \tag{2.22}$$

Moreover, if I_X is of the form

$$I_X(\varphi) = \begin{cases} \int_0^1 f(\dot\varphi(t))\,dt & \text{if } \varphi \in AC_0[0,1]^p \\ \infty & \text{otherwise} \end{cases}, \tag{2.23}$$

for some convex function $f : \mathbb{R}^p \to [0,\infty]$, then $r = f$.

Proof: The first assertion follows easily from Lemma 2.4 and the Contraction Principle, once it is seen that $S_n = X_n(1)$, $n = 1, 2, \ldots$. Next, by Jensen's inequality, we have

$$\begin{aligned} r(x) &= \inf\{I_X(\psi) : x = \psi(1),\ \psi \in D[0,1]^p\} \\ &= \inf\left\{\int_0^1 f(\dot\psi(t))\,dt : x = \psi(1),\ \psi \in AC_0[0,1]^p\right\} \\ &\geq \inf\left\{f(\int_0^1 \dot\psi(t)\,dt) : x = \psi(1),\ \psi \in AC_0[0,1]^p\right\} \\ &= \inf\{f(\psi(1)) : x = \psi(1),\ \psi \in AC_0[0,1]^p\} \\ &= f(x),\ x \in \mathbb{R}^p. \end{aligned}$$

This bound is achieved by the element ψ in $AC_0[0,1]^p$ given by $\psi(t) = tx$ for t in $[0,1]$, and the second assertion follows. ■

We now state a few results which will allow us to consider the LDP for partial sum processes in more suitable spaces or topologies.

We begin with the equivalence of the LDP for partial sum processes on $D[a,b]^p$ and $D^Q[a,b]^p$. Indeed, because

$$\mathbf{P}\left[X_n(\cdot) \in D^Q[0,T]^p\right] = \mathbf{P}\left[X_n^{T,-}(\cdot) \in D_l^Q[0,T]^p\right] = 1, \quad n = 1, 2, \ldots$$

with the sets $D^Q[0,T]^p$ and $D_l^Q[0,T]^p$ being closed (Lemma 2.12), Lemma 1.11 yields the following useful proposition.

Proposition 2.16 *For each $T > 0$, the family $\{X_n^T(\cdot),\ n = 1, 2, \ldots\}$ (resp. $\{X_n^{T,-}(\cdot),\ n = 1, 2, \ldots\}$) satisfies the LDP in $(D[0,T]^p, \tau_0)$ (resp. $(D_l[0,T]^p, \tau_0)$) with rate function I if and only if it satisfies the LDP in $(D^Q[0,T]^p, \tau_0)$ (resp. $(D_l^Q[0,T]^p, \tau_0)$) with rate function I^Q. Moreover, we have*

$$I(\varphi) = \begin{cases} I^Q(\varphi) & \text{if } \varphi \in D^Q[0,T]^p \ \text{(resp. } D_l^Q[0,T]^p\text{)} \\ \infty & \text{otherwise,} \end{cases} \tag{2.24}$$

and the rate function I is good if and only if the rate function I^Q is good.

In view of this proposition, we have the choice in stating our LDP results for partial sum processes in either setting. We choose to state them in $D[0,T]^p$ and $D_l[0,T]^p$, in an effort to be consistent with the relevant literature.

Finally, it is reassuring to know that the LDP for partial sum processes on $(D[a,b]^p, \tau_0)$ and $(D[a,b]^p, \tau_\infty)$ are in most cases equivalent. Because the topology τ_∞ is finer that τ_0, the following result is a simple consequence of Lemma 1.12.

Proposition 2.17 *Let $\{X_n(\cdot),\ n=1,2,\ldots\}$ be a sequence of random elements in $(D[a,b]^p,\tau_\infty)$.*

1. *If the sequence $\{X_n(\cdot),\ n=1,2,\ldots\}$ satisfies the LDP in $(D[a,b]^p,\tau_\infty)$ with good rate function $I: D[a,b]^p \to [0,\infty]$, then it satisfies the LDP in $(D[a,b]^p,\tau_0)$ with the same rate function.*

2. *If the sequence $\{X_n(\cdot),\ n=1,2,\ldots\}$ satisfies the LDP in $(D[a,b]^p,\tau_0)$ with rate function (resp. good rate function) I such that $I(\varphi)=\infty$ for φ not in $C[a,b]^p$, then it satisfies the LDP in $(D[a,b],\tau_\infty)$ with the same rate function (resp. good rate function).*

2.2.2 LDP for $X_n^T(\cdot)$ in the space $D[0,T]$

We begin with two simple preliminary Lemmas.

Lemma 2.18 *Let $r:\mathbb{R}^p\to[0,\infty]$ be a rate function associated with the LDP of a collection of probability measures $\{\mu_n,\ n=1,2,\ldots\}$. Then, for all a,b such that $0\le a\le b$, we have*

$$\inf\left\{\int_a^b r(\dot{\psi}(t))\,dt : \psi\in AC[a,b]^p\right\} = 0.$$

Proof: By considering the family of functions $\{\psi_c,\ c\in\mathbb{R}^p\}$ in $AC[a,b]^p$ defined by $\psi_c(t)=ct,\ a\le t\le b$, we get the upper bound

$$\begin{aligned}\inf\left\{\int_a^b r(\dot{\psi}(t))\,dt : \psi\in AC[a,b]^p\right\} &\le \inf_{c\in\mathbf{R}}\int_a^b r(\dot{\psi}_c(t))\,dt\\ &= (b-a)\inf_{c\in\mathbf{R}^p} r(c).\end{aligned}$$

The result now follows from the fact that the rate function r is non-negative with $\inf_{c\in\mathbf{R}} r(c)=0$ (Lemma 1.3). ■

Lemma 2.19 *Let the sequence of partial sum processes $\{X_n(\cdot),\ n=1,2,\ldots\}$ in $D[0,1]^p$ satisfy the LDP in $(D[0,1]^p,\tau_0)$ with good rate function $I_X: D[0,1]^p\to[0,\infty]$. Then for each $K=1,2,\ldots$, the subsequence $\{X_{nK}(\cdot),\ n=1,2,\ldots\}$ satisfies the LDP in $(D[0,1]^p,\tau_0)$ with good rate function $K\,I_X$.*

Proof: Fix $K=1,2,\ldots$. For any subset B in the Borel σ-field of $(D[0,1]^p,d_0)$, we have

$$\limsup_{n\to\infty}\frac{1}{n}\ln\,\mathbf{P}\left[X_{nK}(\cdot)\in B\right] = K\limsup_{n\to\infty}\frac{1}{nK}\ln\,\mathbf{P}\left[X_{nK}(\cdot)\in B\right]$$

$$\leq \; K \limsup_{n\to\infty} \frac{1}{n} \ln \, \mathbf{P}[\, X_n(\cdot) \in B \,] \,,$$

and

$$\liminf_{n\to\infty} \frac{1}{n} \ln \, \mathbf{P}[\, X_{nK}(\cdot) \in B \,] \;\; \geq \;\; K \liminf_{n\to\infty} \frac{1}{n} \ln \, \mathbf{P}[\, X_n(\cdot) \in B \,] \,.$$

It is then plain from these inequalities that the existence of a LDP for the sequence $\{X_n(\cdot),\ n = 1, 2, \ldots\}$ in $D[0,1]^p$ with good rate function I_X translates into one for the subsequence $\{X_{nK}(\cdot),\ n = 1, 2, \ldots\}$ in $D[0,1]^p$ with good rate function $K\, I_X$. ■

We now show that the LDP for a partial sum process in $D[0,1]^p$ yields the LDP in $D[0,K]^p$ for its extension on $[0,K]$, with $K = 1, 2, \ldots$. This result will be later extended to $D[0,T]$ for any irrational $T > 0$, and under some additional continuity assumptions on the elements of the effective domain of the rate function, for any rational $T > 0$.

For each $T > 0$ and each mapping $\varphi : [0,T] \to \mathbb{R}^p$, define the mapping $\varphi_T : [0,1] \to \mathbb{R}^p$ by

$$\varphi_T(t) \;=\; \frac{1}{T}\varphi(T\,t), \quad t \in [0,1]. \tag{2.25}$$

Proposition 2.20 *Assume the family $\{X_n(\cdot),\ n = 1, 2, \ldots\}$ to satisfy the LDP in $(D[0,1]^p, \tau_0)$ with good rate function $I_X : D[0,1]^p \to [0,\infty]$.*

Then, for each $K = 1, 2, \ldots$, the family $\{X_n^K(\cdot),\ n = 1, 2, \ldots\}$ satisfies the LDP in $(D[0,K]^p, \tau_0)$ with good rate function $I_X^K : D[0,K]^p \to [0,\infty]$ given by

$$I_X^K(\varphi) \;\equiv\; K I_X(\varphi_K), \quad \varphi \in D[0,K]^p. \tag{2.26}$$

Proof: Fix $K = 1, 2, \ldots$ and $n = 1, 2, \ldots$; we have

$$\begin{aligned} X_n^K(t) \;&=\; \frac{1}{n}\sum_{i=1}^{\lfloor nt \rfloor} x_i \\ &=\; K\frac{1}{nK}\sum_{i=1}^{\lfloor nK\frac{t}{K} \rfloor} x_i, \quad t \in [0,K], \end{aligned}$$

so that we can write

$$X_n^K(\cdot) \;=\; F(X_{nK}(\cdot)) \tag{2.27}$$

where the mapping $F : D[0,1]^p \to D[0,K]^p$ is defined by

$$F(z)(t) = K\, z(\frac{t}{K}), \quad t \in [0,K], \quad z \in D[0,1]^p \,.$$

By Lemma 2.9, F is S-continuous. Therefore, in view of (2.27), the LDP for the family $\{X_n^K(\cdot),\ n=1,2,\ldots\}$ in $D[0,K]^p$ follows easily from that for the family $\{X_{nK}(\cdot),\ n=1,2,\ldots\}$ (Lemma 2.19) and the Contraction Principle. The corresponding rate function $I_X^K : D[0,K]^p \to [0,\infty]$ is given by

$$\begin{aligned} I_X^K(\varphi) &= \inf\left\{K\, I_X(\psi) : \varphi(t) = K\psi(\frac{t}{K}),\ t\in[0,K],\ \psi \in D[0,1]^p\right\} \\ &= \inf\left\{K\, I_X(\psi) : \psi(u) = \frac{1}{K}\varphi(Ku),\ u\in[0,1],\ \psi \in D[0,1]^p\right\} \\ &= K\, I_X(\varphi_K), \quad \varphi \in D[0,K]^p, \end{aligned}$$

which completes the proof. ■

The expression for the new rate function can be explicitly computed when the rate function I_X is of the usual form

$$I_X(\varphi) = \begin{cases} \int_0^1 r(\dot\varphi(t))\, dt & \text{if} \quad \varphi \in AC_0[0,1]^p \\ \infty & \text{otherwise} \end{cases}, \tag{2.28}$$

for some Borel-measurable function $r : \mathbb{R}^p \to [0,\infty]$.

Corollary 2.21 *If I_X is of the form (2.28), then*

$$I_X^K(\varphi) = \begin{cases} \int_0^K r(\dot\varphi(t))\, dt & \textit{if} \quad \varphi \in AC_0[0,K]^p \\ \infty & \textit{otherwise} \end{cases}. \tag{2.29}$$

Proof: The result follows from a change of variable in the integral, after substitution of I_X in the expression (2.26) for I_X^K. ■

It is interesting to note that Proposition 2.20 actually holds on $D[a,b]$ for any interval $[a,b]$ provided the restriction mapping from $D[0,\lceil b\rceil]$ into $D[a,b]$ is continuous on the effective domain of $I_X^{\lceil b\rceil}$. We give the result on $D[0,T]$ for arbitrary $T>0$. An extra assumption on the continuity of the elements of the effective domain of I_X is required, as the restriction mapping from $D[0,\lceil T\rceil]$ into $D[0,T]$ is not always S-continuous. However, this assumption is automatically satisfied when T is irrational, as well as when I_X is of the form (2.28).

Corollary 2.22 *Assume the family of partial sum processes $\{X_n(\cdot),\ n=1,2,\ldots\}$ to satisfy the LDP in $(D[0,1]^p, \tau_0))$ with good rate function $I_X : D[0,1]^p \to [0,\infty]$ and let $T>0$. If T is rational, assume further that every element of the effective domain of I_X is continuous at $t = \frac{T}{\lceil T\rceil}$.*

Then the family of partial sum processes $\{X_n^T(\cdot),\ n = 1,2,\ldots\}$ *satisfies the LDP in* $(D[0,T]^p, \tau_0)$ *with good rate function* $I_X^T : D[0,T]^p \to [0,\infty]$ *given by*

$$I_X^T(\varphi) \equiv \inf_{\psi \in D[0,\lceil T\rceil]^p} \left\{ \lceil T\rceil I_X(\psi_{\lceil T\rceil}) : \psi = \varphi \ \ on\ [0,T] \right\}, \quad \varphi \in D[0,T]^p. \quad (2.30)$$

Proof: Under the enforced assumptions, Proposition 2.20 yields the LDP for $\{X_n^{\lceil T\rceil}(\cdot),\ n = 1,2,\ldots\}$ with good rate function $I_X^{\lceil T\rceil} : D[0,\lceil T\rceil]^p \to [0,\infty]$ given by

$$I_X^{\lceil T\rceil}(\psi) = \lceil T\rceil\, I_X(\psi_{\lceil T\rceil}), \quad \psi \in D[0,\lceil T\rceil]^p,$$

with $\psi_{\lceil T\rceil}$ as defined by (2.25). We immediately see from this last expression that $\mathcal{D}_{I_X^{\lceil T\rceil}}$ is in a one-to-one correspondence with $\mathcal{D}_{I_X}$ and that ψ belongs to $\mathcal{D}_{I_X^{\lceil T\rceil}}$ if and only if $\psi_{\lceil T\rceil}$ belongs to $\mathcal{D}_{I_X}$.

Now consider the restriction $r_T : D[0,\lceil T\rceil]^p \to D[0,T]^p$, and note that

$$X_n^T(\cdot) = r_T(X_n^{\lceil T\rceil}),\ n = 1,2,\ldots\ .$$

By Lemma 2.7, r_T is Borel-measurable. In order to establish S-continuity of r_T on the effective domain of $I_X^{\lceil T\rceil}$, we distinguish two cases.

For T rational, by assumption every mapping in $\mathcal{D}_{I_X}$ is continuous at $t = \frac{T}{\lceil T\rceil}$. Thus, from the expression (2.25) of $\psi_{\lceil T\rceil}$, it is plain that each ψ in $\mathcal{D}_{I_X^{\lceil T\rceil}}$ is continuous at $t = T$, so that by Lemma 2.7, the restriction r_T is S-continuous on the effective domain of $I_X^{\lceil T\rceil}$.

For T irrational, r_T is S-continuous on $D^Q[0,\lceil T\rceil]^p$ by Lemma 2.7, thus on $\mathcal{D}_{I_X^{\lceil T\rceil}}$ because the effective domain of $I_X^{\lceil T\rceil}$ is a subset of $D^Q[0,\lceil T\rceil]^p$ (Proposition 2.16).

Therefore, for T rational or irrational, the LDP for $\{X_n^T(\cdot),\ n = 1,2,\ldots\}$ becomes a simple consequence of the Contraction Principle and the corresponding good rate function $I_X^T : D[0,T]^p \to [0,\infty]$ is given by (2.30). ■

We stress that in view of Proposition 2.20 and Corollary 2.22, the LDP for $\{X_n(\cdot),\ n = 1,2,\ldots\}$ in $(D[0,1]^p, \tau_0)$ with good rate function yields that for $\{X_n^T(\cdot),\ n = 1,2,\ldots\}$ in $(D[0,T]^p, \tau_0)$ for T **integer** or **irrational**, without any additional assumptions. However, we were not able to derive the LDP for $\{X_n^R(\cdot),\ n = 1,2,\ldots\}$ for R rational under the same assumptions. It is unclear to us at this point whether it actually holds without further assumptions. However, in the "good" cases, i.e., when the rate function I_X is of the integral form, we do get the LDP in $D[0,T]^p$ for any $T > 0$.

Corollary 2.23 *Assume the family of partial sum processes* $\{X_n(\cdot),\ n=1,2,\ldots\}$ *to satisfy the LDP in* $(D[0,1]^p, \tau_0)$ *with good rate function* $I_X : D[0,1]^p \to [0,\infty]$ *of the form (2.28) for some rate function* $r : \mathbb{R}^p \to [0,\infty]$.

Then, for any $T > 0$, *the family of partial sum processes* $\{X_n^T(\cdot),\ n = 1,2,\ldots\}$ *satisfies the LDP in* $(D[0,T]^p, \tau_0)$ *with good rate function* $I_X^T : D[0,T]^p \to [0,\infty]$ *given by*

$$I_X^T(\varphi) = \begin{cases} \int_0^T r(\dot\varphi(t))\,dt & \text{if } \ \varphi \in AC_0[0,T]^p \\ \infty & \text{otherwise} \end{cases} . \tag{2.31}$$

Proof: Fix $T > 0$. Because $\mathcal{D}_{I_X} = AC_0[0,1]^p$, each element of $\mathcal{D}_{I_X}$ is continuous at $t = \frac{T}{\lceil T\rceil}$. By applying Corollary 2.22, we obtain the LDP for $\{X_n^T(\cdot),\ n = 1,2,\ldots\}$ with good rate function I_X^T given by (2.30). Furthermore, in this particular case, a closed form expression for $I_X^{\lceil T\rceil}$ was obtained in Corollary 2.21, namely

$$I_X^{\lceil T\rceil}(\psi) = \begin{cases} \int_0^{\lceil T\rceil} r(\dot\psi(t))\,dt & \text{if } \ \psi \in AC_0[0,\lceil T\rceil]^p \\ \infty & \text{otherwise} \end{cases} . \tag{2.32}$$

Now, if φ does not belong to $AC_0[0,T]^p$, then any ψ in $D[0,\lceil T\rceil]^p$ such that $\psi = \varphi$ on $[0,T]$ certainly does not belong to $AC_0[0,\lceil T\rceil]^p$. Thus, we already find from (2.30) and (2.32) that $I_X^T(\varphi) = \infty$ for φ not in $AC_0[0,T]^p$. Finally, for φ in $AC_0[0,T]^p$, we have

$$\begin{aligned} I_X^T(\varphi) &= \inf_{\psi\in AC_0[0,\lceil T\rceil]^p}\left\{\int_0^{\lceil T\rceil} r(\dot\psi(t))\,dt : \varphi = \psi \ \text{ on } [0,T]\right\} \\ &= \int_0^T r(\dot\varphi(t))\,dt + \inf_{\psi\in AC_0[0,\lceil T\rceil]^p}\left(\int_T^{\lceil T\rceil} r(\dot\psi(t))\,dt\right) \\ &= \int_0^T r(\dot\varphi(t))\,dt \end{aligned}$$

where the last step is validated by Lemma 2.18. ■

The assumption that r be a rate function is required in order to establish the closed-form expression for I_X^T; Borel-measurability of r is not enough.

2.2.3 LDP for $\left(X_n^T(\cdot), X_n^{T,-}(\cdot)\right)$ in $D[0,T] \times D_l[0,T]$

When the sequence $\{x_n,\ n = 1,2,\ldots\}$ is stationary, it turns out that the LDP for the family $\{X_n(\cdot),\ n = 1,2,\ldots\}$ implies that for the family of joint partial sum processes $\{(X_n^T(\cdot), X_n^{T,-}(\cdot),\ n = 1,2,\ldots\}$.

The product of two closed sets being closed in the product topology [12, Proposition 7 p. 47], we have from Lemma 1.11 the following equivalence in LDPs.

Proposition 2.24 *For each $T > 0$, the family $\left\{ \left(X_n^T(\cdot), X_n^{T,-}(\cdot) \right),\ n = 1, 2, \ldots \right\}$ satisfies the LDP in $(D^Q[0,T]^p, \tau_0) \times (D_l^Q[0,T]^p, \tau_0)$ with rate function (resp. good rate function) I^Q if and only if it satisfies the LDP in $(D[0,T]^p, \tau_0) \times (D_l[0,T]^p, \tau_0)$ with rate function I (resp. good rate function) given by*

$$I(\varphi_1, \varphi_2) = \begin{cases} I^Q(\varphi_1, \varphi_2) & \text{if} \quad \varphi_1 \in D^Q[0,T]^p, \quad \varphi_2 \in D_l^Q[0,T]^p \\ \infty & \text{otherwise.} \end{cases} \tag{2.33}$$

We are now ready to state and prove our results. We begin with a special case. A continuity assumption on the elements of the effective domain of the original rate function is required in order to overcome the fact that the natural projection mapping is not necessarily S-continuous.

Proposition 2.25 *Let $\{x_n,\ n = 0, \pm 1, \pm 2, \ldots\}$ be a stationary $\mathbb{R}^p$-valued random sequence and assume $\{X_n(\cdot),\ n = 1, 2, \ldots\}$ to satisfy the LDP in $(D[0,1]^p, \tau_0)$ with good rate function $I_X : D[0,1]^p \to [0, \infty]$. Assume further that every element of the effective domain of I_X is continuous at $t = \frac{1}{2}$.*

Then for each $K = 1, 2, \ldots$, the family $\left\{ \left(X_n^K(\cdot), X_n^{K,-}(\cdot) \right),\ n = 1, 2, \ldots \right\}$ satisfies the LDP in the product space $(D[0,K]^p, \tau_0) \times (D_l[0,K]^p, \tau_0)$ with good rate function $I_{X,X^-}^K : D[0,K]^p \times D_l[0,K]^p \to [0, \infty]$ given by

$$I_{X,X^-}^K(\varphi_1, \varphi_2) \equiv \begin{cases} 2K \inf\left\{ I_X(\frac{1}{2K}\varphi + c) : c \in \mathbb{R} \right\} & \text{if} \quad \varphi_1(0) = \varphi_2(0) = 0 \\ \infty & \text{otherwise} \end{cases}, \tag{2.34}$$

for φ_1 in $D[0,K]^p$, φ_2 in $D_l[0,K]^p$ and $\varphi : [0,1] \to \mathbb{R}$ given by

$$\varphi(t) = \begin{cases} -\varphi_2(K - 2Kt), & t \in [0, \frac{1}{2}] \\ \varphi_1(2Kt - K), & t \in [\frac{1}{2}, 1] \end{cases}. \tag{2.35}$$

Note that for φ_1 in $D[0,K]^p$ and φ_2 in $D_l[0,K]^p$, the mapping φ defined by (2.35) is indeed right-continuous with left-hand limits, and is therefore an element of $D[0,1]^p$.

Proof: Fix $K = 1, 2, \ldots$; for $n = 1, 2, \ldots$, we have

$$\left(X_n^K(\cdot), X_n^{K,-}(\cdot) \right) = \left(\frac{1}{n} \sum_{i=1}^{\lfloor n\cdot \rfloor} x_i, \frac{1}{n} \sum_{i=1-\lceil n\cdot \rceil}^{0} x_i \right)$$

$$=_{st} \quad \left(\frac{1}{n} \sum_{i=nK+1}^{\lfloor n\cdot \rfloor + nK} x_i \, , \, \frac{1}{n} \sum_{i=nK+1-\lceil n\cdot \rceil}^{nK} x_i \right) \tag{2.36}$$

where the last step follows from the stationarity of $\{x_n,\ n = 0, \pm 1, \pm 2, \ldots\}$ and the fact that the shift (nK) does not depend on t.

Next, for each t in $[0, K]$, we find after some simple algebra that

$$nK + \lfloor nt \rfloor \ = \ \lfloor nK + nt \rfloor \ = \ \lfloor n(t+K) \rfloor$$

and

$$nK - \lceil nt \rceil \ = \ \lfloor nK - nt \rfloor \ = \ \lfloor n(K-t) \rfloor \, .$$

Therefore, from (2.36) we get that

$$\begin{aligned} \left(X_n^K(\cdot), X_n^{K,-}(\cdot) \right) \ &=_{st} \ \left(\frac{1}{n} \sum_{i=nK+1}^{\lfloor n(K+\cdot) \rfloor} x_i \, , \, \frac{1}{n} \sum_{i=1+\lfloor n(K-\cdot) \rfloor}^{nK} x_i \right) \\ &= \ \left(\frac{1}{n} \sum_{i=1}^{\lfloor n(K+\cdot) \rfloor} x_i - \frac{1}{n} \sum_{i=1}^{nK} x_i \, , \, \frac{1}{n} \sum_{i=1}^{nK} x_i - \frac{1}{n} \sum_{i=1}^{\lfloor n(K-\cdot) \rfloor} x_i \right) . \end{aligned} \tag{2.37}$$

Finally, upon noting that

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^{\lfloor n(K \pm s) \rfloor} x_i \ &= \ 2K \frac{1}{2nK} \sum_{i=1}^{\lfloor 2nK \frac{K \pm s}{2K} \rfloor} x_i \\ &= \ 2K \, X_{2nK}\left(\frac{K \pm s}{2K}\right), \quad s \in [0, K], \end{aligned}$$

we can rewrite (2.37) as

$$\begin{aligned} &\left(X_n^K(\cdot), X_n^{K,-}(\cdot) \right) \\ &=_{st} \ \left(2K \, X_{2nK}\left(\frac{K+\cdot}{2K}\right) - 2K \, X_{2nK}\left(\frac{1}{2}\right) \, , \, 2K \, X_{2nK}\left(\frac{1}{2}\right) - 2K \, X_{2nK}\left(\frac{K-\cdot}{2K}\right) \right) \\ &= \ G(X_{2nK}(\cdot)) \end{aligned} \tag{2.38}$$

where the mapping $G : D[0,1]^p \to D[0,K]^p \times D_l[0,K]^p$ is defined by

$$G(z)(t) \ \equiv \ \begin{pmatrix} 2K \, z\left(\frac{K+t}{2K}\right) - 2K \, z\left(\frac{1}{2}\right) \\ 2K \, z\left(\frac{1}{2}\right) - 2K \, z\left(\frac{K-t}{2K}\right) \end{pmatrix}, \quad t \in [0, K], \quad z \in D[0,1]^p \, .$$

We now argue that G is Borel-measurable and $\mathcal{S}$-continuous on $\mathcal{D}_{I_X}$. By [27, p. 55] and Proposition I in [12, p. 44], it suffices to show Borel-measurability and

$\mathcal{S}$-continuity of the coordinate mappings of G. Because of the similarity between the two coordinate mappings, we only consider $G_1 : z \to 2K\, z(\frac{K+\cdot}{2K}) - 2K\, z(\frac{1}{2})$.

By Lemma 2.4, the mapping $z \to z(\frac{1}{2})$ is Borel-measurable, and under the assumption that the elements of $\mathcal{D}_{I_X}$ are continuous at $t = \frac{1}{2}$, it is $\mathcal{S}$-continuous on $\mathcal{D}_{I_X}$. By Lemma 2.9 the mapping $z \to 2K\, z(\frac{K+\cdot}{2K})$ is $\mathcal{S}$-continuous (hence Borel-measurable), and it follows easily by [27, p. 55] and Proposition I in [12, p. 44] that the mapping $z \to \left(2K\, z(\frac{K+\cdot}{2K}), z(\frac{1}{2})\right)$ is Borel-measurable and $\mathcal{S}$-continuous on $\mathcal{D}_{I_X}$. Finally, upon noting that the constant mapping $z(\frac{1}{2})$ does not have any discontinuity point, Lemma 2.8 yields Borel-measurability and $\mathcal{S}$-continuity of G_1 on D_{I_X}.

Next, by Lemma 2.19 the family $\{X_{2nK}(\cdot),\ n = 1, 2, \ldots\}$ satisfies the LDP in $(D[0,1]^p, \tau_0)$ with good rate function $2K\, I_X$. Thus, in view of (2.38), and of the Borel-measurability and $\mathcal{S}$-continuity of G on $\mathcal{D}_{I_X} = \mathcal{D}_{2K I_X}$, a simple application of the Contraction Principle yields the LDP for $\{G(X_{2nK}(\cdot)),\ n = 1, 2, \ldots\}$, whence for the (stochastically equivalent) processes $\{(X_n^K(\cdot), X_n^{K,-}(\cdot)),\ n = 1, 2, \ldots\}$ in $(D[0,K]^p, \tau_0) \times (D_l[0,K]^p, \tau_0)$ with good rate function $I^K_{X,X^-} : D[0,K]^p \times D_l[0,K]^p \to [0, \infty]$ given by

$$I^K_{X,X^-}(\varphi_1, \varphi_2) = \inf_{\psi \in D[0,1]^p} \{2K\, I_X(\psi) : (\varphi_1, \varphi_2) = G(\psi)\}, \quad \varphi_1 \in D[0,K]^p, \quad \varphi_2 \in D_l[0,K]^p. \tag{2.39}$$

Finally, for φ_1 in $D[0,K]^p$, φ_2 in $D_l[0,K]^p$ and ψ in $D[0,1]^p$, we note that $(\varphi_1, \varphi_2) = G(\psi)$ if and only if

$$\begin{cases} \varphi_1(t) &= 2K\psi(\frac{K+t}{2K}) - 2K\psi(\frac{1}{2}) \\ \varphi_2(t) &= 2K\psi(\frac{1}{2}) - 2K\psi(\frac{K-t}{2K}) \end{cases}, \quad t \in [0, K],$$

or equivalently,

$$\begin{cases} \varphi_1(0) &= \varphi_2(0) \ = \ 0 \\ \psi(t) &= \psi(\frac{1}{2}) + \frac{1}{2K}\varphi_1(2Kt - K), \quad t \in [\frac{1}{2}, 1] \\ \psi(t) &= \psi(\frac{1}{2}) - \frac{1}{2K}\varphi_2(K - 2Kt), \quad t \in [0, \frac{1}{2}] \end{cases}$$

and (2.39) yields the desired expression for I^K_{X,X^-}. ■

In the good cases (i.e., when the rate function is of the form (2.28)), the continuity assumption on the elements of the effective domain of I is automatically satisfied, and we have the following Corollary to Proposition 2.25.

Corollary 2.26 *If the good rate function I_X is of the form (2.28), then*

$$I^K_{X,X^-}(\varphi_1,\varphi_2) = \begin{cases} \int_0^K r(\dot\varphi_1(t))\,dt + \int_0^K r(\dot\varphi_2(t))\,dt & \text{if } \varphi_1,\varphi_2 \in AC_0[0,K]^p \\ \infty & \text{otherwise} \end{cases} . \tag{2.40}$$

Proof: Because $D_{I_X} = AC_0[0,1]^p$, the elements of $\mathcal{D}_{I_X}$ are automatically continuous at $t=\frac{1}{2}$, and Proposition 2.25 yields the LDP for $\{(X^K_n(\cdot), X^{K,-}_n(\cdot)),\ n = 1,2,\ldots\}$ with good rate function I^K_{X,X^-} given by (2.34).

Next, we observe from the definition of φ in (2.35) that φ_1 and φ_2 belong to $AC_0[0,K]^p$ if and only if φ belongs to $AC_0[0,1]^p$. Thus, from the expression (2.28) we have $I_X(\frac{1}{2K}\varphi+c)=\infty$ for φ not in $AC_0[0,1]^p$, and $I^K_{X,X^-}(\varphi_1,\varphi_2)=\infty$ whenever φ_1 or φ_2 is not in $AC_0[0,K]$.

Finally, for φ_1,φ_2 in $AC_0[0,K]^p$ and c in $\mathbb{R}$ we have

$$\begin{aligned} 2K\, I_X(\frac{1}{2K}\varphi + c) &= 2K\int_0^1 r(\frac{1}{2K}\dot\varphi(t))\,dt \\ &= 2K\int_0^{\frac{1}{2}} r(\dot\varphi_2(K-2Kt))\,dt \;+\; 2K\int_{\frac{1}{2}}^1 r(\dot\varphi_1(2Kt-K))\,dt \\ &= \int_0^K r(\dot\varphi_2(u))\,du \;+\; \int_0^K r(\dot\varphi_1(u))\,du\,. \end{aligned}$$

■

It is interesting to know that Proposition 2.25 carries over to the spaces $D[a,b]^p \times D_l[a,b]^p$, **without any additional assumptions** , for arbitrary **irrational** $0 \le a < b$, and under certain continuity assumptions on the elements of the effective domain D_{I_X} if a or b is rational. We give the result for the particular case where $[a,b]=[0,T]$ for arbitrary $T>0$. When I_X is of the form (2.28), we have $D_{I_X} = AC_0[0,1]^p$, and the additional assumptions on the elements of D_{I_X} needed when T is rational are automatically satisfied.

Corollary 2.27 *Let $T>0$, and assume $\{X_n(\cdot),\ n=1,2,\ldots\}$ to satisfy the LDP in $(D[0,1]^p,\tau_0)$ with good rate function $I_X:\ D[0,1]^p \to [0,\infty]$. Assume further that every element of the effective domain $\mathcal{D}_{I_X}$ of I_X is continuous at $t=\frac{1}{2}$, as well as at $t=\frac{1}{2}\pm\frac{T}{2\lceil T\rceil}$ if T is rational.*

Then, the family $\{(X^T_n(\cdot), X^{T,-}_n(\cdot)),\ =1,2,\ldots\}$ satisfies the LDP in the product space $(D[0,T]^p,\tau_0)\times(D_l[0,T]^p,\tau_0)$ with good rate function $I^T_{X,X^-}: D[0,T]^p \times$

$D_l[0,T]^p \to [0,\infty]$ *given by*

$$I^T_{X,X^-}(\varphi_1,\varphi_2) = \inf_{\substack{\psi_1\in D[0,\lceil T\rceil]^p\\ \psi_2\in D_l[0,\lceil T\rceil]^p}} \left\{ I^{\lceil T\rceil}_{X,X^-}(\psi_1,\psi_2) : \begin{array}{c}\varphi_1=\psi_1\\ \varphi_2=\psi_2\end{array} \quad on\ [0,T] \right\}, \qquad (2.41)$$

for φ_1 *in* $D[0,T]^p$ *and* φ_2 *in* $D_l[0,T]^p$ *where* $I^{\lceil T\rceil}_{X,X^-}$ *is given in Proposition 2.25.*

Proof: Under the assumption that $\{X_n(\cdot),\ n=1,2,\ldots\}$ satisfy the LDP with each of the elements of D_{I_X} being continuous at $t=\frac{1}{2}$, Proposition 2.25 (with $K=\lceil T\rceil$) yields the LDP for $\left\{\left(X_n^{\lceil T\rceil}(\cdot), X_n^{\lceil T\rceil,-}(\cdot)\right),\ n=1,2,\ldots\right\}$ in $D[0,T]^p \times D_l[0,T]^p$ with good rate function $I^{\lceil T\rceil}_{X,X^-} : D[0,\lceil T\rceil]^p \times D_l[0,\lceil T\rceil]^p \to [0,\infty]$ given by

$$I^{\lceil T\rceil}_{X,X^-}(\varphi_1,\varphi_2) \equiv \begin{cases} 2\lceil T\rceil \inf\left\{I_X(\frac{1}{2\lceil T\rceil}\varphi + c) : c\in\mathbb{R}\right\} & \text{if} \quad \varphi_1(0)=\varphi_2(0)=0 \\ \infty & \text{otherwise}\end{cases},$$

for φ_1 in $D[0,\lceil T\rceil]^p$, φ_2 in $D_l[0,\lceil T\rceil]^p$ and with φ satisfying

$$\varphi(t) = \begin{cases} -\varphi_2(\lceil T\rceil - 2\lceil T\rceil t), & t\in[0,\frac{1}{2}] \\ \varphi_1(2\lceil T\rceil t - \lceil T\rceil), & t\in[\frac{1}{2},1]\end{cases}. \qquad (2.42)$$

It should be clear that if (φ_1,φ_2) belongs to the effective domain of $I^{\lceil T\rceil}_{X,X^-}$, then there exits c_0 in $\mathbb{R}$ such that $\varphi+c_0$ belongs to $\mathcal{D}_{I_X}$. Thus if the elements of $\mathcal{D}_{I_X}$ are continuous at $t=\frac{1}{2}\pm\frac{T}{2\lceil T\rceil}$, then we immediately see from (2.42) that φ_1 and φ_2 are continuous at $t=T$.

Consider the restriction mapping $\tilde{r}_T : D[0,\lceil T\rceil]^p \times D_l[0,\lceil T\rceil]^p \to D[0,T]^p \times D_l[0,T]^p$. By [27, p. 55] and Lemma 2.7, $\tilde{r}_T$ is Borel-measurable. By Proposition 2.24, the effective domain of $I^{\lceil T\rceil}_{X,X^-}$ is a subset of $D^Q[0,\lceil T\rceil]^p \times D^Q_l[0,\lceil T\rceil]^p$, so that, if T is irrational, each of its element is continuous at $t=T$. On the other hand, if T is rational, then from the continuity assumptions on the elements of $\mathcal{D}_{I_X}$ and the comments made above, we get that each pair (φ_1,φ_2) in the effective domain of $I^{\lceil T\rceil}_{X,X^-}$ is also continuous at $t=T$. Thus, by Lemma 2.7 and Proposition I in [12, p. 49], the mapping $\tilde{r}_T$ is S-continuous on the effective domain of $I^{\lceil T\rceil}_{X,X^-}$ for any $T>0$.

Therefore, because we trivially have

$$\tilde{r}_T\left(X_n^{\lceil T\rceil}(\cdot), X_n^{\lceil T\rceil,-}(\cdot)\right) = \left(X_n^T(\cdot), X_n^{T,-}(\cdot)\right), \quad n=1,2,\ldots,$$

the Contraction Principle yields the LDP for $\{(X_n^T(\cdot), X_n^{T,-}(\cdot)),\ n=1,2,\ldots\}$ in $(D[0,T]^p,\tau_0)\times(D_l[0,T]^p,\tau_0)$ with good rate function $I^T_{X,X^-} : D[0,T]^p \times D_l[0,T]^p \to [0,\infty]$ given by (2.41). ∎

Corollary 2.28 *Assume* $\{X_n(\cdot),\ n = 1, 2, \ldots\}$ *to satisfy the LDP in* $(D[0,1]^p, \tau_0)$ *with good rate function* $I_X : D[0,1]^p \to [0,\infty]$ *of the form (2.28), with* r *being a rate function.*

Then, for any $T > 0$, *the family* $\{(X_n^T(\cdot), X_n^{T,-}(\cdot)),\ = 1, 2, \ldots\}$ *satisfies the LDP in the product space* $(D[0,T]^p, \tau_0) \times (D_l[0,T]^p, \tau_0)$ *with good rate function* $I_{X,X^-}^T : D[0,T]^p \times D_l[0,T]^p \to [0,\infty]$ *given by*

$$I_{X,X^-}^T(\varphi_1, \varphi_2) = \begin{cases} \int_0^T r(\dot{\varphi}_1(t))\,dt + \int_0^T r(\dot{\varphi}_2(t))\,dt & \text{if } \varphi_1, \varphi_2 \in AC_0[0,T]^p \\ \infty & \text{otherwise} \end{cases}. \tag{2.43}$$

In view of Corollary 1.9, this last result shows that the joint process of the past and future has the same large deviations behavior as two independent processes with same marginal distributions. In other words, in terms of sample path large deviations, the past and future of a stationary process satisfying a sample path LDP in $D[0,1]$ with good rate function of the form (2.28) with r being itself a rate function, are **independent**.

Proof: As $\mathcal{D}_{I_X} = AC_0[0,1]^p$, the elements of $\mathcal{D}_{I_X}$ are automatically continuous at $t = \frac{1}{2}$ and $t = \frac{1}{2} \pm \frac{T}{2\lceil T \rceil}$. Thus, by Corollary 2.27, we already find that the desired LDP holds with good rate function given by (2.41). Moreover, a closed-form expression for $I_{X,X^-}^{\lceil T \rceil}$ is available through Corollary 2.26, namely,

$$I_{X,X^-}^{\lceil T \rceil}(\varphi_1, \varphi_2) = \begin{cases} \int_0^{\lceil T \rceil} r(\dot{\varphi}_1(t))\,dt + \int_0^{\lceil T \rceil} r(\dot{\varphi}_2(t))\,dt & \text{if } \varphi_1, \varphi_2 \in AC_0[0,\lceil T \rceil]^p \\ \infty & \text{otherwise} \end{cases}.$$

Using Lemma 2.18 and the expression above, we easily simplify (2.41) to the desired expression (2.43). ■

As we shall see in the next chapter, the LDP for the family of partial sum processes $\{X_n^T(\cdot),\ n = 1, 2, \ldots\}$ or $\left\{\left(X_n^T(\cdot), X_n^{T,-}(\cdot)\right), n = 1, 2, \ldots\right\}$ in $D[0,T]^p$ and $D[0,T]^p \times D_l[0,T]^p$ for all irrational $T > 0$ in turn yields the LDP in $D[0,\infty)^p$ and $D[0,\infty)^p \times D_l[0,\infty)^p$ for their respective extensions on $[0,\infty)$.

Chapter 3

LDP for Partial Sum Processes in $D[0,\infty)$ and $D_l[0,\infty)$

It turns out that the LDP for $\{X_n^T(\cdot),\ n = 1,2,\ldots\}$ in $D[0,T]^p$ for all $T > 0$ yields the LDP in $D[0,\infty)^p$ for the extension of the partial sum process on $[0,\infty)$. In view of the results of Chapter 2, the LDP in $D[0,\infty)^p$ for the extension of the partial sum process on $[0,\infty)$ is thus a consequence of the LDP for the partial sum process in $D[0,1]^p$. This property carries over to the LDP for the negative partial sum process, as well as jointly for the partial sum process and the negative partial sum process.

The results obtained here are used in Chapter 4 to establish the LDP for some functionals on the inputs to a single-server queueing system.

The chapter is organized as follows. We begin with reviewing the extended Skorohod topology on $D[0,\infty)$ and $D_l[0,\infty)$, as well as some topological properties of theses spaces. Then, we state and prove the main theorems on the LDP for the extensions of the partial sum processes on $D[0,\infty)$ and $D_l[0,\infty)$. Finally we review cases where a LDP for partial sum processes indeed holds.

3.1 The spaces $D[0,\infty)$ and $D_l[0,\infty)$

We begin this section by introducing a suitable topology on the spaces $D[0,\infty)$ and $D_l[0,\infty)$. As discussed in Section 1.5.1, we would like to work on a regular topological space which is separable, and this motivates our choice of the extended Skorohod topology. Indeed, it turns out that the Skorohod topology can be extended to the space $D[0,\infty)$ and that the extension makes $D[0,\infty)$ into a separable metrizable space, thus fulfilling our earlier requirements when considering random elements in that space.

We begin with reviewing the extension of the Skorohod topology on $D[0,\infty)$ and with introducing its equivalent on $D_l[0,\infty)$.

3.1.1 The extended Skorohod topology

The Skorohod topology on $D[a,b]$ was extended to $D[0,\infty)$, in the context of weak convergence, by Stone [60] and later by Lindvall [45] and Whitt [66]. In fact, as we shall see, the topologies defined in these three papers coincide. Both Lindvall and Whitt exhibit a metric on $D[0,\infty)$. Today, an account of the Skorohod topology on $D[0,\infty)$, often based on the introduction of a metric derived from the Skorohod metric on $D[a,b]$, can be found in many treatises on weak convergence [31, 39, 44, 55]. The following is Stone's definition of the extended J_1-convergence on $D[0,\infty)$. Let Λ_∞ be the space of continuous one-to-one and strictly increasing mappings from $[0,\infty)$ onto itself. For each λ in Λ_∞, we automatically have $\lambda(0)=0$ and $\lim_{t\to\infty}\lambda(t)=+\infty$.

Definition 3.1 (**J_1-convergence [60]**) *A sequence $\{x_n,\ n=1,2,\ldots\}$ in the space $D[0,\infty)$ is J_1-convergent to x if there exists a sequence $\{\lambda_n,\ n=1,2,\ldots\}$ in Λ_∞ such that for all $T>0$,*

$$\lim_{n\to\infty}\sup_{0\le t\le T}|x_n(t)-x\circ\lambda_n(t)| \;=\; \lim_{n\to\infty}\sup_{0\le t\le T}|\lambda_n(t)-t| \;=\; 0. \tag{3.1}$$

We adopt Whitt's definition of the extended Skorohod topology on $D[0,\infty)$ [66], which we denote by τ_0^e. To that end recall that a topology can be defined by its mode of convergence, provided the mode of convergence is a convergence class. The details of such a construction are beyond the scope of this dissertation. However, the interested reader will find all the relevant material in [40, Chapter 2]. The extended Skorohod topology τ_0^e is then defined as follows. Recall that a net is a mapping from a directed set into a topological space [40, p. 65].

Definition 3.2 (**S-convergence [66]**) *A net $\{x_\alpha,\ \alpha\in I\}$ is said to be S-convergent to x in the topology τ_0^e if the restrictions of x_α in $(D[a,b],\tau_0)$ are S-convergent to the restriction of x in $(D[a,b],\tau_0)$ for each compact interval $[a,b]$ contained in $[0,\infty)$ such that either a and b are continuity points of x or $a=0$.*

It is easily checked that S-convergence indeed defines a convergence class [40, p. 74]. Definition 3.2 also defines a topology on $D_l[0,\infty)$, which we shall refer to as the *extended* Skorohod topology on $D_l[0,\infty)$.

On the other hand, Definition 3.1 defines a convergence on $D_l[0,\infty)$, which we shall refer to as the extended J_1-convergence on $D_l[0,\infty)$.

For any $t > 0$, let d_0^t be the d_0 metric on $D[0,t]$, and let $R_t : D[0,\infty) \to D[0,t]$ denote the restriction to $[0,t]$, defined by

$$R_t(x)(s) = x(s), \quad 0 \le s \le t, \quad x \in D[0,\infty).$$

The metric $d_0^e : D[0,\infty) \times D[0,\infty) \to \mathbb{R}_+$ defined by

$$d_0^e(x,y) \equiv \int_0^\infty e^{-t}\left[d_0^t\left(R_t(x), R_t(y)\right) \wedge 1\right] dt, \quad x,y \in D[0,\infty) \tag{3.2}$$

makes $D[0,\infty)$ into a complete separable metric space and induces the extended Skorohod topology [66, Theorem 2.5 & 2.6].

Similarly, we can define a metric on $D_l[0,\infty)$, still denoted by d_0^e, by setting

$$d_0^e(x,y) \equiv \int_0^\infty e^{-t}\left[d_0^t\left(R_t^l(x), R_t^l(y)\right) \wedge 1\right] dt, \quad x,y \in D_l[0,\infty)$$

where $R_t^l : D_l[0,\infty) \to D_l[0,t]$ is the restriction to the interval $[0,t]$, and d_0^t denotes the metric d_0^l on $D_l[0,t]$ introduced in Section 2.1.1. By adapting the proofs of Theorems 2.5 and 2.6 in [66], we can easily check that d_0^e is a metric on $D_l[0,\infty)$ which induces the extended Skorohod topology on $D_l[0,\infty)$, and that $(D_l[0,\infty), d_0^e)$ is a Polish space.

Other metrics on $D[0,\infty)$ inducing the extended Skorohod topology can be found in [31, p. 117], [39, p. 294] and [55, p. 123].

It is reassuring to know that Stone's definition of convergence coincides with that of Whitt, both in $D[0,\infty)$ and in $D_l[0,\infty)$. We use upper cases for the projection and restriction mappings on $D[0,\infty)$ or $D_l[0,\infty)$ to distinguish them from their analogs in $D[a,b]$ and $D_l[a,b]$ which are denoted by lower cases.

Lemma 3.3 *A sequence $\{x_n,\ n = 1,2,\ldots\}$ in $D[0,\infty)$ (resp. in $D_l[0,\infty)$) converges to x in the extended Skorohod topology τ_0^e (Whitt) if and only if it converges to x according to Stone's definition of the extended J_1-convergence.*

Proof: Let $\{x_n,\ n = 1,2,\ldots\}$ be a sequence in $D[0,\infty)$ (resp. $D_l[0,\infty)$), which is J_1-converging to x according to Definition 3.1. Then there exists a sequence of mappings $\{\lambda_n,\ n = 1,2,\ldots\}$ in Λ_∞ such that (3.1) is satisfied. Let $[a,b]$ be a subset of $[0,\infty)$ such that a,b are either continuity points of x or $a = 0$, and consider the restrictions $R_{ab}(\lambda_n)$ of the mappings λ_n on $[a,b]$. Although we might not have $\lambda_n(a) = a$ and $\lambda_n(b) = b$, by continuity of x at a and b, for n large enough, it is possible to slightly modify λ_n so that $R_{ab}(\lambda_n)$ belongs to Λ_{ab} while still satisfying

$$\lim_{n\to\infty} \sup_{a\le t\le b} |x_n(t) - x \circ \lambda_n(t)| = \lim_{n\to\infty} \sup_{a\le t\le b} |\lambda_n(t) - t| = 0. \tag{3.3}$$

With the sequence of modified $\{\lambda_n,\ n = 1, 2, \ldots\}$ (3.3) yields S-convergence of the restrictions $R_{ab}(x_n)$ (resp. $R^l_{ab}(x_n)$) to the restriction $R_{ab}(x)$ (resp. $R^l_{ab}(x)$) for any interval $[a, b]$ such that a, b are continuity points of x or $a = 0$. Thus the sequence $\{x_n,\ n = 1, 2, \ldots\}$ converges to x in the extended Skorohod topology.

Conversely, assume the sequence $\{x_n,\ n = 1, 2, \ldots\}$ converges to x in the extended Skorohod topology. Then, x being in $D[0, \infty)$ (resp. $D_l[0, \infty)$), it admits at most countably many discontinuity points. Therefore, it is possible to construct a sequence of intervals $[a_k, a_{k+1}]$, $k = 0, 1, \ldots$ such that $a_0 = 0$, $\lim_{k\to\infty} a_k = \infty$ and for each $k = 0, 1, \ldots$, a_k is a continuity point for x. Then, by definition of S-convergence, for each $k = 0, 1, \ldots$, the restrictions of x_n on $[a_k, a_{k+1}]$ converge to the restrictions of x. Hence there exists a family of mappings $\{\lambda^k_n,\ n = 1, 2, \ldots,\ k = 0, 1, \ldots\}$ such that for each $k = 0, 1, \ldots$, the sequence $\{\lambda^k_n,\ n = 1, 2, \ldots\}$ belongs to $\Lambda_{a_k a_{k+1}}$, and satisfies

$$\lim_{n\to\infty} \sup_{a_k \le t \le a_{k+1}} |x_n(t) - x(\lambda^k_n(t))| \;=\; \lim_{n\to\infty} \sup_{a_k \le t \le a_{k+1}} |\lambda^k_n(t) - t| \;=\; 0. \tag{3.4}$$

In particular, for each $k = 0, 1, \ldots$, $\lambda^k_n(a_k) = a_k$ and $\lambda^k_n(a_{k+1}) = a_{k+1}$, so that for each $n = 1, 2, \ldots$, the mapping $\lambda_n : [0, \infty) \to [0, \infty)$ defined by

$$\lambda_n(t) \;=\; \lambda^k_n(t), \quad t \in [a_k, a_{k+1}], \quad k = 0, 1, \ldots .$$

belongs to Λ_∞. Moreover, for any $T > 0$, there exists an integer k_T such that $T \le a_{k_T}$ and (3.4) applied for the finitely many $k = 1, 2 \ldots, k_T$ yields

$$\lim_{n\to\infty} \sup_{0 \le t \le T} |x_n(t) - x \circ \lambda_n(t)| \;=\; 0 \qquad \text{and} \qquad \lim_{n\to\infty} \sup_{0 \le t \le T} |\lambda_n(t) - t| \;=\; 0,$$

so that x_n J_1-converges to x in $D[0, \infty)$ (resp. $D_l[0, \infty)$). ■

In view of this last Lemma and the different results available in the cited references, we find that all the extended Skorohod topologies defined on $D[0, \infty)$ in [31, 39, 44, 45, 55, 60, 61, 66] indeed coincide. Thus, for now on we shall use either one of the two characterizations of Skorohod convergence in $D[0, \infty)$ and $D_l[0, \infty)$.

It should be clear that all the properties or definition given so far apply to the multi-dimensional spaces $D[0, \infty)^p$ (resp. $D_l[0, \infty)^p$) of right continuous (resp. left continuous) mappings with left-hand limits (resp. right-hand limits) taking on values in $\mathbb{R}^p$. The metric d^e_0 on these spaces is defined similarly to that in the one-dimensional case by taking the product norm $|\cdot|_p$ in $\mathbb{R}^p$ defined by

$$|x - y|_p \;=\; \max_{i=1,\ldots,p} |x_i - y_i|$$

for $x = (x_1, \ldots, x_p)$ and $y = (y_1, \ldots, y_p)$ in $\mathbb{R}^p$. For the sake of clarity, we keep the same notations for the metrics and topologies on $D[0,\infty)^p$ and $D_l[0,\infty)^p$ than on $D[0,\infty)$ and $D_l[0,\infty)$.

On the other hand, when referring to the Cartesian product space $(D[0,\infty))^p$ (resp. $(D_l[0,\infty))^p$), we mean the metric space with metric d^p (which induces the product topology) defined by

$$d^p(x,y) = \max_{i=1,\ldots,p} d_0^e(x_i, y_i), \tag{3.5}$$

for $x = (x_1, \ldots, x_p)$ and $y = (y_1, \ldots, y_p)$ and x_i, y_i in $D[0,\infty)$ (resp. $D_l[0,\infty)$) for $i = 1, \ldots, p$.

We shall come back in more details to theses spaces in Section 3.2.3 where we discuss the LDP for partial sum processes in the product space $(D[0,\infty), \tau_0^e)^p$ versus that in the multi-dimensional space $(D[0,\infty)^p, \tau_0^e)$.

3.1.2 Topological properties on $D[0,\infty)$ and $D_l[0,\infty)$

In this section, we review the topological properties of $(D[0,\infty)^p, \tau_0^e)$ and their counterparts in $D_l[0,\infty)^p$ that will be needed in the forthcoming sections and chapters.

The approach we took in defining the Skorohod topology on $D_l[0,\infty)^p$ will prove very useful in establishing the LDP in that space, as we shall see in Sections 3.1.3 and 3.2. Unfortunately, it is of little help in deriving topological properties of $D_l[0,\infty)^p$ from those of $D[0,\infty)^p$, in particular the characterization of the Borel σ-field and the S-continuity of some functionals.

Because Whitt's study of the space of right continuous functions with left-hand limits encompasses the space $D(-\infty,0]$, it is still possible by using his results [66] (although most of the time the transposition from finite time intervals to infinite ones is not done explicitly), to overcome these difficulties by introducing an isometry between $D_l[0,\infty)^p$ and $D(-\infty,0]^p$. We define the mapping $\Phi : D_l[0,\infty)^p \to D(-\infty,0]^p$ by setting

$$\Phi(x)(t) = x(-t), \quad t \in (-\infty, 0], \quad x \in D_l[0,\infty)^p.$$

It is plain that Φ is a bijection, and a review of [66, p. 72] shows that if d_0^e also denotes Whitt's extended Skorohod metric on $D(-\infty,0]^p$, then we have

$$d_0^e(\Phi(x), \Phi(y)) = d_0^e(x,y), \quad x, y \in D_l[0,\infty)^p,$$

thus making $(D_l[0,\infty)^p, d_0^e)$ isometric to $(D(-\infty,0]^p, d_0^e)$.

We begin with the characterization of the Borel σ-fields on the spaces $D[0,\infty)^p$ and $D_l[0,\infty)^p$.

Proposition 3.4 *The Borel σ-fields on $(D[0,\infty)^p, \tau_0^e)$ and $(D_l[0,\infty)^p, \tau_0^e)$ coincide respectively with the projection σ-fields $\mathcal{P}_\infty$ and $\mathcal{P}_\infty^l$ defined as the smallest σ-fields on which the natural projections $\Pi_t : D[0,\infty)^p \to \mathbb{R}^p, \quad t \geq 0$ and $\Pi_t^l : D_l[0,\infty)^p \to \mathbb{R}^p, \quad t \geq 0$ are measurable.*

Proof: The result for $D[0,\infty)^p$ is formally proved as Theorem 6 in [55, p. 127]. Although the metric used there does not coincide with the metric d_0^e, it is known to induce the extended Skorohod topology as defined in Definition 3.2 [55, Problem 2 p. 137].

As for the space $D_l[0,\infty)^p$, the result can be shown either by using the isometry above and Whitt's results [66, Lemma 2.7], or by adapting Pollard's Theorem 6 to the $D_l[0,\infty)$ setting. ■

We now focus on two subspaces of $D[0,\infty)^p$ and $D_l[0,\infty)^p$ that we shall use in Section 3.2 and Chapter 4. We define first the subspace $D^*[0,\infty)$ of $D[0,\infty)$ by setting

$$D^*[0,\infty) \equiv \{x \in D[0,\infty) : \lim_{t\to\infty} x(t) \text{ exists and is finite}\}.$$

As shown in [45], there exists a metric h on $D^*[0,\infty)$ which makes it isometric to the Polish space $(D^*[0,1], d_0)$, where $D^*[0,1] \equiv \{x \in D[0,1] : x(1^-) = x(1)\}$. Furthermore, the metric h is shown [45, Lemma 1] to induce on $D^*[0,\infty)$ Stone's [60] definition of the extended J_1-convergence, whence the extended Skorohod topology by Lemma 3.3.

This interesting property translates to the subspace $D_l^*[0,\infty)$ of $D_l[0,\infty)$ defined by

$$D_l^*[0,\infty) \equiv \{x \in D_l[0,\infty) : \lim_{t\to\infty} x(t) \text{ exists and is finite}\}.$$

We also define $D_l^*[0,1] \equiv \{x \in D_l[0,1] : x(0^+) = x(0)\}$.

Lemma 3.5 *Define the mappings $\widehat{\varphi} : [0,1] \to [0,\infty]$, $\widehat{\Phi} : D_l^*[0,\infty) \to D_l^*[0,1]$ and $h : D_l^*[0,\infty) \times D_l^*[0,\infty) \to \mathbb{R}_+$ by setting*

$$\widehat{\varphi}(t) = \begin{cases} -\ln t & 0 < t \leq 1 \\ \infty & t = 0 \end{cases}, \qquad \widehat{\Phi}(x) = x \circ \widehat{\varphi}, \quad x \in D_l^*[0,\infty), \tag{3.6}$$

and

$$h(x,y) = d_0(\widehat{\Phi}(x), \widehat{\Phi}(y)), \quad x, y \in D_l^*[0,\infty).$$

Then h is a metric, $(D_l^*[0,\infty), h)$ is a Polish space isometric to $(D_l^*[0,1], d_0)$, and h induces the extended Skorohod topology on $D_l^*[0,\infty)$.

Proof: The proof is an easy transposition to the space $D_l[0,\infty)$ of the arguments given by Lindvall in [45, Lemma 1, p. 110] and can be found in Appendix A.6. ■

Next, we denote respectively by $D^Q[0,\infty)^p$ and $D_l^Q[0,\infty)^p$ the subspaces of $D[0,\infty)^p$ and $D_l[0,\infty)^p$ of functions which are continuous at each **irrational** t in $[0,\infty)$. We then have the following simple extension of Lemma 2.12.

Lemma 3.6 *The sets $D^Q[0,\infty)^p$ and $D_l^Q[0,\infty)^p$ are closed subsets (under the metric d_0^e) of $D[0,\infty)^p$ and $D_l[0,\infty)^p$, respectively. Thus $(D^Q[0,\infty)^p, d_0^e)$ and $(D_l^Q[0,\infty)^p, d_0^e)$ are Polish spaces.*

Proof: Let $\{x_n,\ n = 1,2,\ldots\}$ be a sequence in the space $(D^Q[0,\infty)^p, d_0^e)$ (resp. $(D_l^Q[0,\infty)^p, d_0^e)$) which converges to x in $D[0,\infty)^p$ (resp. $D_l[0,\infty)^p$). Let t be an irrational in $[0,\infty)$ and let b be a continuity point of x such that $b \geq t$ (such b exists since any element of $D[0,\infty)^p$ or $D_l[0,\infty)^p$ has at most countably many discontinuity points). By definition of the extended Skorohod topology, the restrictions $\{R_{0b}(x_n),\ n = 1,2,\ldots\}$ do S-converge to $R_{0b}(x)$ in $D[0,b]^p$. The space $D^Q[0,b]^p$ being Polish, thus closed in $D[0,b]^p$ (Lemma 2.12), $R_{0b}(x)$ belongs to $D^Q[0,b]^p$, and x is thus continuous at t. Similarly, $R_{0b}^l(x_n)$ is S-convergent to $R_{0b}^l(x)$ in the closed space $D_l^Q[0,b]^p$ and continuity of x at t follows.

Therefore, x belongs to $D^Q[0,\infty)^p$ (resp. $D_l^Q[0,\infty)^p$), and $(D^Q[0,\infty)^p, d_0^e)$ as well as $(D_l^Q[0,\infty)^p, d_0^e)$ are Polish spaces. ■

Lemma 3.6 will be of great help in circumventing the lack of S-continuity of the restrictions, when using projective limits to derive the LDP for the extension of the partial sum processes in $D[0,\infty)^p$ from that in $D[0,T]^p$ for arbitrary $T > 0$.

We complete this section with measurability and S-continuity results for functionals on the spaces $D[0,\infty)^p$ and $D_l[0,\infty)^p$.

Lemma 3.7 *For t_0 in $[a,\infty)$, define the projection mapping $\Pi_{t_0} : D[a,\infty)^p \to \mathbb{R}^p$ by*

$$\Pi_{t_0}(x) = x(t_0), \quad x \in D[a,\infty)^p.$$

Then Π_a is S-continuous, and for t_0 in (a, ∞), Π_{t_0} is Borel-measurable and S-continuous at x if and only if x is continuous at t_0.

A similar result holds true on $D_l[a,\infty)^p$ for the projection $\Pi_{t_0} : D_l[a,\infty)^p \to \mathbb{R}^p$.

Proof: Borel-measurability of Π_t is in fact shown in the proof of Proposition 3.4, see Theorem 6 in [55, p. 127]. As for the S-continuity of Π_t, fix t in $[a, \infty)$ and consider a sequence $\{x_n,\ n = 1, 2, \ldots\}$ in $D[a,\infty)^p$ which is S-convergent to x. If $b > t$ is a continuity point of x, then by definition of the extended Skorohod topology, $R_{ab}(x_n)$ is S-convergent to $R_{ab}(x)$ in $D[a,b]^p$, and because

$$\Pi_t(x) \;=\; \pi_t \circ R_{ab}(x),$$

we conclude that Π_t is continuous at x if and only if π_t is continuous at $R_{ab}(x)$, or by Lemma 2.4, if and only if either t is a continuity point of $R_{ab}(x)$ (thus of x) or $t = a$. ■

Lemma 3.8 *Let $[c,d] \subseteq [a,\infty)$, and define the restriction $R_{cd} : D[a,\infty)^p \to D[c,d]^p$ by*

$$R_{cd}(x)(t) \;=\; x(t), \quad t \in [c,d], \quad x \in D[a,\infty)^p.$$

Then R_{cd} is Borel-measurable, and is S-continuous at x in $D[a,\infty)^p$ if and only if d is a continuity point of x and either $c = a$, or c is a continuity point of x.

The result holds true on $D_l[a,\infty)^p$ for the restriction $R^l_{cd} : D_l[a,\infty)^p \to D_l[c,d]^p$.

Proof: Borel-measurability of R_{cd} is proved as Lemma 2.3 in [66].

To show S-continuity, let x in $D[0,\infty)^p$ and assume c, d to be continuity points of x (the case $c = a$ is trivial). Let $\{x_n,\ n = 1, 2, \ldots\}$ be a sequence in $D[0,\infty)^p$ which is S-converging to x. Then by definition of the extended S-convergence, $R_{cd}(x_n)$ S-converges to $R_{cd}(x)$, so that R_{cd} is S-continuous at x.

Conversely, suppose for instance that d is not a continuity point of x, and yet R_{cd} is S-continuous at x. Then, as we have the relation

$$\Pi_d \;=\; \pi_d^{cd} \circ R_{cd},$$

Lemma 2.4 yields S-continuity of Π_d at x, a contradiction with Lemma 3.7, and R_{cd} is therefore not S-continuous at x.

The results for R^l_{cd} can be shown by transposing the proofs to the $D_l[0,\infty)^p$ setting, or by using the isometry Φ. ■

Lemma 3.9 *The addition mapping $S : D[a,\infty)^p \times D[a,\infty)^p \to D[a,\infty)^p$ (resp. $S^l : D_l[a,\infty)^p \times D_l[a,\infty)^p \to D_l[a,\infty)^p$) is Borel-measurable, and S-continuous at those points (x,y) which do not have common discontinuity points.*

Proof: A proof for the addition on $D[a,\infty)^p \times D[a,\infty)^p$ can be found in [66, Theorem 4.1].

The same result is easily checked to hold on $D_l[a,\infty)^p \times D_l[a,\infty)^p$, once it is seen that

$$S^l(x,y) \;=\; \Phi^{-1}\left(S\left(\Phi(x),\Phi(y)\right)\right), \quad x,y \in D_l[a,b]^p$$

and upon noting that x and y have no common discontinuity points if and only if $\Phi(x)$ and $\Phi(y)$ have no common discontinuity points. ■

In what follows, the supremum is to be understood component-wise.

Lemma 3.10 *The functional $M : D[0,\infty)^p \to D[0,\infty)^p$ (resp. $D_l[0,\infty)^p \to D_l[0,\infty)^p$) defined by $M(x)(t) = \sup_{0\le s\le t} x(s)$ for all t in $[0,\infty)$ is d_0^e-Lipschitz, hence S-continuous in $D[0,\infty)^p$ (resp. $D_l[0,\infty)^p$).*

Proof: The result on $D[0,\infty)$ is shown as Theorem 6 in [66], and is easily seen to hold on $D_l[0,\infty)$. ■

3.1.3 Projective limits

In this section we review the notion of projective limit (or inverse limit) and show that the space $D^{\mathbb{Q}}[0,\infty)^p$ is homeomorphic to the projective limit of some projective system.

We begin by reviewing inverse systems of sets [11, Section 7 p. 191]. Let $(I,\le)$ be a partially ordered, right-filtering set, and let $(\mathcal{X}_\alpha,\ \alpha\in I)$ be a family of sets indexed by I. For each α,β in I such that $\alpha\le\beta$, let $f_{\alpha\beta} : \mathcal{X}_\beta \to \mathcal{X}_\alpha$ be a mapping such that:

- $f_{\alpha\gamma} = f_{\alpha\beta}\circ f_{\beta\gamma}$, whenever $\alpha\le\beta\le\gamma$,

- $f_{\alpha\alpha}$ is the identity mapping on $\mathcal{X}_\alpha$ for all α in I.

The collection $(\mathcal{X}_\alpha, f_{\alpha\beta})$ is called an **inverse system of sets** relative to the index set I.

Let $\mathcal{X}$ denote the subset of the Cartesian product $\prod_{\alpha\in I}\mathcal{X}_\alpha$ consisting of all the elements x satisfying

$$\mathrm{pr}_\alpha(x) = f_{\alpha\beta}\left(\mathrm{pr}_\beta(x)\right), \quad \alpha \leq \beta,$$

where for each α in I, $\mathrm{pr}_\alpha : \prod_{\alpha\in I}\mathcal{X}_\alpha \to \mathcal{X}_\alpha$ denote the coordinate projection mapping. The set $\mathcal{X}$ is called the **inverse limit of the family** $(\mathcal{X}_\alpha)$ with respect to the family of mappings $(f_{\alpha\beta})$, in short the **inverse limit of** $(\mathcal{X}_\alpha, f_{\alpha\beta})$, and is denoted by

$$\mathcal{X} = \varprojlim \mathcal{X}_\alpha.$$

The restriction $f_\alpha : \mathcal{X} \to \mathcal{X}_\alpha$ of the projection pr_α to $\mathcal{X}$ is called the canonical mapping, and is easily seen to satisfy the relation

$$f_\alpha = f_{\alpha\beta} \circ f_\beta, \quad \alpha \leq \beta.$$

We point out that these definitions do not require any topological structures on the sets $\mathcal{X}_\alpha$.

In case each set $\mathcal{X}_\alpha$ is a topological space, it is possible to consider the inverse limit $\mathcal{X}$ as a topological space by endowing it with the coarsest topology on $\mathcal{X}$ for which the mappings $(f_\alpha,\ \alpha \in I)$ are continuous, i.e., with the initial topological structure on $\mathcal{X}$ [12, Prop. 4 p. 30]. This topology is called the **inverse limit of the topologies of the spaces** $(\mathcal{X}_\alpha)$ and is easily seen to coincide with the product topology induced on $\mathcal{X}$. However, the topology on $\mathcal{X}$ inherits much nicer properties if in addition the mapping $f_{\alpha\beta} : \mathcal{X}_\beta \to \mathcal{X}_\alpha$ is continuous for all pair (α, β) such that $\alpha \leq \beta$.

When the sets $(\mathcal{X}_\alpha)$ are topological spaces and the mappings $(f_{\alpha\beta})$ are continuous for all pair (α, β) such that $\alpha \leq \beta$, the inverse system of sets $(\mathcal{X}_\alpha, f_{\alpha\beta})$ is called an **inverse system of topological spaces** [12, Section 4 p. 48] or a **projective system** [24, p. 144]. The space $\mathcal{X}$ endowed with the inverse limit of the topologies of the spaces $(\mathcal{X}_\alpha)$ is called the **inverse limit of the inverse system of topological spaces** $(\mathcal{X}_\alpha, f_{\alpha\beta})$, or **the projective limit of the projective system** $(\mathcal{X}_\alpha, f_{\alpha\beta})$. When the spaces $(\mathcal{X}_\alpha)$ are Hausdorff, the projective limit is Hausdorff and is a closed subspace of the product space $\prod_{\alpha\in I}\mathcal{X}_\alpha$ [12, Corollary 2 p. 78].

We illustrate the use of projective limit by showing that the space $D^Q[0,\infty)^p$ endowed with the extended Skorohod topology is homeomorphic to the projective limit of the spaces $D^Q[0,T]$, a result that will be used in the next section.

Let ε be a fixed irrational in $(0,1)$ and for each $K = 1, 2, \ldots$, let $K_\varepsilon \equiv K + \varepsilon$ so that K_ε is itself irrational. For each $K \leq L$, we use the generic notation $f_{K_\varepsilon L_\varepsilon}$ to denote the restriction mapping from $D^Q[0, L_\varepsilon]^p$ into $D^Q[0, K_\varepsilon]^p$ as well as from $D_l^Q[0, L_\varepsilon]^p$ into $D_l^Q[0, K_\varepsilon]^p$.

Lemma 3.11 *The spaces $D^Q[0,\infty)^p$ and $D_l^Q[0,\infty)^p$ endowed with the extended Skorohod topology are homeomorphic to the projective limits of the projective systems $(D^Q[0, K_\varepsilon]^p, f_{K_\varepsilon L_\varepsilon})$ and $(D_l^Q[0, K_\varepsilon]^p, f_{KL}^\varepsilon)$, respectively, where for each $K = 1, 2, \ldots$, $D^Q[0, K_\varepsilon]^p$ and $D_l^Q[0, K_\varepsilon]^p$ are endowed with the Skorohod topologies.*

Proof: For $K \leq L$, we note that K_ε being irrational, it is a continuity point for any x in $D^Q[0, L_\varepsilon]^p$ and $D_l^Q[0, L_\varepsilon]^p$, so that by Lemma 2.7 the restriction mapping $f_{K_\varepsilon L_\varepsilon}$ is S-continuous. Therefore, because $f_{K_\varepsilon K_\varepsilon}$ is the identity mapping on $D[0, K_\varepsilon]^p$ and

$$f_{K_\varepsilon M_\varepsilon} = f_{K_\varepsilon L_\varepsilon} \circ f_{L_\varepsilon M_\varepsilon}, \quad K \leq L \leq M,$$

$(D^Q[0, K_\varepsilon]^p, f_{K_\varepsilon L_\varepsilon})$ and $(D_l^Q[0, K_\varepsilon]^p, f_{K_\varepsilon L_\varepsilon})$ are indeed projective systems.

We now restrict the proof to the space $D^Q[0,\infty)^p$; it is easily checked that the exact same arguments hold on the space $D_l^Q[0,\infty)^p$ as well. In order to lighten the notations, for each $K = 1, 2, \ldots$, we denote by $\mathcal{X}_{K_\varepsilon}$ the space $D^Q[0, K_\varepsilon]^p$. Let $\mathcal{X} \equiv \varprojlim \mathcal{X}_{K_\varepsilon}$ be the projective limit of the projective system $(\mathcal{X}_{K_\varepsilon}, f_{K_\varepsilon L_\varepsilon})$, and for each $K = 1, 2, \ldots$, let $f_{K_\varepsilon} : \mathcal{X} \to \mathcal{X}_{K_\varepsilon}$ denote the canonical mapping.

We define the mapping $\Phi : \mathcal{X} \to D^Q[0,\infty)^p$ by setting

$$\Phi\left((x_{1+\varepsilon}, x_{2+\varepsilon}, \ldots)\right) = x, \quad (x_{1+\varepsilon}, x_{2+\varepsilon}, \ldots) \in \mathcal{X},$$

where the mapping $x : [0,\infty) \to \mathbb{R}^p$ is defined by

$$x(t) = x_{K_\varepsilon}(t), \quad t \in [0, K_\varepsilon], \quad K = 1, 2, \ldots . \tag{3.7}$$

From the equality $x_{K_\varepsilon} = f_{K_\varepsilon L_\varepsilon}(x_{L_\varepsilon})$, or equivalently

$$x_{K_\varepsilon}(t) = x_{L_\varepsilon}(t), \quad t \in [0, K_\varepsilon], \quad K \leq L,$$

we immediately see that the relation (3.7) uniquely defines a mapping, and that the mapping belongs to $D^Q[0,\infty)^p$.

We now argue that Φ is one-to-one and onto: For each $K = 1, 2, \ldots$, let $R_{K_\varepsilon} : D^Q[0,\infty)^p \to \mathcal{X}_{K_\varepsilon}$ denote the restriction mapping to $[0, K_\varepsilon]$. Then, for each x in $D^Q[0,\infty)^p$ and all $K = 1, 2, \ldots$, $R_{K_\varepsilon}(x)$ belongs to $\mathcal{X}_{K_\varepsilon}$. Moreover, from the relation $R_{K_\varepsilon} = f_{K_\varepsilon L_\varepsilon} \circ R_{L_\varepsilon}$ valid for all $K \leq L$, we conclude that the element $(R_{1+\varepsilon}(x_1), R_{2+\varepsilon}(x_2), \ldots)$ belongs to $\mathcal{X}$. Finally, because we trivially have

$$x(t) \;=\; R_{K_\varepsilon}(x)(t), \quad t \in [0, K_\varepsilon], \quad K = 1, 2, \ldots,$$

we see that in fact

$$\Phi\left((R_{1+\varepsilon}(x), R_{2+\varepsilon}(x), \ldots)\right) \;=\; x, \quad x \in D^Q[0,\infty)^p$$

and surjectivity of Φ follows. The mapping Φ is also injective, for if

$$\Phi\left((x_{1+\varepsilon}, x_{2+\varepsilon}, \ldots)\right) \;=\; \Phi\left((y_{1+\varepsilon}, y_{2+\varepsilon}, \ldots)\right)$$

for $(x_{1+\varepsilon}, x_{2+\varepsilon}, \ldots)$ and $(y_{1+\varepsilon}, y_{2+\varepsilon}, \ldots)$ in $\mathcal{X}$, then from (3.7) we obtain that

$$x_{K_\varepsilon}(t) \;=\; y_{K_\varepsilon}(t), \quad t \in [0, K_\varepsilon], \quad K = 1, 2, \ldots,$$

whence

$$(x_{1+\varepsilon}, x_{2+\varepsilon}, \ldots) \;=\; (y_{1+\varepsilon}, y_{2+\varepsilon}, \ldots).$$

In order to check continuity of Φ, consider a sequence $\{x^n,\ n = 1, 2, \ldots\}$ in $\mathcal{X}$ with $x^n = (x^n_{1+\varepsilon}, x^n_{2+\varepsilon}, \ldots)$ for each $n = 1, 2, \ldots$, which converges (in the product topology) to $x \equiv (x_{1+\varepsilon}, x_{2+\varepsilon}, \ldots)$. For each $n = 1, 2, \ldots$, define y_n in $D[0,\infty)^p$ by setting $y_n \equiv \Phi(x^n)$. We note that the projective limit being closed in the product topology, x does belong to $\mathcal{X}$. By continuity of the canonical mapping f_{K_ε}, for each $K = 1, 2, \ldots$, the sequence $\{f_{K_\varepsilon}(x^n),\ n = 1, 2, \ldots\}$ is S-convergent to $f_{K_\varepsilon}(x)$ in $D^Q[0, K_\varepsilon]^p$.

Let $y \equiv \Phi(x)$ and let a, b be continuity points of y or $a = 0$. Then, there exists an integer K_b such that $[a, b] \subseteq [0, K_b + \varepsilon]$, and the restriction $r_{ab} : D^Q[0, K_b + \varepsilon]^p \to D^Q[a, b]^p$ is then S-continuous at y (Lemma 2.7). Thus, upon noting that

$$r_{ab}(y_n) \;=\; r_{ab}\left(\Phi(x^n)\right) \;=\; r_{ab}\left(f_{K_b+\varepsilon}(x^n)\right), \quad n = 1, 2, \ldots$$

and

$$r_{ab}(y) \;=\; r_{ab}\left(\Phi(x)\right) \;=\; r_{ab}\left(f_{K_b+\varepsilon}(x)\right),$$

from the convergence of $f_{K_b+\varepsilon}(x^n)$ in $D^Q[0, K_b+\varepsilon]^p$, we get convergence of $r_{ab}(y_n)$ to $r_{ab}(y)$ in $D^Q[a, b]^p$, for any continuity points a, b of y or $a = 0$. Therefore y_n converges to y in the extended Skorohod topology on $D^Q[0,\infty)^p$, i.e., $\Phi(x^n)$ is S-convergent to $\Phi(x)$ and continuity of Φ follows.

Finally Φ^{-1} is also continuous: Let $\{y_n,\ n = 1,2,\ldots\}$ be a sequence in $D^Q[0,\infty)^p$ which converges to y. Then, because K_ε is irrational and y belongs to $D^Q[0,\infty)$ (which is closed by Lemma 3.6), S-continuity at y of the restriction mapping R_{K_ε} (Lemma 3.8) yields S-convergence of the sequence $\{R_{K_\varepsilon}(y_n),\ n = 1,2,\ldots\}$ to $R_{K_\varepsilon}(y)$ for each $K = 1,2,\ldots$. Therefore, by [12, Proposition 10 p. 74], the sequence $\{(R_{1+\varepsilon}(y_n), R_{2+\varepsilon}(y_n),\ldots)\ n = 1,2,\ldots\}$ will S-converge to $(R_{1+\varepsilon}(y), R_{2+\varepsilon}(y),\ldots)$ in the product topology, i.e., $\Phi^{-1}(y_n)$ S-converges to $\Phi^{-1}(y)$ in $\mathcal{X}$. ■

A careful review of this last proof shows that continuity of the mapping $f_{\alpha\beta}$ is indeed required: it is hidden in the fact that the projective limit of Hausdorff spaces is closed.

Lemma 3.11 can easily be generalized as follows. The arguments are similar to those of Lemma 3.11 and are therefore omitted. Let ε be a fixed irrational in $(0,1)$, and for $K \le L$, let the restriction $h_{K_\varepsilon L_\varepsilon} : D^Q[0,L_\varepsilon]^p \times D_l^Q[0,L_\varepsilon]^p \to D^Q[0,K_\varepsilon]^p \times D_l^Q[0,K_\varepsilon]^p$ be defined by

$$h_{K_\varepsilon L_\varepsilon}(x,y)(t) = (x(t),y(t)), \quad t \in [0,K_\varepsilon], \quad x \in D^Q[0,L_\varepsilon]^p, \quad y \in D_l^Q[0,L_\varepsilon]^p.$$

Lemma 3.12 *The space $D^Q[0,\infty)^p \times D_l^Q[0,\infty)^p$ endowed with the product of the extended Skorohod topologies is homeomorphic to the projective limit of the projective system $(D^Q[0,K_\varepsilon]^p \times D_l^Q[0,K_\varepsilon]^p\,,\ h_{K_\varepsilon L_\varepsilon})$, where for each $K = 1,2,\ldots$, $D^Q[0,K_\varepsilon]^p$ and $D_l^Q[0,K_\varepsilon]^p$ are endowed with the Skorohod topologies.*

We point out that because the restriction mappings are not S-continuous on $D[0,T]^p$, we **cannot** use the projective limit approach to construct the whole space $D[0,\infty)^p$ from the inverse system of topological spaces $(D[0,K]^p, r_{KL})$, where for $K \le L$, $r_{KL} : D[0,L]^p \to D[0,K]^p$ is the restriction mapping to $[0,K]$.

3.2 LDP for partial sum processes in $D[0,\infty)$ and $D_l[0,\infty)$

In this section we present one of the main results of this dissertation, and perhaps the most interesting one, namely that for partial sum processes, the LDP in $D[0,1]^p$ implies the LDP in the full space $D[0,\infty)^p$.

Let $\{x_n,\ n = 0,\pm1,\pm2,\ldots\}$ be a $\mathbb{R}^p$-valued bi-infinite stationary random sequence on the probability space $(\Omega,\mathcal{F},\mathbf{P})$. For each $n = 1,2,\ldots$, let $X_n^\infty(\cdot)$

and $X_n^{\infty,-}(\cdot)$ denote respectively the extensions of $X_n^T(\cdot)$ and $X_n^{T,-}(\cdot)$ on $[0,\infty)$, i.e.,

$$X_n^{\infty}(t) = \begin{cases} \sum_{i=1}^{\lfloor nt \rfloor} x_i & \text{if } \lfloor nt \rfloor \geq 1 \\ 0 & \text{otherwise} \end{cases}, \quad t \in [0,\infty), \quad n = 1,2,\ldots$$

and

$$X_n^{\infty,-}(t) = \begin{cases} \sum_{i=1-\lceil nt \rceil}^{0} x_i & \text{if } 1 \leq \lceil nt \rceil \\ 0 & \text{otherwise} \end{cases}, \quad t \in [0,\infty), \quad n = 1,2,\ldots .$$

As shown below, for each $n = 1,2,\ldots$, $X_n^{\infty}(\cdot)$ and $X_n^{\infty,-}(\cdot)$ are random elements in $(D[0,\infty)^p, \tau_0)$ and $(D_l[0,\infty)^p, \tau_0)$, respectively.

Lemma 3.13 *For each $n = 1,2,\ldots$, the mappings $X_n^{\infty}(\cdot) : \Omega \to D[0,\infty)^p$ and $X_n^{\infty,-}(\cdot) : \Omega \to D_l[0,\infty)^p$ are $\mathcal{F}/\mathcal{B}_{\tau_0^e}$ and $\mathcal{F}/\mathcal{B}^l_{\tau_0^e}$-measurable, respectively, where $\mathcal{B}_{\tau_0^e}$ and $\mathcal{B}^l_{\tau_0^e}$ are respectively the Borel σ-fields of $D[0,\infty)^p$ and $D_l[0,\infty)^p$ with respect to the extended Skorohod topologies.*

Proof: We prove only the measurability of the mapping $X_n^{\infty}(\cdot)$, as that of $X_n^{\infty,-}(\cdot)$ would be proved similarly.

Fix $n = 1,2,\ldots$. For all t in $[0,\infty)$, the mapping $\Pi_t \circ X_n^{\infty}(\cdot)$ is measurable, as the sum of finitely many random variables, so that

$$X_n^{\infty}(\cdot)^{-1}\left(\Pi_t^{-1}(A)\right) \in \mathcal{F}, \quad t \in [a,b], \quad A \in \mathcal{B}_{\mathrm{R}}.$$

Moreover, as the collection $\{B \subset D[0,\infty)^p : X_n^{\infty}(\cdot)^{-1}(B) \in \mathcal{F}\}$ is a σ-field on $D[0,\infty)^p$ containing the set $\Pi_t(A)$ for all t in $[a,b]$ and all A in $\mathcal{B}_{\mathrm{R}}$, it must contain the projection σ-field $\mathcal{P}_{\infty}$, i.e.,

$$\mathcal{P}_{\infty} \subset \left\{B : X_n^{\infty}(\cdot)^{-1}(B) \in \mathcal{F}\right\}$$

and measurability of $X_n^{\infty}(\cdot)$ becomes a trivial consequence of Proposition 3.4. ■

In fact, as both $X_n^{\infty}(\cdot)$ and $X_n^{\infty,-}(\cdot)$ are continuous at each irrational t in $[0,\infty)$, following Lemma 2.14, it is easily seen that $X_n^{\infty}(\cdot)$ and $X_n^{\infty,-}(\cdot)$ are random elements in $D^Q[0,\infty)^p$ and $D_l^Q[0,\infty)^p$ respectively.

Furthermore, because

$$\mathbf{P}\left[X_n^{\infty}(\cdot) \in D^Q[0,\infty)^p\right] = \mathbf{P}\left[X_n^{\infty,-}(\cdot) \in D_l^Q[0,\infty)^p\right] = 1, \quad n = 1,2,\ldots,$$

Lemmas 1.11 and 3.6, together with the fact that the product of closed sets is closed [12, Proposition 7 p. 47], yield the following proposition.

Proposition 3.14 **1.** *The family* $\{X_n^\infty(\cdot),\ n=1,2,\ldots\}$ *satisfies the LDP in* $(D^Q[0,\infty)^p, \tau_0^e)$ *with rate function* $I^Q : D^Q[0,\infty)^p \to [0,\infty]$ *if and only if it satisfies the LDP in* $(D[0,\infty)^p, \tau_0^e)$ *with rate function* $I : D[0,\infty)^p \to [0,\infty]$. *Moreover, the rate functions are related through the relation*

$$I(\varphi) = \begin{cases} I^Q(\varphi) & \text{if } \ \varphi \in D^Q[0,\infty)^p \\ \infty & \text{otherwise,} \end{cases}$$

and the rate function I *is good if and only if the rate function* I^Q *is good.*

2. *The family of partial sum processes* $\{(X_n^\infty(\cdot), X_n^{\infty,-}(\cdot)),\ n=1,2,\ldots\}$ *satisfies the LDP in* $(D^Q[0,\infty)^p, \tau_0^e) \times (D_l^Q[0,\infty)^p, \tau_0^e)$ *with rate function* I^Q *if and only if it satisfies the LDP in* $(D[0,\infty)^p, \tau_0^e) \times (D_l[0,\infty)^p, \tau_0^e)$ *with rate function* I. *Furthermore, the rate function* I *is good if and only if the rate function* I^Q *is good, and we have*

$$I(\varphi_1, \varphi_2) = \begin{cases} I^Q(\varphi_1, \varphi_2) & \text{if } \ \varphi \in D^Q[0,\infty)^p, \quad \varphi_2 \in D_l^Q[0,\infty)^p \\ \infty & \text{otherwise.} \end{cases}$$

As we have in mind to apply the projective limit approach to establish the LDP in $D[0,\infty)^p$, this last result will be of a great help, since by Lemma 3.11, the space $D^Q[0,\infty)^p$ is homeomorphic to the projective limit of the spaces $\left(D^Q[0,K_\varepsilon]^p\right)$.

Indeed, the LDP in a projective limit can be deduced from that on the projective system, as illustrated by the next result, due to Dawson and Gärtner.

Let $(\mathcal{X}_\alpha, f_{\alpha\beta})$ be an inverse system of topological spaces with inverse limit $\mathcal{X}$, and denote by f_α its canonical mapping. Finally, let $\mathcal{B}_\alpha$ be the Borel σ-field of the topological space $\mathcal{X}_\alpha$, and let $\{\mu_n,\ n=1,2,\ldots\}$ be a family of probability measures defined on some σ-field $\mathcal{B}$ of $\mathcal{X}$.

Theorem 3.15 (Dawson-Gärtner) *Let* $\{\mu_n,\ n = 1,2,\ldots\}$ *be a family of probability measures on the projective limit* $\mathcal{X}$ *such that for any* $\alpha \in I$, *the Borel probability measures* $\{\mu_n \circ f_\alpha^{-1},\ n=1,2,\ldots\}$ *satisfy the LDP in* $\mathcal{X}_\alpha$ *with good rate function* $I_\alpha : \mathcal{X}_\alpha \to [0,\infty]$. *Then* $\{\mu_n,\ n=1,2,\ldots\}$ *satisfy the LDP in* $\mathcal{X}$ *with good rate function* $I_\mathcal{X} : \mathcal{X} \to [0,\infty]$ *given by*

$$I_\mathcal{X}(x) = \sup_{\alpha \in I} I_\alpha(f_\alpha(x)), \quad x \in \mathcal{X}.$$

A careful review of the proof of the Dawson-Gärtner Theorem [24, p. 144] shows that the following assumptions are required:

i) For all α in I, $\mathcal{B}$ contains $f_\alpha^{-1}(\mathcal{B}_\alpha)$,

ii) For all α in I and each $n = 1, 2, \ldots$, $\mu_n \circ f_\alpha^{-1}$ is a measure on $\mathcal{B}_\alpha$.

Using the Dawson-Gärtner Theorem, we now show that the LDP for the family $\{X_n^\infty(\cdot),\ n = 1, 2, \ldots\}$ in $D[0,\infty)^p$ is a consequence of the LDP for the family $\{X_n(\cdot),\ n = 1, 2, \ldots\}$ in $D[0,1]^p$.

3.2.1 LDP for $X_n^\infty(\cdot)$ in the space $D[0,\infty)$

We start with the most general case: we only assume that the family $\{X_n(\cdot),\ n = 1, 2, \ldots\}$ satisfies the LDP in $(D[0,1]^p, \tau_0)$ with good rate function I_X. The assumption that the rate function is good is required as the Contraction Principle may not hold otherwise. We derive the LDP for $\{X_n^\infty(\cdot),\ n = 1, 2, \ldots\}$ in $(D[0,\infty)^p, \tau_0^e)$ and obtain an expression for the associated rate function, for which a closed-form expression can be found in most situations, as illustrated by Corollary 3.17.

As we have already noticed, because the restriction r_{TU} are not $\mathcal{S}$-continuous, we cannot use the projective limit approach directly to the spaces $(D[0,T]^p)$. Instead, we use a trick which consists in considering the good spaces on which the restriction mappings are $\mathcal{S}$-continuous. In view of Propositions 2.16 and 3.14, the spaces $(D^{\mathbb{Q}}[0,T]^p)$ are good candidates for the job since the LDP for partial sum processes on those spaces is equivalent to that in $D[0,T]^p$ for all $T > 0$. Furthermore, as we have seen in Lemma 3.11, by adequately choosing the index set $\mathcal{T}$, we can turn the inverse system of sets $(D^{\mathbb{Q}}[0,T]^p, r_{TU})_{T\in\mathcal{T}}$ into a projective system.

This trick is not new: It is an adaptation of one used by Deuschel and Stroock [25, p. 178] in the context of uniform large deviations for continuous-time Markov processes. In their setup, the processes are continuous at each t in $\mathbb{Z}$, and they consider the projective system $(D[-K,0], r_{KL})$.

For each $U > 0$ and each mapping $\varphi : \Gamma \to \mathbb{R}^p$, where Γ is a subset of $[0,\infty)$ containing $[0,U]$, we define the mapping $\varphi_U : [0,1] \to \mathbb{R}^p$ by

$$\varphi_U(t) = \frac{1}{U}\varphi(Ut), \quad t \in [0,1]. \tag{3.8}$$

Theorem 3.16 *Assume the family $\{X_n(\cdot),\ n = 1, 2, \ldots\}$ to satisfy the LDP in $(D[0,1]^p, \tau_0)$ with good rate function $I_X : D[0,1]^p \to [0,\infty]$.*

Then the family $\{X_n^\infty(\cdot),\ n = 1, 2, \ldots\}$ satisfies the LDP in $(D[0,\infty)^p, \tau_0^e)$ with good rate function $I_X^\infty : D[0,\infty)^p \to [0,\infty]$ given by

$$I_X^\infty(\varphi) = \sup\left\{ I_X^T(R_T(\varphi)) : T \in (0,\infty) \cap \mathbb{Q}^c \right\}$$

$$= \lim_{\substack{T\to\infty \\ T\in\mathbf{Q}^c}} I_X^T(R_T(\varphi)), \quad \varphi \in D[0,\infty)^p \tag{3.9}$$

where $R_T : D[0,\infty)^p \to D[0,T]^p$ *is the restriction to* $[0,T]$ *and* I_X^T *is given by*

$$I_X^T(R_T(\varphi)) = \inf_{\psi\in D[0,\lceil T\rceil]^p} \left\{ \lceil T\rceil\, I_X(\psi_{\lceil T\rceil}) : \varphi = \psi \text{ on } [0,T] \right\}, \quad \varphi \in D[0,\infty)^p, \tag{3.10}$$

with $\psi_{\lceil T\rceil}$ *as given by (3.8) with* $U = \lceil T\rceil$.

Proof: Let ε be a fixed irrational number in $(0,1)$, and for each $K = 1,2,\ldots$, let K_ε denote $K+\varepsilon$. Consider, as in Section 3.1.3 (Lemma 3.11), the projective system $(\mathcal{X}_{K_\varepsilon}, f_{K_\varepsilon L_\varepsilon})$, where for each $K = 1,2,\ldots$, $\mathcal{X}_{K_\varepsilon}$ is the topological space $(D^Q[0,K+\varepsilon]^p, \tau_0)$ and for each $K \le L$, $f_{K_\varepsilon L_\varepsilon} : \mathcal{X}_{L_\varepsilon} \to \mathcal{X}_{K_\varepsilon}$ is the restriction mapping. Let $\mathcal{X}$ denote the projective limit, and recall that for each $K = 1,2,\ldots$, $f_{K_\varepsilon} : \mathcal{X} \to \mathcal{X}_{K_\varepsilon}$ denotes the canonical mapping.

Define the mapping $Y_n(\cdot) : \Omega \to \mathcal{X}$ by setting

$$Y_n(\cdot) \equiv (X_n^{1+\varepsilon}(\cdot), X_n^{2+\varepsilon}(\cdot), \ldots), \quad n = 1,2,\ldots .$$

By Lemma 2.12, for each $K = 1,2,\ldots$, the space $\mathcal{X}_{K_\varepsilon}$ is separable. Thus for each $n = 1,2,\ldots$, $Y_n(\cdot)$ is a random element in the product space $\prod \mathcal{X}_{K_\varepsilon}$. Since $Y_n(\cdot)$ takes on values in $\mathcal{X}$, the mapping $Y_n(\cdot)$ is in fact $\mathcal{F}/\mathcal{B}_{\mathcal{X}}$-measurable, where $\mathcal{B}_{\mathcal{X}}$ is the Borel σ-field of $\mathcal{X}$, and it follows that $Y_n(\cdot)$ is a random element in $\mathcal{X}$. For each $n = 1,2,\ldots$, the distribution law of $Y_n(\cdot)$ is a measure on $\mathcal{B}_{\mathcal{X}}$, and by continuity of f_{K_ε} we have

$$(f_{K_\varepsilon})^{-1}\left(\mathcal{B}_{\mathcal{X}_{K_\varepsilon}}\right) \subset \mathcal{B}_{\mathcal{X}}, \quad K = 1,2,\ldots .$$

Requirement **i)** of the Dawson-Gärtner Theorem is thus satisfied.

Now, because K_ε is irrational, for each $K = 1,2,\ldots$, Corollary 2.22 yields, under the enforced assumptions, the LDP for the family of partial sum processes

$$\left\{X_n^{K_\varepsilon}(\cdot),\ n = 1,2,\ldots\right\} = \left\{f_{K_\varepsilon}(Y_n(\cdot)),\ n = 1,2,\ldots\right\}$$

in $D[0,K_\varepsilon]^p$ with good rate function $I_X^{K_\varepsilon} : D[0,K_\varepsilon]^p \to [0,\infty]$ given by (2.30). By Proposition 2.16 this LDP in turn yields the LDP in $\mathcal{X}_{K_\varepsilon} = D^Q[0,K_\varepsilon]^p$ and we obtain the following expression

$$I_X^{K_\varepsilon}(\varphi) = \inf_{\psi\in D[0,\lceil K_\varepsilon\rceil]^p} \left\{ \lceil K_\varepsilon\rceil I_X(\psi_{\lceil K_\varepsilon\rceil}) : \varphi = \psi \text{ on } [0,K_\varepsilon] \right\}, \quad \varphi \in D^Q[0,K_\varepsilon]^p,$$

for the restriction of the rate function $I_X^{K_\varepsilon}$ to $D^Q[0,K_\varepsilon]^p$ where $\psi_{\lceil K_\varepsilon\rceil} : [0,1] \to \mathbb{R}^p$ is as given by (3.8) with $U = \lceil K_\varepsilon\rceil$.

Because $f_{K_\varepsilon}(Y_n(\cdot)) = X_n^{K_\varepsilon}(\cdot)$ is a random element in $\mathcal{X}_{K_\varepsilon}$, requirement **ii)** of the Dawson-Gärtner Theorem is also satisfied.

Therefore, by the Dawson-Gärtner Theorem, the family $\{Y_n(\cdot),\ n = 1, 2, \ldots\}$ satisfies the LDP in the projective limit $\mathcal{X}$ with good rate function $I : \mathcal{X} \to [0, \infty]$ given by

$$I((x_{1+\varepsilon}, x_{2+\varepsilon}, \ldots)) \equiv \sup_{K=1,2,\ldots} I_X^{K_\varepsilon}(x_{K+\varepsilon}), \quad x_{K+\varepsilon} \in D^Q[0, K_\varepsilon]^p, \quad K = 1, 2, \ldots .$$

[In this last expression, $x_{K+\varepsilon}$ is a generic element of $D^Q[0, K_\varepsilon]$, and is **not** related to the notation (3.8).]

If Φ is the homeomorphism in Lemma 3.11, then it is plain that

$$\begin{aligned} X_n^\infty(\cdot) &= \Phi((R_{1+\varepsilon}(X_n^\infty(\cdot)), R_{2+\varepsilon}(X_n^\infty(\cdot)), \ldots)) \\ &= \Phi(Y_n(\cdot)), \quad n = 1, 2, \ldots \end{aligned} \tag{3.11}$$

where for each $K = 1, 2, \ldots$, $R_{K_\varepsilon} : D[0, \infty)^p \to D[0, K_\varepsilon]^p$ is the restriction mapping to $[0, K_\varepsilon]$. Thus, in view of the LDP obtained for $\{Y_n(\cdot),\ n = 1, 2, \ldots\}$ and S-continuity of Φ, the Contraction Principle yields the LDP for the family $\{X_n^\infty(\cdot),\ n = 1, 2, \ldots\}$ in $D^Q[0, \infty)^p$ with good rate function $I^Q : D^Q[0, \infty)^p \to [0, \infty]$ given by

$$\begin{aligned} I^Q(\varphi) &= I\left(\Phi^{-1}(\varphi)\right) \\ &= I((R_{1+\varepsilon}(\varphi), R_{2+\varepsilon}(\varphi), \ldots)) \\ &= \sup_{K=1,2,\ldots} I_X^{K_\varepsilon}(R_{K_\varepsilon}(\varphi)), \quad \varphi \in D^Q[0, \infty)^p. \end{aligned}$$

Finally Proposition 3.14 enables us to lift up the LDP for $\{X_n^\infty(\cdot),\ n = 1, 2, \ldots\}$ to the space $D[0, \infty)^p$. Because

$$\sup_{K=1,2,\ldots} I_X^{K_\varepsilon}(R_{K_\varepsilon}(\varphi)) = \infty, \quad \varphi \notin D^Q[0, \infty)^p,$$

the associated good rate function $I_X^\infty : D[0, \infty)^p \to [0, \infty]$ is given by

$$I_X^\infty(\varphi) = \sup_{K=1,2,\ldots} I_X^{K_\varepsilon}(R_{K_\varepsilon}(\varphi)), \quad \varphi \in D[0, \infty)^p. \tag{3.12}$$

From the Contraction Principle and uniqueness of the rate function, it not hard to see that for any irrational $T_2 \geq T_1 > 0$, we have

$$I_X^{T_1}(R_{T_1}(\varphi)) \leq I_X^{T_2}(R_{T_2}(\varphi)), \quad \varphi \in D[0, \infty)^p,$$

so that the argument of the supremum in (3.12) is actually an increasing function of K_ε, and we get

$$I_X^\infty(\varphi) = \lim_{\substack{T\to\infty \\ t\in\mathbf{Q}^c}} I_X^T(R_T(\varphi)) = \sup\left\{I_X^T(R_T(\varphi)) : T \in (0,\infty)\cap\mathbf{Q}^c\right\}.$$

It is reassuring to note that, as expected, the final expression for the rate function does not depend on ε. ■

Remark: For any irrational $T > 0$, the expression (3.10) of the rate function I_X^T yields

$$I_X^T(R_T(\varphi)) \le I_X^{\lceil T\rceil}\left(R_{\lceil T\rceil}(\varphi)\right), \quad \varphi \in D[0,\infty)^p$$

so that

$$\begin{aligned} I_X^\infty(\varphi) &\le \sup_{K=1,2,\ldots} I_X^K(R_K(\varphi)) \\ &= \sup_{K=1,2,\ldots} K\, I_X(\varphi_K), \quad \varphi\in D[0,\infty)^p, \end{aligned} \tag{3.13}$$

with φ_K as given by (3.8) with $U = K$.

At this point, we do not have a way to obtain the LDP in $D[0,K]^p$ from that in $D[0,T]^p$ for $T > K$, and therefore we cannot get equality in (3.13). Similarly, we cannot say whether the quantity $K\, I_X(\varphi_K)$ is increasing in K, which prevents us from replacing the supremum in (3.13) by the limit as $K \to \infty$.

However, when the rate function I_X is of the usual integral form, the equality holds, as demonstrated by the following Corollary.

Corollary 3.17 *If I_X is of the form*

$$I_X(\varphi) = \begin{cases} \int_0^1 r(\dot\varphi(t))\,dt & \text{if } \varphi\in AC_0[0,1]^p \\ \infty & \text{otherwise} \end{cases}, \tag{3.14}$$

for some rate function $r : \mathbb{R}^p \to [0,\infty]$, then we have

$$I_X^\infty(\varphi) = \begin{cases} \int_0^\infty r(\dot\varphi(t))\,dt & \text{if } \varphi\in AC_0[0,\infty)^p \\ \infty & \text{otherwise} \end{cases} \tag{3.15}$$

with

$$\int_0^\infty r(\dot\varphi(t))\,dt = \sup_{K=1,2,\ldots}\int_0^K r(\dot\varphi(t))\,dt = \sup_{T>0}\int_0^T r(\dot\varphi(t))\,dt. \tag{3.16}$$

Proof: The expression for I_X^∞ is a direct consequence of Theorem 3.16 and Corollary 2.23 upon noting from the non-negativity of r that

$$\sup_{T>0}\left\{\int_0^T r(\dot\varphi(t))\,dt : T\in\mathbf{Q}^c\right\} = \sup_{K=1,2,\ldots}\int_0^K r(\dot\varphi(t))\,dt = \int_0^\infty r(\dot\varphi(t))\,dt\,.$$

■

The same technique can be applied to obtain the LDP jointly for the family $(X_n^\infty(\cdot), X_n^{\infty,-}(\cdot)),\ n=1,2,\ldots\}$ in the space $D[0,\infty)^p\times D_l[0,\infty)^p$.

3.2.2 LDP for $(X_n^\infty(\cdot), X_n^{\infty,-}(\cdot))$ in the space $D[0,\infty)^p\times D_l[0,\infty)^p$

In this section, we present our most general result on the LDP for partial sum processes on infinite time intervals. Its generality enables us to establish LDPs for many functionals of partial sum processes.

The proof is very similar to that of Theorem 3.16, but for the sake of completeness, we present it here. For each $T>0$, let $\tilde R_T : D[0,\infty)^p\times D_l[0,\infty)^p\to D[0,T]^p\times D_l[0,T]^p$ be the restriction mapping to $[0,T]$.

Theorem 3.18 *Assume the family $\{X_n(\cdot),\ n=1,2,\ldots\}$ to satisfy the LDP in $(D[0,1]^p, d_0)$ with good rate function $I_X : D[0,1]^p\to[0,\infty]$. Assume further that every element of the effective domain of I_X is continuous at $t=\frac12$.*

Then the family $\{(X_n(\cdot), X_n^{\infty,-}(\cdot)),\ n=1,2,\ldots\}$ satisfies the LDP in the product space $(D[0,\infty)^p,\tau_0^e)\times(D_l[0,\infty)^p,\tau_0^e)$ with good rate function $I_{X,X^-}^\infty : D[0,\infty)^p\times D_l[0,\infty)^p\to[0,\infty]$ given by

$$\begin{aligned} I_{X,X^-}^\infty(\varphi_1,\varphi_2) &= \sup\left\{I_{X,X^-}^T\left(\tilde R_T(\varphi_1,\varphi_2)\right) : T\in(0,\infty)\cap\mathbf{Q}^c\right\} \\ &= \lim_{\substack{T\to\infty\\ T\in\mathbf{Q}^c}} I_{X,X^-}^T\left(\tilde R_T(\varphi_1,\varphi_2)\right) \end{aligned}\tag{3.17}$$

where for each irrational $T>0$, I_{X,X^-}^T is given by

$$I_{X,X^-}^T(\varphi_1,\varphi_2) = \inf_{\substack{\psi_1\in D[0,\lceil T\rceil]^p\\ \psi_2\in D_l[0,\lceil T\rceil]^p}}\left\{I_{X,X^-}^{\lceil T\rceil}(\psi_1,\psi_2) : \begin{array}{c}\varphi_1=\psi_1\\ \varphi_2=\psi_2\end{array}\ \text{on } [0,T]\right\}\tag{3.18}$$

for φ_1 in $D[0,\infty)^p$, φ_2 in $D_l[0,\infty)^p$ and where for each $K=1,2,\ldots$,

$$I_{X,X^-}^K(\psi_1,\psi_2) \equiv \begin{cases} 2K\inf\left\{I_X(\frac{1}{2K}\varphi+c) : c\in\mathbb{R}\right\} & \text{if } \psi_1(0)=\psi_2(0)=0\\ \infty & \text{otherwise}\end{cases},\tag{3.19}$$

for ψ_1 *in* $D[0,K]^p$, ψ_2 *in* $D_l[0,K]^p$ *and* $\psi : [0,1] \to \mathbb{R}$ *given by*

$$\psi(t) = \begin{cases} -\psi_2(K - 2Kt), & t \in [0, \frac{1}{2}] \\ \psi_1(2Kt - K), & t \in [\frac{1}{2}, 1] \end{cases} . \tag{3.20}$$

Proof: Fix an irrational number ε in $(0,1)$, and for each $K = 1, 2, \ldots$, let K_ε denote $K + \varepsilon$. Consider, as in Lemma 3.12, the projective system $(\mathcal{Y}_{K_\varepsilon}, h_{K_\varepsilon L_\varepsilon})$, where for each $K = 1, 2, \ldots$, $\mathcal{Y}_{K_\varepsilon}$ is the topological space $(D^Q[0,K_\varepsilon]^p, \tau_0) \times (D_l^Q[0,K_\varepsilon]^p, \tau_0)$ and for each $K \le L$, $h_{K_\varepsilon L_\varepsilon} : \mathcal{Y}_{L_\varepsilon} \to \mathcal{Y}_{K_\varepsilon}$ is the restriction mapping. Let $\mathcal{Y}$ denote the projective limit of the projective system $(\mathcal{Y}_{K_\varepsilon}, h_{K_\varepsilon L_\varepsilon})$, and for each $K = 1, 2, \ldots$, let $h_{K_\varepsilon} : \mathcal{Y} \to \mathcal{Y}_{K_\varepsilon}$ denote the canonical mapping.

We define the mapping $Z_n(\cdot) : \Omega \to \mathcal{Y}$ by setting

$$Z_n(\cdot) \equiv \left(\left(X_n^{1+\varepsilon}(\cdot), X_n^{1+\varepsilon,-}(\cdot) \right), \left(X_n^{2+\varepsilon}(\cdot), X_n^{2+\varepsilon,-}(\cdot) \right), \ldots \right), \quad n = 1, 2, \ldots .$$

By Lemma 2.12, for each $K = 1, 2, \ldots$, the spaces $D^Q[0,K_\varepsilon]^p$ and $D_l^Q[0,K_\varepsilon]^p$ are separable. Since the closure of a product set is the product of the closures [12, Proposition 7 p. 47], the product set $\mathcal{Y}_{K_\varepsilon}$ is itself separable.

It follows that for each $n = 1, 2, \ldots$, $Z_n(\cdot)$ is a random element in the product space $\prod \mathcal{Y}_{K_\varepsilon}$. Because $Z_n(\cdot)$ takes on values in $\mathcal{Y}$, the mapping $Z_n(\cdot)$ is in fact $\mathcal{F}/\mathcal{B}_\mathcal{Y}$-measurable, where $\mathcal{B}_\mathcal{Y}$ is the Borel σ-field of $\mathcal{Y}$, and $Z_n(\cdot)$ is a random element in $\mathcal{Y}$. For each $n = 1, 2, \ldots$, the distribution law of $Z_n(\cdot)$ is a measure on $\mathcal{B}_\mathcal{Y}$, and by continuity of h_{K_ε}, we have

$$(h_{K_\varepsilon})^{-1} \left(\mathcal{B}_{\mathcal{Y}_{K_\varepsilon}} \right) \subset \mathcal{B}_\mathcal{Y}, \quad K = 1, 2, \ldots .$$

Requirement **i)** of the Dawson-Gärtner Theorem is thus satisfied.

Now, because K_ε is irrational, for each $K = 1, 2, \ldots$, Corollary 2.27 yields, under the enforced assumptions, the LDP for the family of partial sum processes

$$\left\{ \left(X_n^{K_\varepsilon}(\cdot), X_n^{K_\varepsilon,-}(\cdot) \right), \, n = 1, 2, \ldots \right\} = \left\{ h_{K_\varepsilon}(Z_n(\cdot)), \, n = 1, 2, \ldots \right\}$$

in $D[0,K_\varepsilon]^p \times D_l[0,K_\varepsilon]^p$ with good rate function $I_{X,X^-}^{K_\varepsilon} : D[0,K_\varepsilon]^p \times D_l[0,K_\varepsilon]^p \to [0,\infty]$ given by (2.41). By Proposition 2.24 this LDP in turn yields the LDP in $\mathcal{Y}_{K_\varepsilon} = D^Q[0,K_\varepsilon]^p \times D_l^Q[0,K_\varepsilon]^p$, and the restriction of the rate function $I_{X,X^-}^{K_\varepsilon}$ to $\mathcal{Y}_{K_\varepsilon}$ is given by

$$I_{X,X^-}^{K_\varepsilon}(\varphi_1, \varphi_2) = \inf_{\substack{\psi_1 \in D[0,\lceil K_\varepsilon \rceil]^p \\ \psi_2 \in D_l[0,\lceil K_\varepsilon \rceil]^p}} \left\{ I_{X,X^-}^{\lceil K_\varepsilon \rceil}(\psi_1, \psi_2) : \begin{array}{l} \varphi_1 = \psi_1 \\ \varphi_2 = \psi_2 \end{array} \text{ on } [0, K_\varepsilon] \right\},$$

for φ_1 in $D[0,K_\varepsilon]^p$, φ_2 in $D_l[0,K_\varepsilon]^p$. Here, for each $K=1,2,\ldots$, we have

$$I^K_{X,X^-}(\psi_1,\psi_2) \equiv \begin{cases} 2K\inf\left\{I_X(\frac{1}{2K}\psi+c): c\in\mathbb{R}\right\} & \text{if} \quad \psi_1(0)=\psi_2(0)=0 \\ \infty & \text{otherwise} \end{cases},$$

for ψ_1 in $D[0,K]^p$, ψ_2 in $D_l[0,K]^p$ and $\psi:[0,1]\to\mathbb{R}$ given by

$$\psi(t) = \begin{cases} -\psi_2(K-2Kt), & t\in[0,\frac{1}{2}] \\ \psi_1(2Kt-K), & t\in[\frac{1}{2},1] \end{cases}.$$

Because $h_{K_\varepsilon}(Z_n(\cdot)) = \left(X_n^{K_\varepsilon}(\cdot), X_n^{K_\varepsilon,-}(\cdot)\right)$ is a random element in $\mathcal{Y}_{K_\varepsilon}$, requirement **ii)** of the Dawson-Gärtner Theorem is also satisfied.

Therefore, by the Dawson-Gärtner Theorem, the family $\{Z_n(\cdot),\ n=1,2,\ldots\}$ satisfies the LDP in the projective limit $\mathcal{Y}$ with good rate function $I:\mathcal{Y}\to[0,\infty]$ given by

$$I\left((y_{1+\varepsilon},y_{2+\varepsilon},\ldots)\right) = \sup_{K=1,2,\ldots} I_X^{K_\varepsilon}(y_{K+\varepsilon}), \quad y_{K+\varepsilon}\in\mathcal{Y}_{K_\varepsilon}, \quad K=1,2,\ldots .$$

[In the last expression, $y_{K+\varepsilon}$ denotes a generic element of $\mathcal{Y}_{K_\varepsilon}$, and is **not** related to the notation (3.8).]

Next, if $\tilde{\Phi}$ is the homeomorphism in Lemma 3.12, then we have

$$\begin{aligned} \left(X_n^\infty(\cdot), X_n^{\infty,-}(\cdot)\right) &= \tilde{\Phi}\left(\tilde{R}_{1+\varepsilon}\left(X_n^\infty(\cdot),X_n^{\infty,-}(\cdot)\right), \tilde{R}_{2+\varepsilon}\left(X_n^\infty(\cdot),X_n^{\infty,-}(\cdot)\right),\ldots\right) \\ &= \tilde{\Phi}\left(Z_n(\cdot)\right), \quad n=1,2,\ldots . \end{aligned}$$

Thus, in view of the LDP obtained for $\{Z_n(\cdot),\ n=1,2,\ldots\}$ and S-continuity of $\tilde{\Phi}$ (Lemma 3.12), the Contraction Principle yields the LDP for the family $\{(X_n^\infty(\cdot), X_n^{\infty,-}(\cdot)),\ n=1,2,\ldots\}$ in $D^Q[0,\infty)^p\times D_l^Q[0,\infty)^p$ with good rate function $I^Q: D^Q[0,\infty)^p\times D_l^Q[0,\infty)^p\to[0,\infty]$ given by

$$\begin{aligned} I^Q(\varphi_1,\varphi_2) &= I\left(\tilde{\Phi}^{-1}(\varphi_1,\varphi_2)\right) \\ &= I\left(\tilde{R}_{1+\varepsilon}(\varphi_1,\varphi_2), \tilde{R}_{2+\varepsilon}(\varphi_1,\varphi_2),\ldots\right) \\ &= \sup_{K=1,2,\ldots} I^{K_\varepsilon}_{X,X^-}\left(\tilde{R}_{K_\varepsilon}(\varphi_1,\varphi_2)\right), \quad \varphi_1\in D^Q[0,\infty)^p, \quad \varphi_2\in D_l^Q[0,\infty)^p. \end{aligned}$$

Finally, Proposition 3.14 enables us to lift up the LDP to the space $D[0,\infty)^p\times D_l[0,\infty)^p$. We easily find that the associated good rate function $I^\infty_{X,X^-}: D[0,\infty)^p\times D_l[0,\infty)^p\to[0,\infty]$ is given by

$$I^\infty_{X,X^-}(\varphi_1,\varphi_2) = \sup_{K=1,2,\ldots} I^{K_\varepsilon}_{X,X^-}\left(\tilde{R}_{K_\varepsilon}(\varphi_1,\varphi_2)\right), \quad \varphi_1\in D[0,\infty)^p, \quad \varphi_2\in D_l[0,\infty)^p.$$

The Contraction Principle and uniqueness of the rate function easily imply that for any irrational $T_2 \geq T_1 > 0$ we have

$$I^{T_1}_{X,X^-}\left(\tilde{R}_{T_1}(\varphi_1,\varphi_2)\right) \leq I^{T_2}_{X,X^-}\left(\tilde{R}_{T_2}(\varphi_1,\varphi_2)\right), \quad \varphi \in D[0,\infty)^p, \quad \varphi_2 \in D_l[0,\infty)^p,$$

whence

$$\begin{aligned} I^{\infty}_{X,X^-}(\varphi_1,\varphi_2) &= \sup\left\{I^{T}_{X,X^-}\left(\tilde{R}_{T}(\varphi_1,\varphi_2)\right) : T \in (0,\infty) \cap \mathbf{Q}^c\right\} \\ &= \lim_{\substack{T\to\infty \\ T\in\mathbf{Q}^c}} I^{T}_{X,X^-}\left(\tilde{R}_{T}(\varphi_1,\varphi_2)\right) \end{aligned}$$

where for each irrational $T > 0$, I^T_{X,X^-} is as defined in the statement of the theorem. ■

As in the one-dimensional case, we can easily check that the inequality

$$I^{\infty}_{X,X^-}(\varphi_1,\varphi_2) \leq \sup_{k=1,2,\ldots} I^{K}_{X,X^-}\left((\varphi_1)_K,(\varphi_2)_K\right), \quad \varphi_1 \in D[0,\infty)^p, \quad \varphi_2 \in D_l[0,\infty)^p,$$

holds, but we do not have a way to prove equality in the general case. However, here too we have equality when the rate function I_X is of the integral form.

Corollary 3.19 *If the good rate function I_X is of the form (3.14) for some rate function $r : \mathbb{R}^p \to [0,\infty]$, then we have*

$$I^{\infty}_{X,X^-}(\varphi_1,\varphi_2) = \begin{cases} \int_0^\infty r(\dot{\varphi}_1(t))\,dt + \int_0^\infty r(\dot{\varphi}_2(t))\,dt & \text{if } \varphi_1,\varphi_2 \in AC_0[0,1]^p \\ \infty & \text{otherwise} \end{cases} \tag{3.21}$$

with

$$\int_0^\infty r(\dot{\varphi}(t))\,dt = \sup_{K=1,2,\ldots} \int_0^K r(\dot{\varphi}(t))\,dt = \sup_{T>0} \int_0^T r(\dot{\varphi}(t))\,dt, \tag{3.22}$$

for φ in $AC_0[0,\infty)^p$.

Thus, in terms of large deviations, for a stationary sequence satisfying the sample path LDP in $D[0,1]$ with good rate function of the form (3.14) for some rate function r, the entire past and future are independent.

Proof: This Corollary is a direct consequence of Theorem 3.18 and Corollary 2.28. Equality (3.22) is immediate from non-negativity of the rate function r. ■

We complete this section with a simple application of Theorem 3.18 that will be needed in Chapter 4. The assumptions in the following corollary differ slightly from what we had before in that the rate function r associated with I_X is here required to be good.

Corollary 3.20 *Assume the family $\{X_n(\cdot),\ n = 1, 2, \ldots\}$ to satisfy the LDP in $(D[0,1]^p, \tau_0)$ with good rate function $I_X : D[0,1]^p \to [0,\infty]$ of the form (3.14) for some good rate function $r : \mathbb{R}^p \to [0,\infty]$.*

Then the family $\{(X_n(\cdot), X_n^{\infty,-}(\cdot)),\quad n = 1, 2, \ldots\}$ satisfies the LDP in the product space $(D[0,1]^p, \tau_0) \times (D_l[0,\infty)^p, \tau_0^e)$ with good rate function $I^{1,\infty}_{X,X^-} : D[0,1]^p \times D_l[0,\infty)^p \to [0,\infty]$ given by

$$I^{1,\infty}_{X,X^-}(\varphi_1, \varphi_2) = \begin{cases} \int_0^1 r(\dot\varphi_1(t))\,dt + \int_0^\infty r(\dot\varphi_2(t))\,dt & \text{if } \begin{array}{l}\varphi_1 \in AC_0[0,1]^p \\ \varphi_2 \in AC_0[0,\infty)^p\end{array} \\ \infty & \text{otherwise.} \end{cases} \tag{3.23}$$

Proof: Under the enforced assumptions, Theorem 3.18 and Corollary 3.19 yield the LDP for the family $\{(X_n^\infty(\cdot), X_n^{\infty,-}(\cdot))\,,\ n = 1, 2, \ldots\}$ in $D[0,\infty)^p \times D_l[0,\infty)^p$ with good rate function $I^\infty_{X,X^-} : D[0,\infty)^p \times D_l[0,\infty)^p \to [0,\infty]$ given by (3.21).

Define the restriction mapping $F : D[0,\infty)^p \times D_l[0,\infty)^p \to D[0,1]^p \times D_l[0,\infty)^p$ in the obvious way. Then by [27, p. 55], Proposition I in [12, p. 44] and Lemma 2.7, it is plain that F is Borel-measurable and $\mathcal{S}$-continuous on the effective domain of I^∞_{X,X^-}.

Therefore, upon noting that

$$\left(X_n(\cdot), X_n^{\infty,-}(\cdot)\right) = F\left(X_n^\infty(\cdot), X_n^{\infty,-}(\cdot)\right), \quad n = 1, 2, \ldots,$$

the Contraction Principle yields the desired LDP. The associated good rate function $I^{1,\infty}_{X,X^-} : D[0,1]^p \times D_l[0,\infty)^p \to [0,\infty]$ is given by

$$\begin{aligned} I^{1,\infty}_{X,X^-}(\varphi_1, \varphi_2) &= \inf_{\substack{\psi_1 \in D[0,\infty)^p \\ \psi_2 \in D_l[0,\infty)^p}} \left\{ I^\infty_{X,X^-}(\psi_1, \psi_2) : \begin{array}{l} \varphi_1 = \psi_1 \text{ on } [0,1] \\ \varphi_2 = \psi_2 \end{array} \right\} \\ &= \inf_{\psi_1 \in D[0,\infty)^p} \left\{ I^\infty_{X,X^-}(\psi_1, \varphi_2) : \varphi_1 = \psi_1 \text{ on } [0,1] \right\}, \end{aligned} \tag{3.24}$$

for φ_1 in $D[0,1]^p$ and φ_2 in $D_l[0,\infty)^p$.

Whenever φ_2 does not belong to $AC_0[0,\infty)^p$ we have from the expression (3.21) that $I^\infty_{X,X^-}(\psi_1, \varphi_2) = \infty$ for all ψ_1 in $D[0,\infty)^p$. Similarly, if φ_1 does not belong to $AC_0[0,1]^p$, then any ψ_1 in $D[0,\infty)^p$ satisfying $\psi_1 = \varphi_1$ on $[0,1]$ is certainly not in $AC_0[0,\infty)^p$ either, so that $I^\infty_{X,X^-}(\psi_1, \varphi_2) = \infty$. Therefore we already find that

$$I^{1,\infty}_{X,X^-}(\varphi_1, \varphi_2) = \infty, \quad \varphi_1 \notin AC_0[0,1]^p \quad \text{or} \quad \varphi_2 \notin AC_0[0,\infty)^p.$$

On the other hand, for φ_1 in $AC_0[0,1]^p$ and φ_2 in $AC_0[0,\infty)^p$, we observe from (3.21) and (3.24) that

$$\begin{aligned} & I_{X,X}^{1,\infty}(\varphi_1,\varphi_2) \\ &= \inf_{\psi_1 \in AC_0[0,\infty)^p} \left\{ \int_0^\infty r(\dot\psi_1(t))\,dt + \int_0^\infty r(\dot\varphi_2(t))\,dt : \varphi_1 = \psi_1 \text{ on } [0,1] \right\} \\ &= \int_0^1 r(\dot\varphi_1(t))\,dt \;+\; \int_0^\infty r(\dot\varphi_2(t))\,dt \;+\; \inf_{\psi_1 \in AC[1,\infty)^p} \int_1^\infty r(\dot\psi_1(t))\,dt. \end{aligned}$$

The rate function r being good, by Lemma 1.3, there exists c in $\mathbb{R}^p$ such $r(c) = 0$. Thus, by considering the mapping ψ in $AC[1,\infty)$ defined by $\psi(t) = ct$ for all $t \geq 1$, the infimum in the last equality is easily seen to satisfy

$$\begin{aligned} 0 \;\leq\; \inf_{\psi_1 \in AC[1,\infty)^p} \int_1^\infty r(\dot\psi_1(t))\,dt \;&\leq\; \int_1^\infty r(\dot\psi(t))\,dt \\ &= \int_1^\infty r(c)\,dt \\ &= 0, \end{aligned}$$

and the desired expression (3.23) is easily obtained. ■

We complete this section on LDP by discussing the LDP for partial sum processes in the space $(D[0,\infty)^p, \tau_0^e)$ versus that in the product space $(D[0,\infty), \tau_0^e)^p$.

3.2.3 LDP in $(D[0,\infty)^p, \tau_0^e)$ vs. LDP in $(D[0,\infty), \tau_0^e)^p$

The purpose of this section is to discuss the differences which arise from considering a multi-dimensional partial sum process as a random element in $(D[0,\infty)^p, \tau_0^e)$ versus a random element in $(D[0,\infty), \tau_0^e)^p$. Because of the similarities (for that matter) in the spaces $D[a,b]^p$ and $D[a,\infty)^p$, we use the generic notation $D(\mathcal{T}, \mathbb{R}^p)$ in lieu of both spaces, with $\mathcal{T}$ denoting either $[a,b]$ or $[a,\infty)$, for any $0 \leq a < b$.

Consider the $\mathbb{R}^p$-valued random sequence $\{x_n,\ n = 1,2,\ldots\}$, where for each $n = 0,1,\ldots$, $x_n = (x_n^1,\ldots,x_n^p)$. For the duration of this section only, for each $n = 1,2,\ldots$, let $X_n(\cdot)$ denote the restriction of the partial sum process $X_n^\infty(\cdot)$ to $\mathcal{T}$. As we have already seen, for each $n = 1,2,\ldots$, $X_n(\cdot)$ is a random element in $D(\mathcal{T}, \mathbb{R}^p)$. However, we could also in principle define a $\mathbb{R}^p$-valued partial sum process from the same sequence $\{x_n,\ n = 1,2,\ldots\}$ as follows (the definition is to be understood sample path wise). For each $n = 1,2,\ldots$ and each $l = 1,\ldots,p$, we define the mapping $X_n^l(\cdot) : \mathcal{T} \to \mathbb{R}$ by

$$X_n^l(t) \;=\; \frac{1}{n} \sum_{i=1}^{\lfloor nt \rfloor} x_i^l, \quad t \in \mathcal{T}.$$

Following the arguments in Lemma 2.13, it is easily seen that $\mathcal{F}/\mathcal{B}_{\mathrm{R}}$-measurability of x_n yields $\mathcal{F}/\mathcal{B}_{D(\mathcal{T},\mathrm{R})}$-measurability of $X_n^l(\cdot)$ for each $n = 1, 2, \ldots$, and each $l = 1, \ldots, p$, and we conclude by [27, p. 55] that the p-uple $(X_n^1(\cdot), \ldots, X_n^p(\cdot))$ is a random element in the product space $(D(\mathcal{T}, \mathbb{R}), \tau_0)^p$. It is plain that there is a one-to-one correspondence between the latter and $X_n(\cdot)$, and the question naturally arises whether the LDP for one implies that for the other.

To help answering this question, we define the mapping $H : D(\mathcal{T}, \mathbb{R}^p) \to D(\mathcal{T}, \mathbb{R})^p$ as follows. For each z in $D(\mathcal{T}, \mathbb{R}^p)$ with $z(t) = (z_1(t), \ldots, z_p(t))$, we set

$$H(z) = (\tilde{z}_1, \ldots, \tilde{z}_p),$$

where for each $l = 1, \ldots, p$, we have defined the mapping $\tilde{z}_l : [0,1] \to \mathbb{R}$ by $\tilde{z}_l(t) = z_l(t)$ for all t in $\mathcal{T}$. Clearly, the mapping H is one-to-one and onto, and the elements of $D(\mathcal{T}, \mathbb{R}^p)$ can be identified to those of the product space $D(\mathcal{T}, \mathbb{R})^p$.

However, for the Skorohod topology, the image by H of the topology of $D(\mathcal{T}, \mathbb{R}^p)$ is strictly finer than the product topology of $(D(\mathcal{T}, \mathbb{R}), \tau_0)^p$ [39, p. 293] so that H is $\mathcal{S}$-continuous while H^{-1} is not. This unfortunate fact prevents us to completely identify one topological space with the other.

Nevertheless, we still have the following equivalence in the LDPs. Let $C(\mathcal{T}, \mathbb{R}^p)$ be the subspace of $D(\mathcal{T}, \mathbb{R}^p)$ of continuous mapping.

Proposition 3.21 *If the family of partial sum processes* $\{X_n(\cdot),\ n = 1, 2, \ldots\}$ *satisfies the LDP in* $(D(\mathcal{T}, \mathbb{R}^p), \tau_0)$ *with good rate function* $I : D(\mathcal{T}, \mathbb{R}^p) \to [0, \infty]$*, then the family* $\{H(X_n(\cdot)),\ n = 1, 2, \ldots\}$ *satisfies the LDP in* $(D(\mathcal{T}, \mathbb{R}), \tau_0)^p$ *with good rate function* $I_p : D(\mathcal{T}, \mathbb{R})^p \to [0, \infty]$ *given by*

$$I_p\left((\varphi_1, \varphi_2, \ldots,)\right) = I\left(H^{-1}(\varphi_1, \varphi_2, \ldots)\right), \quad (\varphi_1, \varphi_2, \ldots) \in \mathcal{D}(\mathcal{T}, \mathbb{R})^p.$$

Conversely, if the family $\{H(X_n(\cdot)),\ n = 1, 2, \ldots\}$ *satisfies the LDP in the product space* $(D(\mathcal{T}, \mathbb{R}), \tau_0)^p$ *with good rate function* $I_p : \mathcal{D}(\mathcal{T}, \mathbb{R})^p \to [0, \infty]$ *such that* $\mathcal{D}_{I_p}$ *is contained in* $C(\mathcal{T}, \mathbb{R})^p$*, then the family* $\{X_n(\cdot),\ n = 1, 2, \ldots\}$ *satisfies the LDP in* $(D(\mathcal{T}, \mathbb{R}^p), \tau_0)$*, with good rate function* $I : D(\mathcal{T}, \mathbb{R}^p) \to [0, \infty]$ *given by*

$$I(\varphi) = I_p(H(\varphi)), \quad \varphi \in \mathcal{D}(\mathcal{T}, \mathbb{R}^p).$$

Proof: The first assertion follows easily from the Contraction Principle and continuity of H.

Next, because the image of the (Skorohod) Borel σ-field of $D(\mathcal{T}, \mathbb{R})$ coincide with the Borel σ-field of the product space $(D(\mathcal{T}, \mathbb{R}), \tau_0)^p$ [39, p. 293] H^{-1} is Borel-measurable. Furthermore, the space $(C(\mathcal{T}, \mathbb{R}^p), \tau_\infty)$ of continuous mapping is homeomorphic to the product space $(C(\mathcal{T}, \mathbb{R}), \tau_\infty)^p$, so that in view of the fact that the Skorohod topology coincide with the uniform topology on the subspace of continuous functions, we get that H^{-1} is $\mathcal{S}$-continuous when restricted to $C(\mathcal{T}, \mathbb{R})^p$. Therefore, H^{-1} is Borel-measurable and $\mathcal{S}$-continuous on the effective domain of I_p and the second assertion follows also from the Contraction Principle. ■

Proposition 3.21 can be further generalized to encompass the case of the joint LDP for $\{(X_n^\infty(\cdot), X_n^{\infty,-}(\cdot)),\ n = 1, 2, \ldots\}$. To this end, let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two time intervals of either type $[a, b]$ or $[a, \infty)$ and let $D_l(\mathcal{T}_2, \mathbb{R}^p)$ denote the space of left continuous with right-hand limits functions from $\mathcal{T}_2$ into $\mathbb{R}^p$. For the sake of clarity, in the remaining of this section only, for each $n = 1, 2, \ldots$, we denote by $X_n(\cdot)$ (resp. $X_n^-(\cdot)$) the restriction of $X_n^\infty(\cdot)$ (resp. $X_n^{\infty,-}(\cdot)$) to $\mathcal{T}_1$ (resp. $\mathcal{T}_2$).

Define the mapping $\widetilde{H} : D(\mathcal{T}_1, \mathbb{R}^p) \times D_l(\mathcal{T}_2, \mathbb{R}^p) \to D(\mathcal{T}_1, \mathbb{R})^p \times D_l(\mathcal{T}_2, \mathbb{R})^p$ by setting

$$\widetilde{H}(z_1, z_2) = (H(z_1), H_l(z_2)), \quad (z_1, z_2) \in D(\mathcal{T}_1, \mathbb{R}^p) \times D_l(\mathcal{T}_2, \mathbb{R}^p)$$

where the mapping $H_l : D_l(\mathcal{T}, \mathbb{R}^p) \to D_l(\mathcal{T}, \mathbb{R})^p$ is defined similarly to H. It is not hard to check that the properties of H (and their counterparts for H_l) translate into $\mathcal{S}$-continuity of $\widetilde{H}$ and Borel-measurability of $\widetilde{H}^{-1}$. The arguments given in the proof of Proposition 3.21 then easily transpose to this new setup, and we get the following result.

Proposition 3.22 *If the family $\{(X_n(\cdot), X_n^-(\cdot)),\ n = 1, 2, \ldots\}$ satisfies the LDP in $(D(\mathcal{T}_1, \mathbb{R}^p), \tau_0) \times (D_l(\mathcal{T}_2, \mathbb{R}^p), \tau_0)$, with good rate function $I : D(\mathcal{T}_1, \mathbb{R}^p) \times D_l(\mathcal{T}_2, \mathbb{R}^p) \to [0, \infty]$, then the family $\{\widetilde{H}(X_n(\cdot), X_n^-(\cdot)),\ n = 1, 2, \ldots\}$ satisfies the LDP in $(D(\mathcal{T}_1, \mathbb{R}), \tau_0)^p \times (D(\mathcal{T}_2, \mathbb{R}), \tau_0)^p$ with good rate function $I_p : D(\mathcal{T}_1, \mathbb{R})^p \times D(\mathcal{T}_2, \mathbb{R})^p \to [0, \infty]$.*

Conversely, if the family $\{\widetilde{H}(X_n(\cdot), X_n^-(\cdot)),\ n = 1, 2, \ldots\}$ satisfies the LDP in $(D(\mathcal{T}_1, \mathbb{R}), \tau_0)^p \times (D(\mathcal{T}_2, \mathbb{R}), \tau_0)^p$ with good rate function $I_p : D(\mathcal{T}_1, \mathbb{R})^p \times D(\mathcal{T}_2, \mathbb{R})^p \to [0, \infty]$ such that $\mathcal{D}_{I_p}$ is contained in $C(\mathcal{T}_1, \mathbb{R})^p \times C(\mathcal{T}_2, \mathbb{R})^p$, then the family $\{(X_n(\cdot), X_n^-(\cdot)),\ n = 1, 2, \ldots\}$ satisfies the LDP in $(D(\mathcal{T}_1, \mathbb{R}^p), \tau_0) \times (D_l(\mathcal{T}_2, \mathbb{R}^p), \tau_0)$, with good rate function $I : D(\mathcal{T}_1, \mathbb{R}^p) \times D_l(\mathcal{T}_2, \mathbb{R}^p) \to [0, \infty]$.

The rate functions I and I_p are related by the relation

$$I(\varphi_1, \varphi_2) = I_p(\widetilde{H}(\varphi, \varphi_2)), \quad \varphi_1 \in D(\mathcal{T}_1, \mathbb{R}^p), \quad \varphi_2 \in D(\mathcal{T}_2, \mathbb{R}^p).$$

Propositions 3.21 and 3.22 will enable us in Chapter 4 to derive the LDP for partial sum processes in the product space by using the results (in the multi-dimensional space) of this chapter.

In view of the bijection H, in the sequel, whenever no confusion is possible, we identify the elements of $D(\mathcal{T}, \mathbb{R}^p)$ with those of $D(\mathcal{T}, \mathbb{R})^p$ and use the same notation to denote partial sum processes in both spaces. However, we always state explicitly in which topological space the LDP is satisfied.

We conclude this chapter with a brief review of random sequences whose associated partial sum process satisfy the LDP in $(D[0,1]^p, \tau_0)$.

3.3 Examples of LDP for partial sum processes

We begin with a general result from [23] which in some situations makes the sample path LDP a consequence of the LDP for the sample mean sequence. We first record some assumptions.

Let $\{x_n,\ n = 1,2,\dots\}$ be an $\mathbb{R}^p$-valued random sequence defined on some probability space $(\Omega, \mathcal{F}, \mathbf{P})$. For each $n = 1,2,\dots$, $m = 1,2,\dots$ and each real $0 < t_1 < \dots < t_m \leq 1$, define the $\mathbb{R}^{pm}$-valued rv $Z_n(t_1, \dots, t_m)$ by

$$Z_n(t_1,\dots,t_m) \;\equiv\; (X_n(t_1), X_n(t_2) - X_n(t_1), \dots, X_n(t_m) - X_n(t_{m-1}))\,.$$

Let $\|\cdot\|$ denote the Euclidean norm in $\mathbb{R}^p$.

Assumption (DZ-1) *For each $m = 1,2,\dots$ and each real $0 < t_1 < \dots < t_m \leq 1$, the random sequence $\{Z_n(t_1,\dots,t_m),\ n = 1,2,\dots\}$ satisfies the LDP in $\mathbb{R}^{pm}$ with good rate function $r_Z^m : \mathbb{R}^{pm} \to [0,\infty]$ given by*

$$r_Z^m(z_1,\dots,z_m) \;=\; \sum_{i=1}^{m} (t_i - t_{i-1})\, r_X\left(\frac{z_i}{t_i - t_{i-1}}\right), \qquad (z_1,\dots z_m) \in \mathbb{R}^{pm}$$

where $r_X : \mathbb{R}^p \to [0,\infty)$ is the convex good rate function associated with the LDP for the sample mean sequence $\{X_n(1),\ n = 1,2,\dots\}$.

Assumption (DZ-2) *For all $\gamma \geq 0$ and $R \geq 0$,*

$$g_R(\gamma) \;\equiv\; \sup_{\substack{k,m=1,2,\dots \\ k\in[0,Rm]}} \frac{1}{m} \ln \mathbf{E}\left[\exp\left(\gamma \left\| \sum_{i=k+1}^{k+m} x_i \right\|\right)\right] \;<\; \infty,$$

and

$$\sup_{\gamma \geq 0} \left(\limsup_{R\to\infty} \frac{1}{R}\, g_R(\gamma)\right) \;<\; \infty.$$

Theorem 3.23 ([23, Theorem 2]) *Under Assumptions* (**DZ-1**) *and* (**DZ-2**), *the family of partial sum processes* $\{X_n(\cdot),\ n = 1, 2, \ldots\}$ *satisfies the LDP in* $(D[0,1]^p, \tau_\infty)$ *with convex good rate function* $I_X : D[0,1]^p \to [0, \infty]$ *given by*

$$I_X(\varphi) = \begin{cases} \int_0^1 r_X(\dot{\varphi}(t))\, dt & \text{if } \varphi \in AC_0[0,1]^p \\ \infty & \text{otherwise} \end{cases}.$$

The assumptions required for Theorem 3.23 to hold are quite strong. They do simplify for the particular case of iid random sequence, but in that case the result already holds with weaker assumptions. Theorem 3.23 is used in [23] to establish, under suitable conditions, the sample path LDP for stationary and hyper-mixing sequences.

The most common cases investigated in the literature are that of independent and stationary sequences, which we briefly review next.

3.3.1 Sample path LDP for random walks

The LDP for the partial sum processes associated with a sequence of iid $\mathbb{R}^p$-valued rvs with finite moment generating function was proved by Varadhan [64]. Building upon Borovkov's results [9], Mogulskii [48] extended the results to different scalings and to moment generating functions finite only in a neighborhood of the origin.

We present below the result with minimal assumptions; a proof is available in [48]. A proof of the same result, but under stronger assumptions, namely that the logarithmic moment generating function is finite on $\mathbb{R}^p$, based on projective limits and relying on the polygonal approximations of the partial sum process is presented in [24, Theorem 5.1.2, p. 152].

Theorem 3.24 (Mogulskii [48]) *Let* $\{x_n,\ n = 1, 2, \ldots\}$ *be a sequence of iid* $\mathbb{R}^p$*-valued rvs. Assume that there exists* $\delta > 0$ *such that*

$$\Lambda(\theta) \equiv \log \mathbf{E}\left[e^{\langle \theta, x_1 \rangle}\right] < \infty, \quad \theta \in \mathbb{R}^p, \quad ||\theta|| < \delta.$$

Then the family of partial sum processes $\{X_n(\cdot),\ n = 1, 2, \ldots\}$ *satisfies the LDP in* $(D[0,1]^p, \tau_\infty)$ *with convex good rate function* $I_X : D[0,1]^p \to [0, \infty]$ *given by*

$$I_X(\varphi) = \begin{cases} \int_0^1 \Lambda^*(\dot{\varphi}(t))\, dt & \text{if } \varphi \in AC_0[0,1]^p \\ \infty & \text{otherwise} \end{cases}$$

where Λ^* *is the Legendre-Fenchel transform of* Λ.

The generality of Theorem 3.24 yields the joint sample path LDP for the arrivals and capacity sequences of a discrete-time single-server queue with Poisson arrivals and capacity. The following corollary will be used in Chapter 6 to stress that our approach to the large deviations behavior of single-server queues is applicable to certain queues for which previous results could not be applied.

Corollary 3.25 *Let $\{a_{t+1},\ t=0,1,\ldots\}$ and $\{c_{t+1},\ t=0,1,\ldots\}$ be the arrivals and capacity sequences of a discrete-time single-server queue with Poisson arrivals and capacity, and let λ and μ be respectively the arrival and capacity rate. Then the family of partial sum processes $\{(A_n(\cdot), C_n(\cdot)),\ n=1,2,\ldots\}$ satisfies the LDP in $(D[0,1],\tau_0)^2$ with good rate function $I_{AC}: D[0,1]^2 \to [0,\infty]$ given by*

$$I_{AC}(\varphi_1,\varphi_2) = \begin{cases} \int_0^1 (\Lambda_a^*(\dot{\varphi}_1(t)) + \Lambda_c^*(\dot{\varphi}_2(t)))\,dt & \text{if } \varphi_1,\varphi_2 \in AC_0[0,1] \\ \infty & \text{otherwise} \end{cases}$$

with

$$\Lambda_a^*(x) = \begin{cases} x\ln\frac{x}{\lambda} - x + \lambda, & x>0 \\ 0, & x=0 \\ +\infty & x<0 \end{cases} \quad \text{and} \quad \Lambda_c^*(x) = \begin{cases} x\ln\frac{x}{\mu} - x + \mu, & x>0 \\ 0, & x=0 \\ +\infty, & x<0. \end{cases}$$

Proof: In this particular setup, a_{t+1} represents the number of arrivals in the time interval $[t,t+1)$, and is Poisson distributed with mean λ, i.e., for each $t=0,1,\ldots$,

$$\mathbf{P}[a_{t+1}=n] = e^{-\lambda}\frac{\lambda^n}{n!}, \quad n=1,2,\ldots\,.$$

The capacity c_{t+1} is the number of units served in the time interval $[t,t+1)$ and is also Poisson distributed with rate μ, i.e., for each $t=1,2,\ldots$,

$$\mathbf{P}[c_{t+1}=n] = e^{-\mu}\frac{\mu^n}{n!}, \quad n=1,2,\ldots\,.$$

Simple algebra then yields,

$$\begin{aligned} \Lambda_a(\theta) &= \ln \mathbf{E}\left[e^{\theta a_1}\right] \\ &= \ln\left(\sum_{n=1}^{\infty} e^{\theta n} e^{-\lambda}\frac{\lambda^n}{n!}\right) \\ &= \lambda(e^\theta - 1), \quad \theta\in\mathbb{R}. \end{aligned}$$

To compute the Legendre-Fenchel transform, we note that for $x \le 0$, the function $\theta \to \theta x - \Lambda_a(\theta)$ is strictly decreasing so that

$$\Lambda_a^*(x) = \sup_{\theta\in\mathbb{R}} (\theta x - \Lambda_a(\theta)) = \lim_{\theta\to-\infty} (\theta x - \Lambda_a(\theta)) = +\infty, \quad x<0,$$

while for $x > 0$, its derivative vanishes at $\theta = \ln \frac{x}{\lambda}$ and after some algebra, we find

$$\Lambda_a^*(x) \;=\; x \ln \frac{x}{\lambda} - x + \lambda, \quad x > 0.$$

The formulas for Λ_c and Λ_c^* are then easily obtained by substituting μ for λ in the expressions of Λ_a and Λ_a^*, and we get

$$\Lambda_c^*(x) \;=\; \begin{cases} x \ln \frac{x}{\mu} - x + \mu, & x > 0 \\ 0, & x = 0 \\ +\infty, & x < 0. \end{cases}$$

By independence of $\{a_{t+1},\ t = 1, 2, \ldots\}$ and $\{c_{t+1},\ t = 1, 2, \ldots\}$, a direct application of Theorem 3.24 and Corollary 1.9 yields the desired LDP with the good rate function $I_{AC} : D[0,1]^2 \to [0, \infty]$ as given in the statement of the Corollary. ■

There exists more general results for sequence of rvs which are only independent. The interested reader is referred to [23] and [58].

Proposition 3.26 ([23, Corollary 1]) *Let $\{x_n,\ n = 1, 2, \ldots\}$ be a sequence of independent $\mathbb{R}^p$-valued rvs, and for each $n = 1, 2, \ldots,$ define $\Lambda_n : \mathbb{R}^p \to \mathbb{R} \cup \{\pm\infty\}$ by $\Lambda_n(\theta) \equiv \ln \mathbf{E}\left[e^{\langle\theta, x_n\rangle}\right]$ for all θ in $\mathbb{R}^p$. Assume that for each θ in $\mathbb{R}^p$, the limit*

$$\Lambda(\theta) \;\equiv\; \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \Lambda_i(\theta)$$

exists in $\mathbb{R}$ and that Λ is differentiable on $\mathbb{R}^p$.

Then the family of partial sum processes $\{X_n(\cdot),\ n = 1, 2, \ldots\}$ satisfies the LDP in $(D[0,1]^p, \tau_\infty)$ with good rate function $I_X : D[0,1]^p \to [0, \infty]$ given by

$$I_X(\varphi) \;=\; \begin{cases} \int_0^1 \Lambda^*(\dot{\varphi}(t))\, dt & \text{if } \varphi \in AC_0[0,1]^p \\ \infty & \text{otherwise} \end{cases}$$

where Λ^* is the Legendre-Fenchel transform of Λ.

3.3.2 Sample path LDP for stationary and mixing random sequences

To our knowledge, the only result on sample path LDP for stationary dependent random sequences is that of [23] which we reproduce below in its full generality.

Let $\{x_n,\ n = 0, \pm 1, \pm 2, \ldots\}$ be a $\mathbb{R}^p$-valued stationary random sequence, and for k, l in $\mathbb{Z}$ such that $k \leq l$, let $\mathcal{F}_k^l$ denote the σ-field generated by the collection of random variables $\{x_i,\ k \leq i \leq l\}$. We begin by stating the two main assumptions.

Assumption (DZ-3) *For every $C < \infty$, there exists a non-decreasing sequence $\{l(n),\ n = 1, 2, \ldots\}$ in $\mathbb{N}$ with $\sum_{n=1}^{\infty} \frac{l(n)}{n(n+1)} < \infty$, such that*

$$\sup_{\substack{k_1=0,1,\ldots \\ k_2=0,1,\ldots}} \left\{ \mathbf{P}[A]\ \mathbf{P}[B] - e^{l(n)}\, \mathbf{P}[A \cap B] \ : A \in \mathcal{F}_0^{k_1},\ B \in \mathcal{F}_{k_1+l(n)}^{k_1+k_2+l(n)} \right\} \leq e^{-Cn}$$

as well as

$$\sup_{\substack{k_1=0,1,\ldots \\ k_2=0,1,\ldots}} \left\{ \mathbf{P}[A \cap B] - e^{l(n)}\, \mathbf{P}[A]\ \mathbf{P}[B] \ : A \in \mathcal{F}_0^{k_1},\ B \in \mathcal{F}_{k_1+l(n)}^{k_1+k_2+l(n)} \right\} \leq e^{-Cn}.$$

Assumption (DZ-4) *There exist $C > 0$, scalar l and $\alpha \geq 1$ such that for each $k = 2, 3, \ldots$, for all $i \leq k$, and for any real valued, bounded and $\mathcal{F}_{a_i}^{b_i}$-measurable rv W_i with $a_1 \leq b_1 \leq a_2 \leq \ldots$ and $a_{i+1} - b_i \geq l$, the following inequality holds*

$$\mathbf{E}\left[\prod_{i=1}^{k} |W_i| \right] \ \leq\ C^k \prod_{i=1}^{k} \left(\mathbf{E}[\,|W_i|^\alpha\,]\, \right)^{\frac{1}{\alpha}}.$$

Needless to say, checking that these last two Assumptions are verified will not be in general an easy task. Therefore, we rely on the comment in [23] stating that Assumptions **(DZ-3)** and **(DZ-4)** are satisfied when the sequence $\{x_n,\ n = 0, \pm 1, \pm 2, \ldots\}$ is ψ-mixing, in order to apply the following Proposition. A detailed account of the different types of mixing conditions for random sequences can be found in [13, 14].

Proposition 3.27 *Assume that Assumptions* **(DZ-3)** *and* **(DZ-4)** *hold and that $\mathbf{E}\left[e^{\gamma \|x_1\|} \right] < \infty$ for all γ in $\mathbb{R}$.*

Then, the family of partial sum processes $\{X_n(\cdot),\ n = 1, 2, \ldots\}$ satisfies the LDP in $(D[0,1]^p, \tau_\infty)$ with good rate function $I_X : D[0,1]^p \to [0, \infty]$ given by

$$I_X(\varphi) \ = \ \begin{cases} \int_0^1 \Lambda^*(\dot{\varphi}(t))\, dt & \textit{if} \quad \varphi \in AC_0[0,1]^p \\ \infty & \textit{otherwise} \end{cases}$$

where Λ^* is the Legendre-Fenchel transform of the logarithmic moment generating function $\Lambda : \mathbb{R}^p \to \mathbb{R} \cup \{\pm\infty\}$ defined by

$$\Lambda(\theta) \ \equiv\ \lim_{n\to\infty} \frac{1}{n} \ln \mathbf{E}\left[e^{\sum_{i=1}^{n} \langle \theta, x_i \rangle} \right], \quad \theta \in \mathbb{R}^p.$$

In view of Proposition 2.17, the LDP for a family of partial sum processes in $(D[0,1], \tau_\infty)$ also holds in $(D[0,1], \tau_0)$, so that the results presented in the previous sections and Chapter 2 apply to the particular examples reviewed here.

In the upcoming Chapter, we make use of the general result derived here to establish the LDP for some functionals on the inputs to a queueing system. These LDPs will be used in Chapter 5 and Chapter 6 to characterize the large deviations behavior of Lindley processes and single-server queues.

Chapter 4

LDP for Some Functionals of the Inputs to a Queueing System

Under suitable assumptions we derive the LDP for some functionals of the inputs from a general single server queueing system. These results will be used in Chapter 6 to derive the LDP for the partial sum process associated with the steady-state output process of a single-server queue from that of the partial sum process associated with the joint sequence of arrivals and capacities. Our approach makes use of the LDP for the input partial sum process on the whole space $D_l[0,\infty)$, as derived in Chapter 3.

The methodology underlying the proof can in fact be applied to establish the LDP for any Borel-measurable function of a partial sum process satisfying itself the LDP in $D[0,1]$ with rate function I, provided the mapping is $\mathcal{S}$-continuous on the effective domain $\mathcal{D}_I$ of the rate function I.

The chapter is organized as follows. The notation and assumptions are introduced in Section 4.1, while preliminary results are presented in Section 4.2. The proof of the main result is presented in Section 4.3, while the (partial) computation of the rate function is done in Section 4.4. Simple consequences of the main Theorem on the LDP for other functionals are given in Section 4.5.

4.1 Notation and assumptions

Let $\{x_n,\ n = 0,\pm1,\pm2,\ldots\}$ and $\{y_n,\ n = 0,\pm1,\pm2,\ldots\}$ be two $\mathbb{R}$-valued random sequences defined on a common, possibly enlarged, probability space $(\Omega,\mathcal{F},\mathbf{P})$. The sequences $\{x_n,\ n = 0,\pm1,\pm2,\ldots\}$ and $\{y_n,\ n = 0,\pm1,\pm2,\ldots\}$ will later represent respectively the arrivals and capacity sequences in a discrete-time $\mathbf{G/G/1}$ queue, but there is no need at this point for defining them more precisely.

We use here the notation relevant to partial sum processes, as defined in Chapter 2, Section 2.2.1. In addition, for each $n = 1, 2, \ldots$, we introduce the following quantities

$$m_n(t) \equiv \inf_{0 \le s \le t} (Y_n(s) - X_n(s)), \quad t \in [0, 1],$$

$$M_n(t) \equiv \sup_{0 \le s \le t} \left(Y_n^{\infty,-}(s) - X_n^{\infty,-}(s)\right), \quad t \in [0, \infty)$$

and

$$M_n^\infty(t) \equiv \sup_{s \ge 0} \left(Y_n^{\infty,-}(s) - X_n^{\infty,-}(s)\right) = \lim_{s \to \infty} M_n(s), \quad t \in [0, 1] .$$

In view of the discussions and results of Chapter 2 and 3, the partial sum processes $(X_n(\cdot), Y_n(\cdot))$ and $(X_n^{\infty,-}(\cdot), Y_n^{\infty,-}(\cdot))$ are random elements in the spaces $D[0,1]^2$ and $D_l[0,\infty)^2$, respectively, endowed with their respective Skorohod topology. As previously noted, the benefits of using the separable Skorohod topology are twofold: First, n-uples of random elements can be seen as random element in the product space. Secondly, the whole large deviations artillery can be applied to Borel probability measures on Skorohod spaces or product of Skorohod spaces, thus in particular to the distribution laws of random elements in those spaces.

From the Borel-measurability of the addition, infimum and supremum mappings (Lemmas 2.8, 2.10, 3.9 and 3.10), $m_n(\cdot)$ and $M_n(\cdot)$ are in fact random elements in $(D[0,1], \tau_0)$ and $(D_l[0,\infty), \tau_0^e)$, respectively. That $M_n^\infty(\cdot)$ is also a random element in $(D[0,1], \tau_0)$ will become plain later in view of Lemma 4.4.

Our goal is to derive the LDP for the family $\{(X_n(\cdot), m_n(\cdot), M_n^\infty(\cdot)),\ n = 1, 2, \ldots\}$ in $(D[0,1], d_0)^3$. It will be clear from the derivation that a similar LDP holds for the family $\{(Y_n(\cdot), m_n(\cdot), M_n^\infty(\cdot)),\ n = 1, 2, \ldots\}$.

The motivation for establishing this LDP is to obtain the LDP for the transient and stationary output process of a queueing system. Indeed, as we shall see in Chapter 6, the partial sum processes associated with both output processes can be expressed as a simple (continuous) function of these quantities.

Throughout we make the following assumption.

Assumption (E) *The random sequence $\{(x_n, y_n),\ n = 0, \pm 1, \pm 2, \ldots\}$ is stationary, metrically transitive with $\mathbf{E}[y_1] < \mathbf{E}[x_1]$, and the associated family of partial sum processes $\{(X_n(\cdot), Y_n(\cdot)),\ n = 1, 2, \ldots\}$ satisfy the LDP in $(D[0,1], d_0)^2$ with good rate function $I_{X,Y} : D[0,1]^2 \to [0, \infty]$.*

Under this assumption, by Proposition 3.21 the sample path LDP holds in the two-dimensional space $(D[0,1]^2, \tau_0)$, so that in view of Lemma 2.15, the sample mean sequence $\{(X_n(1), Y_n(1)),\ n = 1, 2, \ldots\}$ also satisfies a LDP in $\mathbb{R}^2$.

In addition, in order to compute explicitly the rate function corresponding to the LDP of $\{(X_n(\cdot), m_n(\cdot), M_n^\infty(\cdot)),\ n = 1, 2, \ldots\}$, we also assume that the good rate function $I_{X,Y}$ in Assumption **(E)** is of the special form given by

Assumption (E1)

$$I_{X,Y}(\varphi_1, \varphi_2) = \begin{cases} \int_0^1 r_{x,y}(\dot{\varphi}_1(t), \dot{\varphi}_2(t))\, dt & \text{if } \varphi_1, \varphi_2 \in AC_0[0,1] \\ \infty & \text{otherwise} \end{cases} \tag{4.1}$$

where $r_{x,y} : \mathbb{R}^2 \to [0, \infty]$ is the good rate function associated with the LDP satisfied by $\{(X_n(1), Y_n(1)),\ n = 1, 2, \ldots\}$.

To our knowledge, this is the only known form for a rate function associated with the LDP of a partial sum process, and thus does not appear to be at all restrictive.

Recall from Lemma 2.15 that under (4.1), it suffices that $r_{x,y}$ be convex to ensure that it is the good rate function associated with the LDP for the sequence $\{(X_n(1), Y_n(1)),\ n = 1, 2, \ldots\}$.

Define the mapping $r_{y-x} : \mathbb{R} \to [0, \infty)$ by setting

$$r_{y-x}(c) \equiv \inf_{x,y \in \mathbb{R}} \{r_{x,y}(x, y) : y = x + c\}, \quad c \in \mathbb{R}, \tag{4.2}$$

and let C_+ denote the subset of $D[0,1]$ consisting of all non-negative constant functions with generic element c_A, i.e., $c_A(t) = A$ for all t in $[0,1]$ and $A \geq 0$.

The purpose of this chapter is to prove the following Theorem.

Theorem 4.1 *Let Assumptions* **(E)** *and* **(E1)** *hold and assume that r_{y-x} is convex with $\inf_{c>0} \frac{1}{c} r_{y-x}(c) > 0$.*

Then, the collection of random elements $\{(X_n(\cdot), m_n(\cdot), M_n^\infty(\cdot)),\ n = 1, 2, \ldots\}$ satisfies the LDP in $(D[0,1], \tau_0)^3$ with good rate function $J : D[0,1]^3 \to [0, \infty]$ given by

$$J(\varphi_1, \varphi_2, \varphi_3) = \begin{cases} J_m(\varphi_1, \varphi_2) + J_M(\varphi_3) & \text{if } \varphi_1, \varphi_2 \in AC_0[0,1], \quad \varphi_3 \in C_+ \\ \infty & \text{otherwise} \end{cases}, \tag{4.3}$$

with

$$J_m(\varphi_1, \varphi_2) \tag{4.4}$$
$$\equiv \inf_{\psi \in AC_0[0,1]} \left\{ \int_0^1 r_{x,y}\left(\dot{\varphi}_1(t), \dot{\psi}(t)\right) dt : \varphi_2(t) = \inf_{0 \le s \le t} (\psi(s) - \varphi_1(s)),\ t \in [0,1] \right\}$$

and

$$J_M(c_A) = A \inf_{c>0} \frac{1}{c} r_{y-x}(c), \quad c_A \in \mathcal{C}_+. \tag{4.5}$$

The proof, presented in Section 4.3, is based on the LDP for partial sum processes in the space $D_l[0,\infty)$, obtained from the LDP in $D[0,1]$ in Chapter 3.

4.2 Preliminary results

In this section we derive a few technical results which are used in the proof of Theorem 4.1. We begin with the following general Lemma.

Lemma 4.2 *Let $\{z_n,\ n = 0, \pm 1, \pm 2, \ldots\}$ be a stationary and metrically transitive $\mathbb{R}^p$-valued random sequence, and assume that it satisfies the LDP in $\mathbb{R}^p$ with good rate function $r : \mathbb{R}^p \to [0,\infty]$.*

Then we have $r(\mathbf{E}[z_1]) = 0$.

Proof: Let $\{S_n,\ n = 1, 2, \ldots\}$ denote the sample mean sequence, i.e.,

$$S_n = \frac{1}{n} \sum_{i=1}^{n} z_i, \quad n = 1, 2, \ldots .$$

The sequence $\{z_n,\ n = 0, \pm 1, \pm 2, \ldots\}$ being stationary and metrically transitive, by the Ergodic Theorem [26, Theorem 2.1 p. 465], S_n converges $\mathbf{P}$-a.e., hence in probability, to its mean $m = \mathbf{E}[z_1]$. Thus, for any $\varepsilon > 0$, we have

$$\lim_{n\to\infty} \mathbf{P}[\,|S_n - m| > \varepsilon\,] = 0,$$

or equivalently,

$$\lim_{n\to\infty} \mathbf{P}\left[S_n \in \overline{B_\varepsilon(m)} \right] = 1$$

where $B_\varepsilon(m)$ denotes the open ball of radius ε and center m in $\mathbb{R}^p$. It is then readily seen that

$$\limsup_{n\to\infty} \frac{1}{n} \ln \mathbf{P}\left[S_n \in \overline{B_\varepsilon(m)} \right] = 0, \quad \varepsilon > 0,$$

so that by the LDP upper bound for the sequence $\{z_n,\ n = 1, 2, \ldots\}$ and non-negativity of the rate function r (Lemma 1.3), we obtain

$$\inf_{c \in \overline{B_\varepsilon(m)}} r(c) = 0, \quad \varepsilon > 0.$$

By definition of the infimum, for each $\varepsilon > 0$, there exists c_ε in $\overline{B_\varepsilon(m)}$ such that $r(c_\varepsilon) < \varepsilon$. Taking ε to be $\frac{1}{n}$ then yields the existence of a sequence $\{c_n,\ n = 1, 2, \ldots\}$ converging to m and such that $r(c_n) < \frac{1}{n}$ for all $n = 1, 2, \ldots$. The rate function r being non-negative and lower semi-continuous, we finally get

$$0 \leq r(m) \leq \liminf_{n \to \infty} r(c_n) = 0,$$

and the proof of the Lemma is complete. ■

The next proposition is an easy consequence of Assumptions **(E)** and **(E1)**, and will be useful in computing the rate functions of the desired LDP.

Proposition 4.3 *Under Assumptions* **(E)** *and* **(E1)**, *we have the following facts:*

1. *The mapping $r_{y-x} : \mathbb{R} \to [0, \infty]$ is the good rate function associated with the LDP for the sample mean sequence $\{Y_n(1) - X_n(1),\ n = 1, 2, \ldots\}$;*

2. *If $r_{x,y}$ is convex, so is r_{y-x};*

3. *We have $r_{x,y}(m_x, m_y) = 0$, and $r_{y-x}(m_y - m_x) = 0$ where $m_x \equiv \mathbf{E}[x_1]$ and $m_y \equiv \mathbf{E}[y_1]$.*

Proof: The first assertion follows easily from the assumptions and the Contraction Principle, while the second one is a consequence of Theorem 5.7 in [57, p. 38]. The third one follows easily from the assumptions, Lemma 4.2 and the fact that stationarity and metric transitivity of $\{(x_n, y_n),\ 0, \pm 1, \pm 2, \ldots\}$ imply that of the sequence $\{y_n - x_n,\ n = 0, \pm 1, \pm 2, \ldots\}$ [26, p. 458]. ■

The next result relates $M_n^\infty(\cdot) = \lim_{s \to \infty} M_n(s)$ to Loynes' variable associated with the stochastic recursive equation driven by the sequence $\{y_n - x_n,\ n = 1, 2, \ldots\}$.

Lemma 4.4 *Under Assumption* **(E)**, *we have*

$$M_n^\infty(t) = \lim_{s \to \infty} M_n(s) = \frac{1}{n} M_\infty, \quad t \in [0, 1], \quad n = 1, 2, \ldots$$

where M_∞ is Loynes' variable associated with the stochastic recursive equation

$$z_{n+1} = [z_n + y_{n+1} - x_{n+1}]^+, \quad n = 0, 1, \ldots$$

and is **P**-*a.e. finite.*

Proof: Fix $n = 1, 2, \ldots$. From the definition of $M_n(\cdot)$ we get

$$\begin{aligned} M_n(K) &= \sup_{0 \le s \le K} \left(Y_n^{\infty,-}(s) - X_n^{\infty,-}(s) \right) \\ &= \left[\max_{l=0,\ldots,nK-1} \left(\frac{1}{n} Y(-l, 0) - \frac{1}{n} X(-l, 0) \right) \right]^+, \quad K = 1, 2, \ldots, \end{aligned}$$

as we recall from Section 3.2 that

$$Y_n^{\infty,-}(s) - X_n^{\infty,-}(s) = \frac{1}{n} Y(1 - \lceil ns \rceil, 0) - \frac{1}{n} X(1 - \lceil ns \rceil, 0), \quad s \in [0, \infty).$$

Thus,

$$\lim_{K \to \infty} M_n(K) = \frac{1}{n} M_\infty, \tag{4.6}$$

where we have set

$$M_\infty \equiv \left[\max_{l=0,1,\ldots} \left(Y(-l, 0) - X(-l, 0) \right) \right]^+.$$

The bounds

$$M_n(\lfloor s \rfloor) \le M_n(s) \le M_n(\lceil s \rceil), \quad s \in [0, \infty)$$

are immediate by monotonicity of $M_n(\cdot)$, and (4.6) thus implies

$$\lim_{s \to \infty} M_n(s) = \frac{1}{n} M_\infty.$$

We note that M_∞ is Loynes' variable [3, p. 76], [10, 47], associated with the stochastic recursive equation

$$z_{n+1} = [z_n + y_n - x_n]^+, \; n = 0, 1, \ldots$$

where under Assumption **(E)**, the driving sequence $\{y_n - x_n, \; n = 1, 2, \ldots\}$ is stationary and metrically transitive with negative drift. By Theorem 3 in [47], the random variable M_∞ is **P**-a.e. finite, and the desired result is obtained. ■

Because M_∞ is a random variable on $(\Omega, \mathcal{F}, \mathbf{P})$, $\pi_t \circ M_n^\infty(\cdot) = \frac{1}{n} M_\infty$ is $\mathcal{F}/\mathcal{B}_{\mathbb{R}}$-measurable for all $t \ge 0$, and the arguments given in Lemma 3.13 establish $\mathcal{F}/\mathcal{B}_{D[0,1]}$-measurability of $M_n^\infty(\cdot)$ for each $n = 1, 2, \ldots$. In short, for each $n = 1, 2, \ldots$, $M_n^\infty(\cdot)$ is a random element in $(D[0, 1], \tau_0)$.

The last lemma reviews some properties of the infimum mapping, which are easily transposed to the supremum mapping; the proof can be found in Appendix A.7.

Lemma 4.5 **1.** *For each z in $D[a,b]$ (resp. $D_l[a,b]$), m_z belongs to $D[a,b]$ (resp. $D_l[a,b]$) and is monotone non-increasing. Moreover, if z is non-increasing, then $m_z = z$.*

2. *If z belongs to $AC_0[a,b]$, so does m_z.*

3. *If $m_z(t) < z(t)$ and z is continuous at t, then m_z is differentiable at t and $\dot{m}_z(t) = 0$.*

4. *If z is continuous, m_z is differentiable and $\dot{m}_z$ has isolated zeros, then $m_z = z$.*

4.3 A proof of Theorem 4.1 through a LDP in $D_l[0,\infty)$

In this section, we use the Contraction Principle to derive the LDP for the family $\{(X_n(\cdot), m_n(\cdot), M_n^\infty(\cdot)),\ n = 1,2,\ldots\}$ in $(D[0,1],\tau_0)^3$ from that for the family $\{(X_n(\cdot), Y_n(\cdot)),\ n = 1,2,\ldots\}$ in $(D[0,1],\tau_0)^2$. For the sake of clarity, the proof is divided into Lemmas.

We define the subset $AC_0^+[0,\infty)$ of $AC_0[0,\infty)$ of functions which are non-decreasing, so that each mapping in $AC_0^+[0,\infty)$ is non-negative.

We shall use the following notation: For each x in $D_l[0,\infty)$, let $M_x = M(x)$ be the supremum function and for each y in $D[0,1]$, let $m_y = m(y)$ be the infimum function, i.e., the mappings $M_x : [0,\infty) \to \mathbb{R}$ and $m_y : [0,1] \to \mathbb{R}$ are defined by

$$M_x(t) \equiv \sup_{0\le s\le t} x(s), \quad t \in [0,\infty)$$

and

$$m_y(t) \equiv \inf_{0\le s\le t} y(s), \quad t \in [0,1].$$

Lemma 4.6 *Under Assumptions* **(E)** *and* **(E1)**, *the family of random elements $\{(X_n(\cdot), m_n(\cdot), M_n(\cdot)),\ n = 1,2,\ldots\}$ satisfies the LDP in the product space $(D[0,1],\tau_0)^2 \times (D_l[0,\infty),\tau_0^e)$ with good rate function $I : D[0,1]^2 \times D_l[0,\infty) \to [0,\infty]$ given by*

$$I(\varphi_1,\varphi_2,\varphi_3) = \begin{cases} I_m(\varphi_1,\varphi_2) + I_M(\varphi_3) & \text{if } \varphi_1,\varphi_2 \in AC_0[0,1],\ \varphi_3 \in AC_0^+[0,\infty) \\ \infty & \text{otherwise} \end{cases} \tag{4.7}$$

with

$$I_m(\varphi_1,\varphi_2) \equiv \inf_{\psi\in AC_0[0,1]} \left\{ \int_0^1 r_{x,y}\left(\dot\varphi_1(t),\dot\psi(t)\right) dt : \varphi_2 = m_{\psi-\varphi_1} \right\} \tag{4.8}$$

and

$$I_M(\varphi_3) = \inf_{\psi_1,\psi_2 \in AC_0[0,\infty)} \left\{ \int_0^\infty r_{x,y}\left(\dot{\psi}_2(t), \dot{\psi}_2(t) + \dot{\psi}_1(t)\right) dt : \varphi_3 = M_{\psi_1} \right\}. \quad (4.9)$$

Proof: By Proposition 3.21, Assumptions **(E)** and **(E1)** translate into the existence of the LDP for the family $\{(X_n(\cdot), Y_n(\cdot)),\ n = 1, 2, \ldots\}$ (as an element of $D[0,1]^2$) in the two-dimensional space $(D[0,1]^2, \tau_0)$. Thus, by Corollary 3.20, the family $\{(X_n(\cdot), Y_n(\cdot), X_n^{\infty,-}(\cdot), Y_n^{\infty,-}(\cdot)),\ n = 1, 2, \ldots\}$ (as a random element in the space $D[0,1]^2 \times D_l[0,\infty)^2$) satisfies the LDP in $(D[0,1]^2, \tau_0) \times (D_l[0,\infty)^2, \tau_0)$, which in view of Proposition 3.22, finally yields the LDP in the product space $(D[0,1], \tau_0)^2 \times (D_l[0,\infty), \tau_0)^2$. The corresponding good rate function $I^{+,-}_{X,Y}$ is given by

$$I^{+,-}_{X,Y}(\psi_1, \psi_2, \psi_3, \psi_4) = \int_0^1 r_{x,y}\left(\dot{\psi}_1(t), \dot{\psi}_2(t)\right) dt + \int_0^\infty r_{x,y}\left(\dot{\psi}_3(t), \dot{\psi}_4(t)\right) dt, \quad (4.10)$$

if ψ_1, ψ_2 belong to $AC_0[0,1]$ and ψ_3, ψ_4 belong $AC_0[0,\infty)$, while

$$I^{+,-}_{X,Y}(\psi_1, \psi_2, \psi_3, \psi_4) = \infty \quad (4.11)$$

otherwise.

Next, consider the mapping $F : D[0,1]^2 \times D_l[0,\infty)^2 \to D[0,1]^2 \times D_l[0,\infty)$ defined by

$$F(z_1, z_2, z_3, z_4) \equiv (z_1, m_{z_2 - z_1}, M_{z_4 - z_3}).$$

By Lemmas 2.8 and 3.9, the two mappings $(z_1, z_2, z_3, z_4) \to (z_2 - z_1)$ and $(z_1, z_2, z_3, z_4) \to (z_4 - z_3)$ are Borel-measurable and S-continuous on $AC_0[0,1]^2 \times AC_0[0,\infty)^2$. Because the mapping $(z_1, z_2, z_3, z_4) \to z_1$ is trivially S-continuous (as a coordinate mapping), it follows from [27, p. 55] and Proposition I in [12, p. 44] that the mapping $(z_1, z_2, z_3, z_4) \to (z_1, z_2 - z_1, z_4 - z_3)$ is Borel-measurable and S-continuous on $AC_0[0,1]^2 \times AC_0[0,\infty)^2$.

The infimum and supremum mappings $z \to m_z$ and $z \to M_z$ being S-continuous on $D[0,1]$ and $D_l[0,\infty)$, respectively (Lemmas 2.10 and 3.10), we finally conclude by [27, p. 55] and Corollary I in [12, p. 44] that the mapping F is itself Borel-measurable and S-continuous on the space $AC_0[0,1]^2 \times AC_0[0,\infty)^2$ which contains the effective domain of the rate function $I^{+,-}_{X,Y}$.

Therefore, upon noting that

$$(X_n(\cdot), m_n(\cdot), M_n(\cdot)) = F\left(X_n(\cdot), Y_n(\cdot), X_n^{\infty,-}(\cdot), Y_n^{\infty,-}(\cdot)\right), \quad n = 1, 2, \ldots,$$

the LDP for $\{(X_n(\cdot), m_n(\cdot), M_n(\cdot)),\ n = 1, 2, \ldots\}$ in $(D[0,1], \tau_0)^2 \times (D_l[0,\infty), \tau_0^e)$ is obtained from that for $\{(X_n(\cdot), Y_n(\cdot), X_n^{\infty,-}(\cdot), Y_n^{\infty,-}(\cdot)),\ n = 1, 2, \ldots\}$ [through the Contraction Principle]. The associated good rate function $I : D[0,1]^2 \times D_l[0,\infty) \to [0,\infty]$ is given by

$$I(\varphi_1,\varphi_2,\varphi_3) \equiv \inf_{\substack{\psi_1,\psi_2\in D[0,1]\\ \psi_3,\psi_4\in D_l[0,\infty)}} \left\{ I^{+,-}_{X,Y}(\psi_1,\psi_2,\psi_3,\psi_4) : \begin{array}{l} \varphi_1 = \psi_1 \\ \varphi_2 = m_{\psi_2-\psi_1} \\ \varphi_3 = M_{\psi_4-\psi_3} \end{array} \right\} \quad (4.12)$$

for φ_1, φ_2 in $D[0,1]$ and φ_3 in $D_l[0,\infty)$.

From (4.10)-(4.11), separability in the variables (ψ_1,ψ_2) and (ψ_3,ψ_4) in the optimization (4.12), and the fact that m_z and M_z are absolutely continuous if z is itself absolutely continuous (Lemma 4.5), we readily obtain the simpler expression

$$I(\varphi_1,\varphi_2,\varphi_3) = \begin{cases} I_m(\varphi_1,\varphi_2) + I_M(\varphi_3) & \text{if } \varphi_1,\varphi_2 \in AC_0[0,1],\ \varphi_3 \in AC_0[0,\infty) \\ \infty & \text{otherwise} \end{cases}$$

with I_m as given by (4.8), and I_M given by

$$I_M(\varphi_3) = \inf_{\psi_3,\psi_4\in AC_0[0,\infty)} \left\{ \int_0^\infty r_{x,y}\left(\dot\psi_3(t),\dot\psi_4(t)\right) dt : \varphi_3 = M_{\psi_4-\psi_3} \right\}.$$

It is plain from a change of the dummy variables ψ_3,ψ_4 in this last expression that for all φ_3 in $AC_0[0,\infty)$,

$$I_M(\varphi_3) = \inf_{\psi_1,\psi_2\in AC_0[0,\infty)} \left\{ \int_0^\infty r_{x,y}\left(\dot\psi_2(t),\dot\psi_2(t)+\dot\psi_1(t)\right) dt : \varphi_3 = M_{\psi_1} \right\}.$$

Because M_{ψ_1} belongs to $AC_0^+[0,\infty)$ for ψ_1 in $AC_0[0,\infty)$ (Lemma 4.5), it is necessary for the constraint $\varphi_3 = M_{\psi_1}$ to be satisfied that φ_3 belongs also to $AC_0^+[0,\infty)$. Consequently, we find that $I_M(\varphi_3) = \infty$ for φ_3 not in $AC_0^+[0,\infty)$, while I_M is given by (4.9) for φ_3 in $AC_0^+[0,\infty)$. ∎

Because r_{y-x} is a good rate function, it is Lebesgue-measurable, and for any ψ_1 and ψ_2 in $AC_0[0,\infty)$, the inequality

$$\begin{aligned} \int_0^\infty r_{x,y}\left(\dot\psi_2(t),\dot\psi_2(t)+\dot\psi_1(t)\right) dt &\geq \int_0^\infty \inf_{x\in\mathrm{R}} r_{x,y}(x, x+\dot\psi_1(t))\, dt \\ &= \int_0^\infty r_{y-x}(\dot\psi_1(t))\, dt \end{aligned}$$

holds. Therefore, after taking the infimum over ψ_2, we obtain

$$\inf_{\psi_2\in AC_0[0,\infty)} \left\{ \int_0^\infty r_{x,y}\left(\dot\psi_2(t),\dot\psi_2(t) = \dot\psi_1(t)\right) dt \right\} \geq \int_0^\infty r_{y-x}(\dot\psi_1(t))\, dt$$

and the lower bound

$$I_M(\varphi_3) \;\geq\; \inf_{\psi_1 \in AC_0[0,\infty)} \left\{ \int_0^\infty r_{y-x}(\dot{\psi}_1(t))\, dt : \varphi_3 = M_{\psi_1} \right\} \tag{4.13}$$

is immediate.

However, we were unable to show the reverse inequality under the natural assumption that $r_{x,y}$ be convex. In particular, we were not able to show the existence of a measurable selection, which would have enabled us to show equality. In [16], the author uses the fact that the infimum can be taken inside the integral in a similar setup, but does not provide any justification. Nevertheless, as we will see later, this does not prevent us from explicitly computing J_M.

We eventually want to establish the LDP jointly for the family of random elements $\{(X_n(\cdot), m_n(\cdot), M_n^\infty(\cdot)),\ n = 1, 2, \ldots\}$, where for each $n = 1, 2, \ldots$, $M_n^\infty = \lim_{t\to\infty} M_n(t)$. In view of our results of Lemmas 4.4 and 4.6, a natural avenue to derive this LDP is to use the Contraction Principle together with the natural projection mapping at infinity, which maps an element of $D_l[0,\infty)$ into its limit as t goes to infinity. The problem is that we can only consider the projection mapping at infinity on the subspace $D_l^*[0,\infty)$ of functions which admit a (finite) limit as $t \to \infty$. Thus, we first need to establish the LDP for $\{(X_n(\cdot), m_n(\cdot), M_n(\cdot)),\ n = 1, 2, \ldots\}$ in the subspace $(D[0,1], \tau_0)^2 \times (D_l^*[0,\infty), \tau_0^e)$. Although the application of the Contraction Principle with the projection mapping at infinity will not require other assumptions that **(E)** and **(E1)**, we could only derive the LDP for $\{(X_n(\cdot), m_n(\cdot), M_n(\cdot)),\ n = 1, 2, \ldots\}$ in the subspace $(D[0,1], \tau_0)^2 \times (D_l^*[0,\infty), \tau_0^e)$ under additional assumptions on the good rate function r_{y-x}.

Lemma 4.7 *Let Assumptions* **(E)** *and* **(E1)** *hold and assume that r_{y-x} is convex with $\inf_{c>0} \frac{1}{c} r_{y-x}(c) > 0$.*

Then, the family of random elements $\{(X_n(\cdot), m_n(\cdot), M_n(\cdot)),\ n = 1, 2, \ldots\}$ satisfies the LDP in the product space $(D[0,1], \tau_0)^2 \times (D_l^*[0,\infty), \tau_0^e)$ with good rate function the restriction $I^* : D[0,1]^2 \times D_l^*[0,\infty) \to [0,\infty]$ of the rate function I given in Lemma 4.6.

Proof: By Lemma 4.6, the family $\{(X_n(\cdot), m_n(\cdot), M_n(\cdot)),\ n = 1, 2, \ldots\}$ satisfies the LDP in $(D[0,1], \tau_0)^2 \times (D_l^*[0,\infty), \tau_0^e)$ with good rate function I given by (4.7). By Lemma 4.4, we have the relation

$$\mathbf{P}\left[(X_n(\cdot), m_n(\cdot), M_n(\cdot)) \in D[0,1]^2 \times D_l^*[0,\infty) \right] \;=\; 1, \quad n = 1, 2, \ldots,$$

and the desired result will follow from Lemma 1.11 once we show that the effective domain $\mathcal{D}_I$ of the rate function I is contained in the subspace $D[0,1]^2 \times D_l^*[0,\infty)$.

To this end, let φ_1, φ_2 in $D[0,1]$, and φ_3 in $D_l[0,\infty]$ such that $I(\varphi_1, \varphi_2, \varphi_3) < \infty$. In view of the expression (4.7) of I and the fact that I_m is non-negative, we readily obtain that $I_M(\varphi_3) < \infty$. Thus, from the expression (4.9) of I_M, we already conclude that φ_3 necessarily belongs to the space $AC_0^+[0,\infty)$, and it remains to show that $\varphi_3(t)$ admits a limit as $t \to \infty$, or equivalently, that $\sup_{s \geq 0} \varphi_3(s) < \infty$.

From the lower bound (4.13), we immediately see that

$$\inf_{\psi_1 \in AC_0[0,\infty)} \left\{ \int_0^\infty r_{y-x}(\dot{\psi}_1(t))\, dt : \varphi_3 = M_{\psi_1} \right\} \;\leq\; I_M(\varphi_3) \;<\; \infty.$$

Now, pick $C > \inf_{\psi_1 \in AC_0[0,\infty)} \left\{ \int_0^\infty r_{y-x}(\dot{\psi}_1(t))\, dt : \varphi_3 = M_{\psi_1} \right\}$, so that, by definition of the infimum, there exists ψ^* in $AC_0[0,\infty)$ satisfying $\varphi_3 = M_{\psi^*}$ and

$$\int_0^\infty r_{y-x}(\dot{\psi}^*(t))\, dt \;\leq\; C \;<\; \infty. \tag{4.14}$$

Fix $s \geq 0$. By assumption $\theta^* > 0$, and as we clearly have $\theta^* c \leq r_{y-x}(c)$ for $c > 0$ by definition of θ^*, non-negativity of the rate function r_{y-x} finally yields

$$\theta^* c \;\leq\; r_{y-x}(c), \quad c \in \mathbb{R}. \tag{4.15}$$

Next, r_{y-x} being assumed convex, it is the pointwise supremum of the affine functions which are below it [57, Theorem 12.1 p. 102]. Therefore, (4.15) and non-negativity of the rate function r_{y-x} yield

$$\begin{aligned}
\int_0^s \theta^* \dot{\psi}^*(t)\, dt \;&\leq\; \int_0^s \sup\left\{ h\left(\dot{\psi}^*(t)\right) : h \text{ affine},\ h \leq r_{y-x} \right\} dt \\
&=\; \int_0^s r_{y-x}\left(\dot{\psi}^*(t)\right) dt \\
&\leq\; \int_0^\infty r_{y-x}\left(\dot{\psi}^*(t)\right) dt.
\end{aligned}$$

Finally, this last inequality combined with (4.14) and the fact that $\psi^*(0) = 0$ implies

$$\psi^*(s) \;\leq\; \frac{C}{\theta^*},$$

and we easily conclude that $\varphi_3 = M_{\psi^*}$ is bounded above, or equivalently, that $\lim_{t\to\infty} \varphi_3(t)$ exists and is finite. In short, φ_3 belongs to $D_l^*[0,\infty)$, and $\mathcal{D}_I \subset D[0,1]^2 \times D_l^*[0,\infty)$. As announced at the beginning of the proof, the desired result then follows from Lemma 1.11 (3). ∎

Equipped with this preliminary result, we can now prove Theorem 4.1.

Proof: (Theorem 4.1)

Existence of the LDP for $\{(X_n(\cdot), m_n(\cdot), M_n^\infty(\cdot)),\ n = 1, 2, \ldots\}$:

Under the assumptions of Theorem 4.1, Lemma 4.6 yields the LDP for the family $\{(X_n(\cdot), m_n(\cdot), M_n(\cdot)),\ n = 1, 2, \ldots\}$ in $(D[0,1], \tau_0)^2 \times (D_l^*[0,\infty), \tau_0^e)$. We write

$$(X_n(\cdot), m_n(\cdot), M_n^\infty(\cdot)) \;=\; F\left(X_n(\cdot), m_n(\cdot), M_n(\cdot)\right),$$

where the mapping $F : D[0,1]^2 \times D_l^*[0,\infty) \to D[0,1]^3$ is defined by

$$F(z_1, z_2, z_3) \;=\; (z_1, z_2, z_3^\infty)$$

and $z_3^\infty(t) \equiv \lim_{t\to\infty} z_3(t)$ for t in $[0,1]$.

We first argue that F is continuous in the desired product topology: By continuity of the identity mapping and Corollary I of [12, p. 44], it suffices to show that the coordinate mapping F_3 which assigns the constant function z_3^∞ to any function z_3 in $D_l^*[0,\infty)$ is S-continuous. Recall from Section 3.1.2 that $(D_l^*[0,\infty), h)$ is isometric (with isometry denoted by $\widehat{\Phi}$) to the subset $(D_l^*[0,1], d_0)$ of $(D_l[0,1], d_0)$, where

$$D_l^*[0,1] \;=\; \{x \in D_l[0,1] : x(0+) = x(0)\},$$

and that the metric h induces the relative Skorohod topology on the subspace $D_l^*[0,\infty)$ (Lemma 3.5). The isometry $\widehat{\Phi}$ is thus also continuous when $D_l^*[0,\infty)$ is equipped with the relative Skorohod topology, i.e., $\widehat{\Phi}$ is S-continuous. From the definition (3.6) of the isometry $\widehat{\Phi}$, we have

$$\begin{aligned} F_3(z_3) &= \lim_{t\to\infty} z_3(t) \\ &= \widehat{\Phi}(z_3)(0) \\ &= \pi_0^* \circ \widehat{\Phi}(z_3), \quad z_3 \in D_l^*[0,\infty), \end{aligned}$$

where the mapping $\pi_0^* : D_l^*[0,1] \to \mathbb{R}$ is defined by

$$\pi_0^*(x)(t) = x(0), \quad t \in [0,1], \quad x \in D_l^*[0,1].$$

The projection π_0^* is S-continuous as the restriction to $D_l^*[0,1]$ of the natural projection π_0 which was shown to be S-continuous in Lemma 2.4, and S-continuity of F_3 then follows from that of π_0^* and $\widehat{\Phi}$.

The LDP for $\{(X_n(\cdot), m_n(\cdot), M_n^\infty(\cdot)),\ n = 1, 2, \ldots\}$ is now easily obtained by the Contraction Principle with good rate function $J : D[0,1]^3 \to [0,\infty]$ given by

$$J(\varphi_1, \varphi_2, \varphi_3) \;=\; \inf_{\substack{\psi_1, \psi_2 \in D[0,1] \\ \psi_3 \in D_l^*[0,\infty)}} \left\{ I(\psi_1, \psi_2, \psi_3) : \begin{array}{l} \varphi_1 = \psi_1 \\ \varphi_2 = \psi_2 \\ \varphi_3 = \lim_{t\to\infty} \psi_3(t) \end{array} \right\} \qquad (4.16)$$

for $\varphi_1, \varphi_2, \varphi_3$ in $D[0,1]$ and with I given in Lemma 4.6.

It is plain from (4.16) that J is infinite if φ_3 is not constant, and after substituting in (4.16) the expression of I given in Lemma 4.6, we find

$$J(\varphi_1,\varphi_2,\varphi_3) = \begin{cases} J_m(\varphi_1,\varphi_2) + J_M(\varphi_3) & \text{if} \quad \varphi_1,\varphi_2 \in AC_0[0,1], \quad \varphi_3 \in \mathcal{C}_+ \\ \infty & \text{otherwise} \end{cases}$$

where $J_m = I_m$ is given by (4.8), and

$$J_M(c_A) = \inf_{\psi_3 \in AC_0^*[0,\infty)} \left\{ I_M(\psi_3) : A = \lim_{t\to\infty} \psi_3(t) \right\}, \quad c_A \in \mathcal{C}_+, \tag{4.17}$$

with $AC_0^*[0,\infty) \triangleq \{x \in AC_0[0,\infty) : \lim_{t\to\infty} x(t)$ exists and is finite$\}$ and I_M given by (4.9).

Computation of J_M:

We complete the proof by computing the rate function J_M given by (4.17). We begin with the case $A = 0$.

Under Assumptions **(E)** and **(E1)**, Proposition 4.3 yields $r_{x,y}(m_x, m_y) = 0$ with $m_x \equiv \mathbf{E}[x_1]$, $m_y \equiv \mathbf{E}[y_1]$, and $m_y < m_x$. By appropriately choosing the mappings ψ_1 and ψ_2 in the expression (4.9) of I_M, and ψ_3 in the expression (4.17) of J_M, we can derive an upper bound for $J_M(c_0)$. To this end, pick $\psi_3 = c_0$, and define the functions $\psi_1^0, \psi_2^0 : [0,\infty) \to \mathbb{R}$ by $\psi_1^0(t) = (m_y - m_x)\, t$ and $\psi_2^0(t) = m_x\, t$, for all $t \geq 0$. Clearly, ψ_1^0 and ψ_2^0 belong to $AC_0[0,\infty)$, and because $m_y < m_x$, we have $M_{\psi_1^0} = 0$, so that from the expression (4.9) of I_M we already obtain

$$0 \leq I_M(c_0) \leq \int_0^\infty r_{x,y}(m_x, m_y)\, dt = 0.$$

Next, from (4.17) (with $A = 0$), we get

$$0 \leq J_M(c_0) \leq I_M(c_0) = 0$$

and we conclude $J_M(c_0) = 0$.

We now fix $A > 0$ and proceed similarly to derive an upper bound for $J_M(c_A)$. We consider the collections $\{\psi_T^1,\ T > 0\}$ and $\{\psi_{T,x}^2,\ x \in \mathbb{R},\ T > 0\}$ of functions in $AC_0[0,\infty)$ defined by

$$\psi_T^1(t) = \begin{cases} \dfrac{A}{T}\, t, & 0 \leq t \leq T \\ (m_y - m_x)(t - T) + A, & t > T \end{cases}$$

and

$$\psi_{T,x}^2(t) = \begin{cases} x\, t, & 0 \leq t \leq T \\ m_x(t - T) + xT, & t > T \end{cases}$$

for all x in $\mathbb{R}$, and all $T > 0$. Finally, for each $T > 0$, define the mapping $\psi_T^3 : [0, \infty) \to \mathbb{R}$ by setting $\psi_T^3 \equiv M_{\psi_T^1}$.

Clearly, for any $T > 0$, ψ_T^3 belongs to $AC_0^*[0, \infty)$, and satisfies

$$\lim_{t\to\infty} \psi_T^3(t) \;=\; \sup_{s\geq 0} \psi_T^1(s) \;=\; A,$$

so that in view of (4.17), we get

$$J_M(c_A) \;\leq\; \inf_{T>0} I_M(\psi_T^3).$$

On the other hand, we have from (4.9), by taking $\psi_1 = \psi_T^1$ and $\psi_2 = \psi_{T,x}^2$, that

$$\begin{aligned}
I_M(\psi_T^3) &\leq \inf_{x\in\mathbb{R}} \int_0^\infty r_{x,y}\left(\dot\psi_{T,x}^2(t), \dot\psi_{T,x}^2(t) + \dot\psi_T^1(t)\right) dt \\
&= \inf_{x\in\mathbb{R}} \left(\int_0^T r_{x,y}(x, x + \frac{A}{T})\, dt + \int_T^\infty r_{x,y}(m_x, m_y)\, dt \right) \\
&= \inf_{x\in\mathbb{R}} T\, r_{x,y}(x, x + \frac{A}{T}) \\
&= T\, r_{y-x}(\frac{A}{T}), \quad T > 0
\end{aligned}$$

and we conclude

$$J_M(c_A) \;\leq\; \inf_{T>0} T\, r_{y-x}(\frac{A}{T}) \;=\; A \inf_{c>0} \frac{1}{c}\, r_{y-x}(c). \tag{4.18}$$

In order to obtain a matching lower bound, we first derive a general lower bound for J_M by using (4.13). Let $AC_0^1[0, \infty)$ denote the subspace of $AC_0[0, \infty)$ of functions which admit a finite supremum.

For any ψ_3 in $AC_0^*[0, \infty)$ satisfying $\lim_{t\to\infty} \psi_3(t) = A$ and any ψ_1 in $AC_0[0, \infty)$ with $\psi_3 = M_{\psi_1}$, there exists a ψ in $AC_0^1[0, \infty)$ ($\psi \equiv \psi_1$) such that $\sup_{s\geq 0} \psi(s) = \lim_{t\to\infty} \psi_3(t) = A$, and we immediately conclude that

$$\begin{aligned}
&\inf_{\psi\in AC_0^1[0,\infty)} \left\{ \int_0^\infty r_{y-x}(\dot\psi(t))\, dt : A = \sup_{s\geq 0} \psi(s) \right\} \\
&\leq \inf_{\substack{\psi_1\in AC_0[0,\infty) \\ \psi_3\in AC_0^*[0,\infty)}} \left\{ \int_0^\infty r_{y-x}(\dot\psi_1(t))\, dt : \psi_3 = M_{\psi_1},\ A = \lim_{t\to\infty} \psi_3(t) \right\} \\
&= \inf_{\psi_3\in AC_0^*[0,\infty)} \left\{ \inf_{\psi_1\in AC_0[0,\infty)} \left\{ \int_0^\infty r_{y-x}(\dot\psi_1(t))\, dt : \psi_3 = M_{\psi_1} \right\} : A = \lim_{t\to\infty} \psi_3(t) \right\} \\
&\leq \inf_{\psi_3\in AC_0^*[0,\infty)} \left\{ I_M(\psi_3) : A = \lim_{t\to\infty} \psi_3(t) \right\} \\
&= J_M(c_A)
\end{aligned} \tag{4.19}$$

where we have used the lower bound (4.13).

Pick now any function ψ in $AC_0^1[0,\infty)$ such that $\sup_{s\geq 0}\psi(s) = A$. From the definition of the supremum, there exists a sequence $\{s_n,\ n = 1,2,\ldots\}$ such that $\psi(s_n) \to A$. Moreover, because A is positive and $\psi(0) = 0$, this sequence $\{s_n,\ n = 1,2,\ldots\}$ can be chosen such that both s_n and $\psi(s_n)$ are positive for all $n = 1,2,\ldots$.

The rate function r_{y-x} being non-negative and convex by assumption, Jensen's inequality yields

$$\begin{aligned}
\int_0^\infty r_{y-x}(\dot\psi(t))\,dt &\geq \int_0^{s_n} r_{y-x}(\dot\psi(t))\,dt \\
&\geq s_n\, r_{y-x}(\frac{\psi(s_n)}{s_n}) \\
&= \psi(s_n)\,\frac{s_n}{\psi(s_n)}\,r_{y-x}(\frac{\psi(s_n)}{s_n}) \\
&\geq \psi(s_n)\,\inf_{n=1,2,\ldots}\frac{s_n}{\psi(s_n)}\,r_{y-x}(\frac{\psi(s_n)}{s_n}) \\
&\geq \psi(s_n)\,\inf_{c>0}\frac{1}{c}\,r_{y-x}(c), \quad n = 1,2,\ldots, \qquad (4.20)
\end{aligned}$$

and upon letting $n \to \infty$ in (4.20) we conclude that

$$\int_0^\infty r_{y-x}(\dot\psi(t))\,dt \geq A\,\inf_{c>0}\frac{1}{c}\,r_{y-x}(c),$$

for all ψ in $AC_0^1[0,\infty)$ satisfying $\sup_{s\geq 0}\psi(s) = A$. Recalling (4.19), we get

$$J_M(c_A) \geq \inf_{\psi\in AC_0^1[0,\infty)}\left\{\int_0^\infty r_{y-x}(\dot\psi(t))\,dt : c_A = \sup_{s\geq 0}\psi(s)\right\} \geq A\,\inf_{c>0}\frac{1}{c}\,r_{y-x}(c),$$

which combined with the upper bound (4.18) yields the desired expression for $J_M(c_A)$. ∎

4.4 Computation of the rate function J_m

In this section we focus on obtaining a closed form expression for J_m. Throughout, a.e. refers to almost everywhere with respect to the Lebesgue measure λ on $[0,1]$. Unfortunately, we were only able to compute explicitly the rate function J_m for a certain class of mappings.

Recall from Theorem 4.1 that for φ_1, φ_2 in $AC_0[0,1]$,

$$\begin{aligned} J_m(\varphi_1,\varphi_2) &= \inf_{\psi_1,\psi_2 \in AC_0[0,1]} \left\{ \int_0^1 r_{x,y}\left(\dot\psi_1(t), \dot\psi_2(t)\right) dt : \begin{array}{l} \varphi_1 = \psi_1 \\ \varphi_2 = m_{\psi_2-\psi_1} \end{array} \right\} \\ &= \inf_{\psi \in AC_0[0,1]} \left\{ \int_0^1 r_{x,y}\left(\dot\varphi_1(t), \dot\varphi_1(t) + \dot\psi(t)\right) dt : \varphi_2 = m_\psi \right\} \quad (4.21) \end{aligned}$$

where the last step is obtained through a change of variables, and $J_m(\varphi_1,\varphi_2) = \infty$ otherwise. Let $AC_0^-[0,1]$ denote the subset of $AC_0[0,1]$ of functions which are non-increasing, so that for any φ in $AC_0^-[0,1]$, we have $\dot\varphi \le 0$ a.e. and $\varphi \le 0$.

Before going any further, we note from the Contraction Principle that the mapping $r_{x,y-x} : \mathbb{R}^2 \to [0,\infty]$ defined by

$$r_{x,y-x}(c_1,c_2) \equiv r_{x,y}(c_1, c_1+c_2), \quad c_1, c_2 \in \mathbb{R}$$

is the good rate function associated with the LDP for $\{(x_n, y_n - x_n),\ n = 1,2,\ldots\}$. Furthermore, by Theorem 5.7 in [57, p. 38], the mapping $r_{x,y-x}$ is convex whenever $r_{x,y}$ is itself convex. The rate function J_m can then be expressed in the somewhat simpler way

$$J_m(\varphi_1,\varphi_2) = \inf_{\psi \in AC_0[0,1]} \left\{ \int_0^1 r_{x,y-x}\left(\dot\varphi_1(t), \dot\psi(t)\right) dt : \varphi_2 = m_\psi \right\}, \quad (4.22)$$

The mapping m_z being non-increasing for any z, we already conclude from (4.21) that

$$J_m(\varphi_1,\varphi_2) = \infty \quad \text{if} \quad \varphi_1 \notin AC_0[0,1] \quad \text{or} \quad \varphi_2 \notin AC_0^-[0,1]. \quad (4.23)$$

Next, by Lemma 4.5 (**1**), for all φ_2 in $AC_0^-[0,1]$, there exists ψ in $AC_0[0,1]$ (e.g., $\psi = \varphi_2$) such that $\varphi_2 = m_\psi$, and we already conclude that

$$J_m(\varphi_1,\varphi_2) \le \int_0^1 r_{x,y}\left(\dot\varphi_1(t), \dot\varphi_1(t) + \dot\varphi_2(t)\right) dt, \quad \begin{array}{l} \varphi_1 \in AC_0[0,1] \\ \varphi_2 \in AC_0^-[0,1] \end{array}. \quad (4.24)$$

We now prove that we have equality above for the class of mappings φ_2 satisfying $\dot\varphi_2 < 0$ a.e.. Indeed, if $\dot\varphi_2 < 0$ a.e., then there exists a subset T of $[0,1]$ such that $\lambda(T^c) = 0$ and φ_2 is differentiable on T with $\dot\varphi_2(t) < 0$ for all t in T. Let ψ in $AC_0[0,1]$ such that $m_\psi = \varphi_2$. Then for all t in T, we have $\dot m_\psi(t) < 0$, and it follows by Lemma 4.5 (**3**) that $\dot\psi(t) = \dot m_\psi(t) = \dot\varphi_2(t)$ for all t in T. Thus, since $\lambda(T^c) = 0$, we finally obtain

$$\begin{aligned} \int_0^1 r_{x,y}\left(\dot\varphi_1(t), \dot\varphi_1(t) + \dot\psi(t)\right) dt &= \int_T r_{x,y}\left(\dot\varphi_1(t), \dot\varphi_1(t) + \dot\psi(t)\right) dt \\ &= \int_T r_{x,y}\left(\dot\varphi_1(t), \dot\varphi_1(t) + \dot\varphi_2(t)\right) dt \\ &= \int_0^1 r_{x,y}\left(\dot\varphi_1(t), \dot\varphi_1(t) + \dot\varphi_2(t)\right) dt, \end{aligned}$$

for any ψ in $AC_0[0,1]$ with $m_\psi = \varphi_2$, and the infimum in (4.21) is therefore achieved by $\psi = \varphi_2$, i.e.,

$$J_m(\varphi_1, \varphi_2) = \int_0^1 r_{x,y}\left(\dot\varphi_1(t), \dot\varphi_1(t) + \dot\varphi_2(t)\right) dt, \quad \begin{array}{l} \varphi_1 \in AC_0[0,1] \\ \varphi_2 \in AC_0^-[0,1], \quad \dot\varphi_2 < 0 \text{ a.e.} \end{array}$$

We were unsuccessful in establishing equality in (4.24) for all φ_2 in $AC_0^-[0,1]$, even under the assumption that $r_{x,y}$ be convex.

4.5 LDP for other functionals of the inputs

In this Section, we present simple Corollaries to Theorem 4.1.

Corollary 4.8 *Let Assumptions* **(E)** *and* **(E1)** *hold and assume that r_{y-x} is convex with $\inf_{c>0} \frac{1}{c} r_{y-x}(c) > 0$.*

Then, the family $\{(X_n(\cdot), m_n(\cdot)),\ n = 1,2,\ldots\}$ satisfy the LDP in $(D[0,1], \tau_0)^2$ with good rate function $I_{X,m} : D[0,1]^2 \to [0,\infty]$ given by

$$I_{X,m}(\varphi_1, \varphi_2) = \begin{cases} \int_0^1 r_{x,y}\left(\dot\varphi_1(t), \dot\varphi_1(t) + \dot\varphi_2(t)\right) dt & \text{if } \varphi_1, \varphi_2 \in AC_0[0,1] \\ \infty & \text{otherwise} \end{cases}. \tag{4.25}$$

Proof: Under the enforced assumptions, the desired LDP for the sequence $\{(X_n(\cdot), m_n(\cdot)),\ n = 1,2,\ldots\}$ is a simple consequence of Theorem 4.1 and the Contraction Principle. The associated good rate function $I_{X,m} : D[0,1]^2 \to [0,\infty]$ is given by

$$I_{X,m}(\varphi_1, \varphi_2) = \inf_{\psi_1,\psi_2,\psi_3 \in D[0,1]} \left\{ J(\psi_1,\psi_2,\psi_3) : \begin{array}{c} \varphi_1 = \psi_1 \\ \varphi_2 = \psi_2 \end{array} \right\}, \tag{4.26}$$

with J given by (4.3).

Thus, from the expression of J, we easily find that $I_{X,m}(\varphi_1,\varphi_2) = \infty$ if either φ_1 or φ_2 does not belong to $AC_0[0,1]$. Finally, for φ_1, φ_2 in $AC_0[0,1]$, we readily obtain from (4.5), (4.4) and (4.26) that

$$\begin{aligned} I_{X,m}(\varphi_1,\varphi_2) &= J_m(\varphi_1,\varphi_2) + \inf_{A\ge 0}\left(A \inf_{c>0} \frac{1}{c} r_{y-x}(c)\right) \\ &= J_m(\varphi_1,\varphi_2). \end{aligned}$$

■

Corollary 4.9 *Let Assumptions* **(E)** *and* **(E1)** *hold and assume that r_{y-x} is convex with $\inf_{c>0} \frac{1}{c} r_{y-x}(c) > 0$.*

Then, the family $\{M_n^\infty(\cdot),\ n = 1, 2, \ldots\}$ satisfy the LDP in $(D[0,1], \tau_0)$ with good rate function $I_M : D[0,1] \to [0, \infty]$ given by

$$I_M(\varphi) = \begin{cases} A \inf_{c>0} \frac{1}{c} r_{y-x}(c) & \text{if } \varphi = c_A \in \mathcal{C}_+, \\ \infty & \text{otherwise} \end{cases}. \tag{4.27}$$

Proof: Under the enforced assumptions, the LDP for $\{M_n^\infty(\cdot),\ n = 1, 2, \ldots\}$ follows easily from Theorem 4.1 and the Contraction Principle. The associated good rate function $I_\infty : D[0,1] \to [0, \infty]$ is given by

$$\begin{aligned} I_\infty(\varphi) &= \inf_{\substack{\psi_i \in D[0,1] \\ i=1,2,3}} \{I_m(\psi_1, \psi_2) + I_M(\psi_3) : \varphi = \psi_3) \\ &= \begin{cases} \inf_{\psi_1, \psi_2 \in D[0,1]} I_m(\psi_1, \psi_2) + A \inf_{c>0} \frac{1}{c} r_{y-x}(c) & \text{if } \varphi = c_A \in \mathcal{C}_+, \\ \infty & \text{otherwise} \end{cases} \\ &= \begin{cases} A \inf_{c>0} \frac{1}{c} r_{y-x}(c) & \text{if } \varphi = c_A \in \mathcal{C}_+, \\ \infty & \text{otherwise} \end{cases}, \end{aligned}$$

where the last step follows from the fact that I_m is itself a good rate function (Corollary 4.8) and Lemma 1.3. ■

In the next Chapters, we apply Theorem 4.1 and Corollaries 4.8 and 4.9 to characterize the large deviations behavior of Lindley processes and **G/G/1** queues.

Chapter 5

Asymptotics and Large Deviations of Lindley Processes

In this chapter, building upon our earlier results, we study the large deviations asymptotics of a class of processes very useful in queueing theory, namely Lindley processes.

We consider a driving sequence satisfying the sample path LDP, and study its associated Lindley process, in particular its asymptotics. Under the usual stability conditions on the driving sequence, we show that the stationary version of the Lindley process satisfies the LDP in the function space $D[0,1]$, which in turn yields large deviations asymptotics for its steady-state.

The chapter is organized as follows: Lindley processes are introduced and their stability reviewed in Section 5.1, while the asymptotics are derived in Section 5.2 where we emphasize the difference between the traditional derivation and our sample path approach. Finally, we show in Section 5.3 that a LDP for the steady-state Lindley process can be obtained directly from asymptotics of a special form.

5.1 Lindley processes

We assume all random variables to be defined on the common, possibly enlarged probability space $(\Omega, \mathcal{F}, \mathbf{P})$.

If $\{\xi_{t+1},\ t = 0, 1, \ldots\}$ is a sequence of $\mathbb{R}$-valued random variables, then the corresponding Lindley process is the sequence of $\mathbb{R}$-valued random variables $\{x_t,\ t = 0, 1, \ldots\}$ generated through the recursion

$$x_0 = x; \qquad x_{t+1} = [x_t + \xi_{t+1}]^+, \quad t = 0, 1, \ldots, \tag{5.1}$$

where the initial condition x is some $\mathbb{R}_+$-valued random variable. Such processes naturally arise in the study of queueing systems, in particular single-server queues. The driving sequence ξ_{t+1} is then a simple function of the arrivals and service sequences, and x_t represent the queue length or buffer content. In that context, the first question which comes to mind is the existence of a steady-state or stationary regime for the system under study, which we discuss next.

5.1.1 Stability

It is well known [3, 10, 15, 47] that if the driving sequence $\{\xi_{t+1},\ t = 0, 1, \ldots\}$ is (strictly) stationary and metrically transitive with $\mathbf{E}[\xi_0] < 0$, then the system is stable in the sense that $x_t \Longrightarrow_t x_\infty$ for some $\mathbf{P}$-a.e. finite $\mathbb{R}_+$-valued random variable x_∞. This result is originally due to Loynes [47] and has since then be generalized to driving sequences which are asymptotically stationary in some sense.

In order to state the general stability results, we need to introduce some additional notation. Let ξ denote the driving sequence $\{\xi_{t+1},\ t = 0, 1, \ldots\}$, and for each $n = 1, 2, \ldots$, let ξ^n denote the sequence $\{\xi_{t+1+n},\ t = 0, 1, \ldots\}$. Because ξ_t takes on values in $\mathbb{R}$ which is separable, the random sequences ξ and ξ^n can be seen as random elements in $\mathbb{R}^{\mathrm{N}}$, the one-sided infinite Cartesian product of copies of $\mathbb{R}$. As such, ξ and for each $n = 1, 2, \ldots$, ξ^n admit a distribution, which we denote respectively by $\mathcal{L}(\xi)$ and $\mathcal{L}(\xi^n)$.

Borovkov [10, p. 13] noticed that under some additional natural assumptions, it suffices that $\mathcal{L}(\xi^n)$ weakly converges to the distribution $\mathcal{L}(\xi^0)$ of some stationary and metrically transitive random element $\xi^0 \equiv \{\xi^0_{t+1},\ t = 0, 1, \ldots\}$ in $\mathbb{R}^{\mathrm{N}}$ in order for the queue to be stable.

This result was further generalized by Szczotka [62, 63]: For each $n = 1, 2, \ldots$, let x^n be the sequence generated by the recursion (5.1) with driving sequence ξ^n and initial condition x. Szczotka [62] proved under some natural assumptions that if the driving sequence ξ^n is such that the sequence of distributions $\mathcal{L}(\xi^n)$ converges in one of six ways, then the sequence of distributions $\{\mathcal{L}(\mathrm{x}^n, \xi^n),\ n = 1, 2, \ldots\}$ converges in the same way, independently of the initial condition. The six ways of convergence considered in [62] are weak convergence, weak convergence in the mean, strong convergence, strong convergence in the mean, convergence in variation, and convergence in variation in mean.

We note that the limit (in any of the six ways of convergence mentioned earlier) of the sequence of distributions $\{\mathcal{L}(\xi^n),\ n = 1, 2, \ldots\}$ is necessarily stationary. Thus, following [62], we refer to the limiting distribution as the stationary version of the sequence $\xi = \{\xi_{t+1},\ t = 0, 1, \ldots\}$. The sequence ξ (resp.

(x, ξ)) is said to be weakly asymptotically stationary, strongly asymptotically stationary, or asymptotically stationary in variation if the sequence of distributions $\{\mathcal{L}(\xi^n),\ n = 1, 2, \ldots\}$ (resp. $\{\mathcal{L}(\mathrm{x}^n, \xi^n),\ n = 1, 2, \ldots\}$) respectively weakly converges, strongly converges, or converges in variation. If $\mathcal{L}(\mathrm{x}^0, \xi^0)$ denotes the limiting distribution, then (x^0, ξ^0) is also called the stationary version of (x, ξ).

To help understand the assumptions required in the stability results obtained by Szczotka, we develop a representation of the sequence $\{x_t,\ t = 1, 2, \ldots\}$ generated by the stochastic recursive equation (5.1) in terms of the input sequence $\{\xi_{t+1},\ t = 0, 1, \ldots\}$. Recall that for each s, t in $\mathbb{Z}$, $\Xi(s,t)$ and $\Xi^0(s,t)$ denote respectively the random variables

$$\Xi(s,t) = \begin{cases} \sum_{i=s}^{t} \xi_i & \text{if } s \le t \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \Xi^0(s,t) = \begin{cases} \sum_{i=s}^{t} \xi_i^0 & \text{if } s \le t \\ 0 & \text{otherwise} . \end{cases}$$

Upon iterating (5.1) it is easy to see that the relation

$$x_t = \max\left\{0\,,\, x + \Xi(1,t),\, \max_{s=2,\ldots,t} \Xi(s,t)\right\}, \quad t = 0, 1, \ldots$$

holds.

Recall that a one-sided stationary sequence can always be embedded into a bi-infinite sequence, possibly by enlarging the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ [26, p. 456]. **We shall use this fact throughout without further reference.**

Theorem 5.1 (Szczotka [62]) *Let x be a $\mathbb{R}_+$-valued random variable, and assume that the driving sequence $\xi = \{\xi_{t+1},\ t = 0, 1, \ldots\}$ satisfies*

$$\lim_{k\to\infty} \lim_{n\to\infty} \mathbf{P}\left[\max\left\{x + \Xi(1,n)\,,\, \max_{k \le j \le n} \Xi(n-j+1, n)\right\} > 0\right] = 0, \qquad (5.2)$$

and is either (i) weakly asymptotically stationary, (ii) strongly asymptotically stationary, or (iii) asymptotically stationary in variation. Assume further that in all cases its stationary version $\xi^0 \equiv \{\xi_t^0,\ t = 0, \pm 1, \pm 2, \ldots\}$ is such that $\Xi^0(-t, 0) \to -\infty$ $\mathbf{P}$-a.e. as $t \to \infty$.

Then, under the initial condition x, the joint sequence (x, ξ) is respectively weakly asymptotically stationary, strongly asymptotically stationary, or asymptotically stationary in variation. Furthermore, in all cases, its stationary version (x^0, ξ^0) is independent of the initial condition x and is given by

$$\left(\mathrm{x}^0, \xi^0\right) \equiv \left\{\left(x_{t+1}^0, \xi_{t+1}^0\right),\ t = 0, 1, \ldots\right\}$$

where

$$x_0^0 = \left[\max_{t=0,1,\dots} \Xi^0(-t,0)\right]^+,$$
$$x_{t+1}^0 = \left[x_t^0 + \xi_{t+1}^0\right]^+, \quad t = 0,1,\dots .$$

Proof: This result is given as Theorems 1a, 2a and 3a in [62], with a weaker condition that (5.2) in cases (ii) and (iii). ■

Theorem 5.1 holds true [62] when the sequence $\xi = \{\xi_{t+1},\ t = 0,1,\dots\}$ is (iv) weakly asymptotically stationary in mean, (v) strongly asymptotically stationary in mean, or (vi) asymptotically stationary in variation in mean.

In case (ii), (iii), (v) and (vi) of convergence, condition (5.2) can be replaced by ergodicity of the stationary version.

Corollary 5.2 (Szczotka [62]) *Let the driving sequence $\xi = \{\xi_{t+1},\ t = 0,1,\dots\}$ be either strongly asymptotically stationary, strongly asymptotically stationary in mean, asymptotically stationary in variation, or asymptotically stationary in variation in mean, and let its stationary version $\xi^0 \equiv \{\xi_t^0,\ t = 0,\pm1,\pm2,\dots\}$ be ergodic with $\mathbf{E}[\xi_0^0] < 0$. Then the assertions of Theorem 5.1 hold.*

We note that Loynes' original result is easily seen to be a consequence of this last Corollary.

Similar results hold, in the context of continuous-time single-server queues, for the sequences of distributions $\{\mathcal{L}(\mathrm{l}^n, \mathrm{x}^n, \xi^n),\ n = 1,2,\dots\}$, where for each $n = 1,2,\dots$, l^n is the sequence of number of units in the system generated by the input sequence ξ^n [63].

In the context of single-server queues, it is not clear from Szczotka's results that the assumptions made on the input sequence carry over to the output sequence. This will lead us to consider stability of Lindley's processes under stronger assumptions for which this requirement is satisfied, namely that the driving sequence couples with its stationary and metrically transitive version.

Definition 5.3 *Two random sequences $\{y_t,\ t = 0,1,\dots\}$ and $\{z_t,\ t = 0,1,\dots\}$ defined on the same probability space $(\Omega, \mathcal{F}, \mathbf{P})$ are said to couple if there exists an $\mathbb{N}$-valued random variable α on $(\Omega, \mathcal{F}, \mathbf{P})$, such that*

$$y_{\alpha+t} = z_{\alpha+t}, \quad t = 0,1,\dots .$$

The random variable α is called the coupling time.

For more details on the method of coupling, the reader is referred to the monograph [46]. It is immediate from the definition that if two or finitely many $\mathbb{R}^p$-valued random sequences couple, then they jointly couple with coupling time the maximum of their coupling time. In the sequel, we use this elementary fact without further reference.

We note that if the process $\{z_{t+1},\ t = 0,1,\ldots\}$ couples with each of the stationary and metrically transitive processes $\{z^\star_{t+1},\ t=0,1,\ldots\}$ and $\{z^{\star\star}_{t+1},\ t = 0,1,\ldots\}$, then $\{z^\star_{t+1},\ t=0,1,\ldots\} =_{st} \{z^{\star\star}_{t+1},\ t=0,1,\ldots\}$. In other words, if the process $\{z_{t+1},\ t = 0,1,\ldots\}$ couples with a stationary and metrically transitive process $\{z^\star_{t+1},\ t=0,1,\ldots\}$, then the latter is essentially unique within the class of stationary and metrically transitive processes, and $\{z^\star_{t+1},\ t=0,1,\ldots\}$ can be viewed as the stationary and metrically transitive version of $\{z_{t+1},\ t=0,1,\ldots\}$.

We then have the following result [3].

Proposition 5.4 *Assume the driving sequence $\{\xi_{t+1},\ t = 0,1,\ldots\}$ to couple with a stationary and metrically transitive sequence $\{\widehat{\xi}_{t+1},\ t=0,1,\ldots\}$. Extend the latter to the bi-infinite stationary and metrically transitive sequence $\{\widehat{\xi}_t,\ t = 0,\pm1,\pm2,\ldots\}$. If $\mathbf{E}\left[\widehat{\xi}_1\right] < 0$, then the system is stable in the sense that $x_t \Longrightarrow_t x_\infty$ for some $\mathbb{R}_+$-valued random variable x_∞. Furthermore, for any $\mathbb{R}_+$-valued initial condition x, the sequence $\{(\xi_{t+1}, x_{t+1}),\ t = 0,1,\ldots\}$ couples with the stationary and metrically transitive sequence $\{(\widehat{\xi}_{t+1}, \widehat{x}_{t+1}),\ t=0,1,\ldots\}$, where $\{\widehat{x}_{t+1},\ t=0,1,\ldots\}$ is defined by*

$$\widehat{x}_0 = \widehat{x}_{st}; \qquad \widehat{x}_{t+1} = \left[\widehat{x}_t + \widehat{\xi}_{t+1}\right]^+, \quad t=0,1,\ldots, \tag{5.3}$$

with

$$\widehat{x}_{st} = \left[\max_{t=0,1,\ldots} \widehat{\Xi}(-t,0)\right]^+ =_{st} x_\infty. \tag{5.4}$$

Proof: This result is shown in [3] in the θ_t framework.

Joint stationarity and metric transitivity of $\{(\widehat{\xi}_{t+1}, \widehat{x}_{t+1}),\ t=0,1,\ldots\}$ follows from that of $\{\widehat{\xi}_{t+1},\ t=0,1,\ldots\}$, the definition of $\{\widehat{x}_t,\ t=0,1,\ldots\}$ and Lemma A.1.2.7 in [15, p. 303].

Next, by assumption $\{\xi_{t+1},\ t = 0,1,\ldots\}$ couples with $\{\widehat{\xi}_{t+1},\ t = 0,1,\ldots\}$. Also, from (4.2.6) in [3, p. 91], any two solutions (with **P**-a.e. finite and non-negative initial condition) of a stable stochastic recursive equation couple. Thus, for any $\mathbb{R}_+$-valued initial condition x, the sequence $\{x_{t+1},\ t = 0,1,\ldots\}$ couples with the stationary and metrically transitive sequence $\{\widehat{x}_{t+1},\ t = 0,1,\ldots\}$ as defined in the statement of the Proposition. Therefore, the joint sequence

$\{(\xi_{t+1}, x_{t+1}),\ t = 0,1,\ldots\}$ couples with $\{(\widehat{\xi}_{t+1}, \widehat{x}_{t+1}),\ t = 0,1,\ldots\}$ with coupling time the maximum of the two coupling times of $\{\xi_{t+1},\ t = 0,1,\ldots\}$ and $\{x_{t+1},\ t = 0,1,\ldots\}$.

Moreover, by (4.1.1) in [3, p. 87], the sequence $\{\{x_{t+1+n},\ t = 0,1,\ldots\},\ n = 1,2,\ldots\}$ converges in total variation norm to the stationary sequence $\{\widehat{x}_{t+1},\ t = 0,1,\ldots\}$ as n tends to infinity. Since convergence in total variation norm implies weak convergence, we conclude that $x_t \Longrightarrow_t x_\infty$ with $x_\infty =_{st} \widehat{x}_{st}$. ■

In view of the coupling inequality [46, p. 12], it is plain that the coupling assumption is stronger that the assumption that the driving sequences $\xi^n \equiv \{\xi_{t+1+n},\ t = 0,1,\ldots\}$ converges in variation to its stationary version as $n \to \infty$.

With those general stability results in mind, we are now ready to make the operational assumptions required to characterize the asymptotics and the large deviations of the process.

5.1.2 Assumptions

Because the results derived in this chapter will eventually be applied in the context of series of queues, it seems natural to require the stability assumptions to propagate from the input to the queue to its output. In other words, in order for each queue to be stable, we would need the output process to be asymptotically stationary in some sense. As previously noted, it is unclear to us how to use Szczotka's results to fulfill this requirement. Therefore, we only consider input sequences which couple with their stationary version, as this assumption will be shown in Chapter 6 to carry over to the output sequence.

Assumption (L1) *The random sequence $\{\xi_{t+1},\ t = 0,1,\ldots\}$ couples with a stationary and metrically transitive sequence $\{\widehat{\xi}_{t+1},\ t = 0,1,\ldots\}$ satisfying* $\mathbf{E}\left[\widehat{\xi}_1\right] < 0$.

Under the assumption made above, the Lindley process defined by (5.1) admits a stationary regime, in the sense that Proposition 5.4 is satisfied.

When studying the asymptotics for Loynes variable x_∞, the results of Chapter 4 will be useful. This leads us to consider the following assumption.

Assumption (L2) *The family of partial sum processes $\{\widehat{\Xi}_n(\cdot),\ n = 1,2,\ldots\}$ associated with the sequence $\{\widehat{\xi}_{t+1},\ t = 0,1,\ldots\}$ satisfies the LDP in $(D[0,1], \tau_0)$*

with (convex) good rate function $I_\Xi : D[0,1] \to [0,\infty]$ *given by*

$$I_\Xi(\varphi) = \begin{cases} \int_0^1 r_\xi(\dot{\varphi})\, dt & \text{if } \varphi \in AC_0[0,1] \\ \infty & \text{otherwise} \end{cases} \tag{5.5}$$

where $r_\xi : \mathbb{R} \to [0,\infty]$ *is the convex good rate function associated with the LDP for the sample mean sequence* $\{\widehat{\Xi}_n(1),\ n = 1,2,\ldots\}$. *Moreover, we assume* $\inf_{y>0} \frac{1}{y} r_\xi(y) > 0$.

5.2 Asymptotics for Lindley processes

In this section, we focus on deriving under suitable assumptions the asymptotics for the tail probabilities $\mathbf{P}[x_\infty > b]$.

To help understand the assumptions made in the traditional derivation of the asymptotics, we first review sufficient conditions for a sequence of probability measures to satisfy the LDP in $\mathbb{R}^p$.

5.2.1 Conditions of existence of a LDP in $\mathbb{R}^p$

We start with some definitions and notation.

Consider a sequence of independent identically distributed $\mathbb{R}^d$-valued random variables $\{X_n,\ n = 1,2,\ldots\}$ with common distribution law μ. The **logarithmic moment generating function** associated with μ is defined as

$$\begin{aligned} \Lambda(\theta) &\triangleq \ln \mathbf{E}\left[e^{\langle\theta, X_1\rangle}\right] \\ &= \ln \int_{\mathbb{R}^d} e^{\langle\theta, x\rangle}\, \mu(dx), \quad \theta \in \mathbb{R}^d. \end{aligned} \tag{5.6}$$

The **Legendre-Fenchel transform** of a function $\Lambda : \mathbb{R}^d \to \mathbb{R} \cup \{\pm\infty\}$, is the function $\Lambda^* : \mathbb{R}^d \to \mathbb{R} \cup \{\pm\infty\}$, defined by

$$\Lambda^*(x) \triangleq \sup_{\theta \in \mathbb{R}^d} \left(\langle\theta, x\rangle - \Lambda(\theta)\right), \quad x \in \mathbb{R}^d. \tag{5.7}$$

More details on the properties of the Legendre-Fenchel transform can be found in [57]. Finally, define the empirical mean sequence $\{S_n,\ n = 1,2,\ldots\}$ by

$$S_n \equiv \frac{1}{n}(X_1 + \ldots + X_n), \quad n = 1,2,\ldots .$$

The most famous result in the theory of large deviations is probably the following Theorem due to Cramér [24, p. 27].

Theorem 5.5 (Cramér) *Let $\{X_n,\ n = 1, 2, \ldots\}$ be an $\mathbb{R}$-valued iid random sequence. The empirical mean sequence $\{S_n,\ n = 1, 2, \ldots\}$ satisfies the LDP with convex rate function Λ^*, the Legendre-Fenchel transform of the logarithmic moment generating function Λ associated with X_1.*

There is an extension to Cramér's Theorem for the d-dimensional case.

Proposition 5.6 ([24, Corollary 6.1.6, p. 229]) *Let $\{X_n,\ n = 1, 2, \ldots\}$ be a sequence of $\mathbb{R}^d$-valued iid random variables. Then the empirical mean sequence $\{S_n,\ n = 1, 2, \ldots\}$ satisfies the weak LDP with the convex rate function Λ^*. Moreover, if 0 belongs to $\mathcal{D}_\Lambda^\circ$, then $\{S_n,\ n = 1, 2, \ldots\}$ satisfies the full LDP with convex good rate function Λ^*.*

The following existence theorem [24, p. 45] gives sufficient conditions for dependent random variables to satisfy the LDP. It is the Theorem often used in the derivation of the effective bandwidth through large deviations. A proper convex function $\Lambda : \mathbb{R} \to \mathbb{R}$ with domain D_Λ is said to be **essentially smooth** if

i) D_Λ° is non empty,

ii) Λ is differentiable on D_Λ°,

iii) Λ is steep, i.e. $\lim_{n\to\infty} |\nabla f(x_n)| = \infty$ for any sequence $\{x_n,\ n = 1, 2, \ldots\}$ in D_Λ° converging to a boundary point of D_Λ°.

Theorem 5.7 (Gärtner-Ellis) *Let $\{Z_n,\ n = 1, 2, \ldots\}$ be a sequence of $\mathbb{R}^d$-valued random with logarithmic moment generating function*

$$\Lambda_n(\theta) \triangleq \ln \mathbf{E}\left[e^{\langle\theta, Z_n\rangle}\right], \quad \theta \in \mathbb{R}^d, \quad n = 1, 2, \ldots.$$

Assume that

1. *for each θ in $\mathbb{R}^d$, the limit $\Lambda(\theta) \triangleq \lim_{n\to\infty} \frac{1}{n}\Lambda_n(n\theta)$ exists, possibly as an extended real number,*

2. *the origin belongs to the interior of $\mathcal{D}_\Lambda \triangleq \{\theta \in \mathbb{R}^d : \Lambda(\theta) < \infty\}$,*

3. *Λ is essentially smooth and lower semi-continuous.*

Then $\{Z_n,\ n = 1, 2, \ldots\}$ satisfies the full LDP in $\mathbb{R}^d$ with good rate function Λ^*, the Legendre-Fenchel transform of Λ.

With these theorems in mind, we now present the traditional approach to the derivation of the asymptotics for Lindley processes.

5.2.2 Traditional derivation

In the context of effective bandwidth, there has been recently considerable interest in estimating the tail probabilities $\mathbf{P}[x_\infty > b]$. These asymptotics made use of the representation (5.4), and rely on the existence of large deviations estimates for the driving sequence. The main result along this line is summarized below, and was obtained in various degrees of generality by several authors [16, 20, 22, 29, 35]. We state it here with a minimal set of assumptions, and for the sake of completeness we present a proof whose arguments follow those given in [29]. We write

$$\Lambda_t(\theta) \equiv \frac{1}{t} \ln \mathbf{E}\left[e^{\theta(\widehat{\xi}_1+\ldots+\widehat{\xi}_t)}\right], \quad \theta \in \mathbb{R} \tag{5.8}$$

for each $t = 1, 2, \ldots$.

Proposition 5.8 *Assume Assumption* **(L1)** *to hold and the sequence* $\{\widehat{\xi}_{t+1},\ t = 0, 1, \ldots\}$ *to satisfy the following conditions:*

1. *The limit* $\Lambda(\theta) \equiv \lim_{t\to\infty} \Lambda_t(\theta)$ *exists (possibly as an extended real number) for all* θ *in* $\mathbb{R}$*;*

2. *The set* $\Theta \equiv \{\theta > 0 : \Lambda(\theta) < 0\}$ *is non-empty, and* $\Lambda_t(\theta) < \infty$ *for all* θ *in* Θ *and* $t = 1, 2, \ldots$*;*

3. *The process* $\{t^{-1}\widehat{\Xi}(1,t),\ t = 1, 2, \ldots\}$ *satisfies the LDP lower bound with rate function* $r_\xi : \mathbb{R} \to [0, \infty]$.

Then

$$\liminf_{b\to\infty} \frac{1}{b} \ln \mathbf{P}[x_\infty > b] \geq -\inf_{y>0} \left(\frac{1}{y} \inf_{z>y} r_\xi(z)\right) \tag{5.9}$$

and

$$\limsup_{b\to\infty} \frac{1}{b} \ln \mathbf{P}[x_\infty > b] \leq -\sup\{\theta > 0 : \Lambda(\theta) < 0\}. \tag{5.10}$$

Proof: We begin with the lower bound. Fix $y > 0$. From the construction of Loynes variable, the stationarity of $\{\widehat{\xi}_{t+1},\ t = 0, 1, \ldots\}$, and Proposition 5.4, we have [47]

$$x_\infty =_{st} \left[\max_{t=0,1,\ldots} \widehat{\Xi}(-t, 0)\right]^+ =_{st} \left[\max_{t=0,1,\ldots} \widehat{\Xi}(0, t)\right]^+$$

so that

$$\begin{aligned}
\liminf_{b\to\infty} \frac{1}{b} \ln \mathbf{P}[x_\infty > b] &= \liminf_{b\to\infty} \frac{1}{b} \ln \mathbf{P}\left[\max_{t=0,1,\ldots} \widehat{\Xi}(0,t) > b\right] \\
&\geq \liminf_{b\to\infty} \frac{1}{b} \ln \mathbf{P}\left[\frac{1}{y}\widehat{\Xi}(0, \lceil \frac{b}{y} \rceil) > \frac{b}{y}\right]
\end{aligned}$$

$$
\begin{aligned}
&\geq \liminf_{b\to\infty} \frac{1}{b} \ln \mathbf{P}\left[\frac{1}{y} \widehat{\Xi}(0, \lceil \frac{b}{y} \rceil) > \lceil \frac{b}{y} \rceil \right] \\
&= \liminf_{b\to\infty} \frac{\lceil \frac{b}{y} \rceil}{b} \frac{1}{\lceil \frac{b}{y} \rceil} \ln \mathbf{P}\left[\frac{\widehat{\Xi}(0, \lceil \frac{b}{y} \rceil)}{\lceil \frac{b}{y} \rceil} > y \right] \\
&\geq \limsup_{b\to\infty} \frac{\lceil \frac{b}{y} \rceil}{b} \liminf_{b\to\infty} \frac{1}{\lceil \frac{b}{y} \rceil} \ln \mathbf{P}\left[\frac{\widehat{\Xi}(0, \lceil \frac{b}{y} \rceil)}{\lceil \frac{b}{y} \rceil} > y \right] \\
&= \frac{1}{y} \liminf_{n\to\infty} \frac{1}{n} \ln \mathbf{P}\left[\frac{\widehat{\Xi}(0, n)}{n} > y \right] \\
&\geq -\frac{1}{y} \inf_{z>y} r_\xi(z)
\end{aligned}
$$

where the last step follows from Assumption **(3)**. Because this last inequality holds for any $y > 0$, we finally obtain the lower bound

$$
\liminf_{b\to\infty} \frac{1}{b} \ln \mathbf{P}\left[x_\infty > b \right] \geq -\inf_{y>0} \left(\frac{1}{y} \inf_{z>y} r_\xi(z) \right).
$$

The upper bound requires more work. We begin with the easy inequality

$$
\begin{aligned}
\mathbf{P}\left[x_\infty > b \right] &= \mathbf{P}\left[\max_{t=0,1,\ldots} \widehat{\Xi}(0, t) > b \right] \\
&\leq \sum_{t=0}^{\infty} \mathbf{P}\left[\widehat{\Xi}(0, t) > b \right] \\
&= \sum_{t=0}^{n-1} \mathbf{P}\left[\widehat{\Xi}(0, t) > b \right] + \sum_{t=n}^{\infty} \mathbf{P}\left[\widehat{\Xi}(0, t) > b \right] \\
&\leq n \max_{t=0,\ldots,n-1} \mathbf{P}\left[\widehat{\Xi}(0, t) > b \right] + \sum_{t=n}^{\infty} \mathbf{P}\left[\widehat{\Xi}(0, t) > b \right] \quad (5.11)
\end{aligned}
$$

which holds for any $n = 1, 2, \ldots$.

Now fix θ in Θ, and let $\varepsilon > 0$ such that $\Lambda(\theta) + \varepsilon < 0$. [By Assumption **(2)**, Θ is non-empty]. Thus, from Assumptions **(1)** and **(2)**, there exists t_ε such that

$$
\Lambda_t(\theta) \leq \Lambda(\theta) + \varepsilon, \quad t = t_\varepsilon, t_\varepsilon + 1, \ldots .
$$

By Chebycheff inequality, we then get

$$
\begin{aligned}
\mathbf{P}\left[\widehat{\Xi}(0, t) > b \right] &\leq e^{-\theta b} \mathbf{E}\left[e^{\theta \widehat{\Xi}(0,t)} \right] \\
&\leq e^{-\theta b} e^{t \frac{1}{t} \ln \mathbf{E}\left[e^{\theta \widehat{\Xi}(0,t)} \right]} \\
&= e^{-\theta b} e^{t \Lambda_t(\theta)} \\
&\leq e^{-\theta b} e^{t(\Lambda(\theta)+\varepsilon)}, \quad t = t_\varepsilon, t_\varepsilon + 1, \ldots . \quad (5.12)
\end{aligned}
$$

Because $\Lambda(\theta)+\varepsilon<0$, it follows that

$$\begin{aligned}\sum_{t=t_\varepsilon}^{\infty} \mathbf{P}\left[\widehat{\Xi}(0,t)>b\right] &\leq e^{-\theta b}\sum_{t=t_\varepsilon}^{\infty} e^{t(\Lambda(\theta)+\varepsilon)} \\ &= e^{-\theta b}\,\frac{e^{t_\varepsilon(\Lambda(\theta)+\varepsilon)}}{1-e^{\Lambda(\theta)+\varepsilon}} \\ &= e^{-\theta b}\,\alpha \qquad (5.13)\end{aligned}$$

where we have set $\alpha \equiv \dfrac{e^{t_\varepsilon(\Lambda(\theta)+\varepsilon)}}{1-e^{\Lambda(\theta)+\varepsilon}} < \infty$.

By applying Chebycheff inequality to $\mathbf{P}\left[\widehat{\Xi}(0,t)>b\right]$ for $t=0,\ldots,t_\varepsilon$ as well, we obtain from (5.11) (with $n=t_\varepsilon$) and (5.13)

$$\begin{aligned}&\limsup_{b\to\infty}\frac{1}{b}\ln\,\mathbf{P}\left[x_\infty>b\right] \\ &\leq \left(\limsup_{b\to\infty}\frac{1}{b}\ln\left(t_\varepsilon e^{-\theta b}\max_{t=0,\ldots,t_\varepsilon-1}\mathbf{E}\left[e^{\theta\widehat{\Xi}(0,t)}\right]\right)\right)\vee\left(\limsup_{b\to\infty}\frac{1}{b}\ln\left(e^{-\theta b}\,\alpha\right)\right) \\ &= \left(-\theta+\limsup_{b\to\infty}\frac{1}{b}\ln\max_{t=0,\ldots,t_\varepsilon-1}\mathbf{E}\left[e^{\theta\widehat{\Xi}(0,t)}\right]\right)\vee(-\theta). \qquad (5.14)\end{aligned}$$

Therefore, upon noting from Assumption **(2)** that $\max_{t=0,\ldots,t_\varepsilon-1}\mathbf{E}\left[e^{\theta\widehat{\Xi}(0,t)}\right]$ is bounded, we can rewrite (5.14) as

$$\limsup_{b\to\infty}\frac{1}{b}\ln\,\mathbf{P}\left[x_\infty>b\right] \leq -\theta.$$

Because this last inequality holds for any θ in $\Theta=\{\theta>0:\Lambda(\theta)<0\}$, we finally obtain the upper bound

$$\limsup_{b\to\infty}\frac{1}{b}\ln\,\mathbf{P}\left[x_\infty>b\right] \leq -\sup\{\theta>0:\Lambda(\theta)<0\} \qquad (5.15)$$

which completes the proof. ■

We note that the lower bound is obtained trough the large deviations behavior of the sample mean sequence of the driving sequence $\{\widehat{\xi}_{t+1},\ t=0,1,\ldots\}$, while the upper bound only relies on the assumption relative to the logarithmic moment generating function Λ. In the literature, Proposition 5.8 is always presented with a stronger set of assumptions which ensures that the lower and upper bounds coincide. In particular, it is assumed that the sample mean sequence of $\{\widehat{\xi}_{t+1},\ t=0,1,\ldots\}$ satisfies the LDP with good rate function, the Legendre-Fenchel transform of Λ. As shown below, we then get

$$\inf_{y>0}\frac{1}{y}\left(\inf_{z>y}\Lambda^*(z)\right) = \inf_{y>0}\frac{\Lambda^*(y)}{y},$$

but this assumption is still not enough to ensure that

$$\sup\{\theta > 0 : \Lambda(\theta) < 0\} \;=\; \inf_{y>0} \frac{\Lambda^*(y)}{y}.$$

The next corollary, inspired from [22], gives conditions under which the equality above holds. In addition, it gives necessary and sufficient conditions on the statistics of the driving sequence $\{\widehat{\xi}_{t+1},\ t = 0, 1, \ldots\}$ for the asymptotics to be held below a certain level. This interesting result will be investigated in more details in Section 7.2.1, where it is used to derive effective bandwidths.

Corollary 5.9 *Assume Assumption* **(L1)** *to hold and the sequence* $\{\widehat{\xi}_{t+1},\ t = 0, 1, \ldots\}$ *to satisfy the following conditions:*

1. *The limit* $\Lambda(\theta) \equiv \lim_{t\to\infty} \Lambda_t(\theta)$ *exists (possibly as an extended real number) for all* θ *in* $\mathbb{R}$*;*
2. *The set* $\Theta \equiv \{\theta > 0 : \Lambda(\theta) < 0\}$ *is non-empty, and* $\Lambda_t(\theta) < \infty$ *for all* θ *in* Θ *and* $t = 1, 2, \ldots$*;*
3. *Either the process* $\{t^{-1}\widehat{\Xi}(1,t),\ t = 1, 2, \ldots\}$ *satisfies the LDP with strictly convex good rate function* Λ^**, the Legendre-Fenchel transform of* Λ*, or the conditions of the Gärtner-Ellis Theorem are satisfied.*

Then, the asymptotics

$$\lim_{b\to\infty} \frac{1}{b} \ln \mathbf{P}[x_\infty > b] \;=\; -\sup\{\theta > 0 : \Lambda(\theta) < 0\} \;=\; -\inf_{y>0} \frac{\Lambda^*(y)}{y} \tag{5.16}$$

hold and for all $\delta > 0$*, we have*

$$\lim_{b\to\infty} \frac{1}{b} \ln \mathbf{P}[x_\infty > b] \;\le\; \delta \quad \Longleftrightarrow \quad \Lambda(\delta) \;\le\; 0. \tag{5.17}$$

Proof: In view of Proposition 5.8 it suffices to show that the bounds (5.10) and (5.9) coincide in order to establish the limit (5.16). Because Λ^* is convex with $\Lambda^*(\,\mathbf{E}\left[\widehat{\xi}_1\right]\,) = \inf_{x\in\mathbb{R}} \Lambda^*(x) = 0$ and $\mathbf{E}\left[\widehat{\xi}_1\right] < 0$ (Lemma 4.2), we readily see that Λ^* is non-decreasing on $\mathbb{R}_+$. Therefore, we have

$$\inf_{z>y} \Lambda^*(z) \;=\; \Lambda^*(y^+), \quad y > 0$$

so that

$$\inf_{y>0} \frac{1}{y}\left(\inf_{z>y} \Lambda^*(z)\right) \;=\; \inf_{y>0} \frac{1}{y}\Lambda^*(y^+) \;=\; \inf_{y>0} \frac{1}{y}\Lambda^*(y).$$

The fact that the equality

$$\sup\{\theta > 0 : \Lambda(\theta) < 0\} \;=\; \inf_{y>0} \frac{\Lambda^*(y)}{y}$$

holds under the assumptions of the Corollary is shown in [22, Theorem 3.1], together with the equivalence (5.17). ■

In view of the Gärtner-Ellis Theorem, the assumptions of Corollary 5.9 will be in particular satisfied if:

i) The limit $\Lambda(\theta) \equiv \lim_{t\to\infty} \Lambda_t(\theta)$ exists (possibly as an extended real number) for all θ in $\mathbb{R}$;

ii) The origin belongs to the interior of $\mathcal{D}_\Lambda \equiv \{\theta \in \mathbb{R} : \Lambda(\theta) < \infty\}$;

iii) Λ is differentiable on $\mathcal{D}_\Lambda$ and steep;

iii) The set $\Theta \equiv \{\theta > 0 : \Lambda(\theta) < 0\}$ is non-empty, and $\Lambda_t(\theta) < \infty$ for all θ in Θ and $t = 1, 2, \ldots$.

This set of assumptions will not in general be easily checked. Furthermore, in the context of **G/G/1** queues in series, we do not know a priori if these assumptions will be satisfied by the output process of the queue, a requirement in order to obtain the buffer asymptotics of the next queue.

This suggests the use of the sample path LDP assumption on the input processes, as this assumption will be shown in Chapter 6 to propagate to the output process. In addition, we note that all the existing results on the LDP for the output process of a **G/G/1** queue do require some kind of sample path large deviations behavior [7, 16, 18, 20].

5.2.3 Sample path approach

For each $n = 1, 2, \ldots$, set $\widehat{x}_n(t) = \frac{\widehat{x}_{st}}{n}$ for all t in $[0, 1]$, with $\widehat{x}_{st}$ as given by (5.4).

In view of Lemma 4.4 and the proof of Theorem 4.1 presented in Chapter 4, it should be clear to the reader that Assumptions **(L1)** and **(L2)** yields a result similar to that of Corollary 4.9, and we then have the following theorem.

Recall that $\mathcal{C}_+$ denotes the subset of $D[0,1]$ consisting of all non-negative constant functions, and that c_A denotes its generic element, i.e., $c_A(t) = A$ for all t in $[0, 1]$.

Theorem 5.10 *Under Assumptions* **(L1)** *and* **(L2)**, *the sequence* $\{\widehat{x}_n(\cdot),\ n = 1, 2, \ldots\}$ *satisfies the LDP in* $(D[0,1], \tau_0)$ *and in* $(D[0,1], \tau_\infty)$ *with good rate function* $I_x : D[0,1] \to [0, \infty]$ *given by*

$$I_x(\varphi) = \begin{cases} A \inf_{y>0} \frac{1}{y} r_\xi(y) & \text{if } \varphi \equiv c_A \in \mathcal{C}_+, \\ \infty & \text{otherwise} \end{cases} \tag{5.18}$$

where r_{ξ} is the convex good rate function associated with the LDP of the sample mean $\{\hat{\Xi}_n(1),\ n=1,2,\ldots\}$.

The LDP for $\{\hat{x}_n(\cdot),\ n=1,2,\ldots\}$ in turn yields the buffer asymptotics. This is a consequence of the following general result.

Theorem 5.11 *Let x be a non-negative $\mathbb{R}^p$-valued random variable, and assume that the sequence $\{\frac{1}{n}x,\ n=1,2,\ldots\}$ satisfies the LDP in $(D[0,1],\tau_0)$ (or for that matter in $(D[0,1],\tau_\infty)$ with good rate function $I_x : D[0,1] \to \infty$ given by*

$$I_x(\varphi) = \begin{cases} A\,\theta^* & \text{if } \varphi = c_A \in \mathcal{C}_+ \\ \infty & \text{otherwise} \end{cases}$$

for some $\theta^* > 0$.

Then, we have

$$\lim_{b\to\infty} \frac{1}{b} \ln \mathbf{P}[\,x > b\,] = -\theta^*. \tag{5.19}$$

Proof: For $\delta > 0$ and φ in $C[0,1]$, let $\bar{B}_\delta(\varphi)$ denote the closed ball of center φ and radius δ in $(C[0,1],d_\infty)$, i.e., $\bar{B}_\delta(\varphi) \equiv \{\psi \in C[0,1] : d_\infty(\varphi,\psi) \le \delta\}$. Recall that the topology induced on $C[0,1]$ by the Skorohod topology τ_0 is the uniform topology. Thus, because D_{I_x} is a subset of $C[0,1]$ and $\mathbf{P}\left[\frac{1}{n}x \in C[0,1]\right] = 1$ for all $n=1,2,\ldots$, under the enforced assumptions, Lemma 1.11 **(3)** yields the LDP for the sequence $\{\frac{1}{n}x,\ n=1,2,\ldots\}$ in $(C[0,1],d_\infty)$.

We have

$$\limsup_{b\to\infty} \frac{1}{b} \ln \mathbf{P}[\,x > b\,] \le \limsup_{b\to\infty} \frac{\lfloor b\rfloor}{b} \frac{1}{\lfloor b\rfloor} \ln \mathbf{P}[\,x > \lfloor b\rfloor\,]\ .$$

Because $\ln \mathbf{P}[\,x > \lfloor b\rfloor\,] \le 0$, we obtain from this inequality

$$\begin{aligned}
\limsup_{b\to\infty} \frac{1}{b} \ln \mathbf{P}[\,x > b\,] &\le \lim_{B\to\infty} \sup_{b\ge B} \left(\frac{\lfloor b\rfloor}{b} \frac{1}{\lfloor b\rfloor} \ln \mathbf{P}[\,x > \lfloor b\rfloor\,] \right) \\
&\le \lim_{B\to\infty} \sup_{b\ge B} \left(\inf_{b_1\ge B} \left(\frac{\lfloor b_1\rfloor}{b_1} \right) \frac{1}{\lfloor b\rfloor} \ln \mathbf{P}[\,x > \lfloor b\rfloor\,] \right) \\
&= \liminf_{b\to\infty} \frac{\lfloor b\rfloor}{b} \limsup_{b\to\infty} \frac{1}{\lfloor b\rfloor} \ln \mathbf{P}[\,x > \lfloor b\rfloor\,] \\
&= \limsup_{n\to\infty} \frac{1}{n} \ln \mathbf{P}\left[\frac{x}{n} > 1 \right] \\
&= \limsup_{n\to\infty} \frac{1}{n} \ln \mathbf{P}\left[\frac{x}{n} > 1,\ t \in [0,1] \right]
\end{aligned}$$

$$
\begin{aligned}
&= \limsup_{n\to\infty} \frac{1}{n} \ln \mathbf{P}\left[\frac{x}{n} \in (\bar{B}_1(0))^c \right] \\
&\leq - \inf_{\varphi \in (\bar{B}_1(0))^c} I_x(\varphi),
\end{aligned}
$$

where the last step follows from the upper bound in the LDP for $\{\frac{1}{n}x,\ n = 1, 2, \ldots\}$ in $(C[0,1], \tau_\infty)$.

Next, we readily see from the expression of I_x that

$$
\begin{aligned}
\inf_{\varphi \in (\bar{B}_1(0))^c} I_x(\varphi) &= \inf \left\{ I_x(\varphi) : \varphi \in \mathcal{C}_+ \cap \overline{(\bar{B}_1(0))^c} \right\} \\
&= \inf_{A \geq 1} A\,\theta^* \\
&= \theta^*,
\end{aligned}
$$

and we obtain the upper bound

$$
\limsup_{b\to\infty} \frac{1}{b} \ln \mathbf{P}[x > b] \leq -\theta^*. \tag{5.20}
$$

Similarly, we get the lower bound by noting that

$$
\begin{aligned}
\liminf_{b\to\infty} \frac{1}{b} \ln \mathbf{P}[x > b] &\geq \liminf_{b\to\infty} \frac{\lceil b \rceil}{b} \frac{1}{\lceil b \rceil} \ln \mathbf{P}[x > \lceil b \rceil] \\
&\geq \limsup_{b\to\infty} \frac{\lceil b \rceil}{b} \liminf_{b\to\infty} \frac{1}{\lceil b \rceil} \ln \mathbf{P}[x > \lceil b \rceil] \\
&= \liminf_{n\to\infty} \frac{1}{n} \ln \mathbf{P}\left[\frac{x}{n} > 1 \right] \\
&= \liminf_{n\to\infty} \frac{1}{n} \ln \mathbf{P}\left[\frac{x}{n} \in \bar{B}_1(0)^c \right] \\
&\geq - \inf_{\varphi \in \bar{B}_1(0)^c} I_x(\varphi) \\
&= - \inf_{A > 1} A\,\theta^* \\
&= -\theta^*
\end{aligned}
$$

which combined with (5.20) yield the desired asymptotics (5.19). ■

Combining Theorem 5.10 and Theorem 5.11, we get the following result:

Proposition 5.12 *Under Assumptions* **(L1)** *and* **(L2)**, *the following asymptotics hold*

$$
\lim_{b\to\infty} \frac{1}{b} \ln \mathbf{P}[x_\infty > b] = - \inf_{y > 0} \frac{1}{y} r_\xi(y). \tag{5.21}
$$

This last Proposition is remarkable, as it directly yields the same lower and upper bounds. Furthermore, it will prove very useful in Chapter 7 to build up the notion of effective bandwidth at the sample path level, thus simplifying greatly the assumptions needed in the traditional derivations. Proposition 5.12 also suggests that the lower bound obtained in Proposition 5.8 is in fact the tight bound, and that only in some cases (e.g., under the assumptions of Corollary 5.9) does the upper bound equals the lower one.

We stress that besides convexity of the good rate function r_ξ, this result requires the assumption $\inf_{y>0} \frac{1}{y} r_\xi(y) > 0$.

5.3 LDP for Loynes variable

Its is quite interesting to know that buffer asymptotics of the form

$$\lim_{b\to\infty} \frac{1}{b} \ln \mathbf{P}\left[x_\infty > b\right] = -\theta^*, \tag{5.22}$$

for some $\theta^* > 0$ in turn yields a sample path LDP for the family of processes $\{\widehat{x}_n(\cdot),\ n = 1,2,\ldots\}$ in $(D[0,1], \tau_\infty)$.

The result is made precise by the following Theorem. Recall that

$$\widehat{x}_n(t) = \frac{\widehat{x}_{st}}{n}, \quad t \in [0,1], \quad n = 1,2,\ldots$$

with $\widehat{x}_{st} =_{st} x_\infty$.

Theorem 5.13 *Under (5.22), the sequence $\{\widehat{x}_n(\cdot),\ n = 1,2,\ldots\}$ satisfies the LDP in $(D[0,1], \tau_\infty)$ and in $(D[0,1], \tau_0)$ with good rate function $I_x : D[0,1] \to [0,\infty]$ given by*

$$I_x(\varphi) = \begin{cases} A\,\theta^* & \text{if } \varphi = c_A, \quad A \geq 0 \\ \infty & \text{otherwise.} \end{cases} \tag{5.23}$$

The proof passes through a series of easy Lemmas which we present first. Let $B_\alpha(\varphi)$ denote the open ball of radius α centered at φ in $C[0,1]$, for the uniform norm.

Lemma 5.14 *Assume (5.22) to hold. For $0 < x_- < x_+$, we have*

$$\lim_{n\to\infty} \frac{1}{n} \ln \mathbf{P}\left[x_\infty < nx_+\right] = 0 \tag{5.24}$$

and

$$\lim_{n\to\infty} \frac{1}{n} \ln \mathbf{P}\left[nx_- < x_\infty < nx_+\right] = -\theta^* x_-. \tag{5.25}$$

Proof: The limits (5.24)-(5.25) are simple consequences of (5.22) and of the elementary fact

$$\lim_{n\to\infty} \frac{1}{n} \ln\left(1 - e^{-nz}\right) = 0, \quad z > 0. \tag{5.26}$$

A detailed argument can be found in Appendix A.8. ∎

Lemma 5.15 *The sequence* $\{\hat{x}_n(\cdot),\ n = 1, 2, \ldots\}$ *is exponentially tight in the space* $(C[0,1], d_\infty)$.

Proof: We define

$$K_\alpha \ \equiv\ B_\alpha(0) \cap C_+, \quad \alpha > 0 \tag{5.27}$$

and note that

$$[\hat{x}_n(\cdot) \in K_\alpha^c] \ =\ [x_\infty > n\alpha] \cup [\hat{x}_n(\cdot) \notin C_+].$$

Hence, under (5.22),

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{n} \ln\ \mathbf{P}[\hat{x}_n(\cdot) \in K_\alpha^c] \\
&\quad\leq\ \limsup_{n\to\infty} \frac{1}{n} \ln(\mathbf{P}[x_\infty > n\alpha] + \mathbf{P}[\hat{x}_n(\cdot) \notin C_+]) \\
&\quad=\ \lim_{n\to\infty} \frac{1}{n} \ln\ \mathbf{P}[x_\infty > n\alpha] \\
&\quad=\ -\theta^*\alpha,
\end{aligned} \tag{5.28}$$

and the exponential tightness follows once it is seen that the set K_α is homeomorphic to the compact interval $[0, \alpha]$, thus compact in $(C[0,1], d_\infty)$. ∎

For φ in $C[0,1]$ and $\delta > 0$, we set

$$\varphi_+^\delta \equiv \inf_{t\in[0,1]} \varphi(t) + \delta \quad \text{and} \quad \varphi_-^\delta \equiv \sup_{t\in[0,1]} \varphi(t) - \delta\ . \tag{5.29}$$

Lemma 5.16 *For* φ *in* $C[0,1]$ *and* $\delta > 0$, *we have*

$$\mathcal{L}(B_\delta(\varphi)) \equiv -\lim_{n\to\infty} \frac{1}{n} \ln\ \mathbf{P}[\hat{x}_n(\cdot) \in B_\delta(\varphi)] \ = \begin{cases} \theta^*\varphi_-^\delta & \text{if } 0 < \varphi_-^\delta < \varphi_+^\delta \\ 0 & \text{if } \varphi_-^\delta \leq 0 < \varphi_+^\delta \\ \infty & \text{otherwise} \end{cases} \tag{5.30}$$

with φ_-^δ *and* φ_+^δ *defined by (5.29).*

Proof: Using the continuity of φ, we note the equality

$$[\widehat{x}_n(\cdot) \in B_\delta(\varphi)] = [n\varphi_-^\delta < \widehat{x}_{st} < n\varphi_+^\delta] \tag{5.31}$$

for each $n = 1, 2, \ldots$, so that

$$\mathbf{P}\left[\,\widehat{x}_n(\cdot) \in B_\delta(\varphi)\,\right] = \begin{cases} 0 & \text{if } \varphi_+^\delta \le 0 \\ 0 & \text{if } 0 < \varphi_+^\delta \le \varphi_-^\delta \\ \mathbf{P}\left[x_\infty < n\varphi_+^\delta\right] & \text{if } \varphi_-^\delta \le 0 < \varphi_+^\delta \\ \mathbf{P}\left[n\varphi_-^\delta < x_\infty < n\varphi_+^\delta\right] & \text{if } 0 < \varphi_-^\delta < \varphi_+^\delta\,. \end{cases} \tag{5.32}$$

We readily obtain (5.30) from (5.32) with the help of Lemma 5.14. ■

Proof: (Theorem 5.13)

Clearly, $\mathcal{A} \equiv \{B_\delta(\varphi) : \varphi \in C[0,1], \delta > 0\}$ is a base for $(C[0,1], d_\infty)$. Therefore, by Theorem 4.1.11 of [24, p. 106] and Lemma 5.16, the sequence $\{\widehat{x}_n(\cdot),\ n = 1, 2, \ldots\}$ satisfies the weak LDP in $(C[0,1], d_\infty)$ with rate function

$$I_x(\varphi) = \sup\{\mathcal{L}(B_\delta(\psi)) : \varphi \in B_\delta(\psi),\ \psi \in C[0,1],\ \delta > 0\} \tag{5.33}$$

where $\mathcal{L}(B_\delta(\psi))$ is given by (5.30). In computing $I_x(\varphi)$ for each φ in $C[0,1]$, several cases arise:

1. $\varphi \ge 0$, i.e., $\varphi(t) \ge 0$ for all t in $[0,1]$: In that case, whenever φ lies in the open ball $B_\delta(\psi)$ for some ψ in $C[0,1]$ and $\delta > 0$, we find $\psi(t) - \delta < \varphi(t) < \psi(t) + \delta$ for all t in $[0,1]$, so that by continuity,

$$\psi_+^\delta \equiv \inf_{t\in[0,1]} \psi(t) + \delta > \inf_{t\in[0,1]} \varphi(t) \ge 0\,. \tag{5.34}$$

Two sub-cases then arise:

1.a. $\inf_{t\in[0,1]} \varphi(t) < \sup_{t\in[0,1]} \varphi(t)$, i.e., φ is not a constant: If we take $\psi = \varphi$ and δ such that $0 < \delta < \frac{1}{2}\left(\sup_{t\in[0,1]} \varphi(t) - \inf_{t\in[0,1]} \varphi(t)\right)$, then φ belongs to $B_\delta(\psi)$, and we have

$$\begin{aligned} \psi_-^\delta &= \sup_{t\in[0,1]} \psi(t) - \delta \\ &= \sup_{t\in[0,1]} \varphi(t) - \delta \\ &> 2\delta + \inf_{t\in[0,1]} \varphi(t) - \delta = \psi_+^\delta. \end{aligned} \tag{5.35}$$

Therefore, $\psi_+^\delta < \psi_-^\delta$, so that $\mathcal{L}(B_\delta(\psi)) = \infty$ by Lemma 5.16, and $I_x(\varphi) = \infty$ follows.

1.b. $\inf_{t\in[0,1]}\varphi(t) = \sup_{t\in[0,1]}\varphi(t)$, i.e., $\varphi \equiv A \geq 0$: In that case, whenever φ lies in the open ball $B_\delta(\psi)$ for some ψ in $C[0,1]$ and $\delta > 0$, we find $A - \delta < \psi(t) < A + \delta$ for all t in $[0,1]$, whence by continuity, we see that

$$\psi_-^\delta = \sup_{t\in[0,1]} \psi(t) - \delta < A < \inf_{t\in[0,1]} \psi(t) + \delta = \psi_+^\delta. \tag{5.36}$$

In short, $\psi_-^\delta < \psi_+^\delta$, so that $\mathcal{L}(B_\delta(\psi)) = \theta^*(\psi_-^\delta)^+$ by Lemma 5.16, and (5.33) reduces to

$$I_x(\varphi) = \sup\{\theta^*(\psi_-^\delta)^+ : \varphi \in B_\delta(\psi),\ \psi \in C[0,1],\ \delta > 0\}. \tag{5.37}$$

It is now plain from (5.36) and (5.37) that $I_x(\varphi) \leq \theta^* A$. On the other hand, upon taking $\psi = \varphi = A$ in (5.36), we get $(\psi_-^\delta)^+ = (A-\delta)^+$ for all $\delta > 0$, so that $I_x(\varphi) \geq \theta^* A$, and we conclude $I_x(\varphi) = \theta^* A$.

2. $\varphi(s) < 0$ for some s in $[0,1]$: If we take $\psi = \varphi$, then φ lies in $B_\delta(\varphi)$ and $\inf_{t\in[0,1]}\psi(t) < 0$. Upon selecting $\delta > 0$ such that $\psi_+^\delta = \inf_{t\in[0,1]}\psi(t) + \delta < 0$, we get $\mathcal{L}(B_\delta(\psi)) = \infty$ by Lemma 5.16, whence $I_x(\varphi) = \infty$.

The expression (5.23) for I_x is obtained by combining the various cases. To conclude, we recall that d_∞ is the induced metric on the closed set $(C[0,1], d_\infty)$ of $(D[0,1], d_\infty)$ (or for that matter, d_∞). The sequence $\{\widehat{x}_n(\cdot),\ n = 1,2,\ldots\}$ is exponentially tight by Lemma 5.15, and therefore it satisfies the (strong) LDP with **good** rate function I_x in $(C[0,1], d_\infty)$ (Lemma 1.6). Therefore, $C[0,1]$ being a closed subset of $(D[0,1], d_\infty)$ with $\mathbf{P}[\widehat{x}_n(\cdot) \in C[0,1]] = 1$ for all $n = 1,2,\ldots$, the desired result follows from Lemma 1.11. ■

In the next Chapter, we apply the results derived here to characterize the large deviations behavior of the single-server queue.

Chapter 6

Large Deviations Behavior of the Single-Server Queue

We analyze here the large deviation behavior of the discrete-time single-server queue. Under the assumption that the arrivals and capacity sequences jointly satisfy the LDP in the space $(D[0,1],\tau_0)^2$, the results of Chapter 4 enable us to derive a sample path LDP for the steady-state and transient output processes of the queue. In addition, under the same assumption, we apply the results of Chapter 5 and obtain the LDP for the steady-state queue length, together with queue length asymptotics. To the best of our knowledge, those results are new.

We begin with the description of the model used for the discrete-time **G/G/1** queue, and then derive the buffer asymptotics results. Finally, the results concerning the sample path LDP of the transient and stationary output processes are presented. In particular, conditions under which the associated rate functions coincide are investigated.

6.1 The discrete-time G/G/1 queue

We consider a discrete-time **G/G/1** queue with arrival and capacity sequences $\{a_{t+1},\ t=0,1,\ldots\}$ and $\{c_{t+1},\ t=0,1,\ldots\}$, where a_{t+1} (resp. c_{t+1}) denotes the number of arrivals (resp. capacity) in the time interval $[t,t+1),\ t=0,1,\ldots$. All random variables are assumed to be defined on a common, possibly enlarged probability space $(\Omega,\mathcal{F},\mathbf{P})$. The queue length sequence $\{q_t,\ t=0,1,\ldots\}$, where q_t represent the queue length at time t, is then generated through the Lindley recursion

$$q_0 = q; \qquad q_{t+1} = [q_t + a_{t+1} - c_{t+1}]^+, \quad t=0,1,\ldots, \tag{6.1}$$

where the initial queue length q is an $\mathbb{R}_+$-valued random variable.

As announced in Chapter 5, in order to prepare ourselves for the applicability of the results to queues in series, we make the following assumption on the arrival and capacity sequence of the discrete-time **G/G/1** queue.

Assumption (A1) *The random sequence* $\{(a_{t+1}, c_{t+1}),\ t = 0, 1, \ldots\}$ *couples with a stationary and metrically transitive sequence* $\{(\widehat{a}_{t+1}, \widehat{c}_{t+1}),\ t = 0, 1, \ldots\}$ *satisfying* $\mathbf{E}[\,\widehat{a}_0\,] < \mathbf{E}[\,\widehat{c}_0\,]$.

As shown below, Assumption **(A1)** is in fact stronger that the assumption needed for stability, namely that the driving sequence $\{a_{t+1} - c_{t+1},\ t = 0, 1, \ldots\}$ couples with its stationary version. However, it will be required later in Section 6.3 in order to show that the output process also couples with its stationary and ergodic version.

Lemma 6.1 *Under Assumption* **(A1)**, *the sequence* $\{a_{t+1} - c_{t+1},\ t = 0, 1, \ldots\}$ *couples with the stationary and metrically transitive sequence* $\{\widehat{a}_{t+1} - \widehat{c}_{t+1},\ t = 0, 1, \ldots\}$.

Proof: Stationarity and metric transitivity of $\{\widehat{a}_{t+1} - \widehat{c}_{t+1},\ t = 0, 1, \ldots\}$ follow from the assumptions and Lemma A.1.2.7 in [15, p. 303]. Coupling is a trivial consequence of Assumption **(A1)**. ∎

In view of this last result, it is plain that Proposition 5.4 yields coupling of the joint sequence $\{(q_{t+1}, a_{t+1} - c_{t+1}),\ t = 0, 1, \ldots\}$ with its stationary version $\{(\widehat{q}_{t+1}, \widehat{a}_{t+1} - \widehat{c}_{t+1}),\ t = 0, 1, \ldots\}$. Nevertheless, we shall need the following stronger result when studying the steady-state behavior of the output process in Section 6.3. Recall that under the stationarity assumption made above, the sequence $\{(\widehat{a}_{t+1}, \widehat{c}_{t+1}),\ t = 0, 1, \ldots\}$ can always be embedded into a bi-infinite stationary sequence $\{(\widehat{a}_t, \widehat{c}_t),\ t = 0, \pm 1, \pm 2, \ldots\}$, possibly by enlarging the original probability space $(\Omega, \mathcal{F}, \mathbf{P})$.

Proposition 6.2 *Under Assumption* **(A1)**, *for any* $\mathbb{R}_+$*-valued and* $\mathbf{P}$*-a.e. finite initial condition* q, *the sequence* $\{(q_{t+1}, a_{t+1}, c_{t+1}),\ t = 0, 1, \ldots\}$ *couples with the jointly stationary and metrically transitive sequence* $\{(\widehat{q}^{st}_{t+1}, \widehat{a}_{t+1}, \widehat{c}_{t+1}),\ t = 0, 1, \ldots\}$, *where* $\{\widehat{q}^{st}_{t+1},\ t = 0, 1, \ldots\}$ *is the solution of the stochastic recursive recursion*

$$\widehat{q}_0 = q; \quad \widehat{q}_{t+1} = [\widehat{q}_t + \widehat{a}_{t+1} - \widehat{c}_{t+1}]^+, \quad t = 0, 1, \ldots, \tag{6.2}$$

with initial condition

$$q = \widehat{q}_{st} \equiv \left[\max_{t=0,1,\ldots} \left(\widehat{A}(-t, 0) - \widehat{C}(-t, 0)\right)\right]^+. \tag{6.3}$$

Moreover, $q_t \Longrightarrow_t q_\infty$*, for some* $\mathbf{P}$*-a.e. finite random variable* q_∞ *such that* $q_\infty =_{st} \widehat{q}_{st}$.

Proof: Again, joint stationarity and metric transitivity of $\{(\widehat{q}^{st}_{t+1}, \widehat{c}_{t+1}, \widehat{a}_{t+1}),\ t = 0, 1, \ldots\}$ follow from that of $\{(\widehat{a}_{t+1}, \widehat{c}_{t+1}),\ t = 0, 1, \ldots\}$, the definition of $\widehat{q}^{st}_t$ and Lemma A.1.2.7 in [15, p. 303].

By Proposition 5.4 the sequences $\{\widehat{q}^{st}_{t+1},\ t = 0, 1, \ldots\}$ and $\{q_{t+1},\ t = 0, 1, \ldots\}$ couple, and the desired coupling is easily obtained by considering as a coupling time, the maximum of that of $\{q_{t+1},\ t = 0, 1, \ldots\}$ and $\{(a_{t+1}, c_{t+1}),\ t = 0, 1, \ldots\}$. ■

It is clear from this last Proposition, that under Assumption **(A1)**, the steady-state queue length of the original queue (6.1) is identical to that of the single-server queue (6.2) with arrivals $\{\widehat{a}_{t+1},\ t = 0, 1, \ldots\}$ and capacity $\{\widehat{c}_{t+1},\ t = 0, 1, \ldots\}$. In the sequel, we shall refer to the queue (6.2) as the **stationary version** of the original queue (6.1).

We shall establish the LDP for the output process, by making use of the results of Chapter 4. To that end, we require that the stationary and metrically transitive versions of the arrivals and capacity sequences satisfy Assumptions **(E)** and **(E1)** of Section 4.1.

Assumption (A2) *The family of partial sum processes* $\{(\widehat{A}_n(\cdot), \widehat{C}_n(\cdot)),\ n = 1, 2, \ldots\}$ *associated with the sequence* $\{(\widehat{a}_{t+1}, \widehat{c}_{t+1}),\ t = 0, 1, \ldots\}$ *of the arrivals and capacity sequences satisfies the LDP in* $(D[0,1], \tau_0)^2$ *with (convex) good rate function* $I_{AC} : D[0,1]^2 \to [0, \infty]$ *given by*

$$I_{AC}(\varphi_1, \varphi_2) = \begin{cases} \int_0^1 r_{a,c}(\dot{\varphi}_1(t), \dot{\varphi}_2(t))\, dt, & \varphi_1, \varphi_2 \in AC_0[0,1] \\ \infty & \text{otherwise} \end{cases}, \tag{6.4}$$

where $r_{a,c} : \mathbb{R}^2 \to [0, \infty]$ *is the good rate function associated with the LDP for the sample mean sequence* $\{(\widehat{A}_n(1), \widehat{C}_n(1)),\ n = 1, 2, \ldots\}$*. Furthermore, the good rate function* $r_{a-c} : \mathbb{R} \to [0, \infty]$ *defined by*

$$r_{a-c}(z) \equiv \inf_{x,y \in \mathbb{R}} \{r_{a,c}(x, y) : z = x - y\}, \quad z \in \mathbb{R}$$

is convex and satisfies $\inf_{y>0} \frac{1}{y}\, r_{a-c}(y) > 0$.

Note from Corollary 1.9 and Proposition 2.17 that Assumption **(A2)** is indeed satisfied for mutually independent arrivals and capacity sequences with each satisfying the LDP in $(D[0,1], \tau_\infty)$ with good rate functions of the integral form, an assumption which has been made in (almost) all existing results on effective bandwidths of the departure process of **G/G/1** queues [16, 17, 18, 20, 28, 50, 51]. The exception is for the case of the continuous-time **G/GI/1** queue [7] where a technical assumption is made. We shall come back to this example in some detail in Section 6.3.1. Examples of stationary and metrically sequences whose partial sum process satisfies the LDP in $(D[0,1], \tau_\infty)$ were given in Section 3.3. In particular, by Corollary 3.25, the arrivals and capacity sequences of a stable discrete-time queue with Poisson arrivals and capacities satisfy Assumption **(A2)**, whereas it does not satisfy the assumption of bounded arrivals in [16, 18, 20].

Finally, in view of Proposition 3.21, Assumption **(A2)** will also be satisfied if the $\mathbb{R}^2$-valued sequence $\{(\widehat{a}_{t+1}, \widehat{c}_{t+1}),\ t = 0, 1, \ldots\}$ satisfies the sample path LDP in $(D[0,1]^2, \tau_0)$ with good rate function of the usual integral form.

6.2 Buffer asymptotics

In this section we show that under Assumptions **(A1)** and **(A2)**, the steady-state queue length satisfies the LDP, which in turn yields buffer asymptotics.

To our knowledge, the sample path LDP for the steady-state queue length is a new result. Although large deviations have been used to derive buffer asymptotics under the assumption that the input sequences satisfy the conditions of the Gärtner-Ellis Theorem [22, 29, 36, 42], there is only one instance [50] where the LDP for the transient queue length of a discrete-time **G/G/1** queue with constant capacity is obtained. The method used there combines sample path Large Deviations with the Contraction Principle to establish the desired LDP. However, it is different from ours in that the author derives the desired LDP from that of the **polygonal** approximation of the partial sum process associated with the inputs to the model. The functional used to map this polygonal approximation into the desired queue length is defined implicitly, and the proof of its continuity is not provided in the paper. Furthermore, additional technical assumptions are needed in order to establish the LDP for the polygonal approximation from that of the partial sum process [52].

We begin by relating the steady-state queue length to the functional $M_n^\infty(\cdot)$, as defined in Section 4.1. We adopt the notation on partial sum processes introduced in Section 2.2.1 and Section 3.2.

For each $n = 1, 2, \ldots$, define the random variable $\widehat{q}_n(\cdot)$ by setting

$$\widehat{q}_n(t) = \sup_{s \geq 0} \left(\widehat{A}_n^{\infty,-}(s) - \widehat{C}_n^{\infty,-}(s) \right), \quad t \in [0,1], \tag{6.5}$$

where $\widehat{A}_n^{\infty,-}(\cdot)$ and $\widehat{C}_n^{\infty,-}(\cdot)$ are the extensions on $[0,\infty)$ of the negative partial sum processes associated respectively with $\{\widehat{a}_{t+1},\ t=0,1,\ldots\}$ and $\{\widehat{c}_{t+1},\ t=0,1,\ldots\}$. By Lemma 4.4, we then have

$$\widehat{q}_n(t) = \frac{\widehat{q}_{st}}{n}, \quad t \in [0,1]$$

where $\widehat{q}_{st}$ is Loynes variable (as given by (6.3)) associated with the stationary queue (6.2).

By applying Corollary 4.9, we readily obtain the following Theorem.

Theorem 6.3 *Under Assumptions* **(A1)** *and* **(A2)**, *the sequence* $\{\widehat{q}_n(\cdot),\ n=1,2,\ldots\}$ *satisfies the LDP in* $(D[0,1],\tau_0)$ *and in* $(D[0,1],\tau_\infty)$ *with good rate function* $I_q : D[0,1] \to [0,\infty]$ *given by*

$$I_q(\varphi) = \begin{cases} A \inf_{y>0} \frac{1}{y} r_{a-c}(y) & \text{if } \varphi \equiv c_A \in \mathcal{C}_+, \\ \infty & \text{otherwise} \end{cases} \tag{6.6}$$

where r_{a-c} *is the convex good rate function associated with the LDP of the sample mean* $\{\widehat{A}_n(1) - \widehat{C}_n(1),\ n=1,2,\ldots\}$.

In fact we would have preferred to obtain this last Theorem through our earlier results on Lindley processes. Unfortunately, it turns out that we cannot directly apply Theorem 5.10 to obtain the result above. Indeed, although under Assumptions **(A1)** and **(A2)**, the Contraction Principle yields the LDP for the family $\{(\widehat{A}_n(\cdot) - \widehat{C}_n(\cdot)),\ n=1,2,\ldots\}$, we do not know at this point that the associated rate function is of the integral form (5.5) with $r_\xi = r_{a-c}$, and $\inf_{y>0} \frac{1}{y} r_{a-c}(y) > 0$, an assumption required in Theorem 5.10. This would be the case if we knew that the equality

$$\inf_{\psi \in AC_0[0,1]} \int_0^1 r_{a,c}\left(\dot{\psi}(t), \dot{\psi}(t) + \dot{\varphi}(t)\right) dt = \int_0^1 \left(\inf_{x \in \mathbf{R}} r_{a,c}(x, x + \dot{\varphi}(t))\right) dt$$

holds for all φ in $AC_0[0,1]$, but we have not been able to prove it yet.

Theorem 5.11 then yields the following Corollary to Theorem 6.3.

Recall from Proposition 6.2 that q_∞ is the limit (in distribution) as $t \to \infty$ of the queue length q_t, i.e., the steady-state queue length.

Corollary 6.4 *Under Assumptions* **(A1)** *and* **(A2)**, *we have*

$$\limsup_{b \to \infty} \frac{1}{b} \ln \mathbf{P}\left[q_\infty > b\right] = -\inf_{y>0} \frac{1}{y} r_{a-c}(y). \tag{6.7}$$

6.3 Output process

The problem considered here is of characterizing the large deviations behavior of the output process $\{b_{t+1},\ t=0,1,\ldots\}$ of the discrete-time single-server queue. If for each $t=0,1,\ldots$, b_{t+1} represents the number of departures in the time interval $[t,t+1)$, it is easily seen that the sequence $\{b_{t+1},\ t=1,2,\ldots 0\}$ satisfies the relation

$$b_{t+1} = a_{t+1}-(q_{t+1}-q_t), \quad t=0,1,\ldots . \tag{6.8}$$

It is plain that different solutions $\{q_{t+1},\ t=1,2,\ldots\}$ to the recursion (6.1) will generate different output processes. We distinguish in particular the **transient** output process of the original queue 6.1 as the process obtained from (6.8) with $\{q_{t+1},\ t=1,2,\ldots\}$ being the solution of the recursion (6.1) with initial condition $q_0=0$. The second output process of interest is of course the steady-state or **stationary** output process obtained from the stationary queue length. However, in our setup steady state is only achieved for the stationary version (6.2) of the queue, which prevents us from defining the stationary output process of the queue (6.1) from (6.8). Nevertheless, the transient output process of the original queue (6.1) does have a stationary regime, as proved by the following proposition.

Proposition 6.5 *Under Assumption* **(A1)**, *the* $\mathbb{R}^4$*-valued random sequence* $\{(b_{t+1}, q_{t+1}, a_{t+1}, c_{t+1}),\ t=0,1,\ldots\}$ *couples with the stationary and metrically transitive sequence* $\{(\widehat{b}^{st}_{t+1}, \widehat{q}^{st}_{t+1}, \widehat{a}_{t+1}, \widehat{c}_{t+1}),\ t=0,1,\ldots\}$ *with* $\{\widehat{b}^{st}_{t+1},\ t=0,1,\ldots\}$ *given by*

$$\widehat{b}^{st}_{t+1} = \widehat{a}_{t+1}-(\widehat{q}^{st}_{t+1}-\widehat{q}^{st}_t),\ t=0,1,\ldots . \tag{6.9}$$

where $\{\widehat{q}^{st}_{t+1},\ t=0,1,\ldots\}$ *is as defined in Proposition 6.2, i.e., is the solution of the recursion (6.2) with initial condition* $\widehat{q}_0=\widehat{q}_{st}$ *with* $\widehat{q}_{st}$ *given by (6.3).*

Proof: Let α denote the coupling time of $\{(q_{t+1}, a_{t+1}, c_{t+1}),\ t=0,1,\ldots\}$ with its stationary and metrically transitive version (Proposition 6.2). It is then plain from (6.8) that

$$\begin{aligned} b_{t+1} &= a_{t+1}-(q_{t+1}-q_t) \\ &= \widehat{a}_{t+1}-(\widehat{q}^{st}_{t+1}-\widehat{q}^{st}_t) \\ &= \widehat{b}^{st}_{t+1}, \quad t=\alpha,\alpha+1,\ldots \end{aligned}$$

with $\{\widehat{b}^{st}_{t+1}, =0,1,\ldots\}$ as defined by (6.9), and the desired coupling follows.

Stationarity and metric transitivity of $\{(\widehat{b}^{st}_{t+1}, \widehat{q}^{st}_{t+1}, \widehat{a}_{t+1}, \widehat{c}_{t+1}),\ t=0,1,\ldots\}$ follows from that of the sequence $\{(\widehat{q}^{st}_{t+1}, \widehat{a}_{t+1}, \widehat{c}_{t+1}),\ t=0,1,\ldots\}$ (Proposition 6.2) and Lemma A.1.2.7 in [15, p. 303]. ■

It is plain from its definition that the sequence $\{\widehat{b}^{st}_{t+1},\ t = 0, 1, \ldots\}$ is in fact the stationary output process of the stationary queue (6.2). Thus, Proposition 6.5 states that the stationary version of the transient output process of the original queue (6.1) is in fact the stationary output process of the stationary queue (6.2). It should be stressed that the transient processes of the original queue and of its stationary version are in general different since the driving sequences of the two queues are not the same. We also point out that the transient output process of the stationary queue (6.2) is not in general stationary. Finally Proposition 6.5 fits well into the framework of series of queues, as it already shows that the output process satisfies stability assumptions similar to that of the arrivals.

The problem of obtaining the sample path LDP for the output process of a **G/G/1** queue has already been addressed in the literature. The underlying motivation is to obtain effective bandwidth results for queues in series or intree networks. We begin by reviewing the existing results on this subject.

6.3.1 Literature survey

The first work on that matter appears to be that of de Veciana et al. [20]. The system considered there is a discrete-time single-server queue with constant capacity, and bounded stationary ergodic arrival process. The authors establish the LDP for the stationary version of the output process, and use the expression they obtain for the rate function to derive the effective bandwidth. However, the argument leading to the derivation of this LDP is not rigorously formalized, and the assumptions required on the arrival process for the result to hold are not made precise. Moreover, it is not shown how the assumptions made on the arrival sequence propagate to the departure sequence, and it is therefore unclear how those results can be applied for feed-forward networks, as claimed by the authors.

Those ideas were formalized in a more rigorous way by Chang [16] in the context of intree networks of discrete-time single-server queue with constant capacity, and bounded stationary ergodic arrivals. In [16], the author uses the formalism of the sample path LDP together with the Contraction Principle to establish the LDP for the transient departure process, i.e., the sequence of departures from the queue with empty initial queue length. Although the author only establishes the sample path LDP for the transient output process, by using some very technical results, he obtains the asymptotics of the steady-state queue length at each stage of the intree network. However, in final analysis, it is not too clear what is really the set of assumptions required for the result to hold for intree networks.

In [18], Chang and Zajic extend those results to the stationary output process of a discrete-time single-server queue with time varying capacities. Both arrivals and capacity sequences are assumed to be bounded, stationary and ergodic and to satisfy technical conditions stronger than a sample path LDP. In addition, a mixing condition is required on both arrivals and capacity sequences in order to decouple the past and present in the expression of the stationary output process. We overcame the same difficulty by establishing in Chapter 3 the joint LDP for the past and the future of the sample path process (Corollary 3.20). In [18], the authors only provide a heuristic proof of the LDP for the stationary output process. Furthermore, their assumptions are not shown to propagate to the output process, thus preventing them from applying their results inductively to queues in series. Finally, their heuristic derivation lead them to make the conjecture that in general the effective bandwidth for the stationary and transient output process are different, although they note that it is the same when the capacity is constant.

Duffield and O'Connell [28] establish the upper bound of the LDP for the transient output process as well as for the unused service in a single-server queue with stationary arrivals and service. Their assumptions are basically those of the Gärtner-Ellis Theorem for the sample mean arrival and capacity sequences.

Bertsimas, Paschalidis and Tsitsiklis [7] obtain through an intuitive approach the LDP for the output process of a continuous-time $\mathbf{G/G/1}$ queue with i.i.d. service times. Their assumptions are:

i) The sample mean sequence of inter-arrival times $\{\frac{1}{n}A(1,n),\ n=1,2,\ldots\}$ satisfies the LDP with good rate function Λ_A^*, the Legendre-Fenchel transform of the logarithmic moment generating function (see Chapter 7),

ii) The sample mean sequence of the service times $\{\frac{1}{n}B(1,n),\ n=1,2,\ldots\}$ satisfies the requirements of the Gärtner-Ellis Theorem (with Λ_B^* the Legendre-Fenchel transform of the log-moment generating), and

iii) For every $\varepsilon_1, \varepsilon_2, a > 0$, there exist integers M_A and M_B such that for $n \geq M_A$

$$e^{-n(\Lambda_A^*(a)+\varepsilon_2)} \leq \mathbf{P}\left[A(1,i) - ia \leq \varepsilon_1 n,\ i=1,\ldots,n\right],$$

and for $n \geq M_B$,

$$\begin{aligned} e^{-n(\Lambda_B^*(a)+\varepsilon_2)} &\leq \mathbf{P}\left[B(i,j) - (j-i+1)a \leq \varepsilon_1 n,\ 1 \leq i \leq j \leq n\right], \\ e^{-n(\Lambda_B^*(a)+\varepsilon_2)} &\leq \mathbf{P}\left[B(i,j) - (j-i+1)a \geq -\varepsilon_1 n,\ 1 \leq i \leq j \leq n\right]. \end{aligned}$$

These last three assumptions can be seen as accounting for a sample path LDP of the associated partial sum process. Under these assumptions, the authors derive the LDP for the sample mean of the stationary inter-departure time sequence and show that the sequence satisfies the same assumption that the inter-arrival time sequences, thus enabling them to apply their results inductively to series of queues. The arguments given in the paper give more insight that our general sample path level approach, but the results obtained are weaker than ours. We note that Assumption (iii) is implied by the sample path LDP for the inter-arrival and service time sequences. Also, the derivation given in [7] holds only for i.i.d. service times, which would transpose to i.i.d. arrivals in the discrete-time queue setup.

Finally, we became aware only recently of a revised version of [51]. In the original version we had, the author was deriving the sample path LDP for the transient output process of a discrete-time **G/G/1** queue with stationary arrivals and capacities, from that of the arrivals and capacity sequences. However, many key steps in the arguments were left unspecified. The new version is basically a new paper as the only common part is the definition of the problem. The results obtained there are weaker than ours, and the method is different. The system under consideration is a single-server queue with multi-dimensional stationary arrival process and stationary service process. The author derives the sample path LDP for the transient and stationary departure processes as well as the LDP for the transient and steady-state queue length. The assumptions required are joint sample path LDP for arrivals and service processes and bounded exponential moments, precisely

$$\sup_{t=0,1,\ldots} \mathbf{E}\left[e^{\gamma(a_{t+1}+c_{t+1})}\right] < \infty, \quad \gamma \in \mathbb{R}.$$

In addition, it is required that the rate function of the joint sample LDP of the arrivals and service processes be the sum of the two rate functions associated with the sample path LDP of each of the sequences. Finally, in order to obtain simpler expressions of the rate functions for the LDP of the output processes, it is assumed that the rate functions associated with the LDP of the arrivals and service processes are of the usual integral form (Assumption **(E1)** in Section 4.1). However, some key arguments in [50] appear to be incorrect: Throughout the author uses probability measures built on n-uples of random elements in the space L^∞ equipped with the supremum norm which is **not** separable. Therefore, as pointed out in Section 1.5.1, some of the large deviations techniques used in the paper, such as exponential equivalence **cannot** be used as the requirement that the σ-field on which the measure is defined be contained in the Borel σ-field is violated. Furthermore, a key step in the derivation of the sample path LDP for the stationary output process seems incorrect in that the author uses the LDP of the marginals of a pair of random elements to derive the LDP jointly for the pair. Indeed, the LDP for two sequences on a given space does not in general yield the LDP for the joint sequence in the product space. It does so when the sequences are independent and the spaces separable (Proposition 1.8).

6.3.2 LDP for the output process

In this section, we formally prove that both the stationary version of the transient output process of the original queue (6.1) and the transient output process of the stationary queue (6.2) satisfy the sample path LDP.

We begin with a representation of the output process in terms of the basic input random variables of the model. As previously noted, the stationary version of the transient output processes of the original queue coincide with the stationary output process of the stationary queue. Thus, as we also want to establish the LDP for the transient output process of the stationary queue, we work on the model of the stationary queue only. Then, the only difference in the analytical expression for the stationary and the transient output process lies in the value of the initial condition $\widehat{q}_0$ in the recursion (6.2). For the transient output process $\widehat{q}_0 \equiv 0$, while for the stationary one, $\widehat{q}_0 \equiv \widehat{q}_{st}$.

First, upon iterating (6.2) it is easy to see that for each $t = 0, 1, \ldots$, the relations

$$\begin{aligned}\widehat{q}_t &= \max\left\{0,\ \widehat{q}_0 + \widehat{A}(1,t) - \widehat{C}(1,t),\ \max_{s=2,\ldots,t}\left(\widehat{A}(s,t) - \widehat{C}(s,t)\right)\right\} \\ &= \widehat{A}(1,t) - \widehat{C}(1,t) + \widehat{q}_0 - \min\left\{0,\ \widehat{q}_0 + \min_{s=1,\ldots,t}\left(\widehat{A}(1,s) - \widehat{C}(1,s)\right)\right\}\end{aligned}$$

hold, and simple algebra then readily shows that

$$\begin{aligned}\widehat{B}(1,t) &= \sum_{i=1}^{t} \widehat{b}_i \\ &= \sum_{i=1}^{t} [\widehat{a}_i - (\widehat{q}_i - \widehat{q}_{i-1})] \\ &= \widehat{A}(1,t) - (\widehat{q}_t - \widehat{q}_0) \\ &= \widehat{C}(1,t) + \min\left\{0,\ \widehat{q}_0 + \min_{s=1,\ldots,t}\left(\widehat{A}(1,s) - \widehat{C}(1,s)\right)\right\}.\end{aligned}$$

As usual, the minimum over the empty set is taken to be ∞, a convention which is consistent with our earlier definitions.

In order to write this last expression more compactly, we define the random variables $\{\widehat{m}(1,t),\ t = 0, 1, \ldots\}$ by

$$\widehat{m}(1,t) \equiv \min_{s=1,\ldots,t}\left(\widehat{A}(1,s) - \widehat{C}(1,s)\right), \quad t = 0, 1, \ldots$$

and note that

$$\widehat{B}(1,t) = \widehat{C}(1,t) + \min\{0\,,\ \widehat{q}_0 + \widehat{m}(1,t)\}, \quad t = 0, 1, \ldots . \tag{6.10}$$

Because we have in mind to apply our results from Chapter 4, we need to consider the partial sum processes associated with the random variables of the model. We use here the notations relevant to partial sum processes introduced in Section 2.2.1. Using (6.10), for each $n = 1, 2, \ldots$, we can write

$$\frac{\widehat{B}(1, \lfloor nt \rfloor)}{n} = \frac{\widehat{C}(1, \lfloor nt \rfloor)}{n} + \min\left\{0, \frac{\widehat{q}_0}{n} + \frac{\widehat{m}(1, \lfloor nt \rfloor)}{n}\right\}, \quad t \in [0,1] \tag{6.11}$$

where

$$\begin{aligned}\frac{\widehat{m}(1, \lfloor nt \rfloor)}{n} &= \min_{s=1,\ldots,\lfloor nt \rfloor} \left(n^{-1}(\widehat{A}(1,s) - \widehat{C}(1,s))\right) \\ &= \inf_{0 \le s \le t} \left(n^{-1}(\widehat{A}(1, \lfloor ns \rfloor) - \widehat{C}(1, \lfloor ns \rfloor))\right) \\ &= \inf_{0 \le s \le t} \left(\widehat{A}_n(s) - \widehat{C}_n(s)\right), \quad t \in [0,1] .\end{aligned}$$

Therefore, defining the family of random processes $\{\widehat{m}_n(\cdot),\ n = 1, 2, \ldots\}$ by

$$\widehat{m}_n(t) \equiv \inf_{0 \le s \le t} \left(\widehat{A}_n(s) - \widehat{C}_n(s)\right), \quad t \in [0,1], \quad n = 1, 2, \ldots,$$

we can rewrite (6.11) as

$$\widehat{B}_n(\cdot) = \widehat{C}_n(\cdot) + \min\left\{0, \widehat{m}_n(\cdot) + \frac{\widehat{q}_0}{n}\right\}, \quad n = 1, 2, \ldots , \tag{6.12}$$

where $\widehat{q}_0$ is the initial queue length.

From this representation and the results derived in Chapter 4, we can establish the LDPs for the stationary and transient output processes of the stationary queue. We first distinguish the two output processes of interest. We denote by $\{\widehat{B}_n^{st}(\cdot),\ n = 1, 2, \ldots\}$ the partial sum process associated with the stationary output process and by $\{\widehat{B}_n^{tr}(\cdot),\ n = 1, 2, \ldots\}$ that associated with the transient output process. Then $\widehat{B}_n^{st}(\cdot)$ is obtained from (6.12) with $\widehat{q}_0 = q_{st}$, and $\frac{\widehat{q}_0}{n} = \widehat{q}_n(\cdot)$, while $\widehat{B}_n^{tr}(\cdot)$ is obtained from the same equation with $\widehat{q}_0 = 0$.

Recall that $AC_0^-[0,1]$ denotes the subspace of $AC_0[0,1]$ of functions which are non-increasing, so that $\psi \le 0$ for all ψ in $AC_0^-[0,1]$.

Theorem 6.6 *Under Assumptions* **(A1)** *and* **(A2)**, *the family of partial sum processes* $\{\widehat{B}_n^{st}(\cdot),\ n = 1, 2, \ldots\}$ *associated with the stationary version of the output process satisfies the LDP in* $(D[0,1], \tau_0)$ *and in* $(D[0,1], \tau_\infty)$ *with good rate function* $I_B^{st} : D[0,1] \to [0, \infty]$ *given by*

$$I_B^{st}(\varphi) = \inf_{\substack{\psi_1 \in AC_0[0,1] \\ \psi_2 \in AC_0^-[0,1] \\ A \ge 0}} \left\{ I_m(\psi_1, \psi_2) + A \inf_{y>0} \frac{1}{y}\, r_{a-c}(y) : \varphi = \psi_1 + \min\{0, \psi_2 + A\} \right\} \tag{6.13}$$

for φ in $AC_0[0,1]$, and where for ψ_1 in $AC_0[0,1]$ and ψ_2 in $AC_0^-[0,1]$, I_m is given by

$$I_m(\psi_1,\psi_2) \;=\; \inf_{\psi\in AC_0[0,1]}\left\{\int_0^1 r_{a,c}\left(\dot\psi_1(t);\dot\psi_1(t)+\dot\psi(t)\right)dt : \psi_2 = m_\psi\right\}, \qquad (6.14)$$

while $I_B^{st}(\varphi)=\infty$ otherwise.

Proof: Under Assumptions **(A1)** and **(A2)**, Theorem 4.1 applied with $X_n(\cdot)\equiv\widehat{C}_n(\cdot)$ and $Y_n(\cdot)\equiv\widehat{A}_n(\cdot)$ yields the LDP for the family $\{(\widehat{C}_n(\cdot),\widehat{m}_n(\cdot),\widehat{q}_n(\cdot)),\ n=1,2,\ldots\}$ in $(D[0,1],,\tau_0)^3$ with good rate function $I: D[0,1]^3\to[0,\infty]$ given by

$$I(\varphi_1,\varphi_2,\varphi_3) \;=\; \begin{cases} I_m(\varphi_1,\varphi_2)+I_q(c_A) & \text{if } \begin{array}{l}\varphi_1\in AC_0[0,1]\\ \varphi_2\in AC_0^-[0,1]\end{array},\quad \varphi_3\equiv c_A\in\mathcal{C}_+ \\ \infty & \text{otherwise}\end{cases}.$$

where I_m is as given in (6.14), and

$$I_q(c_A) \;=\; A\inf_{y>0}\frac{1}{y}\,r_{a-c}(y), \quad c_A\in\mathcal{C}_+.$$

Define the mapping $F: D[0,1]^3\to D[0,1]$ by

$$F(z_1,z_2,z_3) \;=\; z_1+\min\{0,z_2+z_3\}.$$

By Lemmas 2.8 and 2.11 the mapping F is Borel-measurable and S-continuous on $AC_0[0,1]^3$ which contains $\mathcal{D}_I$ the effective domain of I. Therefore, upon noting that (6.12) can be rewritten as

$$\widehat{B}_n^{st}(\cdot) \;=\; F\left(\widehat{C}_n(\cdot),\widehat{m}_n(\cdot),\widehat{q}_n(\cdot)\right), \quad n=1,2,\ldots,$$

the Contraction Principle yields the LDP for the family of partial sum processes $\{\widehat{B}_n^{st}(\cdot),\ n=1,2,\ldots\}$ in $(D[0,1],\tau_0)$ with good rate function $I_B^{st}: D[0,1]\to[0,\infty]$ given by

$$I_B^{st}(\varphi) \;=\; \inf_{\psi_1,\psi_2,\psi_3\in D[0,1]}\{I(\psi_1,\psi_2,\psi_3):\varphi=\psi_1+\min\{0,\psi_2+\psi_3\}\}, \quad \varphi\in D[0,1].$$

We note that for ψ_1 in $AC_0[0,1]$, ψ_2 in $AC_0^-[0,1]$, and c_A in $\mathcal{C}_+$, the mapping $\psi_1+\min\{0,\psi_2+c_A\}$ is itself in $AC_0[0,1]$. Thus, from the expression of I above we get that $I_B^{st}(\varphi)=\infty$ for φ not in $AC_0[0,1]$, while for φ in $AC_0[0,1]$, we obtain (6.13). By Proposition 2.17, the LDP $\{\widehat{B}_n^{st}(\cdot),\ n=1,2,\ldots\}$ also holds in $(D[0,1],\tau_\infty)$. ■

Theorem 6.7 *Under Assumptions* **(A1)** *and* **(A2)**, *the family* $\{\widehat{B}_n^{tr}(\cdot),\ n = 1,2,\ldots\}$ *of partial sum processes associated with the transient output process of the stationary queue (6.2), satisfies the LDP in* $(D[0,1],\tau_0)$ *and in* $(D[0,1],\tau_\infty)$ *with good rate function* $I_B^{tr} : D[0,1] \to [0,\infty]$ *given by*

$$I_B^{tr}(\varphi) = \begin{cases} \inf_{\psi \in AC_0^-[0,1]} I_m(\varphi - \psi, \psi) & \text{if } \varphi \in AC_0[0,1] \\ \infty & \text{otherwise} \end{cases} \tag{6.15}$$

where I_m *is given by (6.14).*

Proof: The proof follows the same lines as that of Theorem 6.6. Under the enforced assumptions, Corollary 4.8 yields the LDP for $\{(\widehat{C}_n(\cdot),\widehat{m}_n(\cdot)),\ n = 1,2,\ldots\}$ in $(D[0,1],\tau_0)^2$ with good rate function $I_{C,m} : D[0,1]^2 \to [0,\infty]$ given by

$$I_{C,m}(\varphi_1,\varphi_2) = \begin{cases} I_m(\varphi_1,\varphi_2) & \text{if } \varphi_1 \in AC_0[0,1], \quad \varphi_2 \in AC_0^-[0,1] \\ \infty & \text{otherwise} \end{cases} \tag{6.16}$$

with I_m given by (6.14). The mapping $G : D[0,1]^2 \to [0,\infty]$ defined by

$$G(z_1,z_2) = z_1 + \min\{0,z_2\}$$

is Borel-measurable and S-continuous on the space $AC_0[0,1]^2$ which contains the effective domain $\mathcal{D}_{I_{C,m}}$ of $I_{C,m}$ (Lemmas 2.8 and 2.11). Therefore, from (6.12) we have

$$B_n^{tr}(\cdot) = G\left(\widehat{C}_n(\cdot),\widehat{m}_n(\cdot)\right), \quad n = 1,2,\ldots,$$

and the desired LDP becomes a simple consequence of the Contraction Principle. The associated good rate function $I_B^{tr} : D[0,1] \to [0,\infty]$ is given by

$$I_B^{tr}(\varphi) = \inf_{\psi_1,\psi_2 \in D[0,1]} \{I_{C,m}(\psi_1,\psi_2) : \varphi = \psi_1 + \min\{0,\psi_2\}\}, \quad \varphi \in D[0,1].$$

It is plain that for ψ_1 in $AC_0[0,1]$ and ψ_2 in $AC_0^-[0,1]$, the mapping $\psi_1 + \min\{0,\psi_2\} = \psi_1 + \psi_2$ is itself in $AC_0[0,1]$ and we deduce immediately from (6.16) that $I_B^{tr}(\varphi) = \infty$ for φ not in $AC_0[0,1]$, while

$$\begin{aligned} I_B^{tr}(\varphi) &= \inf_{\substack{\psi_1 \in AC_0[0,1] \\ \psi_2 \in AC_0^-[0,1]}} \{I_m(\psi_1,\psi_2) : \varphi = \psi_1 + \psi_2\} \\ &= \inf_{\psi \in AC_0^-[0,1]} I_m(\varphi - \psi,\psi), \quad \varphi \in AC_0[0,1] \end{aligned}$$

and the result (6.15) follows easily. By Proposition 2.17, the LDP for $\{\widehat{B}_n^{tr}(\cdot),\ n = 1,2,\ldots\}$ also holds in the space $(D[0,1],\tau_\infty)$. ∎

Unfortunately, the expression for J_m obtained in Section 4.4 for certain classes of mappings is not enough to compute I_B^{st} and I_B^{tr} explicitly.

As we can already see, the rate functions associated with the LDP for the two output processes are a priori different.

6.3.3 Stationary vs. transient

The conjecture of Chang and Zajic [18] translates in our set up to the properties of the rate functions associated with the transient and stationary output processes of the stationary queue (6.2).

We begin with the following easy Lemma.

Lemma 6.8 *For all φ in $D[0,1]$, we have*

$$I_B^{st}(\varphi) \leq I_B^{tr}(\varphi).$$

Proof: The desired inequality trivially holds when φ is not in $AC_0[0,1]$, for then $I_B^{st}(\varphi) = I_B^{tr}(\varphi) = \infty$. Now, fix φ in $AC_0[0,1]$, and let ψ_1 and ψ_2 in $AC_0[0,1]$ and $AC_0^-[0,1]$, respectively, such that $\varphi = \psi_1 + \psi_2$. It is immediate (take $A = 0$ in (6.13)) that

$$\begin{aligned} I_B^{st}(\varphi) &= \inf_{\substack{\psi_1,\psi_2 \in AC_0[0,1] \\ A \geq 0}} \left\{ I_m(\psi_1, \psi_2) + A \inf_{y>0} \frac{1}{y}\, r_{a-c}(y) : \varphi = \psi_1 + \min\{0, \psi_2 + A\} \right\} \\ &\leq I_m(\psi_1, \psi_2) \end{aligned}$$

and it follows that

$$I_B^{st}(\varphi) \leq \inf_{\substack{\psi_1 \in AC_0[0,1] \\ \psi_2 \in AC_0^-[0,1]}} \{ I_m(\psi_1, \psi_2) : \varphi = \psi_1 + \psi_2 \} = I_B^{tr}(\varphi), \quad \varphi \in AC_0[0,1].$$

■

We next give a condition under which the rate functions coincide. The condition is quite strong and will not often be satisfied. In order to state the result, we need a definition.

Definition 6.9 ([24, p. 114]) *Two families $\{X_n,\ n = 1, 2, \ldots\}$ and $\{Y_n,\ n = 1, 2, \ldots\}$ of random elements on a common probability space $(\Omega, \mathcal{F}, \mathbf{P})$, and taking on values in a metric space $(\mathcal{X}, d)$ are said to be exponentially equivalent in*

$(\mathcal{X}, d)$ *if for each* $\delta > 0$ *and* $n = 1, 2, \ldots$, *the subset* $\{d(X_n, Y_n) > \delta\}$ *of* Ω *is* $\mathcal{F}$*-measurable and*

$$\limsup_{n\to\infty} \frac{1}{n} \ln \mathbf{P}\left[d(X_n, Y_n) > \delta \right] = -\infty.$$

Proposition 6.10 *If* $\mathbf{E}\left[e^{\theta \widehat{q}_{st}}\right] < \infty$ *for all* $\theta > 0$, *then* $\{\widehat{B}_n^{st}(\cdot),\ n = 1, 2, \ldots\}$ *and* $\{\widehat{B}_n^{st}(\cdot),\ n = 1, 2, \ldots\}$ *are exponentially equivalent. Thus, if one satisfies the LDP, so does the other with the same rate function.*

Proof: It is readily seen from (6.12) that

$$\begin{aligned} d_0(\widehat{B}_n^{st}(\cdot), \widehat{B}_n^{tr}(\cdot)) &\leq d_\infty(\widehat{B}_n^{st}(\cdot), \widehat{B}_n^{tr}(\cdot)) \\ &\leq \frac{\widehat{q}_{st}}{n}, \quad n = 1, 2, \ldots . \end{aligned} \tag{6.17}$$

Thus, from Chebycheff inequality, for any $\delta > 0$, we have

$$\begin{aligned} \mathbf{P}\left[d_0(\widehat{B}_n^{st}(\cdot), \widehat{B}_n^{tr}(\cdot)) > \delta \right] &\leq \mathbf{P}\left[\widehat{q}_{st} > n\,\delta \right] \\ &\leq e^{-n\delta\theta}\ \mathbf{E}\left[e^{\theta \widehat{q}_{st}} \right], \quad \theta > 0 . \end{aligned} \tag{6.18}$$

Taking the logarithm, dividing by n and letting n goes to infinity in the last inequality yields

$$\limsup_{n\to\infty} \frac{1}{n} \ln \mathbf{P}\left[d_0(\widehat{B}_n^{st}(\cdot), \widehat{B}_n^{tr}(\cdot)) > \delta \right] \leq -\delta\theta, \quad \theta > 0 , \tag{6.19}$$

where the last step is obtained from the finiteness of the exponential moment. Exponential equivalence of $\{\widehat{B}_n^{st}(\cdot),\ n = 1, 2, \ldots\}$ and $\{\widehat{B}_n^{tr}(\cdot),\ n = 1, 2, \ldots\}$ then follows easily upon letting θ go to infinity in (6.19). Finally, by Theorem 4.2.13 in [24, p. 114], if one family satisfies the LDP, so does the other with the same good rate function. ■

Because we were not able to obtain explicit expressions for the rate functions governing the large deviations of the output processes, we cannot carry out the verification of Chang and Zajic's conjecture. For the same reasons, we cannot show that all the assumptions on the input process of the queue propagate to the output. However, we have obtained an important new result, in the sense that both the stationary and transient output process satisfy the sample path LDP. The missing step in propagating the result is Lemma 4.7, where the actual form of the rate function combined with the condition $\inf_{c>0} \frac{1}{c} r_{y-x}(c) > 0$ is needed to establish the LDP in the appropriate subspace.

Our conviction is that the full assumption in fact propagates. Indeed, as the rate functions derived from the Contraction Principle are always expressed as the infimum of the original rate functions under some constraint, we feel that, under the stability condition of the next queue, the effective domain of the output rate function is indeed contained in the subspace $D[0,1]^2 \times D_l^*[0,\infty)$, as required in Lemma 4.7. Moreover, the quantity $\inf_{c>0} \frac{1}{c} r_{y-x}(c) > 0$ is intrinsically related to the buffer asymptotics, and the condition that it be positive only prevents the decay rate of the buffer overflow probability to be 0, and should propagate.

In the next Chapter, we apply the results obtained here to derive effective bandwidths for **G/G/1** queues in series.

Chapter 7

Effective Bandwidth for Single-Server Queues

In this chapter, we review the notion of effective bandwidth and give new results on the effective bandwidth of **G/G/1** queues, based on our sample path large deviations results of the earlier chapters, in particular those of Chapters 5 and 6. In addition, we attempt to generalize our results to queues in series, and pinpoint the problems encountered.

The chapter is organized as follows. We begin by reviewing some example of effective bandwidths, as it first appeared in the literature. The derivation of the effective bandwidth is done in Section 7.2 where both the traditional results and our sample path approach are presented. Finally, the case of queues in series is investigated in Section 7.3.

7.1 Examples and literature survey

Consider a network of K deterministic links, each of capacity C_k, $k = 1 \ldots, K$. Suppose there exist J types of connection which can be carried over the network and that a connection of type j, $j = 1 \ldots, J$ requires a capacity α_{jk} at link k. Then the network is able to carry n_j connections of type j simultaneously, if and only if, for each link k, $k = 1, \ldots, K$,

$$\sum_{j=1}^{J} n_j \alpha_{jk} \leq C_k \, . \tag{7.1}$$

In many operational situations, the required capacity of each connection is not known deterministically, but varies randomly over the lifetime of the connection. A good illustration is an ATM network, where the required resources of each connection are unknown, and as statistical multiplexing takes place at each node, the network may experience an important degradation of the performance on some connections.

It has recently been shown that in some cases a condition of the type (7.1) is necessary and sufficient to ensure a given level of performance at a node. Hui [38] first showed, for a model of unbuffered resource, that the probability of resource overload could be held below a desired level by requiring the number n_j of connections accepted for a source of type j to lie in an admissible region of the type

$$\sum_{j=1}^{J} n_j \alpha_j \leq C, \tag{7.2}$$

where C represents the capacity of the resource, and α_j is determined by the statistical characteristics of the source of type j.

This result was later extended by Kelly [41] to models with buffered resource: For the **M/G/1** queue fed by sources of different type j, Kelly showed that both the mean delay and the probability that the delay exceeds a given threshold could be held below a certain level if the number n_j of calls accepted from source j satisfied a linear constraint of type (7.2). He referred to the constant α_j as the **effective bandwidth** of a source of type j. An effective bandwidth for the **D/G/1** queue with tail probabilities constraint is also derived in [41].

Gibbens and Hunt [32] found an effective bandwidth for the Uniform Arrivals and Services (UAS) or fluid model [2]: They consider a single communication channel with deterministic capacity, fed by a superposition of on/off fluid sources of different types. They show that there exists effective bandwidths $\alpha_i(\xi)$ such that the set $\{n : \mathbf{P}[x \geq b] \leq p\}$ of number of admissible calls for which the buffer overflow probability remains below a certain level satisfies

$$\left\{n : \sum_i \alpha_i(\xi) n_i < c\right\} \subseteq \{n : \mathbf{P}[x \geq b] \leq p\} \subseteq \left\{n : \sum_i \alpha_i(\xi) n_i \leq c\right\}, \tag{7.3}$$

when $b \to \infty$ and $\frac{\ln p}{b} \to \xi$.

Independently, Guérin et al. [37] derived an effective bandwidth for the fluid flow model, which they refer to as the **equivalent capacity**, for both individual and multiplexed connections under a buffer overflow probability constraint. The expression obtained is quite complex and approximations are used to compute the equivalent capacity.

At the same time, Elwalid and Mitra [30] were able to compute the effective bandwidth for Markov modulated fluid sources and Markov modulated Poisson sources. They consider the general Markov Modulated fluid model and show that there exists an effective bandwidth associated with the asymptotics of the performance criterion $\mathbf{P}[W > B] \leq p$, i.e., for p and B such that $\frac{\log p}{B} \to \xi$ in $[-\infty, 0]$. They find the effective bandwidth to be the maximal eigenvalue of the matrix $\Lambda - \frac{1}{\xi}M$ where M is the Markov chain generator, and Λ is the diagonal matrix of the input rates.

We point out that all the results discussed so far only rely on algebraic computations. The derivation of effective bandwidths via the theory of large deviations, namely through the existence of a LDP for the traffic stream, was introduced afterwards by Kesidis et al. [42].

In [42], a general approach to effective bandwidths under the buffer overflow probability constraint $\mathbf{P}[X > B] \leq e^{-B\delta}$ is presented. However the authors only give heuristics arguments. Those were later made rigorous by Glynn and Whitt [35] and Duffield and O'Connell [29], and general conditions for the existence of effective bandwidths were given. We shall come back to this large deviations approach to effective bandwidth in more details in Section 7.2.

As can be seen from this short review of the existing results, there is no formal definition for effective bandwidths. The reason is that the notion of effective bandwidth as well as its existence depends on the characteristics of the source, as well as on the model and the performance criterion chosen.

The most important property of the effective bandwidth is its separability among independent sources. Indeed, the separability property is the key to the derivation of a linear constraint on the number of streams of each type, which, when satisfied, ensures that the performance of the system is held above a certain level. The fact that the constraint is linear among the independent sources then allows the use of effective bandwidths to perform congestion control in networks such as ATM networks. We do not discuss here the use of effective bandwidth as a means of performing congestion control in data networks, and rather refer the interested reader to [21, 22, 30, 37].

Recently, researchers have focused on the effective bandwidth of sources fed to a single-server queueing system with deterministic capacity, under the buffer overflow probability criterion. In fact, in that case, the effective bandwidth is intrinsically linked to the buffer asymptotics. The available results are reviewed in more details in Section 7.2.

In principle, the effective bandwidth could be defined for sources fed to more general queueing systems under different performance criterion. In Section 7.2 we pursue this idea and show that an effective bandwidth for stationary and ergodic sources (satisfying a sample path LDP) fed to a single-server queueing system with time-varying capacities can be defined.

7.2 Effective bandwidth

In this section, we apply the results obtained on the asymptotics of Lindley processes to derive the effective bandwidth of traffic streams fed to a work-conserving single-server queue.

The following Proposition, from [57] will be useful in proving separability of the effective bandwidth. We first need a definition.

The **infimal convolution** [57, p. 34] of m convex functions $f_1, \ldots, f_m : \mathbb{R}^p \to [-\infty, \infty]$ is the mapping $g : \mathbb{R}^p \to [-\infty, \infty]$ defined by

$$g(x) = \inf_{\substack{x_i \in \mathbb{R}^p \\ i=1,\ldots,m}} \{f_1(x_1) + \cdots + f_m(x_m) : x_1 + \cdots + x_m = x\}, \quad x \in \mathbb{R}^p.$$

The infimal convolution of $f_1, \ldots, f_m$ is denoted by $f_1 \diamond \cdots \diamond f_m$, and is a convex mapping [57, Theorem 5.4 p. 33] whenever the convex mappings $f_1, \ldots, f_m$ are proper.

Proposition 7.1 ([57, Theorem 16.4 p. 145]) *For $m = 1, 2, \ldots$, and each $i = 1, \ldots, m$, let $f_i : \mathbb{R}^p \to [-\infty, \infty]$ be a proper convex function. Then, we have the following relationships on the Legendre-Fenchel transforms of $f_1, \ldots, f_m$ and of their infimal convolution:*

$$(f_1 \diamond \cdots \diamond f_m)^* = f_1^* + \cdots + f_m^*, \tag{7.4}$$

and

$$(\mathrm{cl} f_1 + \cdots + \mathrm{cl} f_m)^* = \mathrm{cl}(f_1^* \diamond \cdots \diamond f_m^*) \tag{7.5}$$

where for each $i = 1, \ldots, m$, $\mathrm{cl} f_i$ denotes the closure of f_i.

7.2.1 Traditional approach

Consider the traffic stream $\{a_{t+1},\ t = 0, 1, \ldots\}$, with the usual interpretation that a_{t+1} cells arrive at a network node during time slot $[t, t+1)$, $t = 0, 1, \ldots$. We offer this arrival stream to a fictitious work-conserving single server queue with constant release rate of c cells/slot and infinite buffer capacity. Under these operational assumptions, the buffer content sequence $\{q_t,\ t = 0, 1, \ldots\}$ evolves according to the Lindley recursion

$$q_0 = q; \quad q_{t+1} = [q_t + a_{t+1} - c]^+, \quad t = 0, 1, \ldots . \tag{7.6}$$

This recursion is of the form (5.1) discussed in Chapter 5, with driving sequence $\{a_{t+1} - c,\ t = 0, 1, \ldots\}$. By Proposition 5.4, if the arrival process $\{a_{t+1},\ t =$

$0, 1, \ldots\}$ couples with a stationary and metrically transitive sequence $\{\widehat{a}_{t+1},\ t = 1, 2, \ldots\}$ with $\mathbf{E}[\widehat{a}_1] < c$, then steady state is eventually reached in the sense that $q_t \Longrightarrow_t q_\infty$ for some $\mathbb{R}_+$-valued rv q_∞. Specializing Corollary 5.9 to this setup, under appropriate conditions we can obtain the buffer asymptotics

$$\lim_{b \to \infty} \frac{1}{b} \ln \mathbf{P}[q_\infty > b] = -\theta_a^\star \tag{7.7}$$

for some positive constant $\theta_a^\star$. More precisely, we write

$$\Lambda_t^a(\theta) \equiv \frac{1}{t} \ln \mathbf{E}[\exp(\theta(\widehat{a}_1 + \ldots + \widehat{a}_t))], \quad \theta \in \mathbb{R}. \tag{7.8}$$

Corollary 7.2 *Assume the sequence $\{\widehat{a}_{t+1},\ t = 0, 1, \ldots\}$ to satisfy the following conditions:*

1. *The limit $\Lambda_a(\theta) \equiv \lim_{t\to\infty} \Lambda_t^a(\theta)$ exists (possibly as an extended real number) for all θ in $\mathbb{R}$;*

2. *The set $\Theta_a \equiv \{\theta > 0 : \Lambda_a(\theta) < c\theta\}$ is non-empty, and $\Lambda_t^a(\theta) < \infty$ for all θ in Θ_a and $t = 1, 2, \ldots$;*

3. *Either the process $\{t^{-1}\widehat{A}(1, t),\ t = 1, 2, \ldots\}$ satisfies the LDP with strictly convex good rate function $\Lambda_a^\star$, or the conditions of the Gärtner-Ellis Theorem are satisfied.*

Then, the buffer asymptotics (7.7) hold with $\theta_a^\star$ given by

$$\theta_a^\star = \sup\{\theta > 0 : \Lambda_a(\theta) < c\theta\} = \inf_{y>0} \frac{\Lambda_a^\star(c + y)}{y}, \tag{7.9}$$

and for all $\delta > 0$,

$$\lim_{b\to\infty} \frac{1}{b} \ln \mathbf{P}[q_\infty > b] \le -\delta \iff \Lambda_a(\delta) \le c\delta. \tag{7.10}$$

Proof: We apply Corollary 5.9 with $\xi_{t+1} = a_{t+1} - c$ for each $t = 1, 2, \ldots$. From the definitions of the limits Λ and Λ_a, we clearly have

$$\Lambda(\theta) = \Lambda_a(\theta) - c\theta, \quad \theta \in \mathbb{R},$$

and the result easily follows upon noting that

$$\Lambda^\star(y) = \sup_{\theta\in\mathbb{R}} (y\theta - \Lambda(\theta)) = \sup_{\theta\in\mathbb{R}} (y\theta + c\theta - \Lambda_a(\theta)) = \Lambda_a^\star(y + c), \quad y \in \mathbb{R}.$$

■

With this in mind, the effective bandwidth $\alpha_a(\cdot)$ of the traffic stream $\{a_{t+1},\ t = 0,1,\ldots\}$ is then simply

$$\alpha_a(\delta) \equiv \frac{\Lambda_a(\delta)}{\delta}, \quad \delta > 0, \tag{7.11}$$

so that in view of Corollary 7.2, we obtain

$$\lim_{b\to\infty} \frac{1}{b} \ln \mathbf{P}[q_\infty > b] \leq -\delta \quad \Longleftrightarrow \quad \alpha(\delta) \leq c, \quad \delta > 0.$$

The effective bandwidth $\alpha(\delta)$ then represents the smallest release rate that supports the QoS level characterized by

$$\mathbf{P}[x_\infty > b] \sim \exp(-b\delta), \quad b \to \infty . \tag{7.12}$$

Finally, the notion of effective bandwidth would not be complete unless we had separability for independent sources, which we show next.

To this end, we consider a collection of independent traffic sources $\{\{a^k_{t+1},\ t = 0,1,\ldots\},\ k = 1,\ldots,K\}$. We offer the aggregate stream to a work-conserving single-server queue with constant capacity c. If each of the sources $\{a^k_{t+1},\ t = 0,1,\ldots\}$ couples with its stationary and metrically transitive version $\{\widehat{a}^k_{t+1},\ t = 0,1,\ldots\}$, then the aggregate source also couples with a stationary and metrically transitive random sequence: Stationarity follows from the fact that, by independence, the sources are jointly stationary, and metric transitivity is easily seen to hold through the Ergodic Theorem. Under these assumptions, the workload $\{q_{t+1},\ t = 0,1,\ldots\}$ of the queue weakly converges to the steady-state queue length q_∞ (Proposition 5.4), and the question naturally arises whether we have an equivalence of the type

$$\sum_{k=1}^{K} \alpha_k(\delta) \leq c \quad \Longleftrightarrow \quad \lim_{b\to\infty} \frac{1}{b} \ln \mathbf{P}[q_\infty > b] \leq -\delta, \quad \delta > 0$$

for some function α_k of the statistics of the k^{th}-source. A partial answer is given by the following Proposition. We write

$$\Lambda^k_t(\theta) \equiv \frac{1}{t} \ln \mathbf{E}\left[e^{\theta(\widehat{a}^k_1+\ldots+\widehat{a}^k_t)}\right], \quad \theta \in \mathbb{R}, \quad k = 1,\ldots,K,$$

$\Lambda_k(\theta) \equiv \lim_{t\to\infty} \Lambda^k_t(\theta)$, whenever the limit exists, and $\Lambda^\Sigma_t(\theta) \equiv \sum_{k=1}^K \Lambda^k_t(\theta)$ for all θ in $\mathbb{R}$.

Proposition 7.3 *Assume the following conditions to hold:*

1. *The limit* $\Lambda_\Sigma(\theta) \equiv \lim_{t\to\infty} \Lambda_t^\Sigma$ *exists (possibly as an extended number) for all* θ *in* $\mathbb{R}$ *;*

2. *The set* $\Theta_\Sigma \equiv \{\theta > 0 : \Lambda_\Sigma(\theta) < c\theta\}$ *is non-empty and* $\Lambda_t^\Sigma(\theta) < \infty$ *for all* θ *in* Θ_Σ *and* $t = 1, 2, \ldots$*;*

3. *Either, for each* $k = 1, \ldots, K$*, the sample mean sequence* $\{t^{-1}\widehat{A}^k(1,t),\ t = 1, 2, \ldots\}$ *satisfies the LDP with strictly convex good rate function* Λ_k^**, and* Λ_k *is lower semi-continuous, or the conditions of the Gärtner-Ellis Theorem are satisfied for the sample mean sequence associated with each source* $k = 1, \ldots, K$.

Then, for any $\delta > 0$*, we have*

$$\lim_{b\to\infty} \frac{1}{b} \ln \mathbf{P}[q_\infty > b] \ \leq \ \delta \quad \Longleftrightarrow \quad \sum_{k=1}^{K} \alpha_k(\delta) < c \tag{7.13}$$

where for each $k = 1, \ldots, K$, $\alpha_k(\delta) \equiv \dfrac{\Lambda_k(\delta)}{\delta}$.

Proof: We apply Corollary 5.9 with

$$\xi_{t+1} \ = \ \sum_{k=1}^{K} a_{t+1}^k + c, \quad t = 0, 1, \ldots.$$

Clearly, from the enforced assumptions, it suffices to show that Assumption **(3)** of Corollary 5.9 is satisfied. If the conditions of the Gärtner-Ellis Theorem hold for each source, it is plain from the equality

$$\Lambda_\Sigma(\theta) \ = \ \sum_{k=1}^{K} \Lambda_k(\theta), \quad \theta \in \mathbb{R},$$

that they also hold for the aggregate source, and the desired result follows easily.

On the other hand, if the LDP holds for each source, then $\mathbb{R}$ being Polish, and the sources independent, Corollary 1.9 yields the LDP jointly for the sample mean sequences in the product space $\mathbb{R}^K$. Therefore, by the Contraction Principle, the sample mean of the aggregate traffic stream satisfies the LDP with good rate function $r_\Sigma : \mathbb{R} \to [0, \infty]$ given by

$$r_\Sigma(z) \ = \ \inf\left\{ \sum_{k=1}^{K} \Lambda_k^*(x_k) : z = x_1 + \ldots + x_k,\ x_1, \ldots, x_K \in \mathbb{R} \right\}, \quad z \in \mathbb{R}$$

i.e.,

$$r_\Sigma \ = \ \Lambda_1^* \diamond \cdots \diamond \Lambda_K^*.$$

By Proposition 7.1, we find from lower semi-continuity and convexity of Λ_k^* that [57, Theorem 12.2 p. 104]

$$r_\Sigma^* \ = \ \sum_{k=1}^{K} \Lambda_k^{**} \ = \ \sum_{k=1}^{K} \Lambda_k \ = \ \Lambda_\Sigma,$$

so that r_Σ being also lower semi-continuous, and convex as the infimal convolution of convex proper functions, we finally get [57, Theorem 12.2 p. 104]

$$r_\Sigma = r_\Sigma^{**} = \Lambda_\Sigma^*.$$

Corollary 5.9 then easily yields the desired result. ∎

This last Proposition is given as Corollary 3.1 in [22], but the arguments given in the proof seem incomplete. In particular, it is unclear for us how the assumptions **(1)** and **(2)** of Corollary 5.9 satisfied for each traffic stream, translate to the same assumption for the aggregate traffic stream. Also, in the situation where each of the stream satisfies the LDP, it is not a priori obvious that the rate function associated with the LDP for the aggregate traffic stream is the Legendre-Fenchel transform of the logarithmic moment generating function of the aggregate stream, as implicitly assumed in [22]. Hence our additional assumption that Λ_k be lower semi-continuous in that case.

In view of Proposition 7.3, the performance of the multiplexer can be evaluated by the means of a linear constraint on the statistics of the sources. This very interesting separability property is really the key property of the effective bandwidth, which renders that notion so useful in performing admission control in data networks.

In principle, the notion of effective bandwidth could be defined, under suitable assumptions, for a multiplexer with time-varying capacities. The effective bandwidth result would then read as

$$\lim_{b\to\infty} \frac{1}{b} \ln \mathbf{P}[q_\infty > b] \le -\delta \iff \sum_{k\in\mathcal{K}} n_k \alpha_k(\delta) \le \frac{-\Lambda_c(-\delta)}{\delta}, \quad \delta > 0$$

where Λ_c is the logarithmic moment generating function associated with the stationary and metrically transitive version of the capacity sequence $\{c_{t+1},\ t = 0, 1, \ldots\}$. The computation is carried over in the sample path situation, in the next section.

7.2.2 Sample path approach

This section is devoted to a sample path approach to effective bandwidth, as we show that this notion exists under the assumption that the input sequence satisfy a sample path LDP with some mild conditions on the rate function. The following Lemma is the key to the derivation of the effective bandwidth in the sample path framework.

Lemma 7.4 *Assume that the buffer asymptotics*

$$\lim_{b\to\infty} \frac{1}{b} \ln \mathbf{P}[x_\infty > b] = -\inf_{y>0} \frac{1}{y} r(y)$$

hold for some rate function $r : \mathbb{R} \to [0, \infty]$.

Then we have the equivalence

$$\lim_{b\to\infty} \frac{1}{b} \ln \mathbf{P}[x_\infty > b] \le -\delta \iff r^*(\delta) \le 0, \quad \delta > 0 \tag{7.14}$$

where r^* *is the Legendre-Fenchel transform of the rate function* r.

Proof: Fix $\delta > 0$. From the buffer asymptotics, we readily obtain

$$\begin{aligned} \lim_{b\to\infty} \frac{1}{b} \ln \mathbf{P}[x_\infty > b] \le -\delta &\iff \inf_{y>0} \frac{1}{y} r(y) \ge \delta \\ &\iff r(y) \ge \delta y, \quad y > 0 \\ &\iff \delta y - r(y) \le 0, \quad y > 0 \\ &\iff \sup_{y>0} (\delta y - r(y)) \le 0 . \end{aligned} \tag{7.15}$$

Next, we note that for $y \le 0$, we have $\delta y \le 0$, so that, in view of the fact that r is a rate function and is therefore non-negative, we get

$$\delta y - r(y) \le -r(y) \le 0, \quad y \le 0,$$

or equivalently,

$$\sup_{y\le 0} (\delta y - r(y)) \le 0.$$

As this last inequality always holds, we readily obtain from (7.15) that

$$\begin{aligned} \lim_{b\to\infty} \frac{1}{b} \ln \mathbf{P}[x_\infty > b] \le -\delta &\iff \sup_{y\in\mathbb{R}} (\delta y - r(y)) \le 0 \\ &\iff r^*(\delta) \le 0. \end{aligned}$$

■

We begin with the somewhat simpler situation consisting of a single source fed to a multiplexer with **time-varying** capacity independent of the source. Let $\{a_{t+1},\ t = 0, 1, \ldots\}$ and $\{c_{t+1},\ t = 0, 1, \ldots\}$ denote respectively the arrivals sequence of the source, and the capacity sequence of the multiplexer. The queue length sequence $\{q_{t+1},\ t = 0, 1, \ldots\}$ is then given by a recursion of the form

(7.6). As usual, we assume that each of the arrivals and capacity sequences couples with its respective stationary and metrically transitive version, denoted by $\{\widehat{a}_{t+1},\ t = 0,1,\ldots\}$ and $\{\widehat{c}_{t+1},\ t = 0,1,\ldots\}$. Under these assumptions and the stability condition $\mathbf{E}[\widehat{a}_1] < \mathbf{E}[\widehat{c}_1]$, the queue length at the multiplexer admits a steady-state regime in the sense that the sequence $\{q_{t+1},\ t = 0,1,\ldots\}$ weakly converges to a non-negative random variable q_∞ (Proposition 6.2). Under the additional assumption that both sequences satisfy the sample path LDP, the next Proposition gives the effective bandwidth result.

We say that a $\mathbb{R}$-valued random sequence $\{x_n,\ n = 1,2,\ldots\}$ satisfies the sample path LDP with good rate function of the integral form, if its associated family of partial sum processes $\{X_n(\cdot),\ n = 1,2,\ldots\}$ satisfies the LDP in $(D[0,1],\tau_\infty)$ with good rate function $I_X : D[0,1] \to [0,\infty]$ given by

$$I_X(\varphi) = \begin{cases} \int_0^1 r_x(\dot{\varphi}(t))\,dt & \text{if } \varphi \in AC_0[0,1] \\ \infty & \text{otherwise} \end{cases} \tag{7.16}$$

where $r_x : \mathbb{R} \to [0,\infty]$ is the **convex** good rate function governing the LDP of the sample mean sequence $\{n^{-1}X(1,n),\ n = 1,2,\ldots\}$. We refer to the rate function r_x as the integrand of the rate function I_X.

Corollary 7.5 *Assume the sequences $\{\widehat{a}_{t+1},\ t = 0,1,\ldots\}$ and $\{\widehat{c}_{t+1},\ t = 0,1,\ldots\}$ each satisfies the sample path LDP with good rate function I_A and I_C of the integral form (7.16) with integrand r_a and r_c, respectively. Let $r_{a-c} : \mathbb{R} \to [0,\infty]$ be the (convex) good rate function associated with the sample mean sequence $\{\widehat{A}_n(1) - \widehat{C}_n(1),\ n = 1,2,\ldots\}$, i.e.,*

$$r_{a-c}(z) \equiv \inf_{x,y\in\mathbb{R}} \{r_{a,c}(x,y) : z = x - y\}, \quad z \in \mathbb{R}, \tag{7.17}$$

and assume that $\inf_{y>0} \frac{1}{y}\, r_{a-c}(y) > 0$.

Then the following effective bandwidth result hold:

$$\lim_{b\to\infty} \frac{1}{b} \ln \mathbf{P}[q_\infty > b] \le -\delta \iff r_a^*(\delta) \le -r_c^*(-\delta), \quad \delta > 0. \tag{7.18}$$

Proof: Under the enforced assumptions, Assumptions **(A1)** and **(A2)** of Section 6.1 are satisfied. Therefore, Corollary 6.4 together with Lemma 7.4, yields

$$\lim_{b\to\infty} \frac{1}{b} \ln \mathbf{P}[q_\infty > b] \le -\delta \iff r_{a-c}^*(\delta) \le 0, \quad \delta > 0$$

where the good rate function r_{a-c} is given by (7.17). under the independence assumption, we have $r_{a,c}(x,y) = r_a(x) + r_c(y)$ for all x, y in $\mathbb{R}$, and by a simple change of the dummy variable y into $-y$ in the infimum above, we get

$$r_{a-c}(z) = \inf_{x,y \in \mathbb{R}} \{r_a(x) + r_c(-y) : z = x + y\}, \quad z \in \mathbb{R}$$

and Proposition 7.1 (with $f_1 = r_a$ and $f_2(y) = r_c(-y)$, y in $\mathbb{R}$) yields

$$r^*_{a-c} = r^*_a + g^*.$$

The result now follows from the relation

$$g^*(\theta) = r^*_c(-\theta), \quad \theta \in \mathbb{R}.$$

which is readily established by some algebra. ■

If we define the effective bandwidth of the source by $\alpha(\delta) \equiv \frac{r^*_a(\delta)}{\delta}$, Corollary 7.5 then states that

$$\lim_{b \to \infty} \frac{1}{b} \ln \mathbf{P}[q_\infty > b] \leq -\delta \iff \alpha(\delta) \leq \frac{-r^*_c(-\delta)}{\delta}, \quad \delta > 0.$$

In particular, if the capacity c_{t+1} is constant, then by Theorem 3.24, the family $\{\widehat{C}_n(\cdot),\ n = 1, 2, \ldots\}$ satisfies the assumption of Corollary 7.5 with r_c being the Legendre-Fenchel transform of the logarithmic moment generating function $\Lambda_c(\theta) = c\theta$, θ in $\mathbb{R}$. Therefore $r^*_c = \Lambda^{**} = \Lambda$, and we obtain the traditional formulation of the effective bandwidth, i.e.,

$$\lim_{b \to \infty} \frac{1}{b} \ln \mathbf{P}[q_\infty > b] \leq -\delta \iff \alpha(\delta) \leq c, \quad \delta > 0.$$

To complete the definition of this sample path effective bandwidth, it remains to prove its separability property, which we do next.

At this point, we have not been able to prove that the rate function associated with the sample path LDP of the summation of independent sources is of the integral form (7.16) with integrand the good rate function of the LDP associated with the sample mean of the aggregate traffic stream. This unfortunate glitch forces us to make stronger assumptions than simply assuming that each source satisfies the sample path LDP to establish the separability property of the effective bandwidths of independent sources.

We consider K independent streams fed to the same multiplexer as in the single source situation. We make the usual coupling assumptions, and we denote by $\{\widehat{a}^k_{t+1},\ t = 0, 1, \ldots\}$ the corresponding stationary and metrically transitive versions. We also assume the stability condition

$$\sum_{k=1}^{K} \mathbf{E}\left[\widehat{a}^k_1\right] < \mathbf{E}[\widehat{c}_1] .$$

Proposition 7.6 *Assume that the sequence $\{\widehat{c}_{t+1},\ t = 0, 1, \ldots\}$, and for each $k = 1, \ldots, K$, the sequence $\{\widehat{a}^k_{t+1},\ t = 0, 1, \ldots\}$, each satisfies the sample path LDP with good rate function $I_C, I_{A_k} : D[0, \infty] \to [0, \infty]$ of the integral form (7.16) with integrand $r_c, r_{a_k} : \mathbb{R} \to [0, \infty]$, respectively. Assume further that*

$$\inf_{\substack{\psi_i \in AC_0[0,K] \\ i=1,\ldots,K}} \left\{ \int_0^1 \sum_{k=1}^{K} r_{a_k}\left(\dot{\psi}_k(t)\right) dt : \varphi = \psi_1 + \ldots + \psi_K \right\} = \int_0^1 r_a(\dot{\varphi}(t))\, dt \quad (7.19)$$

for all φ in $AC_0[0, 1]$, where $r_a : \mathbb{R} \to [0, \infty]$ is given by

$$r_a = r_{a_k} \diamond \cdots \diamond r_{a_K}.$$

Then the following effective bandwidth result holds:

$$\lim_{b\to\infty} \frac{1}{b} \ln \mathbf{P}[q_\infty > b] \le -\delta \iff \sum_{k=1}^{K} \alpha_k(\delta) \le \frac{-r_c^*(-\delta)}{\delta}, \quad \delta > 0$$

with for each $k = 1, \ldots, K$, $\alpha_k(\delta) \equiv \dfrac{r^*_{a_k}(\delta)}{\delta}$, $\delta > 0$.

Proof: Under the enforced assumptions, by Proposition 2.17 and independence of the sources, Corollary 1.9 yields the LDP for the family $\{(\widehat{A}^1_n(\cdot), \ldots, \widehat{A}^K_n(\cdot)),\ n = 1, 2, \ldots\}$ in the product space $(D[0, 1], \tau_0)^K$ with good rate function $I : D[0, 1]^K \to [0, \infty]$ given by

$$I(\varphi_1, \ldots, \varphi_K) = I_{A_1}(\varphi_1) + \ldots + I_{A_K}(\varphi_K), \quad \varphi_1, \ldots, \varphi_K \in D[0, 1].$$

By Lemma 2.8, it is plain that the mapping $(z_1, \ldots, z_K) \to z_1 + \ldots + z_K$ is Borel-measurable (with respect to the product Skorohod and Skorohod topologies) and S-continuous on $AC_0[0, 1]^K$. Therefore, as the effective domain of I is contained in $AC_0[0, 1]^K$, the Contraction Principle yields the LDP for the family $\{\widehat{A}_n(\cdot),\ n = 1, 2, \ldots\}$ where we have set $a_{t+1} \equiv a^1_{t+1} + \ldots + a^K_{t+1}$, $t = 0, 1, \ldots$, so that for each $n = 1, 2, \ldots$, $\widehat{A}_n(\cdot) = \widehat{A}^1_n(\cdot) + \ldots + \widehat{A}^K_n(\cdot)$. The associated good rate function $I_A : D[0, 1] \to [0, \infty]$ is given by

$$I_A(\varphi) = \inf_{\substack{\psi_i \in D[0,1] \\ i=1,\ldots,K}} \{I_{A_1}(\psi_1) + \ldots + I_{A_K}(\psi_K) : \varphi = \psi_1 + \ldots + \psi_K\}, \quad \varphi \in D[0, 1],$$

and in view of the assumption (7.19) and of the expression of the rate functions I_{A_k} for $k = 1, \ldots, K$, we easily obtain

$$I_A(\varphi) = \begin{cases} \int_0^1 r_a(\dot{\varphi}(t))\, dt & \text{if} \quad \varphi \in AC_0[0,1] \\ \infty & \text{otherwise} \end{cases}$$

where r_a is as given in the statement of the Proposition.

Now, by a simple application of Corollary 7.5, we obtain the equivalence

$$\lim_{b \to \infty} \frac{1}{b} \ln \mathbf{P}[q_\infty > b] \leq -\delta \iff r_a^*(\delta) \leq -r_c^*(-\delta), \quad \delta > 0.$$

Finally, Proposition 7.1 yields the formula

$$r_a^*(\delta) = \sum_{k=1}^{K} r_{a_k}^*(\delta), \quad \delta > 0,$$

and the desired equivalence follows. ■

We note that in a different setup, but under similar conditions, the author in [16] uses the fact (without stating any justification) that the equality (7.19) holds.

In view of the results obtained in this chapter, it is plain that large deviations techniques are very useful in deriving buffer asymptotics or effective bandwidths of sources fed to a work-conserving single-server multiplexer with constant or time-varying capacities. However, although the results obtained are interesting, and certainly promising for the dimensioning of buffers, the assumptions required for the effective bandwidth to exist and the separability property to hold are quite strong, and not easily checked. It turns out that a sample path approach does simplify greatly the set of assumptions needed, but a sample path LDP is also a stronger assumption than a LDP in $\mathbb{R}$. The sample path approach yields, under a single assumption (the sample path LDP with good rate function of the integral form for the source), all the results which already existed in the literature, except the separability property which requires an additional technical assumption.

We note that the effective bandwidth of a stream is completely defined by the rate function associated with the LDP for the sample mean sequence of the stream. Thus, in the next section, we attempt to use the results derived in Chapter 6 to establish buffer asymptotics and effective bandwidths for queues in series.

7.3 Queues in series

The purpose of this section is to lay the grounds for the application of our large deviations and effective bandwidths to queues in series. We begin by introducing the analytical model, and then discuss the applicability of our earlier results on buffer asymptotics and effective bandwidths the queues in series. Finally, we present the conclusions of our investigation, and some axis of further research.

7.3.1 Model and assumptions

Of interest in this section is a system consisting of K tandem queues in series, numbered from 1 to K. For $k = 1, \ldots, K$, the output of the k^{th} queue is directly fed into the $k+1^{\text{th}}$ one.

Each queue is a discrete-time single-server queue with capacity sequence $\{c_{t+1}^k,\ t = 0, 1, \ldots\}$, $k = 1, \ldots, K$, and the input to the first queue is a traffic stream, represented by the $\mathbb{R}$-valued random sequence $\{a_{t+1}^1,\ t = 0, 1, \ldots\}$. For $k = 1, \ldots, K$, we denote the input to the k^{th} queue by $\{a_{t+1}^k,\ k = 0, 1, \ldots\}$ and its output by $\{b_{t+1}^k,\ t = 0, 1, \ldots\}$, so that we immediately have the relations

$$a_{t+1}^{k+1} = b_{t+1}^k, \quad t = 0, 1, \ldots, \quad k = 1, \ldots, K-1. \tag{7.20}$$

The queue length sequence $\{q_{t+1}^k,\ t = 0, 1, \ldots\}$ of the k^{th} queue is governed by the recursion

$$q_0^k = q^k; \quad q_{t+1}^k = \left[q_t^k + a_{t+1}^k - c_{t+1}^k\right]^+, \quad t = 0, 1, \ldots, \quad k = 1, \ldots, K \tag{7.21}$$

where the initial condition q^k is a $\mathbb{R}_+$-valued random variable. In vector notation, we have

$$Q_0 = Q; \quad Q_{t+1} = [Q_t + \xi_{t+1}]^+, \quad t = 0, 1, \ldots \tag{7.22}$$

with $Q \equiv (q^1, \ldots, q^K)$, and for each $t = 1, 2, \ldots$, $Q_t \equiv (q_t^1, \ldots, q_t^K)$ and $\xi_t \equiv (a_t^1 - c_t^1, \ldots, a_t^K - c_t^K)$.

To avoid trivial situations, we impose the system to be stable, and as we have in mind to apply the stability results induced by Proposition 5.4, we assume that the arrivals and capacities sequences couple with their stationary versions, i.e.,

Assumption (S1)
The $\mathbb{R}^{K+1}$-valued sequence $\{(a_{t+1}^1, c_{t+1}^1, \ldots, c_{t+1}^K),\ t = 0, 1, \ldots\}$ couples with the stationary and metrically transitive sequence $\{(\widehat{a}_{t+1}^1, \widehat{c}_{t+1}^1, \ldots, \widehat{c}_{t+1}^K),\ t = 0, 1, \ldots\}$, and $\mathbf{E}[\widehat{a}_1^1] < \min_{k=1,\ldots,K} \mathbf{E}\left[\widehat{c}_1^k\right]$. Furthermore, the arrivals and

capacities sequences are mutually independent.

We emphasize the requirement that the stationary and metrically transitive versions of the arrivals and capacities sequences be **jointly** stationary and metrically transitive. Indeed, even in presence of independence, although joint stationarity follows from stationarity of each sequence, metric transitivity of each of them does not necessarily imply joint metric transitivity.

In addition, following the terminology introduced in Section 7.2.2, we assume that the arrivals and capacity sequences satisfy the sample path LDP.

Assumption (S2) *The sequences $\{\widehat{a}^1_{t+1},\ t=0,1,\ldots\}$ and for each $k=1,\ldots,K$ $\{\widehat{c}^k_{t+1},\ t=0,1,\ldots\}$, each satisfies the sample path LDP with good rate functions $I_A, I_{C_k} : D[0,1] \to [0,\infty]$ of the integral form (7.16) with integrand $r_a, r_{c_k} : \mathbb{R} \to [0,\infty]$, respectively.*

Although we chose here to consider independent queues in series, the assumption that the arrivals and capacities sequences jointly satisfy the sample path LDP in the appropriate product space would yield similar results.

With the model defined, and the assumptions in mind, we now carry out the steady-state analysis of the system.

7.3.2 Buffer asymptotics and Effective bandwidth

The first step in studying the buffer asymptotics at each queue consists in studying the steady-state queue length. From Proposition 6.5, we infer that the coupling assumptions will propagate along the queues, thus enabling us to conclude stability from the usual joint coupling of the input and capacity sequences with their stationary and metrically transitive versions. This is made precise by the following Proposition. For each $t=0,1,\ldots$, we denote by $\mathbf{c}_{t+1}$ the vector $(c^1_{t+1},\ldots,c^K_{t+1})$.

Proposition 7.7 *Under Assumptions* **(S1)**, *for each $k=1,\ldots,K$, the $\mathbb{R}^{K+2}$-valued random sequence $\{(a^k_{t+1}, \mathbf{c}_{t+1}, q^k_{t+1}),\ t=0,1,\ldots\}$ couples with its stationary and metrically transitive version $\{(\widehat{a}^k_{t+1}, \widehat{\mathbf{c}}_{t+1}, \widehat{q}^k_{t+1}),\ t=0,1,\ldots\}$ where the stationary queue length $\{\widehat{q}^k_{t+1},\ t=0,1,\ldots\}$ is defined by the recursion*

$$\widehat{q}^k_0 = \widehat{q}^k_{st}; \quad \widehat{q}^k_{t+1} = \left[\widehat{q}^k_t + \widehat{a}^k_{t+1} - \widehat{c}^k_{t+1}\right]^+, \quad t=0,1,\ldots \tag{7.23}$$

with

$$\widehat{q}^k_{st} \equiv \max_{t=0,1,\ldots} \left(\widehat{A}^k(-t,0) - \widehat{C}^k(-t,0)\right).$$

Consequently, for each $k = 1, \ldots, K$, $q_t^k \Longrightarrow_t q_\infty^k$ *for some* $\mathbb{R}_+$*-valued rv* q_∞^k *with* $q_\infty^k =_{st} \widehat{q}_{st}^k$.

Proof: We use an induction argument on k. For $k = 1$, the coupling follows easily by Proposition 6.2, whereas stationarity and metric transitivity of $\{(\widehat{a}_{t+1}^1, \widehat{c}_{t+1}, \widehat{q}_{t+1}^1),\ t = 0, 1, \ldots$ follows from that of $\{(\widehat{a}_{t+1}^1, \widehat{c}_{t+1}),\ t = 0, 1, \ldots$, the expression of $\{\widehat{q}_{t+1}^1,\ t = 0, 1, \ldots\}$, and Lemma A.1.2.7 in [15, p. 303].

We next assume that the statement of the Proposition is true for $k = l$, and show that it is then also true for $k = l + 1$. Indeed, if the statement holds for $k = l$, then the output process of the l^{th} queue couples with its stationary version $\{\widehat{b}_{t+1}^l,\ t = 0, 1, \ldots\}$ (Proposition 6.5) given by

$$\widehat{b}_{t+1}^l \;=\; \widehat{a}_{t+1}^l - (\widehat{q}_{t+1}^l - \widehat{q}_t^l), \quad t = 0, 1, \ldots \tag{7.24}$$

with $\{\widehat{q}_{t+1}^l,\ t = 0, 1, \ldots\}$ as given by (7.23) with $k = l$. Therefore, from the induction hypothesis on the coupling and (7.20), we readily see that the sequence $\{(a_{t+1}^{l+1}, c_{t+1}),\ t = 0, 1, \ldots\}$ couples with $\{(\widehat{a}_{t+1}^{l+1}, \widehat{c}_{t+1}),\ t = 0, 1, \ldots\}$, where for each $t = 0, 1, \ldots$, $\widehat{a}_{t+1}^{l+1} = \widehat{b}_{t+1}^l$, as given in (7.24). Stationarity and metric transitivity of $\{(\widehat{a}_{t+1}^{l+1}, \widehat{c}_{t+1}),\ t = 0, 1, \ldots\}$ follows from that of $\{(\widehat{a}_{t+1}^l, \widehat{c}_{t+1}, \widehat{q}_{t+1}^l),\ t = 0, 1, \ldots\}$, the expression for a_{t+1}^{l+1}, and Lemma A.1.2.7 in [15, p. 303]. Therefore, from (7.20) and (7.24) we also have

$$\mathbf{E}\left[\widehat{a}_1^{l+1}\right] = \mathbf{E}\left[\widehat{b}_1^l\right] \;=\; \mathbf{E}\left[\widehat{a}_1^l\right] = \ldots = \mathbf{E}\left[\widehat{a}_1^1\right] < \mathbf{E}\left[\widehat{C}_1^{l+1}\right]$$

and under Assumption **(S1)**, we conclude by Proposition 6.2 that the queue length sequence $\{q_{t+1}^{l+1},\ t = 1, 2, \ldots\}$ couples with the sequence $\{\widehat{q}_{t+1}^{l+1},\ t = 0, 1, \ldots\}$ defined by the recursion

$$\widehat{q}_0^{l+1} \;=\; \widehat{q}_{st}^{l+1}; \quad \widehat{q}_{t+1}^{l+1}(t) \;=\; \left[\widehat{q}_t^{l+1} + \widehat{a}_{t+1}^{l+1} - \widehat{c}_{t+1}^{l+1}\right]^+, \quad t = 0, 1, \ldots$$

with

$$\widehat{q}_{st}^{l+1} \;=\; \max_{t=0,1,\ldots} \left(\widehat{A}^{l+1}(-t, 0) - \widehat{C}^{l+1}(-t, 0)\right).$$

In view of the coupling of $\{(a_{t+1}^{l+1}, c_{t+1}),\ t = 0, 1, \ldots\}$ with $\{(\widehat{a}_{t+1}^{l+1}, \widehat{c}_{t+1}),\ t = 0, 1, \ldots\}$, we then easily obtain that of $\{(a_{t+1}^{l+1}, c_{t+1}, q_{t+1}^{l+1}),\ t = 0, 1, \ldots\}$ with $\{(\widehat{a}_{t+1}^{l+1}, \widehat{c}_{t+1}, \widehat{q}_{t+1}^{l+1}),\ t = 0, 1, \ldots\}$, and stationarity and metric transitivity of the latter follows from that of $\{(\widehat{a}_{t+1}^{l+1}, \widehat{c}_{t+1}),\ t = 0, 1, \ldots\}$, the expression for $\widehat{q}_{t+1}^{l+1}$ and Lemma A.1.2.7 in [15, p. 303].

The statement of the Proposition thus holds true for $l = k + 1$, which completes the proof. ■

One of the nice features of a sample path approach to the analysis of the large deviations behavior of $G/G/1$ queues, is that, as demonstrated in Section 6.3, a sample path LDP jointly on the arrivals and capacity sequences carries over to the **stationary** version of the output process.

However, as pointed out at the end of Chapter 6, we do not know at this point whether the associated rate function is of the integral form with convex integrand r_b satisfying $\inf_{y>0} \frac{1}{y} r_b(y) > 0$, which **prevents** us from applying the results inductively on the queues.

Although the integral form of the rate function of the output is required in order to obtain the buffer asymptotics of the next queue and therefore the existence of an effective bandwidth, it could very well be the case that the sample path LDP for the input propagates along the queues without this special form of the rate function.

To summarize, two technical problems prevent us at this point to apply our sample path approach to buffer asymptotics and effective bandwidth inductively to queues in series:

i) The equality in (7.19), which is related to the existence of measurable selections;

ii) The derivation of an integral closed form for the rate function of the stationary output process.

Chapter 8

Conclusions and Future Research

The results obtained in this Dissertation are for the most part of a theoretical nature, and can be classified in two categories. The first category encompasses results on large deviations for extensions of partial sum processes on the entire time axis $(-\infty, +\infty)$, while the second one consists of results on the large deviations behavior of Lindley processes or single-server queues. In addition, in a somewhat more applied context, we have derived the effective bandwidth of a general class of queueing systems, under a simple assumption, namely the sample path LDP of the inputs to the system.

We come back to the results obtained in Chapters 2 and 3: There, it was shown that under the sole assumption that a family of partial sum processes $\{X_n(\cdot),\ n = 1, 2, \ldots\}$ satisfies the LDP in $D[0,1]$ with good rate function $I : D[0,1] \to [0,\infty]$, the family $\{X_n^\infty(\cdot),\ n = 1, 2, \ldots\}$ of its extensions on the entire half-line $[0,\infty)$ satisfies the LDP in the space $D[0,\infty)$ with good rate function $I_\infty : D[0,\infty) \to [0,\infty]$. Furthermore, if the rate function I is of the usual integral form

$$I(\varphi) = \begin{cases} \displaystyle\int_0^1 r\left(\dot{\varphi}(t)\right) dt & \text{if} \quad \varphi \in AC_0[0,1] \\ \infty & \text{otherwise,} \end{cases} \tag{8.1}$$

for some rate function $r : \mathbb{R} \to [0,\infty]$, then the new rate function can be **explicitly** computed, and is given by

$$I^\infty(\varphi) = \begin{cases} \displaystyle\int_0^\infty r\left(\dot{\varphi}(t)\right) dt & \text{if} \quad \varphi \in AC_0[0,\infty) \\ \infty & \text{otherwise.} \end{cases}$$

For a stationary random sequence, this last result transposes to one even more interesting: If the family $\{X_n(\cdot),\ n = 1, 2, \ldots\}$ satisfies the LDP in $D[0,1]$ with good rate function $I : D[0,1] \to [0,\infty]$ such that the elements of the effective domain are continuous at $t = \frac{1}{2}$, then the family $\{(X_n^\infty(\cdot), X_n^{\infty,-}(\cdot)),\ n =$

$1, 2, \ldots\}$ of its extensions on the **entire past and future** jointly satisfies the LDP in the product space $D[0,\infty) \times D_l[0,\infty)$ with good rate function $I^{\infty}_{+,-}$: $D[0,\infty) \times D_l[0,\infty)$. In case the rate function I is of the form (8.1), then the rate function $I^{\infty}_{+,-}$ is given by

$$I^{\infty}_{+,-}(\varphi_1, \varphi_2) = \begin{cases} \int_0^{\infty} r(\dot{\varphi}_1(t))\, dt + \int_0^{\infty} r(\dot{\varphi}_2(t))\, dt & \text{if} \quad \varphi_1, \varphi_2 \in AC_0[0,\infty) \\ \infty & \text{otherwise,} \end{cases}$$

so that, in term of large deviations, the stationary sequence **behaves exactly as if the past was independent of the future**. So far rate functions of the type (8.1) have only been derived for i.i.d. or stationary hyper-mixing random sequences, so that this last result is perhaps not too surprising. Our result then seems to imply a converse in the sense that if a stationary sequence satisfies a sample path LDP with good rate function of the type (8.1), then the process presents some kind of mixing properties.

Further research on this matter could consist in identifying a generic expression for the rate function associated with the LDP of a sample path process, and deriving the corresponding rate function of its extension on $D[0,\infty)$. In addition, some technical points were left unanswered in Chapter 2, in particular whether the extension of a partial sum process on intervals of the type $[0, R]$ for R **rational** also satisfies the LDP in $D[0, R]$ without any continuity assumptions of the elements of the effective domain.

On the (very) technical aspects of this first part, we conclude that the Skorohod topology has very nice properties which in the end yield global results as nice as what the uniform topology would yield. However, in the process, one has to go through many subtle technicalities, and caution is required when establishing continuity properties of functionals.

Beyond the results specifically obtained in the second part of the Dissertation, the approach we have taken there shows that a functional approach at the sample path level, combined with the Contraction Principle, yields in principle many large deviations results on processes of interest in queueing theory. However, it should be pointed out that the use of the Contraction Principle does not in general yield an explicit form of the rate function. Even in the "good cases", i.e., when starting with a rate function of the integral form, the computation of the infimum obtained via the Contraction Principle is not easily carried out.

However, we did obtain some nice new results on the large deviations behavior of G/G/1 queues. In particular, the fact that the stationary output process of a discrete-time single-server queue with time-varying capacities satisfies the sample path LDP with good rate function, under the assumption that the arrivals and capacities jointly satisfy the LDP with good rate function of the integral form with some mild natural conditions on r. Unfortunately, we fell short of obtaining the results for queues in series, as we were unable to show that those natural conditions on the form of the rate function of the inputs propagate to the rate function of the output process. In particular, because we could not derive an explicit expression for the rate function associated with the LDP for the output process, we could not, from our results, check the conjecture made by Chang and Zajic [18] that the rate functions for the stationary and transient output process are in general different and coincide when the capacity is constant.

We also established the LDP for the steady-state queue length in the function space $D[0, 1]$, under two different assumptions. The first one is the natural assumption that the inputs to the system jointly satisfies the sample path LDP. The other one consists of buffer asymptotics of the usual form (which are also a consequence of the previous assumption). Under the latter assumption, the LDP is obtained bare-handed through algebraic and probabilistic arguments.

Finally, we proved that the notion of effective bandwidth could be derived for a very general class of systems, under the sole assumption that the inputs satisfies the sample path LDP with good rate function of the integral form and satisfying additional natural conditions. Although an unsolved technical point prevented us to fully establish the separability properties of the effective bandwidth, we strongly feel that this point can eventually be solved, which would then really yields a nice and simple way of obtaining the effective bandwidth.

In term of further research on this second part, many avenues can be investigated. The most natural one is of course to try to explicitly compute the rate functions, or at least to show that the rate function of the output process is of the integral form and satisfies the conditions needed for its propagation through queues in series. A key step missing in order to complete those computations is the existence of a measurable selection in presence of convexity, in order to justify taking the infimum inside the integral. If that cannot be done, the next natural step is to try to express the conditions we had in a different way in order to show that they indeed propagate. Also, conditions under which the rate functions of the stationary and transient output process coincide are worth investigating.

Solving the issue of convergence of the departures process of an infinite series of queues also relies on the propagation of the LDP sample path from the input to the output. In case this missing step can be solved, it will be very interesting to investigate the properties of the input-output mapping between the rate functions, as well as that between the effective bandwidths, and in particular their convergence.

To summarize, we have built the first major steps of a difficult and promising problem, namely that of characterizing the relationship between the large deviations of the input and of those of the output of a general single-server queue, which once completed, will yield a fistful of interesting results, together with new areas of exploration.

Appendix A

Proofs

A.1 A proof of Proposition 1.7

(1.) : We follow the lines of Exercise 1.2.19 in [24, p. 9]. Fix $\alpha < \infty$ and let $\Psi_I(\alpha)$ be the corresponding level set of the good rate function I. Since $\mathcal{X}$ is locally compact, each element x in $\mathcal{X}$ admits a neighborhood with compact closure. Thus, we can write

$$\Psi_I(\alpha) \subset \bigcup_{x \in \Psi_I(\alpha)} O_x,$$

where O_x is open set containing x and $\overline{O}_x$ is compact.

Because I is a good rate function, its level set $\Psi_I(\alpha)$ is compact, and from its open cover $\bigcup_{x \in \Psi_I(\alpha)} O_x$ we can extract a finite sub-cover, i.e., there exist $x_1, x_2, \ldots, x_n$ in $\Psi_I(\alpha)$ such that

$$\Psi_I(\alpha) \subset \bigcup_{i=1}^{n} O_{x_i} \subset \bigcup_{i=1}^{n} \overline{O}_{x_i},$$

or equivalently,

$$\left(\bigcup_{i=1}^{n} \overline{O}_{x_i} \right)^c \subset \Psi_I(\alpha)^c.$$

Therefore, from the formulation (1.2.7) in [24, p. 6] of the LDP upper bound for $\{\mu_n,\ n = 1, 2, \ldots\}$, we get

$$\limsup_{n \to \infty} \frac{1}{n} \ln \mu_n \left[\left(\bigcup_{i=1}^{n} \overline{O}_{x_i} \right)^c \right] < -\alpha,$$

and the desired result follows upon noting that $\bigcup_{i=1}^{n} \overline{O}_{x_i}$ is compact as a finite union of compact sets [12, Proposition 5 p. 86].

(2.): This result is shown as Theorem (P) in [56].

A.2 A proof of Proposition 1.8

Lemma A.1 *Let $\{X_n,\ n = 1, 2, \ldots\}$ and $\{Y_n,\ n = 1, 2, \ldots\}$ be two families of random elements in the separable regular topological spaces $\mathcal{X}$ and $\mathcal{Y}$, respectively. If $\{X_n,\ n = 1, 2, \ldots\}$ and $\{Y_n,\ n = 1, 2, \ldots\}$ are each exponentially tight, then the family $\{(X_n, Y_n),\ n = 1, 2, \ldots\}$ is exponentially tight in $\mathcal{X} \times \mathcal{Y}$.*

Proof: For each $n = 1, 2, \ldots$, let μ_n be the joint distribution law of (X_n, Y_n) and let μ_n^X and μ_n^Y denote the marginal distribution laws of X_n and Y_n, respectively.

Fix $\alpha < \infty$. From the exponential tightness of $\{X_n,\ n = 1, 2, \ldots\}$ and $\{Y_n,\ n = 1, 2, \ldots\}$, there exist compact sets K_α^1 in $\mathcal{X}$ and K_α^2 in $\mathcal{Y}$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \ln \mu_n^X \left((K_\alpha^1)^c\right) < -\alpha$$

and

$$\limsup_{n\to\infty} \frac{1}{n} \ln \mu_n^X \left((K_\alpha^2)^c\right) < -\alpha.$$

By Tychonoff Theorem [12, Theorem 3 p. 88], $K_\alpha^1 \times K_\alpha^2$ is compact in $\mathcal{X} \times \mathcal{Y}$. Moreover, upon noting that

$$\begin{aligned} \mu_n \left[(K_\alpha^1 \times K_\alpha^2)^c\right] &= \mu_n \left[\left((K_\alpha^1)^c \times \mathcal{X}\right) \cup \left((K_\alpha^2)^c \times \mathcal{Y}\right)\right] \\ &\leq \mu_n^X \left[(K_\alpha^1)^c\right] + \mu_n^Y \left[(K_\alpha^2)^c\right], \quad n = 1, 2, \ldots, \end{aligned}$$

we get

$$\begin{aligned} \limsup_{n\to\infty} \frac{1}{n} \mu_n \left[(K_\alpha^1 \times K_\alpha^2)^c\right] &\leq \left(\limsup_{n\to\infty} \frac{1}{n} \mu_n^X \left[(K_\alpha^1)^c\right]\right) \vee \left(\limsup_{n\to\infty} \frac{1}{n} \mu_n^X \left[(K_\alpha^2)^c\right]\right) \\ &< (-\alpha) \vee (-\alpha) \\ &= -\alpha. \end{aligned}$$

■

A proof of Proposition 1.8:

Our argument follows the lines of Exercise 4.2.7 in [24, p. 113].

Let $\tau_{\mathcal{X}}$ and $\tau_{\mathcal{Y}}$ denote respectively the topologies of $\mathcal{X}$ and $\mathcal{Y}$. For each $n = 1, 2, \ldots$, let μ_n be the (joint) distribution law of (X_n, Y_n), and let μ_n^X and μ_n^Y denote the marginal distribution laws of X_n and Y_n, respectively.

We note that the separability assumption ensures that for each $n = 1, 2, \ldots$, the pair (X_n, Y_n) is a random element in $\mathcal{X} \times \mathcal{Y}$ and that its distribution law is defined on the Borel σ-field of the product space, thus enabling us to use all large deviations techniques presented in [24].

We begin by establishing the weak LDP for $\{(X_n, Y_n),\ n = 1, 2, \ldots\}$ through the indirect approach of considering a base on the product topology [24, Theorem 4.1.11, p. 106].

It is well known [12, p. 44] that the set

$$\mathcal{A} \equiv \{O_1 \times O_2 : O_1 \in \tau_{\mathcal{X}},\ O_2 \in \tau_{\mathcal{Y}}\}$$

is a base for the product topology on $\mathcal{X} \times \mathcal{Y}$. Following [24, Theorem 4.1.11, p. 106], we define

$$\mathcal{L}_A \equiv -\liminf_{n\to\infty} \frac{1}{n} \ln \mu_n(A), \quad A \in \mathcal{A},$$

and

$$I(x,y) \equiv \sup_{\substack{A\in\mathcal{A} \\ (x,y)\in A}} \mathcal{L}_A, \quad x \in \mathcal{X}, \quad y \in \mathcal{Y}.$$

Each A in $\mathcal{A}$ can be expressed as $A = O_1 \times O_2$, with O_1 in $\tau_{\mathcal{X}}$ and O_2 in $\tau_{\mathcal{Y}}$. Hence, from the definition of $\mathcal{L}_A$, and the independence of X_n and Y_n, we obtain

$$\begin{aligned}
\mathcal{L}_A &= -\liminf_{n\to\infty} \frac{1}{n} \ln \mu_n(O_1 \times O_2) \\
&= -\liminf_{n\to\infty} \frac{1}{n} \left(\ln \mu_n^X(O_1) + \ln \mu_n^Y(O_2)\right) \\
&\leq -\liminf_{n\to\infty} \frac{1}{n} \ln \mu_n^X(O_1) - \liminf_{n\to\infty} \frac{1}{n} \ln \mu_n^Y(O_2). \qquad \text{(A.1)}
\end{aligned}$$

Because $\tau_{\mathcal{X}}$ and $\tau_{\mathcal{Y}}$ are trivial bases for the topologies on $\mathcal{X}$ and $\mathcal{Y}$ respectively, the LDP for $\{X_n,\ n = 1, 2, \ldots\}$ and $\{Y_n,\ n = 1, 2, \ldots\}$, and Theorem 4.1.18 in [24, p. 108] together yield the equalities

$$\begin{aligned}
I_X(x) &= \sup_{\substack{O_1\in\tau_{\mathcal{X}} \\ x\in O_1}} \left(-\liminf_{n\to\infty} \frac{1}{n} \ln \mu_n(O_1)\right) \\
&= \sup_{\substack{O_1\in\tau_{\mathcal{X}} \\ x\in O_1}} \left(-\limsup_{n\to\infty} \frac{1}{n} \ln \mu_n(O_1)\right), \quad x \in \mathcal{X} \qquad \text{(A.2)}
\end{aligned}$$

and

$$\begin{aligned} I_Y(y) &= \sup_{\substack{O_2 \in \tau_{\mathcal{Y}} \\ y \in O_2}} \left(-\liminf_{n\to\infty} \frac{1}{n} \ln \mu_n(O_2) \right) \\ &= \sup_{\substack{O_2 \in \tau_{\mathcal{Y}} \\ y \in O_2}} \left(-\limsup_{n\to\infty} \frac{1}{n} \ln \mu_n(O_2) \right), \quad y \in \mathcal{Y}. \end{aligned} \tag{A.3}$$

Keeping in mind these expressions, we readily obtain from (A.1) that

$$\begin{aligned} I(x,y) &= \sup_{\substack{A \in \mathcal{A} \\ (x,y) \in A}} \mathcal{L}_A \\ &\leq \sup_{\substack{O_1 \in \tau_{\mathcal{X}} \\ x \in O_1}} \left(-\liminf_{n\to\infty} \frac{1}{n} \ln \mu_n^X(O_1) \right) + \sup_{\substack{O_2 \in \tau_{\mathcal{Y}} \\ y \in O_2}} \left(-\liminf_{n\to\infty} \frac{1}{n} \ln \mu_n^Y(O_2) \right) \\ &= I_X(x) + I_Y(y), \quad x \in \mathcal{X}, \quad y \in \mathcal{Y}. \end{aligned} \tag{A.4}$$

On the other hand,

$$\begin{aligned} -\limsup_{n\to\infty} \frac{1}{n} \ln \mu_n(A) &\doteq -\limsup_{n\to\infty} \frac{1}{n} \ln \mu_n(O_1 \times O_2) \\ &\geq -\limsup_{n\to\infty} \frac{1}{n} \ln \mu_n^X(O_1) - \limsup_{n\to\infty} \frac{1}{n} \ln \mu_n^Y(O_2), \end{aligned}$$

whence, in view of (A.2) and (A.3),

$$\begin{aligned} &\sup_{\substack{A \in \mathcal{A} \\ (x,y) \in A}} \left(-\limsup_{n\to\infty} \frac{1}{n} \ln \mu_n(A) \right) \\ &\quad\geq \sup_{\substack{O_1 \in \tau_{\mathcal{X}} \\ x \in O_1}} \left(-\limsup_{n\to\infty} \frac{1}{n} \ln \mu_n^X(O_1) \right) + \sup_{\substack{O_2 \in \tau_{\mathcal{Y}} \\ y \in O_2}} \left(-\limsup_{n\to\infty} \frac{1}{n} \ln \mu_n^Y(O_2) \right) \\ &\quad= I_X(x) + I_Y(y), \quad x \in \mathcal{X}, \quad y \in \mathcal{Y}. \end{aligned} \tag{A.5}$$

Collecting (A.4) and (A.5), we finally find that

$$I_X(x) + I_Y(y) \leq \sup_{\substack{A \in \mathcal{A} \\ (x,y) \in A}} \left(-\limsup_{n\to\infty} \frac{1}{n} \ln \mu_n(A) \right) \leq I(x,y) \leq I_X(x) + I_Y(y)$$

for all x in $\mathcal{X}$ and all y in $\mathcal{Y}$, and the weak LDP for $\{(X_n, Y_n),\ n = 1, 2, \ldots\}$ becomes a simple consequence of Theorem 4.1.11 in [24, p. 106]. The corresponding rate function $I : \mathcal{X} \times \mathcal{Y} \to [0, \infty]$ is given by

$$I(x,y) = I_X(x) + I_Y(y), \quad x \in \mathcal{X}, \quad y \in \mathcal{Y}.$$

The proof is completed upon noting that by Lemma 1.6, exponential tightness of the family $\{(X_n, Y_n),\ n = 1, 2, \ldots\}$ (Lemma A.1) transforms the weak LDP for $\{(X_n, Y_n),\ n = 1, 2, \ldots\}$ with rate function I into a full LDP with good rate function I.

A.3 A proof of Theorem 1.10

We begin with an elementary Lemma.

Lemma A.2 *Let $(\mathcal{X}, \tau_{\mathcal{X}})$ and $(\mathcal{Y}, \tau_{\mathcal{Y}})$ be two topological spaces, and assume the mapping $f : \mathcal{X} \to \mathcal{Y}$ to be continuous on the subset D of $\mathcal{X}$. Then for any subset Γ of $\mathcal{Y}$ we have the inclusions*

$$\overline{f^{-1}(\Gamma)} \cap D \subset f^{-1}(\overline{\Gamma}) \qquad \text{and} \qquad f^{-1}(\Gamma^\circ) \cap D \subset \left(f^{-1}(\Gamma)\right)^\circ.$$

Proof: Let Γ be a subset of $\mathcal{Y}$, and let x in $\overline{f^{-1}(\Gamma)} \cap D$. Then, by definition of the closure,

$$O \cap f^{-1}(\Gamma) \neq \emptyset, \quad x \in O \in \tau_{\mathcal{X}}. \tag{A.6}$$

Because f is continuous on D which contains x, for any neighborhood V of $f(x)$, there exists a neighborhood U of x such that $f(U) \subset V$. By (A.6) (with $O = U$), there exists x_U in U with $f(x_U)$ in Γ, whence

$$V \cap \Gamma \supset f(U) \cap \Gamma \supset \{f(x_U)\} \neq \emptyset,$$

so that $f(x)$ belongs to $\overline{\Gamma}$, or equivalently x belongs to $f^{-1}(\overline{\Gamma})$.

Now, if x is in $f^{-1}(\Gamma^\circ) \cap D$. Then $f(x)$ belongs to Γ° which is open, and from the continuity of f at x, there exists a neighborhood U of x such that $f(U) \subset \Gamma^\circ \subset \Gamma$. Therefore, $U \subset f^{-1}(\Gamma)$ and x belongs to $(f^{-1}(\Gamma))^\circ$. ■

A proof of Theorem 1.10: We follow the proof sketched in [24, Theorem 4.2.1 p. 110], and begin by showing that I' is a good rate function. We first note that

$$\begin{aligned} y \in \mathcal{D}_{I'} &\Leftrightarrow \inf\{I(x) : x \in \mathcal{X},\ y = f(x)\} < \infty \\ &\Leftrightarrow \exists x \in \mathcal{X} \text{ s.t. } y = f(x) \text{ and } I(x) < \infty \\ &\Leftrightarrow y \in f(\mathcal{D}_I) \end{aligned}$$

so that $\mathcal{D}_{I'} = f(\mathcal{D}_I)$ and $\mathcal{D}_I \subset f^{-1}(\mathcal{D}_{I'})$.

Next, fix $\alpha \geq 0$ and consider the level sets $\Psi_{I'}(\alpha)$ and $\Psi_I(\alpha)$. Clearly, if y belongs to $f(\Psi_I(\alpha))$, then there exists x in $\Psi_I(\alpha)$ such that $y = f(x)$ and by definition of I', we get

$$I'(y) \leq I(x) \leq \alpha,$$

so that y belongs to $\Psi_{I'}(\alpha)$ and the inclusion $f(\Psi_I(\alpha)) \subset \Psi_{I'}(\alpha)$ follows.

On the other hand, if y belongs to $\Psi_{I'}(\alpha)$, then by definition of the infimum, for any $n = 1, 2, \ldots$, there exists x_n in $\mathcal{X}$ such that $f(x_n) = y$ and

$$I'(y) \le I(x_n) < I'(y) + \frac{1}{n}. \tag{A.7}$$

Clearly, the sequence $\{x_n,\ n = 1, 2, \ldots\}$ belongs to the compact set $\Psi_I(1 + \alpha)$, thus contains a subsequence $\{x_{n_k},\ k = 1, 2, \ldots\}$ converging to some x^* in $\Psi_I(1 + \alpha)$. Moreover, upon letting $n \to \infty$ in (A.7), we readily obtain that $I(x_n) \to I'(y)$, and lower semi-continuity of I then yields

$$I(x^*) \le \liminf_{n\to\infty} I(x_n) = I'(y),$$

so that x^* belongs in fact to $\Psi_I(\alpha)$. By continuity of f at x, we also have

$$f(x^*) = \lim_{k\to\infty} f(x_{n_k}) = y$$

and the conclusion $I'(y) \le I(x^*)$ follows. In view of the reverse inequality obtained earlier, we finally get $I'(y) = I(x^*)$, and y thus belongs to $f(\Psi_I(\alpha))$. The equality $\Psi_{I'}(\alpha) = f(\Psi_I(\alpha))$ is then immediate.

Next, to show compactness of the level set $\Psi_{I'}(\alpha)$, consider an open covering $\cup_\alpha O_\alpha$ of $\Psi_{I'}(\alpha) = f(\Psi(\alpha))$ in $\mathcal{Y}$. From the continuity of f on $\mathcal{D}_I$, Lemma A.2 implies that $\cup_\alpha (f^{-1}(O_\alpha))^\circ$ is an open covering of $\Psi_I(\alpha)$ because

$$\begin{aligned} \Psi_I(\alpha) &= \Psi_I(\alpha) \cap \mathcal{D}_I \\ &\subset \bigcup_\alpha \left(f^{-1}(O_\alpha) \cap D_I\right) \\ &\subset \bigcup_\alpha \left(f^{-1}(O_\alpha)\right)^\circ. \end{aligned}$$

The rate function I being good, $\Psi_I(\alpha)$ is compact, and there exists $\alpha_1, \ldots, \alpha_n$ such that

$$\Psi_I(\alpha) \subset \bigcup_{i=1}^n \left(f^{-1}(O_{\alpha_i})\right)^\circ \subset \bigcup_{i=1}^n f^{-1}(O_{\alpha_i}),$$

or, equivalently,

$$\Psi_{I'}(\alpha) = f(\Psi_I(\alpha)) \subset \bigcup_{i=1}^n O_{\alpha_i}.$$

Therefore, for each $\alpha \ge 0$, $\Psi_{I'}(\alpha)$ is compact and I' is a good rate function.

Before establishing the LDP bounds, we note from the expression of the rate function I' that

$$\inf_{y\in\Gamma} I'(y) = \inf_{x\in f^{-1}(\Gamma)} I(x), \quad \Gamma \subset \mathcal{Y}. \tag{A.8}$$

Now, let Γ be a Borel set in $\mathcal{Y}$. By Borel-measurability of f, $f^{-1}(\Gamma)$ is a Borel set in $\mathcal{X}$, and we get from the upper bound in the LDP for $\{\mu_n,\ n=1,2,\ldots\}$

$$\limsup_{n\to\infty} \frac{1}{n} \ln \mu_n \circ f^{-1}(\Gamma) \leq - \inf_{x \in \overline{f^{-1}(\Gamma)}} I(x). \tag{A.9}$$

From the definition of the effective domain, Lemma A.2 (f is continuous on $\mathcal{D}_I$), and (A.8), we obtain

$$\begin{aligned} \inf\left\{I(x) : x \in \overline{f^{-1}(\Gamma)}\right\} &= \inf\left\{I(x) : x \in \overline{f^{-1}(\Gamma)} \cap \mathcal{D}_I\right\} \\ &\geq \inf\left\{I(x) : x \in f^{-1}\left(\overline{\Gamma}\right)\right\} \\ &= \inf\left\{I'(y) : y \in \overline{\Gamma}\right\} \end{aligned} \tag{A.10}$$

and the LDP upper bound for $\{\mu_n \circ f^{-1},\ n=1,2,\ldots\}$ follows from (A.9) and (A.10).

Similarly, from the lower bound in the LDP for $\{\mu_n,\ n=1,2,\ldots\}$, we get

$$\liminf_{n\to\infty} \frac{1}{n} \ln \mu_n \circ f^{-1}(\Gamma) \geq - \inf_{x \in (f^{-1}(\Gamma))^\circ} I(x), \tag{A.11}$$

and by Lemma A.2 and (A.8) we see that

$$\begin{aligned} \inf\left\{I(x) : x \in \left(f^{-1}(\Gamma)\right)^\circ\right\} &\leq \inf\left\{I(x) : x \in f^{-1}(\Gamma^\circ) \cap \mathcal{D}_I\right\} \\ &= \inf\left\{I(x) : x \in f^{-1}(\Gamma^\circ)\right\} \\ &= \inf\left\{I'(y) : y \in \Gamma^\circ\right\}. \end{aligned} \tag{A.12}$$

The LDP lower bound for $\{\mu_n \circ f^{-1},\ n=1,2,\ldots\}$ then becomes an immediate consequence of (A.11) and (A.12).

A.4 A proof of Lemma 1.11

The following elementary Lemma is used to show that the goodness of the rate function is preserved when restricted to $\mathcal{E}$ or extended to $\mathcal{X}$.

Lemma A.3 *Let $(\mathcal{X}, \tau_{\mathcal{X}})$ be a topological space, and let $\mathcal{E}$ a subspace of $\mathcal{X}$ endowed with the topology $\tau_{\mathcal{E}}$ induced by $(\mathcal{X}, \tau_{\mathcal{X}})$. Then, a set $A \subset \mathcal{E}$ is compact in $(\mathcal{E}, \tau_{\mathcal{E}})$ if and only if it is compact in $(\mathcal{X}, \tau_{\mathcal{X}})$.*

Proof: Let $A \subset \mathcal{E}$, and assume that A is compact in $(\mathcal{E}, \tau_{\mathcal{E}})$. Let $\cup_\alpha O_\alpha$ be an open covering of A in $\tau_{\mathcal{X}}$, so that we have

$$A = A \cap \mathcal{E} \subset \bigcup_\alpha (O_\alpha \cap \mathcal{E}).$$

Because $\cup_\alpha (O_\alpha \cap \mathcal{E})$ is an open covering in $(\mathcal{E}, \tau_{\mathcal{E}})$ of the compact set A, there exists $\alpha_1, \ldots \alpha_n$ such that

$$A \subset \bigcup_{i=1}^{n} (O_{\alpha_i} \cap \mathcal{E}) \subset \bigcup_{i=1}^{n} O_{\alpha_i}$$

and compactness of A in $(\mathcal{X}, \tau_{\mathcal{X}})$ readily follows.

Conversely, assume A is compact in $(\mathcal{X}, \tau_{\mathcal{X}})$, and let $\cup_\alpha U_\alpha$ be an open covering of A in $\tau_{\mathcal{E}}$. By definition of the induced topology, for each α, we can write $U_\alpha = O_\alpha \cap \mathcal{E}$, where O_α is open in $\mathcal{X}$, so that

$$A \subset \bigcup_\alpha (O_\alpha \cap \mathcal{E}) \subset \bigcup_\alpha O_\alpha.$$

Therefore, the union $\cup_\alpha O_\alpha$ being an open covering in $\mathcal{X}$ of the compact set A, it admits a finite sub-covering, and we easily obtain

$$A = A \cap \mathcal{E} \subset \bigcup_{i=1}^{n} (O_{\alpha_i} \cap \mathcal{E}) = \bigcup_{i=1}^{n} U_{\alpha_i},$$

from which we conclude that A is compact in $(\mathcal{E}, \tau_{\mathcal{E}})$. ∎

A proof of Lemma 1.11: Let $\tau_{\mathcal{X}}$ and $\tau_{\mathcal{E}}$ denote respectively the topology of $\mathcal{X}$ and that induced by $\mathcal{X}$ on $\mathcal{E}$.

(1): From the LDP lower bound for the family $\{\mu_n,\ n = 1, 2, \ldots\}$ applied to the Borel-measurable set $\mathcal{E}^c$, we obtain

$$\inf_{x \in \mathrm{int}_{\tau_{\mathcal{X}}}(\mathcal{E}^c)} I(x) \geq \liminf_{n \to \infty} \frac{1}{n} \ln \mu_n(\mathcal{E}^c) = \infty$$

and we conclude immediately that $\mathrm{int}_{\tau_{\mathcal{X}}}(\mathcal{E}^c) \subset \mathcal{D}_I^c$, or equivalently, that $\mathcal{D}_I \subset \mathrm{cl}_{\tau_{\mathcal{X}}}(\mathcal{E})$.

(2): Let Γ be a Borel set in $\mathcal{X}$. As the intersection $\mathrm{cl}_{\tau_{\mathcal{X}}}(\Gamma) \cap \mathcal{E}$ is closed in $\mathcal{E}$, the upper bound in the LDP for $\{\mu_n,\ n = 1, 2, \ldots\}$ in $\mathcal{E}$ yields

$$\begin{aligned} \limsup_{n \to \infty} \frac{1}{n} \ln \mu_n(\Gamma) &\leq \limsup_{n \to \infty} \frac{1}{n} \ln \mu_n \left(\mathrm{cl}_{\tau_{\mathcal{X}}}(\Gamma)\right) \\ &= \limsup_{n \to \infty} \frac{1}{n} \ln \mu_n \left(\mathrm{cl}_{\tau_{\mathcal{X}}}(\Gamma) \cap \mathcal{E}\right) \end{aligned}$$

$$
\begin{aligned}
&\le -\inf_{x\in \mathrm{cl}_{\tau_{\mathcal{X}}}(\Gamma)\cap\mathcal{E}} I(x)\\
&= -\inf_{x\in \mathrm{cl}_{\tau_{\mathcal{X}}}(\Gamma)\cap\mathcal{E}} I'(x)\\
&= -\inf_{x\in \mathrm{cl}_{\tau_{\mathcal{X}}}(\Gamma)} I'(x)
\end{aligned}
$$

where the last two steps follow from the expression (1.15) of I'.

Similarly, as $\mathrm{int}_{\tau_{\mathcal{X}}}(\Gamma)\cap\mathcal{E}$ is open in $\mathcal{E}$, the lower bound in the LDP for $\{\mu_n,\ n=1,2,\ldots\}$ in $\mathcal{E}$ yields

$$
\begin{aligned}
\liminf_{n\to\infty}\frac{1}{n}\ln\mu_n(\Gamma) &\ge \liminf_{n\to\infty}\frac{1}{n}\ln\mu_n\left(\mathrm{int}_{\tau_{\mathcal{X}}}(\Gamma)\right)\\
&= \liminf_{n\to\infty}\frac{1}{n}\ln\mu_n\left(\mathrm{int}_{\tau_{\mathcal{X}}}(\Gamma)\cap\mathcal{E}\right)\\
&\ge -\inf_{x\in \mathrm{int}_{\tau_{\mathcal{X}}}(\Gamma)\cap\mathcal{E}} I(x)\\
&= -\inf_{x\in \mathrm{int}_{\tau_{\mathcal{X}}}(\Gamma)} I'(x)
\end{aligned}
$$

which completes the bounds in the LDP for $\{\mu_n,\ n=1,2,\ldots\}$ in $\mathcal{X}$.

To see that I' is a rate function in $\mathcal{X}$, we first note from the definition of I' that

$$
\Psi_{I'}(\alpha) = \{x\in\mathcal{X} : I'(x)\le\alpha\} = \{x\in\mathcal{E} : I(x)\le\alpha\} = \Psi_I(\alpha).
$$

Next, by lower semi-continuity of I in $\mathcal{E}$, the level set $\Psi_I(\alpha)$ is closed in $\mathcal{E}$, and can therefore be expressed as $\Psi_I(\alpha) = F\cap\mathcal{E}$, with F closed in $\mathcal{X}$. Thus, as $\mathcal{E}$ is closed in $\mathcal{X}$, we find that $\Psi_{I'}(\alpha) = \Psi_I(\alpha)$ is itself closed in $\mathcal{X}$. Finally, by Lemma A.3, I' is good whenever I is.

(3): Let Γ be a Borel set of $\mathcal{E}$. Clearly, $\mathrm{cl}_{\tau_{\mathcal{E}}}(\Gamma)$ is a closed set in $\mathcal{E}$, so that, by the definition of the induced topology, we can write $\mathrm{cl}_{\tau_{\mathcal{E}}}(\Gamma) = F\cap\mathcal{E}$ for some closed set F in $\mathcal{X}$. Let $I_{\mathcal{E}}$ denote the restriction of I to $\mathcal{E}$. By the upper bound in the LDP for $\{\mu_n,\ n=1,2,\ldots\}$ in $\mathcal{X}$, we then get

$$
\begin{aligned}
\limsup_{n\to\infty}\frac{1}{n}\ln\mu_n(\Gamma) &\le \limsup_{n\to\infty}\frac{1}{n}\ln\mu_n\left(\mathrm{cl}_{\tau_{\mathcal{E}}}(\Gamma)\right)\\
&= \limsup_{n\to\infty}\frac{1}{n}\ln\mu_n\left(F\cap\mathcal{E}\right)\\
&= \limsup_{n\to\infty}\frac{1}{n}\ln\mu_n(F)\\
&\le -\inf_{x\in F} I(x) \qquad\qquad (A.13)\\
&= -\inf_{x\in F\cap\mathcal{E}} I(x) \qquad\qquad (A.14)
\end{aligned}
$$

$$
\begin{aligned}
&= -\inf_{x\in F\cap\mathcal{E}} I_{\mathcal{E}}(x) \\
&= -\inf_{x\in \mathrm{cl}_{\tau_{\mathcal{E}}}(\Gamma)} I_{\mathcal{E}}(x)
\end{aligned}
$$

where the link from (A.13) to (A.14) comes from the assumption that $\mathcal{D}_I \subset \mathcal{E}$.

Similarly, because $\mathrm{int}_{\tau_{\mathcal{E}}}(\Gamma)$ is open in $\mathcal{E}$, we have $\mathrm{int}_{\tau_{\mathcal{E}}}(\Gamma) = O \cap \mathcal{E}$ for some open O in $\mathcal{X}$, and we obtain from the lower bound in the LDP for $\{\mu_n,\ n = 1, 2, \ldots\}$ in $\mathcal{X}$ that

$$
\begin{aligned}
\liminf_{n\to\infty} \frac{1}{n} \ln \mu_n(\Gamma) &\geq \liminf_{n\to\infty} \frac{1}{n} \ln \mu_n\,(O\cap\mathcal{E}) \\
&= \limsup_{n\to\infty} \frac{1}{n} \ln \mu_n(O) \\
&\geq -\inf_{x\in O} I(x) \\
&= -\inf_{x\in O\cap\mathcal{E}} I_{\mathcal{E}}(x) \\
&= -\inf_{x\in \mathrm{int}_{\tau_{\mathcal{E}}}(\Gamma)} I_{\mathcal{E}}(x).
\end{aligned}
$$

Furthermore, because $\mathcal{D}_I \subset \mathcal{E}$, we have the equality

$$
\Psi_{I_{\mathcal{E}}}(\alpha) = \Psi_I(\alpha) \cap \mathcal{E}, \quad \alpha \in \mathbb{R}
$$

with $\Psi_I(\alpha)$ closed in $\mathcal{X}$, whence $\Psi_{I_{\mathcal{E}}}(\alpha)$ is closed in $\mathcal{E}$, and $I_{\mathcal{E}}$ is lower semicontinuous in $\mathcal{E}$, which completes the derivation of the LDP for $\{\mu_n,\ n = 1, 2, \ldots\}$ in $\mathcal{E}$. Finally, by Lemma A.3, the rate function $I_{\mathcal{E}}$ is good whenever I is good.

A.5 A proof of Lemma 2.6

Let $\{x_n,\ n = 1, 2, \ldots\}$ be a sequence in $D[a, b]^p$ which is S-converging to x, and let c in (a, b) be a continuity point of x.

By definition of the S-convergence, $\{x_n,\ n = 1, 2, \ldots\}$ J_1-converges to x, and there exists a sequence of mapping $\{\lambda_n,\ n = 1, 2, \ldots\}$ in Λ_{ab} such that

$$
\lim_{n\to\infty} \sup_{t\in[a,b]} |x_n(t) - x\circ\lambda_n(t)| = \lim_{n\to\infty} \sup_{t\in[a,b]} |\lambda_n(t) - t| = \lim_{n\to\infty} \sup_{t\in[a,b]} |\lambda_n^{-1}(t) - t| = 0. \tag{A.15}
$$

Because we may not have $\lambda_n(c) = c$, the restriction of λ_n to $[a, c]$ (resp. $[c, b]$) may not be in Λ_{ac} (resp. Λ_{cb}), thus preventing us from obtaining the convergence of the restrictions of $\{x_n,\ n = 1, 2, \ldots\}$ on $D[a, c]^p$ and $D[c, b]^p$ directly from (A.15).

However, it is possible to construct from $\{\lambda_n,\ n=1,2,\ldots\}$ a sequence $\{\mu_n,\ n=1,2,\ldots\}$ in Λ_{ab} with $\mu_n(c)=c$ for all $n=1,2,\ldots$, and satisfying

$$\lim_{n\to\infty}\sup_{t\in[a,b]}|x_n(t)-x\circ\mu_n(t)| \;=\; \lim_{n\to\infty}\sup_{t\in[a,b]}|\mu_n(t)-t| \;=\; 0.$$

For each $n=1,2,\ldots$, we define c_n and C_n by

$$c_n \;\equiv\; \min\{c-\frac{1}{n},\lambda_n^{-1}(c-\frac{1}{n})\} \quad\text{and}\quad C_n \;\equiv\; \max\{c+\frac{1}{n},\lambda_n^{-1}(c+\frac{1}{n})\}. \tag{A.16}$$

Fix $n=1,2,\ldots$ large enough so that $[c_n,C_n]\subset[a,b]$. We construct the mapping $\mu_n:[a,b]\to[a,b]$ as follows: If $\lambda_n(c)<c$, we set $\mu_n=\lambda_n$ on $[a,c-\frac{1}{n}]$ and on $[\lambda_n^{-1}(c+\frac{1}{n}),b]$, and complete μ_n on $[c-\frac{1}{n},c]$ and $[c,\lambda_n^{-1}(c+\frac{1}{n})]$ by a linear interpolation passing through the point (c,c). On the other hand, if $\lambda_n(c)>c$, we set $\mu_n=\lambda_n$ on $[a,\lambda_n^{-1}(c-\frac{1}{n})]$ and $[c+\frac{1}{n},b]$, and complete μ_n similarly by a linear interpolation passing trough the point (c,c).

It is then easily checked that for those n where μ_n is defined, we have (by taking into account both constructions)

$$\begin{aligned}
&\sup_{t\in[a,b]}|\lambda_n(t)-\mu_n(t)|\\
&\quad\le \sup_{t\in[c_n,C_n]}|\lambda_n(t)-\mu_n(t)|\\
&\quad\le |\lambda_n(C_n)-\lambda_n(c_n)|\\
&\quad= \left|\max\{c+\frac{1}{n},\lambda_n(c+\frac{1}{n})\} \;-\; \min\{c-\frac{1}{n},\lambda_n(c-\frac{1}{n})\}\right|.
\end{aligned} \tag{A.17}$$

Thus, because

$$\begin{aligned}
|\lambda_n(c\pm\frac{1}{n})-(c\mp\frac{1}{n})| &\le |\lambda_n(c\pm\frac{1}{n})-(c\pm\frac{1}{n})|+|c\pm\frac{1}{n}-(c\mp\frac{1}{n})|\\
&\le \sup_{t\in[a,b]}|\lambda_n(t)-t|+\frac{2}{n},
\end{aligned}$$

(A.15) and (A.17) yield

$$\lim_{n\to\infty}\sup_{t\in[a,b]}|\lambda_n(t)-\mu_n(t)| \;=\; 0,$$

so that

$$\lim_{n\to\infty}\sup_{t\in[a,b]}|\mu_n(t)-t| \;\le\; \lim_{n\to\infty}\sup_{t\in[a,b]}|\lambda_n(t)-t|+\lim_{n\to\infty}\sup_{t\in[a,b]}|\lambda_n(t)-\mu_n(t)| \;=\; 0. \tag{A.18}$$

Now, by definition of the supremum, we can write

$$\sup_{t\in[c_n,C_n]}|x\circ\lambda_n(t)-x\circ\mu_n(t)| \;\le\; \frac{1}{n}+|x\circ\lambda_n(t_n)-x\circ\mu_n(t_n)| \tag{A.19}$$

where for each $n = 1, 2, \ldots, t_n$ belongs to $[c_n, C_n]$. We immediately see from the definition (A.16), the inequality

$$|\lambda_n^{-1}(c \pm \frac{1}{n}) - (c \pm \frac{1}{n})| \le \sup_{t\in[a,b]} |\lambda_n^{-1}(t) - t|,$$

and (A.15) that $\lim_{n\to\infty} t_n = c$, so that in view of the bounds

$$|\lambda_n(t_n) - c| \le \sup_{t\in[a,b]} |\lambda_n(t) - t| + |t_n - c| \text{ and } |\mu_n(t_n) - c| \le \sup_{t\in[a,b]} |\mu_n(t) - t| + |t_n - c|,$$

we finally deduce that $\lim_{n\to\infty} \lambda_n(t_n) = \lim_{n\to\infty} \mu_n(t_n) = c$. Therefore, upon letting $n \to \infty$ in (A.19), continuity of x at c finally yields

$$\lim_{n\to\infty} \sup_{t\in[c_n,C_n]} |x \circ \lambda_n(t) - x \circ \mu_n(t)| = |x(c) - x(c)| = 0. \tag{A.20}$$

Consequently, as we have

$$\begin{aligned} &\sup_{t\in[a,b]} |x_n(t) - x \circ \mu_n(t)| \\ &\quad \le \sup_{t\in[a,b]} |x_n(t) - x \circ \lambda_n(t)| + \sup_{t\in[a,b]} |x \circ \lambda_n(t) - x \circ \mu_n(t)| \\ &\quad \le \sup_{t\in[a,b]} |x_n(t) - x \circ \lambda_n(t)| + \sup_{t\in[c_n,C_n]} |x \circ \lambda_n(t) - x \circ \mu_n(t)|, \end{aligned}$$

for those n where μ_n is defined, we get from (A.15) and (A.20) that

$$\lim_{n\to\infty} \sup_{t\in[a,b]} |x_n(t) - x \circ \mu_n(t)| = 0. \tag{A.21}$$

The desired result then follows easily (after renumbering of μ_n) from (A.18) and (A.21) by considering the restrictions of $\{\mu_n,\ n = 1, 2, \ldots\}$ on $[a, c]$ and $[c, b]$, once it is recalled that for each $n = 1, 2, \ldots, \mu_n$ belongs to Λ_{ab} with $\mu_n(c) = c$.

A.6 A proof of Lemma 3.5

It is plain from the definitions that both $\widehat{\varphi}$ and $\widehat{\Phi}$ are bijections. The mapping h is therefore a metric on $D_l^*[0, \infty)$ and we readily see from the relation

$$h(x, y) = d_0(\widehat{\Phi}(x), \widehat{\Phi}(y)), \quad x, y \in D_l^*[0, \infty)$$

that $(D_l^*[0, \infty), h)$ is isometric to $(D_l^*[0, 1], d_0^l)$. Next, because $D^*[0, 1]$ is a Polish space [45, p. 111], and Φ_{01} is an isometry (see Section 2.1.1), the space $D_l^*[0, 1] =$

$\Phi_{01}^{-1}(D^*[0,1])$ is itself Polish. By the isometry $\widehat{\Phi}$, this translates to $(D_l^*[0,\infty), h)$ being a Polish space.

In view of Lemma 3.3, the proof will be completed if we show that a sequence $\{x_n,\ n = 1, 2, \ldots\}$ in $D_l^*[0,\infty)$ J_1-converges to x if and only if $h(x_n, x) \to 0$ as $n \to \infty$.

To this end, let $\{x_n,\ n = 1, 2, \ldots\}$ be a sequence in $D_l^*[0,\infty)$, such that

$$\lim_{n\to\infty} h(x_n, x) \ = \ 0,$$

for some x in $D_l^*[0,\infty)$, or equivalently

$$\lim_{n\to\infty} d_0(\widehat{\Phi}(x_n), \widehat{\Phi}(x)) \ = \ 0.$$

From the definition of d_0, this last expression is equivalent to the existence of a sequence $\{\lambda_n,\ n = 1, 2, \ldots\}$ in Λ_{01} such that

$$\begin{cases} \lim_{n\to\infty} \sup_{t\in[0,1]} |x_n \circ \widehat{\varphi}(t) - x \circ \widehat{\varphi} \circ \lambda_n(t)| & = \ 0 \\ \lim_{n\to\infty} \sup_{t\neq s\in[0,1]} |\ln \dfrac{\lambda_n(t) - \lambda_n(s)}{t - s}| & = \ 0. \end{cases} \tag{A.22}$$

For each $n = 1, 2, \ldots$, define the mapping μ_n in Λ_∞ by $\mu_n = \widehat{\varphi} \circ \lambda_n \circ \widehat{\varphi}^{-1}$. As $\widehat{\varphi}$ is a bijection, we have

$$\begin{aligned} \sup_{t\in[0,\infty)} |x_n(t) - x \circ \mu_n(t)| \ &= \ \sup_{t\in[0,\infty)} |x_n \circ \widehat{\varphi} \circ \widehat{\varphi}^{-1}(t) - x \circ \widehat{\varphi} \circ \lambda_n \circ \widehat{\varphi}^{-1}(t)| \\ &= \ \sup_{t\in[0,1]} |x_n \circ \widehat{\varphi}(t) - x \circ \widehat{\varphi} \circ \lambda_n(t)| \, . \end{aligned} \tag{A.23}$$

Furthermore, we observe that

$$\begin{aligned} \sup_{t\in[0,\infty)} |\mu_n(t) - t| \ &= \ \sup_{t\in[0,\infty)} |\widehat{\varphi} \circ \lambda_n \circ \widehat{\varphi}^{-1}(t) - \widehat{\varphi} \circ \widehat{\varphi}^{-1}(t)| \\ &= \ \sup_{t\in(0,1]} |\widehat{\varphi} \circ \lambda_n(t) - \widehat{\varphi}(t)| \\ &= \ \sup_{t\in(0,1]} |\ln t - \ln(\lambda_n(t))| \\ &= \ \sup_{t\in(0,1]} |\ln \frac{\lambda_n(t) - \lambda_n(0)}{t - 0}| \\ &\leq \ \sup_{t\neq s\in[0,1]} |\ln \frac{\lambda_n(t) - \lambda_n(s)}{t - s}|. \end{aligned} \tag{A.24}$$

Therefore, collecting (A.22), (A.23) and (A.24), we conclude that

$$\begin{cases} \lim_{n\to\infty} \sup_{t\in[0,\infty)} |x_n(t) - x \circ \mu_n(t)| & = \ 0 \\ \lim_{n\to\infty} \sup_{t\in[0,\infty)} |\mu_n(t) - t| & = \ 0, \end{cases} \tag{A.25}$$

and $\{x_n,\ n = 1, 2, \ldots\}$ J_1-converges to x.

Conversely, if $\{x_n,\ n = 1, 2, \ldots\}$ J_1-converges to x, then there exists a sequence $\{\mu_n,\ n = 1, 2, \ldots\}$ in Λ_∞ such that (A.25) holds. For each $n = 1, 2, \ldots$, we set $\lambda_n = \widehat{\varphi}_n^{-1} \circ \mu_n \circ \widehat{\varphi}_n$, so that λ_n belongs to Λ_{01}. We note that

$$\begin{aligned}\sup_{t\in[0,1]} |\lambda_n(t) - t| &= \sup_{t\in[0,1]} |\widehat{\varphi}^{-1} \circ \mu_n \circ \widehat{\varphi}(t) - \widehat{\varphi}^{-1} \circ \widehat{\varphi}(t)| \\ &= \sup_{t\in[0,\infty)} |\widehat{\varphi}^{-1} \circ \mu_n(t) - \widehat{\varphi}^{-1}(t)| \\ &\leq \sup_{t\in[0,\infty)} |\mu_n(t) - t|, \end{aligned} \tag{A.26}$$

where in the last step we have used the fact that $|(\widehat{\varphi}^{-1})'(t)| = e^{-t}$ is bounded by 1 for all t in $[0, \infty)$. Using (A.23), we finally get

$$\begin{cases} \lim_{n\to\infty} \sup_{t\in[0,1]} |x_n \circ \widehat{\varphi}(t) - x \circ \widehat{\varphi} \circ \lambda_n(t)| &= 0 \\ \lim_{n\to\infty} \sup_{t\in[0,1]} |\lambda_n(t) - t| &= 0. \end{cases}$$

In other words, $\{\widehat{\Phi}(x_n),\ n = 1, 2, \ldots\}$ J_1-converges to $\widehat{\Phi}(x)$ in $D_l^*[0, 1]$, so that in view of Lemma 2.3, $h(x_n, x) = d_0(\widehat{\Phi}(x_n), \widehat{\Phi}(x)) \to 0$, which completes the proof.

A.7 A proof of Lemma 4.5

(1.): Monotonicity of m_z is easily seen from its definition and yields the existence of all one-sided limits. It is easily checked that right continuity (resp. left continuity) of z at t implies that of m_z, and it follows that m_z belongs to $D[a, b]$ (resp. $D_l[a, b]$) for z in $D[a, b]$ (resp. $D_l[a, b]$).

(2.): Let z be in $AC_0[a, b]$ and fix $\varepsilon > 0$. Clearly, $z(0) = 0$ implies $m_z(0) = 0$. From the absolute continuity of z, there exists $\delta > 0$ such that for each sequence of non-overlapping intervals $\{(a_i, b_i),\ i = 1, \ldots, m\}$ of $[0, 1]$,

$$\sum_{i=1}^{m} |b_i - a_i| < \delta \ \Rightarrow\ \sum_{i=1}^{m} |z(b_i) - z(a_i)| < \frac{\varepsilon}{2}. \tag{A.27}$$

Let $\{(a_i, b_i),\ i = 1, \ldots, m\}$ be a sequence of non overlapping intervals in $[0, 1]$ with $\sum_{i=1}^{m} |b_i - a_i| < \delta$ and fix i in $\{1, \ldots, m\}$. Then we have

$$\begin{aligned}|m_z(b_i) - m_z(a_i)| &= |\inf_{0\leq s\leq b_i} z(s) - \inf_{0\leq s\leq a_i} z(s)| \\ &= \left|\min\left\{\inf_{0\leq s\leq a_i} z(s),\ \inf_{a_i<s\leq b_i} z(s)\right\} - \inf_{0\leq s\leq a_i} z(s)\right|\end{aligned}$$

$$
\begin{aligned}
&= \left| \min\left\{0\,,\ \inf_{a_i<s\le b_i} z(s) - \inf_{0\le s\le a_i} z(s)\right\}\right| \\
&= \max\left\{0\,,\ \inf_{0\le s\le a_i} z(s) - \inf_{a_i<s\le b_i} z(s)\right\} \\
&\le \max\left\{0\,,\ z(a_i) - \inf_{a_i<s\le b_i} z(s)\right\} \qquad \text{(A.28)}
\end{aligned}
$$

Next, from the definition of the infimum, there exists s_i in $(a_i, b_i]$ such that

$$\inf_{a_i<s\le b_i} z(s) + \frac{\varepsilon}{2m} \ge z(s_i)\ .$$

Hence, it follows from (A.28) that

$$
\begin{aligned}
|m_z(b_i) - m_z(a_i)| &\le \max\left\{0\,,\ z(a_i) - z(s_i) + \frac{\varepsilon}{2m}\right\} \\
&\le |z(a_i) - z(s_i)| + \frac{\varepsilon}{2m},
\end{aligned}
$$

so that upon summing up over all $i = 1, \ldots, m$,

$$\sum_{i=1}^{m} |m_z(b_i) - m_z(a_i)| \le \sum_{i=1}^{m} |z(s_i) - z(a_i)| + m\frac{\varepsilon}{2m}. \qquad \text{(A.29)}$$

Since $\{(a_i, s_i),\ i = 1 \ldots, m\}$ is a collection of non overlapping intervals of $[0, 1]$ with

$$\sum_{i=1}^{m} |s_i - a_i| = \sum_{i=1}^{m} (s_i - a_i) \le \sum_{i=1}^{m} (b_i - a_i) < \delta,$$

we get $\sum_{i=1}^{m} |z(s_i) - z(a_i)| < \frac{\varepsilon}{2}$ from (A.27), and the results follow from (A.29).

(3.): Pick t in $[0, 1]$ such that $m_z(t) < z(t)$. By continuity of z, there exists $\delta > 0$ such that

$$z(u) > m_z(t),\ u \in (t - \delta, t + \delta).$$

It is then plain that

$$m_z(u) = m_z(t),\ u \in (t - \delta, t + \delta)$$

so that m_z is differentiable on $(t - \delta, t + \delta)$ and

$$\dot{m}_z(u) = 0,\ u \in (t - \delta, t + \delta).$$

(4.): Let $D \equiv \{t \in [0, 1] : \dot{m}_z(t) = 0\}$. Since m_z is non-increasing, $\dot{m}_z(t) < 0$ for all t in D^c, so that by **(3)**, $m_z(t) = z(t)$ for all t in D^c.

Let t in D. Since t is isolated in D, there exists a $\delta > 0$ such that $(t - \delta, t + \delta) \cap D = \{t\}$. Thus $\dot{m}_z(u) < 0$ for all $u \neq t$ in $(t - \delta, t + \delta)$. By **(3)**, we necessarily have $m_z(u) = z(u)$ for all $u \neq t$ in $(t - \delta, t + \delta)$, and continuity of z and m_z then implies $m_z(t) = z(t)$.

A.8 A proof of Lemma 5.14

Write $F(b) = \mathbf{P}[q_\infty > b]$ for all $b \geq 0$. By Assumption **(E)**, we get

$$\lim_{n\to\infty} \frac{1}{n} \ln F(nx_\pm) = -\gamma^* x_\pm,$$

so that for every $\varepsilon > 0$ there exists an integer $n(\varepsilon)$ such that

$$-\varepsilon \leq \frac{1}{n} \ln F(nx_\pm) + \gamma^* x_\pm \leq \varepsilon, \quad n \geq n(\varepsilon)$$

or equivalently,

$$e^{-n(\gamma^* x_\pm + \varepsilon)} \leq F(nx_\pm) \leq e^{-n(\gamma^* x_\pm - \varepsilon)}, \quad n \geq n(\varepsilon). \tag{A.30}$$

Upon selecting ε such that $0 < \varepsilon < \gamma^* x_+$ in (A.30), we readily conclude from (5.26) that

$$\lim_{n\to\infty} \frac{1}{n} \ln \mathbf{P}[q_\infty \leq nx_+] = 0$$

and the desired conclusion (5.24) immediately follows from the trivial bounds

$$\mathbf{P}[q_\infty \leq n(x_+ - \delta)] \leq \mathbf{P}[q_\infty \leq nx_+] \leq \mathbf{P}[q_\infty \leq n(x_+ + \delta)], \quad n = 1, 2, \ldots$$

where we have chosen δ such that $0 < \delta < x_+$.

In a similar manner, to prove (5.25), we note that it suffices to establish

$$\lim_{n\to\infty} \frac{1}{n} \ln \mathbf{P}[nx_- < q_\infty \leq nx_+] = -\gamma^* x_-. \tag{A.31}$$

This is a direct consequence of the bounds

$$\mathbf{P}[nx_- < q_\infty \leq n(x_+ - \delta)] \leq \mathbf{P}[nx_- < q_\infty < nx_+] \leq \mathbf{P}[nx_- < q_\infty \leq nx_+]$$

where we have chosen δ such that $0 < \delta < x_+$. From the inequalities (A.30) we readily conclude that

$$e^{-n(\gamma^* x_- + \varepsilon)} \Gamma_n^-(\varepsilon) \leq F(nx_-) - F(nx_+) \leq e^{-n(\gamma^* x_- - \varepsilon)} \Gamma_n^+(\varepsilon)$$

where we have set

$$\Gamma_n^\pm(\varepsilon) \equiv 1 - e^{-n(\gamma^*(x_+ - x_-) \pm 2\varepsilon)}.$$

Therefore,

$$\frac{1}{n} \ln (F(nx_-) - F(nx_+)) \leq -(\gamma^* x_- - \varepsilon) + \frac{1}{n} \ln \Gamma_n^+(\varepsilon) \tag{A.32}$$

while

$$\frac{1}{n}\ln\left(F(nx_-) - F(nx_+)\right) \;\geq\; -(\gamma^* x_- + \varepsilon) + \frac{1}{n}\ln\Gamma_n^-(\varepsilon) \tag{A.33}$$

Finally, select ε so that $0 < 2\varepsilon < \gamma^*(x_+ - x_-)$ and let n go to infinity in (A.32)–(A.33). We obtain via (5.26) that

$$\limsup_{n\to\infty}\frac{1}{n}\ln\left(F(nx_-) - F(nx_+)\right) \;\leq\; -\gamma^* x_- + \varepsilon$$

and

$$\liminf_{n\to\infty}\frac{1}{n}\ln\left(F(nx_-) - F(nx_+)\right) \;\geq\; -\gamma^* x_- + \varepsilon$$

and the desired conclusion (A.31) is obtained by letting ε go to zero in the last two inequalities.

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