

THESIS REPORT

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H^∞ Robust Adaptive Control

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Abstract

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Traditional adaptive control algorithms are not robust to dynamic uncertainties. The robust adaptive control algorithms developed previously to deal with dynamic uncertainties do not facilitate quantitative design to predefined robustness specifications. Their design procedures require time-consuming trial-and-error. We proposed a new robust adaptive control algorithm with a systematic and quantitative design procedure. The adaptive controller consists of an H^∞ suboptimal control law and a robust parameter estimator. Stability and robustness analysis is based on a general frozen time analysis framework. Global boundedness of the adaptive control system in the presense of parametric uncertainty, unmodeled dynamics, and bounded noises is proved. This condition is based on the shifted H^∞ norm of the frozen time system and is used as the underlying theoretical foundation of the H^∞ robust adaptive controller. Numerical examples showing the effectiveness of the H^∞ adaptive scheme are provided.

H^∞ Robust Adaptive Control

by

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Dedication

To my dear wife, Lichen

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Chapter 1

Introduction

Traditional adaptive control schemes are designed to be robust to the uncertainties in the parameters of the plant assuming we have fairly good knowledge about the structure of the plant. However, they are notorious for being non-robust to unmodeled dynamics or dynamic uncertainties. Many schemes have been proposed to enhance robustness of adaptive controllers to unmodeled dynamics and bounded disturbances [1, 2, 3, 4]. However, these schemes all have a common problem in real applications. It is that they cannot quantify the stability robustness levels of the closed-loop systems. In particular, if a stability robustness requirement is given in terms of the size of the unmodeled dynamics, there is no systematic procedure to design an adaptive control system to satisfy this requirement using existing robust adaptive schemes. Robustness levels of adaptive control systems design using these schemes can only be evaluated through extensive simulations, so the designer has to go through a tedious trial-and-error process. The development of a systematic design procedure which facilitates quantitative robustness analysis is an important problem in adaptive control.

A typical adaptive controller is composed of two parts: a parameter estimator and a control law. Most robust adaptive control schemes employ special param-

eter estimators to achieve robustness, but they still use the non-robust control laws used in the conventional non-robust adaptive controllers. Model reference control (MRC) [1, 2, 4], linear quadratic gaussian (LQG) control, and pole placement control (PPC) are the most common control laws used in these algorithms. The capability of the control law part has not been exploited to improve robustness of adaptive control systems. It is interesting to find out whether using some kind of robust control law can further improve the robustness properties of adaptive controllers.

Based on the above observations of the drawbacks of traditional adaptive (and robust adaptive) control schemes, we started this research with two goals:

1. Develop a quantitative and systematic robust adaptive control design method.
2. Investigate the usefulness of H^∞ control laws when applied to adaptive control.

It turns out that these two goals are related. Using robust control laws enables us to do quantitative analysis of robustness properties for adaptive control systems, which cannot be done easily with non-robust control laws.

We set up the design problem as follows. Suppose we are given the structure of the nominal linear time-invariant plant model along with a set of possible parameters, the size of unmodeled dynamics, and the bound on the magnitude of the disturbances. We want to design an adaptive controller such that all signals in the control loop remain bounded for any bounded input regardless of the uncertainty in the parameters of the nominal plant and the presense of the non-parametric uncertainty.

The systems considered in this research are single-input-single-output (SISO)

linear time-invariant (LTI) systems with coprime factor uncertainty descriptions. The coprime factor uncertainty model can represent a larger class of systems than the additive or multiplicative uncertainty models do. In particular, it allows the nominal system and the true system to have different numbers of unstable poles and zeros, while the multiplicative or the additive uncertainty models do not.

We developed a new sufficient condition to guarantee that every signal in the adaptive control system is bounded. This sufficient condition is expressed in terms of the shifted H^∞ norm of the estimated nominal plant model.

We developed a new robust adaptive control design scheme which has a systematic and quantitative design procedure using the boundedness condition we derived. The proposed adaptive control scheme is based on the frozen time approach, i.e. the plant parameters are estimated by the parameter estimator, then the controller parameters are computed by the control law as if the estimated plant parameters are the true plant parameters at each sample time.

The control law used in the proposed adaptive control scheme is the H^∞ robust control law for the coprime factor uncertainty. This H^∞ control law can be computed easily by solving two algebraic Riccati Equations. The simplicity of the computation of the control law is important to adaptive controllers because the control law is updated in real-time. The other reason we choose this control law is its continuity property. Lipschitz continuity of the control law with respect to the plant parameters is important for adaptive control algorithms based on the frozen time approach because if the controller parameters are not Lipschitz continuous with respect to the plant parameters, then the overall adaptive system may vary arbitrarily fast even when the plant parameters vary slowly, and the frozen time analysis cannot be applied. The H^∞ robust control law for the

coprime factor uncertainty model used in our algorithm is Lipschitz continuous with respect to the plant [5, 6, 7, 8, 9, 10].

There were some other results that applied H^∞ control laws in adaptive controllers [11, 12, 13, 14, 15, 16, 9, 17, 18, 8, 10, 19], but none of them has completed theoretic analysis of boundedness of the adaptive systems. For example, [11, 12, 13, 19] did not consider the effects of unmodeled dynamics, so they are indeed ideal-case results. [16, 9, 18, 8, 10] ignored the interaction between control laws and parameter estimators when unmodeled dynamics exist, which is very critical to stability of adaptive control systems. [17] relied on the persistent excitation conditions to achieve boundedness. None of these results facilitated systematic and quantitative design procedures as ours.

The parameter estimator used in the proposed adaptive controller was developed by Lemaire [20]. It was also used in [21, 22]. This estimator is the discrete time version of a continuous time estimator introduced in [23]. It uses a recursive least squares-type identification algorithm with dead-zone and projection modifications. The projection modification of the estimator is used to constrain the parameter estimate within a feasible set to ensure the solvability of the control law. The dead zone is used to turn off the updating when the prediction error can not be distinguished from the errors due to the external disturbance and non-parametric uncertainty. Hence the parameter estimate is updated only when the prediction error is really due to a large parametric error. The dead zone is normalized by the prediction error to guarantee the time varying adaptation gain is bounded no matter how large the prediction error is. This estimator provides important properties required for proving global boundedness of the adaptive loop.

Many simulations have been performed to validate the properties of the proposed adaptive control scheme. The results confirmed robustness of the control scheme. The design parameters of the adaptive controller were chosen based on concise rules once the specifications on the size of uncertainties was given. No trial-and-error procedures are needed.

In order to further improve the performance of the proposed adaptive control scheme, we proposed a modified scheme which allows frequency-dependent weighting functions to be included in the coprime factor uncertainty model of the plant. These weighting functions, when chosen properly, can give more freedom on the control design. Hence it is easier to achieve both good robustness and better performance simultaneously. Global boundedness of the weighted scheme was also proved. Same examples used in the non-weighted scheme were re-designed using the weighted scheme. Significant performance improvement was observed.

The result of this dissertation is unique as no one has done global boundedness analysis on H^∞ adaptive controllers and no one has been able to design an H^∞ adaptive controller to meet predefined specifications on the parametric uncertainty, unmodeled dynamics, and bounded disturbances. The main contributions of this research are:

- A systematic and quantitative design procedure for adaptive control systems is proposed. This can not be done with most previous adaptive control algorithms.
- The H^∞ robust control laws is incorporated in the adaptive controller with a rigorous theoretic support. The complexity of the control law is the same as that of the adaptive LQG controller.

- This adaptive controller can achieve global boundedness and stability robustness in the presence of parametric uncertainties, bounded external disturbances and unmodeled dynamics.

The rest of this dissertation is divided into seven chapters. Chapter 2 reviews the development of the stability results for adaptive control systems and some relevant recent results. Chapter 3 introduces the notations and analysis tools used in this dissertation. Chapter 4 formulates the problem that we try to solve. The parameter estimator used in the proposed adaptive control scheme is also introduced in this chapter. Chapter 5 reviews some sufficient conditions for global boundedness of the signals in an adaptive control system. The adaptive l^1 controller developed by Voulgaris *et al.* [21, 22] using these boundedness conditions is reviewed, and its disadvantages are explained. Chapter 6 is the main theoretical result of this research. We derive a new boundedness condition based on the shifted H^∞ norms of frozen time systems. The algorithm of the adaptive H^∞ controller is presented. Chapter 7 presents some numerical examples to demonstrate the adaptive H^∞ control design. Chapter 8 introduces an extension of the adaptive H^∞ control scheme which includes frequency-dependent weighting functions in the uncertainty model. Chapter 9 concludes this dissertation. The MATLAB scripts used to simulating the examples in Chapter 7 are listed in Appendix A.

Chapter 2

Background

This chapter reviews the development of stability and robustness results for adaptive control systems and literature related to this research.

2.1 Stability and Robustness Results

This section reviews the development of stability and robustness results in adaptive control developed in the last three decades.

Most early adaptive control designs were heuristic and focused on the performance issue, that is how to adjust the controller parameters to minimize a performance index, without rigorous consideration of stability. In 1966 Parks [24] demonstrated that gradient-based adaptive systems, such as the MIT rule-based adaptive control, could be unstable, and he showed that an adaptive control design based on the Lyapunov method could make a class of systems globally stable. Researchers then concentrated on the stability issues.

It is well known that adaptive control systems are nonlinear even when the controlled plants are linear, so it is more difficult to analyze stability of adaptive systems than linear systems. Stability problems of nonlinear systems are usually divided into local stability problems and global stability problems. Lo-

cal stability problems consider stability about a particular trajectory, which is determined by the external input signal and the initial conditions. Hence, an adaptive control system designed to achieve local stability for a particular input signal may not be stable for other input signals or initial conditions. Global stability problems consider stability of adaptive control systems for a whole family of input signals, e.g. L^∞ -signals, and any bounded initial conditions.

In 1979 and 1980 several groups [25, 26, 27, 28] established so-called “ideal-case” global stability results for model reference adaptive control systems almost simultaneously. Their main result says that every signal in the adaptive loop will remain bounded and the output error between the plant output and the reference model output will go to zero if the external reference signal is bounded and the following ideal assumptions hold:

1. The time delay of the plant is known (for the discrete-time case).
2. Upper bounds on the degrees of the numerator and the denominator are known.
3. The plant is minimum-phase.
4. The sign of the high frequency gain is known.
5. There are no external disturbances

Assumptions 2 and 5 are not realistic because physical systems are usually more complicated than the mathematical models used in the control design and physical systems often suffer from some kind of disturbances or noises. Many people soon discovered that these ideal-case results had serious robustness problems.

Egardt [25] showed that even small bounded disturbances could cause instability in adaptive control systems. Rohrs *et al.* [29, 30] also demonstrated by simulations that mild violations of the ideal assumptions could lead to instability. Since then the robustness problem has been a focus of research in adaptive control.

We now explain the stability and robustness problem of adaptive control system using a general framework.

Given a SISO LTI plant

$$M_0 y(t) = N_0 u(t) \quad (2.1)$$

where

$$\begin{aligned} N_0(z^{-1}) &= b_1 z^{-1} + b_2 z^{-2} + \cdots + b_m z^{-m} \\ M_0(z^{-1}) &= 1 + a_1 z^{-1} + \cdots + a_n z^{-n}. \end{aligned}$$

We let

$$\begin{aligned} \theta_p &\triangleq [-a_1, \dots, -a_n, b_1, \dots, b_m]^T \\ \phi(t) &\triangleq [y(t-1), \dots, y(t-n), u(t-1), \dots, u(t-m)]^T, \end{aligned}$$

then we have

$$y(t) = \phi^T(t) \theta_p.$$

The error equation of a typical discrete time adaptive control system can be described as follows:

$$e(t) = H(z^{-1}) \{ \phi^T(t) \tilde{\theta}(t-1) \} \quad (2.2)$$

$$\tilde{\theta}(t+1) = \tilde{\theta}(t) - \epsilon \phi(t) H(z^{-1}) \{ \phi^T(t) \tilde{\theta}(t+1) \}. \quad (2.3)$$

where $\tilde{\theta}(t) = \theta_p - \hat{\theta}(t)$ is the error between the true parameter θ_p and the parameter estimate $\hat{\theta}(t)$, $\phi(t)$ is the regressor vector, $e(t)$ is the prediction error, ϵ is the adaptation gain, and $H(s)$ is a stable linear system depending on the algorithm (see [1] for detailed definitions). The $\{\cdot\}$ in (2.2) indicates that $H(z^{-1})$ operates on the signal $\phi^T(t)\tilde{\theta}(t-1)$ to produce $e(t)$ and similarly in (2.3) $H(z^{-1})$ operates on the signal $\phi^T(t)\tilde{\theta}(t+1)$. Notice that (2.3) is a non-causal form of the parameter updating law, which is not directly implementable. Sometimes the non-causal form is more convenient for analysis than the causal form. We will show that it can be converted into a causal form which is implementable.

We now show that the non-causal form (2.5) can be converted into a causal form by some algebraic manipulations. For simplicity, we choose $H(z^{-1}) = 1$. Then the adaptive system can be described as follows:

$$e(t) = \phi^T(t)\tilde{\theta}(t-1) = y(t) - \phi^T(t)\hat{\theta}(t-1) \quad (2.4)$$

$$\tilde{\theta}(t+1) = \tilde{\theta}(t) - \epsilon\phi(t)(\phi^T(t)\tilde{\theta}(t+1)). \quad (2.5)$$

This is the so-called normalized least mean squares (LMS) algorithm [1].

$$(I + \epsilon\phi(t)\phi^T(t))\tilde{\theta}(t+1) = \tilde{\theta}(t), \quad (2.6)$$

$$\Rightarrow \tilde{\theta}(t+1) = (I + \epsilon\phi(t)\phi^T(t))^{-1}\tilde{\theta}(t), \quad (2.7)$$

Recall the Matrix Inversion Lemma:

Let A , C , and $C^{-1} + DA^{-1}B$ be nonsingular matrices. Then

$$(A + BCD)^{-1} = A^{-1} - A^{-1}B(C^{-1} + DA^{-1}B)^{-1}DA^{-1}.$$

Applying the Matrix Inversion lemma to (2.5) with

$$A = I, \quad B = \phi(t), \quad C = \epsilon, \quad D = \phi^T(t),$$

we can get

$$\tilde{\theta}(t+1) = [I - \phi(t)(\epsilon^{-1} + \phi^T(t)\phi(t))^{-1}\phi^T(t)]\tilde{\theta}(t), \quad (2.8)$$

$$\Rightarrow \tilde{\theta}(t+1) = \left[I - \epsilon \frac{\phi(t)\phi^T(t)}{1 + \epsilon\phi^T(t)\phi(t)} \right] \tilde{\theta}(t). \quad (2.9)$$

Equation (2.9) is in the causal form.

The averaging theory, which was previously used in Ljung's ODE approach [31], was a very popular and powerful tool in studying local stability of systems like (2.9) in the 1980's. One example use [1] of averaging in adaptive control described here. Consider the non-causal form of the error equation (2.5) again. Assume H is a general LTI system with $H(z) = d + c(zI - A)^{-1}b$ and that A is Hurwitz (i.e. all the eigenvalues of A lie in the open unit disk of the complex plane). Then the adaptive system (2.3) has the following state space representation:

$$\tilde{\theta}(t+1) = \tilde{\theta}(t) - \epsilon\phi(t)(d\phi^T(t)\tilde{\theta}(t+1) + c^T x(t)) \quad (2.10)$$

$$x(t+1) = Ax(t) + b\phi^T(t)\tilde{\theta}(t+1) \quad (2.11)$$

$$e(t) = Cx(t) + d\phi^T(t)\tilde{\theta}(t+1) \quad (2.12)$$

Under certain conditions there exists a Lyapunov transformation $L(k, \epsilon)$ that transforms the system (2.10-2.11) into a new system with decoupled slow-fast dynamics as below [1]:

$$\tilde{\theta}(t+1) = (I - \epsilon g(k, \epsilon)\phi(t)v^T(k, \epsilon))\tilde{\theta}(t) + \epsilon g(k, \epsilon)\phi(t)c^T z(t) \quad (2.13)$$

$$z(t+1) = (A + \epsilon g(k, \epsilon)(L(k+1, \epsilon) - b\phi^T(t))\phi(t)c^T)z(t) \quad (2.14)$$

where $g(k, \epsilon) = 1/(1 + \epsilon d\phi^T(t)\phi(t))$, and $v^T(k, \epsilon) = d\phi^T(t) + c^T L(k, \epsilon)$. Notice that the Lyapunov transformation will preserve the stability properties, and

for sufficiently small ϵ the fast z -subsystem is exponentially stable because A is Hurwitz ($H(s)$ is stable). Hence for sufficiently small ϵ the adaptive system (2.10-2.11) has the same stability properties as the slow system (2.13). To consider Lyapunov stability of the slow $\tilde{\theta}$ -subsystem (2.13) we can drop the forcing term and just analyze the unforced part:

$$\tilde{\theta}(t+1) = (I - \epsilon g(k, \epsilon) \phi(t) v^T(k, \epsilon)) \tilde{\theta}(t) \quad (2.15)$$

It is a fact that for sufficiently small ϵ we have $g(k, \epsilon) \approx 1$ and $v^T(k, \epsilon) \approx v_0^T(t) = d\phi^T(t) + c^T L_0(t)$, where $L_0(t) = \sum_{j=-\infty}^k A^{j-1} b \phi^T(j)$, so the stability properties of the slow subsystem (2.13) can be determined by the stability properties of the simplified system

$$\tilde{\theta}(t+1) = (I - \epsilon \phi(t) v_0^T(t)) \tilde{\theta}(t) \quad (2.16)$$

The intuition is that $\phi(t)$ in (2.16) usually varies much faster than $\tilde{\theta}(t)$ when ϵ is very small, so the trajectory of the solution $\tilde{\theta}(t)$ of the system (2.16) can be well-approximated by the trajectory of the solution of the averaged system of (2.16).

Let $R(t) = \phi(t) v_0^T(t)$, then (2.16) can be written as

$$\tilde{\theta}(t+1) = (I - \epsilon R(t)) \tilde{\theta}(t). \quad (2.17)$$

Define the sample averages as

$$\bar{R}_i = \frac{1}{K_i} \sum_{k_{i-1}}^{k_i-1} R(t), \quad k_i = k_{i-1} + K_i \quad (2.18)$$

over the interval $0 < K_m \leq K_i \leq K_M < \infty$. The averaging theory says if there exists a constant positive definite matrix $P = P^T > 0$ satisfying

$$P \bar{R}_i + \bar{R}_i^T P \geq I, \quad \forall i$$

then there exists an $\epsilon^* > 0$ such that the system (2.17) is exponentially stable for all $\epsilon \in (0, \epsilon^*)$. When ϕ is periodic, we have $\bar{R}_i = \bar{R}$, $\forall i$, so very simple and useful stability and instability criteria can be obtained.

Åström [32], Kokotovic *et al.* [33, 34], Fu *et al.* [35], and Kosut *et al.* [36] used the averaging theory to analyze adaptive control systems in the presence of unmodeled dynamics, to explain the instability mechanisms, to determine the boundary between the stable and unstable regions, and to estimate the convergence of the adaptation. A thorough discussion of the application of averaging in adaptive control can be found in [1, 4].

Other useful tools for analyzing local stability of adaptive control systems are linearization [37], total stability [1], Lyapunov theory, persistence of excitation, time scale separation, small gain theorem [38], integral manifold [1], passivity, and positivity ([1, 39, 4, 40, 41]).

When high frequency disturbances or unmodeled dynamics exist, the estimator will constantly be pushing the parameter estimate to match the high frequency dynamics. Hence the parameter estimate may drift. Parameter drift is the main cause of instability when the ideal-case adaptive controller is applied to a non-ideal system. Notice an unbounded parameter estimate will almost surely cause instability of the adaptive control system in the presence of high frequency unmodeled dynamics. This is because, as the coefficients of the estimated transfer function model grow larger, the adaptive control law will "think" the plant has a large bandwidth. Thus a high gain controller is applied. When the relative degree of the actual transfer function is greater than two, the phase shift will be greater than 180° , and the closed-loop system will become unstable when the loop gain is too high. Therefore, eliminating the parameter drift is the

first step to achieving robustness of adaptive control systems.

Once the instability mechanism was understood, researchers started looking for adaptation laws which would be robust to bounded disturbances and unmodeled dynamics. We review some of the most popular ways to improve robustness of adaptive controllers next. We separate them into two different cases, one for external bounded disturbances, and one for internal disturbances due to unmodeled dynamics. For simplicity, we consider the simple case of $H(z^{-1}) = 1$.

Robustness to bounded disturbances Suppose a bounded external disturbance d_1 is injected into a SISO linear time-invariant system as follows

$$M_0 y(t) = N_0 u(t) + d_1(t) \quad (2.19)$$

where

$$N_0(z^{-1}) = b_1 z^{-1} + b_2 z^{-2} + \cdots + b_m z^{-m}$$

$$M_0(z^{-1}) = 1 + a_1 z^{-1} + \cdots + a_n z^{-n}$$

We let

$$\theta_p \triangleq [-a_1, \dots, -a_n, b_1, \dots, b_m]^T$$

$$\phi(t) \triangleq [y(t-1), \dots, y(t-n), u(t-1), \dots, u(t-m)]^T,$$

then the error equation of the adaptive system becomes

$$y(t) = \phi^T(t) \theta_p + d_1(t).$$

There are four popular techniques to achieve robustness to bounded disturbances:

1. Persistent excitation: The regressor $\phi(t)$ is said to be persistently exciting (PE) if there exist positive constants T, α, β such that for all T

$$0 < \alpha I \leq \sum_{t=0}^{T-1} \phi(t)\phi^T(t) \leq \beta I < \infty. \quad (2.20)$$

When ϕ is PE, the equilibrium of (2.9) is a point and $\tilde{\theta}(t)$ will converge exponentially to the equilibrium. When ϕ is not PE, the equilibrium is a manifold instead of a point. The parameter estimate $\hat{\theta}$ might drift to infinity along this manifold. Narendra and Annaswamy [37] showed that if the degree of persistent excitation is sufficiently large compared to the magnitude of the disturbances, then all the signals in the loop will be bounded. This approach is not very practical because it requires the reference signals to have the PE property, but in many applications the set point is not persistently exciting and it is usually undesirable to inject additional perturbations to the plant.

2. Projection: Egardt [25], Kreisselmeier and Narendra [42] showed that global boundedness of all signals in the closed-loop system in the presence of bounded disturbances can be achieved by projecting the parameter estimates onto a compact convex set containing the true parameter vector.
3. Dead-zone: Egardt [25], Peterson and Narendra [43], and Samson [44] used dead-zone modifications, which have a form like the following:

$$\tilde{\theta}(t+1) = \tilde{\theta}(t) - \epsilon f(e(t)) \frac{\phi(t)}{1 + \epsilon \phi^T(t)\phi(t)} \quad (2.21)$$

where

$$f(e(t)) = \begin{cases} e(t) - D_1 & \text{if } e(t) > D_1 \\ 0 & \text{if } |e(t)| \leq D_1 \\ e(t) + D_1 & \text{if } e(t) < -D_1 \end{cases} \quad (2.22)$$

This kind of parameter estimator stops the adaptation when the estimation error is smaller than the disturbances, to assure boundedness of all signals in the closed-loop system.

4. Leakage: This technique is also known as σ -modification [45, 46]. It is mostly used in continuous-time adaptive systems. For example, the gradient algorithm for the ideal case is

$$\dot{\hat{\theta}}(t) = -\epsilon\phi(t)e(t). \quad (2.23)$$

The idea of σ -modification is to add a leakage term $-\sigma\hat{\theta}(t)$ to get a new parameter updating law,

$$\dot{\tilde{\theta}}(t) = -\epsilon\phi(t)e(t) - \sigma\tilde{\theta}(t), \quad (2.24)$$

for which the Lyapunov function

$$V(t) = \tilde{\theta}^T(t)\tilde{\theta}(t), \quad (2.25)$$

has a negative definite derivative when the parameter estimate error is outside a compact region containing the origin. Hence the parameter estimate will remain bounded.

The latter two approaches eliminated the pure integration action of the original adaptation law (see (2.9)) to prevent the parameter drift caused by bounded disturbances.

Robustness to unmodeled dynamics Suppose a SISO linear time-invariant system is modeled as follows

$$(M_0 + \Delta_M)y(t) = (N_0 + \Delta_N)u(t) \quad (2.26)$$

where

$$N_0(z^{-1}) = b_1 z^{-1} + b_2 z^{-2} + \dots + b_m z^{-m}$$

$$M_0(z^{-1}) = 1 + a_1 z^{-1} + \dots + a_n z^{-n}$$

and Δ_M and Δ_N are high frequency unmodeled dynamics. We let

$$\theta_p \triangleq [-a_1, \dots, -a_n, b_1, \dots, b_m]^T$$

$$\phi(t) \triangleq [y(t-1), \dots, y(t-n), u(t-1), \dots, u(t-m)]^T,$$

and $d_2(t) = -\Delta_M y(t) + \Delta_N u(t)$, then the error equation of the adaptive system becomes

$$y(t) = \phi^T(t) \theta_p + d_2(t)$$

where $d_2(t)$ is the term contributed by the disturbance.

When unmodeled dynamics exist, the error equation of the adaptive system becomes

$$e(t) = \phi^T(t) \tilde{\theta}(t) + d_2(t). \quad (2.27)$$

The robustness problem for unmodeled dynamics is more difficult because the internal disturbance $d_2(t)$ could be unbounded even when the unmodeled dynamics is stable. Hence the techniques requiring the assumption of bounded disturbances cannot be directly applied to this case. Praly [47] proposed the idea to divide both sides of the error equation (2.27) by a normalizing signal $n(t)$ with the property that

$$\left| \frac{d_2(t)}{n(t)} \right| \leq M < \infty, \quad \forall k. \quad (2.28)$$

i.e.

$$\frac{e(t)}{n(t)} = \frac{\phi^T(t) \tilde{\theta}(t)}{n(t)} + \frac{d_2(t)}{n(t)}, \quad (2.29)$$

to get a normalized error equation with a bounded disturbance term:

$$e_n(t) = \phi_n^T(t)\tilde{\theta}(t) + d_{2n}(t), \quad (2.30)$$

where $e_n(t) = \frac{e(t)}{n(t)}$, $\phi_n(t) = \frac{\phi(t)}{n(t)}$, and $d_{2n}(t) = \frac{d_2(t)}{n(t)}$. Now $d_{2n}(t)$ is a bounded disturbance signal, so we can use the modification techniques used in the bounded disturbance case, e.g. parameter projection [47, 48], dead zone [3], and leakage [49], with the normalized signals $e_n(t)$, $\phi_n(t)$, and $d_{2n}(t)$ to achieve global boundedness for the normalized signals. To show the non-normalized signals are also bounded we need to show the normalizing signal $n(t)$ is bounded. This is usually done by using the Gronwall-Bellman lemma.

Persistent excitation of the regressor signals provides the same robustness effect to unmodeled dynamics as it does to the bounded disturbances. It was showed in [50, 51, 52, 36, 1] that if the regressor ϕ is persistently exciting, then the closed-loop adaptive control system is exponentially stable, hence it is robust to sufficiently small unmodeled dynamics.

It was shown recently that parameter projection alone is enough to guarantee global boundedness of all signals in the closed-loop system in the presence of bounded disturbances and small unmodeled uncertainties [53].

2.2 Adaptive Control Schemes Using Robust Control Laws

Most of the above results were derived for model reference adaptive controllers (MRAC). In fact, only MRAC have been studied extensively. However, the model reference controller lacks robustness because it attempts to make the closed-loop system match the model at high frequencies, where the modeling

error is large. It is interesting to investigate whether other control laws can improve robustness or not. Some attempts to use linear robust control laws in adaptive controllers have been made recently [54, 11, 12, 13, 55, 56, 14, 15, 16, 57, 21, 9, 17, 18, 8, 10, 58, 19, 59, 22, 60]. This section reviews some adaptive control schemes that used robust control laws or ideas developed in the linear robust control theory.

Ueng *et al.* [56] used a coprime factor uncertainty model

$$P(z^{-1}) = (M_0 + \Delta_M)^{-1}(N_0 + \Delta_N), \quad (2.31)$$

which is an uncertainty model first used in the linear robust control literature, to describe the multi-input multi-output (MIMO) plant to be controlled. They used the LQG control law and a gradient-type parameter estimator with dead-zone and projection modifications.

Grimble *et al.* [11, 12, 13, 19] proposed a self-tuning regulator (in the stochastic setting) using an H^∞ control law that minimizes a cost function of the following form

$$J = \|\Psi_{\psi\psi}(z^{-1})\|_\infty \quad (2.32)$$

where $\psi(t) = P_c(z^{-1})e(t) + F_c(z^{-1})u(t)$, P_c and F_c are weighting transfer functions for e and u , respectively, and $\Psi_{\psi\psi}$ is the power spectral density matrix for ψ . The computation of this control law for this cost function is simpler than that of the H^∞ control law for the conventional mixed sensitivity cost function, but the physical meaning of the new cost function is not as intuitive as the mixed sensitivity. They used the regular least squares type (non-robustified) parameter estimator and did not study the effects of unmodeled dynamics on stability of the adaptive system.

Dahleh and Dahleh [55] used the l^1 optimal controller law and a parameter estimator similar to that in [56] to get an adaptive control design which is robust to bounded disturbances. Voulgaris *et al.* [21, 22] extended the results of [55] to accommodate unmodeled dynamics. The l^1 optimal controller, in general, is not Lipschitz continuous with respect to the plant parameters, so the slowly varying assumption for the stability condition may be violated.

Krause *et al.* [57] assumed that a robust control law attaining a certain level of robustness is available and then studied robust stability and asymptotic performance of the closed-loop systems. They developed a continuous time gradient-type parameter estimator with a dead-zone which turns off the adaptation when the shifted L^2 -norm of the prediction error is smaller than a certain threshold. However, they did not provide a specific construction of the control law, and it is not clear how to compute the performance measure they defined.

Wang and Zames [14] used an H^∞ suboptimal control law for the sensitivity minimization problem in an adaptive control scheme. They did not specify the parameter estimator or consider stability of the adaptive loop.

Wang designed time varying (non-adaptive) controllers to achieve l^∞ - [9] and l^2 -stability [8, 10] for slowly varying systems in the presence of the gap metric uncertainty [5]. This is not an adaptive control problem, but it is related to this research because the gap metric uncertainty is similar to the coprime factor uncertainty used in this research, and the AAK (V. M. Adamjan, D.Z. Arov, M.G. Krein [61]) construction of the suboptimal controller used in [9, 8, 10] is also equivalent to the state space construction of the suboptimal controller [62] used in this research.

Owen and Zames [18] extended the optimal robust disturbance attenuation

problem in linear robust control (two-disk problem) to the case that identification data are added as time goes on to enhance the performance and the signal space is l^2 . However, they assume the adaptive system has reached convergence, so they did not consider the interaction between the control law and the parameter estimator.

Jacobson and Tadmor [17] combined an H^∞ frozen time complete information controller with a robust identification scheme. They required the regressor to be persistently exciting, hence not in l^2 , but they assumed the signal space is l^2 during the proof of stability and performance results, so their assumptions are not consistent.

Lall and Glover [59] use a receding horizon H^∞ control law in an adaptive controller. They studied the performance of the adaptive system assuming the system is stable. They did not give any method for actually determining the model.

H^∞ control designs for linear systems with known nominal models have been studied extensively, but using H^∞ control laws in adaptive control systems is still an on-going research topic [63, 14, 18, 17]. There are at least two difficulties when applying H^∞ control laws to adaptive control. First, linear H^∞ control considers signals with bounded energy, i.e. l^2 signals, but most adaptive control problems involve signals with bounded magnitude, i.e. l^∞ signals. For example, a key ingredient for convergence of parameter estimators is persistently exciting signals, which are in l^∞ but not in l^2 . Secondly, for LTI operators on $l^2(-\infty, \infty)$ Parseval's theorem provides an isometry between the kernel (impulse response) and (Laplace) transformation representations, so we can estimate the $l^2(-\infty, \infty)$ time-domain behavior from the frequency-domain properties. How-

ever similar relations do not exist in the situation of a general LTV operator on $l^\infty(-\infty, \infty)$, so we can not directly incorporate H^∞ control laws in adaptive controllers without a rigorous theoretic support.

Zames and Wang [14, 16] established fundamental relations between the norms of frozen time systems and of global LTV systems. They showed that an asymptotic isometry exists between a special frequency domain norm and the l^1 norm for slowly varying systems in the sense that the supremum of the shifted H^∞ norms of all frozen time (LTI) systems is getting closer to the l^1 norm of the overall linear time varying system as the variation rate is getting smaller. Their results inspired this research, and their theory is used as the machinery to prove the effectiveness of the proposed robust adaptation control scheme.

2.3 Quantitative and Systematic Adaptive Control Design

Most robust adaptive control results reviewed in section 2 are qualitative. There is no way to estimate the robustness level of an adaptive control system, and there are no systematic design procedures to achieve global robustness for a given plant structure and some uncertainty specifications. For example, a typical robust stability result for adaptive control system is the following: (Theorem 4.1 of [49])

Theorem 1 [49] *Consider the SISO plant*

$$\frac{y(s)}{u(s)} = G(s) = G_0(s)[1 + \mu\Delta_2(s)] + \mu\Delta_1(s) \quad (2.33)$$

with the controller specified in the paper [49]. There exists a μ^ such that for each $\mu \in [0, \mu^*]$ all the signals in the loop are bounded for any initial condition*

and bounded input signal.

This kind of result is not very useful for the synthesis of robust adaptive controllers because it does not provide a way to calculate the μ^* or a procedure to design the adaptive controller given a bound on μ .

How to get a systematic design procedure which facilitates global quantitative robustness and performance analysis is an important problem in the synthesis of robust adaptive controllers. This section reviews some recent results which try to solve this problem. Some recent papers proposed interesting results in quantifying the stability robustness level and the performance robustness level of the adaptive controllers. These results allow us to get systematic design procedures for robust adaptive controllers. This section reviews some of these quantitative robustness results.

Dahleh and Dahleh [55] derived a boundedness condition in terms of the l^1 norm of time varying sensitivity functions in the case of bounded disturbances. They proposed a procedure to design an adaptive controller that satisfies the boundedness condition via frozen time optimal l^1 control. Voulgaris *et al* [21, 22] extended the result of [55] to accommodate unmodeled dynamics. There are two disadvantages to their design procedures:

1. The l^1 optimal controller in general is not Lipschitz continuous with respect to the plant parameter, so the slowly varying assumption for the stability condition may be violated.
2. The l^1 norm and the l^1 optimal controller are more difficult to calculate than the H^2 or H^∞ controllers.

Krause et al [57] used the supremum of the ratio between the L^2 -norms of the output and input signals, which is an extension of the H^∞ robust performance in the non-adaptive robust control literature, as the performance measure. They characterize the control law which can achieve robust performance, but they did not provide a specific control law that satisfies this characterization. The performance measure they defined is very difficult, if not impossible, to compute.

A. Datta and S. P. Bhattacharyya [54] developed a direct model reference robust adaptive control scheme which facilitates quantitative analysis of the robustness of the closed-loop system. They used the results from the area of robust parametric stability to derive a procedure for verifying a condition which guarantee the boundedness of all signals in the closed-loop system. This procedure involves calculating the worst case shifted H^∞ norms over certain one parameter families. Their controller structure and boundedness condition are different those proposed in this dissertation.

Chapter 3

Preliminaries

3.1 Global-local Double Algebra

This section introduces the Global-Local Double Algebras defined by Zames and Wang [16]. This theory is used later to prove the boundedness condition of the adaptive control system. The importance of this theory is that it relates the l^1 kernel norm to the frequency domain H^∞ norm. Therefore it allows us to apply an H^∞ control law, which is usually defined in the l^2 signal space, to adaptive control problems which involves the l^∞ signal space.

Let \mathcal{R} , \mathcal{C} , \mathcal{Z} denote the real, complex numbers, and integers. Let \mathcal{R}^n and $\mathcal{R}^{n \times n}$ denote n -tuples and $n \times n$ matrices over \mathcal{R} . \mathcal{C}^n and $\mathcal{C}^{n \times n}$ are defined similarly with \mathcal{C} replacing \mathcal{R} . Let $l_\sigma^p[a, b]$, $1 \leq p \leq \infty$, $\sigma \geq 1$, denote the space of sequences $u(t)$, $t = a, a+1, \dots, b$ either of vectors in \mathcal{C}^n or matrices in $\mathcal{C}^{n \times n}$ for which

$$\|u\|_{l_{\sigma[a,b]}^p} \triangleq \begin{cases} \left[\sum_{t=a}^b (|u(t)|\sigma^t)^p \right]^{1/p} < \infty & \text{for } 1 \leq p < \infty \\ \sup_{a \leq t \leq b} |u(t)|\sigma^t < \infty & \text{for } p = \infty \end{cases} \quad (3.1)$$

The dimension n will usually be suppressed. When $\sigma = 1$, σ will also be suppressed. When $\|u\|_{l_{\sigma[a,b]}^p}$ exists for $a = -\infty$ and $b = \infty$, it will be written as $u \in l_\sigma^p$. Let H^∞ denote the space of rational transfer functions which are ana-

lytic and essentially bounded on the region $|z| > 1$. For $K \in H^\infty$ the norm on H^∞ is defined as

$$\|K\|_{H^\infty} \triangleq \sup_{|z| \geq 1} |K(z)|$$

where $|K(z)|$ denotes the largest singular value of $K(z)$. For $\sigma > 1$, define the σ -shifted H^∞ space, denoted by H_σ^∞ , to be the space of rational transfer functions which are analytic and essentially bounded on the region $|z| > 1/\sigma$. H_σ^p is the subspace of H^p whose elements have decay rates no less than σ . For $K \in H_\sigma^\infty$ the norm on H_σ^∞ is called the σ -shifted H^∞ norm and is defined as

$$\|K\|_{H_\sigma^\infty} \triangleq \sup_{|z| \geq 1/\sigma} |K(z)|$$

Let B be the space of bounded linear causal operators from $l^\infty(-\infty, \infty)$ to $l^\infty(-\infty, \infty)$ which can be represented as convolution sums, i.e., for any $K \in B$ there is a kernel (impulse response) $k(t, \cdot) \in l^1(-\infty, \infty)$ such that

$$(Ku)(t) = \sum_{\tau=-\infty}^t k(t, \tau)u(\tau), \quad t \in \mathcal{Z}, \quad u \in l^\infty \quad (3.2)$$

and $k(t, \tau) = 0$ whenever $t < \tau$. The operator norm on B is defined as follows:

$$\|K\|_B = \sup_{t \in \mathcal{Z}} \|k(t, \cdot)\|_{l^1} < \infty \quad (3.3)$$

where

$$\|k(t, \cdot)\|_{l^1} \triangleq \sum_{\tau=-\infty}^{\infty} |k(t, \tau)|. \quad (3.4)$$

It is easy to show that B is a normed space. Furthermore, each element in B is characterized by an l^1 kernel, and the l^1 space is a complete space, so B is also a complete space. Hence B is a Banach space.

Let B_e be the extended space of B , i.e. B_e contains sequences whose truncations lie in B . The introduction of B_e allows us to deal with some unstable linear time-varying operators.

Given an operator K in B , the local system or frozen-time system corresponding to K at time α is the time-invariant operator, denoted by K_α , with the same domain as K satisfying

$$(K_\alpha u)(t) = \sum_{\tau=-\infty}^{\infty} k(\alpha, \alpha - (t - \tau)) u(\tau), \quad t \in \mathcal{Z}. \quad (3.5)$$

The system corresponding to the operator $K \in B$ is called the global system.

The definition of a local system here is chosen for the following reason. For a given input $\{u(t) : t \in \mathcal{Z}\}$ the output at time α produced by the local system frozen at time α is the same as the output at time α produced by the global system, i.e.

$$(K_\alpha u)(t) = (Ku)(t), \quad \text{if } t = \alpha. \quad (3.6)$$

Other definitions might be more appropriate in other cases. For example, as Zames and Wang [16] pointed out, if the global system varies fast but its averaged operator varies slowly, then it is more appropriate to define a local system based on the averaged system corresponding to the global system.

Let \hat{F}_α be the transfer function of the frozen-time system F_α , the frozen time system corresponding to F , in H^∞ . Call \hat{F}_α the local transfer function corresponding to the global operator F at time α . \hat{F} will denote the sequence $\{\hat{F}_\alpha\}, \alpha \in \mathcal{Z}$.

Define two products on B_e :

Definition 1 For any F and K in B_e define the global product \cdot to be the usual composition product, i.e.

$$(F \cdot K)u(t) = \sum_{i=-\infty}^t f(t, i) \left[\sum_{\tau=-\infty}^i k(i, \tau) u(\tau) \right], \quad (3.7)$$

where f and k are the kernels of F and K , respectively. The local product $F \otimes K$ of F and K is the unique operator satisfying

$$(F \otimes K)_\alpha = F_\alpha K_\alpha, \quad \forall \alpha \in \mathcal{Z}, \quad (3.8)$$

where F_α , K_α is the frozen-time (or local) system of F , K at time α , respectively, i.e.

$$(F_\alpha K_\alpha)u(t) = \sum_{i=-\infty}^t f(\alpha, \alpha - (t - i)) \left[\sum_{\tau=-\infty}^i k(\alpha, \alpha - (i - \tau))u(\tau) \right]. \quad (3.9)$$

The symbol for the global product will often be suppressed, i.e. $F \cdot K = F K$.

Definition 2 A local-global double algebra is any subspace in B_e which is equipped with global and local products and is an algebra with respect to each.

B_e itself is a double algebra. Another new operator is the *product difference* binary operator Δ :

$$F \nabla K \triangleq FK - F \otimes K.$$

Given $\sigma > 1$, two kinds of norms are defined on a double algebra.

$$\|K\|_{(\sigma)} \triangleq \sup_t \sum_{\tau=-\infty}^t |k(t, \tau) \sigma^{(t-\tau)}| \quad (3.10)$$

$$\mu_\sigma(\hat{K}) \triangleq \sup_t \|\hat{K}_t\|_{H_\sigma^\infty} = \sup_t \|\hat{K}_t(\sigma(\cdot))\|_{H^\infty} \quad (3.11)$$

where $\sigma(\cdot)$ means σ times the argument, \hat{K}_t is the transfer function of the frozen-time system K_t , the frozen time system corresponding to K , in H^∞ , and \hat{K} means the sequence $\{\hat{K}_t\}$, $t \in \mathcal{Z}$.

Given $\sigma > 1$ the following linear spaces are defined:

$$\mathcal{E}_\sigma \triangleq \{K \in B : \|K\|_{(\sigma)} < \infty\}$$

$$\underline{\mathcal{E}}_\sigma \triangleq \{K \in B : \|K\|_{(\sigma_0)} < \infty \text{ for some } \sigma_0 > \sigma\}$$

$$\hat{\underline{\mathcal{E}}}_\sigma \triangleq \{K \in B : \mu_{\sigma_0}(\hat{K}) < \infty \text{ for some } \sigma_0 > \sigma\}$$

$$\overline{\mathcal{E}}_\sigma \triangleq \{K \in B : \mu_\sigma(\hat{K}) < \infty\}$$

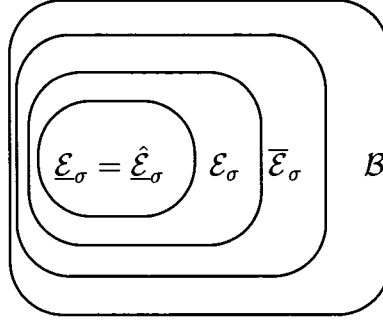


Figure 3.1: The relations between the subspaces

$\hat{\mathcal{E}}_\sigma$ and $\overline{\mathcal{E}}_\sigma$ are useful in dealing with operators initially specified in the frequency domain. These spaces have the following relations:

- For $K \in \mathcal{E}_\sigma$, \hat{K}_τ is in H_σ^∞ for all τ . Every stable causal operator in \mathcal{E}_σ has local transfer functions in H_σ^∞ , but the reverse is not true.
- $\underline{\mathcal{E}}_\sigma$ is identical to $\hat{\mathcal{E}}_\sigma$. An operator in $\underline{\mathcal{E}}_\sigma$ has both frequency domain and time domain specifications.
- For $\sigma > 1$, $\underline{\mathcal{E}}_\sigma \subset \mathcal{E}_\sigma \subset \overline{\mathcal{E}}_\sigma \subset \mathcal{B}$.
- $\mathcal{E}_{\sigma_1} \subset \overline{\mathcal{E}}_{\sigma_1} \subset \mathcal{E}_{\sigma_2}$, if $\sigma_1 > \sigma_2$.

The relations between these subspaces are illustrated in Fig. 3.1.

Two inverses are defined for an operator K in a double algebra :

Definition 3 *The global inverse K^{-1} of K is an operator satisfying $K^{-1} \cdot K = K \cdot K^{-1} = I$. The local inverse K^\ominus of K is an operator satisfying $K^\ominus \otimes K = K \otimes K^\ominus = I$.*

If $K \in B_e$, then K^{-1} (and K^\ominus) exists in B_e if and only if $k(t, t)$ (the diagonal element of the Hankel matrix of K) is invertible for all time t . Conditions for

local invertibility in B_e are identical to the global case. Therefore, if $K^\ominus \in B_e$, then $K^{-1} \in B_e$, too. In the normed double algebra B this is not true. If $K^\ominus \in B$, it is not necessary that $K^{-1} \in B$. An additional condition is needed to carry local invertibility over to global invertability in B . The following theorem gives a condition for local invertibility to imply global invertability and the relation between these two inverses.

Theorem 2 [16] *Let $K \in B$ with a local inverse $K^\ominus \in \mathcal{B}$.*

1. *If $\|K^\ominus \nabla K\|_B < 1$, then the global inverse $K^{-1} \in B$, and*

$$\begin{aligned} K^{-1} &= (K^\ominus K)^{-1} K^\ominus = (I + K^\ominus \nabla K)^{-1} K^\ominus \\ \|K^{-1}\|_B &\leq \|K^\ominus\|_B (1 - \|K^\ominus \nabla K\|_B)^{-1} \end{aligned}$$

2. *If $\|K \nabla K^\ominus\|_B < 1$, then the global inverse $K^{-1} \in B$, and*

$$\begin{aligned} K^{-1} &= K^\ominus (K K^\ominus)^{-1} = K^\ominus (I + K \nabla K^\ominus)^{-1} \\ \|K^{-1}\|_B &\leq \|K^\ominus\|_B (1 - \|K \nabla K^\ominus\|_B)^{-1} \end{aligned}$$

Proof

See [16]. □

Note that the condition on $\|K \nabla K^\ominus\|_B$ is related to the slowness of the variation of the kernel in the time domain, or of the transfer function in the frequency domain. Define two measures of the variation rates of linear time-varying operators.

Definition 4 *Given $\sigma > 1$ define the time-domain variation rate as:*

$$d_{(\sigma)}(K) \triangleq \|KT - TK\|_{(\sigma)},$$

where T is the shift operator satisfying $(Tu)(t) = u(t-1)$, $t \in \mathcal{Z}$. K is said to commute approximately with the shift if $d_{(\sigma)}(K) < \|K\|_{(\sigma)}$.

Definition 5 Given $\sigma > 1$ define the frequency-domain variation rate as:

$$\partial_{\sigma}(\hat{K}) \triangleq \sup_{t \in \mathcal{Z}} \|\hat{K}_{t+1} - \hat{K}_t\|_{H_{\sigma}^{\infty}}.$$

K has slowly varying local transfer functions, or \hat{K} is slowly varying if $\partial_{\sigma}(\hat{K}) < \mu_{\sigma}(\hat{K})$.

A time-domain frozen-time stability condition is given as follows:

Theorem 3 [16] Let $G, K \in \mathcal{E}_{\sigma}$, $\sigma > 1$, and either G has no memory or K is shift-invariant, then the existence of $(I + G \otimes K)^{\ominus} \in \mathcal{E}_{\sigma}$ implies $(I + G K)^{-1} \in B$, provided that

$$d_{(1)}(G \otimes K) < (e \ln \sigma) \|(I + G \otimes K)^{\ominus} \otimes [(1 - \alpha) I - \alpha G \otimes K]\|_{(\sigma)}^{-1}$$

for some $\alpha \in \mathcal{R}$.

Proof

See [16]. □

The time-domain variation rate needed in order to apply Theorem 3 involves the l^1 kernel norm of an inverse, so it is in general difficult to compute or measure. Therefore, Zames and Wang turned to a frequency-domain approach. The idea is that if the global system K varies slowly, the transfer function of its local system will vary slowly, too. The relation between these two variation rates is expressed explicitly in the following lemma:

Lemma 1 For $\sigma_0 > \sigma > 1$, $d_{\sigma}(K) \leq \sigma \kappa_{\sigma_0/\sigma} \partial_{\sigma}(\hat{K})$, where $\kappa_{\sigma_0/\sigma} := (1 - (\sigma_0/\sigma)^{-2})^{-1/2}$.

Proof

See [16]. □

In the time-invariant $l^2(-\infty, \infty)$ case Parseval's theorem provides an isometry between the time domain and the frequency domain representations, so the $l^2(-\infty, \infty)$ time-domain behavior can be estimated from the frequency-domain properties. However this cannot be done in the situation of time-varying $l^\infty(-\infty, \infty)$ with the time-domain norm $\|\cdot\|_{a(\sigma)}$ and the frequency-domain norm $\mu_\sigma(\cdot)$.

Zames and Wang defined an operator norm $\|\cdot\|_{a(\sigma)}$, called the auxiliary operator norm, which is equivalent to the B norm and show the existence of an approximate isometry between the auxiliary operator norm and the frequency-domain μ_σ norm for slowly varying systems. They first define the auxiliary signal norm $\|\cdot\|_{a(\sigma)}$ on $l^\infty(-\infty, \infty)$ by

$$\|u\|_{a(\sigma)} := \begin{cases} \kappa_\sigma^{-1} \sup_{t \in \mathbb{Z}} \left(\sum_{\tau=-\infty}^t |u(\tau) \sigma^{-(t-\tau)}|^2 \right)^{1/2}, & \sigma < \infty \\ \|u\|_{l^\infty}, & \sigma = \infty \end{cases},$$

where $\kappa_\sigma = (1 - \sigma^{-2})^{-\frac{1}{2}}$ and $\Pi_{(t)}$ is the truncation operator which maps $f(\tau)$ to a function $g(t)$ with

$$g(\tau) = \begin{cases} f(\tau) & \text{if } \tau \leq t \\ 0 & \text{otherwise} \end{cases}$$

Don't confuse the truncation operator $\Pi_{(t)}$ with the frozen time system corresponding to a general operator Π . The factor κ_σ^{-1} is introduced to make $\|u\|_{a(\sigma)} = 1$ when $u(\tau) = 1$ for all τ .

The induced operator norm of $\|\cdot\|_{a(\sigma)}$ is called the auxiliary operator norm:

$$\|K\|_{a(\sigma)} := \sup\{\|Ku\|_{a(\sigma)} : u \in l^\infty(-\infty, \infty), \|u\|_{a(\sigma)} \leq 1\}$$

This auxiliary norm is equivalent to $\|\cdot\|_B$ and $\|K\|_{a(\infty)} = \sup_t \|k(t, \cdot)\|_{l^1}$, where $K \in B$ and k is the kernel of K .

An important property of this auxiliary norm is

$$\mu_\sigma(\hat{K}) - \alpha \leq \|K\|_{a(\sigma)} \leq \mu_\sigma(\hat{K}) + \beta \quad (3.12)$$

for some non-negative α, β . It can be shown that $\beta \rightarrow 0$ as $\partial_\sigma(\hat{K}) \rightarrow 0$, and $\alpha \rightarrow 0$ as $\partial_\sigma(\hat{K}) \rightarrow 0$ and $\sigma \rightarrow 1$.

Define a seminorm called the *recent past seminorm*:

$$\|K\|_{a(\sigma;t)} := \kappa_\sigma^{-1} \sigma^{-t} \sup\{\|\Pi_{(t)} K u\|_{l^2_\sigma} : u \in l^\infty(-\infty, \infty), \|u\|_{a(\sigma)} \leq 1.\}$$

$\|\cdot\|_{a(\sigma;t)}$ is called the recent past seminorm because the definition of the signal norm $\|\cdot\|_{l^2_\sigma}$ gives larger weightings to more recent values of the signal. Note that $\|K\|_{a(\sigma)} = \sup_t \|K\|_{a(\sigma;t)}$.

With this machinery in hand a frequency domain frozen-time stability condition is given in the following theorem.

Theorem 4 [16] *Let $G, K \in \bar{\mathcal{E}}_\sigma$. Assume G is time-varying with $\mu_\sigma(\hat{G}) = 1$, and K is time-invariant. The global closed loop operator $(I + G K)^{-1}$ is l^∞ -bounded if the local closed loop operator $(I + G \otimes K)^{-1}$ is l^∞ -bounded and*

$$\partial_\sigma(\hat{G}) < \frac{\sigma - 1}{\gamma \mu_\sigma(\hat{K})(1 + \gamma \|G\|_{a(\sigma)} \mu_\sigma(\hat{K}))}. \quad (3.13)$$

When G has no memory, the variation bound becomes

$$\partial_\sigma(\hat{G}) < \frac{\sigma - 1}{\gamma \mu_\sigma(\hat{K})(1 + \gamma \mu_\sigma(\hat{K}))}. \quad (3.14)$$

Proof

See [16]. □

This is a significant step in the frozen-time analysis since the frequency-domain type variation bound allows us to consider a larger class of problems, for example, the robust adaptive control problem which considers the presence of unmodeled dynamics. Furthermore, currently available H^∞ optimization techniques can be applied to slowly varying systems. The inequality (3.14) only contains frequency-domain norms, so it is easier to check, but (3.13) still involves the auxiliary operator norm $\|G\|_{a(\sigma)}$ which is usually not very tractable. If $G \in \mathcal{E}_\sigma$ then, by using the following fact [16]:

$$\|G\|_{a(\sigma)} \leq \mu_\sigma(\hat{G}) + \frac{\partial_\sigma(\hat{G})}{\sigma - 1}, \quad (3.15)$$

Replace (3.13) with the following slightly more conservative pure frequency-domain inequality:

$$\partial_\sigma(\hat{G}) \leq \frac{\sigma - 1}{\gamma\mu_\sigma(\hat{K}) + \gamma^2\mu_\sigma^2(\hat{K})[\mu_\sigma(\hat{G}) + \frac{\partial_\sigma(\hat{G})}{\sigma - 1}]}. \quad (3.16)$$

$$= \frac{\sigma - 1}{\gamma\mu_\sigma(\hat{K}) + \gamma^2\mu_\sigma^2(\hat{K})[1 + \frac{\partial_\sigma(\hat{G})}{\sigma - 1}]}. \quad (3.17)$$

since $\mu_\sigma(\hat{G}) = 1$ by assumption. (3.17) is even easier to check than (3.13).

The following theorem links the recent past seminorm of global operators with the shifted H^∞ norms of local operators are stated below.

Theorem 5 [16] *Suppose $K \in \mathcal{E}_\sigma$ and \hat{K} has slowly varying local transfer functions, which means*

$$\partial_\sigma(\hat{K}) \triangleq \sup_{t \in \mathbb{Z}} \|\hat{K}_{t+1} - \hat{K}_t\|_{H_\sigma^\infty} < \mu_\sigma(\hat{K}).$$

Then the following relation holds:

$$\|K\|_{a(\sigma,t)} - \|\hat{K}_t\|_{H_\sigma^\infty} \leq \kappa_\sigma^{(\infty)} \partial_\sigma(\hat{K}), \quad (3.18)$$

where $\kappa_\sigma^{(\infty)} = (\sigma - 1)^{-1}$.

Proof

See [16]. □

3.2 Robust Stabilization for Systems with Coprime Factor Uncertainties

This section introduces the design of discrete-time linear H^∞ controllers for systems described by coprime factor uncertainty models. This design will be extended for the adaptive H^∞ controllers in section 6.3.

Consider a SISO LTI plant $P(z^{-1})$ for which the input/output relation described by a coprime factor uncertainty model of the following form:

$$(M_0 + \Delta_M)y(t) = (N_0 + \Delta_N)u(t) + d_1(t) \quad (3.19)$$

where M_0 and N_0 are Hurwitz polynomials in z^{-1} , the backward-shift operator or unit delay operator, of the following form

$$\begin{aligned} N_0(z^{-1}) &= b_1 z^{-1} + b_2 z^{-2} + \cdots + b_n z^{-n} \\ M_0(z^{-1}) &= 1 + a_1 z^{-1} + \cdots + a_n z^{-n} \end{aligned}$$

and d_1 is a noise term. Assume $d_1 \in l^2$ here. Notice that in the adaptive robust stabilization problem we will introduce later d_1 will be assume to be in l^∞ instead of l^2 because l^∞ signals are more natural to adaptive control problems. For example, persistent exciting signals are in l^∞ , but not in l^2 .

Define the nominal plant $P_0(z^{-1})$ to be $P_0(z^{-1}) = M_0^{-1}N_0$. Assume the parameters of the nominal system P_0 are known exactly and $\left\| \begin{array}{c} \Delta_N \\ \Delta_M \end{array} \right\|_{H^\infty} \leq 1/\gamma$.

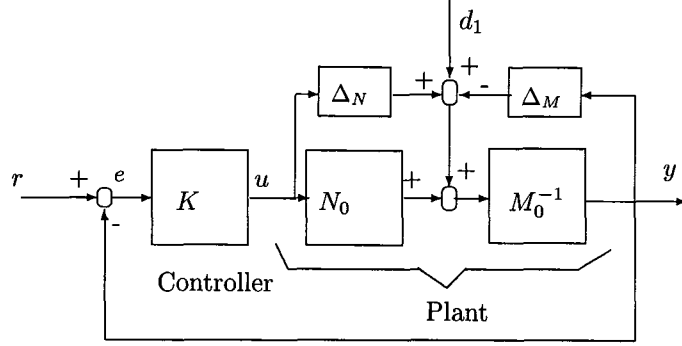


Figure 3.2: The block diagram of the coprime factor robust stabilization problem

The goal is to find a controller K such that the closed-loop system as shown in Figure 3.2 is internally stable. By the small gain theorem we can show that this problem is equivalent to finding a controller such that

$$\left\| \begin{bmatrix} K(1 + P_0 K)^{-1} M_0^{-1} \\ (1 + P_0 K)^{-1} M_0^{-1} \end{bmatrix} \right\|_{H^\infty} < \gamma. \quad (3.20)$$

This control problem can be rewritten as a standard control design problem as in Figure 3.3 with the transfer matrix of the generalized plant G being

$$G = \begin{bmatrix} \begin{bmatrix} 0 \\ M_0^{-1} \\ M_0^{-1} \end{bmatrix} & \begin{bmatrix} I \\ P_0 \\ P_0 \end{bmatrix} \end{bmatrix}. \quad (3.21)$$

For robust stabilization problem, we can let $r = 0$ because it does not affect the stability properties of this linear control system. Let $w = \Delta_N u - \Delta_M y + d_1$ and $z = [u \ y]^T$. Let T_{zw} be the sensitivity function from w to z . Note that

$$T_{zw} = \begin{bmatrix} K(1 + P_0 K)^{-1} M_0^{-1} \\ (1 + P_0 K)^{-1} M_0^{-1} \end{bmatrix}.$$

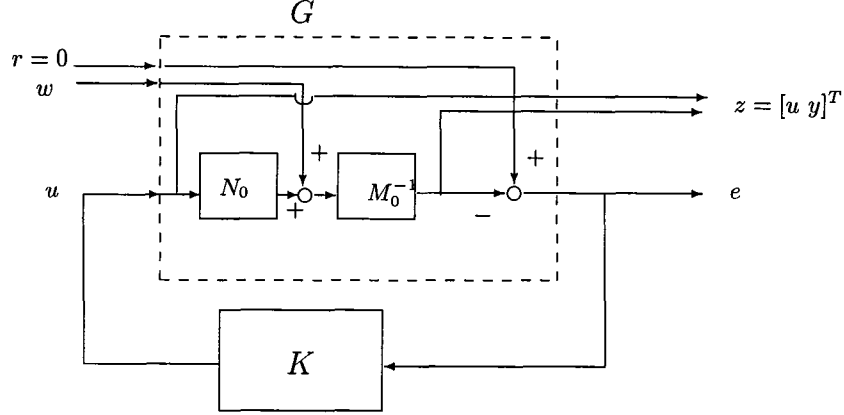


Figure 3.3: The coprime factor robust stabilization problem drawn as the standard form

Therefore, the problem (3.20) will be equivalent to finding a controller K such that $\|T_{zw}\|_{H^\infty} < \gamma$. One efficient algorithm to solve this H^∞ control problem is developed by Doyle *et al.* [64]. In order to use this method, we need to find a state-space realization for the generalized plant G .

In theory we can find the transfer matrix

$$G = \begin{bmatrix} \begin{bmatrix} 0 \\ M_0^{-1} \end{bmatrix} & \begin{bmatrix} I \\ P_0 \end{bmatrix} \\ M_0^{-1} & P_0 \end{bmatrix} \quad (3.22)$$

first, then construct a state-space realization of this transfer matrix. We usually start with a non-minimal state-space realization, and then reduce its order to get a minimal realization. However, in practice this approach often has numerical problems. For example, MATLAB's robust control toolbox often gave us a wrong answer when we tried to do this computation. The numerical problems are probably due to the computation involving high order transfer functions.

Such computations are very sensitive to numerical errors. In order to avoid the numerical problem, we should convert the transfer functions M_0 and N_0 into state-space forms as early as possible during the computation.

We now introduce a numerically robust procedure to construct the state space realization for G . First notice that if the transfer function of P_0 is given first, we can construct a coprime factorization for P_0 as follows. Suppose P_0 has a state-space realization $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$ and L is a matrix such that $A + LC$ is stable (i.e. $\max |\lambda_i(A + LC)| < 1$). Then we can get a coprime factorization $P_0 = M_0^{-1}N_0$ with the state-space realizations

$$N_0 = \left[\begin{array}{c|c} A + LC & B + LD \\ \hline C & D \end{array} \right]$$

$$M_0 = \left[\begin{array}{c|c} A + LC & L \\ \hline C & I \end{array} \right].$$

Then the state-space realization of the generalized plant G is

$$G = \left[\begin{array}{c|cc} A & -L & B \\ \hline \left[\begin{array}{c} 0 \\ C \end{array} \right] & \left[\begin{array}{c} 0 \\ I \end{array} \right] & \left[\begin{array}{c} I \\ D \end{array} \right] \\ C & I & D \end{array} \right]. \quad (3.23)$$

However, in our problem the transfer functions N_0 and M_0 are given first. We cannot choose an arbitrary state-space realization for $P_0 = M_0^{-1}N_0$ and an L , because if we do that, then

$$N_0 = \left[\begin{array}{c|c} A + LC & B + LD \\ \hline C & D \end{array} \right]$$

and

$$M_0 = \left[\begin{array}{c|c} A + LC & L \\ \hline C & I \end{array} \right]$$

may not correspond to the same N_0 and M_0 as in (3.19). Note that the problem defined in (3.19) is strongly dependent on the choice of the coprime factorization. Different coprime factorizations will lead to different solutions.

For a general coprime factorization, it may not be easy to find the right L . However, for the particular parametric structure of M_0 and N_0 as (3.19) we can find A, B, C, D, L by the following procedure:

Step 1. Given two stable coprime transfer function M_0 and N_0 :

$$\begin{aligned} N_0(z^{-1}) &= b_1 z^{-1} + b_2 z^{-2} + \dots + b_n z^{-n} \\ M_0(z^{-1}) &= 1 + a_1 z^{-1} + \dots + a_n z^{-n}. \end{aligned}$$

Step 2. Compute the observability canonical form state-space realizations of M_0 and N_0 :

$$\begin{aligned} N_0 &= \left[\begin{array}{c|c} A_N & B_N \\ \hline C_N & D_N \end{array} \right] \\ M_0 &= \left[\begin{array}{c|c} A_M & B_M \\ \hline C_M & D_M \end{array} \right]. \end{aligned}$$

The choice of the observability canonical form and the parametric model structure (3.24) make D_N always 0, and D_M always 1.

Step 3.

$$A = A_M - LC_M$$

$$B = B_N$$

$$C = C_N$$

$$D = 0$$

$$L = B_M.$$

For example, given $P_0(z) = \frac{2}{z-0.5}$. We can rewrite it as $P_0(z^{-1}) = \frac{2z^{-1}}{1-0.5z^{-1}}$, so we can let $N_0(z) = \frac{2}{z}$ and $M_0 = \frac{z-0.5}{z}$. The observability canonical forms of N_0 and M_0 are

$$N_0 = \left[\begin{array}{c|c} 0 & 2 \\ \hline 1 & 0 \end{array} \right]$$

$$M_0 = \left[\begin{array}{c|c} 0 & -0.5 \\ \hline 1 & 1 \end{array} \right].$$

If we choose $X = 1 + 0.5z^{-1}$ and $Y = 0.125z^{-1}$, then $XM_0 + YN_0 = 1$, so M_0 and N_0 are coprime. From the formula of step 3 above we have

$$A = 0.5$$

$$B = 2$$

$$C = 1$$

$$D = 0$$

$$L = -0.5.$$

Remark 1 In the literature of linear robust control for coprime factor uncertainties, the normalized coprime factorization is the most popular factorization for the following reasons:

- The normalized coprime factorization of an LTI system is unique

- The normalized coprime factor uncertainty model has a strong relation with the gap metric uncertainty model
- The procedure to compute the optimal robust controller for the normalized coprime factor uncertainty is very easy compared to the γ -iteration used in other types of H^∞ robust stabilization problems.

However, in the adaptive control problem studied in this research, we cannot use the normalized factorization because we need to parameterize the nominal plant as coprime factors of the FIR (finite impulse response) forms in order to formulate the parameter estimation law. The normalized coprime factors are in general not FIR transfer functions, so we cannot use them in our problem.

Note that the particular parameterization of the coprime factors we used also guarantees the uniqueness of the factorization. We will only need a suboptimal controller instead of an optimal controller, so the computation complexity is not a problem. Therefore, the advantages of using the normalized coprime factorization are not significant to our problem.

3.3 Discrete-Time H^∞ control

The formula to compute the discrete-time H^∞ controller directly is much less studied than that of the continuous-time H^∞ controller, and the direct computation of the discrete time H^∞ controller is still more involved than those for the continuous-time H^∞ controller. Fortunately, the following lemma provides us a way to compute the discrete-time H^∞ controller using the bilinear transformation and the formula of the continuous-time H^∞ controller.

Lemma 2 [65] *The bilinear transformation preserves the H^∞ norm. Given a discrete-time system $P_1(z^{-1})$ in the H^∞ space of all stable discrete-time transfer functions with $\|P_1(z^{-1})\|_{H^\infty} = \sup_{\omega \in [0, 2\pi]} |P_1(e^{j\omega})|$ and a bilinear transformation Ψ defined by*

$$s = \frac{z + 1}{z - 1}. \quad (3.24)$$

Let $P_2(s) = \Psi[P_1(z^{-1})]$. Then $P(s)$ is in the H^∞ space of all stable continuous-time transfer functions with $\|P_2(s)\|_{H^\infty} = \sup_{\omega \in R} |P_2(j\omega)|$. Furthermore, we have

$$\sup_{\omega \in [0, 2\pi]} |P_1(e^{j\omega})| = \sup_{\omega \in R} |P_2(j\omega)|. \quad (3.25)$$

Therefore, we can use a bilinear transformation to transform the generalized plant $G(z^{-1})$ to the s-domain system, $G(s)$. Then we can use the standard formula for solving the continuous-time H^∞ controller to compute the central H^∞ controller $K(s)$ and then use the inverse bilinear transform to transform $K(s)$ back to the z-domain controller $K(z^{-1})$.

Chapter 4

Adaptive Robust Stabilization Problem

A robust stabilization problem is usually defined as finding a controller which is able to stabilize not only the nominal plant but also a set of possible plants containing the nominal plant. There are many significant results about the synthesis of fixed robust stabilizing controllers with respect to various kinds of uncertainty set specifications assuming the nominal plant is given. For example H^∞ controllers can be designed to stabilize a plant with additive, multiplicative, or coprime factor unstructured uncertainties robustly, and μ -synthesis can be used to design a controller which robustly stabilizes a plant with structured uncertainty.

In practice it is not trivial to obtain the exact knowledge of the nominal plant. The model of the nominal plant is usually derived from a simplified analytic model or identified by some open-loop experiments. Many systems are too complicated to be expressed analytically, and many systems are not allowed to perform open-loop system identification. In these cases, we cannot get the exact knowledge of the nominal plant. An adaptive robust stabilization problem is defined as finding an adaptive controller which is able to stabilize a plant with unknown parameters and unmodeled dynamics.

4.1 Problem Formulation

This section formulates the adaptive robust stabilization problem studied in this dissertation.

We consider a SISO LTI plant $P(z^{-1})$ described by a coprime factor uncertainty model of the following form:

$$(M_0 + \Delta_M)y(t) = (N_0 + \Delta_N)u(t) + d_1(t) \quad (4.1)$$

where M_0 and N_0 are Hurwitz polynomials in z^{-1} , the backward-shift operator or unit delay operator, of the following form

$$\begin{aligned} N_0(z^{-1}) &= b_1 z^{-1} + b_2 z^{-2} + \cdots + b_n z^{-n} \\ M_0(z^{-1}) &= 1 + a_1 z^{-1} + \cdots + a_n z^{-n}. \end{aligned}$$

Define the nominal plant $P_0(z^{-1})$ to be:

$$P_0(z^{-1}) = M_0^{-1} N_0. \quad (4.2)$$

Let

$$\theta_p \triangleq [-a_1, \dots, -a_n, b_1, \dots, b_n]^T. \quad (4.3)$$

Assume

A1. The true parameter θ_p corresponds to a coprime transfer function $P(\theta_p) = M_0^{-1} N_0$, and θ_p lies in a known compact convex set denoted by Θ_p . Furthermore, assume every element θ of Θ_p corresponds to a coprime transfer function $P(\theta)$.

A2. Assume Δ_M and Δ_N are stable, LTI, and possibly infinite-dimensional with

$$\left\| \begin{array}{c} \Delta_N \\ \Delta_M \end{array} \right\|_B \leq D_3.$$

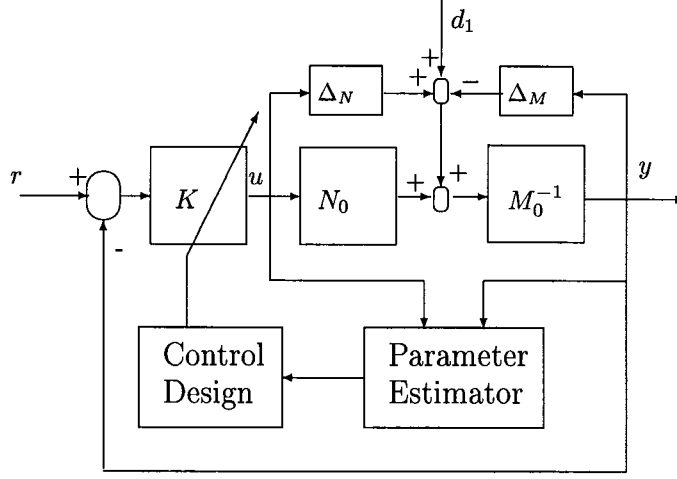


Figure 4.1: The block diagram of the adaptive control system

A3. Assume $d_1 \in l_\infty$ and $|d_1(t)| \leq D_1$.

Consider the adaptive control system shown in Figure 4.1. The design objective is to find an adaptive controller for the plant described in (4.1) so that all signals in the closed-loop system are bounded whenever the command reference $r(t)$ is bounded and assumptions A1-A3 hold.

Remark 2 Assumption A1 is not very conservative. It could allow a large set of possible plants in Θ_p . For example, given a set of continuous-time systems of the following form:

$$P(s) = \frac{s + \eta}{s^2 + 2\zeta\omega_n s + \omega_n^2} \quad (4.4)$$

where $\eta \in [-2, 2]$, $\zeta \in [-0.9, 0.9]$, $\omega_n \in [0.1, 10]$. This parameter set covers systems of dramatically different dynamic behaviors. For each $P(s)$ corresponding some particular η , ζ , and ω_n , we use the zero-order-hold discretization to get a

discrete-time system of the following form:

$$P(z^{-1}) = \frac{b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}. \quad (4.5)$$

The set of possible parameters for $P(z^{-1})$ is $\{(a_1, a_2, b_1, b_2) \mid a_1 \in [-4.335, -0.770], a_2 \in [0.177, 5.629], b_1 \in [0.035, 0.248], b_2 \in [-0.262, -0.037]\}$. No pole/zero cancellation occurs for $P(z^{-1})$ corresponding to any element in this parameter set and this set is obviously compact and convex, so this set satisfies assumption A1.

In some cases the set Θ_p may be quite small. For example, if the nominal plant has poles and zeros that are very close, then a small perturbation in the parameters could result in pole/zero cancellation. Hence Θ_p needs to be small in order to guarantee that assumption A1 holds.

On the other hand, assumption A1 is always satisfied for a sufficiently small convex set Θ_p in the neighborhood of θ_p because M_0 and N_0 are assumed to be coprime and sufficiently small perturbations will not destroy coprimeness.

Some robust adaptation literature do not assume the knowledge of Θ_p , but require (dominantly) persistently exciting signals instead. This assumption is in fact one way to achieve assumption A1 that we used here. The reason is that persistently exciting inputs can drive the parameter estimate to a small neighborhood of the true parameter where assumption A1 holds. We prefer to assume the knowledge of Θ_p because the persistent excitation condition for systems with unknown parameters and non-parametric uncertainty is difficult to check in general.

Remark 3 We restrict N_0 and M_0 to be in the FIR (in z^{-1}) form because of two reasons:

1. this is the most convenient way to parameterize the plant in terms of parameter estimation.
2. FIR polynomials in z^{-1} are stable transfer functions in z , so (4.1) is a stable factor representation of the system $P_0(z^{-1})$, no matter whether $P_0(z^{-1})$ is stable or unstable.

Remark 4 We require the leading coefficients of M_0 and N_0 to be 1 and 0, respectively. With these constraints the N_0 and M_0 corresponding to any given $P_0(z^{-1})$ are unique.

4.2 Robust Parameter Estimator

The focus of this research is on the control law part of the adaptive controller, so instead of inventing a new parameter estimation scheme a good robust parameter estimation law developed by Lemaire [20] is used. It was in turn derived from a continuous-time parameter estimator introduced by Middleton *et al* [23]. This estimator was also used by Voulgaris *et al* [21, 22].

We can rewrite the input/output relation of this plant (Eq. (4.1)) as follows:

$$\begin{aligned}
M_0 y(t) &= N_0 u(t) + \Delta_N u(t) - \Delta_M y(t) + d_1(t) \\
\Rightarrow y(t) &= (1 - M_0) y(t) + N_0 u(t) + \Delta_N u(t) - \Delta_M y(t) + d_1(t) \\
\Rightarrow y(t) &= \phi(t-1)^T \theta_p + d_2(t) + d_1(t)
\end{aligned}$$

where

$$\begin{aligned}
d_2(t) &\triangleq \Delta_N u(t) - \Delta_M y(t), \\
\phi(t-1) &\triangleq [y(t-1), \dots, y(t-n), u(t-1), \dots, u(t-n)]^T.
\end{aligned}$$

The parameter estimator cannot distinguish d_1 and d_2 from the measured input/output data, so we will often combine d_1 and d_2 together as a total perturbation denoted as η :

$$\eta(t) \triangleq d_1(t) + d_2(t),$$

Therefore

$$y(t) = \phi(t)^T \theta_p + \eta(t). \quad (4.6)$$

(4.6) is a standard linear regression form plus an error term. Define

$$\hat{\theta}(t) = [-\hat{a}_1(t), \dots, -\hat{a}_n(t), \hat{b}_1(t), \dots, \hat{b}_n(t)]^T$$

to be the estimate of θ_p at time t . Let

$$\hat{y}(t) = \phi^T(t-1) \hat{\theta}(t-1)$$

be the estimate of $y(t)$ based on the data up to time $t-1$. Let $\tilde{\theta}(t) = \theta_p - \hat{\theta}(t)$.

Let

$$e(t) = y(t) - \hat{y}(t)$$

be the prediction error. Recall

$$d_2(t) = \begin{bmatrix} \Delta_N \\ -\Delta_M \end{bmatrix}^T \begin{bmatrix} u(t) \\ y(t) \end{bmatrix},$$

and

$$\left\| \begin{bmatrix} \Delta_N \\ \Delta_M \end{bmatrix} \right\|_B \leq D_3.$$

Define

$$D_2(t) \triangleq D_3 \max_{0 \leq \tau \leq t} \{|u(\tau)|, |y(\tau)|\},$$

then

$$|d_2(\tau)| \leq D_2(t), \quad \forall 0 \leq \tau \leq t,$$

i.e. D_2 is a bounding function for $|d_2|$.

The estimation law for $\hat{\theta}$ is a least squares algorithm modified with a dead zone and projection [66]:

$$\hat{\theta}(t) = Proj \left\{ \hat{\theta}(t-1) + \frac{v(t)P(t-2)\phi(t-1)}{1 + \phi(t-1)^T P(t-2)\phi(t-1)} e(t) \right\} \quad (4.7)$$

$$P(t-1) = P(t-2) - \frac{v(t)P(t-2)\phi(t-1)\phi(t-1)^T P(t-2)}{1 + \phi(t-1)^T P(t-2)\phi(t-1)} \quad (4.8)$$

$$P(-1) = P(-1)^T > 0, \quad (4.9)$$

where

$$v(t) = \frac{\alpha f(\beta(D_2(t) + D_1), e(t))}{|e(t)|}, \quad (4.10)$$

$$\alpha \in (0, 1), \quad \beta = \sqrt{1/(1 - \alpha)}, \quad (4.11)$$

and f is a dead-zone function defined as

$$f(x, y) = \begin{cases} |y| - |x| & \text{if } |y| > |x| \\ 0 & \text{otherwise} \end{cases}. \quad (4.12)$$

$Proj$ is a projection operator into the set Θ_p to ensure the computability of the control law. If the algorithm leads to a $\hat{\theta}$ outside Θ_p , we just project $\hat{\theta}$ on to the surface of Θ before continuing. The specific choice of the projection operator is not very important as long as that the projected estimate, denoted by $\hat{\theta}'(t) = Proj \hat{\theta}(t)$, is closer to the true parameter θ_p than is $\hat{\theta}$ for $\theta_p \in \Theta_p$, so the nonincreasing property of the Lyapunov function, $V(t) = \tilde{\theta}(t)^T P(t-1)^{-1} \tilde{\theta}(t)$, used in the proof of the convergence properties of the parameter estimator will be maintained. Hence the convergence results will be retained precisely as in the case without projection.

One very popular choice is the orthogonal projection operator. When Θ is a $2n$ -dimension rectangle, for example,

$$\begin{aligned}
& b_{1_{min}} < b_1 < b_{1_{max}} \\
& \dots \\
& b_{n_{min}} < b_n < b_{n_{max}} \\
& a_{1_{min}} < a_1 < a_{1_{max}} \\
& \dots \\
& a_{n_{min}} < a_n < a_{n_{max}}
\end{aligned}$$

we can use a simpler projection scheme as follows:

$$\begin{aligned}
& \text{If } \hat{b}_i < b_{i_{min}}, \text{ then set } \hat{b}_i = b_{i_{min}}. \\
& \text{If } \hat{b}_i > b_{i_{max}}, \text{ then set } \hat{b}_i = b_{i_{max}}. \\
& \text{If } \hat{a}_i < a_{i_{min}}, \text{ then set } \hat{a}_i = a_{i_{min}}. \\
& \text{If } \hat{a}_i > a_{i_{max}}, \text{ then set } \hat{a}_i = a_{i_{max}}.
\end{aligned}$$

This is the scheme we used in the examples of Chapter 7.

The prediction error e is caused by both parametric error and non-parametric error. The purpose of the dead zone is to turn off the updating when the prediction error e cannot be distinguished from the error introduced by the external disturbance and non-parametric uncertainty (when $|e| < \beta(D_2(t) + D_1)$), hence the parameter estimate is updated only when the prediction error e is really due to a large parametric error. Furthermore, the dead zone is normalized by e to guarantee the time-varying adaptation gain $v(t)$ is bounded no matter how large e is. When this estimation procedure is used in an adaptive controller, this adaptation law has the following properties:

Lemma 3 [66]

$$1. \quad \lim_{t \rightarrow \infty} \frac{f^2(\beta(D_2(t) + D_1), |e(t)|)}{1 + \phi(t-1)^T P(t-2) \phi(t-1)} = 0 \quad (4.13)$$

$$2. \quad \lim_{t \rightarrow \infty} \|\hat{\theta}(t) - \hat{\theta}(t-1)\|_2 = 0 \quad (4.14)$$

$$3. \quad P(t) > 0, \quad \bar{\sigma}[P(t)] \leq \bar{\sigma}[P(-1)] < \infty, \quad \forall t \quad (4.15)$$

Proof

See [66]. □

The proof of the above lemma relies on the fact that D_2 is the bounding function of d_2 . Properties (4.13-4.15) are derived without any assumption on stability of the controlled system, so (4.13-4.15) hold no matter whether ϕ and e are bounded or not, and no matter whether the estimator is operated in an open loop or a closed-loop manner. This fact makes them useful in deriving boundedness conditions for the adaptive control system in chapter 5.

Chapter 5

Boundedness Condition

This chapter discussed some sufficient conditions that guarantee the signals in the adaptive loop to be bounded. We will also discuss an adaptive control scheme using the l^1 control law proposed by Voulgaris *et al* [21, 22].

5.1 Boundedness Conditions

The interaction between the estimation law and the control law is the main difficulty in the boundedness or stability proof of an adaptive control system. One can prove the convergence of the parameter estimate (not necessarily to the true parameter) if we assume the signals in the loop are bounded, but the assumption of parameter convergence is often required when proving boundedness. This dilemma must be broken in order to prove boundedness or stability of the adaptive control system.

Goodwin, Ramadge, and Caines [28] proved global boundedness of an ideal case adaptive control system by using control law-independent properties of the estimator, such as (4.13-4.15), and a sufficient condition on the properties of signals in the adaptive loop. Voulgaris *et al.* [21, 22] generalized the boundedness condition of Goodwin *et al.* to the case when external disturbances and

unmodeled dynamics are present. This condition is stated below.

Lemma 4 [21, 22] *Consider the adaptive control system in Fig 4.1 with the plant described by (4.1) and the parameter estimator described by (4.7-4.8). The signals e , u , and y in the adaptive control loop are bounded providing the following conditions hold:*

1.

$$\lim_{t \rightarrow \infty} \frac{f^2(\beta(D_2(t) + D_1), e(t))}{1 + \phi(t-1)^T P(t-2) \phi(t-1)} = 0. \quad (5.1)$$

2. $\exists c_1 \geq 0, c_2 > 0, T_1 \geq 0$ such that $\forall t \geq T_1$,

$$\|\phi(t)\| \leq c_1 + c_2 \max_{T_1 \leq \tau \leq t} |e(\tau)|. \quad (5.2)$$

3. $\exists k_1 \geq 0, 0 < k_2 < 1/\beta$ and $T_2 \geq 0$ such that $\forall t \geq T_2$,

$$D_2(t) \leq k_1 + k_2 \max_{T_2 \leq \tau \leq t} |e(\tau)|. \quad (5.3)$$

Proof¹

Case 1: Suppose $\{e(t)\}$ is a bounded sequence. Hypothesis 2 of the lemma (5.2) implies ϕ is also bounded. By the definition of ϕ , u and y are also bounded.

Case 2: Suppose $\{e(t)\}$ is an unbounded sequence. From the properties of the parameter estimators (lemma 3) the solution for the difference equation (4.7-4.8) exist all time t . Hence, e and ϕ cannot have finite escape time. Without loss of generality, we may assume there exists a subsequence $\{e(t_n)\}$ such that

$$\lim_{t_n \rightarrow \infty} |e(t_n)| = \infty, |e(t)| \leq |e(t_n)|, \forall t \leq t_n, \text{ and } |e(t_1)| > 0.$$

¹This lemma is due to [21], but I give my own proof here

Hence

$$\max_{0 \leq \tau \leq t_n} |e(\tau)| = |e(t_n)|.$$

Along the subsequence $\{e(t_n)\}$ we have

$$\begin{aligned} & \left| \frac{f^2(\beta(D_2(t_n) + D_1), e(t_n))}{1 + \phi^T(t_n - 1)P(t_n - 1)\phi(t_n - 1)} \right| \\ \text{(by (4.15))} & \geq \left| \frac{f^2(\beta(D_2(t_n) + D_1), e(t_n))}{1 + \bar{p}\|\phi(t_n - 1)\|_2^2} \right| \text{ for some } \bar{p} > 0 \quad (5.4) \\ \text{(by definition of } f, (4.12)) & \geq \left| \frac{(|e(t_n)| - \beta(D_2(t_n) + D_1))^2}{1 + \bar{p}\|\phi(t_n - 1)\|_2^2} \right| \quad (5.5) \end{aligned}$$

$$\text{(by (5.2))} \geq \frac{(|e(t_n)| - \beta(D_2(t_n) + D_1))^2}{1 + \bar{p}(c_1 + c_2|e(t_n)|)^2} \quad (5.6)$$

$$\geq \frac{\left(1 - \frac{\beta D_2(t_n)}{|e(t_n)|} - \frac{\beta D_1}{|e(t_n)|}\right)^2}{\frac{1}{|e(t_n)|^2} + \bar{p}\left(\frac{c_1}{|e(t_n)|} + c_2\right)^2}. \quad (5.7)$$

From the hypothesis of the lemma (5.3):

$$D_2(t_n) \leq k_1 + k_2 \max_{T_2 \leq \tau \leq t} |e(\tau)|, \quad \forall t_n, \quad (5.8)$$

and from the nondecreasing assumption on the subsequence $\{e(t_n)\}$, we have

$$D_2(t_n) \leq k_1 + k_2 |e(t_n)| \quad (5.9)$$

$$\Rightarrow \frac{D_2(t_n)}{|e(t_n)|} \leq \frac{k_1}{|e(t_n)|} + k_2. \quad (5.10)$$

By assumption

$$\lim_{t_n \rightarrow \infty} |e(t_n)| = \infty, \quad (5.11)$$

so

$$\lim_{t_n \rightarrow \infty} \frac{D_2(t_n)}{|e(t_n)|} \leq k_2 < \frac{1}{\beta}, \quad (5.12)$$

$$\lim_{t_n \rightarrow \infty} \frac{D_1}{|e(t_n)|} = 0 \quad (5.13)$$

$$\lim_{t_n \rightarrow \infty} \frac{c_1}{|e(t_n)|} = 0 \quad (5.14)$$

$$\lim_{t_n \rightarrow \infty} \frac{1}{|e(t_n)|} = 0. \quad (5.15)$$

Therefore

$$\lim_{t \rightarrow \infty} \frac{f^2(\beta(D_2(t) + D_1), |e(t_n)|)}{1 + \phi(t-1)^T P(t-2) \phi(t-1)} \geq \frac{(1 - \beta k_2)^2}{\bar{p} c_2^2}. \quad (5.16)$$

From the third hypothesis of the lemma we have $k_2 < 1/\beta$, so

$$\frac{(1 - \beta k_2)^2}{\bar{p} c_2^2} > 0. \quad (5.17)$$

This is a contradiction to the first hypothesis. Therefore, the sequence $\{e(t)\}$ must be bounded, and u , y and e are bounded by the same argument as case 1. \square

Lemma 4 is a sufficient condition on the signals of the adaptive loop. It is not explicitly related to the control law, so it is only useful for analysis, but not for design. It gives no clue about how to design an adaptive controller that satisfies the above condition. For design problems we need a sufficient condition on the control law. Voulgaris *et al* [21, 22] derived a sufficient condition based on the B norm of time-varying sensitivity functions. We will introduce their sufficient condition next.

Define the estimated model with

$$\hat{N}_t(z^{-1}) = \hat{b}_1(t)z^{-1} + \hat{b}_2(t)z^{-2} + \dots + \hat{b}_n(t)z^{-n} \quad (5.18)$$

$$\hat{M}_t(z^{-1}) = 1 + \hat{b}_1(t)z^{-1} + \dots + \hat{a}_n(t)z^{-n}. \quad (5.19)$$

Then the error equation can be written as follows

$$\begin{aligned} e(t) &= y(t) - \phi(t-1)^T \hat{\theta}(t-1) \\ &= [(1 - M_0(z^{-1}))y(t) + N_0(z^{-1})u(t) + \eta(t)] \end{aligned} \quad (5.20)$$

$$-[(1 - \hat{M}_t(z^{-1}))y(t) + \hat{N}_t(z^{-1})u(t)] \quad (5.21)$$

$$= \hat{M}_t(z^{-1})y(t) - \hat{N}_t(z^{-1})u(t). \quad (5.22)$$

The last equality is because

$$M_0 y(t) = N_0 u(t) + \eta(t). \quad (5.23)$$

The error equation (5.22) can be thought as a fictitious time-varying system driven by a noise signal $e(t)$ which is due to the estimation error $\tilde{\theta} = \theta_p - \hat{\theta}$, internal disturbance d_2 (caused by the unmodeled dynamics), and the external disturbance d_1 , i.e.

$$e(t) = \phi(t-1)^T \tilde{\theta}(t-1) + d_1(t) + d_2(t). \quad (5.24)$$

Let S^{ue} be the map from $e(t)$ to $u(t)$ and S^{ye} be the map from $e(t)$ to $y(t)$. i.e.

$$u = S^{ue}e, \quad y = S^{ye}e. \quad (5.25)$$

The following lemma gives a boundedness condition in terms of linear time-varying sensitivity operators S^{ue} and S^{ye} . Define a global sensitivity operator S as

$$S = \begin{bmatrix} S^{ue} \\ S^{ye} \end{bmatrix} \quad (5.26)$$

Lemma 5 [21, 22] *Consider the adaptive control system in Fig 4.1 with the plant described by (4.1) and the parameter estimator described by (4.7-4.8). If the control law $u(\tau) = K_t(z^{-1})[y(\tau)]$ can stabilize the fictitious system $\hat{M}_t(z^{-1})y(\tau) - \hat{N}_t(z^{-1})u(\tau) = e(\tau)$ and satisfies*

$$\|S\|_B < \frac{1}{\beta D_3}, \quad (5.27)$$

where β is the design parameter in the parameter estimator (4.7-4.8), then e , u , and y of the original adaptive control system will be bounded.

Proof²

The proof will be done by verifying (5.1–5.3) of lemma 4. First note that (5.1) is a property of the estimation law (4.7–4.8), so it is always fulfilled as long as the estimation law (4.7–4.8) is used. Hence we only need to show that (5.2) and (5.3) are satisfied with the control law.

If a time-varying control law $u(t) = K_t[y(t)]$ is designed such that the system $\hat{M}_t y(\tau) - \hat{N}_t u(\tau) = e(\tau)$ is l^∞ -stable, i.e. maps l^∞ to l^∞ , then (5.2) is satisfied because $\phi(t)$ is a vector of delayed $u(t)$ and $y(t)$. Furthermore, if the control law is designed such that

$$\left\| \begin{array}{c} S^{ue} \\ S^{ye} \end{array} \right\|_B < \frac{1}{\beta D_3}, \quad (5.28)$$

then

$$D_2(t) = \max_{0 \leq \tau \leq t} \{|u(\tau)|, |y(\tau)|\} \leq \left\| \begin{array}{c} S^{ue} \\ S^{ye} \end{array} \right\|_B e(t). \quad (5.29)$$

Thus (5.3) will be satisfied, Hence boundedness of the adaptive control system is proved by lemma 4. \square

(5.27) is formulated as a mixed sensitivity problem often seen in linear robust control problems, but the operator involved is time-varying.

Remark 5 Many papers on indirect adaptive control treat the system $e(t) = \hat{M}_t y(t) - \hat{N}_t u(t)$ as a linear time-varying system without explanation. However, the time-varying polynomials $\hat{M}_t(z^{-1})$ and $\hat{N}_t(z^{-1})$ are in fact functions of the parameter estimate $\hat{\theta}(t)$, so the overall adaptive system is a nonlinear time-varying system. Formally there are two ways to explain why it can be treated

²This lemma is due to [21], but I give my own proof here

as a linear time-varying system. For local stability analysis, the adaptive system can be linearized around a tuned system. For global stability problems assume the difference equation describing the adaptive control system is

$$x(t+1) = F(x(t))x(t) + G(x(t))r(t) \quad (5.30)$$

$$\hat{\theta}(t+1) = \Psi(x(t))x(t). \quad (5.31)$$

With a careful choice of adaptation laws we can guarantee that $\hat{\theta}(t)$ is bounded for all time t no matter whether the state $x(t)$ is bounded or not. Assume that, given an initial condition $x(0)$ and the external input $r(t)$, the solution $x(t)$ and $\hat{\theta}(t)$ of the difference equation (5.30-5.31) exists for $t \in [0, \infty)$. Let $z(0) = x(0)$. Define a linear time-varying system

$$z(t+1) = F(x(t))z(t) + G(x(t))r(t) \quad (5.32)$$

$$\hat{\theta}(t+1) = \Psi(x(t))z(t). \quad (5.33)$$

It is easy to see that the nonlinear time-varying system (5.30-5.31) and the linear time-varying system (5.32-5.33) have the same solution, i.e. $x(t) = z(t)$, $\forall t \geq 0$. Let $F_1(t) = F(x(t))$, $G_1(t) = G(x(t))$, and $\Psi_1(t) = \Psi(x(t))$, then the linear time-varying system (5.32-5.33) can be rewritten as

$$z(t+1) = F_1(t)z(t) + G_1(t)r(t) \quad (5.34)$$

$$\hat{\theta}(t+1) = \Psi_1(t)z(t). \quad (5.35)$$

The nonlinear time-varying system (5.30-5.31) is stable if the linear time-varying system (5.34-5.35) is stable, so we can analyze stability of the adaptive system (5.30-5.31) by analyzing stability of its corresponding linear time-varying system (5.34-5.35).

Remark 6 We can see the advantage and necessity of using a robust control law in adaptive control design from lemma 5. Robust control laws can achieve a specified bound on the norms of the sensitivity functions, while classical control laws, such as MRC, PPC, LQG cannot.

5.2 Frozen Time l^1 Adaptive Control Design

In the adaptive control system \hat{M}_t and \hat{N}_t are estimated as time goes on, so we don't know the complete trajectories of $\hat{M}(t)$ and $\hat{N}(t)$ a priori. Hence a linear time-invariant controller satisfying (5.27) cannot be designed directly. One solution is to design an LTI controller $C(\hat{\theta}(t))$ for the the estimated model (a frozen time system) for each time t .

Notice that if we fix the polynomials \hat{M}_t and \hat{N}_t , the fictitious system

$$\hat{M}_t(z^{-1})y(\tau) = \hat{N}_t(z^{-1})u(\tau) + e(\tau), \quad \tau \in \mathcal{Z} \quad (5.36)$$

is the frozen time system of the overall time-varying adaptive system at time t . Define S_t^{ye} and S_t^{ue} to be the sensitivity operators from e to y and from e to u , respectively, for the frozen time system (5.36) at time t , i.e.

$$y(\tau) = S_t^{ye}e(\tau) \quad (5.37)$$

$$u(\tau) = S_t^{ue}e(\tau), \quad \tau \in \mathcal{Z}. \quad (5.38)$$

Define a local sensitivity operator S^l by

$$S^l = \left\{ \begin{bmatrix} S_t^{ue} \\ S_t^{ye} \end{bmatrix}, \quad t \in \mathcal{Z} \right\}. \quad (5.39)$$

In other word, S^l is the linear time-varying operator constructed by cascading

every frozen time (LTI) sensitivity

$$\begin{bmatrix} S_t^{ue} \\ S_t^{ye} \end{bmatrix} \quad (5.40)$$

in time. It is important to remember that the global sensitivity operator

$$S = \begin{bmatrix} S^{ue} \\ S^{ye} \end{bmatrix} \quad (5.41)$$

is the sensitivity operator of the real time-varying adaptive system . S^l is just an operator introduced to solve the frozen time control problem

$$\|S_t^l\|_{l^1} \leq \epsilon_2, \quad \forall t \quad (5.42)$$

for some positive number ϵ_2 and some norm.

Note that S and S^l are both linear time-invariant operators. When the variation of the system is sufficiently slow, S can be approximated arbitrarily closely by S^l in a sense that will be defined precisely later. Hence, if we can design a controller to bound the norm of S^l , then the controller will also bound the norm of S , i.e. we want to design an adaptive controller satisfying $\|S\|_B \leq \frac{1}{\beta D_3}$ by designing frozen time controllers satisfying $\|S_t^l\|_{l^1} \leq \gamma_1$ for each time t for some positive number γ_1 and for some norm.

Dahleh and Dahleh [55] and Voulgaris et al [21, 22] used l^1 optimal control to design frozen time controllers satisfying $\|S_t^l\|_{l^1} \leq \frac{1}{\beta D_3} - \epsilon_1$, for each time t where ϵ_1 is a small positive number. They showed that frozen time stability implies stability and that $\|S\|_B$ is arbitrarily close to $\sup_t \|S_t^l\|_{l^1}$ if the adaptation gain is sufficiently small. (For the particular adaptation law (4.7-4.8), the estimated model approaches a linear time-invariant system, so the frozen-time stability applies automatically) The detailed algorithm of the frozen time l^1 adaptive controller can be found in Voulgaris et al [21, 22].

5.3 Disadvantages of the l^1 Adaptive Control

The frozen time l^1 adaptive control of [21, 22] has some disadvantages. First, the l^1 optimal controller in general is not Lipschitz continuous with respect to the parameters of the plant except in some rather restrictive cases. As we explained in Chapter 1, if the controller parameter is not Lipschitz continuous with respect to the plant parameter, then the overall adaptive system may vary arbitrarily fast even when the plant parameter varies slowly, and the frozen time analysis cannot be applied. An example showing the discontinuity of the l^1 optimal control law from [60] is given below. Suppose P is a linear time-invariant plant with its frozen time systems described by the sequence $\{P_t\}$ where $P_t(z^{-1}) = D_t(z^{-1})^{-1}N_t(z^{-1})$, $D(z^{-1}) = 1, \forall t$,

$$N_t(z^{-1}) = \begin{cases} 2z^{-1} + 1, & t = 0, 1 \\ 2z^{-1} + (1 + \gamma t), & 2 \leq t \leq T = 1 + \frac{1}{\gamma} \\ 2z^{-1} + (1 + \gamma T), & t > T \end{cases} \quad (5.43)$$

For this P we have $\|N_t - N_\tau\| \leq \gamma|t - \tau|, \forall t, \tau$, so P is slowly varying when γ is small. Consider an one-block problem for the frozen time system at time t :

$$\inf_{Q_t \text{ stabilizing}} \|S_t^{ye}\|_{l^1} = \inf_{Q_t \text{ stabilizing}} \|1 + P_t Q_t\|_{l^1}. \quad (5.44)$$

Computing the l^1 optimal control law from [60] yields

$$S_0^{ye} = 1, S_1^{ye} = 1, \dots, S_{T-1}^{ye} = 1, S_T^{ye} = 0, S_{T+1}^{ye} = 0, \dots$$

for any $\gamma > 0$ no matter how small, so the l^1 optimal control law is discontinuous.

Secondly, the algorithm to compute the l^1 optimal controller is not very efficient. The current technique to compute the l^1 optimal controller is by transforming the original problem into a semi-infinite linear programming problem

using duality. The solution of the latter is then computed to the desired accuracy by truncating the constraints or the variables (Dahleh 1992). When high accuracy is required, it might take a long time to solve the optimization problem. Therefore, it is not very suitable for application requiring real time computation, such as adaptive control.

On the contrary, the H^∞ robust control we used in the proposed adaptive control scheme is Lipschitz continuous with respect to the parameters of the plant, and can be solved more efficiently in real time. Therefore, it is more suitable for adaptive control than the l^1 control law.

The other problem that might cause difficulty in implementing the l^1 adaptive controller is that the order of the l^1 controller is not fixed, while the structure of adaptive controllers are usually required to be fixed. For some systems the orders of the optimal l^1 controllers are very high. One solution for this problem is to choose the highest order among the feasible set as the order of the adaptive controller, but this will have a big overhead to the on-line computation.

Chapter 6

Frozen Time H^∞ Robust Adaptive Control

This chapter introduces a new boundedness condition and the frozen time H^∞ adaptive control design scheme. Based on this new conditions, an H^∞ adaptive controller design scheme will be developed in section 6.3. The reasons we choose the H^∞ control law as the control law design rule in our adaptive control scheme are:

- The central H^∞ suboptimal controller is Lipschitz continuous with respect to the plant [67, 7].
- The H^∞ control theory for LTI systems with known nominal models is fully developed. There are many efficient algorithms and computer packages for the H^∞ control design. In our case the solution for the central H^∞ suboptimal controllers can be computed by solving two Algebraic Riccati Equations (ARE). This is particularly important when the control law is computed on-line.

6.1 A New Boundedness Condition

A new boundedness condition in terms of the shifted H^∞ norms of the frozen time systems of S^{ue} and S^{ye} will be derived in this section.

We want to design an adaptive controller satisfying $\|S\|_B \leq \frac{1}{\beta D_3}$ by designing frozen time controllers satisfying $\|S_t^l\|_{H_\sigma^\infty} \leq \gamma_1$ for each time t for some positive number γ_1 . To do so we need to link the B norm of a linear time-varying operator with the shifted H^∞ norms of its frozen time operators, and find a bound on the time domain B norm of S in terms of the frequency domain shifted H^∞ norm of S_t^l .

Before we start the derivation of the theorem, we need the following lemma which links the B norm with the auxiliary operator norms quantitatively.

Lemma 6 *For any $K \in B$*

$$\frac{1}{\kappa_\sigma} \|K\|_{a(\sigma)} \leq \|K\|_B \leq \kappa_\sigma \|K\|_{a(\sigma)} \quad (6.1)$$

Proof

We will use the fact that the auxiliary norms $\|\cdot\|_{a(\sigma)}$ of l^∞ signals are equivalent to each other and to the l^∞ norm, i.e. $\|u\|_{a(\sigma_1)} \leq \text{constant} \cdot \|u\|_{a(\sigma_2)} \leq \text{constant} \cdot \|u\|_{a(\sigma_1)}$, $1 < \sigma_1 < \sigma_2$, and $\|u\|_{a(\sigma)} \leq \|u\|_{l^\infty} \leq \kappa_\sigma \|u\|_{a(\sigma)}$, $1 < \sigma$. Define three sets,

$$U_1 = \{u \in l^\infty \mid \|u\|_{a(\sigma)} \leq 1\}$$

$$U_2 = \{u \in l^\infty \mid \|u\|_{l^\infty} \leq 1\}$$

$$U_3 = \{u \in l^\infty \mid \kappa_\sigma \|u\|_{a(\sigma)} \leq 1\}.$$

It is easy to see that $U_1 \subset U_2 \subset U_3$. For any $K \in B$, we have

$$\|K\|_B = \sup_{u \in U_2} \frac{\|Ku\|_{l^\infty}}{\|u\|_{l^\infty}} \leq \sup_{u \in U_2} \frac{\kappa_\sigma \|Ku\|_{a(\sigma)}}{\|u\|_{l^\infty}}$$

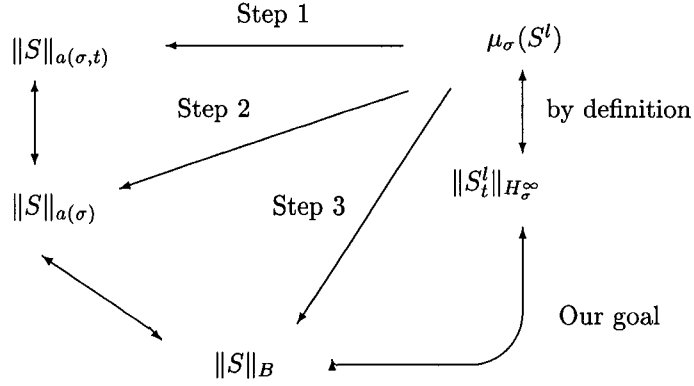


Figure 6.1: Relations between the shifted H^∞ norm, the μ_σ norm, the auxiliary norm, the recent past seminorm, and the B norm.

$$\begin{aligned}
&\leq \sup_{u \in \bar{U}_2} \frac{\kappa_\sigma \|Ku\|_{a(\sigma)}}{\|u\|_{a(\sigma)}} = \kappa_\sigma \sup_{u \in \bar{U}_2} \frac{\|Ku\|_{a(\sigma)}}{\|u\|_{a(\sigma)}} \\
&\leq \kappa_\sigma \sup_{u \in \bar{U}_3} \frac{\|Ku\|_{a(\sigma)}}{\|u\|_{a(\sigma)}} = \kappa_\sigma \sup_{u \in \bar{U}_1} \frac{\|Ku\|_{a(\sigma)}}{\|u\|_{a(\sigma)}} \\
&= \kappa_\sigma \|K\|_{a(\sigma)}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\|K\|_B &= \sup_{u \in \bar{U}_2} \frac{\|Ku\|_{l^\infty}}{\|u\|_{l^\infty}} \geq \sup_{u \in \bar{U}_1} \frac{\|Ku\|_{l^\infty}}{\|u\|_{l^\infty}} \\
&\geq \sup_{u \in \bar{U}_1} \frac{\|Ku\|_{a(\sigma)}}{\|u\|_{l^\infty}} \\
&\geq \sup_{u \in \bar{U}_1} \frac{\|Ku\|_{a(\sigma)}}{\kappa_\sigma \|u\|_{a(\sigma)}} = \frac{1}{\kappa_\sigma} \sup_{u \in \bar{U}_1} \frac{\|Ku\|_{a(\sigma)}}{\|u\|_{a(\sigma)}} \\
&= \frac{1}{\kappa_\sigma} \|K\|_{a(\sigma)}.
\end{aligned}$$

□

In general the B norm of a linear time-varying operator is much more difficult to compute than the H^∞ or the shifted H^∞ norms of its frozen time operators, which are linear time-invariant. In particular, in adaptive control problems, the

overall time-varying system is not known a priori, so it is impossible to evaluate its B norm. However, we have knowledge about the frozen time system starting from the initial time to the current time instants, so we can evaluate their shifted H^∞ norms. The following theorem relates the B norm of a linear time-varying operator to the shifted H^∞ norms of its frozen time operators. It allows us to approximate the B norm by the shifted H^∞ norm.

Theorem 6

$$\|S\|_B \leq \kappa_\sigma \sup_t \|S_t^l\|_{H_\sigma^\infty} + \kappa_\sigma \kappa_\sigma^{(\infty)} \partial_\sigma(S^l). \quad (6.2)$$

Proof

We will prove the theorem using the following steps. First we find a bound of $\|S\|_{a(\sigma,t)}$ in terms of $\mu_\sigma(S^l)$ and the frequency-domain variation rate $\partial_\sigma(S^l)$ of S^l . Then we relate this bound to $\|S\|_{a(\sigma)}$. We then relate this bound to $\|S\|_B$. Finally, we can get a bound for $\|S\|_B$ in terms of $\|S_t^l\|_{H_\sigma^\infty}$ and $\partial_\sigma(S^l)$. Fig. 6.1 illustrates the steps we are going to develop. Detailed derivation of these bounds are explained in the next few paragraphs.

Step1. Find a bound of $\|S\|_{a(\sigma,t)}$ in terms of $\mu_\sigma(S^l)$ and $\partial_\sigma(S^l)$.

By Theorem 5 (which is proposition 3.4 of [16]) we have

$$\|S\|_{a(\sigma,t)} \leq \mu_\sigma(S^l) + \kappa_\sigma^{(\infty)} \partial_\sigma(S^l) \quad (6.3)$$

where $\kappa_\sigma^{(\infty)} = (\sigma - 1)^{-1}$ and $\partial_\sigma(S^l) \triangleq \sup_{t \in \mathbb{Z}} \|S_{t+1}^l - S_t^l\|_{H_\sigma^\infty}$.

Step 2. Find a bound on $\|S\|_{a(\sigma)}$ in terms of $\mu_\sigma(S^l)$ and $\partial_\sigma(S^l)$.

Note that $\|S\|_{a(\sigma)} = \sup_t \|S\|_{a(\sigma,t)}$. Take the supremum on the left hand side of (6.3) over time t to get

$$\|S\|_{a(\sigma)} \leq \mu_\sigma(S^l) + \kappa_\sigma^{(\infty)} \partial_\sigma(S^l). \quad (6.4)$$

Note that (6.4) is independent of any particular time t .

Step 3. Find a bound on $\|S\|_B$ in terms of $\mu_\sigma(S^l)$ and $\partial_\sigma(S^l)$.

We can relate $\|S\|_B$ and the auxiliary norm $\|S\|_{a(\sigma)}$ using Lemma 6:

$$\|S\|_B \leq \kappa_\sigma \|S\|_{a(\sigma)}. \quad (6.5)$$

Then combine (6.4) and (6.5) to get

$$\|S\|_B \leq \kappa_\sigma \mu_\sigma(S^l) + \kappa_\sigma \kappa_\sigma^{(\infty)} \partial_\sigma(S^l). \quad (6.6)$$

The proof is concluded by using the definition of $\mu_\sigma(S^l)$:

$$\mu_\sigma(S^l) = \sup_t \|S_t^l\|_{H_\sigma^\infty}$$

to (6.6). □

Theorem 6 is the key for the design of the frozen time H^∞ adaptive controller. The following corollary is a new boundedness condition in terms of the easy-to-compute shifted H^∞ norm, instead of the B norm, for the H^∞ adaptive controller:

Corollary 1 *The plant described by (4.1) satisfying assumptions (A1-A3) can be stabilized globally by an adaptive controller composed by the parameter estimation law (4.7-4.8) and an H^∞ control law satisfying*

$$\exists T > 0 \ni \|S_t^l\|_{H_\sigma^\infty} < \frac{1}{\kappa_\sigma \beta D_3}, \quad \forall t \geq T. \quad (6.7)$$

Proof

It was shown in [67, 7, 68] that the central suboptimal solutions of the H^∞ control problem

$$\|S_t^l\|_{H_\sigma^\infty} < \frac{1}{\kappa_\sigma \beta D_3},$$

is unique and varies smoothly with respect to the plant because the solution of the associated Riccati equation varies smoothly with the plant, i.e. the solution S_t^l for the H^∞ control problem is Lipschitz continuous with respect to \hat{M}_p and \hat{N}_t . The second property (4.14) of the adaptation law guarantees the variation rate of the estimated model (\hat{M}_t and \hat{N}_t) will go to zero. Therefore the variation rate $\partial_\sigma(S^l)$ goes to zero as time $t \rightarrow \infty$. Stability of the closed-loop system is then proved by using Theorem 6 and Lemma 5. \square

6.2 State-Space Realization of the Shifted Systems

Notice that the σ shifted system of $P(z)$ is $P_1(z) = P(\sigma z)$ because

$$\|P(z)\|_{H^\infty} = \|P(\sigma z)\|_{H^\infty}. \quad (6.8)$$

If P has a state-space realization

$$P = \left[\begin{array}{c|c} A_P & B_P \\ \hline C_P & D_P \end{array} \right], \quad (6.9)$$

then the σ -shifted system $P_\sigma(z) = P(\sigma z)$ has a state-space realization

$$P_\sigma = \left[\begin{array}{c|c} \sigma A_P & B_P \\ \hline \sigma C_P & D_P \end{array} \right]. \quad (6.10)$$

Conversely, if a shifted system $K_\sigma(z) = K(z/\sigma)$ has a state-space realization

$$K_\sigma = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array} \right], \quad (6.11)$$

then the non-shifted system K has a state-space realization

$$K = \left[\begin{array}{c|c} A_K/\sigma & B_K \\ \hline C_K/\sigma & D_K \end{array} \right]. \quad (6.12)$$

Therefore, to solve the shifted H^∞ norm problem, just shift the generalized plant G to G_σ , find the central controller K_σ for the the H^∞ robust stabilization problem for G_σ , then shift K_σ back to K .

6.3 Adaptive H^∞ Controller

Theorem 1 allows us to transform the robust adaptive control design problem into a sequence of frozen time robust stabilization problems:

$$\|S_t^l\|_{H_\sigma^\infty} \leq \frac{1}{\kappa_\sigma \beta D_3}. \quad (6.13)$$

where $\kappa_\sigma = (1 - \sigma^{-2})^{-1/2}$. Besides the shift σ , the above frozen time robust stabilization problem are essentially the same as the linear robust stabilization problem for the coprime factor uncertainty discussed in section 3.2, so we can use the procedures introduced in section 3.2 and 6.2 to solve (6.13) for every time t successively.

The overall algorithm for the H^∞ robust adaptive controller can be summarized as follows:

1. Given the design specifications: Θ_p , D_1 , D_3 and n (see assumptions A1-3).
2. Choose design parameters α and σ .
3. Use the parameter estimation law (4.7-4.8) to get an estimated model \hat{M}_t and \hat{N}_t .
4. Construct the observability canonical forms state-space realizations

$$\hat{N}_t = \left[\begin{array}{c|c} A_N & B_N \\ \hline C_N & D_N \end{array} \right] \quad (6.14)$$

and

$$\hat{M}_t = \left[\begin{array}{c|c} A_M & B_M \\ \hline C_M & D_M \end{array} \right] \quad (6.15)$$

5. Find A , B , C , D , L corresponding to \hat{N}_t , and \hat{M} :

$$A = A_M - LC_M$$

$$B = B_N$$

$$C = C_N$$

$$D = 0$$

$$L = B_M.$$

6. Form the generalized plant

$$G = \left[\begin{array}{c|c} A_G & B_G \\ \hline C_G & D_G \end{array} \right] = \left[\begin{array}{c|cc} A & -L & B \\ \hline \left[\begin{array}{c} 0 \\ C \end{array} \right] & \left[\begin{array}{c} 0 \\ I \end{array} \right] & \left[\begin{array}{c} I \\ D \end{array} \right] \\ C & I & D \end{array} \right]. \quad (6.16)$$

7. Shift $G = \left[\begin{array}{c|c} A_G & B_G \\ \hline C_G & D_G \end{array} \right]$ to $G_\sigma = \left[\begin{array}{c|c} \sigma A_G & B_G \\ \hline \sigma C_G & D_G \end{array} \right]$.

8. Find $K_\sigma = \left[\begin{array}{c|c} A_{K_\sigma} & B_{K_\sigma} \\ \hline C_{K_\sigma} & D_{K_\sigma} \end{array} \right]$ such that $\|T_{zw}\|_{H^\infty} < \gamma$ where $\gamma = \frac{1}{\kappa_\sigma \beta D_3}$.

9. Shift K_σ back to $K = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array} \right] = \left[\begin{array}{c|c} A_{K_\sigma}/\sigma & B_{K_\sigma} \\ \hline C_{K_\sigma}/\sigma & D_{K_\sigma} \end{array} \right]$.

10. Apply the control $u(t) = -Ky(t-1)$, to get $y(t)$.

11. Repeat step 3-10 for time $t+1$.

6.4 Selection of The Design Parameters

We discuss how to select the design parameters α (or, equivalently, β) and σ in this section. Recall that α ($\alpha \in (0, 1)$) is the adaptation gain of the parameter estimator and β is defined as

$$\beta = \sqrt{\frac{1}{1 - \alpha}}. \quad (6.17)$$

The dead zone mechanism of the parameter estimator is defined as

$$v(t) = \frac{\alpha f(\beta(D_2(t) + D_1), e(t))}{|e(t)|},$$

where f is defined as

$$f(x, y) = \begin{cases} |y| - |x| & \text{if } |y| > |x| \\ 0 & \text{otherwise} \end{cases}. \quad (6.18)$$

Also note that $\alpha \in (0, 1)$ hence $\beta > 1$. When α is close to 0, then β is close to 1, and the adaptation will be slow, but the parameter updating mechanism will be turned on more frequently (as soon as $|e|$ is slightly bigger than $D_1 + D_2$). Conversely, when α is close to 1, then β is much bigger than 1, and the adaptation will be faster, but the parameter updating mechanism will be turned on less frequently (only when $|e|$ is β -times bigger than $D_1 + D_2$). In most cases, it is desirable to have a bigger β so the adaptation can be faster.

Recall that $\kappa_\sigma := (1 - \sigma^{-2})^{-1/2}$, $\sigma > 1$, so $\kappa_\sigma > 1$. When σ is close to 1, κ_σ is much greater than 1. When σ is much greater than 1, κ_σ will be close to 1. Also note that

$$\|P(z)\|_{H^\infty_\sigma} = \|P(\sigma z)\|_{H^\infty}.$$

Hence the bigger σ is, the smaller the σ -shifted H^∞ space is. It is usually better to use a smaller σ to reduce the conservatism.

However, we cannot choose β arbitrarily large or σ arbitrarily close to 1. The main limitation on β and σ come from the condition (6.7):

$$\|S_t^l\|_{H_\sigma^\infty} < \frac{1}{\kappa_\sigma \beta D_3}. \quad (6.19)$$

For a given plant, the minimal sensitivity that can be achieved is limited. Therefore, if the product $\kappa_\sigma \beta D_3$ is too large, then the H^∞ sensitivity problem (6.7) may not be solvable for every time t . Given the set of possible parameters, Θ_p , we can perform a worst case analysis to find the maximum of the achievable performance, denoted as γ_{min} , over Θ_p , i.e. $\gamma_{min} = \sup_{\theta \in \Theta_p} [inf_{stabilizing\ K} \|S^l(\theta)\|_{H_\sigma^\infty}]$. Given the size of unmodeled dynamics, D_3 , we have the constraint on the product of $\kappa_\sigma \beta$:

$$\begin{aligned} \frac{1}{\kappa_\sigma \beta D_3} &> \gamma_{min} \\ \Rightarrow \quad \kappa_\sigma \beta &< \frac{1}{\gamma_{min} D_3}. \end{aligned}$$

Therefore, β and σ need to be chosen such that the above inequality holds. If we want to use bigger β to speed up the adaptation (bigger β corresponds to bigger α), then we need a bigger σ in order to get a smaller κ_σ . A bigger σ will limit the allowable size of the set Θ_p because the σ -shifted H^∞ space will be smaller. Therefore, there is a trade-off between the adaptation speed and the size of the feasible set.

On the other hand we can write the inequality above as

$$D_3 < \frac{1}{\gamma_{min} \kappa_\sigma \beta}.$$

Once the set Θ_p and σ are determined, we can find γ_{min} and κ_σ . Along with β we can determine the maximal size of unmodeled dynamics that the adaptive control system can tolerate. Notice that β and κ_σ are always greater than 1,

and the γ achievable by any H^∞ controller for coprime factor uncertainty must be greater than 1. Therefore, D_3 must be less than 1.

6.5 H^∞ Robust Adaptive Control v.s. l^1 Robust Adaptive Control

In this section we will present an example for which the l^1 optimal controller is not continuous with respect to the plant while the H^∞ suboptimal controller is continuous with respect to the plant, which implies the theorem that satisfies the l^1 robust adaptive control will fail for this example while the theoretical bases for the H^∞ robust adaptive control we proposed in this dissertation still applies.

Given

$$W(z) = \frac{49z^2(3z+2)^2}{(4z-1)^2(31z-30)^2}$$

and

$$P(z) = \frac{(z-2)(z-3)}{z^2}.$$

Consider the weighted sensitivity $S = W(1 + PC)^{-1}$. D. G. Meyer [69] showed that for any α between $-1/6$ and 0 , the FIR transfer function:

$$S^*(z) = \alpha - 5\alpha z^{-1} + (1 + 6\alpha)z^{-2}$$

is l^1 optimal, i.e.

$$S^* = \arg \min_{\text{stabilizing } C} \|S\|_{l^1}.$$

The corresponding l^1 optimal controller C^* can be found as

$$C^*(z) = \frac{W(z) - S^*(z)}{P(z)S^*(z)}.$$

For this P and W there are infinitely many l^1 optimal controllers.

M. Dahleh and M. A. Dahleh [55] showed that for any δ , if $P(z)$ is perturbed to

$$\frac{(z - 2 + \delta)(z - 3)}{z^2},$$

then the l^1 optimal weighted sensitivity S^* will become unique and has a degree of 1. In particular, if $\delta = 0.002$, i.e.

$$P(z) = \frac{(z - 1.9982)(z - 3)}{z^2},$$

then

$$S^* = -0.16567 + 0.83036z^{-1}.$$

No matter which l^1 optimal controller for

$$P(z) = \frac{(z - 2)(z - 3)}{z^2}$$

we choose, a small change in the plant could cause a large deviation in the optimal l^1 controller. Therefore, the l^1 robust adaptive controller cannot be used when

$$P(z) = \frac{(z - 2)(z - 3)}{z^2}$$

is in the set of possible nominal plants, Θ_p .

Now we consider the H^∞ suboptimal controller such that

$$\|S\|_{H^\infty} = \|W(1 + PC)^{-1}\|_{H^\infty} \leq 1.$$

If we consider the central suboptimal controller, then the solution is always unique. Using the discrete-time H^∞ controller solver of MATLAB's Robust Control Toolbox we can get the H^∞ central suboptimal controller for

$$P(z) = \frac{(z - 2)(z - 3)}{z^2}$$

is $K_1(z) = (-1.4924 + 0.31786z^{-1} + 1.1786z^{-2} - 0.55861z^{-3} + 0.0066662z^{-4} - 5.3919 \times 10^{-8}z^{-5} + 5.8624 \times 10^{-9}z^{-6}) (1 - 8.2688z^{-1} + 9.3404z^{-2} + 4.5804z^{-3} - 9.9439z^{-4} + 3.685z^{-5} - 0.39997z^{-6})^{-1}$, and the H^∞ central suboptimal controller for

$$P(z) = \frac{(z - 1.998)(z - 3)}{z^2}$$

is $K_2(z) = (-1.4896 + 0.31726z^{-1} + 1.1764z^{-2} - 0.55758z^{-3} + 0.006654z^{-4} - 1.3563 \times 10^{-7}z^{-5} - 1.4712 \times 10^{-8}z^{-6}) (1 - 8.2495z^{-1} + 9.3127z^{-2} + 4.5714z^{-3} - 9.9173z^{-4} + 3.6747z^{-5} - 0.39884z^{-6})^{-1}$.

This shows that a small change in the plant causes a small change in the H^∞ central suboptimal controller, i.e. the H^∞ central suboptimal controller is continuous with respect to the plant, so the H^∞ robust adaptive controller can handle the case when

$$P(z) = \frac{(z - 2)(z - 3)}{z^2}$$

is in the set of possible nominal plants, Θ_p .

Chapter 7

Numerical Examples

We present some simulation results for the frozen time H^∞ robust adaptive control algorithm introduced in the previous chapter. These examples demonstrate that the proposed adaptive control scheme is able to stabilize stable and unstable, low-order and high-order systems with parameteric errors and unmodeled dynamics. The systematic and quantitative design procedure allows us to choose the design parameters without trial-and-error routines which are usually required by other adaptive control schemes.

The MATLAB scripts for Example 1 and 3 are listed in Appendix A. Other examples can be simulated by changing the values of some variables in the scripts. MATLAB's μ -Analysis and Synthesis Toolbox, Control System Toolbox, and Simulink are required in order to run these scripts.

7.1 Example 1: System with a First Order Nominal Plant

Given a discrete-time system:

$$(1 - 0.8z^{-1} + 0.016z^{-2} - 0.014z^{-3})y(t) \quad (7.1)$$

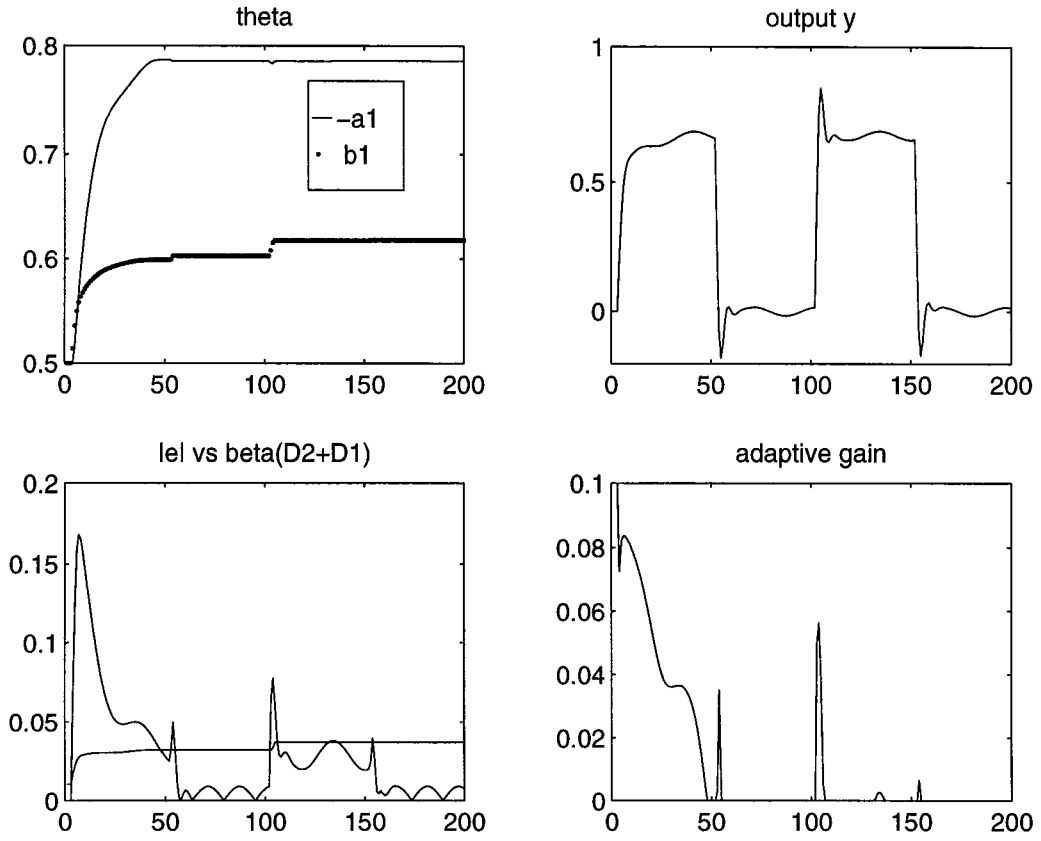


Figure 7.1: Response for example 1 where $P(z^{-1}) = (0.7z^{-1} + 0.018z^{-2} + 0.012z^{-3})(1 - 0.8z^{-1} + 0.016z^{-2} - 0.014z^{-3})^{-1}$, $d_1(t) = 0.01 \sin(0.2t)$ and $\hat{\theta}(0) = (0.5, 0.5)^T$.

$$= (0.7z^{-1} + 0.018z^{-2} + 0.012z^{-3})u(t) + d_1(t)$$

with a sampling period of 0.1 secs. Suppose $d_1(t) = 0.01 \sin(0.2t)$. The true plant model is in the form

$$(M_0 + \Delta_M)y(t) = (N_0 + \Delta_N)u(t) + d_1(t)$$

with the nominal parameters (which are unknown to the designer) be $(a_1, b_1) = (-0.8, 0.7)$, the coprime factors of the nominal plant be $M_0(z^{-1}) = 1 - 0.8z^{-1}$ and $N_0(z^{-1}) = 0.7z^{-1}$ and the uncertainty terms be $\Delta_M = 0.016z^{-2} - 0.014z^{-3}$ and $\Delta_N = 0.018z^{-2} + 0.012z^{-3}$. It is easy to check that $\left\| \begin{array}{c} \Delta_N \\ \Delta_M \end{array} \right\|_B \leq 0.03$.

Assume the only information the designer has is:

1. the structure of the nominal model is of the following form:

$$P_0(z^{-1}) = \frac{b_1 z^{-1}}{1 + a_1 z^{-1}}. \quad (7.2)$$

2. the set of possible parameters of the nominal model is

$$\Theta_p = \{(-a_1, b_1) \mid -1 \leq a_1 \leq -0.1, -0.1 \leq b_1 \leq 1\}. \quad (7.3)$$

3. the size of the unmodeled dynamics expressed in terms of the B -norm is:

$$\left\| \begin{array}{c} \Delta_N \\ \Delta_M \end{array} \right\|_B < D_3 = 0.03. \quad (7.4)$$

4. the magnitude of the noise d_1 is less than or equal to $D_1 = 0.01$, i.e.

$$|d_1(t)| \leq D_1 = 0.01, \forall t.$$

We want to design an adaptive controller to stabilize this system based on the above information.

Assumptions A1-A3 of section 4.1 are true for this example. In fact, for the first order structure we chose for the nominal plant (7.2), Θ_p can be arbitrarily large without any occurrence of pole/zero cancellation. We chose σ to be 1.1 first. Then we performed worst case analysis on Θ_p to find $\gamma_{min} = \sup_{\theta \in \Theta_p} [\inf_{\text{stabilizing } K} \|S\|_{H^\infty}] = 10.0781$. The κ_σ corresponding to $\sigma = 1.1$ is 2.4004. We need to choose β to satisfy

$$\beta D_3 \kappa_\sigma < \frac{1}{\gamma_{min}}.$$

We pick $\beta = 1.0541$ which is corresponding to $\alpha = 0.1$. Figure 7.1 shows the step response of the closed-loop control system using the frozen time H^∞ robust adaptive controller with the plant (7.1) with the initial parameters

$$a_1 = -0.5, b_1 = 0.5.$$

The response plots of most examples in the following sections are arranged in the following format (unless we specify otherwise explicitly). The left-upper subplot is the time history of $\hat{\theta}$, the estimate of the plant parameters. The upper-right subplot is the output of the plant $y(t)$. The lower left subplot shows the trajectories of two variables, the prediction error $e(t)$ and $\beta(D_2(t) + D_1)$. These two variables are used in the dead-zone mechanism of the parameter estimator. When $|e(t)|$ is less than $\beta(D_2(t) + D_1)$ the adaptation is stopped by setting the adaptation gain $v(t)$ to be zero. The lower-right subplot is the history of the adaptive gain $v(t)$. The x-axis of every subplot represent the number of steps which is corresponding to the real time equal to t times the sample period ($t = 1, 2, 3, \dots$).

We would like to emphasize that $\hat{\theta}$ does not necessarily converge to θ_p , the nominal parameters, because of the lack of persistent exciting inputs. The MAT-

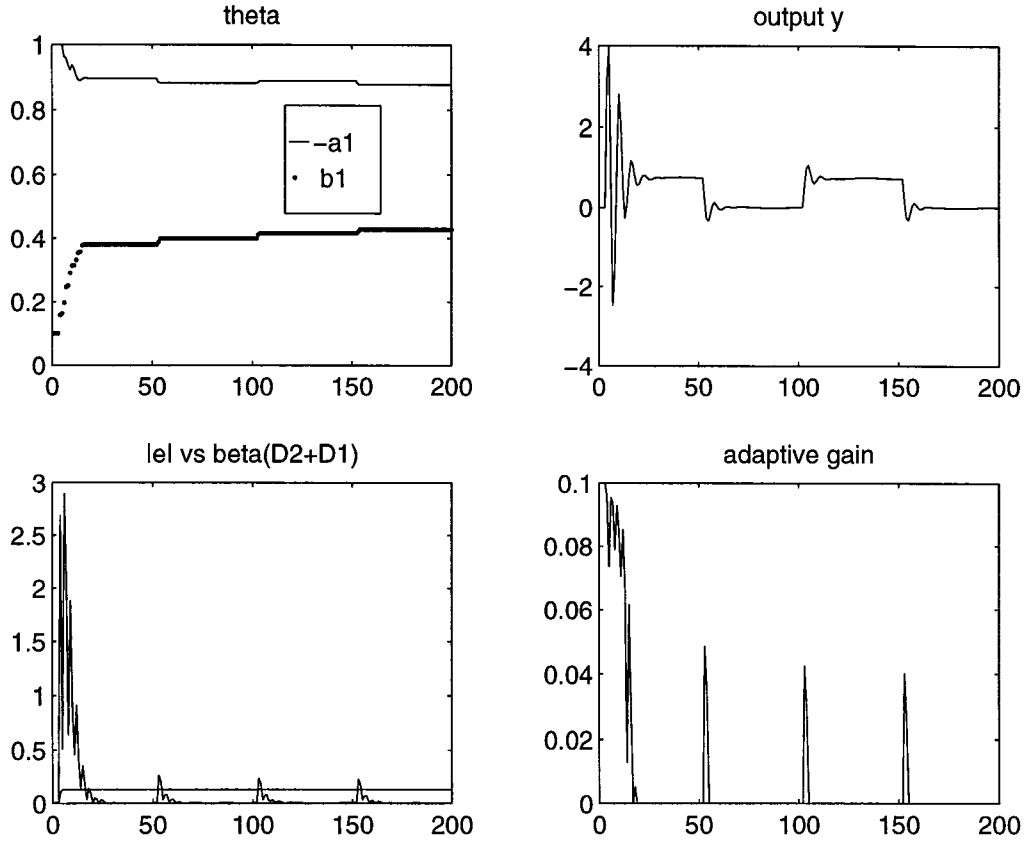


Figure 7.2: Another simulation for example 1 with different initial parameters, $\hat{\theta}(0) = (1, 0.1)^T$, from those for Figure 7.1.

LAB script for simulating this example is listed in Appendix A.

To demonstrate that this adaptive control system is insensitive to the choice of initial parameters, which is important for global stability, a simulation with the same plant but different initial parameters was performed. The initial parameters were chosen as

$$a_1 = -1, b_1 = 0.1.$$

The results are shown in Figure 7.2. Comparing Figure 7.1 and Figure 7.2, we can see a poor guess of the initial parameters only affects the transient response.

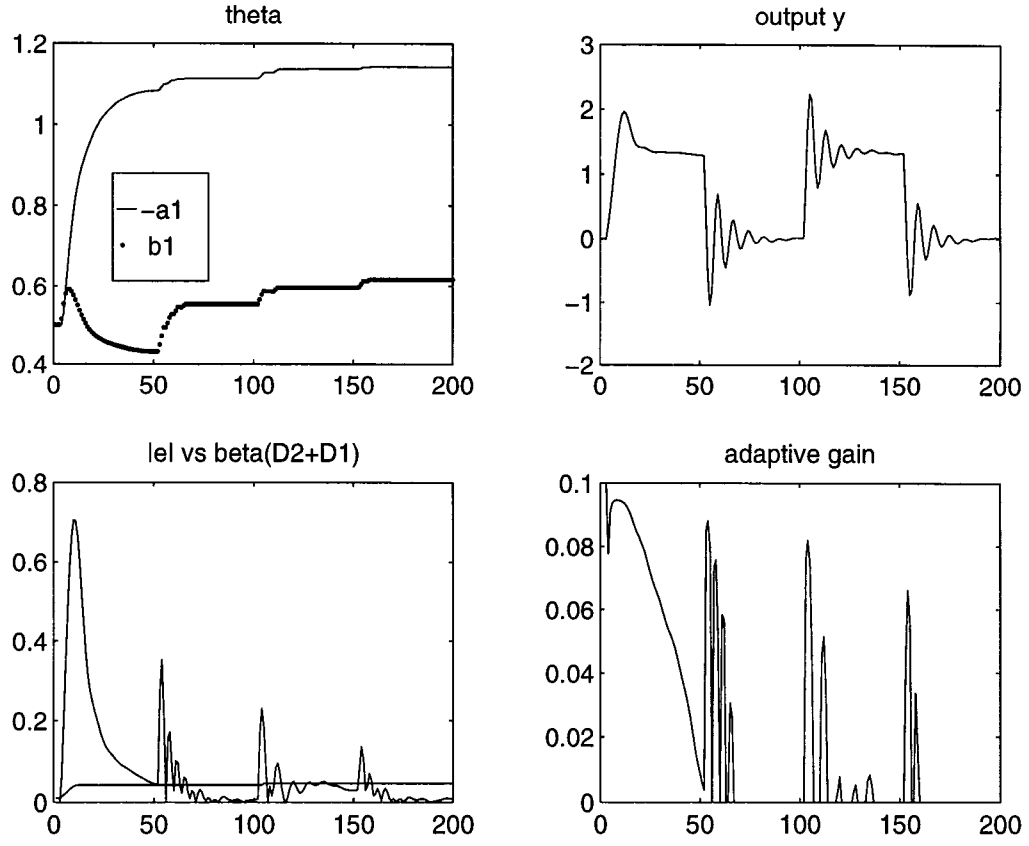


Figure 7.3: Response for example 2 for which the plant $P(z^{-1}) = (0.7z^{-1} + 0.018z^{-2} + 0.012z^{-3}) (1 - 1.2z^{-1} + 0.016z^{-2} - 0.014z^{-3})^{-1}$ is unstable.

The steady state performance of the two cases is almost the same.

7.2 Example 2: Unstable System

The next example demonstrates that the proposed adaptive control law can also stabilize unstable systems. Given a discrete-time system

$$\begin{aligned}
 (1 - 1.2z^{-1} + 0.016z^{-2} - 0.014z^{-3})y(t) \\
 = (0.7z^{-1} + 0.018z^{-2} + 0.012z^{-3})u(t) + d_1(t)
 \end{aligned} \tag{7.5}$$

with a sampling period of 0.1 secs. Suppose $d_1(t) = 0.01 \sin(0.2t)$. The maximal radius of the poles of the true plant is 1.1882, so the true plant is unstable.

Assume the only information the designer has is:

1. the structure of the nominal model is of the following form:

$$P_0(z^{-1}) = \frac{b_1 z^{-1}}{1 + a_1 z^{-1}}. \quad (7.6)$$

2. the set of possible parameters of the nominal model is

$$\Theta_p = \{(-a_1, b_1) \mid -2 \leq a_1 \leq -0.1, 0.1 \leq b_1 \leq 1\}. \quad (7.7)$$

3. the size of the unmodeled dynamics expressed in terms of the B -norm is:

$$\left\| \begin{array}{c} \Delta_N \\ \Delta_M \end{array} \right\|_B < D_3 = 0.015. \quad (7.8)$$

4. the magnitude of the noise d_1 is less than or equal to $D_1 = 0.01$, i.e.

$$|d_1(t)| \leq D_1 = 0.01, \forall t.$$

The worst case analysis gave $\gamma_{min} = 20.0716$. We chose $\alpha = 0.1$, so $\beta = 1.0541$.

We chose $\sigma = 1.1$, so $\kappa_\sigma = 2.4004$. We have verified that assumptions A1-A4 are satisfied, and the design parameters satisfy the inequality

$$\beta D_3 \kappa_\sigma < \frac{1}{\gamma_{min}}.$$

Figure 7.3 shows the step response of the closed-loop control system using the frozen time H^∞ robust adaptive controller with the plant (7.5).

The response in Figure 7.3 is stable although it is more oscillatory than that of example 7.1. This is because of the unstable characteristics of the plant.

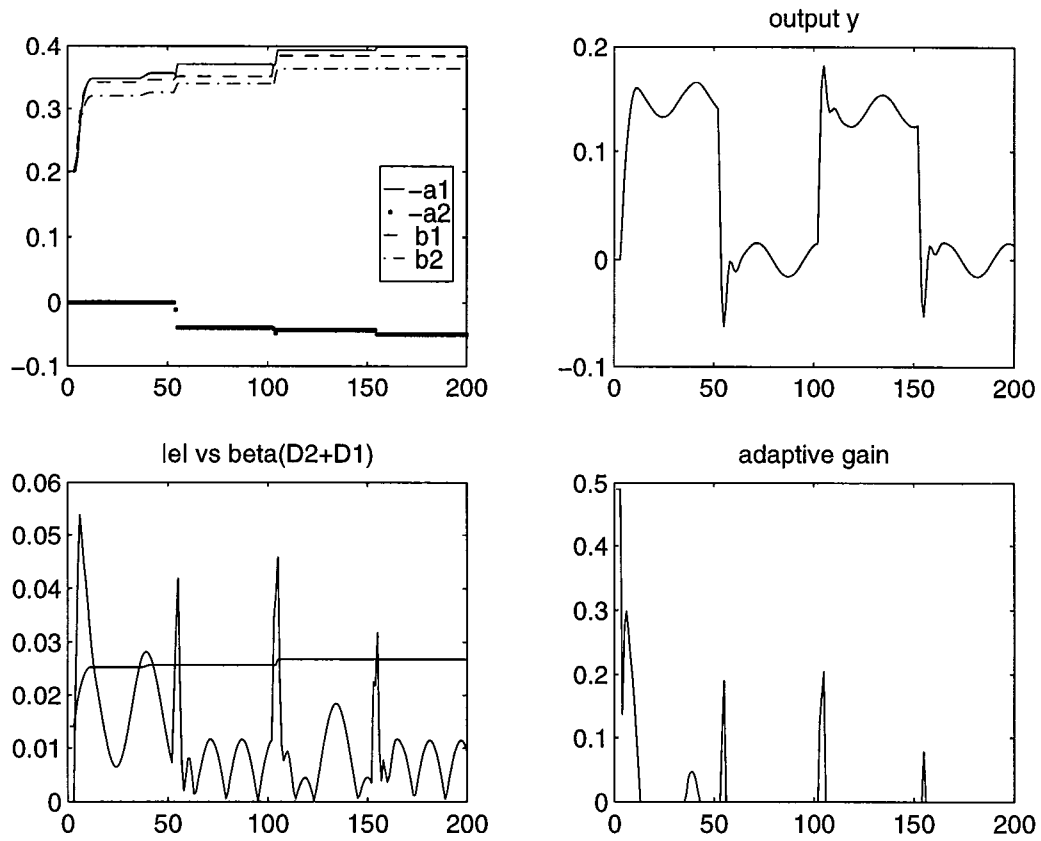


Figure 7.4: Response of example 3 for which the plant $P(z^{-1}) = (0.5z^{-1} + 0.2z^{-2} + 0.0183z^{-3} - 0.0122z^{-4}) (1 - 0.8z^{-1} + 0.4z^{-2} - 0.0244z^{-3} - 0.0152z^{-4})^{-1}$ and $d_1(t) = 0.01 \sin(0.2t)$.

7.3 Example 3: System with a Second Order Nominal Plant

The next example demonstrates the case when the nominal plant has a second order structure. Given a fourth order discrete-time system with the following input/output relation:

$$(1 - 0.8z^{-1} + 0.4z^{-2} - 0.0244z^{-3} - 0.0152z^{-4})y(t) \quad (7.9)$$

$$= (0.5z^{-1} + 0.2z^{-2} + 0.0183z^{-3} - 0.0122z^{-4})u(t) + d_1(t)$$

where $d_1(t) = 0.01 \sin(0.2t)$ is a noise.

Assume the only information the designer has is:

1. the structure of the nominal model is of the following form:

$$P_0(z^{-1}) = \frac{b_1z^{-1} + b_2z^{-2}}{1 + a_1z^{-1} + a_2z^{-2}}. \quad (7.10)$$

2. the set of possible parameters of the nominal model is

$$\begin{aligned} \Theta_p = \{ & (-a_1, -a_2, b_1, b_2) \mid -0.8 \leq a_1 \leq -0.2, \\ & 0 \leq a_2 \leq 1, 0.1 \leq b_1 \leq 1, 0.1 \leq b_2 \leq 1\}. \end{aligned} \quad (7.11)$$

3. the size of the unmodeled dynamics expressed in terms of the B -norm is:

$$\left\| \begin{array}{c} \Delta_N \\ \Delta_M \end{array} \right\|_B < D_3 = 0.05. \quad (7.12)$$

4. the magnitude of the noise d_1 is less than or equal to $D_1 = 0.01$, i.e.

$$|d_1(t)| \leq D_1 = 0.1, \forall t.$$

Assume the nominal parameters of the nominal plant are

$$a_1 = -0.8, a_2 = 0.4, b_1 = 0.5, b_2 = 0.2,$$

then we have

$$\Delta_M = -0.0244z^{-3} - 0.0152z^{-4}$$

and

$$\Delta_N = +0.0183z^{-3} - 0.0122z^{-4}.$$

Assumptions A1-A3 of section 4.1 are true for this example. We chose σ to be 1.1 first. Then we performed worst case analysis on Θ_p to find $\gamma_{min} = \sup_{\theta \in \Theta_p} [\inf_{\text{stabilizing } K} \|S\|_{H^\infty}] = 5.7298$. The κ_σ corresponding to $\sigma = 1.1$ is 2.4004. We need to choose β to satisfy

$$\beta D_3 \kappa_\sigma < \frac{1}{\gamma_{min}}.$$

We pick $\beta = 1.4$ which is corresponding to $\alpha = 0.4898$.

Figure 7.4 shows the step response of the closed control system using the frozen time H^∞ robust adaptive controller with the plant (7.9). The MATLAB script for simulating this example is listed in Appendix A.

The steady sinusoidal variation of the response in the out $y(t)$ (with a period of about 3 secs) is caused by the sinusoidal disturbance, not instability of the closed-loop system. This can be confirmed by a simulation of the noise-free case shown in Figure 7.5.

7.4 Example 4: Adaptive v.s. Fixed H^∞ Controller

This example demonstrates the advantage of the H^∞ adaptive controller over the fixed linear H^∞ controller. Using the same plant as the previous example a linear fixed H^∞ controller treating the initial parameters of the adaptive controller as the nominal parameters was designed to tolerate the same size of unmodeled dynamics. The output of the resulting closed-loop system is shown in Figure 7.6.

The response using the fixed controller is much more oscillatory than the result of the adaptive case. This is because the initial parameters are far from the real nominal parameters. The fixed robust controller cannot tolerate the total

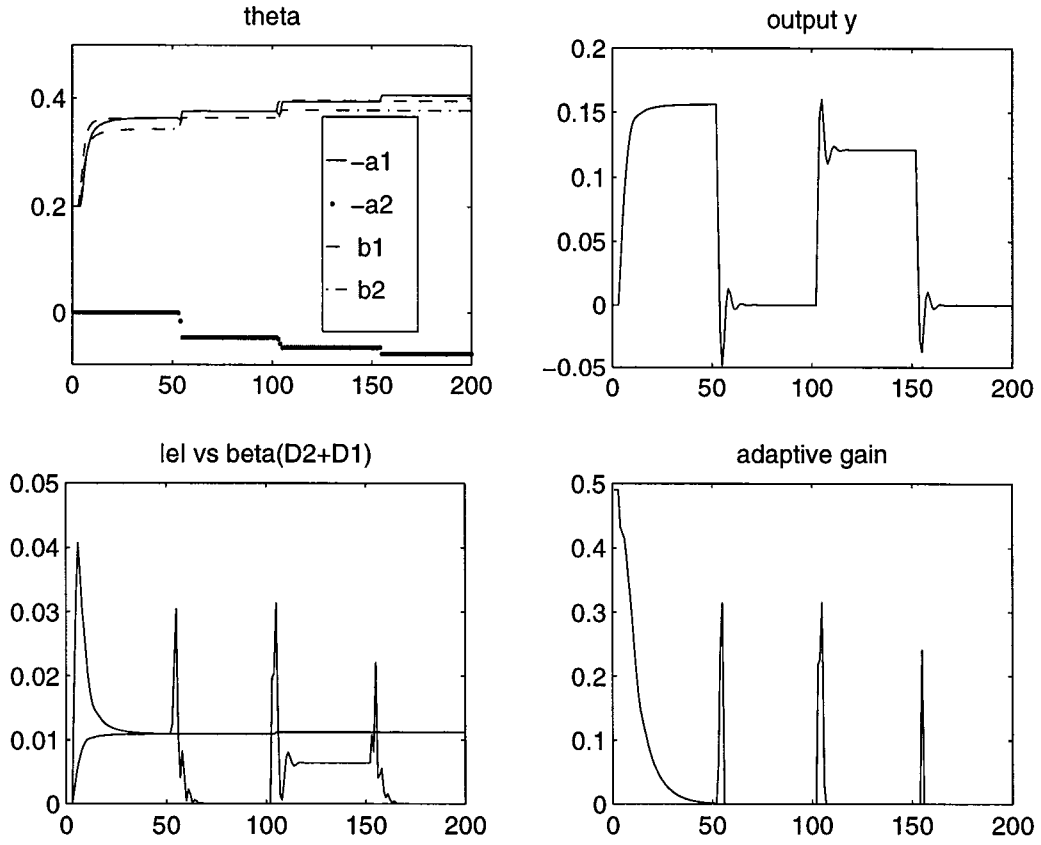


Figure 7.5: Response of example 3 for which the plant $P(z^{-1}) = (0.5z^{-1} + 0.2z^{-2} + 0.0183z^{-3} - 0.0122z^{-4}) (1 - 0.8z^{-1} + 0.4z^{-2} - 0.0244z^{-3} - 0.0152z^{-4})^{-1}$ and no disturbances.

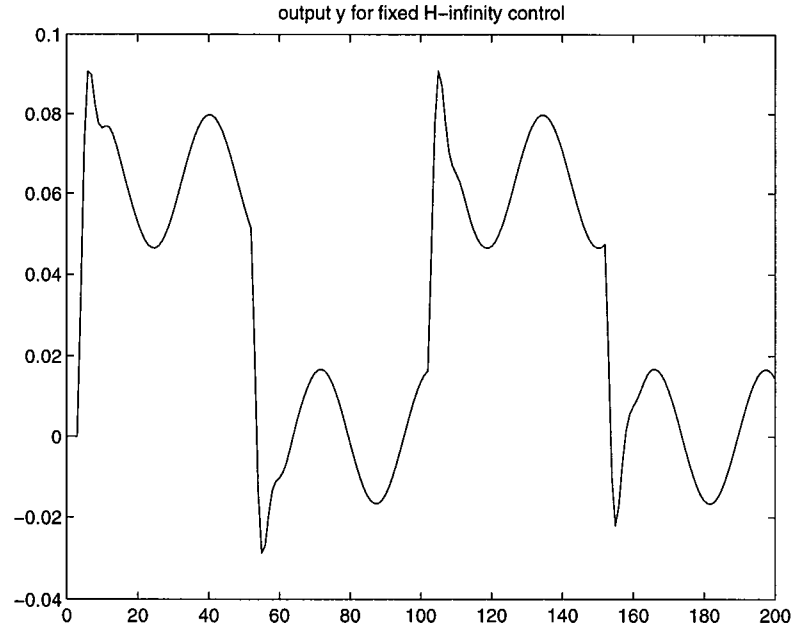


Figure 7.6: Response of example 4 using the fixed H^∞ controller. The response is very oscillatory because the fixed robust controller cannot handle the total uncertainties due to the parametric errors and the unmodeled dynamics.

uncertainties due to parametric errors and unmodeled dynamics, while the H^∞ adaptive controller can reduce the parametric uncertainties on-the-fly. We do not imply that the adaptive H^∞ controller is superior to the fixed H^∞ controller in every aspect. An experienced control designer may be able to design a fixed H^∞ controller with similar performance as the adaptive H^∞ controller using extensive analysis of the system model and detailed off-line system identification. However, it is always desirable to have a control design method which can be used without many years of experience and less requirement on the knowledge of the plant yet still can achieve a good performance. The adaptive H^∞ control algorithm proposed in this dissertation can fulfill this goal. It requires less experience to achieve a control design of moderate performance and very good robustness.

7.5 Discussion

From these examples we can observe shortcomings of the H^∞ adaptive control design. The transient responses of some cases are not very good, and the steady state gains of some cases are not close to one. There are two reasons for these problems. First, we do not require persistent excitation of the reference signals, so we cannot guarantee the parameter estimate will converge to the true nominal parameters. The residual parametric errors plus unmodeled dynamics limit the achievable performance. This phenomenon can be seen in the upper-left subplots of Figure 7.1, 7.2, 7.3, and 7.4.

Secondly, the DC gain or the low frequency gain of the asymptotic H^∞ controller is inadequate. This problem is inherent in the usage of the non-weighted mixed sensitivity formulation (see Lemma 5). Most control systems are designed to be insensitive to low-frequency disturbances, and robust to high-frequency unmodeled dynamics. From Bode's sensitivity theorem, we know that good performance cannot be achieved if the disturbances and the unmodeled dynamics span the same frequency band. We will propose an extension of the H^∞ adaptive controller which can deal with frequency-weighted unmodeled dynamics in the next chapter.

Chapter 8

Extension of the H^∞ adaptive control design

In this chapter we propose a solution to improve the performance of the H^∞ adaptive controller. The basic idea is to extend the formulation of the robust adaptive control problem to include frequency-dependent weighting functions in the coprime factor uncertainty model so the boundedness condition can be expressed as a weighted mixed sensitivity problem. Hence, a robust adaptive controller with better performance can be obtained by a suitable choice of the weighting functions. The details of this extension are discussed in the next section.

8.1 Weighted H^∞ Adaptive Robust Control

We modify the model of the plant as follows:

$$(M_0 + \Delta_M W_M)y(t) = (N_0 + \Delta_N W_N)u(t) + d_1(t) \quad (8.1)$$

where M_0 and N_0 are Hurwitz polynomials of the following form

$$\begin{aligned} N_0(z^{-1}) &= b_1 z^{-1} + b_2 z^{-2} + \cdots + b_n z^{-n} \\ M_0(z^{-1}) &= 1 + a_1 z^{-1} + \cdots + a_n z^{-n}. \end{aligned}$$

W_M and W_N are transfer functions in H^∞ . They are the weighting functions used to model the frequency dependency of the unmodeled dynamics. The same assumptions (A1-A3) as in the original problem are needed. Define

$$\begin{aligned}\theta_p &\triangleq [-a_1, \dots, -a_n, b_1, \dots, b_n]^T, \\ \phi(t-1) &\triangleq [y(t-1), \dots, y(t-n), u(t-1), \dots, u(t-n)]^T, \\ d_2(t) &\triangleq \Delta_N W_N u(t) - \Delta_M W_M y(t), \\ D_2(t) &\triangleq D_3 \max_{0 \leq \tau \leq t} \{|W_N u(\tau)|, |W_M y(\tau)|\},\end{aligned}$$

then $y(t) = \phi(t-1)^T \theta_p + d_2(t) + d_1(t)$. The differences between the new formulation and the original one are in the definitions of d_2 and D_2 and the inclusion of the weighting transfer functions W_N and W_M in the plant model (8.1).

We use the same parameter estimator (4.7-4.8) as before. Note that D_2 is still a bounding function for d_2 as in the original problem formulation so the parameter estimator will still have the desired properties (4.13-4.15) and Lemma 4 still holds.

Following a similar derivation as before, we can get a new boundedness condition:

Lemma 7 *Consider the adaptive control system with the plant described by (8.1) and the parameter estimator described by (4.7-4.8). If the control law $u(\tau) = K_t(z^{-1})[y(\tau)]$ can stabilize the fictitious system $\hat{M}_t(z^{-1})y(\tau) - \hat{N}_t(z^{-1})u(\tau) = e(\tau)$ and satisfies*

$$\left\| \begin{array}{c} W_N S^{ue} \\ W_M S^{ye} \end{array} \right\|_B < \frac{1}{\beta D_3}, \quad (8.2)$$

where β is the design parameter in the parameter estimator (4.7-4.8), then e , u , and y of the original adaptive control system will be bounded.

Proof

The proof will be done by verifying (5.1–5.3) of lemma 4. First note that (5.1) is a property of the estimation law (4.7–4.8), so it is always fulfilled as long as the estimation law (4.7–4.8) is used. Hence we only need to show that (5.2) and (5.3) are satisfied with the control law.

If a time-varying control law $u(t) = K_t[y(t)]$ is designed such that the system $\hat{M}_t y(\tau) - \hat{N}_t u(\tau) = e(\tau)$ is l^∞ -stable, i.e. maps l^∞ to l^∞ , then (5.2) is satisfied because $\phi(t)$ is a vector of delayed $u(t)$ and $y(t)$. Furthermore, if the control law is designed such that

$$\left\| \begin{array}{c} W_N S^{ue} \\ W_M S^{ye} \end{array} \right\|_B < \frac{1}{\beta D_3}, \quad (8.3)$$

then

$$D_2(t) = \max_{0 \leq \tau \leq t} \{|W_N u(\tau)|, |W_M y(\tau)|\} \leq \left\| \begin{array}{c} W_N S^{ue} \\ W_M S^{ye} \end{array} \right\|_B e(t). \quad (8.4)$$

Thus (5.3) will be satisfied, Hence boundedness of the adaptive control system is proved by lemma 4. \square

From the relation of the B norm and the shifted H^∞ norm we derived in section 6.1, we can get a msimilar algorithm for a new frozen time H^∞ robust adaptive controller corresponding to (8.2).

We now give a heuristic discussion of the selection of the weighting functions W_N and W_M . Since the controller is computed from the frozen time system, we can consider the linear time-invariant case. Suppose P and K are linear time-invariant plant and controller, respectively. It is easy to show that

$$\begin{aligned} W_N S^{ue} &= \frac{W_N K M^{-1}}{1 + PK} \\ W_M S^{ye} &= \frac{W_M M^{-1}}{1 + PK}. \end{aligned}$$

If $W_M M^{-1}$ is large at low frequencies and small at high frequencies (i.e. a low pass transfer function), the sensitivity $S = \frac{1}{1+PK}$ will be small at low frequencies, which results in better steady state performance.

On the other hand we need $W_N M^{-1}$ to be small at low frequencies and large at high frequencies (i.e. a high pass transfer function), in order to maintain good robustness with respect to unmodeled dynamics.

Therefore, the inclusion of the weighting functions W_N and W_M gives us more freedom in the control design, and proper selections of W_N and W_M leads to more useful controllers.

8.2 Re-design the Examples

In order to confirm the idea we proposed in the previous section, we re-designed some examples in Chapter 7. The simulations results are very encouraging.

We first re-designed the controller for example 7.1. We choose $W_N(z^{-1})$ as the discretized system (using bilinear transformation) of a continuous-time weighting $W_N(s) = 0.5 \frac{s+1}{0.5s+1}$ and $W_M(z^{-1})$ as the discretized system (using bilinear transformation) of a continuous time weighting $W_M(s) = \frac{0.5s+1}{s+1}$. The response of the re-designed system (shown in Figure 8.1) has much less overshoot than that of Figure 7.1. The MATLAB script for this simulation is listed in Appendix A.

We then re-designed the system of example 7.2, which has an unstable plant. $W_N(z^{-1})$ is chosen as the discretized system of $W_N(s) = 0.5 \frac{s+1}{0.5s+1}$ and $W_M(z^{-1})$ is chosen as the discretized system of $W_M(s) = \frac{0.5s+1}{s+1}$. The response of the re-designed system (shown in Figure 8.1) has significantly better improvement on the transient response over that of Figure 7.3.

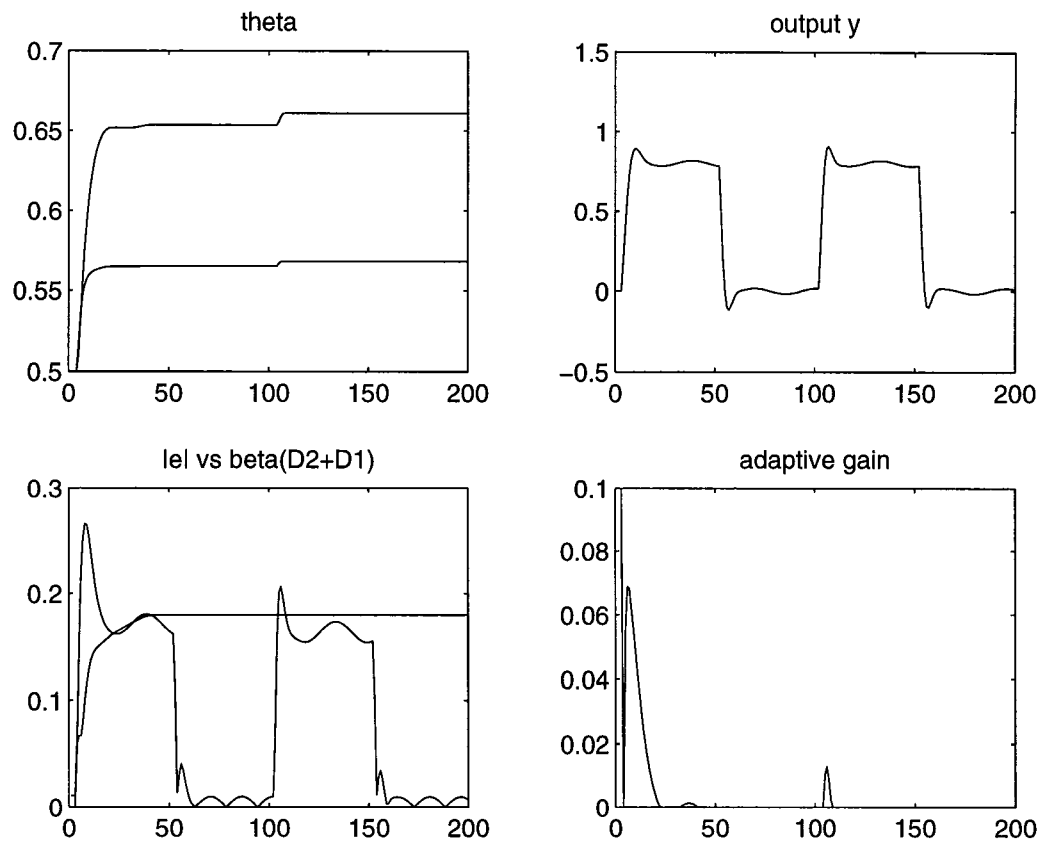


Figure 8.1: Response of the re-designed system for example 7.1 using the weighted uncertainty model. The response is significantly better than that of Figure 7.1.

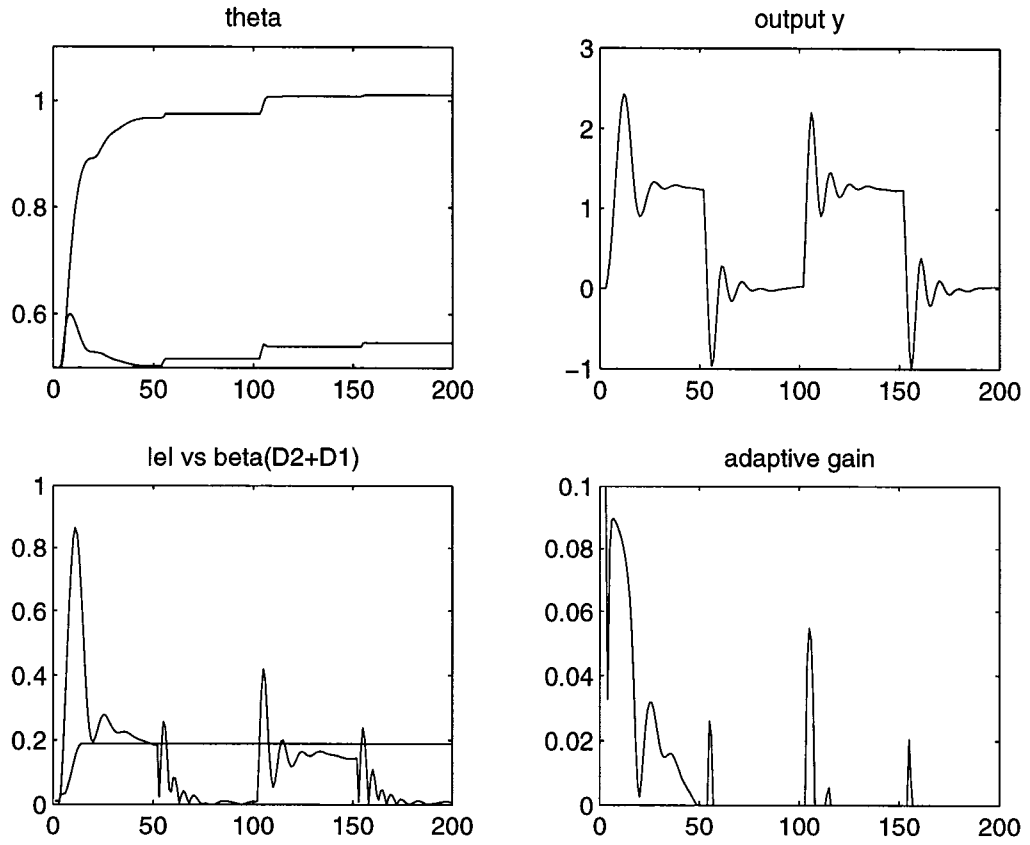


Figure 8.2: Response of the re-designed system for example 7.2 using the weighted uncertainty model. The response (shown in Figure 8.2) is significantly better than that of Figure 7.3. The plant in this example is an unstable system.

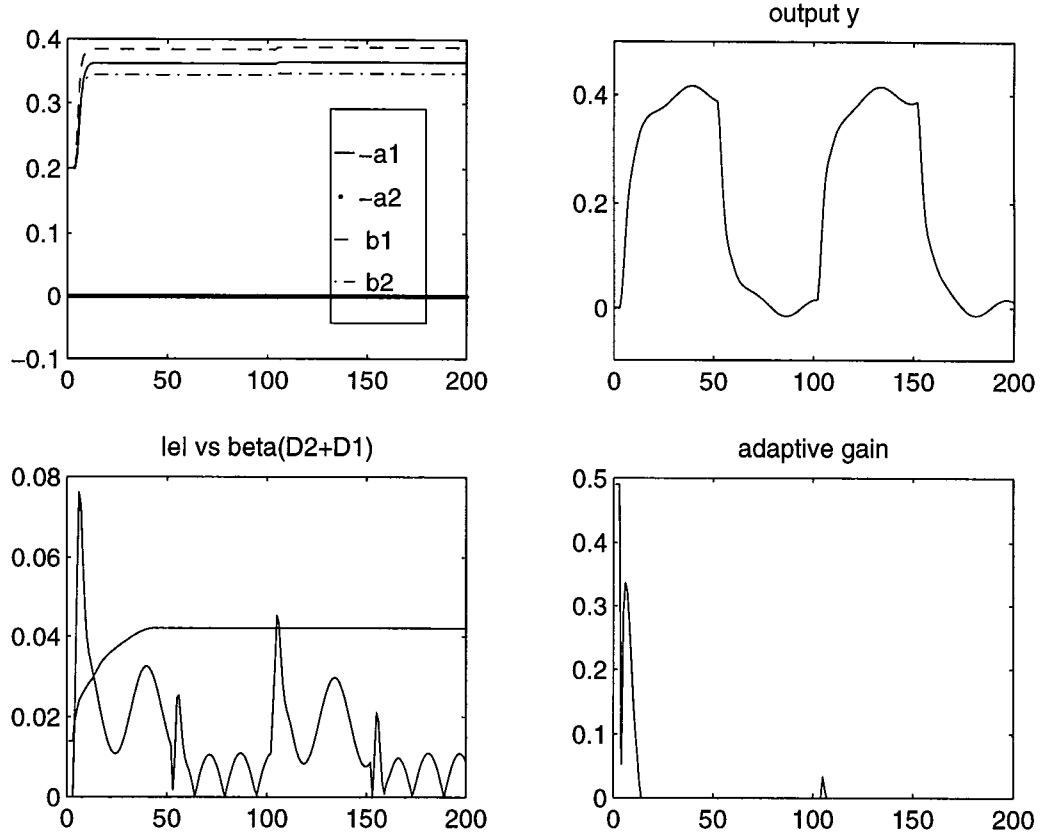


Figure 8.3: Response of the re-designed system for example 7.3 using the weighted uncertainty model. Both the transient response and the steady state gain are much better than that of Figure 7.4.

We also re-design the system of example 7.3, for which the true plant is fourth order and the nominal plant is second order. $W_N(z^{-1})$ is chosen as the discretized system of $W_N(s) = 0.5 \frac{s+1}{0.5s+1}$ and $W_M(z^{-1})$ is chosen as the discretized system of $W_M(s) = \frac{0.5s+1}{s+1}$. The response of the re-designed system is shown in Figure 8.3. Both the transient response and the steady state gain are also greatly improved over that of Figure 7.4. The steady sinusoidal variation of the response in the out $y(t)$ (with a period of about 3 secs) is caused by the sinusoidal dis-

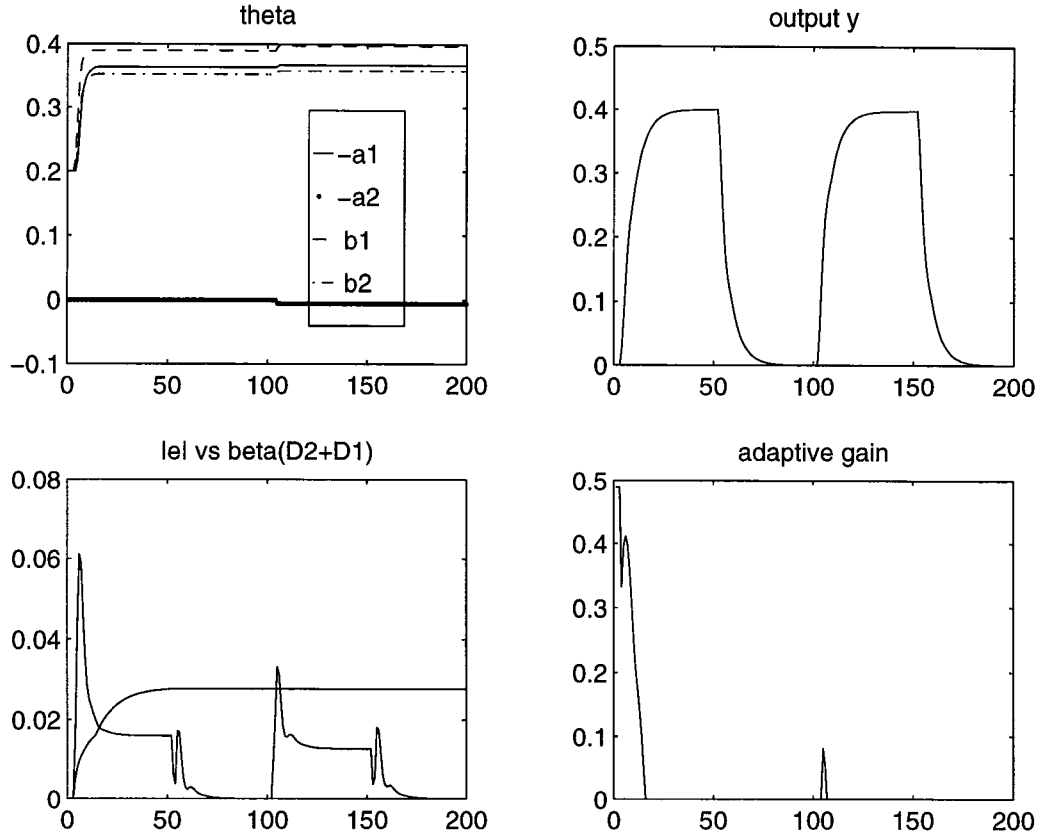


Figure 8.4: Response of the re-designed system for example 7.3 using the weighted uncertainty model with no disturbances.

turbance, not instability of the closed-loop system. This can be confirmed by a simulation of noise-free case shown in Figure 8.4.

In all of the above cases, transient responses are improved to a large extent, but the steady state loop gains are still not big enough in most cases, so the output still is not close to unity when the command input is a unit step. This does not imply that this method can not improve the steady state performance. We think the main difficulty lies in the choice of the weighting functions. Currently, we choose the weighting functions based the rules learned from linear

robust control theory. As the adaptive control system is nonlinear, these rules may not be suitable. Further study on how to select the weighting functions is an important task for future research.

Chapter 9

Conclusions

A novel robust adaptive control design scheme was developed. This scheme integrates an H^∞ robust control law and a robust parameter estimator. Stability properties of the proposed adaptive control scheme have been derived. Several numerical examples are provided to demonstrate the effectiveness of this new adaptive control scheme.

Unlike the heuristic and qualitative design procedures of previous robust adaptive control schemes, the design procedure proposed here is systematic and quantitative. Given the set of possible parameters of the nominal plant, the size of the unmodeled dynamics, and the size of the noise, this scheme can be used to design an H^∞ adaptive controller which can guarantee that every signal in the adaptive system be bounded when the external input is bounded.

A new boundedness condition in terms of the shifted H^∞ norm was derived using frozen time analysis. We developed an H^∞ adaptive robust control algorithm for systems with unweighted coprime factor uncertainty models first. This scheme was further modified to deal with weighted coprime factor uncertainties. Robustness of the proposed adaptive control schemes was then proved by the boundedness condition we derived. Examples showed that both the unweighted

and the weighted schemes have very good robustness, but the weighted scheme has the potential to achieve better performance than the unweighted scheme does.

Appendix A

MATLAB scripts

A.1 Script for example 7.1

```
% H-infinity robust adaptive control algorithm
% require control system toolbox and mu-toolbox
% D3:l-infinity gain (i.e l-1 norm) of the uncertainty
% D1:bound on the external disturbance d(t)
%(Num0d,Den0d):discrete Time nominal model
%(Num1d,Den1d):discrete time true model
%The leading coefficient of Den0d has to be 1
%The leading coefficient of Num0d has to be zero

clear

ts=0.1;    %ts is the sampling time
T=200;    %T is total step to be simulated
D1=0.01;   %D1 is the size of the bounded disturbance
D3=0.1;    %D3 is the size of the coprime factor uncertainty
P=100*[1 0;0 1];%P is the corvarience matrix
```

```

maxuy=0;    %maxuy is used for generating the normalized signal
alpha=0.1;
beta=sqrt(1/(1-alpha));
sigma=1.1;  % shifted amount
kappa1=1/sqrt(1-sigma^(-2));
gamma=1/(kappa1*(beta+0.1)*D3);

Num0d=[0 0.7];Den0d=[1 -0.8];

% Generate purterbation with the l-1 norm equal to D3
% first pick arbitrary impulse response for the numerator and
% denominator purterbation, then normalized them such the
% l-1 norm equal to D3
deln=[0 0 0.9 0.6];DeltaN=D3*deln/norm(deln,1);
deld=[0 0 0.8 -0.7];DeltaD=D3*deld/norm(deld,1);

%True system
Num1d=[Num0d 0 0]+DeltaN;
Den1d=[Den0d 0 0]+DeltaD;

variation=[1;1];
N=length(Den0d);
thnominal=[-Den0d(2:N) Num0d(2:N)]';
thmin=thnominal-variation;
thmax=thnominal+variation;

```



```

thbar=[0;0];
phi=zeros(2,T);
th=[0.5*ones(1,T); 0.5*ones(1,T)];
e=zeros(1,T);
y=zeros(1,T);
u=zeros(1,T);% control signal; initialized to be zero
v=alpha*ones(1,T); %
    % command reference signal
r=[ones(1,T/4) zeros(1,T/4) ones(1,T/4) zeros(1,T/4)];
wd=2;
d=D1*sin(0:wd*ts:wd*(T-1)*ts); % wd=10 <--> 10rad/sec if T=1000
xc=zeros(length(Num0d)-1,T);

for t=4:1:T,

    num=[0 th(2,t-1)];den=[1 -th(1,t-1)];
    [atmp,btmp,ctmp,dtmp]=tf2ss(num,[1 zeros(1,length(den)-1)]);%N
    AN=atmp';BN=ctmp';CN=btmp';DN=dtmp;
    [atmp,btmp,ctmp,dtmp]=tf2ss(den,[1 zeros(1,length(den)-1)]);%M
    AM=atmp';BM=ctmp';CM=btmp';DM=dtmp;
    L=BM;A=AM-L*CM;B=BN;C=CN;D=0;
    %generalized plant
    Ag=A*sigma;Bg=[-L B];Cg=[[zeros(1,length(den)-1)];C;C]*sigma;
    Dg=[[0;1] [1;D];1 D];
    G=pck(Ag,Bg,Cg,Dg);

```

```

[K,Tzw,gfin,ax,ay,hamx,hamy] = ...
dhfsyn(G,1,1,gamma,gamma,0.1,...
0.1,inf,-1,2,1e-10,1e-6);

[Ac,Bc,Cc,Dc]=unpck(K);

% convert the controller from z/sigma back to z
% controller derived by dhfsyn has a negative DC gain,
% so we reverse the sign of Cc and Dc to make it
% confirm to our convention
Ac=Ac/sigma; Cc=-Cc/sigma; Dc=-Dc;

% r ---->+ ----->controller --->system ----->
%      + ^-                                     |
%      |                                     |
%      |_____|
xc(:,t-1)=Ac*xc(:,t-2)+Bc*(-y(t-2)+r(t-2));
u(t-1)=Cc*xc(:,t-2)+Dc*(-y(t-2)+r(t-2));
%y(t)=-Den0d(2)*y(t-1)+Num0d(2)*u(t-1)+d(t-1);
y(t)=-Den1d(2)*y(t-1)-Den1d(3)*y(t-2)-Den1d(4)*y(t-3)+...
      Num1d(2)*u(t-1)+Num1d(3)*u(t-2)+Num1d(4)*u(t-3)+d(t);
phi(:,t-1)=[y(t-1);u(t-1)];
yh=phi(:,t-1)'*th(:,t-1);
e(t)=y(t)-yh;
m=max(abs(u(t-1)),abs(y(t-1)));
if maxuy<m,
    maxuy=m;

```

```

end;
D2(t)=D3*maxuy;
if beta*(D2(t)+D1)<abs(e(t)),
    v(t)=alpha*(abs(e(t))-beta*(D2(t)+D1))/abs(e(t));
else,
    v(t)=0;
end;
P=P-v(t)*P*phi(:,t-1)*phi(:,t-1)'+P/...
(1+phi(:,t-1)'+P*phi(:,t-1));
thbar=th(:,t-1)+v(t)*P*phi(:,t-1)*e(t)/...
(1+phi(:,t-1)'+P*phi(:,t-1));
if thbar(1)<thmin(1), thbar(1)=thmin(1); end;
if thbar(1)>thmax(1), thbar(1)=thmax(1); end;
if thbar(2)<thmin(2), thbar(2)=thmin(2); end;
if thbar(2)>thmax(2), thbar(2)=thmax(2); end;
th(:,t)=thbar;
disp([t]);
end

subplot(221);plot(th');title('theta');
subplot(222);plot(y');title('output y')
subplot(223);
plot(1:T,abs(e'),'y',1:T,beta*(D2+D1*ones(size(D2))), 'r');
title('|e| vs beta(D2+D1)')
subplot(224);plot(v');title('adaptive gain')

```

A.2 Script for example 7.3

```
%Robust H-infinity adaptive control algorithm
%
%D3:l-infinity gain (i.e l-1 norm) of the uncertainty
%D1:bound on the external disturbance d(t)
%(Num0d,Den0d):discrete Time nominal model
%(Num1d,Den1d):discrete time true model
%The leading coefficient of Den0d has to be 1
%The leading coefficient of Num0d has to be zero

clear

ts=0.1; %ts is the sampling time
T=200; %T is total step to be simulated
D1=0.01; %D1 is the size of the bounded disturbance
D3=0.1; %D3 is the size of the coprime factor uncertainty
maxuy=0; %maxuy is used for generating the normalized signal
alpha=0.1;
beta=sqrt(1/(1-alpha));
sigma=1.1; % shifted amount
kappa1=1/sqrt(1-sigma^(-2));
gamma=1/(kappa1*(beta+0.1)*D3);

Num0d=[0 0.5 0.2];Den0d=[1 -0.8 0.4];
%Generate purterbation with the l-1 norm equal to D3
%first pick arbitrary impulse response for the numerator and
```

```

%denominator purterbation, then normalized them such the
% 1-1 norm equal to D3
deln=[0 0 0 0.6 -0.4];
deld=[0 0 0 -0.8 -0.5];
P=100*eye(2*(length(Num0d)-1));%P is the corvarience matrix
l1norm_N=sum(abs(deln));l1norm_D=sum(abs(deld));
l1norm_ND=sqrt(l1norm_N^2+l1norm_D^2);
DeltaN=D3*deln/l1norm_ND;
DeltaD=D3*deld/l1norm_ND;
%True system
Num1d=[Num0d 0 0]+DeltaN;
Den1d=[Den0d 0 0]+DeltaD;

variation=[1;1;1;1];
N=length(Den0d);
thnominal=[-Den0d(2:N) Num0d(2:N)]';
thmin=thnominal-variation;
thmax=thnominal+variation;

thbar=zeros(4,1);
phi=zeros(4,T);
th=0.5*[ones(4,T)];
e=zeros(1,T);
y=zeros(1,T);
u=zeros(1,T);% control signal; initialized to be zero

```

```

v=alpha*ones(1,T); %
r=[ones(1,T/4) zeros(1,T/4) ones(1,T/4) zeros(1,T/4)];
wd=2;
d=D1*sin(0:wd*ts:wd*(T-1)*ts); % wd=10 <--> 10rad/sec if T=1000
xc=zeros(length(Num0d)-1,T);

for t=4:1:T,
    num=[0 th(3,t-1) th(4,t-1)];den=[1 -th(1,t-1) -th(2,t-1)];
    [atmp,btmp,ctmp,dtmp]=tf2ss(num,[1 zeros(1,length(den)-1)]);%N
    AN=atmp';BN=ctmp';CN=btmp';DN=dtmp;
    [atmp,btmp,ctmp,dtmp]=tf2ss(den,[1 zeros(1,length(den)-1)]);%M
    AM=atmp';BM=ctmp';CM=btmp';DM=dtmp;
    L=BM;A=AM-L*CM;B=BN;C=CN;D=0;
    % generalized plant
    Ag=A*sigma;Bg=[-L B];Cg=[[zeros(1,length(den)-1)];C;C]*sigma;
    Dg=[[0;1] [1;D];1 D];
    G=pck(Ag,Bg,Cg,Dg);
    [K,Tzw,gfin,ax,ay,hamx,hamy] = ...
dhfsyn(G,1,1,gamma,gamma,0.1,...
0.1,inf,-1,2,1e-10,1e-6);
    [Ac,Bc,Cc,Dc]=unpck(K);
    %convert the controller from z/sigma back to z
    %controller derived by dhfsyn has a negative DC gain,
    %so we reverse the sign of Cc and Dc to make
    %it confirm to our convention

```

```

Ac=Ac/sigma; Cc=-Cc/sigma; Dc=-Dc;

% r ---->+ ----->controller --->system ----->
%      + ^- |
%      | |
%      |_____|
xc(:,t-1)=Ac*xc(:,t-2)+Bc*(-y(t-2)+r(t-2));
u(t-1)=Cc*xc(:,t-2)+Dc*(-y(t-2)+r(t-2));

y(t)=-Den1d(2)*y(t-1)-Den1d(3)*y(t-2)-Den1d(4)*y(t-3)+...
      Num1d(2)*u(t-1)+Num1d(3)*u(t-2)+Num1d(4)*u(t-3)+d(t);
phi(:,t-1)=[y(t-1);y(t-2);u(t-1);u(t-2)];
yh=phi(:,t-1)'*th(:,t-1);
e(t)=y(t)-yh;
m=max(abs(u(t-1)),abs(y(t-1)));
if maxuy<m,
    maxuy=m;
end;
D2(t)=D3*maxuy;
if beta*(D2(t)+D1)<abs(e(t)),
    v(t)=alpha*(abs(e(t))-beta*(D2(t)+D1))/abs(e(t));
else,
    v(t)=0;
end;
P=P-v(t)*P*phi(:,t-1)*phi(:,t-1)'+P/...

```

```

(1+phi(:,t-1)'*P*phi(:,t-1));

    thbar=th(:,t-1)+v(t)*P*phi(:,t-1)*e(t)/...
(1+phi(:,t-1)'*P*phi(:,t-1));

    if thbar(1)<thmin(1), thbar(1)=thmin(1); end;
    if thbar(1)>thmax(1), thbar(1)=thmax(1); end;
    if thbar(2)<thmin(2), thbar(2)=thmin(2); end;
    if thbar(2)>thmax(2), thbar(2)=thmax(2); end;

    th(:,t)=thbar;

    disp([t]);

end

subplot(221);plot(th');title('theta');
subplot(222);plot(y');title('output y')
subplot(223);
plot(1:T,abs(e'),'y',1:T,beta*(D2+D1*ones(size(D2))), 'r');
title('|e| vs beta(D2+D1)')
subplot(224);plot(v');title('adaptive gain')

```

A.3 Script for Examples in Section 8.2

The procedure to construct the state space realization of the generalized plant G introduced in Section 3.2 can not be applied to the case of the weighted uncertainty model. We use Simulink to construct the generalization plant G and then use the *dlinmod* command to extract the state space model of G . It is very efficient and numerically robust.

The MATLAB m-file to re-design example 7.1 using the weighted uncertainty

model is listed first, followed by the Simulink block diagram of the generalized plant.

```
% H-infinity robust adaptive control algorithm
% assume the plant is represented by a weighted
% non-normalized coprime factorization

%D3:l-infinity gain (i.e l-1 norm) of the uncertainty
%D1:bound on the external disturbance d(t)
%(Num0d,Den0d):discrete Time nominal model
%(Num1d,Den1d):discrete time true model
%The leading coefficient of Den0d has to be 1
%The leading coefficient of Num0d has to be zero

clear

ts=0.1;      %ts is the sampling time

KWN=0.5;NumWN=[1 1];DenWN=[1/2 1];
[NumWNd,DenWNd]=c2dm(KWN*NumWN,DenWN,ts,'tustin');
KWM=1;NumWM=[1/2 1];DenWM=[1/1 1];
[NumWMd,DenWMd]=c2dm(KWM*NumWM,DenWM,ts,'tustin');

T=200;      %T is total step to be simulated
D1=0.01;    %D1 is the size of the bounded disturbance
D3=0.2;     %D3 is the size of the coprime factor uncertainty
P=100*[1 0;0 1];%P is the corvariance matrix
```

```

maxuy=0;    %maxuy is used for generating the normalized signal
alpha=0.1;
beta=sqrt(1/(1-alpha));
sigma=1.1;  % shifted amount
kappa1=1/sqrt(1-sigma^(-2));
gamma=1/(kappa1*(beta+0.1)*D3);

Num0d=[0 0.7];Den0d=[1 -0.8];

%Generate purterbation with the l-1 norm equal to D3
%first pick arbitrary impulse response for the numerator and
%denominator purterbation, then normalized them such the
% l-1 norm equal to D3
deln=[0 0 0.9 0.6];DeltaN=D3*deln/norm(deln,1);
deld=[0 0 0.8 -0.7];DeltaD=D3*deld/norm(deld,1);

%True system
Num1d=[Num0d 0 0]+DeltaN;
Den1d=[Den0d 0 0]+DeltaD;

variation=[1;1];
N=length(Den0d);
thnominal=[-Den0d(2:N) Num0d(2:N)]';
thmin=thnominal-variation;
thmax=thnominal+variation;

```

```

thbar=[0;0];
phi=zeros(2,T);
th=[0.5*ones(1,T); 0.5*ones(1,T)];
e=zeros(1,T);
y=zeros(1,T);
u=zeros(1,T);% control signal; initialized to be zero
yw=zeros(1,T);uw=zeros(1,T);
v=alpha*ones(1,T); %
% command reference signal
r=[ones(1,T/4) zeros(1,T/4) ones(1,T/4) zeros(1,T/4)];
wd=2;
d=D1*sin(0:wd*ts:wd*(T-1)*ts); % wd=10 <--> 10rad/sec if T=1000
%d=D1*(rand([1 T])-0.5*ones(1,T));
xc=zeros(4,T);

for t=4:1:T,
    num=[0 th(2,t-1)];den=[1 -th(1,t-1)];
    [atmp,btmp,ctmp,dtmp]=tf2ss(num,[1 zeros(1,length(den)-1)]);%N
    AN=atmp';BN=ctmp';CN=btmp';DN=dtmp;
    [atmp,btmp,ctmp,dtmp]=tf2ss(den,[1 zeros(1,length(den)-1)]);%M
    AM=atmp';BM=ctmp';CM=btmp';DM=dtmp;
    L=BM;A=AM-L*CM;B=BN;C=CN;D=0;
    %generalized plant
    %Ag=A*sigma;Bg=[-L B];Cg=[[zeros(1,length(den)-1)];C;C]*sigma;
    %Dg=[[0;1] [1;D];1 D];

```

```

[Ag,Bg,Cg,Dg]=dlinmod('aw1g');
G=pck(Ag*sigma,Bg,Cg*sigma,Dg);
[K,Tzw,gfin,ax,ay,hamx,hamy] = ...
dhfsyn(G,1,1,gamma,gamma,0.1,...
0.1,inf,-1,2,1e-10,1e-6);
[Ac,Bc,Cc,Dc]=unpck(K);
% convert the controller from z/sigma back to z
% controller derived by dhfsyn has a negative DC gain,
% so we reverse the sign of Cc and Dc to make it
% confirm to our convention
Ac=Ac/sigma; Cc=-Cc/sigma; Dc=-Dc;

% r ---->+ ----->controller ---->system ----->
%      + ^-                                     |
%      |                                     |
%      |_____|
xc(:,t-1)=Ac*xc(:,t-2)+Bc*(-y(t-2)+r(t-2));
u(t-1)=Cc*xc(:,t-2)+Dc*(-y(t-2)+r(t-2));
%y(t)=-Den0d(2)*y(t-1)+Num0d(2)*u(t-1)+d(t-1);
y(t)=-Den1d(2)*y(t-1)-Den1d(3)*y(t-2)-Den1d(4)*y(t-3)+...
      Num1d(2)*u(t-1)+Num1d(3)*u(t-2)+Num1d(4)*u(t-3)+d(t);
phi(:,t-1)=[y(t-1);u(t-1)];
yh=phi(:,t-1)'+th(:,t-1);
e(t)=y(t)-yh;
uw(t-1)=-DenWNd(2)*uw(t-2)+(NumWNd(1)*u(t-1)+NumWNd(2)*u(t-2));

```

```

yw(t-1)=-DenWMd(2)*yw(t-2)+(NumWMd(1)*y(t-1)+NumWMd(2)*y(t-2));
m=max(abs(uw(t-1)),abs(yw(t-1)));
if maxuy<m,
    maxuy=m;
end;
D2(t)=D3*maxuy;
%disp([abs(beta*(D2(t)+D1)) abs(e(t))])
if beta*(D2(t)+D1)<abs(e(t)),
    v(t)=alpha*(abs(e(t))-beta*(D2(t)+D1))/abs(e(t));
else,
    v(t)=0;
end;
P=P-v(t)*P*phi(:,t-1)*phi(:,t-1)'/...
    (1+phi(:,t-1)'*P*phi(:,t-1));
thbar=th(:,t-1)+v(t)*P*phi(:,t-1)*e(t)/...
(1+phi(:,t-1)'*P*phi(:,t-1));
if thbar(1)<thmin(1), thbar(1)=thmin(1); end;
if thbar(1)>thmax(1), thbar(1)=thmax(1); end;
if thbar(2)<thmin(2), thbar(2)=thmin(2); end;
if thbar(2)>thmax(2), thbar(2)=thmax(2); end;
th(:,t)=thbar;
%disp([t th(:,t)' u(t-1) y(t-1) yh]);
disp([t]);
end

```

```

subplot(221);plot(th');title('theta');
subplot(222);plot(y');title('output y')
subplot(223);
plot(1:T,abs(e'),'y',1:T,beta*(D2+D1*ones(size(D2))), 'r');
title('|e| vs beta(D2+D1)')
subplot(224);plot(v');title('adaptive gain')

```

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