TECHNICAL RESEARCH REPORT

Institute for Systems Research

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The Institute for Systems Research is supported by the National Science Foundation Engineering Research Center Program (NSFD CD 8803012), Industry and the University

TR 93-11

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February 3, 1993

Abstract

We consider a single-queue system with multiple servers that are non-identical. Our interest is in applying the technique of perturbation analysis to estimate derivatives of mean steady-state system time. Because infinitesimal perturbation analysis yields biased estimates for this problem, we apply smoothed perturbation analysis to get unbiased estimators. In the most general cases, the estimators require additional simulation, so we propose an approximation to eliminate this. For two servers, we give an analytical proof of unbiasedness in steady state for the Markovian case. We provide simulation results for both Markovian and non-Markovian examples, and compare the performance with regenerative likelihood ratio estimators.

Keywords: Sensitivity analysis, gradient estimation, queues.

1 Introduction

Estimation of gradients of performance measures with respect to parameters of underlying distributions in a single simulation has recently become an important topic of research. Two major applications are sensitivity analysis and stochastic optimization (Fu 1994). Overviews of the state-of-the-art in gradient estimation can be found in Ho and Cao (1991) and L'Ecuyer (1991). The easiest technique to apply and the one generally possessing the lowest variance is infinitesimal perturbation analysis (IPA); see, e.g., Ho and Cao (1991) and Glasserman (1991). Unfortunately, IPA does not work universally (see also Cao 1985, Cao 1988, and Heidelberger et al. 1989). One example is the queueing system we consider in this paper: a single-queue (unlimited capacity), two-server system, where the servers are not assumed to be identical. Taking the performance measure of interest to be mean steady-state system time and considering a parameter in the underlying service time distributions, it can be easily shown that IPA in general gives a biased estimate. In fact, Fu and Hu (1990) prove that for the Markovian case, the bias is proportional to the square of the difference in the mean service times. Experimental results for non-exponential cases found in Fu, Hu, and Nagi (1992) seem to indicate that this proportionality holds for non-exponential cases, as well.

In this paper, we apply the technique of smoothed perturbation analysis (SPA), introduced by Gong and Ho (1987), to develop estimators for single-queue, multi-server queueing systems with non-identical servers. In particular, we apply the results of Fu and Hu (1992), where a general framework for applying SPA is established. As is noted there, the resulting estimator is not always easily implementable in practice without incurring often prohibitively large amounts of additional simulation. Thus, in the spirit of Fu and Hu (1993), where a more difficult problem of second derivative estimation is tackled for the identical server system, we propose an approximation which requires no additional simulation. Simulation experiments are used to test this approximation. For the Markovian case, the approximation is exact, and we give an analytical proof of unbiasedness in steady state. We then investigate the performance of the estimators for various interarrival time and service time distributions. An alternative technique of gradient estimation is the likelihood ratio (LR) or score function (SF) approach, which is also described in Ho and Cao (1991) and L'Ecuyer (1991). We derive regenerative likelihood ratio gradient estimators and via numerous simulation experiments compare the performance of the SPA estimators to these estimators, as well as to finite difference estimates.

The rest of the paper is organized as follows. In Section 2, we present the SPA estimators. In Section 3, we give a proof of unbiasedness in steady state for the Markovian case. In Section 4, we investigate the performance of the estimators by conducting numerous simulation experiments and comparing with finite difference estimates and likelihood ratio estimators. We present some conclusions in Section 5.

2 The Estimator

We consider a single-queue, first-come, first-served (FCFS) queueing system with m non-identical servers, a general renewal arrival process and general independent service time distributions. Steady-state system time, denoted by T, is our performance measure of interest, and we wish to estimate $dET/d\theta$, where θ is a parameter of the service time distribution. Consider an arbitrary busy period of the system, and let $X_i(\theta)$ be the service time of the ith (to arrive) customer in the busy period and A_i be the interarrival time between the (i-1)th and ith customers in the busy period. We will take time 0 to be the arrival of the first customer. Let $g(\cdot)$ and $G(\cdot)$ be the respective p.d.f. and c.d.f. of the interarrival times, and $f_j(\cdot)$ and $F_j(\cdot)$ be the respective p.d.f. and c.d.f. of the service times at server j, $j = 1, \ldots, m$.

We first present the IPA estimator, which will be one part of the full SPA estimator. To describe the IPA estimator, we introduce the important concept of a server's **local** busy period: *the interval between two adjacent idle times of the (same) server.* Using this idea of local busy periods, and defining the set of customers preceding i in the same local busy period $L(i) = \{j < i : j \text{ in the same local busy period as } i\}$, we have (see Fu and Hu 1991, for details)

$$\frac{dT_i}{d\theta} = \sum_{j \in L(i)} \frac{dX_j}{d\theta} + \frac{dX_i}{d\theta}, \tag{1}$$

where T_i is the system time of the ith customer (in the busy period), and $dX/d\theta$ is the derivative of the service time random variable given by (cf. Suri and Zazanis 1988)

$$\frac{dX}{d\theta} = \frac{-\partial F/\partial\theta}{\partial F/\partial x}, \tag{2}$$

where $F(x;\theta)$ is differentiable with respect to both x and θ. Note that the subscript on F indicating the server has been omitted here for notational convenience. Equation (1) can also be written in recursive form:

$$\frac{dT_i}{d\theta} = \begin{cases} dX_i/d\theta & \text{if } i \text{ initiates local busy period,} \\ dT_{\hat{i}}/d\theta + dX_i/d\theta & \text{otherwise.} \end{cases} \tag{3}$$

where $\hat{i} = \max_{j \in L(i)} j$, i.e., customer $\hat{i}$ is the index of the customer preceding i in the local busy period. Intuitively, the change in system time of customer i is the sum of the change in system time of the customer just preceding him in the same local busy period (if any such customer exists) plus the change in customer i's own service time.

The intuitive reason why IPA does not work is that small perturbations in the parameter could cause a switching of the assignment of servers from which customers receive service. Of course, in the identical server case, such a switching has no effect on the performance measure of interest, since the service times would be i.i.d. Even in the case of non-identical servers, there are many situations in which there is a symmetry that gives a cancelling out of the expected effect of the switching (e.g., exactly two customers switching between their respective two servers). However, there are two cases where this cancelling out does not hold. In the terminology of Fu and Hu (1993), there are two types of critical adjacent event pairs, which we describe as follows:

(1) $m+1$ in system, departure at server j, followed by departure at server $\hat{j} \neq j$;

(2) $< m$ in system, departure at server j, followed by arrival.

We now give intuitive explanations as to why (1) and (2) are critical. In (1), there is a single customer in queue (since there are $m+1$ in the system). In the sequence of events described, the customer would go to server j, whereas were the events to change order, the customer would go to server $\hat{j}$, causing a finite change in that customer's system time. On the other hand, in (2), there is at least one server idle. In the sequence of events described, the arriving customer may be served by server j (depending on the serving policy for the assignment of idle servers), whereas were the events to change order, the customer would have no chance of going to server j, since it would be busy. Thus, a finite change in the customer's system time occurs with the probability that the customer be served by server j in the original sequence.

More formally, we define

$$\begin{aligned}
D_i &= \text{the departure time of customer } i, \\
S(i) &= \text{the server of customer } i, \\
N(t) &= \text{the number of customers in the system at time } t, \\
\sigma(t) &= \text{next event after time } t \in \{\alpha, \beta_j, j = 1, \ldots, m\}, \\
\text{where } \alpha &= \text{arrival event}, \\
\beta_j &= \text{departure event at server } j.
\end{aligned}$$

We then have the following sets:

$$\begin{aligned}
L_1^{(j)} &= \{i : S(i) = j, N(D_i^+) = m, \sigma(D_i) = \beta_{\hat{j}}, \hat{j} \neq j\}, \\
L_2^{(j)} &= \{i : S(i) = j, N(D_i^+) < m - 1, \sigma(D_i) = \alpha\}.
\end{aligned}$$

In words, the first set is the set of indices of customers who depart server j leaving m in the system and such that the next event is a departure at another server, whereas the second set is the set of indices of customers who depart server j leaving less than $m-1$ in the system and such that the next event is an arrival.

We now specialize to the two-server case for expositional and notational ease. Let p be the probability that a customer arriving to an empty system goes to server 1, and $q = 1 - p$ be the probability it goes to server 2. We let θ_1 be a parameter in $F_1(\cdot)$ and let θ_2 be a parameter in $F_2(\cdot)$. Then our estimators are given by

$$\left(\frac{dT}{d\theta_1}\right)_{SPA,n} = \frac{1}{n}\left(\sum_{i=1}^{n}\frac{dT_i}{d\theta_1} + p\sum_{i\in L_2^{(1)}}\frac{dT_i}{d\theta_1}\frac{g(z_i)}{1-G(z_i)}\delta T_{i,1} + \sum_{i\in L_1^{(1)}}\frac{dT_i}{d\theta_1}\frac{f_2(x_i)}{F_2(x_i+y_i)-F_2(x_i)}\delta T_{i,1}\right), \quad (4)$$

$$\left(\frac{dT}{d\theta_2}\right)_{SPA,n} = \frac{1}{n}\left(\sum_{i=1}^{n}\frac{dT_i}{d\theta_2} + q\sum_{i\in L_2^{(2)}}\frac{dT_i}{d\theta_2}\frac{g(z_i)}{1-G(z_i)}\delta T_{i,2} + \sum_{i\in L_1^{(2)}}\frac{dT_i}{d\theta_2}\frac{f_1(x_i)}{F_1(x_i+y_i)-F_1(x_i)}\delta T_{i,2}\right), \quad (5)$$

where
$$\begin{aligned}
\xi_0 &= \text{age of interarrival time},\\
\xi_1 &= \text{age of service time at server 1},\\
\xi_2 &= \text{age of service time at server 2},\\
x_i &= \text{age of the service time at other server at } D_i,\\
y_i &= \text{minimum of the residual service time and the residual interarrival time},\\
z_i &= \text{age of the interarrival time at } D_i,\\
L_1^{(j)} &= \{i : S(i) = j, N(D_i^+) = 2, \sigma(D_i) = \beta_{\hat{j}}, \hat{j} \neq j\}, j = 1,2,\\
L_2^{(j)} &= \{i : S(i) = j, N(D_i^+) = 0\}, j = 1,2.
\end{aligned}$$

Note that $L_2^{(j)}$ is notationally simplified for the two-server case, because satisfying the original set of conditions means the departure leaves the system empty; hence, the next event must be an arrival. Intuitively, the summands are a product of two terms: the $\delta T_{i,j}$ terms representing the effect due to a critical adjacent event order change and the other two terms representing the rate at which the change occurs. To precisely define these quantities, we define the augmented state by taking the physical states $0, (0,1), (1,0), 2, 3, \ldots$ and adding to it the appropriate ages of the random variables. We then condition on the state at D_i as follows:

$$\delta T_{i,1} = \left(\sum_{k=i}^{n} T_k \Bigg| [(0,1) : \xi_0 = z_i, \xi_2 = 0] - \sum_{k=i}^{n} T_k \Bigg| [(1,0) : \xi_0 = z_i, \xi_1 = 0]\right),$$

$$\delta T_{i,2} = \left(\sum_{k=i}^{n} T_k \Bigg| [(1,0) : \xi_0 = z_i, \xi_1 = 0] - \sum_{k=i}^{n} T_k \Bigg| [(0,1) : \xi_0 = z_i, \xi_2 = 0]\right).$$

Thus, the "initial" states here depend on the age of the interarrival time. For more than two servers, there would be a dependence on the ages of service times at other servers, as well.

For these estimators, we have

Theorem 1. The SPA estimators $(dT/d\theta_i)_{SPA,n}, i = 1,2$, are unbiased for all n.

Proof. Under some mild conditions, the result follows from Theorem 3 in Fu and Hu (1993). □

We are actually interested in steady-state performance. For large n, we can simplify the expressions for $\delta T_{i,j}$ to the following:

$$\begin{aligned} \delta T_{i,j} &= S_{i,j} - M_{i,j}E[T], j = 1,2, \\ S_{i,j} &= E\left[\sum_{I(i,j)} T_k\right], \\ M_{i,j} &= E\left[|I(i,j)|\right], \end{aligned}$$

where $I(i,1)$ denotes a sample path starting from state $[(0,1) : \xi_0 = z_i, \xi_2 = 0]$ and ending the first time it hits state $[(1,0) : \xi_0 = z_i, \xi_1 = 0]$, and similarly $I(i,2)$ denotes a sample path starting from state $[(0,1) : \xi_0 = z_i, \xi_2 = 0]$ and ending the first time it hits state $[(1,0) : \xi_0 = z_i, \xi_1 = 0]$. The $|I(i,j)|$ notation indicates the number of service completions in $I(i,j)$ (in some sense its "length"). However, it is clear that this "simplification" actually leads to intractability, since the probability of hitting such a single state from an uncountable state space is 0 in the general case.

On the other hand, for the Markovian case, the state is completely defined by the physical state (without the age) since the distributions are all memoryless. As long as the queue is stable (hence ergodic), the probability of returning to a given state is 1. Thus, we actually have independence from i. Defining

$$\begin{aligned} S_1 &= \text{expected sum of system times from state (0,1) to (1,0)}, \\ S_2 &= \text{expected sum of system times from state (1,0) to (0,1)}, \\ M_1 &= \text{expected number of service completions from state (0,1) to (1,0)}, \\ M_2 &= \text{expected number of service completions from state (1,0) to (0,1)}, \\ \Delta_j &= S_j - M_j E[T],\ j = 1,2, \end{aligned}$$

we have $E[\delta T_{i,j}] = \Delta_j$, $j = 1,2$. In addition, we also have $S_1 = -S_2$, $M_1 = -M_2$, and hence $\Delta_1 = -\Delta_2$. We can thus get estimates from the sample path itself whenever the appropriate physical states are encountered, regardless of the values of the ages of the other random variables. Thus, instead of doing a separate estimate *each* time a critical adjacent event is encountered, we simply estimate two terms simultaneously, i.e., our estimator is as follows:

$$\left(\frac{dT}{d\theta_1}\right)_{SPA} = \frac{1}{n}\left(\sum_{i=1}^{n}\frac{dT_i}{d\theta_1} + \left[p\sum_{i\in L_2^{(1)}}\frac{dT_i}{d\theta_1}\frac{g(z_i)}{1-G(z_i)} + \sum_{i\in L_1^{(1)}}\frac{dT_i}{d\theta_1}\frac{f_2(x_i)}{F_2(x_i+y_i)-F_2(x_i)}\right]\hat{\Delta}_1\right) \quad (6)$$

$$\left(\frac{dT}{d\theta_2}\right)_{SPA} = \frac{1}{n}\left(\sum_{i=1}^{n}\frac{dT_i}{d\theta_2} + \left[q\sum_{i\in L_2^{(2)}}\frac{dT_i}{d\theta_2}\frac{g(z_i)}{1-G(z_i)} + \sum_{i\in L_1^{(2)}}\frac{dT_i}{d\theta_2}\frac{f_1(x_i)}{F_1(x_i+y_i)-F_1(x_i)}\right]\hat{\Delta}_2\right) \quad (7)$$

where $\widehat{\Delta}_j, j = 1,2$ are estimates of the quantities of $\Delta_j, j = 1,2$, which can be estimated on the given sample path without the need for additional simulation. This is the general idea in Fu and Hu (1991) for any Markov chain.

The approximation is to use (6) and (7) for more general distributions, instead of estimating separate $\delta T_{i,j}$ for each value of age z_i encountered, as in (4) and (5). Thus, over the entire sample path, we are in some sense "averaging" over the possible initial state vectors. The approximation eliminates the need for additional simulation. We first prove that these estimators are unbiased in steady state for the Markovian case, and then check empirically the bias of the estimators for the general case.

3 Unbiasedness for the Markovian Case

We have

Theorem 2. In steady state, the SPA estimators $(dT/d\theta_i)_{SPA}, i = 1,2$, are unbiased when the arrival process is Poisson and service times are exponential.

Proof. We calculate analytically the expected value of the SPA estimator in steady state, and show that it equals the derivative of mean steady-state system time. We do the proof for θ_1, with a symmetrical proof following for θ_2.

Let λ be the rate of the Poisson arrival process, and $\mu_1 = 1/\theta_1$ and $\mu_2 = 1/\theta_2$ be the service rates at the exponential servers 1 and 2, respectively. Recall that if both servers are available, then the first server is chosen with probability p and the second server with probability $q = 1 - p$.

We introduce the following notation:

$$\begin{aligned}
\widehat{\mu} &= \mu_1 + \mu_2 \\
\rho &= \lambda/\widehat{\mu} \\
\mu^* &= \frac{\mu_1\mu_2}{\mu_1+\mu_2}\left(\frac{\mu_1+\mu_2+2\lambda}{p\mu_2+q\mu_1+\lambda}\right)
\end{aligned}$$

Since the system can be represented as a continuous-time Markov chain, the stationary state probabilities $p_s,\ s \in \{0,(1,0),(0,1),2,...\}$, can be easily found to be

$$p_0 = \frac{1-\rho}{1-\rho+\lambda/\mu^*} \tag{8}$$

$$p_{1,0} = \frac{\lambda(p\mu^*+\rho\mu_2)}{\mu^*(\mu_1+\lambda)}p_0 \tag{9}$$

$$p_{0,1} = \frac{\lambda(q\mu^*+\rho\mu_1)}{\mu^*(\mu_2+\lambda)}p_0 \tag{10}$$

$$p_n = \frac{\lambda}{\mu^*}\rho^{n-1}p_0; n = 2,3,\ldots \tag{11}$$

$$(12)$$

Hence, the expected number in the system is given by

$$E[N] = \frac{\lambda/\mu^*}{(1-\rho)(1-\rho+\lambda/\mu^*)}, \tag{13}$$

and applying Little's Law, the expected system time is

$$E[T] = \frac{1/\mu^*}{(1-\rho)(1-\rho+\lambda/\mu^*)} \tag{14}$$

Differentiating (14), we get

$$\frac{dE[T]}{d\theta_1} = \frac{(p+\lambda\theta_2)(\theta_1+\theta_2)}{(1-\rho)(1-\rho+\lambda/\mu^*)(\theta_1+\theta_2+2\lambda\theta_1\theta_2)} - \frac{(1+2\lambda\theta_2)}{(1-\rho)(1-\rho+\lambda/\mu^*)(\theta_1+\theta_2+2\lambda\theta_1\theta_2)\mu^*} +$$

$$\frac{p\theta_1+\lambda\theta_1\theta_2+\theta_2 q}{(1-\rho)(1-\rho+\lambda/\mu^*)(\theta_1+\theta_2+2\lambda\theta_1\theta_2)} + \frac{\lambda{\theta_2}^2}{(\theta_1+\theta_2)^2(1-\rho)^2(1-\rho+\lambda/\mu^*)\mu^*} -$$

$$\frac{\left(-\frac{\lambda{\theta_2}^2}{(\theta_1+\theta_2)^2} + \frac{\lambda(p+\lambda\theta_2)(\theta_1+\theta_2)}{\theta_1+\theta_2+2\lambda\theta_1\theta_2} - \frac{\lambda(1+2\lambda\theta_2)}{(\theta_1+\theta_2+2\lambda\theta_1\theta_2)\mu^*} + \frac{\lambda(p\theta_1+\lambda\theta_1\theta_2+\theta_2 q)}{\theta_1+\theta_2+2\lambda\theta_1\theta_2}\right)}{(1-\rho)(1-\rho+\lambda/\mu^*)^2\mu^*} \tag{15}$$

We proceed to check that in steady state, the expectation of the SPA estimator given by (6) gives the same expression.

We consider a customer with index i^* that arrives in steady state, having system time T. We denote this customer by C_{i^*}. We have

$$\begin{aligned} E\left(\frac{dT}{d\theta_1}\right)_{SPA} &= E\left(\frac{dT}{d\theta_1}\right) + \left\{pP\{i \in L_2^{(1)}\}E\left[\frac{dT_i}{d\theta_1}\middle| i \in L_2^{(1)}\right]E\left[\frac{g(z_i)}{1-G(z_i)}\right]\right. \\ &+ \left. qP\{i \in L_1^{(1)}\}E\left[\frac{dT_i}{d\theta_1}\middle| i \in L_1^{(1)}\right]E\left[\frac{f_2(x_i)}{F_2(x_i+y_i)-F_2(x_i)}\right]\right\}\Delta_1. \end{aligned} \tag{16}$$

We first compute the expectation of the IPA part of (16), $dT/d\theta_1$, given by Equation (1). For exponential services with means θ_1 and θ_2 at the two servers, respectively, we have

$$\frac{dX_j}{d\theta_1} = \begin{cases} X_j/\theta & \text{if } S(j)=1, \\ 0 & \text{if } S(j)=2. \end{cases} \tag{17}$$

Thus, for C_{i^*}, (1) becomes

$$\frac{dT}{d\theta_1} = \frac{1}{\theta_1}\begin{cases} \sum_{j\in L(i^*)} X_j + X_{i^*} & \text{if } S(i^*)=1, \\ 0 & \text{if } S(i^*)=2. \end{cases} \tag{18}$$

But $\sum_{j\in L(i^*)} X_j + X_{i^*}$ is simply the time from the beginning of the local busy period to the departure of C_{i^*}, so referring to Figure 1, we can rewrite (18) as

$$\frac{dT}{d\theta_1} = \frac{1}{\theta_1}\begin{cases} \phi^{(1)} + T & \text{if } S(i^*)=1, \\ 0 & \text{if } S(i^*)=2, \end{cases} \tag{19}$$

where $\phi^{(1)}$ is the length of the local busy period at server 1 at the arrival time of C_{i^*}. Hence,

$$E\left[\frac{dT}{d\theta_1}\right] = \frac{1}{\theta_1}\left(\phi^{(1)} + E[T|S(i^*) = 1]\right) P\{S(i^*) = 1\} \tag{20}$$

We calculate the various quantities in (20) by conditioning on the state found upon the arrival of C_{i^*}. In Fu and Hu (1991), reversibility was used to do similar calculations, but for the case of unequal servers, we can no longer use this technique, due to the unaggregated states (0,1) and (1,0). We instead use the embedded Markov chain and conditioning.

In the embedded Markov chain, "time" is indexed by the occurrence of arrival and departure events. Let n^* be the discrete-time index of the event just prior to the arrival of C_{i^*}, so that C_{i^*}'s arrival is event $n^* + 1$, and let Z_n be the state of the embedded Markov chain at time n. We will be conditioning on Z_{n^*}.

We calculate $\phi^{(1)}$ by first calculating the quantity $\chi^{(1)}$, defined as the length of the local busy period at server at the event just prior to the arrival time of C_{i^*} (see Figure 1). Let $\chi_s = (\chi^{(1)}|Z_{n^*} = s)$, i.e., χ_s is $\chi^{(1)}$ under the condition that the state found upon the arrival of C_{i^*} is s, Similarly, let $\phi_s = (\phi^{(1)}|Z_{n^*} = s)$, i.e., ϕ_s is $\phi^{(1)}$ under the condition that the state found upon the arrival of C_{i^*} is s.

First, by definition, we have

$$\chi_0 = \chi_{0,1} = \phi_0 = \phi_{0,1} = 0. \tag{21}$$

We also have the following relationships between χ_s and ϕ_s for the other states:

$$\phi_{1,0} = \chi_{1,0} + \frac{1}{\mu_1 + \lambda}, \tag{22}$$

$$\phi_n = \chi_n + \frac{1}{\widehat{\mu} + \lambda}, \quad n \geq 2. \tag{23}$$

We now derive expressions for the remaining χ_s by conditioning on the state at n^*-1 and obtaining a set of linear recursive equations similar to Poisson's equation. The desired conditional probabilities are shown in Figure 2, computed simply by Bayes' Rule:

$$P\{Z_{n-1} = t|Z_n = s\} = \frac{P\{Z_n = s|Z_{n-1} = t\}p_t}{p_s}.$$

Thus, we have

$$\chi_2 = \frac{\lambda}{\lambda + \widehat{\mu}}\left(\chi_3 + \frac{1}{\lambda + \widehat{\mu}}\right) + \frac{\lambda}{\lambda + \widehat{\mu}}\left(\frac{p_{1,0}}{p_2}\right)\left(\chi_{1,0} + \frac{1}{\lambda + \mu_1}\right) \tag{24}$$

$$\chi_{1,0} = \frac{\mu_2}{\lambda + \mu_1}\left(\frac{p_2}{p_{1,0}}\right)\left(\chi_2 + \frac{1}{\lambda + \widehat{\mu}}\right), \tag{25}$$

$$\chi_n = \chi_{n-1} + \frac{1}{\widehat{\mu} - \lambda}, \quad n \geq 3. \tag{26}$$

Solving (24), (25), and (26), we obtain

$$\chi_2 = \frac{\lambda}{\mu_1(\lambda+\widehat{\mu})}\left(\frac{\widehat{\mu}+\mu_1}{\widehat{\mu}-\lambda}+\frac{p_{1,0}}{p_2}\right), \tag{27}$$

$$\chi_{1,0} = \frac{\mu_2\widehat{\mu}}{\mu_1(\lambda+\widehat{\mu})(\widehat{\mu}-\lambda)}\left(\frac{p_2}{p_{1,0}}\right)+\frac{\lambda\mu_2}{\mu_1(\lambda+\widehat{\mu})(\lambda+\mu_1)}, \tag{28}$$

$$\chi_n = \chi_2+\frac{n-2}{\widehat{\mu}-\lambda},\ n\geq 3. \tag{29}$$

We now consider the second term in (20), $E[T|S(i^*)=1]$, again by conditioning. Defining $\tau_s = E[T|S(i^*)=1, Z_{n^*}=s]$, we have

$$\tau_0 = \theta_1, \tag{30}$$

$$\tau_{1,0} = 0, \tag{31}$$

$$\tau_{0,1} = \theta_1, \tag{32}$$

$$\tau_n = \frac{n-1}{\widehat{\mu}}+\theta_1,\ n\geq 2. \tag{33}$$

Lastly, we must also express the probability term in (20), $P\{S(i^*)=1\}$, conditioned on $Z_{n^*}=s$. Defining $\mathcal{P}_s = P\{S(i^*)=1|Z_{n^*}=s\}$, we have

$$\mathcal{P} = p, \tag{34}$$

$$\mathcal{P}_{1,0} = 0, \tag{35}$$

$$\mathcal{P}_{0,1} = 1, \tag{36}$$

$$\mathcal{P}_n = \frac{\mu_1}{\mu_1+\mu_2},\ n\geq 2. \tag{37}$$

Finally, unconditioning the terms in (20) by summing over all the states, we get

$$\begin{aligned} E\left(\frac{dT}{d\theta_1}\right) &= \frac{1}{\theta_1}\sum_{s\in\{0,(1,0),(0,1),2,\ldots\}} p_s\left(\phi_s+\tau_s\right)\mathcal{P}_s \\ &= p_0\,p+p_{0,1}+\frac{\mu_1}{\mu_1+\mu_2}\sum_{n=2}^{\infty}p_n\frac{1}{\theta_1}\left(\phi_2+\frac{n-2}{\widehat{\mu}-\lambda}+\frac{n-1}{\widehat{\mu}}+\theta_1\right) \\ &= p_0\,p+p_{0,1}+\frac{\mu_1}{\mu_1+\mu_2}\left(\frac{p_0\lambda}{\mu^*}\right)\left(\frac{\rho}{1-\rho}\right)\left(1+\phi_2\mu_1+\left(\frac{\mu_1\rho}{\widehat{\mu}-\lambda}+\frac{\mu_1}{\widehat{\mu}}\right)\frac{1}{1-\rho}\right) \end{aligned} \tag{38}$$

Now we proceed to the other terms in (16). By definition of the sets of $L_k^{(1)}$, $k=1,2$, and ϕ_s, we have

$$E\left[\frac{dT_i}{d\theta_1}\middle| i\in L_2^{(1)}\right] = \frac{\phi_{1,0}}{\theta_1}, \tag{39}$$

$$E\left[\frac{dT_i}{d\theta_1}\middle| i\in L_1^{(1)}\right] = \frac{\phi_3}{\theta_1}, \tag{40}$$

(41)

Of course, the probability terms in (16) also follow from the definition of the sets of $L_k^{(1)}$, $k = 1, 2$. For $L_1^{(1)}$, the condition requires a visit to state 3, followed by a departure at server 1 and then a departure at server 2. For $L_2^{(1)}$, the condition requires a visit to state $(1,0)$, followed by a departure at server 1 and then an arrival. Thus, we have

$$P\left\{i \in L_2^{(1)}\right\} = \left(p_{1,0}\frac{\mu_1+\lambda}{\lambda}\right)\left(\frac{\mu_1}{\mu_1+\lambda}\right)(1) = \frac{\mu_1(1-\rho)(p\mu^* + \rho\mu_2)}{(\mu_1+\lambda)(\mu^*(1-\rho)+\lambda)}, \tag{42}$$

$$P\left\{i \in L_1^{(1)}\right\} = \left(p_3\frac{\hat{\mu}+\lambda}{\lambda}\right)\left(\frac{\mu_1}{\hat{\mu}+\lambda}\right)\left(\frac{\mu_1}{\hat{\mu}+\lambda}\right) = \frac{\mu_1\mu_2(1-\rho)\rho^2}{(\hat{\mu}+\lambda)(\mu^*(1-\rho)+\lambda)}. \tag{43}$$

Since interarrival times and service times are exponential, we have

$$\frac{g(z_i)}{1-G(z_i)} = \lambda, \tag{44}$$

$$E\left[\frac{f_2(x_i)}{F_2(x_i+y_i) - F_2(x_i)}\right] = E\left[\frac{\mu_2}{1-\exp(-\mu_2 y_i)}\right] = \lambda + \mu_1 + \mu_2, \tag{45}$$

the latter following because by definition of the set $L_1^{(1)}$, $y_i \sim (\min(X_1, A) | \min(X_1, A) > X_2)$, where $A \sim G, X_1 \sim F_1, X_2 \sim F_2$.

Finally, we compute Δ_1 as follows. Let $A_{i \to j}$ be the expectation of the integral of the number in system process from state i to j, and $M_{i \to j}$ be the corresponding expected number of service completions. These quantities can be computed from the Markov chain by conditioning on states and obtaining a set of linear recursive equations, as was done to obtain χ_s. We omit the details here. After solving the equations, we obtain

$$\begin{aligned}
A_{2 \to (1,0)} &= \left(\frac{\mu_1}{(p\mu_2+\lambda)} + \left(2 + \frac{\lambda(3\hat{\mu} - 2\lambda)}{(\hat{\mu}-\lambda)^2}\right)\right) \Big/ \left(\hat{\mu} - \frac{\lambda\mu_1}{(p\mu_2+\lambda)}\right) \\
A_{(0,1) \to (1,0)} &= (1 + \lambda A_{2 \to (1,0)})/(p\mu_2 + \lambda) \\
M_{2 \to (1,0)} &= \left(\hat{\mu} + \frac{1}{\mu_1\mu_2(p\mu_2+\lambda)} + \frac{\lambda}{1-\lambda/\hat{\mu}}\right) \Big/ \left(\hat{\mu} - \frac{\lambda\mu_1}{(p\mu_2+\lambda)}\right) \\
M_{(0,1) \to (1,0)} &= (\mu_2 + \lambda M_{2 \to (1,0)})/(p\mu_2 + \lambda)
\end{aligned}$$

Then,

$$\Delta_1 = A_{(0,1) \to (1,0)} - M_{(0,1) \to (1,0)} E[T]. \tag{46}$$

Using (38), (39), (40), (42), (43), (44), (45), and (46), the expression for expectation SPA estimator, (16), was computed and compared with the analytical derivative given by (15) by using Mathematica, which verified that the expressions were equal. □

4 Simulation Experiments

In the previous section, we proved unbiasedness of the approximate SPA estimator for the Markovian case. In general, the estimator will be biased. In this section, we report the results of numerous simulation experiments designed to test the performance of the approximate SPA estimators. For non-Markovian cases, we compare the SPA estimates to finite difference estimates. In addition, we also compare the performance to estimates derived through an alternative technique for gradient estimations called the likelihood ratio (LR) method, which gives unbiased estimates for our problem.

4.1 Gradient Estimation via Likelihood Ratios

The likelihood ratio (LR) method, also called the Score Function (SF) method, is another technique for gradient estimation in stochastic simulation models (see e.g., Glynn 1990, Reiman and Weiss 1989, or Rubinstein 1989). Here, we briefly present an overview of the LR technique, and derive regenerative LR estimators for our problem. In general, the LR method has wider applicability than IPA, but when IPA works, it usually has much lower variance. Variance comparisons between SPA and LR, on the other hand, seem to be quite problem dependent (see, e.g., Vázquez-Abad and L'Ecuyer 1991).

Let

$$E[L(X)] = \int L(x)dF(\theta,x) \tag{47}$$

be the performance measure of interest, where X is a random vector with joint cumulative distribution function $F(\theta,\cdot)$ and density $f(\theta,\cdot) = dF(\theta,\cdot)/d\cdot$ depending on a parameter (or vector of parameters) θ. Differentiating (47), we have

$$\begin{aligned}\frac{\partial l(\theta)}{\partial\theta} &= \frac{\partial}{\partial\theta}\int L(x)f(\theta,x)dx &&= \int L(x)\frac{\partial f(\theta,x)}{\partial\theta}dx \\ &= \int L(x)\frac{\partial f(\theta,x)}{\partial\theta}\frac{f(\theta,x)}{f(\theta,x)}dx &&= \int L(x)\frac{\partial \ln f(\theta,x)}{\partial\theta}f(\theta,x)dx \\ &= E\left[L(X)\frac{\partial \ln f(\theta,x)}{\partial\theta}\right].\end{aligned} \tag{48}$$

Thus, in a single simulation, one can estimate the derivative of the performance measure along with the performance measure itself. Higher derivatives can be handled in a similar manner. A set of mild assumptions relating to the differentiability of the performance measure (cf., e.g., Rubinstein 1989) allows the interchange of differentiation and integration in the first line. However, as we shall see, the "naive" estimator for (48) leads to unbounded variance for steady-state performance measures.

For the system under consideration, the interarrival times and the service times comprise the random vector. Since these times are all independently generated, the density function f will simply be the product of the density functions of the interarrival and service time distributions. Thus, the joint density over N service completions for the two-server queue would be given by

$$f(\theta, A_1, ..., A_N, X_1, ..., X_N) = \prod_{i=1}^{N} g(A_i) \prod_{S(i)=1, i \leq N} f_1(X_i) \prod_{S(i)=2, i \leq N} f_2(X_i), \tag{49}$$

where A_i, X_i, $i = 1, ..., N$ are the interarrival times and service times, respectively. For example, in the Markovian case with arrival rate λ and service rates μ_1 and μ_2, (49) becomes

$$f(\theta, A_1, ..., A_N, X_1, ..., X_N) = \prod_{i=1}^{N} \lambda e^{-\lambda A_i} \prod_{S(i)=1, i \leq N} \mu_1 e^{-\mu_1 X_i} \prod_{S(i)=2, i \leq N} \mu_2 e^{-\mu_2 X_i}, \tag{50}$$

and we have ($\mu_k = 1/\theta_k$, $k = 1, 2$)

$$\ln f(\theta_1, \theta_2, A_1, ..., A_N, X_1, ..., X_N) = \sum_{i=1}^{N} (\ln \lambda - \lambda A_i) + \sum_{S(i)=1, i \leq N} (\ln \mu_1 - \mu_1 X_i) + \sum_{S(i)=2, i \leq N} (\mu_2 - \mu_2 X_i), \tag{51}$$

and

$$\frac{\partial \ln f}{\partial \theta_k} = \sum_{S(i)=k, i \leq N} \left(\frac{X_i}{\theta_k^2} - \frac{1}{\theta_k} \right), \ k = 1, 2. \tag{52}$$

The natural estimators would then be given by

$$\left(\frac{dT}{d\theta_k} \right)_{LR} = \frac{1}{N} \sum_{i=1}^{N} T_i \sum_{S(i)=k, i \leq N} \left(\frac{X_i}{\theta_k^2} - \frac{1}{\theta_k} \right), \ k = 1, 2. \tag{53}$$

The problem with these estimators is that if they are used to estimate steady state quantities by increasing the horizon length N, then it is obvious that the variance of the estimator will increase linearly with N, resulting quickly in a useless estimator. To resolve this problem, we derive a regenerative estimator instead.

Using regenerative theory, we can express the mean steady-state system time as a ratio of expectations:

$$E[T] = \frac{E[Q]}{E[\eta]}, \tag{54}$$

where η is the number of customers served in a busy period and Q is the sum of the system times of customers served in a busy period. Differentiation of (54) yields

$$\frac{dE[T]}{d\theta_k} = \frac{dE[Q]/d\theta_k}{E[\eta]} - \frac{dE[\eta]/d\theta_k}{E[\eta]} E[T], \ k = 1, 2. \tag{55}$$

Let η_j be the number of customers served in the jth busy period, and $L_{i,j}$ and $X_{i,j}$ be the system time and service time for the ith customer in the jth busy period, respectively. Then, employing (48) in conjunction with (52), we have the following regenerative estimators over M busy periods:

$$\left(\frac{dT}{d\theta_k} \right)_{LR} = \frac{1}{N} \sum_{j=1}^{M} \left\{ \sum_{i=1}^{\eta_j} L_{i,j} \frac{\partial \ln f}{\partial \theta_k} \right\} - \frac{1}{N} \sum_{j=1}^{M} \left\{ \eta_j \sum_{S(i)=k, i \leq N} \frac{\partial \ln f}{\partial \theta_k} \right\} \bar{T}, \ k = 1, 2, \tag{56}$$

where $N = \sum_{j=1}^{M} \eta_j$ is the total number of customers served, and $\bar{T} = \sum_{j=1}^{N} T_j/N$ is the estimate of mean system time. The advantage of these estimators is that the summations are bounded by the length of the busy periods, so as long as the busy periods are not too long, the variance of the estimators should be reasonable. Further reductions in variance can be achieved via variance reduction techniques such as conditional expectation and jackknifing, but will not be pursued here.

4.2 Simulation Results

The following systems were studied:

1. M/M/2
2. M/U/2
3. M/We/2
4. U/M/2
5. U/We/2

For the distributions of service times, the following expressions were used:

	Exponential	Uniform	Weibull
$f_\theta(x)$	$(1/\theta)e^{-x/\theta}$, $x > 0$	$1/2\theta (0 \leq x \leq 2\theta)$	$\beta\theta x^{\theta-1} e^{-\beta x^\theta}$, $x > 0$
$F_\theta(x)$	$1 - e^{-x/\theta}$	$x/2\theta (0 \leq x \leq 2\theta)$; 1 $(x > 2\theta)$	$1 - e^{-\beta x^\theta}$
$\frac{dX}{d\theta} = \frac{-\partial F/\partial\theta}{\partial F/\partial x}$	X/θ	X/θ	$-X(\ln X)/\theta$
$\frac{\partial \ln f_\theta(x)}{\partial\theta}$	$(x - \theta)/\theta^2$		$1/\theta + (1 - \beta x^\theta)\ln x$

The following parameter values were chosen :

$\lambda = 1.0$

$\theta_1 = \theta = 0.2, 0.4, 0.8$ (for exponential and uniform); $\theta = 0.5, 2.0$ (for Weibull)

$\theta_2 = c\ \theta$; c = 1.0, 1.1, 1.15, 1.2, 1.25, 1.3, 1.4, 1.5

$p = 0.0, 0.5, 1.0$

Note that when the servers are nearly equal (i.e., c is close to unity), $\theta = 0.2, 0.4$ and 0.8 correspond to approximate traffic intensities of $\rho = 0.1, 0.2$ and 0.4, respectively, for the cases of exponential and uniform service times. For the Weibull case, $(\theta, \beta) = (0.5, \sqrt{5})$ as well as $(2.0, \pi/0.64)$ correspond to $\rho = 0.2$; these values were chosen because they result in different coefficients of variation.

For systems 1, 2 and 4, 72 experiments were performed, while 48 experiments were performed for systems 3 and 5. Ten replications for each experiment were simulated for 100,000 busy periods, and 90% confidence intervals were constructed.

For the Markovian case, we provide the analytical gradients (denoted EXACT in the Table 1). For the other systems (2-5), estimates of the true derivatives were obtained by two-point finite differencing (FD) using common random numbers (although, for cases 2 and 3, analytical results could also be derived). This entailed additional experiments at perturbed parameter values of $\theta_1 + \Delta\theta$ and $\theta_2 + \Delta\theta$ to obtain the finite difference derivatives for the two servers, respectively; $\Delta\theta$ was chosen to be 0.005. For cases having uniform service distributions, θ is the scale parameter; hence, the service times were uniform on $[0,\ 2\theta_i]$ in the nominal path and $[0,\ 2(\theta_i + 0.005)]$ in the perturbed paths. In these cases, gradient estimation via Likelihood Ratios (LR) is not possible due to the discontinuity of f at 2θ (see, e.g., L'Ecuyer 1990).

The detailed results are presented in tabular form for each of the cases. Mean and deviation estimates are represented in the columns as mean $\pm$ 90% confidence intervals.

We now discuss the results obtained for each of the cases:

1. **M/M/2:** The detailed results are presented in Tables 1 and 2 for the two servers, respectively. The first three columns define the instance of the experiment. Column 4 reports the IPA estimates. It can clearly be observed that IPA is biased for unequal servers (see Fu and Hu, 1990; Fu, Hu and Nagi, 1992). Column 5 reports the SPA estimates. As proven in section 3, SPA is unbiased for this case, which is confirmed by the numerical results. The variance increases with ρ and δ. However, the variance is always lower than LR (column 6) which is also expected to provide unbiased estimates.

2. **M/U/2:** The detailed results are presented in Tables 3 and 4 for the two servers, respectively. Once again, the first three columns define the instance of the experiment. Column 4 reports the IPA estimates. As expected, and shown in Fu, Hu and Nagi (1992), IPA is biased. Column 5 (denoted by SPA^1) reports the SPA estimator presented in equations (6) and (7). In this case, this estimator is only an approximation, and the results are not accurate. For lower traffic intensities, it seems to improve upon the IPA estimator, but for high ρ, it may even worsen the IPA estimator. The reason for this is that $\hat{\Delta}_j, j = 1,2$ turns out to be a poor approximation for $\delta T_{i,j}$.

   Interestingly, in the M/G/2 case due to the arrival process being Markovian and specifically for 2 servers, independence from i can be achieved with the following change in definitions:

$$
\begin{aligned}
S_1 &= \text{expected sum of system times from state } [(0,1):\xi_2 = 0] \text{ to an arrival to state } (1,0),\\
S_2 &= \text{expected sum of system times from state } [(1,0):\xi_1 = 0] \text{ to an arrival to state } (0,1),\\
M_1 &= \text{expected number of service completions from state } [(0,1):\xi_2 = 0] \text{ to an arrival to state } (1,0),\\
M_2 &= \text{expected number of service completions from state } [(1,0):\xi_1 = 0] \text{ to an arrival to state } (0,1).
\end{aligned}
$$

These quantities can once again be estimated from the given sample path without the need for additional simulation. In addition to this case being rather limited, it further fails for: (i) $p = 0$ as S_1 and M_1 cannot be estimated, and (ii) $p = 1$ as S_2 and M_2 cannot be estimated. Thus, we did not think it worthwhile to employ this as an estimator for a fresh set of simulations. Instead, by taking advantage of the accumulations from the previous run (for SPA^1) and running some offline simulations to estimate S_j and $M_j, j = 1, 2$, we obtained the revised SPA estimator which is referred to by SPA^2 (column 6). The offline simulations were conducted as follows. Ten replications for each experiment were simulated for 10,000 instances of the appropriate starting and ending states. Each replication was associated with one replication of the SPA run, and 90% confidence intervals were constructed.

As seen from column 6 of tables 3 and 4, the revised SPA estimator performs well, and provides tighter confidence intervals than FD (column 7). Due to our earlier remark, LR cannot be performed in this case.

3. **M/We/2:** The detailed results are presented in Tables 5 and 6 for the two servers, respectively. The first three columns define the instance of the experiment. Column 4 reports the IPA estimates, which although biased, are reasonably close to FD (column 8). Column 5 (denoted by SPA^1) reports the SPA estimator presented in equations (6) and (7). Again, in this case, this estimator is only an approximation, and the results are rather poor; in fact they are worse than IPA in general. The reason for this is once again attributed to $\hat{\Delta}_j, j = 1, 2$ being a poor approximation for $\delta T_{i,j}$.

   Employing the revised definitions of quantities described in the M/U/2 case and conducting the offline simulations, the revised estimator is presented as SPA^2 in column 6. As earlier, these result in unbiased estimates. When compared to LR (column 7) and FD (column 8), the revised SPA reveals significantly tighter confidence intervals.

4. **U/M/2:** The detailed results are presented in Tables 7 and 8 for the two servers, respectively. Once again, the first three columns define the instance of the experiment. Column 4 reports the IPA estimates, and as expected IPA is biased. Column 5 reports the SPA estimator presented in equations (6) and (7). In this case, this estimator is only an approximation, and the results are not accurate. As in M/U/2, for lower traffic intensities, it seems to improve upon the IPA estimator, but for high ρ, it may worsen the IPA estimator. LR (column 6) is accurate and has lower variance than FD.

5. **U/We/2:** The detailed results are presented in Tables 9 and 10 for the two servers, respectively. Column 4 reports the IPA estimates, which although biased, are reasonably close to FD. Column 5 reports the SPA estimator presented in equations (6) and (7). Again, in this

case, this estimator is only an approximation, and the results are rather poor; in fact they are worse than IPA in general. LR (column 6) appears to be reasonably accurate and has lower variance than FD.

5 Summary and Conclusions

We have applied smoothed perturbation analysis to the problem of derivative estimation for a single-queue system with non-identical servers. The exact estimator is generally not practical except for low traffic conditions, so we propose an approximation similar in spirit to the one in Fu and Hu (1993), where an approximation performed well. As in Fu and Hu (1993), the proposed approximation is exact in the Markovian case. We then performed a set of experiments for more general distributions similar to the cases investigated in Fu and Hu (1993). Unlike the results contained there, however, for this system sometimes the approximation provides a reasonable estimate and sometimes it is quite poor. In fact, sometimes the SPA estimate does worse than the IPA estimate. For the cases where it does work, it does appear to exhibit lower variance than the LR estimate. Overall, we would recommend the use of the much simpler and easier-to-implement IPA estimator over the approximate SPA estimator when the servers are relatively close. In the case of extreme non-identicality, the regenerative LR estimator is to be preferred over the approximate SPA estimator in all but the Markovian cases.

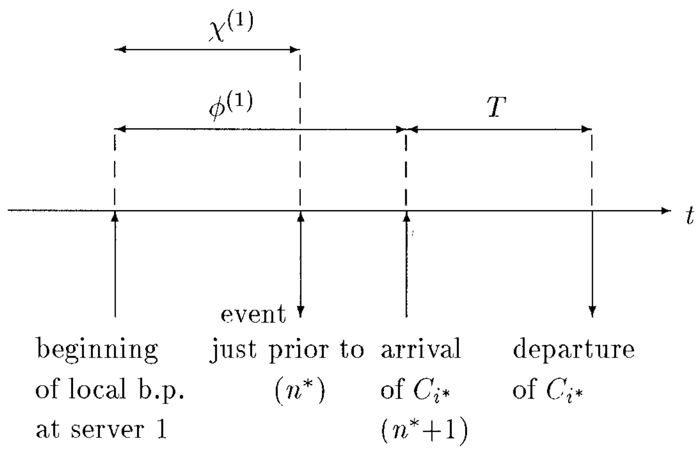

Figure 1: Calculation of IPA Component.

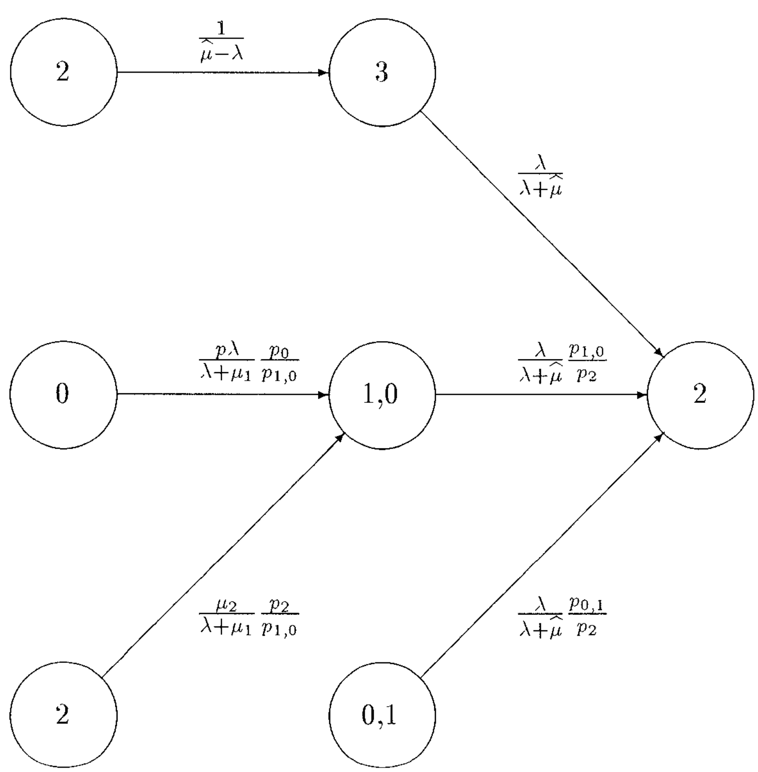

Figure 2: Conditional Probabilities of Embedded Markov Chain.

Table 1
Estimators for Server 1 of the M/M/2 Queue

p	θ	δ	IPA	SPA	LR	EXACT
		0.00	0.168±0.004	0.168±0.004	0.165±0.005	0.171
		0.10	0.184±0.004	0.185±0.004	0.183±0.005	0.188
		0.15	0.192±0.004	0.194±0.004	0.191±0.005	0.196
	0.2	0.20	0.200±0.004	0.202±0.004	0.199±0.005	0.205
		0.25	0.208±0.004	0.210±0.004	0.208±0.006	0.213
		0.30	0.216±0.004	0.219±0.004	0.217±0.006	0.222
		0.40	0.231±0.004	0.236±0.004	0.235±0.006	0.239
		0.50	0.247±0.005	0.253±0.005	0.253±0.007	0.256
		0.00	0.314±0.004	0.314±0.004	0.311±0.007	0.316
		0.10	0.345±0.004	0.348±0.004	0.346±0.008	0.351
		0.15	0.361±0.004	0.365±0.004	0.364±0.007	0.368
0.0	0.4	0.20	0.377±0.004	0.383±0.004	0.382±0.008	0.386
		0.25	0.393±0.004	0.401±0.004	0.397±0.009	0.404
		0.30	0.409±0.004	0.419±0.005	0.414±0.008	0.421
		0.40	0.440±0.005	0.454±0.005	0.449±0.009	0.457
		0.50	0.471±0.005	0.489±0.006	0.484±0.010	0.492
		0.00	0.676±0.008	0.674±0.010	0.666±0.019	0.680
		0.10	0.772±0.008	0.779±0.012	0.775±0.021	0.788
		0.15	0.823±0.009	0.834±0.012	0.832±0.021	0.844
	0.8	0.20	0.874±0.009	0.892±0.013	0.888±0.024	0.901
		0.25	0.928±0.009	0.952±0.014	0.948±0.028	0.960
		0.30	0.982±0.011	1.011±0.016	1.006±0.027	1.020
		0.40	1.092±0.012	1.131±0.018	1.118±0.027	1.145
		0.50	1.209±0.013	1.260±0.019	1.254±0.032	1.273
		0.00	0.514±0.001	0.514±0.002	0.511±0.003	0.515
		0.10	0.522±0.001	0.526±0.001	0.525±0.003	0.527
		0.15	0.525±0.001	0.531±0.001	0.530±0.004	0.533
	0.2	0.20	0.530±0.001	0.538±0.002	0.537±0.004	0.539
		0.25	0.532±0.001	0.543±0.001	0.541±0.003	0.545
		0.30	0.536±0.001	0.549±0.002	0.548±0.003	0.551
		0.40	0.544±0.002	0.562±0.002	0.559±0.005	0.563
		0.50	0.550±0.001	0.573±0.001	0.572±0.004	0.575
		0.00	0.563±0.001	0.562±0.002	0.560±0.005	0.564
		0.10	0.582±0.002	0.591±0.002	0.589±0.006	0.593
		0.15	0.592±0.001	0.605±0.002	0.601±0.005	0.608
0.5	0.4	0.20	0.602±0.002	0.619±0.003	0.617±0.007	0.622
		0.25	0.613±0.002	0.634±0.003	0.630±0.010	0.637
		0.30	0.624±0.002	0.649±0.003	0.645±0.008	0.652
		0.40	0.645±0.003	0.679±0.003	0.677±0.010	0.682
		0.50	0.666±0.003	0.709±0.004	0.706±0.010	0.711
		0.00	0.816±0.006	0.817±0.009	0.808±0.017	0.822
		0.10	0.902±0.007	0.917±0.011	0.913±0.022	0.925
		0.15	0.947±0.007	0.971±0.010	0.963±0.024	0.978
	0.8	0.20	0.993±0.008	1.025±0.012	1.019±0.023	1.034
		0.25	1.040±0.007	1.080±0.012	1.079±0.027	1.090
		0.30	1.089±0.009	1.142±0.015	1.131±0.026	1.148
		0.40	1.190±0.011	1.258±0.018	1.250±0.026	1.268
		0.50	1.295±0.013	1.380±0.022	1.365±0.035	1.392
		0.00	0.862±0.003	0.859±0.001	0.861±0.006	0.860
		0.10	0.865±0.003	0.875±0.002	0.878±0.006	0.876
		0.15	0.867±0.003	0.883±0.002	0.886±0.006	0.885
	0.2	0.20	0.868±0.003	0.891±0.002	0.894±0.006	0.893
		0.25	0.870±0.003	0.900±0.002	0.902±0.006	0.901
		0.30	0.872±0.003	0.908±0.002	0.910±0.006	0.910
		0.40	0.875±0.003	0.924±0.003	0.926±0.006	0.926
		0.50	0.878±0.003	0.939±0.003	0.943±0.006	0.942
		0.00	0.812±0.002	0.810±0.003	0.808±0.004	0.812
		0.10	0.826±0.002	0.844±0.004	0.843±0.004	0.847
		0.15	0.833±0.002	0.861±0.004	0.859±0.004	0.864
1.0	0.4	0.20	0.840±0.002	0.878±0.004	0.876±0.006	0.881
		0.25	0.847±0.002	0.895±0.005	0.893±0.006	0.898
		0.30	0.855±0.002	0.911±0.005	0.910±0.007	0.915
		0.40	0.869±0.002	0.945±0.005	0.944±0.008	0.949
		0.50	0.883±0.002	0.978±0.006	0.978±0.009	0.982
		0.00	0.960±0.005	0.958±0.011	0.953±0.020	0.964
		0.10	1.034±0.005	1.062±0.013	1.067±0.027	1.070
		0.15	1.076±0.006	1.119±0.014	1.112±0.021	1.126
	0.8	0.20	1.118±0.006	1.176±0.015	1.173±0.026	1.183
		0.25	1.163±0.007	1.233±0.017	1.226±0.028	1.241
		0.30	1.207±0.008	1.291±0.020	1.278±0.027	1.300
		0.40	1.297±0.009	1.411±0.022	1.399±0.029	1.421
		0.50	1.395±0.012	1.535±0.026	1.524±0.032	1.547

Table 2
Estimators for Server 2 of the M/M/2 Queue

p	θ	δ	IPA	SPA	LR	EXACT
		0.00	0.862±0.003	0.859±0.001	0.861±0.006	0.860
		0.10	0.849±0.003	0.833±0.001	0.836±0.005	0.834
		0.15	0.843±0.003	0.821±0.001	0.823±0.005	0.822
	0.2	0.20	0.837±0.003	0.809±0.001	0.811±0.005	0.810
		0.25	0.831±0.003	0.797±0.001	0.799±0.005	0.798
		0.30	0.825±0.003	0.786±0.001	0.787±0.005	0.786
		0.40	0.814±0.003	0.763±0.001	0.763±0.005	0.763
		0.50	0.802±0.003	0.741±0.001	0.741±0.004	0.742
		0.00	0.812±0.002	0.810±0.003	0.808±0.004	0.812
		0.10	0.797±0.002	0.776±0.003	0.771±0.004	0.777
		0.15	0.789±0.002	0.759±0.003	0.755±0.005	0.760
0.0	0.4	0.20	0.782±0.002	0.742±0.002	0.738±0.005	0.744
		0.25	0.774±0.002	0.726±0.003	0.724±0.005	0.728
		0.30	0.766±0.002	0.710±0.003	0.708±0.005	0.712
		0.40	0.751±0.002	0.681±0.002	0.678±0.004	0.682
		0.50	0.736±0.002	0.652±0.002	0.649±0.005	0.653
		0.00	0.960±0.005	0.958±0.011	0.953±0.020	0.964
		0.10	0.956±0.005	0.926±0.011	0.918±0.016	0.931
		0.15	0.951±0.005	0.909±0.010	0.897±0.016	0.915
	0.8	0.20	0.948±0.004	0.892±0.011	0.885±0.015	0.900
		0.25	0.946±0.005	0.876±0.011	0.872±0.014	0.884
		0.30	0.943±0.005	0.861±0.011	0.857±0.016	0.869
		0.40	0.934±0.005	0.828±0.012	0.826±0.020	0.840
		0.50	0.925±0.004	0.799±0.011	0.801±0.017	0.812
		0.00	0.515±0.001	0.515±0.001	0.515±0.003	0.515
		0.10	0.511±0.001	0.506±0.001	0.505±0.003	0.507
		0.15	0.510±0.001	0.503±0.001	0.501±0.002	0.502
	0.2	0.20	0.506±0.001	0.497±0.001	0.496±0.004	0.498
		0.25	0.506±0.001	0.494±0.001	0.494±0.003	0.494
		0.30	0.544±0.001	0.519±0.002	0.515±0.005	0.520
		0.40	0.498±0.001	0.481±0.001	0.480±0.002	0.481
		0.50	0.495±0.001	0.473±0.001	0.470±0.003	0.473
		0.00	0.563±0.002	0.563±0.002	0.559±0.006	0.564
		0.10	0.557±0.002	0.548±0.002	0.543±0.004	0.549
		0.15	0.555±0.002	0.542±0.002	0.539±0.006	0.542
0.5	0.4	0.20	0.551±0.002	0.534±0.002	0.529±0.006	0.535
		0.25	0.548±0.001	0.527±0.002	0.524±0.003	0.527
		0.30	0.544±0.001	0.519±0.002	0.515±0.005	0.520
		0.40	0.538±0.002	0.505±0.002	0.500±0.004	0.505
		0.50	0.531±0.002	0.491±0.002	0.488±0.004	0.491
		0.00	0.820±0.006	0.816±0.009	0.812±0.022	0.822
		0.10	0.820±0.007	0.803±0.009	0.795±0.017	0.807
		0.15	0.820±0.007	0.792±0.010	0.786±0.015	0.799
	0.8	0.20	0.820±0.006	0.785±0.009	0.775±0.017	0.791
		0.25	0.819±0.007	0.777±0.009	0.767±0.014	0.783
		0.30	0.820±0.006	0.767±0.009	0.765±0.018	0.774
		0.40	0.817±0.005	0.750±0.009	0.743±0.021	0.757
		0.50	0.812±0.006	0.729±0.009	0.728±0.016	0.739
		0.00	0.168±0.004	0.168±0.004	0.165±0.005	0.171
		0.10	0.167±0.004	0.167±0.004	0.164±0.006	0.169
		0.15	0.167±0.004	0.166±0.004	0.163±0.006	0.168
	0.2	0.20	0.166±0.004	0.165±0.004	0.162±0.006	0.167
		0.25	0.166±0.004	0.164±0.004	0.161±0.006	0.166
		0.30	0.165±0.004	0.163±0.004	0.160±0.006	0.165
		0.40	0.164±0.004	0.161±0.004	0.158±0.006	0.164
		0.50	0.163±0.004	0.160±0.004	0.157±0.006	0.162
		0.00	0.314±0.004	0.314±0.004	0.311±0.007	0.316
		0.10	0.312±0.004	0.309±0.004	0.305±0.008	0.311
		0.15	0.310±0.004	0.306±0.004	0.303±0.007	0.308
1.0	0.4	0.20	0.309±0.004	0.303±0.004	0.299±0.007	0.305
		0.25	0.308±0.004	0.301±0.004	0.297±0.007	0.302
		0.30	0.306±0.004	0.298±0.004	0.294±0.007	0.300
		0.40	0.304±0.004	0.292±0.004	0.289±0.007	0.294
		0.50	0.301±0.004	0.287±0.004	0.283±0.007	0.289
		0.00	0.676±0.008	0.674±0.010	0.666±0.019	0.680
		0.10	0.683±0.009	0.671±0.011	0.655±0.014	0.675
		0.15	0.683±0.008	0.666±0.010	0.659±0.018	0.672
	0.8	0.20	0.684±0.008	0.662±0.011	0.653±0.017	0.669
		0.25	0.684±0.008	0.658±0.011	0.650±0.016	0.665
		0.30	0.684±0.008	0.653±0.010	0.649±0.019	0.661
		0.40	0.685±0.007	0.644±0.009	0.642±0.020	0.652
		0.50	0.683±0.007	0.635±0.010	0.633±0.019	0.642

Table 3
Estimators for Server 1 of the M/U/2 Queue

p	θ	δ	IPA	SPA^1	SPA^2	FD
0.0	0.2	0.00	0.167±0.003	0.169±0.003		0.166±0.004
		0.10	0.182±0.003	0.186±0.003		0.183±0.003
		0.15	0.189±0.003	0.194±0.003		0.191±0.003
		0.20	0.197±0.003	0.203±0.003		0.198±0.002
		0.25	0.205±0.003	0.211±0.003		0.206±0.004
		0.30	0.212±0.003	0.220±0.003		0.216±0.004
		0.40	0.227±0.003	0.237±0.003		0.231±0.003
		0.50	0.242±0.003	0.254±0.003		0.248±0.003
	0.4	0.00	0.296±0.002	0.302±0.002		0.296±0.005
		0.10	0.325±0.002	0.335±0.003		0.330±0.004
		0.15	0.339±0.002	0.351±0.002		0.346±0.005
		0.20	0.353±0.003	0.367±0.003		0.357±0.004
		0.25	0.367±0.003	0.384±0.003		0.376±0.005
		0.30	0.381±0.003	0.400±0.003		0.391±0.005
		0.40	0.409±0.003	0.433±0.003		0.423±0.003
		0.50	0.436±0.003	0.464±0.003		0.458±0.005
	0.8	0.00	0.582±0.005	0.574±0.005		0.581±0.009
		0.10	0.657±0.005	0.651±0.006		0.672±0.011
		0.15	0.696±0.006	0.691±0.006		0.708±0.014
		0.20	0.735±0.006	0.731±0.007		0.754±0.006
		0.25	0.776±0.006	0.772±0.008		0.797±0.017
		0.30	0.816±0.006	0.812±0.008		0.851±0.013
		0.40	0.901±0.007	0.894±0.008		0.941±0.011
		0.50	0.987±0.007	0.976±0.009		1.027±0.009
0.5	0.2	0.00	0.511±0.001	0.510±0.001	0.511±0.001	0.510±0.002
		0.10	0.517±0.001	0.521±0.001	0.521±0.001	0.520±0.004
		0.15	0.520±0.001	0.527±0.001	0.527±0.001	0.527±0.006
		0.20	0.524±0.001	0.532±0.001	0.533±0.001	0.537±0.004
		0.25	0.528±0.001	0.538±0.001	0.539±0.001	0.542±0.007
		0.30	0.531±0.001	0.544±0.001	0.545±0.001	0.549±0.009
		0.40	0.537±0.001	0.555±0.001	0.556±0.001	0.555±0.011
		0.50	0.544±0.001	0.567±0.001	0.567±0.001	0.572±0.010
	0.4	0.00	0.547±0.001	0.544±0.001	0.547±0.002	0.546±0.004
		0.10	0.563±0.001	0.569±0.002	0.572±0.002	0.575±0.005
		0.15	0.572±0.001	0.582±0.002	0.585±0.002	0.600±0.007
		0.20	0.580±0.001	0.594±0.002	0.598±0.002	0.588±0.011
		0.25	0.589±0.001	0.608±0.002	0.612±0.002	0.605±0.006
		0.30	0.598±0.002	0.620±0.002	0.625±0.003	0.611±0.017
		0.40	0.615±0.002	0.646±0.002	0.651±0.003	0.635±0.018
		0.50	0.633±0.002	0.671±0.002	0.677±0.003	0.692±0.021
	0.8	0.00	0.726±0.003	0.693±0.005	0.728±0.006	0.730±0.010
		0.10	0.788±0.004	0.764±0.005	0.809±0.007	0.820±0.018
		0.15	0.822±0.005	0.802±0.006	0.851±0.009	0.855±0.016
		0.20	0.855±0.005	0.839±0.007	0.893±0.009	0.862±0.027
		0.25	0.889±0.005	0.877±0.008	0.937±0.010	0.905±0.019
		0.30	0.924±0.006	0.916±0.008	0.982±0.011	1.015±0.028
		0.40	0.997±0.006	0.994±0.009	1.075±0.013	1.101±0.040
		0.50	1.074±0.007	1.076±0.010	1.167±0.015	1.168±0.039
1.0	0.2	0.00	0.855±0.002	0.817±0.001	0.855±0.002	0.853±0.003
		0.10	0.857±0.002	0.832±0.001	0.871±0.002	0.870±0.004
		0.15	0.859±0.002	0.839±0.001	0.879±0.002	0.877±0.004
		0.20	0.860±0.002	0.847±0.001	0.886±0.002	0.885±0.004
		0.25	0.861±0.002	0.854±0.001	0.894±0.002	0.892±0.004
		0.30	0.863±0.002	0.861±0.001	0.902±0.002	0.901±0.005
		0.40	0.865±0.002	0.876±0.001	0.917±0.002	0.918±0.005
		0.50	0.868±0.002	0.890±0.001	0.932±0.002	0.929±0.005
	0.4	0.00	0.797±0.002	0.736±0.001	0.797±0.002	0.798±0.006
		0.10	0.808±0.002	0.765±0.002	0.828±0.002	0.828±0.005
		0.15	0.813±0.001	0.779±0.002	0.844±0.002	0.841±0.005
		0.20	0.819±0.001	0.793±0.002	0.859±0.002	0.859±0.007
		0.25	0.824±0.001	0.807±0.002	0.874±0.002	0.873±0.005
		0.30	0.829±0.001	0.821±0.002	0.889±0.002	0.889±0.004
		0.40	0.840±0.001	0.849±0.002	0.918±0.002	0.903±0.004
		0.50	0.850±0.001	0.876±0.003	0.947±0.003	0.945±0.008
	0.8	0.00	0.869±0.003	0.750±0.006	0.872±0.006	0.873±0.012
		0.10	0.924±0.003	0.821±0.007	0.956±0.006	0.952±0.013
		0.15	0.953±0.004	0.858±0.008	0.999±0.007	1.012±0.021
		0.20	0.982±0.004	0.897±0.009	1.043±0.008	1.055±0.019
		0.25	1.012±0.004	0.934±0.009	1.088±0.009	1.099±0.016
		0.30	1.042±0.005	0.972±0.010	1.133±0.010	1.146±0.019
		0.40	1.105±0.006	1.049±0.012	1.226±0.011	1.243±0.030
		0.50	1.172±0.007	1.129±0.014	1.321±0.014	1.321±0.026

Table 4
Estimators for Server 2 of the M/U/2 Queue

p	θ	δ	IPA	SPA^1	SPA^2	FD
0.0	0.2	0.00	0.855±0.002	0.817±0.001	0.855±0.002	0.853±0.003
		0.10	0.843±0.002	0.791±0.001	0.829±0.002	0.830±0.004
		0.15	0.837±0.002	0.778±0.002	0.817±0.002	0.816±0.004
		0.20	0.831±0.002	0.765±0.001	0.805±0.002	0.804±0.004
		0.25	0.825±0.002	0.753±0.001	0.792±0.002	0.788±0.002
		0.30	0.819±0.002	0.741±0.001	0.781±0.002	0.781±0.003
		0.40	0.807±0.002	0.718±0.001	0.758±0.002	0.756±0.004
		0.50	0.795±0.002	0.696±0.002	0.736±0.002	0.734±0.004
	0.4	0.00	0.797±0.002	0.736±0.001	0.797±0.002	0.798±0.006
		0.10	0.781±0.001	0.699±0.001	0.762±0.002	0.760±0.006
		0.15	0.773±0.001	0.681±0.001	0.744±0.002	0.744±0.006
		0.20	0.764±0.001	0.662±0.002	0.726±0.002	0.717±0.007
		0.25	0.756±0.001	0.646±0.002	0.710±0.002	0.709±0.003
		0.30	0.748±0.001	0.630±0.002	0.694±0.002	0.694±0.008
		0.40	0.733±0.001	0.599±0.001	0.663±0.002	0.658±0.005
		0.50	0.718±0.001	0.569±0.001	0.634±0.002	0.634±0.008
	0.8	0.00	0.869±0.003	0.750±0.006	0.872±0.006	0.873±0.012
		0.10	0.859±0.003	0.706±0.006	0.835±0.005	0.839±0.011
		0.15	0.854±0.002	0.683±0.006	0.816±0.005	0.802±0.011
		0.20	0.848±0.003	0.663±0.006	0.799±0.005	0.788±0.009
		0.25	0.843±0.003	0.644±0.005	0.782±0.006	0.779±0.013
		0.30	0.837±0.003	0.622±0.006	0.763±0.007	0.767±0.016
		0.40	0.826±0.002	0.586±0.006	0.732±0.007	0.725±0.004
		0.50	0.813±0.003	0.550±0.006	0.702±0.007	0.681±0.011
0.5	0.2	0.00	0.511±0.001	0.511±0.001	0.511±0.001	0.510±0.002
		0.10	0.507±0.001	0.502±0.001	0.503±0.001	0.502±0.005
		0.15	0.505±0.001	0.497±0.001	0.498±0.001	0.495±0.005
		0.20	0.503±0.001	0.493±0.001	0.493±0.001	0.496±0.008
		0.25	0.500±0.001	0.488±0.001	0.489±0.001	0.490±0.007
		0.30	0.498±0.000	0.484±0.001	0.485±0.001	0.483±0.011
		0.40	0.494±0.001	0.475±0.001	0.476±0.001	0.474±0.012
		0.50	0.490±0.001	0.467±0.001	0.468±0.001	0.470±0.018
	0.4	0.00	0.547±0.001	0.544±0.001	0.548±0.001	0.547±0.004
		0.10	0.540±0.001	0.528±0.001	0.532±0.002	0.532±0.005
		0.15	0.536±0.001	0.520±0.001	0.524±0.002	0.529±0.006
		0.20	0.533±0.001	0.512±0.001	0.516±0.002	0.514±0.014
		0.25	0.530±0.001	0.504±0.001	0.508±0.002	0.509±0.011
		0.30	0.526±0.001	0.496±0.001	0.501±0.002	0.499±0.013
		0.40	0.519±0.001	0.481±0.001	0.486±0.001	0.475±0.016
		0.50	0.511±0.001	0.466±0.001	0.471±0.001	0.483±0.027
	0.8	0.00	0.726±0.004	0.694±0.005	0.729±0.006	0.731±0.010
		0.10	0.723±0.004	0.668±0.005	0.709±0.007	0.723±0.014
		0.15	0.721±0.003	0.655±0.004	0.699±0.006	0.697±0.015
		0.20	0.719±0.003	0.643±0.005	0.689±0.008	0.660±0.026
		0.25	0.717±0.004	0.631±0.005	0.681±0.008	0.658±0.023
		0.30	0.714±0.003	0.617±0.005	0.670±0.007	0.704±0.035
		0.40	0.709±0.003	0.592±0.005	0.651±0.008	0.640±0.043
		0.50	0.704±0.004	0.567±0.004	0.633±0.009	0.638±0.036
1.0	0.2	0.00	0.167±0.003	0.169±0.003		0.166±0.004
		0.10	0.165±0.003	0.167±0.003		0.165±0.003
		0.15	0.165±0.003	0.166±0.003		0.163±0.003
		0.20	0.164±0.003	0.165±0.003		0.162±0.003
		0.25	0.164±0.003	0.164±0.003		0.162±0.003
		0.30	0.163±0.003	0.163±0.003		0.161±0.002
		0.40	0.162±0.003	0.161±0.003		0.160±0.002
		0.50	0.161±0.003	0.160±0.003		0.157±0.003
	0.4	0.00	0.296±0.002	0.302±0.002		0.296±0.005
		0.10	0.293±0.002	0.296±0.002		0.291±0.003
		0.15	0.292±0.002	0.293±0.002		0.288±0.003
		0.20	0.290±0.002	0.290±0.002		0.287±0.003
		0.25	0.289±0.002	0.287±0.002		0.285±0.005
		0.30	0.287±0.002	0.284±0.002		0.278±0.005
		0.40	0.284±0.002	0.279±0.002		0.271±0.005
		0.50	0.282±0.002	0.274±0.002		0.270±0.004
	0.8	0.00	0.582±0.005	0.574±0.005		0.581±0.009
		0.10	0.582±0.005	0.561±0.005		0.570±0.014
		0.15	0.582±0.004	0.555±0.005		0.586±0.015
		0.20	0.582±0.004	0.548±0.005		0.571±0.010
		0.25	0.582±0.004	0.543±0.005		0.563±0.010
		0.30	0.581±0.004	0.535±0.005		0.570±0.016
		0.40	0.579±0.005	0.520±0.006		0.553±0.016
		0.50	0.578±0.005	0.506±0.005		0.532±0.013

Table 5
Estimators for Server 1 of the M/We/2 Queue

θ	p	δ	IPA	SPA^1	SPA^2	LR	FD
0.5	0.0	0.00	-0.115±0.004	-0.053±0.015		-0.138±0.017	-0.112±0.006
		0.10	-0.095±0.004	-0.029±0.013		-0.108±0.012	-0.094±0.007
		0.15	-0.092±0.005	-0.020±0.026		-0.101±0.009	-0.091±0.011
		0.20	-0.087±0.005	-0.022±0.022		-0.094±0.009	-0.082±0.005
		0.25	-0.083±0.005	-0.020±0.029		-0.088±0.009	-0.085±0.011
		0.30	-0.080±0.005	-0.011±0.019		-0.083±0.010	-0.081±0.013
		0.40	-0.076±0.005	-0.026±0.023		-0.082±0.009	-0.071±0.008
		0.50	-0.076±0.005	0.062±0.116		-0.082±0.008	-0.074±0.008
	0.5	0.00	-0.197±0.007	-0.173±0.009	-0.196±0.008	-0.224±0.027	-0.194±0.012
		0.10	-0.182±0.007	-0.150±0.007	-0.181±0.007	-0.180±0.022	-0.159±0.041
		0.15	-0.175±0.006	-0.143±0.006	-0.174±0.006	-0.185±0.015	-0.171±0.049
		0.20	-0.166±0.004	-0.127±0.005	-0.164±0.004	-0.179±0.017	-0.162±0.057
		0.25	-0.163±0.007	-0.125±0.006	-0.162±0.007	-0.166±0.013	-0.122±0.052
		0.30	-0.162±0.005	-0.112±0.007	-0.160±0.004	-0.162±0.016	-0.110±0.067
		0.40	-0.151±0.003	-0.102±0.005	-0.150±0.003	-0.162±0.015	-0.161±0.101
		0.50	-0.149±0.004	-0.088±0.020	-0.149±0.004	-0.152±0.013	-0.080±0.112
	1.0	0.00	-0.279±0.010	-0.312±0.010	-0.278±0.011	-0.294±0.028	-0.275±0.013
		0.10	-0.261±0.008	-0.276±0.011	-0.259±0.008	-0.263±0.028	-0.254±0.014
		0.15	-0.254±0.008	-0.257±0.008	-0.252±0.008	-0.259±0.028	-0.257±0.016
		0.20	-0.249±0.007	-0.245±0.007	-0.247±0.007	-0.257±0.027	-0.250±0.014
		0.25	-0.245±0.007	-0.234±0.006	-0.242±0.007	-0.251±0.027	-0.242±0.014
		0.30	-0.241±0.007	-0.223±0.006	-0.239±0.006	-0.249±0.024	-0.246±0.014
		0.40	-0.234±0.007	-0.206±0.008	-0.232±0.006	-0.245±0.026	-0.246±0.014
		0.50	-0.230±0.007	-0.195±0.006	-0.228±0.006	-0.239±0.027	-0.237±0.015
2.0	0.0	0.00	0.046±0.000	-Inf± NaN		0.045±0.001	0.046±0.001
		0.10	0.049±0.000	2.048±0.048		0.049±0.000	0.049±0.001
		0.15	0.050±0.000	1.468±0.023		0.050±0.000	0.050±0.001
		0.20	0.051±0.000	1.205±0.022		0.052±0.000	0.052±0.001
		0.25	0.052±0.000	1.022±0.017		0.053±0.001	0.053±0.001
		0.30	0.054±0.000	0.916±0.017		0.054±0.000	0.055±0.001
		0.40	0.056±0.000	0.772±0.012		0.057±0.000	0.058±0.001
		0.50	0.058±0.000	0.686±0.013		0.059±0.001	0.059±0.001
	0.5	0.00	0.084±0.000	-Inf± NaN	0.083±0.000	0.083±0.001	0.084±0.001
		0.10	0.085±0.000	1.954±0.017	0.086±0.000	0.086±0.001	0.085±0.002
		0.15	0.086±0.000	1.399±0.010	0.087±0.000	0.087±0.001	0.090±0.004
		0.20	0.087±0.000	1.133±0.013	0.088±0.000	0.088±0.001	0.085±0.006
		0.25	0.087±0.000	0.965±0.005	0.090±0.000	0.089±0.001	0.097±0.011
		0.30	0.088±0.000	0.856±0.007	0.091±0.000	0.090±0.001	0.093±0.005
		0.40	0.089±0.000	0.716±0.004	0.093±0.000	0.092±0.001	0.087±0.011
		0.50	0.090±0.000	0.633±0.002	0.094±0.000	0.094±0.001	0.101±0.015
	1.0	0.00	0.122±0.000	-Inf± NaN	0.122±0.000	0.121±0.001	0.124±0.003
		0.10	0.123±0.000	2.603±0.015	0.125±0.000	0.124±0.001	0.122±0.002
		0.15	0.123±0.000	1.856±0.011	0.126±0.000	0.126±0.001	0.126±0.001
		0.20	0.123±0.000	1.483±0.009	0.128±0.000	0.127±0.001	0.128±0.002
		0.25	0.124±0.000	1.255±0.006	0.129±0.000	0.128±0.001	0.129±0.003
		0.30	0.124±0.000	1.101±0.006	0.130±0.000	0.130±0.001	0.129±0.002
		0.40	0.125±0.000	0.915±0.005	0.133±0.000	0.132±0.001	0.134±0.002
		0.50	0.126±0.000	0.803±0.005	0.135±0.000	0.134±0.002	0.134±0.003

Table 6
Estimators for Server 2 of the M/We/2 Queue

θ	p	δ	IPA	SPA^1	SPA^2	LR	FD
0.5	0.0	0.00	-0.279±0.010	-0.312±0.010	-0.278±0.011	-0.294±0.028	-0.275±0.013
		0.10	-0.132±0.007	-0.139±0.008	-0.132±0.008	-0.146±0.021	-0.129±0.008
		0.15	-0.080±0.006	-0.082±0.006	-0.081±0.006	-0.096±0.023	-0.078±0.008
		0.20	-0.040±0.005	-0.039±0.005	-0.040±0.005	-0.056±0.020	-0.048±0.007
		0.25	-0.009±0.005	-0.007±0.005	-0.008±0.005	-0.023±0.019	-0.008±0.011
		0.30	0.018±0.004	0.019±0.004	0.018±0.004	0.003±0.017	0.022±0.007
		0.40	0.056±0.003	0.055±0.003	0.057±0.003	0.048±0.013	0.050±0.010
		0.50	0.081±0.002	0.076±0.003	0.082±0.002	0.077±0.012	0.083±0.007
	0.5	0.00	-0.197±0.005	-0.176±0.006	-0.198±0.004	-0.208±0.020	-0.190±0.007
		0.10	-0.094±0.005	-0.081±0.006	-0.094±0.005	-0.120±0.016	-0.077±0.039
		0.15	-0.059±0.003	-0.048±0.003	-0.059±0.004	-0.075±0.013	-0.121±0.045
		0.20	-0.029±0.003	-0.024±0.003	-0.029±0.003	-0.047±0.013	0.012±0.064
		0.25	-0.007±0.003	-0.006±0.003	-0.007±0.003	-0.024±0.016	0.078±0.039
		0.30	0.012±0.002	0.015±0.004	0.012±0.002	-0.002±0.013	0.079±0.069
		0.40	0.040±0.002	0.033±0.001	0.040±0.002	0.035±0.011	0.172±0.135
		0.50	0.058±0.001	0.047±0.002	0.058±0.001	0.050±0.010	0.136±0.124
	1.0	0.00	-0.115±0.004	-0.053±0.015		-0.138±0.017	-0.112±0.006
		0.10	-0.058±0.004	-0.018±0.020		-0.082±0.018	-0.055±0.006
		0.15	-0.037±0.003	-0.003±0.015		-0.058±0.017	-0.040±0.008
		0.20	-0.019±0.003	-0.004±0.015		-0.037±0.015	-0.020±0.005
		0.25	-0.006±0.003	0.002±0.006		-0.023±0.013	-0.005±0.005
		0.30	0.005±0.003	0.009±0.013		-0.008±0.012	0.006±0.005
		0.40	0.022±0.002	0.014±0.005		0.011±0.010	0.021±0.003
		0.50	0.033±0.002	0.023±0.005		0.023±0.008	0.034±0.003
2.0	0.0	0.00	0.122±0.000	-Inf± NaN	0.122±0.000	0.121±0.001	0.124±0.003
		0.10	0.109±0.000	2.330±0.012	0.107±0.000	0.107±0.001	0.106±0.002
		0.15	0.104±0.000	1.566±0.009	0.101±0.000	0.100±0.001	0.102±0.001
		0.20	0.098±0.000	1.183±0.008	0.095±0.000	0.095±0.001	0.095±0.001
		0.25	0.094±0.000	0.948±0.004	0.090±0.000	0.089±0.001	0.090±0.001
		0.30	0.089±0.000	0.790±0.004	0.085±0.000	0.084±0.001	0.086±0.001
		0.40	0.081±0.000	0.590±0.003	0.076±0.000	0.076±0.001	0.076±0.001
		0.50	0.074±0.000	0.467±0.002	0.068±0.000	0.068±0.001	0.068±0.001
	0.5	0.00	0.084±0.000	-Inf± NaN	0.084±0.000	0.083±0.001	0.084±0.001
		0.10	0.076±0.000	1.728±0.012	0.075±0.000	0.074±0.001	0.075±0.003
		0.15	0.072±0.000	1.167±0.006	0.071±0.000	0.070±0.001	0.070±0.005
		0.20	0.069±0.000	0.885±0.006	0.067±0.000	0.067±0.001	0.063±0.003
		0.25	0.065±0.000	0.711±0.003	0.064±0.000	0.063±0.001	0.064±0.006
		0.30	0.062±0.000	0.595±0.003	0.060±0.000	0.060±0.001	0.056±0.006
		0.40	0.057±0.000	0.446±0.003	0.055±0.000	0.054±0.001	0.052±0.009
		0.50	0.052±0.000	0.356±0.002	0.049±0.000	0.049±0.001	0.055±0.015
	1.0	0.00	0.046±0.000	-Inf± NaN		0.045±0.001	0.046±0.001
		0.10	0.041±0.000	1.669±0.020		0.041±0.001	0.041±0.001
		0.15	0.040±0.000	1.117±0.020		0.039±0.000	0.039±0.001
		0.20	0.038±0.000	0.822±0.012		0.037±0.000	0.037±0.002
		0.25	0.036±0.000	0.647±0.012		0.035±0.000	0.035±0.001
		0.30	0.034±0.000	0.531±0.010		0.033±0.000	0.034±0.001
		0.40	0.031±0.000	0.387±0.008		0.030±0.001	0.031±0.001
		0.50	0.029±0.000	0.298±0.004		0.027±0.001	0.028±0.001

Table 7
Estimators for Server 1 of the U/M/2 Queue

p	θ	δ	IPA	SPA	LR	FD
0.0	0.2	0.00	0.104±0.005	0.103±0.004	0.104±0.006	0.104±0.005
		0.10	0.114±0.005	0.114±0.004	0.115±0.006	0.116±0.006
		0.15	0.120±0.005	0.120±0.004	0.121±0.005	0.120±0.004
		0.20	0.125±0.005	0.126±0.004	0.126±0.005	0.126±0.005
		0.25	0.130±0.004	0.132±0.004	0.131±0.005	0.132±0.004
		0.30	0.136±0.004	0.138±0.004	0.138±0.005	0.138±0.006
		0.40	0.147±0.004	0.150±0.004	0.149±0.004	0.147±0.005
		0.50	0.158±0.004	0.162±0.004	0.160±0.004	0.160±0.005
	0.4	0.00	0.210±0.004	0.205±0.004	0.213±0.005	0.210±0.007
		0.10	0.234±0.004	0.230±0.004	0.236±0.005	0.235±0.006
		0.15	0.245±0.004	0.243±0.003	0.250±0.005	0.247±0.005
		0.20	0.257±0.004	0.256±0.003	0.262±0.006	0.258±0.006
		0.25	0.269±0.004	0.269±0.003	0.274±0.006	0.268±0.005
		0.30	0.281±0.004	0.281±0.003	0.287±0.005	0.282±0.006
		0.40	0.305±0.004	0.308±0.003	0.313±0.005	0.312±0.006
		0.50	0.328±0.004	0.334±0.003	0.338±0.006	0.339±0.006
	0.8	0.00	0.480±0.005	0.419±0.004	0.485±0.012	0.457±0.011
		0.10	0.549±0.006	0.480±0.005	0.561±0.014	0.571±0.019
		0.15	0.585±0.006	0.511±0.006	0.598±0.014	0.596±0.019
		0.20	0.620±0.006	0.543±0.006	0.636±0.014	0.628±0.014
		0.25	0.658±0.007	0.576±0.007	0.679±0.015	0.685±0.017
		0.30	0.697±0.008	0.611±0.008	0.722±0.016	0.723±0.025
		0.40	0.774±0.008	0.678±0.009	0.805±0.019	0.787±0.026
		0.50	0.854±0.009	0.746±0.010	0.892±0.021	0.888±0.025
0.5	0.2	0.00	0.504±0.001	0.503±0.001	0.505±0.004	0.503±0.003
		0.10	0.507±0.001	0.509±0.001	0.509±0.003	0.512±0.005
		0.15	0.510±0.001	0.514±0.001	0.513±0.005	0.514±0.009
		0.20	0.512±0.001	0.516±0.001	0.517±0.005	0.517±0.009
		0.25	0.513±0.001	0.520±0.001	0.520±0.004	0.518±0.009
		0.30	0.516±0.001	0.523±0.001	0.522±0.003	0.516±0.010
		0.40	0.520±0.000	0.530±0.001	0.531±0.003	0.528±0.011
		0.50	0.524±0.001	0.538±0.001	0.536±0.004	0.552±0.018
	0.4	0.00	0.524±0.001	0.519±0.001	0.524±0.006	0.522±0.006
		0.10	0.535±0.001	0.536±0.001	0.542±0.004	0.536±0.010
		0.15	0.541±0.001	0.545±0.001	0.552±0.005	0.548±0.012
		0.20	0.547±0.001	0.554±0.001	0.561±0.005	0.554±0.014
		0.25	0.553±0.001	0.563±0.001	0.572±0.005	0.569±0.014
		0.30	0.559±0.001	0.572±0.001	0.579±0.005	0.588±0.014
		0.40	0.571±0.001	0.590±0.001	0.599±0.006	0.597±0.029
		0.50	0.584±0.001	0.608±0.001	0.615±0.005	0.611±0.035
	0.8	0.00	0.664±0.003	0.609±0.004	0.670±0.012	0.641±0.012
		0.10	0.717±0.004	0.666±0.005	0.733±0.013	0.730±0.013
		0.15	0.746±0.004	0.694±0.005	0.775±0.016	0.746±0.020
		0.20	0.774±0.005	0.724±0.006	0.810±0.016	0.759±0.045
		0.25	0.805±0.005	0.754±0.006	0.846±0.020	0.835±0.024
		0.30	0.835±0.006	0.784±0.007	0.880±0.017	0.901±0.047
		0.40	0.901±0.007	0.851±0.009	0.966±0.018	0.935±0.052
		0.50	0.968±0.008	0.915±0.010	1.047±0.017	1.065±0.062
1.0	0.2	0.00	0.905±0.005	0.906±0.002	0.902±0.003	0.902±0.006
		0.10	0.906±0.005	0.916±0.001	0.912±0.003	0.913±0.005
		0.15	0.906±0.005	0.921±0.001	0.918±0.003	0.920±0.005
		0.20	0.907±0.005	0.926±0.001	0.923±0.003	0.926±0.005
		0.25	0.907±0.005	0.931±0.002	0.928±0.003	0.932±0.006
		0.30	0.908±0.005	0.936±0.002	0.934±0.003	0.938±0.006
		0.40	0.909±0.004	0.946±0.002	0.944±0.003	0.951±0.006
		0.50	0.910±0.004	0.956±0.002	0.954±0.003	0.959±0.006
	0.4	0.00	0.838±0.004	0.833±0.001	0.837±0.004	0.836±0.008
		0.10	0.844±0.003	0.855±0.002	0.861±0.004	0.856±0.008
		0.15	0.846±0.003	0.865±0.002	0.873±0.004	0.881±0.007
		0.20	0.849±0.003	0.876±0.002	0.885±0.004	0.888±0.008
		0.25	0.853±0.003	0.887±0.003	0.897±0.005	0.895±0.011
		0.30	0.856±0.003	0.898±0.003	0.909±0.004	0.906±0.007
		0.40	0.861±0.003	0.920±0.003	0.934±0.005	0.937±0.010
		0.50	0.867±0.003	0.941±0.004	0.957±0.005	0.959±0.010
	0.8	0.00	0.848±0.002	0.787±0.007	0.852±0.010	0.820±0.011
		0.10	0.892±0.003	0.844±0.007	0.923±0.013	0.938±0.025
		0.15	0.915±0.003	0.872±0.009	0.966±0.014	0.964±0.015
		0.20	0.940±0.004	0.903±0.009	1.002±0.016	1.009±0.031
		0.25	0.964±0.004	0.934±0.009	1.041±0.019	1.032±0.024
		0.30	0.991±0.005	0.965±0.010	1.080±0.019	1.072±0.021
		0.40	1.044±0.005	1.028±0.012	1.164±0.022	1.163±0.041
		0.50	1.100±0.007	1.089±0.014	1.249±0.021	1.299±0.039

Table 8
Estimators for Server 2 of the U/M/2 Queue

p	θ	δ	IPA	SPA	LR	FD
0.0	0.2	0.00	0.905±0.005	0.906±0.002	0.902±0.003	0.902±0.006
		0.10	0.895±0.005	0.887±0.002	0.883±0.003	0.882±0.004
		0.15	0.891±0.005	0.878±0.002	0.874±0.003	0.873±0.006
		0.20	0.886±0.004	0.868±0.002	0.864±0.003	0.869±0.005
		0.25	0.882±0.004	0.859±0.002	0.855±0.003	0.852±0.008
		0.30	0.877±0.004	0.850±0.002	0.845±0.002	0.849±0.007
		0.40	0.867±0.004	0.831±0.002	0.826±0.002	0.826±0.009
		0.50	0.858±0.004	0.813±0.002	0.808±0.002	0.806±0.008
	0.4	0.00	0.838±0.004	0.833±0.001	0.837±0.004	0.836±0.008
		0.10	0.821±0.003	0.798±0.001	0.802±0.003	0.803±0.009
		0.15	0.814±0.003	0.781±0.002	0.785±0.003	0.789±0.009
		0.20	0.806±0.003	0.765±0.001	0.769±0.003	0.769±0.011
		0.25	0.798±0.003	0.749±0.001	0.752±0.002	0.752±0.012
		0.30	0.790±0.003	0.732±0.002	0.737±0.002	0.731±0.008
		0.40	0.775±0.003	0.701±0.002	0.706±0.002	0.700±0.011
		0.50	0.759±0.003	0.672±0.002	0.675±0.002	0.668±0.013
	0.8	0.00	0.848±0.002	0.787±0.007	0.852±0.010	0.820±0.011
		0.10	0.837±0.003	0.740±0.006	0.811±0.011	0.832±0.011
		0.15	0.831±0.002	0.717±0.006	0.793±0.012	0.799±0.027
		0.20	0.826±0.002	0.694±0.006	0.776±0.012	0.766±0.024
		0.25	0.820±0.002	0.673±0.007	0.758±0.012	0.758±0.016
		0.30	0.816±0.002	0.654±0.006	0.743±0.011	0.752±0.026
		0.40	0.804±0.003	0.613±0.007	0.707±0.010	0.691±0.025
		0.50	0.792±0.002	0.576±0.008	0.678±0.011	0.661±0.022
0.5	0.2	0.00	0.505±0.001	0.504±0.001	0.501±0.003	0.503±0.003
		0.10	0.502±0.001	0.498±0.001	0.499±0.004	0.499±0.007
		0.15	0.501±0.001	0.495±0.001	0.495±0.003	0.500±0.006
		0.20	0.500±0.001	0.493±0.001	0.491±0.002	0.496±0.008
		0.25	0.498±0.001	0.490±0.001	0.489±0.002	0.496±0.014
		0.30	0.497±0.001	0.487±0.001	0.487±0.003	0.484±0.015
		0.40	0.494±0.001	0.480±0.001	0.480±0.002	0.483±0.019
		0.50	0.490±0.001	0.474±0.001	0.475±0.003	0.478±0.020
	0.4	0.00	0.524±0.001	0.519±0.001	0.525±0.004	0.524±0.007
		0.10	0.519±0.001	0.506±0.001	0.513±0.003	0.508±0.010
		0.15	0.517±0.001	0.500±0.001	0.506±0.004	0.506±0.012
		0.20	0.513±0.002	0.493±0.002	0.501±0.004	0.494±0.018
		0.25	0.511±0.001	0.487±0.001	0.494±0.004	0.492±0.015
		0.30	0.508±0.001	0.480±0.001	0.488±0.004	0.492±0.026
		0.40	0.503±0.001	0.467±0.001	0.478±0.004	0.486±0.032
		0.50	0.498±0.001	0.454±0.001	0.467±0.004	0.476±0.035
	0.8	0.00	0.665±0.003	0.610±0.004	0.667±0.010	0.645±0.012
		0.10	0.664±0.004	0.586±0.004	0.655±0.011	0.656±0.018
		0.15	0.663±0.003	0.574±0.005	0.644±0.011	0.609±0.025
		0.20	0.663±0.003	0.562±0.004	0.638±0.011	0.597±0.026
		0.25	0.661±0.004	0.549±0.005	0.628±0.010	0.623±0.032
		0.30	0.660±0.004	0.538±0.004	0.624±0.011	0.629±0.050
		0.40	0.658±0.004	0.514±0.005	0.609±0.014	0.580±0.022
		0.50	0.654±0.003	0.490±0.004	0.594±0.013	0.600±0.065
1.0	0.2	0.00	0.104±0.005	0.103±0.004	0.104±0.006	0.104±0.005
		0.10	0.103±0.005	0.102±0.004	0.104±0.006	0.103±0.005
		0.15	0.103±0.005	0.102±0.004	0.104±0.006	0.102±0.005
		0.20	0.103±0.005	0.101±0.004	0.104±0.006	0.102±0.005
		0.25	0.103±0.005	0.101±0.004	0.103±0.007	0.102±0.004
		0.30	0.103±0.005	0.100±0.004	0.103±0.007	0.101±0.005
		0.40	0.102±0.004	0.099±0.004	0.103±0.007	0.102±0.004
		0.50	0.102±0.004	0.099±0.004	0.103±0.007	0.101±0.006
	0.4	0.00	0.210±0.004	0.205±0.004	0.213±0.005	0.210±0.007
		0.10	0.209±0.004	0.201±0.004	0.211±0.005	0.207±0.005
		0.15	0.208±0.004	0.199±0.003	0.210±0.005	0.209±0.005
		0.20	0.208±0.004	0.197±0.003	0.209±0.005	0.205±0.006
		0.25	0.207±0.004	0.196±0.003	0.208±0.005	0.205±0.004
		0.30	0.207±0.004	0.193±0.003	0.208±0.005	0.199±0.005
		0.40	0.206±0.004	0.190±0.003	0.205±0.005	0.202±0.006
		0.50	0.205±0.004	0.186±0.003	0.203±0.006	0.199±0.006
	0.8	0.00	0.480±0.005	0.419±0.004	0.485±0.012	0.457±0.011
		0.10	0.483±0.005	0.406±0.004	0.484±0.011	0.483±0.015
		0.15	0.485±0.006	0.400±0.004	0.480±0.012	0.480±0.009
		0.20	0.487±0.005	0.393±0.005	0.480±0.013	0.487±0.020
		0.25	0.488±0.006	0.387±0.005	0.480±0.012	0.474±0.014
		0.30	0.488±0.005	0.379±0.005	0.479±0.011	0.475±0.020
		0.40	0.490±0.006	0.366±0.005	0.474±0.012	0.480±0.023
		0.50	0.490±0.005	0.351±0.005	0.470±0.015	0.463±0.020

Table 9
Estimators for Server 1 of the U/We/2 Queue

θ	p	δ	IPA	SPA	LR	FD
0.5	0.0	0.00	-0.098±0.007	-0.026±0.020	-0.105±0.017	-0.105±0.011
		0.10	-0.080±0.006	-0.011±0.011	-0.084±0.016	-0.080±0.007
		0.15	-0.075±0.006	0.078±0.069	-0.078±0.011	-0.073±0.006
		0.20	-0.071±0.006	-0.006±0.016	-0.073±0.012	-0.072±0.006
		0.25	-0.067±0.006	0.005±0.026	-0.069±0.008	-0.070±0.008
		0.30	-0.064±0.005	0.029±0.043	-0.067±0.007	-0.066±0.010
		0.40	-0.061±0.005	0.028±0.019	-0.066±0.007	-0.061±0.009
		0.50	-0.059±0.005	0.049±0.057	-0.062±0.008	-0.062±0.005
	0.5	0.00	-0.171±0.005	-0.140±0.006	-0.164±0.023	-0.178±0.009
		0.10	-0.159±0.004	-0.122±0.004	-0.172±0.020	-0.136±0.037
		0.15	-0.151±0.004	-0.105±0.007	-0.154±0.019	-0.156±0.064
		0.20	-0.147±0.004	-0.102±0.004	-0.154±0.016	-0.101±0.081
		0.25	-0.142±0.003	-0.089±0.006	-0.141±0.015	-0.171±0.069
		0.30	-0.140±0.004	-0.085±0.006	-0.150±0.017	-0.193±0.088
		0.40	-0.135±0.003	-0.075±0.005	-0.147±0.012	-0.227±0.058
		0.50	-0.130±0.003	-0.059±0.010	-0.135±0.012	-0.214±0.110
	1.0	0.00	-0.249±0.004	-0.315±0.024	-0.257±0.022	-0.250±0.008
		0.10	-0.232±0.004	-0.252±0.007	-0.242±0.023	-0.225±0.018
		0.15	-0.225±0.003	-0.233±0.006	-0.232±0.022	-0.227±0.017
		0.20	-0.221±0.004	-0.217±0.006	-0.227±0.019	-0.212±0.017
		0.25	-0.216±0.003	-0.201±0.006	-0.222±0.017	-0.205±0.011
		0.30	-0.212±0.003	-0.189±0.005	-0.217±0.016	-0.216±0.011
		0.40	-0.206±0.002	-0.169±0.005	-0.212±0.015	-0.210±0.013
		0.50	-0.202±0.003	-0.153±0.004	-0.206±0.016	-0.197±0.009
2.0	0.0	0.00	0.029±0.000	-Inf± NaN	0.029±0.000	0.029±0.000
		0.10	0.031±0.000	1.019±0.027	0.031±0.000	0.030±0.000
		0.15	0.032±0.000	0.731±0.017	0.032±0.000	0.032±0.001
		0.20	0.032±0.000	0.591±0.013	0.033±0.000	0.033±0.000
		0.25	0.033±0.000	0.511±0.013	0.034±0.000	0.033±0.001
		0.30	0.034±0.000	0.453±0.009	0.034±0.000	0.035±0.000
		0.40	0.036±0.000	0.390±0.011	0.036±0.000	0.036±0.001
		0.50	0.037±0.000	0.350±0.008	0.037±0.000	0.038±0.001
	0.5	0.00	0.079±0.000	-Inf± NaN	0.079±0.000	0.080±0.001
		0.10	0.080±0.000	1.128±0.006	0.081±0.000	0.080±0.004
		0.15	0.081±0.000	0.814±0.004	0.081±0.000	0.080±0.002
		0.20	0.081±0.000	0.657±0.004	0.082±0.000	0.081±0.003
		0.25	0.081±0.000	0.566±0.005	0.083±0.000	0.077±0.004
		0.30	0.082±0.000	0.502±0.003	0.083±0.000	0.088±0.008
		0.40	0.082±0.000	0.426±0.002	0.084±0.000	0.079±0.005
		0.50	0.083±0.000	0.381±0.002	0.086±0.001	0.083±0.010
	1.0	0.00	0.130±0.000	-Inf± NaN	0.130±0.001	0.131±0.001
		0.10	0.130±0.000	1.787±0.005	0.132±0.001	0.133±0.001
		0.15	0.131±0.000	1.285±0.004	0.133±0.001	0.133±0.002
		0.20	0.131±0.000	1.033±0.004	0.134±0.001	0.133±0.002
		0.25	0.131±0.000	0.883±0.004	0.135±0.001	0.136±0.001
		0.30	0.131±0.000	0.781±0.003	0.136±0.001	0.134±0.001
		0.40	0.131±0.000	0.655±0.002	0.138±0.001	0.136±0.002
		0.50	0.131±0.000	0.579±0.002	0.139±0.001	0.139±0.001

Table 10
Estimators for Server 2 of the U/We/2 Queue

θ	p	δ	IPA	SPA	LR	FD
0.5	0.0	0.00	-0.249±0.004	-0.315±0.024	-0.257±0.022	-0.250±0.008
		0.10	-0.109±0.004	-0.133±0.006	-0.119±0.018	-0.110±0.009
		0.15	-0.062±0.004	-0.077±0.004	-0.070±0.018	-0.054±0.015
		0.20	-0.023±0.003	-0.031±0.003	-0.032±0.016	-0.029±0.004
		0.25	0.009±0.003	0.005±0.003	-0.000±0.014	0.012±0.007
		0.30	0.033±0.002	0.032±0.002	0.025±0.014	0.033±0.005
		0.40	0.069±0.002	0.069±0.002	0.064±0.011	0.066±0.009
		0.50	0.093±0.001	0.093±0.002	0.088±0.007	0.093±0.007
	0.5	0.00	-0.177±0.005	-0.140±0.010	-0.198±0.030	-0.180±0.009
		0.10	-0.080±0.005	-0.057±0.004	-0.078±0.010	-0.102±0.057
		0.15	-0.048±0.003	-0.029±0.004	-0.057±0.011	-0.065±0.050
		0.20	-0.022±0.003	-0.005±0.003	-0.026±0.014	-0.045±0.044
		0.25	-0.001±0.003	0.012±0.002	-0.011±0.013	0.012±0.064
		0.30	0.017±0.002	0.027±0.003	0.016±0.012	-0.007±0.079
		0.40	0.042±0.002	0.047±0.002	0.041±0.007	0.028±0.069
		0.50	0.057±0.001	0.059±0.001	0.052±0.005	0.068±0.093
	1.0	0.00	-0.098±0.007	-0.026±0.020	-0.105±0.017	-0.105±0.011
		0.10	-0.051±0.006	-0.004±0.008	-0.049±0.012	-0.047±0.005
		0.15	-0.034±0.004	-0.001±0.008	-0.035±0.010	-0.038±0.008
		0.20	-0.021±0.004	0.005±0.011	-0.021±0.010	-0.020±0.004
		0.25	-0.010±0.004	0.009±0.007	-0.010±0.009	-0.011±0.009
		0.30	-0.001±0.003	0.021±0.009	-0.002±0.009	-0.000±0.003
		0.40	0.012±0.002	0.028±0.008	0.012±0.006	0.009±0.005
		0.50	0.021±0.002	0.029±0.003	0.020±0.006	0.021±0.004
2.0	0.0	0.00	0.130±0.000	-Inf± NaN	0.130±0.001	0.131±0.001
		0.10	0.117±0.000	1.626±0.006	0.116±0.001	0.115±0.002
		0.15	0.111±0.000	1.114±0.004	0.109±0.001	0.111±0.001
		0.20	0.106±0.000	0.853±0.003	0.103±0.001	0.103±0.001
		0.25	0.101±0.000	0.694±0.002	0.098±0.001	0.098±0.001
		0.30	0.096±0.000	0.586±0.002	0.093±0.001	0.094±0.001
		0.40	0.087±0.000	0.447±0.001	0.083±0.000	0.083±0.001
		0.50	0.080±0.000	0.361±0.001	0.075±0.000	0.075±0.001
	0.5	0.00	0.079±0.000	-Inf± NaN	0.080±0.000	0.080±0.001
		0.10	0.072±0.000	1.020±0.006	0.071±0.000	0.071±0.004
		0.15	0.069±0.000	0.703±0.004	0.068±0.000	0.068±0.002
		0.20	0.065±0.000	0.538±0.003	0.065±0.000	0.067±0.004
		0.25	0.062±0.000	0.440±0.002	0.061±0.000	0.062±0.003
		0.30	0.060±0.000	0.372±0.002	0.058±0.000	0.060±0.007
		0.40	0.054±0.000	0.285±0.001	0.053±0.000	0.049±0.009
		0.50	0.050±0.000	0.232±0.001	0.048±0.000	0.050±0.005
	1.0	0.00	0.029±0.000	-Inf± NaN	0.029±0.000	0.029±0.000
		0.10	0.026±0.000	0.856±0.014	0.026±0.000	0.026±0.000
		0.15	0.025±0.000	0.565±0.009	0.025±0.000	0.024±0.001
		0.20	0.024±0.000	0.420±0.010	0.024±0.000	0.023±0.001
		0.25	0.023±0.000	0.331±0.007	0.023±0.000	0.022±0.000
		0.30	0.022±0.000	0.270±0.005	0.021±0.000	0.021±0.001
		0.40	0.020±0.000	0.196±0.006	0.020±0.000	0.020±0.001
		0.50	0.018±0.000	0.151±0.003	0.018±0.000	0.018±0.001

References

[1] Cao, X.R., "Convergence of Parameter Sensitivity Estimates in a Stochastic Experiment," *IEEE Transactions on Automatic Control*, **AC-30**, pp.845-853, 1985.

[2] Cao, X.R., "A Sample Performance Function of Closed Jackson Queueing Networks," *Operations Research*, **36**, pp.128-136, 1988.

[3] Fu, M.C., "Optimization Via Simulation: A Review," to appear in *Annals of Operations Research*, 1994.

[4] Fu, M.C. and Hu, J.Q., "Bias Properties of Infinitesimal Perturbation Analysis for Multi-Server Queues," *Proceedings of the Winter Simulation Conference*, pp.377-381, 1990.

[5] Fu, M.C. and Hu, J.Q., "Consistency of Infinitesimal Perturbation Analysis for the GI/G/m Queue," *European Journal of Operational Research*, Vol.54, No.1, pp.121-139, 1991.

[6] Fu, M.C. and Hu, J.Q., "Extensions and Generalizations of Smoothed Perturbation Analysis in a Generalized Semi-Markov Process Framework," *IEEE Trans. Automatic Control*, **37**, pp.1483-1500, 1992.

[7] Fu, M.C. and Hu, J.Q., "Second Derivative Sample Path Estimators for the GI/G/m Queue," to appear in *Management Science*, 1993.

[8] Fu, M.C., Hu, J.Q., and Nagi, R., "Bias Properties of Infinitesimal Perturbation Analysis for Systems with Parallel Servers," *Computers & Operations Research*, Vol.19, No.5, pp.409-423, 1992.

[9] Glasserman, P., *Gradient Estimation Via Perturbation Analysis*, Kluwer Academic, 1991.

[10] Glynn, P. W. "Likelihood Ratio Gradient Estimation for Stochastic Systems", *Communications of the ACM*, **33**, 10 (1990), 75–84.

[11] Gong, W.B. and Ho, Y.C., "Smoothed Perturbation Analysis of Discrete-Event Dynamic Systems," *IEEE Transactions on Automatic Control*, **AC-32**, pp.858-867, 1987.

[12] Heidelberger, P., Cao, X.R., Zazanis, M.S., and Suri, R., "Convergence Properties of Infinitesimal Perturbation Analysis Estimates," *Management Science*, **34**, pp.1281-1302, 1988.

[13] Ho, Y.C. and Cao, X.R., *Discrete Event Dynamic Systems and Perturbation Analysis*, Kluwer Academic, 1991.

[14] L'Ecuyer, P., "An Overview of Derivative Estimation," *Proceedings of the Winter Simulation Conference*, pp.207-217, 1991.

[15] L'Ecuyer, P., "A Unified View of the IPA, SF, and LR Gradient Estimation Techniques," *Management Science*, pp. 1364-1383, 1990.

[16] Reiman, M.I. and Weiss, A., "Sensitivity Analysis for Simulations via Likelihood Ratios," *Operations Research,* **37**, pp.830-844, 1989.

[17] Rubinstein, R.Y., "Sensitivity Analysis and Performance Extrapolation for Computer Simulation Models," *Operations Research,* **37**, pp.72-81, 1989.

[18] Suri, R. and Zazanis, M.A., "Perturbation Analysis Gives Strongly Consistent Sensitivity Estimates for the M/G/1 Queue," *Management Science*, **34**, pp.39-64, 1988.

[19] Vázquez-Abad, F.J. and L'Ecuyer, P., "Comparing Alternative Methods for Derivative Estimation When IPA Does Not Apply Directly," *Proceedings of the Winter Simulation Conference*, pp.1004-1011, 1991.