Technical Research Report

S Y S T E M S
R E S E A R C H
C E N T E R

Supported by the National Science Foundation Engineering Research Center Program (NSFD CD 8803012), the University of Maryland, Harvard University, and Industry

Stabilization of Globally Noninteractive Nonlinear Systems via Dynamic State-Feedback

by S. Battilotti and W.P. Dayawansa

TR 91-63

STABILIZATION OF GLOBALLY NONINTERACTIVE NONLINEAR SYSTEMS VIA DYNAMIC STATE–FEEDBACK

S. Battilotti †
Dipartimento di Informatica e Sistemistica
Via Eudossiana 18 - 00184 - Roma (Italy)

W. P. Dayawansa ††
System Research Center and Department of Electrical Engineering
University of Maryland - College Park - MD 20742 (USA)

Abstract

We consider the problem of semiglobal asymptotic stabilization and noninteracting control via dynamic state–feedback for a class of nonlinear control systems. It is assumed that the plant has been already rendered noninteractive. A sufficient condition for the stabilization of the overall system, without destroying the noninteraction property, is given in terms of stabilizability of certain subsystems.

Keywords : noninteraction, global stability.

(AMS) subject classification: 93

---

† Research was supported by Ministero dell'Universita' e della Ricerca Scientifica e Tecnologica.

†† Research was partially supported by NSF grant ECS 9096121 and the National Science Foundation Engineering Research Center Program NSFD CD 8803012 and a grant from the Minta Martin Fund at the University of Maryland.

I. Introduction

Let us consider a square, affine nonlinear control system

$$\begin{aligned} \dot{x} &= f(x) + \sum_{i=1}^{m} g_i(x) u_i \\ y_i &= h_i(x) \qquad i = 1, \ldots, m, \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ and $u_i(t) \in \mathbb{R}$, f and g_i are smooth vector fields on $\mathbb{R}^n$, $h_i : \mathbb{R}^n \to \mathbb{R}$ are smooth scalar functions for $i = 1, \ldots, m$. We assume that $x_0 = 0$ is such that $f(x_0) = 0$. Moreover, throughout the paper, we set

$$\begin{aligned} h(x) &= (\, h_1(x) \quad \ldots \quad h_m(x) \,)^T \\ g(x) &= (\, g_1(x) \quad \ldots \quad g_m(x) \,) \end{aligned}$$

$$\begin{aligned} G &: \text{ the smooth distribution spanned by the columns of } g(x) \\ K_i &: \text{ the distribution which annihilates } \operatorname{span}\{dh_i\} \,. \end{aligned}$$

We also suppose that G and K_i have constant dimensions m and $n-1$ respectively.

We say that (1) is noninteractive if the i–th input does not influence the j–th output for $j \neq i$. The problem of noninteracting control with stability consists of finding a state–feedback control law such that the closed–loop system resulting from (1) is noninteractive and asymptotically stable at x_0.

Both linear and nonlinear noninteracting control problems have been widely discussed in the literature ([1,3,4,5,6,7,8,9,11,12,13,14,15,16,17,18,19]). Unlike the linear case, only recently some necessary ([20,21]) and sufficient ([22,23]) conditions have been given to guarantee that the resulting noninteractive closed–loop system is also asymptotically stable. In [22] it is shown how to extend to a nonlinear setting the construction given in [3] to solve the problem of linear noninteracting control with stability. In the linear case, we know that if we find a set of m subspaces $R_i \subset \cap_{j \neq i} K_j$ which can be made simultaneously invariant by means of static state–feedback, then the problem of noninteracting control is solvable. Even if these subspaces are stabilizable and the plant is controllable, it is not possible in general to obtain asymptotic stability as well as noninteraction by means of static feedback alone [1]. The idea which underlies [3] is to use dynamic feedback to define an "extended system" and an associated set of m independent controllability subspaces R_i^e which can be made simultaneously invariant and, thus, asymptotically stable.

In this paper, for the class of globally noninteractive systems (1), it is shown that it is possible to carry over to a global setting the results contained in [22]. Additionally the proof given here allows us to state the local results proven in [22] under weaker hypotheses. Roughly speaking, on the original system we can define a set of m subsystems, which essentially correspond to the decoupled channels. In general these subsystems cannot be simultaneously stabilized without destroying the noninteraction property, since they share some common parts. By using dynamic feedback, we can embed these subsystem into a larger state space in order to "separate" them , thus allowing us to stabilize them separately so that stability can be achieved for the overall system without affecting noninteraction.

The paper is organized as follows. In Section II some important concepts are reviewed. In Section III, by using dynamic extension, we define a set of "extended" distributions, which are crucial to the solution of our problem. In Section IV we explicitly construct the dynamic feedback which renders (1) asymptotically stable and noninteractive.

II. Basic notations and concepts

In this section we introduce some concepts and notation we use later (the interested reader is referred to [24, 27, 29, 30]).

In what follows, if θ is any smooth vector field and Δ is any smooth distribution, by $[\theta, \Delta]$ we mean the set $\{[\theta, \tau] : \tau \in \Delta\}$, where $[\cdot, \cdot]$ is the Lie bracket of any two smooth vector fields.

A smooth distribution Δ is said to be globally invariant under a smooth vector field θ if $[\theta, \Delta] \subset \Delta$.

A smooth distribution Δ is said to be globally weakly controlled invariant if

$$\begin{aligned} &[f, \Delta] \subset \Delta + G \\ &[g_j, \Delta] \subset \Delta + G \qquad j = 1, \ldots, m. \end{aligned}$$

For a given set of smooth vector fields $\{\tau_1, \ldots, \tau_m\}$ and a given smooth distribution K, we denote by $\langle \tau_1, \ldots, \tau_m | K \rangle$ the smallest distribution which contains K and is invariant under $f, g_1, \ldots, g_m$ (see [24] for proof of existence)

Lemma II.1. Let us define the following sequence of distributions,

$$\begin{aligned} S_0 &= K \\ S_k &= \sum_{i=1}^{m} [\tau_i, S_{k-1}] + S_{k-1} \ . \end{aligned}$$

If there exists $k^* < n$ such that $S_{k^*} = S_{k^*+1}$ (in this case, we say that $\langle \tau_1, \ldots, \tau_m | K \rangle$ is *finitely computable*), then

$$S_{k^*} = \langle \tau_1, \ldots, \tau_m | K \rangle \ .$$

◇

Actually, in the above sequence, we implicitly suppose that each S_k is a smooth distribution (see [24] for a more general setting). If each S_k is a nonsingular distribution, then $\langle \tau_1, \ldots, \tau_m | K \rangle$ is finitely computable. In this case, supposing that $K = \text{span}\{\tau_j : j \in J \subset \{1, \ldots, m\}\}$ and setting $f = g_0$, by induction, it can be shown easily that the distributions S_k are locally spanned by vector fields in the set

$$\{\theta : \theta = \tau_j \text{ or } \theta = [\tau_{i_k}, \ldots [\tau_{i_1}, \tau_j] \ldots] : 1 \le k \le n-1; 0 \le i_h \le m \ ; \ 1 \le h \le k \text{ and } j \in J\} \ ,$$

and $\langle \tau_1, \ldots, \tau_m | K \rangle$ is involutive. It should be noted, however, that, even in this case, in general it is impossible to find an everywhere linearly independent set of vector fields spanning $\langle \tau_1, \ldots, \tau_m | K \rangle$. This must be taken into account in what follows.

Moreover, throughout the paper, denoting by Δ any smooth nonsingular involutive distribution, we use the following notation.

$$\begin{aligned} \mathcal{L}_x^\Delta &: \text{leaf of } \Delta \text{ passing through } x \\ \mathcal{F}^\Delta &: \text{foliation of } \Delta \,. \end{aligned}$$

We say that (1) is semiglobally stabilizable to x_0 if for all bounded sets $\Omega \ni x_0$ of $I\!R^n$ there exists a static state–feedback (preserving the equilibrium point) which makes x_0 an asymptotically stable equilibrium for the resulting closed–loop system and Ω is contained in the basin of attraction [29].

Finally, (1) is said to be convergent input bounded state (CIBS) if for each control $u(t)$ on $[0, +\infty)$ such that $u(t) \to 0$ as $t \to \infty$ and for each $\bar{x}$, the solution of (1) with $x(0) = \bar{x}$ exists for all $t \geq 0$ and is bounded [30].

Throughout the paper, we also implicitly suppose that any feedback control law we consider preserves the equilibrium point at the origin. In particular, we consider static feedback laws

$$u = \alpha(x) + \beta(x)v \,,$$

with $\alpha(0) = 0$ and $\beta(x)$ nonsingular (in this case the feedback law is said to be *regular*), and dynamic feedback laws

$$\begin{aligned} u &= \alpha(x, w) + \beta(x, w)v \\ \dot{w} &= \gamma(x, w) + \delta(x, w)v \,, \end{aligned}$$

with $\alpha(0, 0) = 0$ and $\gamma(0, 0) = 0$.

III. Fundamental assumptions and definitions

Set

$$\begin{aligned} P_i^* &= \langle f, g_1, \ldots, g_m | \text{span}\{g_j : j \neq i\} \rangle \qquad i = 1, \ldots, m \\ P_0 &= \langle f, g_1, \ldots, g_m | G \rangle \end{aligned}$$

and

$$P^* = \bigcap_{i=1}^{m} P_i^* \,.$$

We assume that P_i^* and P_0 can be computed by means of standard algorithms (see Section II). For this purpose, we assume that each S_k (see Lemma II.1) has constant dimension and throughout the paper, by saying globally finitely computable, we mean this.

We assume that our system has already been rendered noninteractive (without stability). We wish to find conditions under which there exist a dynamic state–feedback law which semiglobally stabilizes the system and at the same time preserves the noninteraction property. A typical case in which regular static state–feedback alone cannot help us is the following. Suppose that the free dynamics of (1) on $\mathcal{L}_0^{P^*}$ (whenever it is well–defined) is unstable at the origin. In this case, it was shown in [17] that in a neighbourhood of the origin we cannot make the system noninteractive and asymptotically stable at the same time by means of regular static state–feedback.

Generally, under suitable regularity assumptions, by changing coordinates in a neighbourhood of the origin, it is possible to put a noninteractive system in a simple standard form (see [17]) . Unfortunately, this is not true anymore in a global setting. As it will be seen later, we require a suitable noninteractive form to be defined globally.

We also denote by $\mathcal{I}$ the Lie ideal generated by the vector fields $\{[g_j, ad_f^k g_i] : i, j = 1, \ldots, m; k \geq 0 \text{ and } i \neq j\}$ in the Lie algebra generated by $\{f, g_1, \ldots, g_m\}$ and define

$$\Delta_{MIX} = \text{span}\{\tau : \tau \in \mathcal{I}\} .$$

Throughout the paper, we assume that Δ_{MIX} is nonsingular. Now $\mathcal{L}_0^{\Delta_{MIX}}$ is an invariant subset of the free dynamics of (1). The restriction of (1) to $\mathcal{L}_0^{\Delta_{MIX}}$ is usually referred to as the Δ_{MIX} dynamics. It has been shown in [20] that, for any dynamic feedback control law such that the closed–loop system resulting from (1) has still some relative degree and is noninteractive, the resulting system admits an invariant subset for the free dynamics , and the restriction of the free dynamics to this invariant set is diffeomorphic to the Δ_{MIX} dynamics (for details see [20,21,22]).

Thus, we collect all the previous facts in the two following assumptions.

Assumption 1. The distributions P_i and P_0 are globally finitely computable and $P_0, P_i, P^*$and Δ_{MIX} $\quad i = 1, \ldots, m$ have constant dimension for all x. Moreover, $\dim P_0 = n$ for all x.

Assumption 2. (1) is globally noninteractive and there exist global coordinates $(x_1, \ldots, x_m)$ such that (1) has the form

$$\begin{aligned}
\dot{x}_i &= f_i(x_i) + g_{ii}(x_i)u_i \qquad i = 1, \ldots, m \\
\dot{x}_{m+1} &= f_{m+1}(x_1, \ldots, x_m, x_{m+1}) + \sum_{j=1}^{m} g_{m+1,j}(x_1, \ldots, x_m, x_{m+1})u_j \\
\dot{x}_{m+2} &= f_{m+2}(x) + \sum_{j=1}^{m} g_{m+2,j}(x)u_j \\
y_i &= h_i(x_i) \qquad i = 1, \ldots, m
\end{aligned} \tag{2}$$

with

$$\begin{aligned}
P_i^* &= \text{span}\{\partial/\partial x_j : j \neq i\} \qquad i = 1, \ldots, m \\
P^* &= \text{span}\{\partial/\partial x_{m+1}, \partial/\partial x_{m+2}\} \\
P_0 &= \text{span}\{\partial/\partial x_i : i = 1, \ldots, m+2\} \\
\Delta_{MIX} &= \text{span}\{\partial/\partial x_{m+2}\} \\
P_i^* \cap G &= \text{span}\{g_i : j \neq i\} \qquad i = 1, \ldots, m \\
P^* \cap G &= 0 .
\end{aligned}$$

The requirement $P^* \cap G = 0$ is a necessary condition for (2) to have some relative degree at x_0 [17], which is a sufficient condition to achieve local noninteracting control [24], and its role will be clear later. However, this assumption can be weakened as in [23], but for simplicity we do not do so here. Note that under Assumption 2 the free dynamics of (2) on $\mathcal{L}_0^{P^*}$ (which in what follows we call P^* *dynamics*) is globally defined and is given by

$$\begin{aligned}\dot{x}_{m+2} &= f_{m+2}(0,\ldots,x_{m+1},x_{m+2})\\ \dot{x}_{m+1} &= f_{m+1}(0,\ldots,x_{m+1})\,.\end{aligned}$$

Also the the Δ_{MIX} *dynamics*, is globally defined and given by

$$\dot{x}_{m+2} = f_{m+2}(0,\ldots,x_{m+2})\,.$$

Note also that, under our assumptions, $\mathcal{L}_x^{P_i^*}, \mathcal{L}_x^{P_0}$ and $\mathcal{L}_x^{P^*}$ are globally diffeomorphic to Euclidean spaces and that $\dim P_0 = n$, i.e. we require strong accessibility at each point (in the linear case this is equivalent to full controllability; for the case $\dim P_0 \neq n$, see [22]). The quotient $\mathbb{R}^n/\mathcal{F}^{\Delta_{MIX}}$ can be identified with $\mathbb{R}^{\widehat{n}}$, where $\widehat{n} = n - \dim(\Delta_{MIX})$, by omitting the x_{m+2} coordinates from $\mathbb{R}^n$. Moreover, there exist vector fields $\widehat{f}$ and $\widehat{g}_j$, $j = 1,\ldots,m$, defined on $\mathbb{R}^n/\mathcal{F}^{\Delta_{MIX}}$ such that, if $\sigma : \mathbb{R}^n \to \mathbb{R}^n/\mathcal{F}^{\Delta_{MIX}}$ is the canonical projection and σ_* its differential,

$$\begin{aligned}\widehat{f}\circ\sigma(x) &= \sigma_*(f(x))\\ \widehat{g}_j\circ\sigma(x) &= \sigma_*(g_j(x)) \qquad j = 1,\ldots,m.\end{aligned}$$

The vector fields $\widehat{f}$ and $\widehat{g}_j$ can be identified with those obtained from f and g_j in (2) by omitting the x_{m+2} coordinates from $\mathbb{R}^n$. Thus, we can define the following set of smooth distributions on $\mathbb{R}^n/\mathcal{F}^{\Delta_{MIX}}$

$$\widehat{R}_i = \langle \widehat{f}, \widehat{g}_1, \ldots, \widehat{g}_m | \mathrm{span}\{\widehat{g}_i\}\rangle \qquad i = 1,\ldots,m.$$

Let $\widehat{f} = \widehat{g}_0$. We will assume that the distributions $\widehat{R}_i$, $i = 1,\ldots,m$, are constant dimensional and finitely computable. Then from Lemma II.1 (see also Lemma I.8.6 in [24] and our remarks in Section II) it follows that there exist vector fields $\{\widehat{X}_{ik}\}$ in the set

$$\{\theta : \theta = \widehat{g}_i \text{ or } \theta = [\widehat{g}_{j_h}, \ldots, [\widehat{g}_{j_1}, \widehat{g}_i]\ldots] \;:\; 1 \le h \le n-1\;;\; 0 \le j_k \le m\;;\; 1 \le k \le h\} \qquad (3)$$

such that

$$\widehat{R}_i = \mathrm{span}\{\widehat{X}_{ik} : 1 \le k \le s_i\} \qquad i = 1,\ldots,m \qquad (4)$$

for some s_i such that $s_i \ge \dim\widehat{R}_i$. Note that s_i is finite, since $\widehat{R}_i$ is finitely computable. Using the global form (2), and setting $\widehat{x} = \sigma(x)$, it is possible to show by induction that

$$\widehat{X}_{ik} = (\partial/\partial x_i)\widehat{Y}_{ik}(x_i) + (\partial/\partial x_{m+1})\widehat{Z}_{ik}(\widehat{x}) \qquad i = 1,\ldots,m, \qquad (5)$$

where $\widehat{Y}_{ik}$ and $\widehat{Z}_{ik}$ are respectively the x_i and x_{m+1} components of $\widehat{X}_{ik}$. In general these vector fields are not everywhere independent. Collecting the above assumptions, we have the following

Assumption 3. The distributions $\widehat{R}_i, \sum_{j\neq i} \widehat{R}_j$, $i = 1, \ldots, m$, and $\sum_{i=1}^m \widehat{R}_i$ are constant dimensional. Moreover, $\widehat{R}_i$ is globally finitely computable and the vector fields $\{\widehat{X}_{ik} : 1 \leq k \leq s_i\}$ are complete.

Note that Assumption 3 does not require the existence of a set of everywhere independent vector fields which span $\widehat{R}_i$.

We need now an auxiliary result. We denote by $\widehat{P}_i$ and $\widehat{P}_0$ the distributions which assign to each point $\sigma(x)$ the subspaces $(\sigma_*)_x(P_i)$ and $(\sigma_*)_x(P_0)$ respectively and by $\widehat{\Delta}_{MIX}$ the distribution on $\mathbb{R}^n/\mathcal{F}^{\Delta_{MIX}}$, defined in the same way as Δ_{MIX} but with f and g_j replaced by $\widehat{f}$ and $\widehat{g}_j$, $j = 1, \ldots, m$.

Lemma III.1.

$$\widehat{P}_0 = \sum_{i=1}^m \widehat{R}_i = \text{span}\{\partial/\partial x_j : j \neq (m+2)\} = \langle \widehat{f}, \widehat{g}_1, \ldots, \widehat{g}_m | \text{span}\{\widehat{g}_i : i = 1, \ldots, m\}\rangle$$

$$\widehat{P}_i = \sum_{j\neq i} \widehat{R}_j = \text{span}\{\partial/\partial x_j : j \notin \{i, (m+2)\}\} = \langle \widehat{f}, \widehat{g}_1, \ldots, \widehat{g}_m | \text{span}\{\widehat{g}_j : j \neq i\}\rangle$$

$$i = 1, \ldots, m$$

$$\widehat{\Delta}_{MIX} = 0\ .$$

◇

Proof. As in [22, Lemma III.1].•

Note that $\dim(\text{span}\{\widehat{g}_1, \ldots, \widehat{g}_m\}) = m$, since otherwise $P^* \cap G \neq 0$ (assumption $P^* \cap G = 0$ is crucial here). Note also that, since $\widehat{R}_i$ is nonsingular, it is also involutive ([23, Lemma 1.8.8]). From Lemma III.1 it follows that $\widehat{R}_i \subset \cap_{j\neq i} \sum_{h\neq j} \widehat{R}_h = \text{span}\{\partial/\partial x_i, \partial/\partial x_{m+1}\}$, hence for each i there exist local coordinates $(x_1^T, \ldots, x_{i-1}^T, x_{i+1}^T, \ldots, x_m^T, \varphi_i^T, \bar{\varphi}_i^T)^T$ (depending on i) such that

$$\widehat{R}_i = \text{span}\{\partial/\partial \varphi_i\}\qquad,$$

(note that the x_j, $j \neq \{i, m+1, m+2\}$, coordinates are the same as in (2) and that φ_i and $\bar{\varphi}_i$ depend only on $\widehat{x}$). This result does not hold in general on all of $\widehat{\mathbb{R}^n}$. With this in mind, we make the following assumption

Assumption 4. For each $i = 1, \ldots, m$ there exist global coordinates

$$(x_1^T, \ldots, x_{i-1}^T, x_{i+1}^T, \ldots, x_m^T, \varphi_i^T, \bar{\varphi}_i^T)^T$$

such that

$$\widehat{R}_i = \text{span}\{\partial/\partial\varphi_i\} \qquad i = 1, \ldots, m$$

and $(x_1^T, \ldots, x_{i-1}^T, x_{i+1}^T, \ldots, x_m^T, \varphi_i^T, \bar{\varphi}_i^T)^T : I\!R^{\widehat{n}} \to I\!R^{\widehat{n}}$ is onto.

We define now certain subsystems, which play a key role in the solution of our problem. Note that, since $\widehat{g}_i \in \widehat{R}_i$, the restriction of $\widehat{g}_i$ to $\mathcal{L}_0^{\widehat{R}_i}$, $i = 1, \ldots, m$, is a well–defined smooth vector fields on $\mathcal{L}_0^{\widehat{R}_i}$. Since $\widehat{f}(0) = 0$ and $\widehat{R}_i$ is invariant under $\widehat{f}$, it follows that $\widehat{f}$ restricts to a well–defined vector field on $\mathcal{L}_0^{\widehat{R}_i}$ as well. Thus, the following subsystems are well–defined

$$\widehat{\Sigma}_i : (\widehat{f}|\mathcal{L}_0^{\widehat{R}_i}, \widehat{g}_i|\mathcal{L}_0^{\widehat{R}_i}) \qquad i = 1, \ldots, m. \tag{6}$$

In the sequel, having set $\widehat{u} = (\, x_1^T, \;\; \cdots \;\; , x_m^T, \;\; x_{m+1}^T \,)^T$ and $\bar{u} = (\, u^T, \;\; \widehat{u}^T \,)^T$, we also refer to the following system

$$\begin{aligned} \dot{x}_{m+2} &= \varphi(x_{m+2}, \bar{u}) = \\ &= f_{m+2}(x_1, \ldots, x_m, x_{m+1}, x_{m+2}) + \sum_{j=1}^{m} g_{m+2,j}(x_1, \ldots, x_m, x_{m+1}, x_{m+2}) u_j \,. \end{aligned} \tag{7}$$

Note that this subsystem is not affine with respect to the input $\bar{u}$. As it can be immediately checked, the free dynamics of (7) corresponds to the Δ_{MIX} dynamics.

In the following sections, for simplicity of notation, unless otherwise stated, we drop the hats.

IV. The dynamic extension

In analogy with [22], we now introduce an "extended" system. Let

$$\begin{aligned} n_i &= \text{dim} x_i \qquad i = 1, \ldots, m \\ n_0 &= \text{dim} x_{m+1} \\ n_{wi} &= n_i + n_0 \qquad i = 1, \ldots, m \\ n_w &= \sum_{i=1}^{m} n_{wi} \\ n^e &= n + n_w \,. \end{aligned}$$

Consider the extended system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ \dot{w} &= u_w \\ y &= h(x) \,, \end{aligned}$$

where $\dim w = n_w$. Having set,

$$w_i = \begin{pmatrix} \lambda_i \\ \mu_i \end{pmatrix} \ , \ \dim\lambda_i = n_i \ , \ \dim\mu_i = n_0 \ , \ i = 1, \ldots, m,$$

$$w = \begin{pmatrix} w_1 \\ \vdots \\ w_m \end{pmatrix},$$

and

$$x^e = \begin{pmatrix} x \\ w \end{pmatrix} \ , \ x_0^e = \begin{pmatrix} x_0 \\ w_0 \end{pmatrix} \ , \ u^e = \begin{pmatrix} u \\ u_w \end{pmatrix} \ ,$$

we can rewrite the extended system in the compact form

$$\begin{aligned} \dot{x}^e &= f^e(x^e) + g^e(x^e)u^e \\ y^e &= h^e(x^e) \ , \end{aligned} \tag{8}$$

with

$$f^e(x^e) = \begin{pmatrix} f(x) \\ 0 \\ \vdots \\ 0 \end{pmatrix} \ , \ g^e(x^e) = (\, g_1^e(x^e) \quad \ldots \quad g_{2m}^e(x^e)\,) \ ,$$

$$h^e(x^e) = h(x) \ ,$$

where the i–th zero block has dimension n_{wi}, $i = 1, \ldots, m$, and

$$g_i^e(x^e) = \begin{pmatrix} g_i(x) \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} , g_{m+i}^e(x^e) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ I_{n_{wi} \times n_{wi}} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad i = 1, \ldots, m.$$

In what follows we need also the following notations:

$$G^e = \mathrm{span}\{g_i^e : i = 1, \ldots 2m\} \ , \ G_{wi} = \mathrm{span}\{g_{m+i}^e\} \text{ and } G_w = \bigoplus_{i=1}^{m} G_{wi} \quad .$$

Now let us consider the set of vector fields contained in (3) and correspondingly define the following "extended" vector fields

$$X_{ik}^e(x^e) = \begin{pmatrix} X_{ik}(x) \\ 0 \\ \vdots \\ X_{ik}^*(x_i, \mu_i) \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad i = 1, \ldots, m, \tag{9.1}$$

where

$$X^*_{ik}(x_i,\mu_i) = \begin{pmatrix} Y_{ik}(x_i) \\ Z_{ik}(x) \end{pmatrix} \Bigg|_{x_j=0 \text{ for } j\neq i \; ; \; x_{m+1}=\mu_i} \tag{9.2}$$

(see (5) for the definition of Y_{ik} and Z_{ik}) and set

$$R^e_i = \text{span}\{X^e_{ik} : 1 \le k \le s_i\} \qquad i = 1,\ldots,m. \tag{10}$$

Note that the distributions defined above are smooth. In what follows, we also refer to the "extended" vector field

$$\bar{g}^e_i(x^e) = \begin{pmatrix} g_i(x) \\ 0 \\ \vdots \\ g^*_i(x_i,\mu_i) \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad i = 1,\ldots,m, \tag{11}$$

where g^*_i is defined as in (9.2). The next Lemma tells us that the distributions R^e_i have some interesting properties.

Lemma IV.1. If $\Delta_{MIX} = 0$, the distributions R^e_i are independent, involutive and dim $R^e_i =$ dim R_i.$\diamond$

Proof. Let $x \in I\!R^n$ and U be an open neighbourhood of x. Denoting by r_i the constant dimension of R_i, let $\{X_{ik} : k \in \{l_{i1},\ldots,l_{ir_i}\} \subset \{1,\ldots,s_i\}\}$ a set of vector fields, contained in (3), which span R_i. On U we have

$$X_{iq} = \sum_{t=1}^{r_i} c_{iql_{it}} X_{il_{it}} \qquad i = 1,\ldots,m \; ; \; q = 1,\ldots,s_i$$

for some unique smooth functions $c_{iql_{it}}$. First, we show that the $c_{iql_{it}}$ depend only on x_i. As a matter of fact, since $\Delta_{MIX} = 0$, we have

$$[X_{iq}, X_{jh}] = 0 \qquad i,j = 1,\ldots,m \; ; \; i \neq j \; ; \; h = 1,\ldots,s_j \; ; \; q = 1,\ldots,s_i,$$

which implies $L_{X_{jh}} c_{il_{it}q} = 0$ for $j \neq i$. Our claim follows from Lemma III.1. Therefore, by construction

$$X^e_{iq} = \sum_{t=1}^{r_i} c_{il_{it}q} X^e_{il_{it}} \qquad i = 1,\ldots,m \; ; \; q = 1,\ldots,s_i.$$

Thus, around any point x^e, in order to span R^e_i, we can take any set of independent vector fields $\{X^e_{ik} : k \in \{l_{i1},\ldots,l_{ir_i}\} \subset \{1,\ldots,s_i\}\}$ such that $\{X_{ik} : k \in \{l_{i1},\ldots,l_{ir_i}\} \subset \{1,\ldots,s_i\}\}$ span R_i around x. Thus, mutual independence of the distributions R^e_i, $i =$

$1,\ldots,m$, and the fact that dim $R_i =$ dim R_i^e follow at once by constuction. On the other hand, involutivity can be shown as in [22].•

Since $[X_{ik}^e, X_{jh}^e] = 0$ for $j \neq i$, the next lemma follows immediately.

Lemma IV.2. The distributions $\sum_{j\neq i} R_j^e$, $i = 1,\ldots,m$, and $\sum_{i=1}^m R_i^e$ are constant dimensional and involutive for all x^e.◇

It is worth noting that

$$\begin{aligned}
&[f^e, R_i^e] \subset G_{wi} + R_i^e \subset G^e + R_i^e \qquad &&, i = 1,\ldots,m \\
&[g_j^e, R_i^e] \subset G_{wi} + R_i^e \subset G^e + R_i^e \qquad &&, i = 1,\ldots,m \; ; \; j = 1,\ldots,2m
\end{aligned}$$

and for all $x^e \in I\!R^{n^e}$, i.e. the distributions R_i^e, $i = 1,\ldots,m$, are globally weakly controlled invariant for the extended system (8).

V. Main result

We define the global noninteracting control problem with stability in the following way:

(GNCPS). Given any bounded set Ω, find a dynamic state–feedback control law

$$\begin{aligned}
u &= \alpha(x^e) + \beta(x^e)v \\
\dot{w} &= \gamma(x^e) + \delta(x^e)v
\end{aligned}$$

such that the resulting closed–loop system is noninteractive and locally asymptotically stable at $x_0^e = 0$ with a basin of attraction which contains Ω.

Moreover, to avoid trivial solutions to **(GNCPS)**, we impose a regularity constraint on the class of dynamic feedback laws considered.

Regularity assumption. The matrix

$$\begin{pmatrix} \beta(x^e) \\ \delta(x^e) \end{pmatrix}$$

has rank m for all x.

The main result is the following

Theorem. Suppose that Assumptions 1 through 4 hold. Then **(GNCPS)** is solvable if

a) the system (7) is CIBS,

and,

b) the subsystems (6) are globally stabilizable via dynamic state–feedback.$\diamond$

Remark 1. In the case of linear systems $\Delta_{MIX} = 0$ always and R_i is a controllability subspace, so that our assumptions are met. Thus, our theorem extends to a global setting a well–known result for linear systems [3], i.e. if the noninteracting control problem is solvable for a linear system , then it is also solvable with stability.

Remark 2. It is worth noting that the subsystems (6) are defined on the system $\widehat{\Sigma}$ obtained from Σ by omitting the x_{m+2} coordinate. In general, it is not possible to state assumption (b) directly on the original system Σ. It should also be noted that dynamic state feedback stabilizability does not imply state feedback stabilizability (see [25] for a counterexample) .

As it will be seen, under our assumptions, it is possible to construct a global set of coordinates for the leaves of the distributions R_i^e, $i = 1, \ldots, m$, (Lemma V.2). These distributions can be made simultaneously invariant under $f^e + g^e\alpha^e$ and $g^e\beta_j^e$ for $j = 1, \ldots, 2m$ (Lemma V.3), so that the resulting closed–loop system is noninteractive (not necessarily stable) with respect to $\{u_1, \ldots, u_m\}$. Unlike the distributions R_i, $i = 1, \ldots, m$, the distributions R_i^e, $i = 1, \ldots, m$, are now independent by construction (Lemma IV.1). Since for each $i = 1, \ldots, m$ the leaves $\mathcal{L}_0^{R_i}$ and $\mathcal{L}_0^{R_i^e}$ are diffeomorphic (Lemma V.1), Assumption 4 and Lemma V.4 allow us to simultaneously stabilize the dynamics on each $\mathcal{L}_0^{R_i^e}$ and the dynamics modulo the foliation of $\sum_{i=1}^m R_i^e$ without destroying the above noninteraction property.

Before proving the theorem, we first give some preliminary results. In what follows, we implicitly suppose that assumptions 1 through 4 hold.

Lemma V.1.

$$\begin{aligned}\mathcal{L}_0^{R_i^e} &= \{x^e \in I\!R^{n^e} \,|\, x_j = 0, \lambda_j = 0, \mu_j = 0 \text{ for } j \neq i\,;\ \lambda_i = x_i\,,\ \mu_i = x_{m+1} \\ &\quad \text{and } (0, \ldots, x_i, 0, \ldots, x_{m+1}) \in \mathcal{L}_0^{R_i}\} \qquad i = 1, \ldots, m.\end{aligned}$$

In particular, $\pi : I\!R^{n^e} \to I\!R^n$, the canonical projection onto the first n coordinates, maps $\mathcal{L}_0^{R_i^e}$ diffeomorphically onto $\mathcal{L}_0^{R_i}$.$\diamond$

Proof. Fix $i \in 1, \ldots, m$. By Chow's Theorem (see [26]), any point in $\mathcal{L}_0^{R_i^e}$ can be joined to the origin by a concatenation of integral curves of $\{X_{ik}^e : 1 \le k \le s_i\}$. Now each such curve satisfies the differential equations,

$$\begin{aligned}\dot{x}_i &= Y_{ik}(x_i) \\ \dot{x}_j &= 0 \qquad j \neq \{i, m+1, m+2\} \\ \dot{x}_{m+1} &= Z_{ik}(x_1, \ldots, x_{m+1}) \\ \dot{\mu}_i &= Z_{ik}(0, \ldots, x_i, 0, \ldots, \mu_i) \\ \dot{\lambda}_i &= Y_{ik}(x_i)\,.\end{aligned}$$

Since the first curve of the concatenation passes through the origin, we conclude at once that at all points on $\mathcal{L}_0^{R_i^e}$

$$\begin{aligned} x_j &= 0 \qquad j \neq \{i, m+1, m+2\} \\ x_i &= \lambda_i \\ x_{m+1} &= \mu_i \ . \end{aligned}$$

Since the x component of the curve satisfies differential equations given by X_{ik}, it follows that the x component lies on $\mathcal{L}_0^{R_i}$. The assertion that π maps $\mathcal{L}_0^{R_i^e}$ diffeomorphically onto $\mathcal{L}_0^{R_i}$ follows at once.•

Lemma V.2. There exist global coordinates $z = (z_1^T \cdots z_{m+1}^T)^T$ such that

$$R_i^e = \text{span}\{\frac{\partial}{\partial z_i}\} \qquad i = 1, \dots, m.$$

◇

Proof. We split the proof into different steps.

a) By Assumption 4, for each $i = 1, \dots, m$ there exist global coordinates

$$(x_1^T, \dots, x_{i-1}^T, x_{i+1}^T, \dots, x_m^T, \varphi_i^T, \bar{\varphi}_i^T)^T$$

(depending on i) such that

$$R_i = \text{span}\{\frac{\partial}{\partial \varphi_i}\} \qquad i = 1, \dots, m.$$

Set $r_i = \dim R_i$. Let us define

$$\begin{aligned} \varphi_i^* &: \mathbb{R}^{n^e} \to \mathbb{R}^{r_i} \qquad i = 1, \dots, m \\ \varphi_i^*(x^e) &= \varphi_i(0, 0, \dots, 0, x_i, 0, \dots, 0, \mu_i) \\ \bar{\varphi}_i^* &: \mathbb{R}^{n^e} \to \mathbb{R}^{n_i - r_i} \qquad i = 1, \dots, m \\ \bar{\varphi}_i^*(x^e) &= \bar{\varphi}_i(0, 0, \dots, 0, x_i, 0, \dots, 0, \mu_i) \end{aligned}$$

and

$$\Psi : \mathbb{R}^{n^e} \to \mathbb{R}^{n^e}$$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_m \\ x_{m+1} \\ \lambda_1 \\ \vdots \\ \lambda_m \\ \mu_1 \\ \vdots \\ \mu_m \end{pmatrix} \mapsto \begin{pmatrix} \varphi_1^* \\ \vdots \\ \varphi_m^* \\ \bar{\varphi}_1^* \\ \vdots \\ \bar{\varphi}_m^* \\ \lambda_1 \\ \vdots \\ \lambda_m \\ x_{m+1} \end{pmatrix} .$$

Clearly Ψ is a diffeomorphism.

b) Suppose now that we have applied the above coordinate change. Take the projection

$$\begin{aligned} \sigma_i : I\!R^{n^e} &\to I\!R^{r_i} \qquad i = 1, \ldots, m \\ x^e &\mapsto \sigma_i(x^e) = \varphi_i^*(x^e) \, . \end{aligned}$$

We claim that the restriction of σ_i to any leaf of R_i^e is a diffeomorphism. First observe that this is trivially true for any leaf passing through a point at which $x_j = 0$ for $j \neq \{i, m+1, m+2\}$, $x_i = \lambda_i$ and $x_{m+1} = \mu_i$. This follows by considering the flows of the vector fields $\{X_{ik}\}_{k=1}^{s_i}$ and $\{X_{ik}^e\}_{k=1}^{s_i}$ and arguing as in Lemma V.1.

Now consider an arbitrary point $\widetilde{x}^e = (\widetilde{x}_1, \ldots, \widetilde{x}_{m+1}, \widetilde{\lambda}_1, \ldots, \widetilde{\lambda}_m, \widetilde{\mu}_1, \ldots, \widetilde{\mu}_m) \in I\!R^{n^e}$. Define

$$\bar{x}^e = (0, \ldots, \widetilde{x}_i, 0, \ldots, \widetilde{\mu}_i, \widetilde{\lambda}_1, \ldots, \widetilde{\lambda}_{i-1}, \widetilde{x}_i, \widetilde{\lambda}_{i+1}, \ldots, \widetilde{\lambda}_m, \widetilde{\mu}_1, \ldots, \widetilde{\mu}_m) \, .$$

From above, $\bar{\sigma}_i = \sigma_i \big| \mathcal{L}_{\bar{x}^e}^{R_i^e} : \mathcal{L}_{\bar{x}^e}^{R_i^e} \to I\!R^{r_i}$ is a diffeomorphism. We will use this to show that $\widetilde{\sigma}_i = \sigma_i \big| \mathcal{L}_{\widetilde{x}^e}^{R_i^e}$ is onto in the following way.

Let us pick an arbitrary point in $I\!R^{r_i}$ and let $\widehat{x}^e$ be its inverse image under $\bar{\sigma}_i$. Now we can join $\bar{x}^e$ to $\widehat{x}^e$ by a concatenation of integral curves of $\{X_{ik}^e\}_{k=1}^{s_i}$. Consider the x_i and μ_i components of the differential equations corresponding to these integral curves. It is easily seen that if we replace the initial condition $\bar{x}^e$ by $\widetilde{x}^e$, these components will have the same initial conditions, and since they are not affected by the other components and the $\{X_{ik}\}_{k=1}^{s_i}$ are complete, it follows that the x_i and μ_i components will be unchanged. Since the image of σ_i depends only on x_i and μ_i components, it now follows that there exists a point $\bar{\bar{x}}^e \in \mathcal{L}_{\widetilde{x}^e}^{R_i^e}$ such that

$$\widetilde{\sigma}_i(\bar{\bar{x}}^e) = \bar{\sigma}_i(\widehat{x}^e) \qquad i = 1, \ldots, m.$$

Thus, we have shown that $\widetilde{\sigma}_i$ is onto.

Now we will show that $\widetilde{\sigma}_i$ is a diffeomorphism by first showing that the map $\varrho_i = (\bar{\sigma}_i)^{-1}\widetilde{\sigma}_i$ is a covering map. Note that ϱ_i maps $\mathcal{L}_{\widetilde{x}^e}^{R_i^e}$ onto $\mathcal{L}_{\bar{x}^e}^{R_i^e}$.

Now fix $q_0 \in \mathcal{L}_{\bar{x}^e}^{R_i^e}$ and consider $q_j \in \varrho_i^{-1}(q_0) \subset \mathcal{L}_{\widetilde{x}^e}^{R_i^e}$ for $j \in A$, where A is some index set. By redefining the indices if necessary, we assume that $\text{span}\{X_{ik}^e : 1 \leq k \leq r_i\} = R_i^e$ in a neighbourhood of q_0. Denoting by Φ_t^X the flow of a complete vector field X, let us define the following maps

$$\begin{aligned} \psi_i^0 : (-\epsilon, \epsilon)^{r_i} &\to \mathcal{L}_{\bar{x}^e}^{R_i^e} \qquad i = 1, \ldots, m \\ (t_1, \ldots, t_{r_i}) &\mapsto \Phi_{t_{r_i}}^{X_{ir_i}^e} \circ \cdots \circ \Phi_{t_1}^{X_{i1}^e}(q_0) \end{aligned} \tag{12}$$

$$\begin{aligned} \psi_i^j : (-\epsilon, \epsilon)^{r_i} &\to \mathcal{L}_{\widetilde{x}^e}^{R_i^e} \qquad i = 1, \ldots, m \\ (t_1, \ldots, t_{r_i}) &\mapsto \Phi_{t_{r_i}}^{X_{ir_i}^e} \circ \cdots \circ \Phi_{t_1}^{X_{i1}^e}(q_j) \, , \end{aligned} \tag{13}$$

where ϵ is chosen sufficiently small for (13) to be an embedding, so that ψ_i^0 is a diffeomorphism onto its image (with respect to the "preferred" coordinates charts [27]). Set

$V_j = \psi_i^j((-\epsilon,\epsilon)^{r_i})$. Note also that (13) is defined for the same ϵ as (12) and this is possible since the vector fields X_{ik} are complete.

The crucial fact is that V_j is also open and that $\varrho_i\big|V_j$ is a diffeomorphism onto V_0. First note that by definition the φ_i^* component of q_0 and q_j are the same. Note also that on $\mathcal{L}_{x^e}^{R_i^e}$ we have $\bar\varphi_l^* = \text{const}$ for $l = 1,\ldots,m$: as amatter of fact, on these leaves $d\bar\varphi_l^*(X_{ik}^e) = 0$ for $l = 1,\ldots,m$, since $d\bar\varphi_l(X_{ik}) = 0$ which implies also $d\bar\varphi_l(X_{ik})\big|_{x_l=0 \text{ for } l\neq i \text{ and } x_{m+1}=\mu_i} = 0$. Thus, also the $\bar\varphi_i^*$ components are equal, since they are constant along leaves of R_i^e and they are equal at $\widetilde{x}^e$ and $\bar{x}^e$. Hence from the definitions $\mu_i(q_0) = \mu_i(q_j)$ and $x_i(q_0) = x_i(q_j)$ (here $\mu_i(q_j)$ and $x_i(q_j)$ means respectively μ_i and x_i components of q_j). Now, by considering the x_i and μ_i components of integral curves of $\{X_{ik}\}_{k=1}^{r_i}$, we conclude that,

$$\sigma_i(\psi_i^0(t_1,\ldots,t_{r_i})) = \sigma_i(\psi_i^j(t_1,\ldots,t_{r_i})) \qquad i = 1,\ldots,m$$

for all $(t_1,\ldots,t_{r_i}) \in (-\epsilon,\epsilon)^{r_i}$. Since ψ_i^0 and $\bar\sigma_i$ are diffeomorphisms, it follows that ψ_i^j is a diffeomorphism onto V_j as well and in particular V_j is open.

Moreover, we have $(\varrho_i)^{-1}(V_0) = \bigcup_{j\in A} V_j$, where A is some index set. As a matter of fact, suppose that there exists $p \in (\varrho_i)^{-1}V_0$ but $\notin V_\alpha$ for any α. Then, $\varrho_i(p) \in V_0$ and $\varrho_i(p) = \psi_i^0(t_{r_i},\ldots,t_1)$ for some $(t_{r_i},\ldots,t_1) \in (-\epsilon,\epsilon)^{r_i}$. Let us consider the point $q_\alpha = \Phi_{-t_1}^{X_{i1}^e} \circ \cdots \circ \Phi_{-t_{r_i}}^{X_{ir_i}^e}(p)$. By construction q_α and q_0 have the same φ_i^* coordinates and thus $\varrho_i(q_\alpha) = q_0$. This clearly gives a contradiction, since it implies that $p \in V_\alpha$.

By standard arguments in manifold theory (see [27]) it now follows that the requirement that the $\{V_j\}_{j\in A}$ be disjoint can be met by making V_0 smaller if necessary. Thus, we have shown that ϱ_i is a covering map.

Since $\mathcal{L}_{\bar{x}^e}^{R_i}$ is simply connected (it is diffeomorphic to $\mathbb{R}^{r_i}$), we conclude at once that ϱ_i and hence $\widetilde\sigma_i$ is a diffeomorphism [27].

c) Since $\widetilde\sigma_i$ is a global diffeomorphism, it follows that

$$z_i = \varphi_i^*(x^e) \qquad i = 1,\ldots,m$$

is a global set of coordinates for $\mathcal{L}_{\widetilde{x}^e}^{R_i^e}$. Thus, $(z_1^T,\ldots,z_m^T)^T$ is a global set of coordinates for the leaves of $\sum_{i=1}^m R_i^e$. As z_{m+1} coordinate, we can choose the intersection between each leaf of $\sum_{i=1}^m R_i^e$ and the axis

$$(\lambda_1^T,\ldots,\lambda_m^T,(\bar\varphi_1^*)^T,\ldots,(\bar\varphi_m^*)^T,x_{m+1}^T)^T\ .$$

This intersection is unique, since otherwise $(z_1^T,\ldots,z_m^T)^T$ would not be a global set of coordinates for that leaf of $\sum_{i=1}^m R_i^e$. Also z_{m+1} depends smoothly on x^e. Thus, we set

$$z_{m+1} = \varphi_{m+1}(x^e)\ .$$

Moreover, from standard properties of foliations, the Jacobian of the map

$$x^e \mapsto (\, z_1^T \quad \cdots \quad z_m^T \quad z_{m+1}^T\,)$$

is nonsingular on $\mathbb{R}^{n^e}$. The proof of the lemma is now complete.•

Suppose now that we have changed coordinates, according to the previous lemma. In general, the distributions R_i^e are not invariant under f^e or g_j^e for $j = 1, \ldots, 2m$, so that we cannot still derive a standard noninteractive form similar to (2). The next lemma tells us how to do it.

Lemma V.3. There exists a globally defined smooth feedback $u^e = \alpha^e(x^e) + \beta^e(x^e)v^e$ such that $\alpha^e(0) = 0$, $\beta^e(x^e)$ is nonsingular on $\mathbb{R}^{n^e}$, the distributions R_i^e, $i = 1, \ldots, m$, are invariant under $\widetilde{f}^e = f^e + g^e\alpha^e$ and $\widetilde{g}_j^e = g^e\beta_j^e$, $j = 1, \ldots, 2m$, where β_j^e is the j–th column of β^e, and $\widetilde{g}_i^e = \bar{g}_i^e$, $i = 1, \ldots, m$. As a consequence, the closed–loop system resulting from (8) is given in z coordinates by

$$\begin{aligned} \dot{z}_i &= \widetilde{f}_i(z_i, z_{m+1}) + \widetilde{g}_{ii}(z_i, z_{m+1})v_i + \widetilde{g}_{i,m+1}(z_i, z_{m+1})v_{m+1} \qquad i = 1, \ldots, m \\ \dot{z}_{m+1} &= \widetilde{f}_{m+1}(z_{m+1}) + \widetilde{g}_{m+1,m+1}(z_{m+1})v_{m+1} \\ y_i &= h_i(z_i, z_{m+1}) \qquad i = 1, \ldots, m \ , \end{aligned} \tag{14}$$

with $v^e = (\, v_1^T \quad \cdots \quad v_{m+1}^T \,)^T$, $\dim v_i = 1$ for $i = 1, \ldots, m$ and $\dim v_{m+1} = n_w$.◇

Proof. The proof consists of four steps.

a) By means of a nonsingular matrix

$$\beta_1^e = \begin{pmatrix} I_{m\times m} & 0 \\ * & I_{n_w \times n_w} \end{pmatrix} ,$$

we can arrange the columns of g^e in such a way that

$$\widetilde{g}_i^e = g^e\beta_{1i}^e = \bar{g}_i^e \in R_i^e \qquad i = 1, \ldots, m$$

(see (11) for the definition of $\bar{g}_i^e$). Since R_i^e is involutive and by construction $[\widetilde{g}_j^e, X_{ik}^e] = 0$ for $j \neq i$ and $j, i = 1, \ldots, m$, it follows that

$$[\widetilde{g}_j^e, R_i^e] \subset R_i^e \qquad i, j = 1, \ldots, m.$$

b) Fix $i = 1, \ldots, m$. By construction we already have for $j = 1, \ldots, m$

$$[g_{m+i}^e, R_j^e] \subset \begin{cases} R_i^e + G_{wi} & \text{if } j = i \\ R_j^e & \text{else} \end{cases} . \tag{15}$$

Let us define

$$\bar{g}_{m+i}^e(z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_{m+1}) = (g_{m+i}^e(z)\big|_{z_i = z_{i0} = \text{const}}) \bmod R_i^e \ . \tag{16}$$

Note that $\bar{g}^e_{m+i}$ has rank n_{wi}, since otherwise $R^e_i \cap G_{wi} \neq 0$. As in [28, Lemma 3.1], it can be shown that

$$\mathrm{span}\{\bar{g}^e_{m+i}\} = \mathrm{span}\{g^e_{m+i}\} \bmod R^e_i \ .$$

This exactly means that there exists a globally defined nonsingular matrix β^e_{2i} such that $\bar{g}^e_{m+i} = g^e_{m+i}\beta^e_{2i} \bmod R^e_i$. Thus, setting $\widetilde{g}^e_{m+i} = g^e_{m+i}\beta^e_{2i}$, we have

$$[\widetilde{g}^e_{m+i}, R^e_i] \subset R^e_i$$

and from (15) and (16) it follows also

$$[\widetilde{g}^e_{m+i}, R^e_j] \subset R^e_j \qquad j = 1, \ldots, m \ ; \ j \neq i \ .$$

Note that

$$\beta^e = \begin{pmatrix} I_{m\times m} & 0 & \cdots & 0 \\ 0 & \beta^e_{21} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \beta^e_{2m} \end{pmatrix}$$

is nonsingular on $\mathbb{R}^{n^e}$.

c) we want now to show that there exists $\alpha^e(x^e) = (\, 0^T_{m\times m} \quad \alpha^T_1(x^e) \quad \cdots \quad \alpha^T_m(x^e)\,)^T$, defined on $\mathbb{R}^{n^e}$, such that

$$[\widetilde{f}^e, R^e_i] = [f^e + \sum_{j=1}^{m} g^e_{m+j}\alpha_j, R^e_i] \subset R^e_i \qquad i = 1, \ldots, m. \tag{17}$$

An explicit expression of α^e in x coordinates is given by

$$\alpha^e = \begin{pmatrix} 0_{m\times m} \\ f_1(x_1) \\ f_{m+1}(x_1, 0, \ldots, 0, \mu_1) \\ \vdots \\ f_m(x_m) \\ f_{m+1}(0, 0, \ldots, x_m, \mu_m) \end{pmatrix} . \tag{18}$$

Note that $\alpha^e(0) = 0$. To show that (18) satisfy (17), it is sufficient to prove it around x^e. Note that, substituting (18) in (17), we have

$$[\widetilde{f}^e, X^e_{il_{it}}] = [f^e + \sum_{j=1}^{m} g^e_{m+j}\alpha_j, X^e_{il_t}] = \begin{pmatrix} [f, X_{il_{it}}] \\ 0 \\ \vdots \\ [f, X_{il_{it}}]^* \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad i = 1, \ldots, m \ ; \ \{l_{i1}, \ldots, l_{ir_i}\} \subset \{1, \ldots, s_i\},$$

where $[f, X_{il_{it}}]^* = [f, X_{il_{it}}](x^e)\Big|_{x_j=0 \text{ for } j\neq i\ ;\ x_{m+1}=\mu_i}$. Moreover, since $[f, X_{il_{it}}] = \sum_{h=1}^{r_i} c_{il_{it}l_{ih}} X_{il_{ih}}$ and using the same arguments as in the proof of Lemma IV.1, it can be shown that the $c_{il_{it}l_{ih}}$ depend only on x_i. This implies that $[\widetilde{f}^e, X^e_{il_{it}}] \in R^e_i$.

d) the form (14) now follows easily, since $\widetilde{g}^e_i \in R^e_i$, $[\widetilde{f}^e, R^e_i] \subset R^e_i, [\widetilde{g}^e_j, R^e_i] \subset R^e_i$ for $j = 1, \ldots, 2m$ and $R^e_i \subset \cap_{j\neq i} K^e_j$, where $K^e_j = \ker dh^e_j(x^e)$.•

The following lemma states an interesting property of the matrix $\widetilde{g}_{m+1,m+1}$.

Lemma V.4. The matrix $\widetilde{g}_{m+1,m+1}$ has full row rank.

Proof. We prove the lemma in the linear case. The nonlinear analogue can be proven by means of pointwise arguments and regularity assumptions. The matrix $\widetilde{g}_{m+1,m+1}$ has n_w columns and $n + n_w - \sum_{i=1}^m r_i \leq n_w$ rows, since in our case $n = \dim(\sum_{i=1}^m R_i)$. Suppose that these rows are not independent. Then, by means of elementary column transformations, we can rearrange g^e in such a way that

$$G^e = \begin{pmatrix} \widetilde{g}_{11} & \cdots & 0 & B_1 & * \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \widetilde{g}_{mm} & B_m & * \\ 0 & \cdots & 0 & 0 & * \end{pmatrix}$$

where $(B_1^T \cdots B_m^T)^T$ has column rank strictly greater than $n_w - (n + n_w - \sum_{i=1}^m r_i) = -n + \sum_{i=1}^m r_i$. Since the independent columns of the matrix

$$\begin{pmatrix} \widetilde{g}_{11} & \cdots & 0 & B_1 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \widetilde{g}_{mm} & B_m \\ 0 & \cdots & 0 & 0 \end{pmatrix}$$

are elements of $(\sum_{i=1}^m R^e_i) \cap G^e$, it follows that it must be $\dim((\sum_{i=1}^m R^e_i) \cap G^e) > m + (\sum_{i=1}^m r_i) - n$. If we show that $\dim((\sum_{i=1}^m R^e_i) \cap G^e) = m + (\sum_{i=1}^m r_i) - n$, our thesis follows by contradiction. It is clearly sufficient to show that $\dim((\sum_{i=1}^m R^e_i) \cap G_w) = \sum_{i=1}^m r_i - n$. For, let us define the subspace $W \subset I\!R^{r_1+\ldots+r_m}$ as

$$W = \{(c_1^T, \ldots, c_m^T)^T \in I\!R^{r_1+\ldots+r_m} : \sum_{i=1}^m \sum_{t=1}^{r_i} X_{il_{it}} c_i = 0\ ,\ \{l_{i1}, \ldots, l_{ir_i}\} \subset \{1, \ldots, s_i\}\} \qquad ,$$

where $\{X_{il_{i1}}, \ldots, X_{il_{ir_i}}\}$ is a basis of R_i. It is clear that this subspace, which is given by the kernel of the matrix

$$X = (X_{1l_{11}} \cdots X_{1l_{1r_i}} \cdots X_{ml_{m1}} \cdots X_{ml_{mr_m}})$$

coincides with the subspace $(\sum_{i=1}^m R_i^e) \cap G_w$. Since from linear algebra we have

$$\sum_{i=1}^{m} r_i = \dim(\ker\{X\}) + \dim(\mathrm{Im}\{X\}) =$$

$$= \dim((\sum_{i=1}^{m} R_i^e) \cap G_w) + n$$

our claim follows.•

Proof (of Main Theorem). Since $\pi_*(\widetilde{f}^e) = f$ and $\pi_*(\widetilde{g}_i^e) = g_i$, $i = 1, \dots, m$, from Lemma V.1 it follows that for each $i = 1, \dots, m$ the subsystem

$$\Sigma_i^e : (\widetilde{f}^e|\mathcal{L}_0^{R_i^e}, \widetilde{g}_i^e|\mathcal{L}_0^{R_i^e}) \tag{19}$$

is diffeomorphic to Σ_i. Thus, from assumption b), it follows that (19) can be globally stabilized via dynamic smooth feedback. This and Lemma V.4 ensure that assumptions of [29, Theorem 6.1] are satisfied, so that, given any bounded set Ω, (14) can be rendered locally asymptotically stable at 0, with a basin of attraction containing Ω, by a suitable feedback

$$\begin{aligned} v_i &= \eta_i(z_i, \psi_i) + \bar{v}_i \qquad i = 1, \dots, m \\ \dot{\psi}_i &= \zeta_i(z_i, \psi_i) \qquad i = 1, \dots, m \\ v_{m+1} &= \widetilde{g}_{m+1,m+1}^{\dagger}[-\widetilde{f}_{m+1}(z_{m+1}) - a z_{m+1}] \, , \end{aligned} \tag{20}$$

where $a > 0$ is a real number depending on Ω and $\widetilde{g}_{m+1,m+1}^{\dagger}$ is the pseudoinverse of $\widetilde{g}_{m+1,m+1}$. Moreover, the resulting closed–loop system is also noninteractive, as it can be easily checked. Now, it is easy to see that the system

$$\dot{x}_{m+2} = f_{m+2}(x_1, \dots, x_{m+1}, x_{m+2}) + \sum_{j=1}^{m} g_{m+2,j}(x_1, \dots, x_{m+1}, x_{m+2}) F_j(x_1, \dots, x_m, x_{m+1}, w)$$

is CIBS (just pick $\bar{u} = (\, x_1^T \;\; \cdots \;\; x_m^T \;\; x_{m+1}^T \;\; F_1^T \;\; \cdots \;\; F_m^T \,)^T$ and recall Assumption (a)) From the above facts and [30], it follows that, given any bounded set Ω, (2) can be rendered locally asymptotically stable at 0, with a basin of attraction which contains Ω, by means of the composition of the feedback law of Theorem V.3 and the feedback law (20). The resulting closed–loop system is also noninteractive, since z_{m+2} does not influence any output.•

Remark 3. It is important to note that the noninteracting structure of the stabilizing feedback (20) is essential to achieve noninteraction of the cloed–loop system.

Remark 4. Note that the dynamic state–feedback (20) possibly increases the dimension of the dynamic extension introduced in Section IV.

IX. Conclusions

In this paper we have given for the first time some sufficient conditions for global noninteracting control with stability by means of dynamic state–feedback. This result generalizes and extends [22], in the sense that locally our sufficient condition can be stated in terms of stability of the Δ_{MIX} dynamics and stabilizability via dynamic state–feedback of the subsystems $\widehat{\Sigma}_i$, $i = 1, \ldots, m$. As a matter of fact, the necessary conditions stated in [22] are too strong since they imply that the subsystems $\widehat{\Sigma}_i$, $i = 1, \ldots, m$, are exponentially stabilizable by means of static state–feedback.

Except for the stability of the Δ_{MIX} dynamics, which has been shown to be necessary for local stability and noninteraction [20], it is still an open question to determine whther the stabilizability of the subsystems $\widehat{\Sigma}_i$, $i = 1, \ldots, m$, via dynamic state–feedback is also necessary to solve our problem.

References

[1] G. Gilbert, The decoupling of multivariable systems by state–feedback, *SIAM Journal on Control and Optimization*, vol.7, pp. 50–63, 1969.

[2] H.J. Sussmann, V.Jurdjevic, Controllability of nonlinear systems, *Journal of Differential Equation*, vol.12, pp.95–116, 1972.

[3] W.M. Wonham, *Linear Multivariable Control: A Geometric Approach*, New York, Springer Verlag, 2nd ed., 1979.

[4] W.A. Porter, Diagonalization and inverse for nonlinear systems, *International Journal of Control*, vol.10. pp.67–76, 1970.

[5] S.N. Singh, W.J.Rugh, Decoupling in a class of nonlinear systems by state variable feedback, *Transactions of the ASME*, vol.94, pp.323–329.

[6] E. Freund, The structure of decoupled nonlinear systems, *International Journal of Control*, vol.21, pp. 651–659, 1975.

[7] S. Singh, Decoupling of invertible nonlinear systems with state–feedback and precompensation, *IEEE Transactions on Automatic Control*, vol. AC–25, pp.1237–1239, 1980.

[8] A. Isidori, A.J.Krener, C.Gori Giorgi, S.Monaco, Nonlinear decoupling via feedback: a differential geometric approach, *IEEE Transactions on Automatic Control*, vol. AC–26, pp.331–345, 1981.

[9] A. Isidori, A.J.Krener, C.Gori Giorgi, S.Monaco, Locally (f,g)-invariant distributions, *Systems and Control Letters*, vol.1, pp.12–15, 1981.

[10] W. Respondek, On decomposition of nonlinear control systems, *Systems and Control Letters*, vol.1, pp.301–308, 1982.

[11] D. Claude, Decoupling of nonlinear systems, *Systems and Control Letters*, vol.1, pp.242–248, 1982.

[12] H. Nijmeijer, J.M.Schumacher, Zeroes at infinity for affine nonlinear systems, *IEEE Transactions on Automatic Control*, vol. AC–30, pp.566–573, 1985.

[13] J. Descusse, C.H.Moog, Decoupling with dynamic compensation for strong invertible affine systems, *International Journal of Control*, vol.42, pp. 1387–1398, 1986.

[14] H. Nijmeijer, J.M.Schumacher, The regular local noninteracting control problem for nonlinear control systems, *SIAM Journal on Control and Optimization*, vol.24, pp.1232–1245, 1986.

[15] H. Nijmeijer, W.Respondek, Decoupling via dynamic compensation for nonlinear control systems, *Proceedings on 25th IEEE Conference on Decision and Control*, pp.192–197, 1986.

[16] J. Descusse, C.H.Moog, Dynamic decoupling for right–invertible nonlinear systems, *Systems and Control Letters*, vol.8, pp.345–349, 1987.

[17] A. Isidori, J.W.Grizzle, Fixed modes and nonlinear noninteracting control with stability, *IEEE Transactions on Automatic Control*, vol.AC–33, pp. 907–914, 1988.

[18] J.W. Grizzle, A.Isidori, Block noninteracting control with stability via static state–feedback, *Mathemathics of Control, Signals and Systems*, pp.315–342, 1989.

[19] H. Nijmeijer, Feedback decomposition of nonlinear control, *IEEE Transaction on Automatic Control*, vol. AC–28, pp.861–862, 1983.

[20] K. Wagner, Nonlinear noninteraction with stability by dynamic state–feedback, to appear on *SIAM Journal on Control and Optimization.*

[21] W. Zahn, T.J.Tarn, A. Isidori, A canonical Dynamic extension for noninteraction with stability for affine nonlinear square systems, preprint.

[22] S. Battilotti, A sufficient condition for noninteracting control with stability via dynamic state–feedback, to appear on *IEEE Transactions on Automatic Control.*

[23] S. Battilotti, A sufficient condition for noninteracting control with stability via dynamic state–feedback : block–partitioned outputs, submitted to *International Journal of Control.*

[24] A. Isidori, *Nonlinear Control Systems*, Springer Verlag, 2nd ed., 1989.

[25] W. Boothby , R. Marino, Feedback stabilization of nonlinear systems, *Systems and Control Letters*, 12, 1989.

[26] W.I. Chow, Uber system von linearen partiellen diffrential gleichungen ester ordnung, *Math. Ann.*, 117, 98–105, 1939.

[27] W. Boothby, *An introduction to Differentiable Manifolds and Riemannian Geometry*, Academic Press, 1975.

[28] W. Dayawansa, D. Cheng, W.B. Boothby, T.J. Tarn , Global (f,g)–invariance of nonlinear systems, *SIAM Journal on Control and Optimization*, Vol. 26, 5, 1988.

[29] H.J. Sussmann, P.V.Kokotovic, Peaking and stabilizability, *Conference on Decision and Control*, Tampa, 1989.

[30] E.D. Sontag, Remarks on stabilization and input–to–state stability, *Conference on Decision and Control*, Tampa, 1989.