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The Pompeiu Problem, What's New?

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Al compadre Roger

In [BZ] we stated the following general (or so we thought) version of the Pompeiu problem. Let X be a locally compact space, μ a non-negative Radon measure on $X, \{C_j\}_{j=1}^N$ a finite family of compact subsets of X, and G a topological group acting on X and letting the measure μ invariant. Consider the *Pompeiu map*

$$P: \; C(X) \to (C(G))^N$$

defined by

$$(P_j f)(g) := \int_{gC_j} f \, d\mu,$$

where P_j is the jth component of P and we denote gx the action of the element $g \in G$ on the point $x \in X$.

We say that the family $\{C_j\}$ has the *Pompeiu property* if P is injective. Note that for this condition to be non-trivial one essentially needs the action of G to be transitive. The *Pompeiu problem* is to decide *as explicitly as possible* whether the family has the Pompeiu property.

In order to spare ourselves the possibility of missing any reference as well as the need to go back over the history of this problem, let us now point out to the reader the existence of a very nice introduction to this question by Zalcman [Z1] and an extensive bibliography compiled by the same author [Z2]. One can also consult the recently translated monograph [A1] for a more functional analytical view of the subject.

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Zalcman and Brown-Schreiber-Taylor showed in the early seventies that if $X = \mathbf{R}^n, G = M(n) =$ Euclidean motion group, and $C_j = B(0, r_j) =$ ball of center 0 and radius $r_j > 0, j = 1, 2$, then

Proposition 1 *P is injective for $\{B(0, r_j)\}_{j=1,2}$ if and only if $r_1/r_2 \notin Z_n = \{\xi/\mu : \ \xi, \mu$ roots of the Bessel equation $J_{n/2}(z) = 0\}$.*

Moreover, for a single bounded open set $\Omega, C = \bar{\Omega}$, if $C^c = \mathbf{R}^n \setminus C$ is connected we have the following equivalence:

Proposition 2 *P is injective if and only if there is no $\alpha > 0$ such that the overdetermined Neumann problem*

$$(\mathcal{N}) \left\{ \begin{array}{rcl} \Delta u \ + \ \alpha u & = & 0 \text{ in } \Omega \\ u \ = \ 1, \frac{\partial u}{\partial n} & = & 0 \text{ on } \partial\Omega \end{array} \right.$$

has a solution.

As shown by Willms, from the work of Cafarelli it follows that

Corollary 3 *If ∂C is Lipschitz but not real analytic everywhere then C has the Pompeiu property in $\mathbf{R}^n$ (for the group $M(n)$).*

It turns out that the key element of the proof of the above two propositions is the reduction of the Pompeiu problem to the spectral synthesis problem in $C_*(\mathbf{R}^n)$, the space of radially symmetric continuous functions in $\mathbf{R}^n$. This observation led to the extension of these three statements to noncompact irreducible symmetric spaces $X = G/K$ of rank 1 in [BZ], [BS]. The difficulty in extending this result to higher rank is the failure of the spectral synthesis for $C(\mathbf{R}^n), n \geq 2$, as shown by the example of Gurevich, extended to symmetric spaces by Gay and me [BG1]. In other words, the Pompeiu problem in G/K, when rank ≥ 2, is at least as difficult as the Pompeiu problem for $X = \mathbf{R}^n$ when $G = \mathbf{R}^n (n \geq 2)$! On the other hand, one can still study particular cases of this type. For instance, in a much misunderstood paper [BT], Taylor and I showed that if $a, b, c > 0$ then the family of three squares Q_a, Q_b, Q_c, of sides parallel to the axes and size a, b, c, satisfy the Pompeiu property in $\mathbf{R}^2$ with respect to the group of translations if and only if none of the quotients $a/b, b/c, c/a$ is rational. (*Three squares theorem.*) This condition is equivalent to

$$\{\xi \in \mathbf{C}^2 : \ \hat{\chi}_a(\xi) = \hat{\chi}_b(\xi) = \hat{\chi}_c(\xi) = 0\} = \phi, \tag{1}$$

where χ_a is the characteristic function of the square Q_a, $\hat{\chi}_a$ its Fourier transform, and the notation is similar for the other squares. The Pompeiu property is just the statement that for any $f \in C(\mathbf{R}^2)$

$$\chi_a * f = \chi_b * f = \chi_c * f = 0 \quad \text{implies } f = 0. \tag{2}$$

The point is that the failure of the spectral synthesis does not guarantee that (1) implies (2) for characteristic functions of three *arbitrary* compact sets C_1, C_2, C_3. On the other hand the statement we proved in [BT] shows that *if* C_1 *is a square* (or a rectangle) *then* (1) *does imply* (2) *for any other collection* C_2, C_3 (or more generally, $C_2, \ldots, C_N$).

We note that if G/K is a non-compact irreducible symmetric space of rank $n \geq 2$ the following simple looking question is open:

Problem 1 *Find necessary and sufficient conditions on* $(n+1)$ $n-$*tuples* $r_1, \ldots, r_{n+1}$ *of positive numbers so that the family of balls* $\{B(0, r_j)\}_{j=1}^{n+1}$ *has the Pompeiu property in* G/K. *(Similar problem for Cartesian products of lower rank spaces).*

Before proceeding further, note that in the formulation (2), if $C_1, C_2, C_3, \ldots, C_N$ are compact sets in $\mathbf{R}^2$ (we stick to dimension 2 to simplify the reasoning), χ_j the respective characteristic functions, and there is a solution $\nu_1, \ldots, \nu_N \in \mathcal{E}(\mathbf{R}^2)$ to the *Bezout equation*

$$\nu_1 * \chi_1 + \cdots + \nu_N * \chi_N = \delta \tag{3}$$

then clearly

$$\chi_1 * f = \cdots = \chi_N * f = 0$$

imply

$$0 = \nu_1 * \chi_1 * f + \cdots + \nu_N * \chi_N * f = \delta * f = f$$

On the other hand, by Fourier transform, (3) is equivalent to

$$\hat{\nu}_1 \hat{\chi}_1 + \cdots + \hat{\nu}_N \hat{\chi}_N = 1 \tag{3'}$$

so that

$$\{\zeta \in \mathbf{C}^2 : \hat{\chi}_1(\zeta) = \cdots = \hat{\chi}_N(\zeta) = 0\} = \phi \tag{1'}$$

is clearly a necessary condition for (3′). Once we pose the problem this way, one can see that my old work with Alain Yger on deconvolution is relevant to the Pompeiu problem for the translation group. For instance, we proved in [BY1] that we could take C_1 to be a square and C_2, C_3 convenient rotations of the same square, so that the family $\{C_1, C_2, C_3\}$ has the Pompeiu property for the translation group. In fact, one should be able to exploit this idea together with our work on exponential-polynomials [BY2] to solve Problem 1 (at least partially).

I hinted at the beginning of this paper that the statement of the Pompeiu problem from [BZ] may be overly restricted. In fact, I think that, for reasons that I'll explain further below, a fundamental contribution of Gay was the discovery of a sharp *local* Pompeiu property. In [BG1], [BG3] we proved (we restrict ourselves as always to $n = 2$ to simplify).

Proposition 4 *Let Ω be a Jordan region with piecewise smooth boundary Γ. Assume that $K = \bar{\Omega}$ has the Pompeiu property (with respect to the Euclidean group) in $\mathbf{R}^2$.* Let $r > 0$ *be such that $K \subseteq \bar{B}(0, r)$. Then if $R > 2r$ and $f \in C(B(0, R))$ is such that*

$$\int_{gK} f(x)dx \;=\; 0 \quad \text{for any } g \in M(n) \text{such that } gK \subseteq B(0, R) \tag{4}$$

It follows that $f = 0$.

Proposition 5 *Let $r_1, r_2 > 0$ be such that $r_1/r_2 \notin Z_2$ (defined in Proposition 1) and let $R > r_1 + r_2$. Assume $f \in C(B(0, R))$ satisfies*

$$\int_{B(x,r_1)} f(y)dy \;=\; 0 \text{ for all } x \text{ such that } |x| < R - r_1$$

and

$$\int_{B(x,r_2)} f(y)dy \;=\; 0 \text{ for all } x \text{ such that } |x| < R - r_2 \tag{5}$$

then $f \equiv 0$.

Note that local deconvolution, i.e., recovering f in $B(0, R)$ from its averages over those disks $B(x, r_1), B(x, r_2)$ contained in $B(0, R)$ cannot be done directly by solving the Bezout equation. Nevertheless in [BGY1] we succeded in doing that explicitly (see also [BGY2] for the local inversion of the three-squares theorem). This local deconvolution for balls has been extended by El Harchaoui [H1], [H2] to balls in real and complex hyperbolic spaces, showing in particular that Proposition 5 holds in these spaces. I believe that Proposition 4 also holds for G/K of rank 1, but as the work of El Harchaoui shows this is probably not altogether trivial.

An interesting corollary of Propositions 4 and 5 is the following:

Corollary 6 *Let $f \in C(\mathbf{R}^2)$ and $R > 0$ satisfy either of the following two conditions:*

(a) There is a Jordan region K with piecewise smooth boundary satisfying the Pompeiu property with respect to the Euclidean motion group such that

$$\int_{gK} f(y)dy \ = \ 0 \text{ if } gK \cap \bar{B}(0,R) \ = \ \phi$$

(b) There are $r_1, r_2 > 0$ such that $r_1/r_2 \notin Z_2$ and $\int_{B(x,r_j)} f(y)dy \ = \ 0$ whenever $B(x,r_j) \cap \bar{B}(0,R) = \phi$

Then $supp(f) \subseteq \bar{B}(0,R)$.

If we consider Proposition 4 in the light of the earlier definition of the Pompeiu property we see that roughly it says that what one needs is not a group G acting transitively an X but just a *sufficiently large* neighborhood U of the unit element $e \in G$. The natural question is whether one needs μ to be invariant, the results in [BP] can be interpreted to say that one can get by without this condition also. Thus, we are naturally·led to the following problem, about which I have lectured occasionally but have never put it down in writing before:

Problem 2 *Let X be a real analytic Riemannian manifold with the property that the radius of injectivity ι_x is bounded below by $\iota_0 > 0$, for every $x \in X$. Show that there is an integer $N > 0$ and a "thin" set $E \subseteq]0, \iota_0/2[^N$ such that if $(r_1, \ldots, r_N) \in]0, \iota_0/2[^N \backslash E$ and $f \in C(X)$ then*

$$\int_{B(x,r_j)} f(y)dv(y) \ = \ 0 \quad \forall x \in X, j = 1, \cdots, N$$

implies that $f \equiv 0$. Here $B(x,r)$ denotes the geodesic ball of center x and radius r and dv denotes the volume element with respect to the metric.

Some evidence for this pseudoconjecture can be found in the following result of Quinto [Q]:

Proposition 7 *Let X be a real analytic Riemannian manifold with the property that $\iota_x \geq \iota_0 > 0$ for every $x \in X$. Let $\mu : X \times X \rightarrow \mathbf{R}^+$ be a real analytic function. Let $f \in C(X)$ and assume there is $0 < r < \iota_0$ such that*

$$\int_{S(x,r)} f(y)\mu(x,y)d\sigma_x(y) = 0 \quad \forall x \in X$$

where $S(x,r)$ is the geodesic sphere of center x and radius r and $d\sigma_x$ is the canonical measure induced by the Riemannian structure on that sphere. If we also know there is some $x_o \in X$ such that $supp f \cap \bar{B}(x_o, r) = \phi$, it follows that $f \equiv 0$.

This is a one-radius theorem but it requires that f has a "hole" in its support. Essentially, it says that if one can prove a rather local Pompeiu property (using more radii) in such a general situation, then it is automatically globally true. The main point of the proof of Quinto is the use of microlocal analysis, as he had done earlier with Boman to study the Radon transform [BQ].

It is well-known that the Pompeiu problem in $\mathbf{R}^2 = \mathbf{C}$ is related to the Morera theorem [Z1]. Since the motion group $M(\mathbf{C})$ is the collection of transformations $z \mapsto e^{i\theta}z+a$ for some $\theta \in \mathbf{R}, a \in \mathbf{C}$, we obtain from Corollary 3 and Proposition 4 the following version of the theorem of Morera.

Propostion 8 *Let Γ be a piecewise smooth Jordan curve in $B(0,1/2) \subseteq \mathbf{C}$ which is not real analytic everywhere (for instance, a triangle). Let $f \in C(B(0,1))$ such that*

$$\int_{g\Gamma} f(z)dz = 0 \quad \forall g \in M(\mathbf{C}) \text{ such that } g\Gamma \subseteq B(0,1). \tag{6}$$

Then $f \in H(B(0,1))$.

Similarly, we have

Proposition 9 *Let $r_1, r_2 > 0, r_1 + r_2 < \frac{1}{2}, r_1/r_2 \notin Z_2$, then if $f \in C(B(0,1))$ satisfies*

$$\int_{|w-z|=r_1} f(z)dz = 0 \quad \forall w \in B(0, 1-r_1)$$

and

$$\int_{|w-z|=r_2} f(z)dz \;=\; 0 \quad \forall w \in B(0,1-r_2)$$

it follows that $f \in H(B(0,1))$.

Using the same kind of techniques Volchkov [V] proved

Proposition 10 *Let* $f \in C(\bar{B}(0,1))$ *satisfy*

$$\int_\Gamma f(z)dz \;=\; 0$$

for every circle Γ *interior-tangent to the unit circle, then* $f \in H(B(0,1))$.

In a similar vein, the ideas of Proposition 7 lead Globevnik and Quinto [GQ] to the following version of the Morera theorem.

Proposition 11 *Let* A *be a region in* $\mathbf{C}$ *with* $0 \in A$, *define*

$$\Omega := \bigcup_{a \in A} \partial B(a/2, |a|).$$

Let $f \in C(\Omega)$ *which is holomorphic near* 0. *Assume that*

$$a \mapsto \int_{\Gamma(a)} f(z)dz$$

is a real analytic function in A, *then* f *is holomorphic in* Ω.

In contrast, the global Pompeiu property for the symmetric space $X = B(0,1) \subseteq \mathbf{C}$ and $G =$ Moebius group $= SU(1,1)$ leads to the following Morera theorem in the unit disk [B3].

Proposition 12 *Let* Γ *be a Jordan curve in* $B(0,1)$ *which is not real analytic, then if* $f \in \mathbf{C}(B(0,1))$ *and it satisfies*

$$\int_{\sigma\Gamma} f(z)dz = 0 \quad \forall\, \sigma \in SU(1,1) \tag{7}$$

it follows that $f \in H(B(0,1))$

Note that while in Proposition 8 we could have replaced (6) by

$$\int_\Gamma f(gz)dz = 0 \quad \forall\, g \in M(\mathbf{C}) \text{ such that } g^{-1}\Gamma \subseteq B(0,1) \tag{6'}$$

we cannot immediately do that in (7), precisely because the measure dz is not $SU(1,1)$-invariant. In [BP] we nevertheless proved the following:

Proposition 13 *Let Ω be a Jordan domain of class $C^{2+\epsilon}$ in $B(0,1)$, assume that there is no $\alpha > 0$ such that the following overdetermined boundary value problem for the hyperbolic Laplacian Δ_h:*

$$\begin{cases} \Delta_h v + \alpha = 0 \text{ in } \Omega \\ \Delta_h w + \alpha w = 0 \text{ in } \Omega \\ v = w = 0 \text{ on } \partial\Omega \\ \frac{\partial v}{\partial x} + \frac{\partial w}{\partial y} = 0 \text{ on } \partial\Omega \\ \frac{\partial v}{\partial y} - \frac{\partial w}{\partial x} = 1 \text{ on } \partial\Omega \end{cases}$$

admits a solution pair v, w. Then if $f \in C(B(0,1))$ and

$$\int_{\partial\Omega} f(gz)dz = 0 \quad \forall G \in SU(1,1) \tag{8}$$

it follows that $f \in H(B(0,1))$. In particular, if $\partial\Omega$ is not real analytic it is always the case that (8) implies that $f \in H(B(0,1))$.

One can also show that if $\Omega = B(0,r), 0 < r < 1$, Proposition 13 is false *but*, if Ω is a disk *not* centered at 0, then the result still holds (contrary to the Euclidean case!)

There are versions of the Pompeiu problem for L^p functions in general groups (see [Z2] and also *boundary Morera* theorems for L^p functions, e.g. when is an L^p function in the Heisenberg group of class CR as a consequence of the vanishing of integrals over spheres, for which we refer to [ABPZ], [ABC1], [ABC2]. For these cases, being of class L^p is a real restriction. The obstruction to the general Morera type theorem (and Pompeiu problem) in the Heisenberg group seems to be equivalent to proving a local version of these theorems.

To conclude this short review, we would like to discuss the converse of the Pompeiu problem, sometimes called *Schiffer problem.* We have seen in Proposition 2 that for a simply connected Jordan domain Ω with real analytic boundary, the set $\bar{\Omega}$ has the Pompeiu property in $\mathbf{R}^2$(or $\mathbf{R}^n$) with respect to Euclidean group if and only if the problem $(\mathcal{N})$ has an eigenvalue. It is clear that if $\Omega =$ disk then there are infinitely many eigenvalues. The Schiffer problem is just to show that if $(\mathcal{N})$ has an eigenvalue, then $\Omega =$ disk.

In [B1] I had shown that the existence of infinitely many eigenvalues characterizes disks (cf. also earlier [B2] for convex sets). In [BYa1], [BYa2] this result was extended to $\mathbf{R}^n$ and hyperbolic spaces. It turns out that this result fails in $S^n, n \geq 3$. Starting from an example in [BYa2], Karlovitz [Ka] showed that the Pompeiu problem can fail for domains Ω with real analytic boundary in S^n, Ω is topologically a polar cap but cannot be contained in any half sphere, and moreover a slew of other well-known theorems for solutions of similar equations to $(\mathcal{N})$ which are valid in $\mathbf{R}^n$, fail in S^n due to the properties of these domains Ω. In particular, the Schiffer problem fails in S^n. Finally, one can write explicitly the eigenfunctions and eigenvalues for the Laplace operator in Ω. This the right place to point out the related work of Molzon [M] which has escaped mention in [Z2].

Starting from a different and simpler proof of the result in [B1] in the case of convex symmetric domains (see [K1] and references therein), Kobayashi also considered the problem of stability of the ball among domains failing Pompeiu and proved that small perturbations of the ball must be balls if they also fail Pompeiu. More precisely,

Proposition 14 [K2] *Let Ω_j be a sequence of domains in $\mathbf{R}^n$ with real analytic boundary such that as $j \to \infty, \Omega_j \to B(0,R)$. Assume each domain Ω_j has at least one eigenvalue λ_j for $(\mathcal{N})$ and the sequence of λ_j is bounded. Then, for all $j >> 1$ we have that $\Omega_j =$ ball.*

The same result and, furthermore, extended to non-compact symmetric spaces of rank 1 was obtained recently with a simpler proof, by Agranovsky and Semenov [AS]. The key point is the following: any eigenvalue for the problem $(\mathcal{N})$ is also an eigenvalue for the Dirichlet problem (at least in $\mathbf{R}^n$) and one also has:

Lemma 15 *If $X =$ non-compact symmetric space of rank 1 or $X = \mathbf{R}^n$ and there is an eigenvalue for $(\mathcal{N})$ and its multiplicity as Dirichlet eigenvalue is smaller or equal to the dimension of X, then Ω is a geodesic ball.*

Some of the features of the proof appear in [B1] for $X = \mathbf{R}^2$. Earlier, Agranovsky had given a completely different proof of Proposition 14 for families depending analytically on a parameter [A1].

Returning to the Schiffer problem, in $\mathbf{R}^2$ the most important progress has been done recently by Garofalo-Segala and Ebenfelt. Their idea has been to try to pursue the asymptotic approach from [B1] but coupled with a very clever use of the Cauchy theorem. Beyond the references in [Z2] we have [GS] and [E1], [E2]. To simplify we concentrate on the following beautiful general result obtained by them.

Proposition 16 *Let Ω be a Jordan domain with real analytic boundary in* $\mathbf{C}$. *Assume Ω is not a disk and let φ, ψ denote comformal maps*

$$\varphi:\ \Omega \to B(0,1) \quad \psi:\ B(0,1) \to \Omega$$

(e.g., ψ could be taken as φ^{-1}). If either of these two conformal maps is given by a rational function, then Ω has the Pompeiu property for the Euclidean group.

Although I have not mentioned a number of problems in diophantine approximation, mean value theorems, and others related to this question and Gay's work, this seems to be the right point to end this update on the story of the Pompeiu problem. *It looks to me that there is still a lot of work to do...*

References

[A1] M. L. Agranovsky, *Invariant function spaces on homogeneous manifolds of Lie groups and applications*, Amer. Math. Soc., 1994.

[A2] M. L. Agranovsky, On the stability of the spectrum in the Pompeiu problem, J. Math. Anal. Appl. 178 (1993), 237-254.

[AS] M. L. Agranovsky and A. M. Semenov, Deformations of balls in Schiffer's conjecture for Riemannian symmetric spaces, to appear.

[ABC1] M. L. Agranovsky, C. A. Berenstein and D.C. Chang, Morera theorem for holomorphic H^p spaces in the Heisenberg group, J. reine angew. Math. 443 (1993), 49-89.

[ABC2] M. L. Agranovsky, C. A. Berenstein and D.C. Chang, Ergodic and mixing properties of radial measures in the Heisenberg group, in *Fourier analysis, analytic and geometric aspects*, W. O. Bray et al. (eds.), M. Dekker, 1994, 1-15.

[**ABW**] L. Aizenberg, C. A. Berenstein and L. Wertheim, Mean-value characterization of pluriharmonic and separately harmonic functions, to appear in Pacific J. Math.

[**Ba**] E. Badertscher, The Pompeiu problem in locally symmetric spaces, J. Analyse Math. 57 (1991), 250-281.

[**B1**] C. A. Berenstein, An inverse spectral theorem and its applications to the Pompeiu problem, J. Anal. Math. 37 (1980), 128-144.

[**B2**] C. A. Berenstein, On the converse to Pompeiu's problem, Notas e comunicaçoes de Matematica 73 (1976), Univ. Fed. Pernambuco, Recife, Brazil, 16 pages.

[**B3**] C. A. Berenstein, A test for holomorphy in the unit ball of $\mathbf{C}^n$, Proc. Amer. Math. Soc. 90 (1984), 88-89.

[**BCPZ**] C. A. Berenstein, D. C. Chang, D. Pascuas, and L. Zalcman, Variations on the theorem of Morera, Contemp. Math. 137, A. Nagel and E. L. Stout (eds.), Amer. Math. Soc., 1992, 63-78.

[**BG1**] C. A. Berenstein and R. Gay, Sur la synthese spectrale dans les espaces symmetriques, J. Math. Pures. Appl. 65 (1986), 323-333.

[**BG2**] C. A. Berenstein and R. Gay, A local version of the 2-circles theorem, Israel J. Math. 55 (1986), 267-288.

[**BG3**] C. A. Berenstein and R. Gay, Le probleme de Pompeiu local, J. Analyse Math. 52 (1989), 133-166.

[**BGVY**] C. A. Berenstein, R. Gay, A. Vidras, and A. Yger, *Residue currents and Bezout identities*, Birkhäuser, 1993.

[**BGY1**] C. A. Berenstein, R. Gay and A. Yger, Inversion of the local Pompeiu transform, J. Analyse Math. 54 (1990), 259-287.

[**BGY2**] C. A. Berenstein, R. Gay and A. Yger, The three-squares theorem, local version, in *Analysis and partial differential equations*, C. Sadosky (ed.), Marcel Dekker, 1990, 35-50.

[**BP**] C. A. Berenstein and D. Pascuas, Morera and mean-value type theorems in the hyperbolic disk, Israel J. Math. 86 (1994), 61-106.

[**BS**] C. A. Berenstein and M. Shahshahani, Harmonic analysis and the Pompeiu problem, Amer. J. Math. 105 (1983), 1217-1229.

[**BT**] C. A. Berenstein and B. A. Taylor, The three-squares theorem for continuous functions, Arch. Rat. Mech. Anal. 63 (1977), 253-259.

[**BYa1**] C. A. Berenstein and P. Yang, An overdetermined Neumann problem in the unit disk, Advances in Math. 44 (1982), 1-17.

[**BYa2**] C. A. Berenstein and P. Yang, An inverse Neumann problem, J. reine angew. Math. 382 (1987), 1-21.

[**BY1**] C. A. Berenstein and A. Yger, Le probleme de la deconvolution, J. Funct. Anal. 54 (1980), 113-160.

[**BY2**] C. A. Berenstein and A. Yger, Ideals generated by exponential polynomials, Advances in Math. 60 (1986), 1-80.

[**BZ**] C. A. Berenstein and L. Zalcman, The Pompeiu problem in symmetric spaces, Comment. Math. Helvetici 55 (1980), 593-621.

[**BQ**] J. Boman and E. T. Quinto, Support theorems for real analytic Radon transforms, Duke Math. J. 55 (1987), 943-948.

[**C**] M. A. Cumberland, The Pompeiu problem and Schiffer's conjecture, Ph.D. thesis, Univ. of Waterloo, Ontario, 1994.

[**E1**] P. Ebenfelt, Some results on the Pompeiu problem, Ann. Acad. Sci. Fennicae 18 (1993), 323-341.

[**E2**] P. Ebenfelt, Singularities of solutions to a certain Cauchy problem and an application to the Pompeiu problem, Duke Math. J. 71 (1993), 119-142.

[**H1**] M. El Harchaoui, Inversion de la transformation de Pompeiu dans le disque hyperbolique (cas de deux disques), Publ. Matemàtiques 37 (1993).

[H2] M. El Harchaoui, Inversion de la transformation de Pompeiu dans l'espace hyperbolique (cas de deux boules), to appear J. Analyse Math.

[GS] N. Garofalo and F. Segala, Univalent functions and the Pompeiu problem, Trans. Amer. Math. Soc. 346 (1994), 137-146.

[GQ] J. Globevnik and E. T. Quinto, Morera theorems and microlocal analysis, preprint.

[Ka] M. A. Karlovitz, Some solutions to overdetermined boundary value problems on subsets of spheres, Ph.D. thesis, Univ. of Maryland, College Park, 1970.

[K1] T. Kobayashi, Convex domains and the Fourier transform on spaces of contant curvature, CIMPA spring course in WuHan 1992, to appear in Travaux en cours, Hermann, Paris.

[K2] T. Kobayashi, Perturbation of domains in the Pompeiu problem, Communications in Analysis and Geometry 1, (1993), 515-541.

[M] R.Molzon, Symmetry and overdetermined boundary value problems, Forum Math. 3 (1991), 143-156.

[Q] E. T. Quinto, Pompeiu transforms on geodesic spheres in real analytic manifolds, Israel J. Math. 84 (1993), 353-363.

[Vo] M. Vogelius, An inverse problem for the equation $\Delta_u = -cu - d$, Ann. Inst. Fourier 44 (1994), 1181-1209.

[V] V. V. Volchkov, Morera type theorems on the unit disk, Analysis Mathematica 20 (1994), 49-63.

[W] N. B. Willms, Saddle points and overdetermined problems for the Helmholtz equation, Ph.D. thesis, Univ. of Waterloo, Ontario, 1992.

[Z1] L. Zalcman, Offbeat integral geometry, Amer. Math. Monthly 87 (1980), 161-175.

[Z2] L. Zalcman, A bibliographic survey of the Pompeiu problem, in *Approximation by solutions of partial differential equations*, B. Fuglede et al. (eds.), Kluwer Acad. Publ., 1992, 185-194 (and addendum obtainable from the author).

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