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On Adaptive Control of Non-Minimum Phase Nonlinear Systems

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ON ADAPTIVE CONTROL OF NON-MINIMUM PHASE NONLINEAR SYSTEMS

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Abstract

We present a technique of indirect adaptive control for approximate linearization of nonlinear systems based on approximate input-output linearization scheme recently proposed in [HSK92]. The controller can achieve adaptive tracking of reasonable trajectories with small error for slightly non-minimum phase systems. It can also be applied to nonlinear systems where the relative degree is not well defined. Simulation results are provided for the familiar ball and beam experiment under some parameter uncertainty.

I. Introduction

There has been considerable research on the application of nonlinear adaptive control theory for improving the feedback linearization in the input-output response of nonlinear systems under parametric uncertainty. Most of the current research, [KKM91, TKKS91, SI89, TKMK89] among others, is based on feedback linearization [Isi89, Nvds90] and assumes some restrictive conditions such as existence of relative degree, bounded input

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bounded state property of the unobservable subsystem regarding the output as the input, or minimum phase property of the nonlinear system. In this paper, using the results of [HSK92], we provide adaptive *approximate* tracking of a wide class of reference signals for nonlinear systems that fail to meet some of the above conditions *slightly*. Our indirect adaptive controller scheme is motivated by the fact that, with the knowledge of the system parameters, approximate input-output linearization of slightly non-minimum phase systems and systems for which relative degree is not well defined can be produced using state feedback and coordinate changes. With parameter uncertainty, a parameter identifier is used that continuously adjusts the parameter estimates on line based on observation error. *Certainty equivalence principle* suggests that these parameter estimates that are converging to their true values may be employed to approximately linearize the nonlinear system asymptotically. In the next section we review the approximate input-output linearization technique for nonlinear systems [HSK92]. In section III, we present the main results on adaptive approximate linearization. Simulation results for the ball and beam experiment is presented in section IV.

II. Review of Approximate Linearization

Consider the following SISO nonlinear system:

$$\begin{aligned} \dot{x}(t) &= f(x) + g(x) \cdot u \\ y(t) &= h(x(t)) \end{aligned} \tag{1}$$

with relative degree r *outside* an open neighborhood U_ϵ of a singular point x_s. A state x_s is called a singular point for output tracking if $a(x_s) \triangleq \mathcal{L}_g \mathcal{L}_f^{r-1} h(x_s) = 0$ [HD87]. This system has a *robust relative degree* of γ in $U_\epsilon(x_s)$ if there exists smooth functions $\phi_i(x)$, $i = 1, \ldots, \gamma$, such that [HSK92]:

$$
\begin{aligned}
h(x) &= \phi_1(x) + \psi_0(x,u) \\
\mathcal{L}_{f+gu}\phi_i(x) &= \phi_{i+1}(x) + \psi_i(x,u) \qquad i = 1,\ldots,\gamma-1 \\
\mathcal{L}_{f+gu}\phi_\gamma(x) &= \tilde{b}(x) + \tilde{a}(x)\cdot u + \psi_\gamma(x,u)
\end{aligned}
$$

where functions $\psi_i(x,u), i = 0,\ldots,\gamma$ are $O(x,u)^2$ and $\tilde{a}(x)$ is $O(1)$.[1] In U_ϵ we make some approximations (of order ϵ) which by abuse of notation may be written as:

$$
\begin{aligned}
\mathcal{L}_g\mathcal{L}_f^{r-1}h(x) &= \epsilon\psi_{r-1}(x) \\
&\vdots \\
\mathcal{L}_g\mathcal{L}_f^{\gamma-2}h(x) &= \epsilon\psi_{\gamma-2}(x)
\end{aligned}
\tag{2}
$$

but $\mathcal{L}_g\mathcal{L}_f^{\gamma-1}h(x)$ is not of order ϵ and $\mathcal{L}_g\mathcal{L}_f^{\gamma-1}h(x_s) \neq 0$. Consider the following two local diffeomorphism of $x \in \Re^n$:

$$
\begin{aligned}
(\xi^T,\eta^T)^T &= (\xi_i = \mathcal{L}_f^{i-1}h(x), i = 1,2,\ldots,r, \quad \eta_1,\ldots,\eta_{n-r})^T \\
(\tilde{\xi}^T,\tilde{\eta}^T)^T &= (\xi^T, \tilde{\xi}_i = \mathcal{L}_f^{i-1}h(x), i = r+1,\ldots,\gamma, \ \ \tilde{\eta}_1,\ldots,\tilde{\eta}_{n-\gamma})^T
\end{aligned}
\tag{3}
$$

We have for the *true system* when $x \in U_\epsilon(x_s)$:

$$
\begin{aligned}
\dot{\xi}_1 &= \xi_2 \\
&\vdots \\
\dot{\xi}_{r-1} &= \xi_r \\
\dot{\xi}_r &= \tilde{\xi}_{r+1} + \epsilon\psi_{r-1}(x)\cdot u \\
\dot{\tilde{\xi}}_{r+1} &= \tilde{\xi}_{r+2} + \epsilon\psi_r(x)\cdot u \\
&\vdots \\
\dot{\tilde{\xi}}_{\gamma-1} &= \tilde{\xi}_\gamma + \epsilon\psi_{\gamma-2}(x)\cdot u \\
\dot{\tilde{\xi}}_\gamma &= \tilde{b}(x) + \tilde{a}(x)\cdot u \\
\dot{\tilde{\eta}} &= \tilde{q}(\tilde{\xi},\tilde{\eta})
\end{aligned}
\tag{4}
$$

[1] Recall that a function $\psi(z)$ is $O(z)^n$ if $lim_{|z|\to 0}\frac{|\psi(z)|}{|z|^n}$ exists and is not zero. $O(z)^0$ is referred to as $O(1)$. By abuse of notation, as in [HSM89], we will also use the notation $\epsilon\psi(x,u)$ to show an $O(x,u)^n, n \geq 2$ that is uniformly higher order on U_ϵ in the sense of [HSK92].

where $\tilde{a}(x) \triangleq \mathcal{L}_g\mathcal{L}_f^{\gamma-1}h(x), \tilde{b}(x) \triangleq \mathcal{L}_f^{\gamma}h(x)$, and $\tilde{a}(x_s)$ is $O(1)$. The approximate system is (set $\epsilon = 0$):

$$
\begin{aligned}
\dot{\xi}_1 &= \xi_2 \\
&\vdots \\
\dot{\xi}_r &= \tilde{\xi}_{r+1} \\
&\vdots \\
\dot{\tilde{\xi}}_{\gamma-1} &= \tilde{\xi}_\gamma \\
\dot{\tilde{\xi}}_\gamma &= \tilde{b}(x) + \tilde{a}(x)\cdot u \\
\dot{\tilde{\eta}} &= \tilde{q}(\tilde{\xi}, \tilde{\eta})
\end{aligned}
\tag{5}
$$

This represents an *approximate* input-output linearized description of the true system (1) obtained by neglecting some ϵ-order terms in some neighborhood U_ϵ of the singular state x_s (*i.e.* $x \in U_\epsilon(x_s)$). When system (1) is operating in U_ϵ , where (5) is a valid approximation, one may design a feedback control law to achieve approximate output tracking [HSK92]. The control law will, in fact, be the exact tracking control law using the approximate description (5). With the above notation in mind, we say (1) is slightly non-minimum phase if the true system, described by (4), is non-minimum phase but its approximate linearization, described by (5) is minimum phase [HSM89].

Approximate Tracking is achieved by choosing the control law u:

$$u = \frac{1}{\tilde{a}(\xi,\eta)}[-\tilde{b}(\xi,\eta) + v] \tag{6}$$

with:

$$v = y_d^{(\gamma)} + \alpha_{\gamma-1}(y_d^{(\gamma-1)} - \tilde{\xi}_\gamma) + \ldots + \alpha_0(y_d - \tilde{\xi}_1) \tag{7}$$

where α_i are chosen so that $s^\gamma + \alpha_{\gamma-1}s^{\gamma-1} + \ldots + \alpha_0$ is a Hurwitz polynomial. Thus the control law u in (6) approximately linearizes the system (1) from input v to the output y

up to the order ϵ (say $O(x,u)^2$).

III. Adaptive Control

Consider a SISO nonlinear system of the form (1) under parameter uncertainty:

$$\begin{aligned} \dot{x}(t) &= f(x,\theta) + g(x,\theta)\cdot u \\ y(t) &= h(x,\theta) \end{aligned} \tag{8}$$

with relative degree r *outside* an open neighborhood U_ϵ of a singular point x_s and *robust relative degree* γ in $U_\epsilon(x_s)$. Further, assume $f(x), g(x)$ and $h(x)$ have the form:

$$\begin{aligned} f(x,\theta) &= \sum_{i=1}^{n} \theta_i^1 \cdot f_i(x) \\ g(x,\theta) &= \sum_{i=1}^{m} \theta_i^2 \cdot g_i(x) \\ h(x,\theta) &= \sum_{i=1}^{l} \theta_i^3 \cdot h_i(x) \end{aligned} \tag{9}$$

with θ^1, θ^2 and θ^3 vectors of unknown parameters and the $f_i(x), g_i(x)$, and $h_i(x)$ known functions. The estimates of these functions are given by:

$$\begin{aligned} \hat{f}(x) &= \sum_{i=1}^{n} \hat{\theta}_i^1 \cdot f_i(x) \\ \hat{g}(x) &= \sum_{i=1}^{m} \hat{\theta}_i^2 \cdot g_i(x) \\ \hat{h}(x) &= \sum_{i=1}^{l} \hat{\theta}_i^3 \cdot h_i(x) \end{aligned} \tag{10}$$

where $\hat{\theta}_j^i$ are the estimates of the unknown parameters θ_j^i. Now let's replace the control law (6) by:

$$u_{ad} = \frac{1}{L_g \widehat{L_f^{\gamma-1}} h}[-\widehat{L_f^\gamma} h(\xi,\eta) + v_{ad}] \tag{11}$$

with:

$$v_{ad} = y_d^{(\gamma)} + \alpha_{\gamma-1}(y_d^{(\gamma-1)} - \hat{\xi}_\gamma) + \ldots + \alpha_0(y_d - \hat{\xi}_1) \tag{12}$$

where α_i are chosen as before and $\xi_{i-1} = L_f^i h$ are replaced by their estimates $\widehat{L_f^i h}$:

$$
\begin{aligned}
\hat{\xi}_i = \widehat{L_f^{i-1} h} &\triangleq L_{\hat{f}}^{i-1} \hat{h} \\
L_g \widehat{L_f^{\gamma-1} h} &\triangleq L_{\hat{g}} L_{\hat{f}}^{\gamma-1} \hat{h}
\end{aligned}
\tag{13}
$$

As in [SI89], since these estimates are not linear in the unknown parameters θ_i, we define each of the parameter products to be a new parameter. For example:

$$
\widehat{L_f h} = \sum_{i=1}^{l} \sum_{j=1}^{n} \theta_i^3 \theta_j^1 L_{f_j} h_i
$$

and we let $\Theta \in \Re^p$ be the large p-dimensional vector of all multilinear parameter products: $\theta_i^1, \theta_j^2, \theta_k^3, \theta_i^1\theta_j^2, \ldots$. The vector containing all the estimates is denoted by $\hat{\Theta} \in \Re^p$ with $\Phi \triangleq \Theta - \hat{\Theta}$ representing the parameter error. Due to the indirect nature of our approach, this overparametrization does not increase the complexity of the closed loop system since a parameter identifier is to be used to estimate the unknow parameters θ_j^i. The parameter vector Θ is, however, constructed here in order to show the stability of the resulting adaptive system. Using the control law (11) in (5) yields:

$$
\begin{aligned}
\dot{\xi}_\gamma &= L_f^\gamma h + \left[L_g L_f^{\gamma-1} h - L_g \widehat{L_f^{\gamma-1} h}\right] \cdot u_{ad} - \widehat{L_f^\gamma h} + v_{ad} \\
&= \left[L_f^\gamma h - \widehat{L_f^\gamma h}\right] + \left[L_g L_f^{\gamma-1} h - L_g \widehat{L_f^{\gamma-1} h}\right] \cdot u_{ad} + v_{ad}
\end{aligned}
\tag{14}
$$

Subtracting v in (7) from both sides gives:

$$
\begin{aligned}
e^{(\gamma)} + \alpha_{\gamma-1} e^{(\gamma-1)} + \ldots \alpha_0 e &= \left[L_g L_f^{\gamma-1} h - L_g \widehat{L_f^{\gamma-1} h}\right] \cdot u_{ad} + \left[L_f^\gamma h - \widehat{L_f^\gamma h}\right] \\
&\quad + \alpha_{\gamma-1} \left(L_f^{\gamma-1} h - \widehat{L_f^{\gamma-1} h}\right) + \ldots + \alpha_1 \left(L_f h - \widehat{L_f h}\right) \\
&= \Phi^T \cdot w(x, u_{ad}(x))
\end{aligned}
\tag{15}
$$

where: $w^T \triangleq \left[L_{g_i} L_{f_j}^{\gamma-1} h_k u_{ad}(x) | \ldots | L_{f_j} h_k\right]$.

Therefore, in the closed loop, for the approximate system, we have in a compact form:

$$
\begin{aligned}
\dot{e} &= Ae + W^T(x, u_{ad}(x)) \cdot \Phi \\
\dot{\tilde{\eta}} &= \tilde{q}(\tilde{\xi}, \tilde{\eta})
\end{aligned} \tag{16}
$$

where A is a Hurwitz matrix and note that if $\phi \triangleq \theta - \hat{\theta} \rightarrow B_\epsilon$ as $t \rightarrow \infty$, then $\Phi \rightarrow B_\epsilon$ as $t \rightarrow \infty$.

To estimate the unknown parameters, we consider an observer-based identifier proposed in [KN73, Kre77, TKKS91]. First, we rewrite (8) as:

$$
\begin{aligned}
\dot{x} &= (f_1 \ldots f_n \mid g_1 u \ldots g_m u) \cdot \begin{pmatrix} \theta^1 \\ \theta^2 \end{pmatrix} \\
&\triangleq Z^T(x, u_{ad}(x)) \cdot \theta
\end{aligned} \tag{17}
$$

Consider the following identifier system:

$$
\begin{aligned}
\dot{\hat{x}} &= \hat{A} \cdot (\hat{x} - x) + Z^T(x, u_{ad}(x)) \cdot \hat{\theta} \\
\dot{\hat{\theta}} &= -Z^T(x, u) \cdot P \cdot (\hat{x} - x)
\end{aligned} \tag{18}
$$

where $\hat{A}$ is a Hurwitz matrix, $\hat{x}$ is the observer state, x is the plant state in (8), and $P > 0$ is a solution to the Lyapunov equation:

$$
\hat{A}^T P + P\hat{A} = -\lambda \cdot I
$$

with $\lambda > 0$. We assume all the states x in (8) are available and hence $\hat{x}$ and $\hat{\theta}$ are given by (18). We also assume θ is a vector of constant but unknown parameters. Then:

$$
\begin{aligned}
\dot{\hat{e}} &= \hat{A} \cdot \hat{e} + Z^T(x, u) \cdot \phi \\
\dot{\phi} &= -Z^T(x, u) \cdot P \cdot \hat{e}
\end{aligned} \tag{19}
$$

is the observer error system where $\hat{e} \triangleq \hat{x} - x$ is the observer state error and $\phi = \hat{\theta} - \theta$ is the parameter error.

Properties of the observer-based identifier in (19) are [SB89, TKKS91]:

i. $\phi \in \mathcal{L}_\infty$

ii. with $\hat{e}(0) = 0$, $\phi(t) \leq \phi(0)\ \ \forall t \geq 0$

iii. $\hat{e} \in \mathcal{L}_\infty \cap \mathcal{L}_2$

iv. if $Z^T(x, u_{ad})$ is bounded (in particular if x and u are bounded) then $\dot{\hat{e}} \in \mathcal{L}_\infty$ and $\hat{e} \to 0$ as $t \to \infty$.

v. $\hat{e}$ and ϕ converge exponentially to zero if $Z(x, u)$ is sufficiently rich, *i.e.*, $\exists\ \delta_1, \delta_2, \sigma > 0$ such that $\forall t$:

$$\delta_1 I \leq \int_t^{t+\sigma} Z Z^T d\tau \leq \delta_2 I \tag{20}$$

However, since $Z(x, u)$ is a function of state x, the above condition can not be verified explicitly ahead of time.

We are now ready to state the main theorem on approximate tracking for slightly non-minimum phase systems under parameter uncertainty when identifier input is sufficiently rich.

Theorem 1 *Assume that:*

i. the reference trajectory and its first $\gamma - 1$ derivatives (i.e.,$y_d, y_d^{(1)}, \ldots, y_d^{(\gamma-1)}$) are bounded,

ii. the vector fields f, g, and h in (8) are unknown but may be parametrized linearly in unknown parameters in the form (9) where vector fields f_i, g_i, and h_i are known functions of x,

iii. *the zero dynamics of the approximate system (5) are locally exponentially stable and* $\tilde{q}(\tilde{\xi}, \tilde{\eta})$ *is locally Lipschitz in* $\tilde{\xi}$ *and* $\tilde{\eta}$,

iv. *the functions* $\psi(x)u_{ad}(x)$ *in (4) and* $w(x, u_{ad}(x))$ *are locally Lipschitz continuous.*

v. $\phi \rightarrow B_\epsilon(0)$ *as* $t \rightarrow \infty$ *(for example, for observer-based identifier (19),* $Z(x, u_{ad})$ *in (17) is sufficiently rich),*

Then, for ϵ *sufficiently small, the states* x *are bounded and the tracking error is of order* ϵ*; i.e.,*

$$|y - y_d| \leq k\epsilon$$

for some $k < \infty$.

Proof. Using the adaptive approximate control law $u_{ad}(x)$ in (11), the error equation with $\phi = \hat{\theta}(t) - \theta$ and $e = \tilde{\xi} - Y_d$ is:

$$\begin{pmatrix} e_1 \\ \vdots \\ e_\gamma \end{pmatrix} = \begin{pmatrix} \tilde{\xi}_1 \\ \vdots \\ \tilde{\xi}_\gamma \end{pmatrix} - \begin{pmatrix} y_d \\ \vdots \\ y_d^{(\gamma-1)} \end{pmatrix}$$

$$e_1^{(\gamma)} + \alpha_{\gamma-1}e_1^{(\gamma-1)} + \ldots \alpha_0 e_1 = w^T(x, u_{ad}) \cdot \Phi$$

The true error system is given by:

$$\begin{bmatrix} \dot{e}_1 \\ \vdots \\ \dot{e}_r \\ \vdots \\ \dot{e}_{\gamma-1} \\ \dot{e}_\gamma \end{bmatrix} = \begin{bmatrix} 0 & 1 & \dots & 0 & \dots & 0 \\ & & \ddots & & & \\ \vdots & & & 1 & & \vdots \\ & & & & \ddots & \\ 0 & & & & & 1 \\ -\alpha_0 & -\alpha_1 & \dots & -\alpha_{r+1} & \dots & -\alpha_{\gamma-1} \end{bmatrix} \cdot \begin{bmatrix} e_1 \\ \vdots \\ e_r \\ \vdots \\ e_{\gamma-1} \\ e_\gamma \end{bmatrix} + \epsilon \begin{bmatrix} 0 \\ \vdots \\ \psi_{r-1}(x) \\ \vdots \\ \psi_{\gamma-2} \\ 0 \end{bmatrix} u_{ad}(x)$$
$$+ \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ w^T(x,u) \end{bmatrix} \cdot \phi \tag{21}$$

Hence, (8), with the adaptive approximate tracking $u_{ad}(x)$ may be expressed in the following compact form:

$$\begin{aligned} \dot{e} &= A \cdot e + \epsilon \Psi(x) \cdot u_{ad}(x) + W^T(x,u) \cdot \phi \\ \dot{\tilde{\eta}} &= \tilde{q}(\tilde{\xi}, \tilde{\eta}) \end{aligned} \tag{22}$$

From (i):

$$|\tilde{\xi}| \leq |e| + b_d \tag{23}$$

for some b_d. From (iii), a converse Lyapunov theorem assures the existence of a Lyapunov function $v_2(\tilde{\eta})$ for the system:

$$\dot{\tilde{\eta}} = \tilde{q}(0, \tilde{\eta})$$

such that:

$$\begin{aligned} k_1|\tilde{\eta}|^2 \leq \quad v_2(\tilde{\eta}) & \leq k_2|\tilde{\eta}|^2 \\ \frac{\partial v_2}{\partial \tilde{\eta}}\tilde{q}(0,\tilde{\eta}) & \leq -k_3|\tilde{\eta}|^2 \\ \left|\frac{\partial v_2}{\partial \tilde{\eta}}\right| & \leq k_4|\tilde{\eta}| \end{aligned} \tag{24}$$

for some positive constants k_1, k_2, k_3, and k_4.

Since x is a local diffeomorphism of $(\tilde{\xi}, \tilde{\eta})$:

$$\begin{aligned} |x| & \leq l_x(|\tilde{\xi}| + |\tilde{\eta}|) \\ & \leq l_x(|e| + b_d + |\tilde{\eta}|) \end{aligned} \tag{25}$$

From assumption (iv), and (v):

$$\begin{aligned} |\phi(t)| & \leq \rho \\ |2PW^T(x, u_{ad}) \cdot \phi| & \leq (l_w|x| + b_w) \cdot \phi \leq l_w l_x(|e| + b_{dw} + |\tilde{\eta}|) \cdot |\phi| \\ & \leq l_w l_x(|e| + b_{dw} + |\tilde{\eta}|) \cdot \rho \end{aligned} \tag{26}$$

where $b_{dw} \triangleq b_d + \frac{1}{l_w l_x} \cdot b_w$.

From (iii) and (iv), since $\tilde{q}(\tilde{\xi}, \tilde{\eta})$ and $\psi(x)u_{ad}(x)$ are locally Lipschitz with $\psi(0)u_{ad}(0) =$ 0:

$$\begin{aligned} |\tilde{q}(\tilde{\xi}_1, \tilde{\eta}_1) - \tilde{q}(\tilde{\xi}_2, \tilde{\eta}_2)| & \leq l_q(|\tilde{\xi}_1 - \tilde{\xi}2| + |\tilde{\eta}_1 - \tilde{\eta}_2|) \\ |2P\psi(x)u_{ad}(x)| & \leq l_u|x| \end{aligned} \tag{27}$$

Also:

$$\begin{aligned} \frac{\partial v_2}{\partial \tilde{\eta}}\tilde{q}(\tilde{\xi}, \tilde{\eta}) & = \frac{\partial v_2}{\partial \tilde{\eta}}\tilde{q}(0, \tilde{\eta}) + \frac{\partial v_2}{\partial \tilde{\eta}}(\tilde{q}(\tilde{\xi}, \tilde{\eta}) - \tilde{q}(0, \tilde{\eta})) \\ & \leq -k_3|\tilde{\eta}|^2 + k_4 l_q|\tilde{\eta}(|e| + b_d) \end{aligned} \tag{28}$$

In order to show that e and $\tilde{\eta}$ are bounded consider the following Lyapunov candidate function for system (22):

$$\begin{aligned} V(e, \tilde{\eta}) = e^T P e + \mu v_2(\tilde{\eta}) \\ A^T P + PA = -I \end{aligned} \tag{29}$$

where $P > 0$, and $\mu > 0$ to be determined later.

Taking the derivative of V along the trajectories of (22), we have:

$$\begin{aligned}
\dot{V} &= -|e|^2 + 2\epsilon e^T P\psi(x)u_{ad}(x) + 2e^T PW^T\phi + \mu\frac{\partial v_2}{\partial\tilde{\eta}}\tilde{q}(\tilde{\xi},\tilde{\eta}) \\
&\leq -|e|^2 + \epsilon|e|l_u l_x(|e| + b_d + |\tilde{\eta}|) + |e|l_w l_x(|e| + b_{dw} + |\tilde{\eta}|)|\phi| \\
&\quad +\mu(-k_3|\tilde{\eta}|^2 + k_4 l_q|\tilde{\eta}|(|e| + b_d)) \\
&\leq -\left(\tfrac{|e|}{2} - \epsilon l_u l_x b_d\right)^2 + (\epsilon l_u l_x b_d)^2 - \left(\tfrac{|e|}{2} - l_w l_x b_{dw}|\phi|\right)^2 + (l_w l_x b_{dw}|\phi|)^2 \\
&\quad -\left(\tfrac{|e|}{2} - (l_w l_x|\phi| + \epsilon l_u l_x + \mu k_4 l_q)\tilde{\eta}\right)^2 + (l_w l_x|\phi| + \epsilon l_u l_x + \mu k_4 l_q)^2|\tilde{\eta}|^2 \\
&\quad -\mu k_3\left(\tfrac{|\tilde{\eta}|}{2} - \tfrac{k_4 l_q b_d}{k_3}\right)^2 + \mu\tfrac{(k_4 l_q b_d)^2}{k_3} - (\tfrac{1}{4} - \epsilon l_u l_x - l_w l_x|\phi|)|e|^2 - \tfrac{3}{4}\mu k_3|\tilde{\eta}|^2 \qquad (30) \\
&\leq -(\tfrac{1}{4} - \epsilon l_u l_x - l_w l_x|\phi|)|e|^2 - \left((\tfrac{3}{4}\mu k_3 - (l_w l_x|\phi| + \epsilon l_u l_x + \mu k_4 l_q)^2\right)|\tilde{\eta}|^2 \\
&\quad +(\epsilon l_u l_x b_d)^2 + (l_w l_x b_{dw}|\phi|)^2 + \mu\tfrac{(k_4 l_q b_d)^2}{k_3}
\end{aligned}$$

Define:

$$\mu_0 \triangleq \frac{k_3}{4(l_w l_x + k_4 l_q + l_u l_x)} \qquad (31)$$

For $\mu \leq \mu_0$ and $\epsilon \leq min(\mu, \frac{1}{8[l_u l_x + l_w l_x]})$ and $|\phi| \leq \epsilon$, we have:

$$\dot{V} \leq -\frac{|e|^2}{8} - \frac{\mu k_3|\tilde{\eta}|^2}{2} + (\epsilon l_u l_x b_d)^2 + (l_w l_x b_{dw}|\phi|)^2 + \mu\frac{(k_4 l_q b_d)^2}{k_3}$$

Note that $|\phi| < \rho\forall t$, and from assumption (v), we can assume that there exists $T > 0$, such that $|\phi| \leq \epsilon$ for all $t \geq T$. Thus, for all $t \geq T, \dot{V} < 0$ whenever $|\tilde{\eta}|$ or $|e|$ is large which implies that $|\tilde{\eta}|$ and $|e|$ and hence, $|\tilde{\eta}|$ and $|x|$, are bounded.

Now using the continuity of $\psi(x)u_{ad}(x)$ and $W^T(x, u_{ad})\phi$, and boundedness of x, we see that

$$
\begin{aligned}
\dot{e} &= A \cdot e + W^T(x, u_{ad})\phi(t) + \epsilon\psi(x)u_{ad}(x) \\
&= A \cdot e + \begin{bmatrix} 0 \\ \vdots \\ \epsilon\psi_r u_{ad}(x) \\ \vdots \\ \epsilon\psi_{\gamma-1}u_{ad}(x) \\ w^T(x, u_{ad})\phi(t) \end{bmatrix}
\end{aligned} \tag{32}
$$

is an exponentially stable linear system driven by a bounded input that approaches an order ϵ input asymptotically. Therefore, we conclude that the tracking error, e, converges to a ball of order ϵ.

The problem of adaptive stabilization clearly follows. One important special case is when the robust relative degree, γ, is equal to n, *i.e.* the approximate system has no zero dynamics but the true system is non-minimum phase. In this case, the true closed-loop system is exponentially stable. The following theorem summarizes this result:

Theorem 2 *Suppose that the the approximate system (5) has no zero dynamics,* i.e. *(8) has robust relative degree, γ, equal to n, $\psi(x)u_{ad}(x)$ and $w(x, u_{ad})$ are locally Lipschitz in x with $\psi(0)u_{ad}(0) = 0$, unknown parameters θ appear linearly in f, g, and h, and $\phi \to B_\epsilon(0)$ as $t \to \infty$. Then, the adaptive control law u_{ad} in (11) exponentially stablizes (8) with ϵ and $\phi(0)$ sufficiently small.*

Proof. Using control law:

$$
u_{ad} = \frac{1}{\widehat{L_g L_f^{n-1} h}(x)} \left[-\widehat{L_f^n h}(x) - \alpha_{n-1}\hat{\xi}_n - \ldots - \alpha_0 \hat{\xi}_1 \right]
$$

in (8) yields in compact form:

$$\dot{\tilde{\xi}} = A \cdot \tilde{\xi} + \epsilon \Psi(x) \cdot u_{ad}(x) + W^T(x, u_{ad}) \cdot \phi \tag{33}$$

with $A, \Psi(x)$, and $W^T(x, u)$ as before. Choose the following Lyapunov candidate function:

$$V(\tilde{\xi}) = \tilde{\xi}^T P \tilde{\xi} \tag{34}$$

with $P > 0, \lambda > 0$, such that $A^T P + PA = -\lambda \cdot I$. Then, using the bounds similar to those in the proof of theorem (1), we have:

$$\begin{aligned} |x| &\leq l_x |\tilde{\xi}| \\ |\phi(t)| &\leq \rho \\ |2PW^T(x, u_{ad}) \cdot \phi| &\leq l_w \cdot |x| \cdot \rho \\ |2P\psi(x) u_{ad}| &\leq l_u |x| \end{aligned} \tag{35}$$

The derivative of $V(\tilde{\xi})$ along the solution trajectories of (33) is:

$$\begin{aligned} \dot{V} &= -\lambda |\tilde{\xi}|^2 + 2\epsilon \tilde{\xi}^T P \psi(x) u_{ad}(x) + 2\tilde{\xi}^T P W^T \phi \\ &\leq -(\lambda - \epsilon l_x l_u - l_w l_x \rho) |\tilde{\xi}|^2 \end{aligned} \tag{36}$$

Hence, $\dot{V}$ is a negative definite for ϵ and $\phi(0)$ sufficiently small, and consequently, (33) is exponentially stable.

The above design scheme may easily be generalized to the multi-input multi-output (MIMO) case where due to the presence of small terms, the decoupling matrix is almost singular. In this case, approximate linearization is achieved using dynamic extension algorithm [Isi89, Nvds90], and with some modifications, the stability analysis presented in this section can be shown to be true.

IV. Simulation Results

In this section, to demonstrate the adaptive scheme developed in this paper and compare its performance with the non-adaptive control, we consider the ball and beam example from [HSK92] with uncertainty in the mass M of the ball. We first review the controller form derived from the second approximation presented there. The equations of motion are given by:

$$\begin{aligned} \left(\frac{J_b}{R^2} + M\right) \ddot{r} + MgSin\theta - Mr\dot{\theta}^2 &= 0 \\ (Mr^2 + J + J_b)\ddot{\theta} + 2Mr\dot{r}\dot{\theta} + MgrCos\theta &= \tau \end{aligned} \tag{37}$$

where M is the mass of the ball, J is the moment of inertia of the beam, J_b is the moment of inertia of the ball, R is the radius of the ball, g is the acceleration of gravity. θ is the beam angle, r is the position of the ball, and τ is the torque applied to the beam.

With exact knowledge of all parameters, a change of coordinates in the input space is possible by applying a torque in the form of [HSK92]:

$$\tau = 2Mr\dot{r}\dot{\theta} + Mgrcos\theta + (Mr^2 + J + J_b)u \tag{38}$$

where u is a new input. The resulting state-space description is:

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} &= \begin{bmatrix} x_2 \\ B(x_1 x_4^2 - gsinx_3) \\ x_4 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u \\ y &= x_1 \end{aligned} \tag{39}$$

where $B \triangleq M/(\frac{J_b}{R^2} + M)$, $y \triangleq r$, $x = (x_1, x_2, x_3, x_4)^T \triangleq (r, \dot{r}, \theta, \dot{\theta})^T$. The objective is to track a desired trajectory $y_d(t)$.

With $\tilde{\xi}_i = \mathcal{L}_f^{i-1} h(x), i = 1, \ldots, 4$, we have:

$$\begin{aligned}
\dot{\tilde{\xi}}_1 &= x_2 \ = \tilde{\xi}_2 \\
\dot{\tilde{\xi}}_2 &= -Bgsinx_3 + Bx_1x_4^2 \ = \tilde{\xi}_3 \\
\dot{\tilde{\xi}}_3 &= -Bgx_4cosx_3 + Bx_2x_4^2 + 2Bx_1x_4u \ = \tilde{\xi}_4 + \psi_3(x, u) \\
\dot{\tilde{\xi}}_4 &= B^2x_1x_4^4 + Bg(1-B)x_4^2sinx_3 + (-Bgcosx_3 + 2Bx_2x_4)u \\
&= \tilde{b}(x) + \tilde{a}(x)u
\end{aligned} \tag{40}$$

where the origin is the singular state, *i.e.* $a(0) = 0$ with $a(x) \triangleq 2Bx_1x_4$. In this case the neglected nonlinearity is $\psi_3(x, u) = 2Bx_1x_4u$ which is of order ϵ in a neighborhood U_ϵ of the singular state 0. The resulting tracking control law is given by equations (6) and (7):

$$u = \frac{1}{\tilde{a}(\xi)}\left[-\tilde{b}(\xi) + y_d^{(4)} + \alpha_4(y_d^{(3)} - \tilde{\xi}_4) + \ldots + \alpha_0(y_d - \tilde{\xi}_1)\right] \tag{41}$$

where α_i are chosen so that $s^4 + \alpha_3 s^3 + \ldots + \alpha_0$ is a Hurwitz polynomial. This control law achieves approximate output tracking of a desired trajectory $y_d(t)$ up to order ϵ.

When M is not exactly known, control laws (41) and (38) can not be implemented and (39) is no longer a valid description. In this case, we construct an adaptive controller, as developed in the last section, that achieves approximate tracking under parameter uncertainty in mass M of the ball. Although parameter M does not appear linearly in (37), we can still proceed with our design scheme by reparametrizing the system parameters as shown bellow. This is possible mainly due to the indirect nature of our adaptive controller. First, we use the observer-based identifier of (18) to estimate parameter B:

$$\begin{aligned}
\dot{\hat{x}}_2 &= -\sigma(\hat{x}_2 - x_2) + (x_1x_4^2 - gsinx_3)\hat{B} \\
&= -\sigma\hat{e} + Z^T(x)\hat{B} \\
\dot{\hat{B}} &= -Z^T(x)\hat{e} \\
\hat{x_2}(0) &= x_2(0) \ = 0
\end{aligned} \tag{42}$$

Then substitute $\hat{M} = \frac{J_b\hat{B}}{R^2(1-\hat{B})}$ in (38) which gives:

$$\tau_{ad} = 2\hat{M}r\dot{r}\dot{\theta} + \hat{M}grcos\theta + (\hat{M}r^2 + J + J_b)u_{ad} \tag{43}$$

From (37), the new state-space description is:

$$\begin{aligned}
\dot{x_1} &= x_2 \\
\dot{x_2} &= B(x_1x_4^2 - gsinx_3) \\
\dot{x_3} &= x_4 \\
\dot{x_4} &= \tfrac{1}{Mx_1^2+J+J_b}\left[(\hat{M}x_1^2 + J + J_b)u_{ad} + 2(\hat{M} - M)x_1x_2x_4 + (\hat{M} - M)gx_1cosx_3\right] \\
&= f_4(x, \hat{M}, u_{ad}) \\
y &= x_1
\end{aligned} \tag{44}$$

Let $\tilde{\xi}_1 = x_1$. Then, choosing $\tilde{\xi}_i$ at each step as in (40) gives:

$$\begin{aligned}
\dot{\tilde{\xi}}_1 &= x_2 = \tilde{\xi}_2 \\
\dot{\tilde{\xi}}_2 &= -Bgsinx_3 + Bx_1x_4^2 = \tilde{\xi}_3 \\
\dot{\tilde{\xi}}_3 &= -Bgx_4cosx_3 + Bx_2x_4^2 + 2Bx_1x_4f_4(\cdot) = \tilde{\xi}_4 + \psi_3(x, u_{ad})
\end{aligned}$$

$$\begin{aligned}
\dot{\tilde{\xi}}_4 &= B^2x_1x_4^4 + Bg(1-B)x_4^2sinx_3 + (-Bgcosx_3 + 2Bx_2x_4)f_4(\cdot) \\
&= B^2x_1x_4^4 + Bg(1-B)x_4^2sinx_3 + \tfrac{\hat{M}-M}{Mx_1^2+J+J_b}(-Bgcosx_3 + 2Bx_2x_4)(2x_1x_2x_4 + gx_1cosx_3) \\
&+ (-Bgcosx_3 + 2Bx_2x_4)\tfrac{\hat{M}x_1^2+J+J_b}{Mx_1^2+J+J_b} \cdot u_{ad} \\
&\triangleq \tilde{b}(x) + \tilde{a}(x)u_{ad}
\end{aligned} \tag{45}$$

where $\tilde{a}(0) \neq 0$. From (11), the resulting adaptive control law is:

$$u_{ad}(x) = \frac{1}{\hat{\tilde{a}}(x)} \cdot [-\hat{\tilde{b}}(x) + v_{ad}] \tag{46}$$

where:

$$
\begin{aligned}
\hat{\hat{a}}(x) &= 2\hat{B}x_2x_4 - \hat{B}gcosx_3 \\
\hat{\hat{b}}(x) &= \hat{B}^2x_1x_4^4 + \hat{B}g(1-\hat{B})x_4^2sinx_3 \\
v_{ad} &= y_d^{(4)} + \alpha_4(y_d^{(3)} - \hat{\xi}_4) + \ldots + \alpha_0(y_d - \hat{\xi}_1)
\end{aligned}
\tag{47}
$$

For simulation we used $y_d(t) = 3cos\frac{\pi t}{5}, x(0) = (2.9, 0, 0.1698, 0)^T$, $\sigma = 1.5$. $M = 0.05kg, R = 0.01m, J = 0.02kgm^2, J_b = 2 \times 10^{-6}kgm^2$, and $g = 9.81m/s^2$. All closed-loop poles were placed at -5. Figures (1) and (2) show the performance of the adaptive controller with 80% uncertainty in M and initial error $0.1m$. The parameter B converged to the correct value 0.7143 in less than two seconds. The error $|r(t) - y_d|$ was driven to almost zero with maximum magnitude within $\epsilon = 4 \times 10^{-4}m$ ball and the neglected non-linearity $|\psi_3(x, u_{ad})|$ was within 0.01 ball. The performance of the non-adaptive controller for 25% uncertainty in M is shown in figures (3) and (4). The output error was about 30 times more, around $0.015m$, than previous simulation even with less parameter uncertainty. The neglected nonlinearity $|\psi_3(x, u)|$ was much higher in this case, around 0.6, making the approximate linearization result hard to apply. The non-adaptive controller became unstable with 50% uncertainty in M. Clearly, the adaptive controller significantly improved the tracking by stabilizing the system, driving the error closer to zero, and providing a robust approximate feedback linearization for the controller design when nonlinearities can not be canceled due to the lack of exact knowledge of the system parameters.

V. Conclusions

We have presented an adaptive approximate tracking result using approximate input-output linearization approach for nonlinear systems that do not have a well defined relative degree at a point of interest (singular state). This scheme is also applicable to slightly non-minimum phase systems. It is shown that the adaptive controller can achieve output tracking of reasonable trajectories with small error. Our approach was based on certainty equivalence principle with an assumption of parameter identifier convergence. Simulation results were presented for an undergraduate control laboratory experiment, the ball and beam example, discussed in [HSK92]. Simulation results show that under parameter uncertainty in the mass of the ball, our adaptive controller provides good tracking with stability while the non-adaptive controller results in an unstable closed-loop system.

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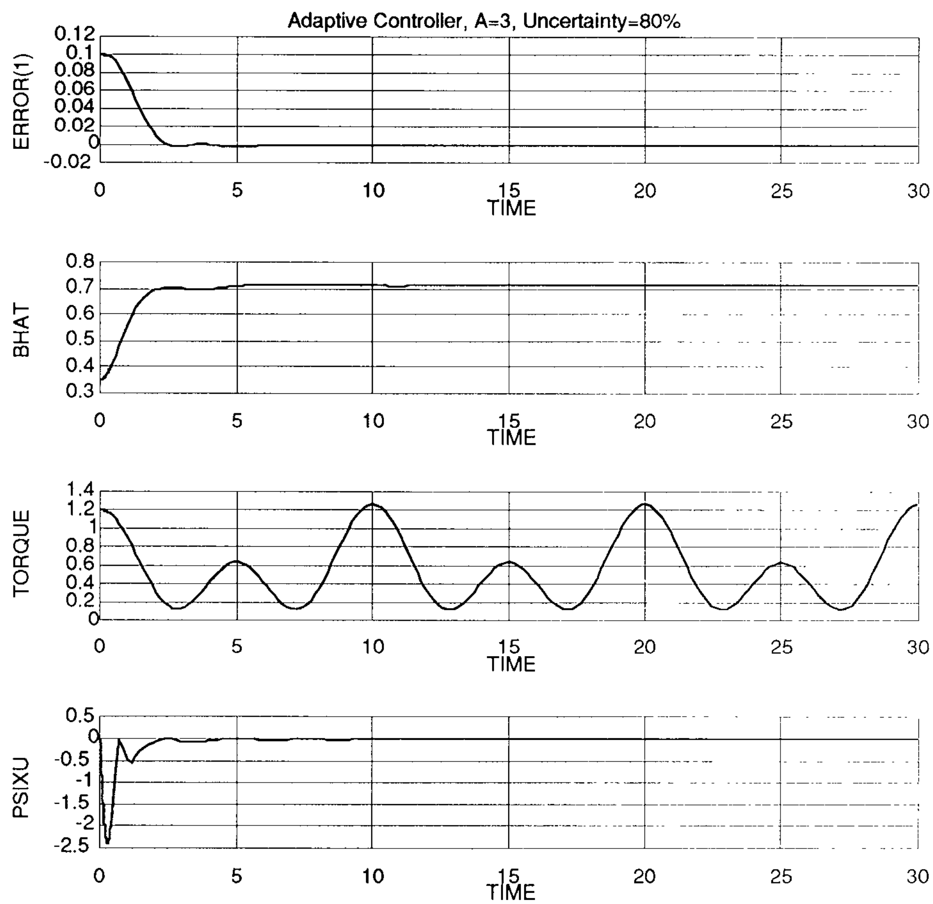

Figure 1: Adaptive Controller: error trajectory with $e(0) = 0.1m$, parameter estimate $\hat{B}$ with initial 80% uncertainty in mass M of the ball, applied torque, and neglected nonlinearity $\psi_3(x, u)$,

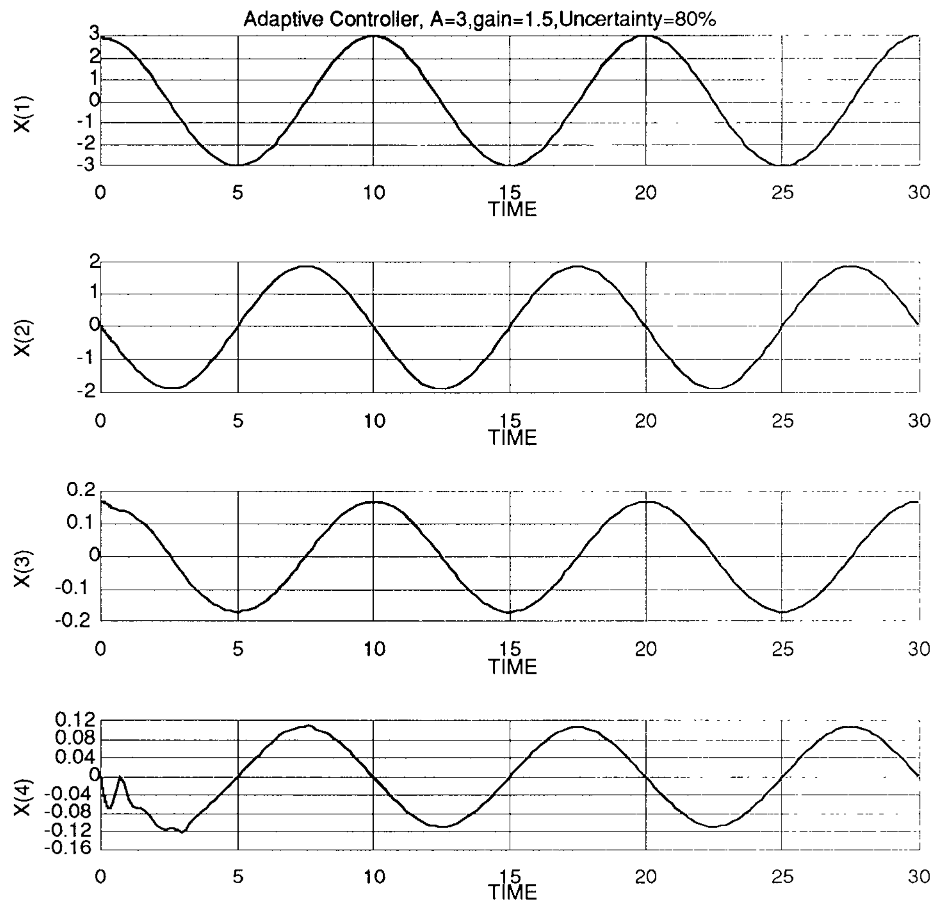

Figure 2: Adaptive Controller: state trajectories $x_i(t)$.

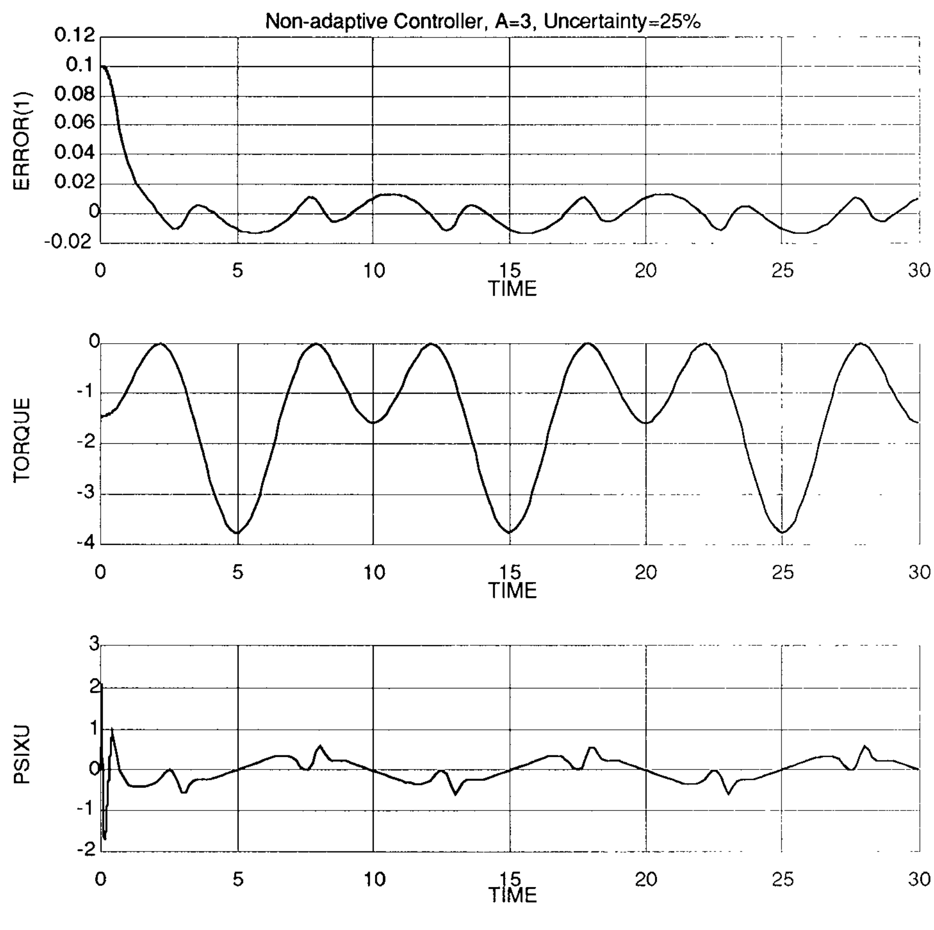

Figure 3: Non-adaptive Controller: error trajectory with $e(0) = 0.1m$, applied torque, and neglected nonlinearity $\psi_3(x, u)$, with 25% uncertainty in mass M of the ball.

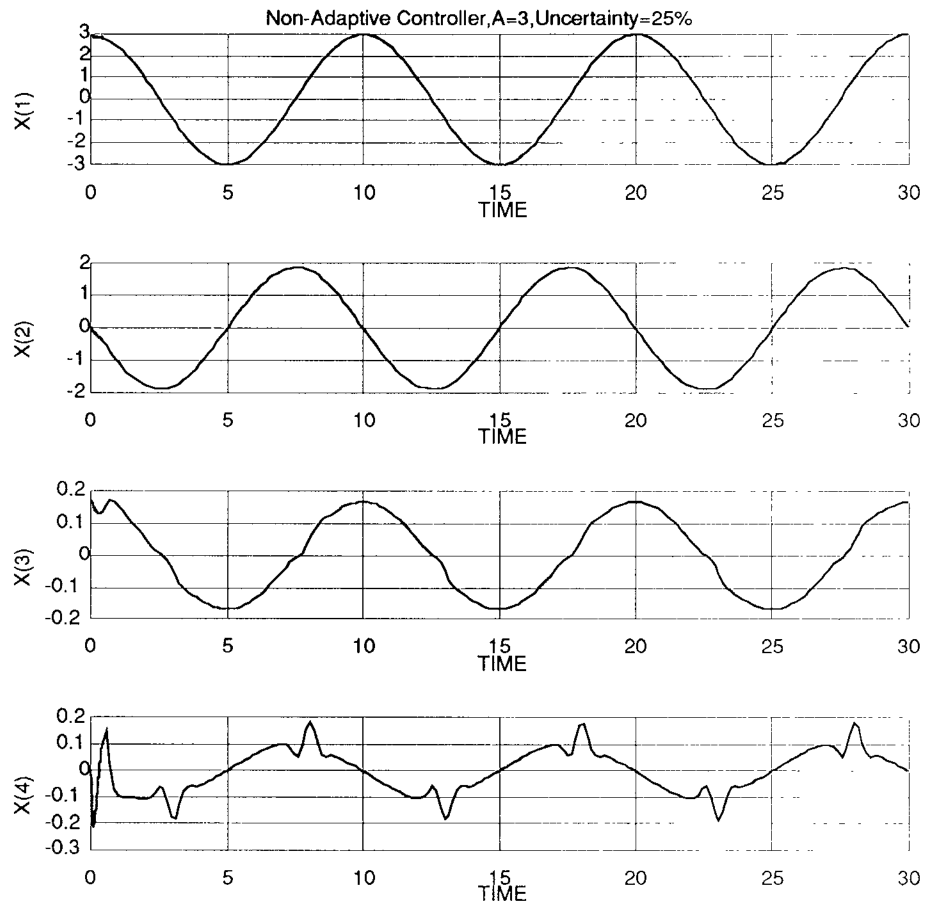

Figure 4: Non-adaptive Controller: state trajectories $x_i(t)$.

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