ABSTRACT

Title of dissertation: TWO-PHASE FLOW OF COMPRESSIBLE VISCOUS DIBLOCK COPOLYMER FLUID

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A diblock copolymer is a linear-chain molecule consisting of two types of monomer. Mathematical models for diblock copolymers can aid researchers in studying the material properties of products as upholstery foam, adhesive tape and asphalt additive. Such models incorporate a variety of factors, including concentration difference, connectivity of the sub-chains, and chemical potential. We consider a flow of two macroscopically immiscible, viscous compressible diblock copolymer fluids.

We first give the derivation of this model on the basis of a local dissipation inequality. Second, we prove that there exist weak solutions to this model. The proof of existence relies on constructing an approximating system by means of time-discretization and vanishing dissipation. We then prove that the solutions to these approximating schemes converge to a solution to the original problem. We also cast thought on the large-time behavior with regularity assumption on the limit.

TWO-PHASE FLOW OF COMPRESSIBLE VISCOUS DIBLOCK COPOLYMER FLUID

by

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Chapter 1: Introduction

We consider a flow of two macroscopically immiscible, viscous compressible diblock copolymer fluids filling a bounded domain $\Omega \subset \mathbb{R}^3$. Diblock copolymers are all around us, found in such products as upholstery foam, adhesive tape and asphalt additives. A diblock copolymer is a a linear-chain molecule consisting of two types of monomers, A and B. This class of macromolecules is produced by joining two or more chemically distinct polymer blocks, each a linear series of identical monomers, that may be thermodynamically incompatible (like oil and vinegar). The monomers are arranged such that there is a subchain of each type of monomers, and those two subchains are grafted together to form a single copolymer chain. A large collection of diblock copolymers is called a polymer melt. Below a critical temperature, even a weak repulsion between unlike monomers A and B induces a strong repulsion between the subchains, causing the subchain to segregate, and this melt will exhibit a phase separation. Because the chains are chemically bonded, a macroscopic segregation whereby the subchains detach from one another can not occur. Rather, in a system of many such macromolecules, the immisibility of these monomers drives the system to form structures which minimize contacts between the unlike monomers and this tendency to separate the monomers into A and B-rich domains is counter balanced by the entropy cost associated with chain stretching. Because of this energetic competition, a phase separation on a mesoscopic scale with A-rich and B-rich domains emerges. The mesoscopic domains which are observed are highly regular periodic structures; for example lamellar, bcc centered spheres, circular tubes, and bicontinuous gyroids. These structures present tremendous potentials for technological applications because they allow for the synthesis of materials with lored mechanical, electrical, and chemical properties (see [5][28][34]) Each geometry could potentially possess different physical characteristics, and thus the ability to readily switch between the phases could allow for materials with tunable properties. Copolymers can be engineered to exhibit specific physical properties which make diblock copolymers of great technological importance.

All block copolymers belong to a broad category of condensed matter sometimes referred to collectively as soft materials, which, in contrast to crystalline solids, are characterized by fluid-like disorder on the molecular scale and a high degree of order at longer length scales. Their complex structure can give block copolymers many useful and desirable properties. The familiar polyurethane foams used in upholstery and bedding are composed of multiblock copolymers known as thermoplastic elastomers that combine high-temperature resilience and low-temperature flexibility.[5]

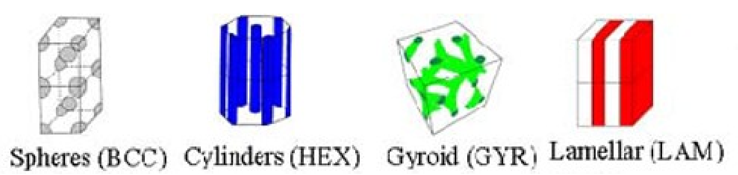

Figure 1.1: Possible geometry of mesoscopic domains for diblock copolymer, from[45]

1.1 Background and Related Work

There are works related to mathematical models for diblock copolymers. Several works are related mean field theories [4][27] in which one must accurately sum the competing energetic contributions of the interaction energy and elastic energy due to chain stretching. Besides mean field theories, a density functional theory was derived by Ohta and Kawasakin in [44] and [38]. This theory uses several approximations to write the free energy exclusively in terms of the (averaged) macroscopic monomer density. That free energy can be rewrititen in a Cahn-Hilliard like functional which is the standard Cahn–Hilliard free energy supplemented with a nonlocal term, reflecting the first order effects of the connectivity of the monomer chains.

Choksi provided a derivation of Nishiura and Ohnishi's nonlocal Cahn–Hilliard-like functional which is accessible to applied mathematicians. We refer the reader to [15][13] and the reference therein. Choksi also ran steady-state simulations starting from random initial data for his model[14]. Simulations based upon minimizing the nonlocal Cahn-Hilliard-like functional suggest minimizers have phase boundaries which resemble constant mean curvature surfaces. The phase structures exhibited in simulations includes lamellar, cylindrical, spherical, double-gyroid. These numerical results on the structures cast a light on the importance of the inclusion of order parameter that is related to connectivity.

There are various works dealing with two-phase flow. The model goes back to Hohenberg and Halperin [37] with the name "model H". Gurtin et al. [33] gave a continuum mechanical derivation based on the concept of microforces. The model is a so-called diffuse interface model, where the difference in concentrations of the two fluids plays the role of the order parameter. To describe a general two-phase flow with droplet formation and coalescence of several droplet, Anderson and McFadden[3] developed diffuse interface models which take a (partial) mixing of the two macroscopically immiscible fluids and a small mesoscopic length-scale into account. By studying a variant of a model by Lowengrub and Truskinovsky[42], which consists of the compressible Navier-Stokes equations governing the motion of the mixture coupled with the Cahn-Hilliard equation for the order parameter, Abels and Feireisl prove existence of global-in-time weak (distributional) solutions of the problem on a bounded domain[2]. Heida et. al developed and generalized Cahn–Hilliard equations within a thermodynamic framework[36][35]. Later, Cherfils and Feireisl et. al establish the existence of global-in-time weak solutions for Cahn–Hilliard–Navier–Stokes system with dynamic boundary conditions[12]. A similar model for incompressible fluids was studied by Boyer [7], Liu and Shen [8], Starovoitov [48], and Abels [1]. Dynmaic boundary condition case was considered and introduced in [29], and well-posedness for the Cahn–Hilliard–Navier–Stokes incompressible model was proved together. Different types of dynamic boundary conditions were considered for the numerical study of the incompressible Navier–Stokes-Cahn–Hilliard equations, see, e.g. Refs. [17], [18], [46], [47] and [49]. There are also numerical simulation works done for incompressible phase-field model of diblock copolymer melt[11][40].

Inspired by Choksi[13] and Abel et. al[2], we will focus on a diffuse interface model for two-phase flow of compressible viscous diblock copolymer fluids, which can help us study the phase behavior of block copolymers in the melt. This model consists of systems of several differential equations: compressible Navier-Stokes equations and modified Cahn-Hilliard type equations. This model differs from Choksi's because it describes the interaction of the diblock copolymer with fluid, so it analyzes the dynamic changes and pattern domain formation of the diblock copolymer melt within a mixture of compressible fluids.

Comparing to Abel-Feireisl's work on Navier-Stokes-Cahn-Hilliard, our model introduces a nonlocal term which reflects the first order effects of the connectivity of the monomer chains, so the dynamics and static state significantly change. Also, this term gives higher nonlinearity, so the complexity increases.

This article is the first one that deals with the existence theory for the interaction of diblock copolymer melt with compressible fluid. All the existing literature on the topic deals with the static problem for diblock copolymers, or the incompressible case. Incompressible case is an idealized situation, while most engineering applications deal with the interaction of compressible or weakly compressible fluids for diblock copolymers. The existence result can can be a great aid during the numerical investigation of the phase separation behaviors of block copolymers.

Because this dissertation focuses on the existence theory and the convergence of solutions in large time, the coefficient for average of concentration difference over space is denoted by M, and the coefficient for intrinsic length scale for minimizer is set to be 1. So in my work, the relation to Fig 1 is not shown. But our existence theory allows people to do numerical investigation with any kind of coefficients, so any kind of mesoscopic domains can be considered.

1.2 Compressible Fluid Model for Diblock Copolymer

In this section we introduce the model that describes the interaction of fluids with diblock copolymer. We study a variant of a model by Lowengrub and Truskinovsky [42] that also extends the model presented by Abels et. al.[2] to two order parameters. This model consists of a system of equations

$$\partial_t \rho + \mathrm{div}_x(\rho \mathbf{u}) = 0 \tag{1.1}$$

$$\begin{aligned} \rho \partial_t \mathbf{u} + \rho \mathbf{u} \cdot \nabla_x \mathbf{u} - \mathrm{div}_x \mathbb{S} + \nabla_x p = -\mathrm{div}_x \left[\nabla_x c \otimes \nabla_x c - \frac{1}{2} |\nabla_x c|^2 \mathbb{I} \right] \\ - \mathrm{div}_x \left[-\nabla_x w \otimes \nabla_x w - (c - M) w \mathbb{I} + \frac{1}{2} |\nabla_x w|^2 \mathbb{I} \right] \end{aligned} \tag{1.2}$$

$$\rho \partial_t c + \rho \mathbf{u} \cdot \nabla_x c = m \Delta \mu \tag{1.3}$$

$$\rho \mu = \rho \frac{\partial f}{\partial c} + w - \Delta c \tag{1.4}$$

$$-\Delta w = c - M \tag{1.5}$$

where $M = \int_\Omega c \, dx$ and $p = \rho^2 \frac{\partial f}{\partial \rho}(\rho, c)$, and

$$\mathbb{S} = 2\nu(c,w)\mathbb{D}(\mathbf{u}_\delta) + \eta(c,w)\mathrm{div}_x \mathbf{u} \mathbb{I}, \tag{1.6}$$

$$\mathbb{D}(\mathbf{u}) = \frac{1}{2}\left(\nabla \mathbf{u} + \nabla \mathbf{u}^T\right) - \frac{1}{3}\mathrm{div}_x \mathbf{u} \mathbb{I} \tag{1.7}$$

for some suitable functions $\nu(c) > 0$, $\eta(c) > 0$ and the free energy density $f(\rho, c)$ to be specified later. Here ρ is total density, $\mathbf{u}$ is the mean velocity of the fluid mixture, p is the pressure, c is the (mass) concentration difference of the two components, and μ is the chemical potential. w is a term related to c via (1.5) and it reflects the first order effects of the connectivity of the monomer chains. The first equation (1.1) is the usual conservation of mass. The second equation (1.2) describes the conservation of linear momentum. In comparison with the compressible Navier-Stokes equation for a single fluid, there is two extra stress contribution in the stress tensor: $\nabla_x c \otimes \nabla_x c - \frac{1}{2}|\nabla_x c|^2 \mathbb{I}$, which describes capillary effect related to the free energy, and $-\nabla_x w \otimes \nabla_x w - (c - M)w\mathbb{I} + \frac{1}{2}|\nabla_x w|^2 \mathbb{I}$, which explains the nature of the joint A and B subchain interactions in the diblock copolymer macromolecule. The free energy for this model is as follow:

$$E_{free} = \int_\Omega \rho f(\rho, c) + \frac{1}{2}|\nabla c|^2 + \frac{1}{2}|\nabla w|^2 \mathrm{d}x \tag{1.8}$$

representing here the surface energy penalizing mixing of the fluids as well as large variations of the concentration difference c and subchain connectivity w. Moreover, (1.3)-(1.5) is a diffusion-convection equation for the concentration difference of modified Cahn-Hilliard equation which takes subchain connectivity into account. The model is derived and explained in more detail in section 2.2.

The system is closed by the boundary and initial conditions

$$\mathbf{u}|_{\partial\Omega} = \nabla c \cdot n|_{\partial\Omega} = \nabla w \cdot n|_{\partial\Omega} = \nabla \mu \cdot n|_{\partial\Omega} = 0, \tag{1.9}$$

$$(\mathbf{u}, c)|_{t=0} = (\mathbf{u}_0, c_0) \tag{1.10}$$

First, to summarize the core hypothesis, we suppose that $\Omega \subset \mathbb{R}^3$ is a bounded domain with C^2-boundary. The viscosity coefficients ν, η are assumed to be continuously differentiable functions of c satisfying

$$0 < \underline{\nu} \leq \nu(c,w) \leq \bar{\nu}, 0 \leq \eta(c,w) \leq \bar{\eta} \text{ for all c.} \tag{1.11}$$

The specific (homogeneous) free energy f takes the form

$$f(\rho,c) = f_e(\rho) + f_0(c), \tag{1.12}$$

and is interrelated to the pressure through the equation of state

$$p(\rho) = \rho^2 \frac{\partial f(\rho,c)}{\partial \rho} = p_e(\rho),\ f_e(\rho) = \int_1^\rho \frac{p(z)}{z^2} dz \tag{1.13}$$

where $p \in C([0,\infty) \cap C^1(0,\infty)$. In what follows, assumption is made that

$$p_e(0) = 0, \underline{p_1}\rho^{\gamma-1} - \underline{p_2} \leq p'(\rho) \leq \bar{p}(1+\rho^{\gamma-1}) \tag{1.14}$$

for a certain isentropic expansion factor $\gamma > \frac{3}{2}$, and $f'(c)$ is Lipschitz with respect to c, and

$$\underline{G}_1 c - \underline{G}_2 \leq f_0'(c) \leq \bar{G}(1+c) \tag{1.15}$$

for all $c \in \mathbb{R}$.

The assumption for p_e (1.14) is in accordance to ideal gas law.

In literature related to diblock copolyer[13][15], $\int_\Omega w\,\mathrm{d}x$ is always a constant. For simplicity, we set

$$\int_\Omega w\, dx = 0. \tag{1.16}$$

the total energy of the system at time $t \in (0,T)$, $t=0$, respectively are denoted by

$$E(t) = \int_\Omega \rho(t)|\mathbf{u}|^2 + \rho f(\rho,c) + \frac{|\nabla_x c|^2}{2} + \frac{|\nabla_x w|^2}{2} \mathrm{d}x \tag{1.17}$$

$$E_0 = \int_\Omega \rho_0^{-1}|\mathbf{m}_0|^2 + \rho_0 f(\rho_0,c_0) + \frac{|\nabla_x c_0|^2}{2} + \frac{|\nabla_x w_0|^2}{2} \mathrm{d}x. \tag{1.18}$$

In addition we set $Q_{(s,t)} = \Omega \times (s,t)$ and $Q_T = Q_{(0,T)}$.

Our main result reads as follows:

Theorem 1.2.1. *Let $0 < T < \infty$, let $\gamma > \frac{3}{2}$, and above assumptions (1.11)-(1.18) be satisfied.*

Then for every non-negative $\rho_0 \in L^\gamma(\Omega)$, measurable $m_0 : \Omega \to \mathbb{R}^3$ with $\rho_0^{-1}|m_0|^2 \in L^1(\Omega)$, $c_0 \in H^1(\Omega)$, and $w_0 \in H^1(\Omega)$ there is a weak solution $\rho \in L^\infty(0,T;L^\gamma(\Omega))$, $\rho \geq 0$, $u \in L^2(0,T;H^1(\Omega;\mathbb{R}^3))$, $c \in L^\infty(0,T;H^1(\Omega))$, $w \in L^\infty(0,T;H^1(\Omega))$ in the following sense:

1. For every $\varphi \in \mathscr{D}\left(\bar{\Omega} \times (0,T);\mathbb{R}^3\right)$

$$\begin{aligned} &-\int_{Q_T} (\rho \mathbf{u} \cdot \partial_t \varphi + (\rho \mathbf{u} \otimes \mathbf{u} + p\mathbb{I} - \mathbb{S}) : \nabla \varphi) \, \mathrm{d}(x,t) \\ &= \int_{Q_T} \left((\nabla c \otimes \nabla c) : \nabla \varphi - \frac{|\nabla c|^2}{2} \operatorname{div} \varphi \right) \mathrm{d}(x,t) \\ &+ \int_{Q_T} \left(-(\nabla w \otimes \nabla w) : \nabla \varphi - (c-M)w : \nabla \varphi + \frac{|\nabla w|^2}{2} \operatorname{div} \varphi \right) \mathrm{d}(x,t) \end{aligned} \tag{1.19}$$

2. ρ is a renormalized solution of (1.1) in the sense of DiPerna and Lions [16], i.e.,

$$\int_{Q_T} (\rho B(\rho) \partial_t \varphi + \rho B(\rho) \mathbf{u} \cdot \nabla \varphi - b(\rho) \operatorname{div} \mathbf{u} \varphi) \, d(x,t) = 0 \tag{1.20}$$

for any test function $\varphi \in \mathscr{D}(\bar{\Omega} \times (0,T))$, and any

$$B(\rho) = B(1) + \int_1^\rho \frac{b(z)}{z^2} dz,$$

where $b \in C^0([0,\infty))$ is a bounded function.

3. For every $\varphi \in \mathscr{D}(\bar{\Omega} \times (0,T))$

$$\int_{Q_T} (\rho c \partial_t \varphi + \rho c \mathbf{u} \cdot \nabla \varphi) \, \mathrm{d}(x,t) = \int_{Q_T} \nabla \mu \cdot \nabla \varphi \, \mathrm{d}(x,t),$$

$$\int_{Q_T} \rho \mu \varphi \, \mathrm{d}(x,t) = \int_{Q_T} \left(\rho \frac{\partial f(\rho,c)}{\partial c} \varphi + w\varphi + \nabla c \cdot \nabla \varphi \right) \mathrm{d}(x,t).$$

and

$$\int_{Q_T} \nabla w \cdot \nabla \varphi \, \mathrm{d}(x,t) = \int_{Q_T} (c - M) \varphi \, \mathrm{d}(x,t)$$

4. The energy inequality

$$E(t) + \int_{Q(s,t)} \mathbb{S} : \nabla_x u + |\nabla_x \mu|^2 d(x,\tau) \leq E(s) \tag{1.21}$$

holds for almost every $0 \leq s \leq t \leq T$ *including* $s = 0$*, where* $E(t)$*,* $E(0) = E_0$ *are determined through (1.17)-(1.18).*

5. ρ*,* ρu*,* c*,* w *are weakly continuous with respect to* $t \in [0,T]$ *with values in* $L^1(\Omega)$ *and* $\rho|_{t=0} = \rho_0$*,* $\rho \mathbf{u}|_{t=0} = \mathbf{m}_0$*,* $c|_{t=0} = c_0$*,* $w|_{t=0} = w_0$*.*

In the theorem, the assumption that viscosity depends on order parameter c and w is made. This fact modifies considerably the relation satisfied by the effective viscous flux that must be handled following the spirit of [20]. The nonlocal term w, which reflects the first order effects of the connectivity of the monomer chains, changes significantly the dynamics and static state. Also, this term gives higher nonlinearity, so the complexity increases. It is challenging to develop converging schemes during the proof for existence. This theorem address the existence of weak solution to modified NSCH system. The importance of weak solutions is that the methods in numerical analysis address weak solutions. The existence provides variational framework for simulations.

The outline of this thesis is as follows: In Chapter 2, we derive the model leading to our system (1.1)-(1.5) on the basis of a local dissipation inequality, which plays the role of the second law of thermodynamics. Moreover, we take advantage of the a priori estimates obtained from the local dissipation inequality and get some preliminary consequences from the estimates. In order to construct the weak solution, a two-level approximation scheme is employed. More precisely, in Chapter 3, we construct solutions to an approximate system to (1.1)-(1.5), where two extra terms are added to the free energy in order to get a better integrability of the density. This is done by using an implicit time discretization of the approximate system. In Chapter 4, we consider the limit of the approximate system to show our main result. Finally, in Chapter 5, we will address large-time behavior of weak solutions to this model under regularity assumptions on the limiting system.

Chapter 2: Derivation of Two-Phase Model and Preliminaries

2.1 Notation

If $\mathbb{A}, \mathbb{B} \in \mathbb{R}^{n\times n}$ are two matrices, then $\mathbb{A} : \mathbb{B} = \sum_{i,j=1}^{n} \mathbb{A}_{ij}\mathbb{B}_{ij}$ denotes their scalar product. If $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, then $\mathbf{a} \otimes \mathbf{b} \in \mathbb{R}^{n\times n}$ is defined by $(a \otimes b)_{ij} = \mathbf{a}_i \mathbf{b}_j$. The characteristic function of a set A is denoted by χ_A. If $\Omega \subseteq \mathbb{R}^n$ is a domain, then $C_0^\infty\left(\Omega;\mathbb{R}^N\right)$ is the set of all smooth and compactly supported functions $f : \Omega \to \mathbb{R}^N$ and $C_0^\infty(\Omega) = C_0^\infty(\Omega;\mathbb{R})$. Moreover, for a general set $A \subseteq \mathbb{R}^n$ we denote $C_{(0)}^\infty\left(A;\mathbb{R}^N\right) = \left\{f \in\ C_0^\infty\left(\mathbb{R}^n;\mathbb{R}^N\right)\right) : \operatorname{supp} f \subseteq A\}$ and $C_{(0)}^\infty(A;\mathbb{R}) = C_{(0)}^\infty(A)$. For short we also write $\mathscr{D}\left(A;\mathbb{R}^N\right) = C_{(0)}^\infty\left(A;\mathbb{R}^N\right)$ and $\mathscr{D}(A) = C_{(0)}^\infty(A)$. The usual Lebesgue spaces are denoted by $L^q(\Omega), 1 \leq q \leq \infty, \|\cdot\|_q$, denotes its norm, and $L^q(\Omega;X)$ denotes the corresponding space of q-integrable X valued functions. The $L^2(\Omega)$-scalar product is denoted by $(.,.)_\Omega$. Furthermore, $W^{s,q}\left(\Omega;\mathbb{R}^N\right), W^{s,q}(\Omega), s \geq 0$, are the Sobolev-Slobodetskii spaces, cf. e.g. [2]. As usual $W_0^{m,q}(\Omega), m \in \mathbb{N}_0$, is the closure of $C_0^\infty(\Omega)$ in $W^{m,q}(\Omega), W^{-m,q}(\Omega) = \left(W_0^{m,q'}(\Omega)\right)', 1 = \frac{1}{q} + \frac{1}{q'}, H^m(\Omega) = W^{m,2}(\Omega)$ and $H_0^m(\Omega) = W_0^{m,2}(\Omega)$. Finally, $C_{\text{weak}}\left([0,T];X\right)$ is the space of all weakly continuous $f : [0,T] \to X$ and $f_n \to f$ in $C_{\text{weak}}\left([0,T];X\right)$ if and only if $\langle f_n(t), x'\rangle_{X,X'} \to_{n\to\infty} \langle f(t), x'\rangle$ uniformly in $t \in [0,T]$ for all $x' \in X'$. Here $\langle\cdot,\cdot\rangle_{X,X'}$ denotes the duality product of X and X'.

2.2 Derivation

In this section we derive a model for a two-component (binary) fluid mixture in which the components are compressible, viscous and immiscible and a nonlocal Cahn-Hilliard diffusion is coupled with internal fluid motion.

2.2.1 Governing equations

We consider two compressible fluids filling a domain $\Omega \subset \mathbb{R}^3$. The mass concentration of the fluids $j = 1,2$ is denoted by $c_j = \frac{M_j}{M}$. Let $\rho_j = \frac{M_j}{V}$ denote the apparent mass density of the fluid j and $\rho = \rho_1 + \rho_2$ the total density. We denote by $\mathbf{u}_j$ the velocity of the fluid $j = 1,2$ and the velocity $\mathbf{u}$ of the mixture is defined as the average velocity given by $\rho\mathbf{u} = \rho_1\mathbf{u}_1 + \rho_2\mathbf{u}_2$.

Our goal in this section is to determine the constitutive relations that describe the physical properties of the mixture. We assume that the principle of mass conservation as well as the conservation of linear and angular momentum with respect to the mean velocity hold, namely

$$\partial_t\rho + \mathrm{div}_x(\rho\mathbf{u}) = 0, \tag{2.1}$$

$$\rho\dot{\mathbf{u}} \equiv \rho\partial_t\mathbf{u} + \rho\mathbf{u}\cdot\nabla_x\mathbf{u} = \mathrm{div}_x\mathbb{T}, \tag{2.2}$$

for a symmetric stress tensor $\mathbb{T} = \mathbb{T}(\rho, c, \nabla_x c, \nabla_x w, \mathbb{D}(\mathbf{u}))$ where $\mathbb{D}(\mathbf{u}) = \frac{1}{2}(\nabla_x\mathbf{u} + \nabla_x\mathbf{u}^\top)$ is the symmetric part of the velocity gradient, $c = c_1 - c_2$ denotes the concentration difference of the two fluids, whereas $c_1 + c_2 = 1$, whereas w is a quantity which will be specified in the sequel.

Without loss of generality we assume for the moment that the exterior forces are zero. The material time derivative of a quantity Λ is given by $\dot{\Lambda} = \partial_t\Lambda + \mathbf{u}\nabla_x\Lambda$. If we denote by

$\mathscr{F}_j$ the mass flux of the fluid j relative to the mean velocity $\mathbf{u}$, then

$$\partial_t \rho_j + \mathrm{div}_x(\rho_j \mathbf{u}) = \mathrm{div}_x \mathscr{F}_j, \tag{2.3}$$

which yields to conservation of mass assuming that $\mathscr{F}_1 + \mathscr{F}_2 = 0$. Therefore, the order parameter c which denotes the concentration difference satisfies the equation

$$\rho \partial_t c + \rho \mathbf{u} \cdot \nabla_x c = \mathrm{div}_x \mathscr{F}, \tag{2.4}$$

where $\mathscr{F} = 2\mathscr{F}_1$, as $\rho_j = \rho c_j$.

Few remarks on the energy of the mixing are now in order. According to a principle of the chemical thermodynamics of fluid mixtures there is a limited miscibility between the so-called immiscible fluid components even at low temperatures. This partial miscibility is characterized by equilibrium concentrations $c_1 \approx 0$ (of the first component in fluid 2) and $c_2 \approx 1$ (of the second component in fluid 2). As the temperature increases, the two equilibrium concentrations approach each other and eventually coincide so that the miscibility gab $c_1 - c_2$ closes at a critical temperature. Above the critical temperature, the system exhibits a continuous sequence of molecular mixtures for all $c \in (0,1)$ and the fluids are considered to be completely miscible. Below the critical temperature, the equilibrium concentrations can be obtained by the standard methods of equilibrium thermodynamics. The particular model of mixing is formulated in terms of the specific free energy $f = f(c)$ which is assumed to be convex if fluids are miscible and non-convex if fluids are partially miscible. The method for determining equilibrium concentration is based on the following common tangent condition

$$\frac{df}{dc}|_{c_1} = \frac{df}{dc}|_{c_2}, \left(f - c\frac{df}{dc}\right)|_{c_1} = \left(f - c\frac{df}{dc}\right)|_{c_2},$$

which is due to Gibbs [31], [32]. For further remarks on the thermodynamics of partially miscible fluids we refer the reader to Landau and Lifschitz (1958).

The relative motion of the fluids can be described by a diffusional model, namely by (2.4)

$$\rho\dot{c} = \mathrm{div}_x \mathscr{F}, \tag{2.5}$$

where the diffusion flux is assumed to specify a *generalized Fick's law*.

Remark 2.2.1. *We introduce the* Helmholtz free energy *of a given volume V of the form:*

$$\int_V F(\rho, c(x), \nabla_x c(x), w(x), \nabla_x w(x))\, dx.$$

where F is defined as following

$$F(c, \nabla_x c, w, \nabla_x w) = \rho f(\rho, c) + \frac{1}{2}|\nabla_x c|^2 + (c - M)\, w - \frac{1}{2}|\nabla w|^2 \tag{2.6}$$

where

$$-\Delta w = c - M, \text{ with B.C.} \tag{2.7}$$

$M = \int_\Omega c(x)dx \in (-1, 1)$. *Here f*

$$f = f_e(\rho) + f_0(c) \tag{2.8}$$

$f_0(c)$ is the free energy. The reason of free energies are in this form will be shown later.

In order to comply with the physical principles of the copolymer melts a modified nonlocal Cahn-Hilliard equation is introduced. In addition to a concentration gradient to the specific free energy the standard Cahn-Hilliard free energy is augmented by a long-range interaction term w associated with the connectivity of the sub-chains in a diblock copolymer macromolecule. Here the function w satisfies a Poisson-type equation.

Now, the diffusion flux $\mathscr{F}$ is assumed to satisfy a generalized Fick's law, namely

$$\mathscr{F} = m\nabla_x\mu, \tag{2.9}$$

and

$$\rho\dot{c} = \mathrm{div}_x(m\nabla_x\mu). \tag{2.10}$$

Chemical potential is defined as

$$\mu = \frac{\delta f}{\delta c} = \frac{\partial f}{\partial c} - \mathrm{div}_x\frac{\partial f}{\partial \nabla_x c} \tag{2.11}$$

and end up with a Cahn-Hilliard type diffusion equation for c:

$$\rho\dot{c} = m\Delta\mu, \tag{2.12}$$

$$\rho\mu \;\; = \;\; \frac{\partial F}{\partial c} - \mathrm{div}_x\frac{\partial F}{\partial \nabla_x c} \tag{2.13}$$

2.2.2 Second law of thermodynamics: local dissipation

Let $V(t)$ be an arbitrary volume that is transported with the flow. Then, the total energy in $V(t)$ is given by

$$\begin{aligned} E(t) &= \int_{V(t)} \rho\frac{(|\mathbf{u}|^2)}{2}\,dx + \int_{V(t)} F(c,\nabla_x c,w)dx \\ &= \int_{V(t)} e(\rho,\mathbf{u},c,\nabla_x c,w)\,dx, \end{aligned}$$

where

$$e(\rho, \mathbf{u}, c, \nabla_x c, w, \nabla_x w) = \rho \frac{|\mathbf{u}|^2}{2} + \rho f(c, \nabla_x c, w).$$

Following Gurtin et al (1996) [33] we assume the dissipation inequality:

$$\frac{d}{dt} \int_{V(t)} e(\rho, \mathbf{u}, c, \nabla_x c, w)\, dx \tag{2.14}$$

$$\leq \int_{\partial V(t)} \mathbb{T}\mathbf{n} \cdot \mathbf{u}\, d\sigma + \int_{\partial V(t)} \dot{c}\mathbf{t} \cdot \mathbf{n}\, d\sigma + \int_{\partial V(t)} \dot{w}\mathbf{s} \cdot \mathbf{n}\, d\sigma + \int_{\partial V(t)} \mu \mathscr{F} \cdot \mathbf{n}\, d\sigma.$$

for every control volume $V(t)$ transported with flow, where σ denotes the two dimensional surface measure. In the above relation the term

$$\int_{\partial V(t)} \mathbb{T}\mathbf{n} \cdot \mathbf{u}\, d\sigma$$

represents the energy carried in $V(t)$ due to *macroscopic* stresses. The term

$$\int_{\partial V(t)} \mu \mathscr{F} \cdot \mathbf{n}\, d\sigma$$

represents the energy due to diffusion, whereas the term

$$\int_{\partial V(t)} \dot{c}\mathbf{t} \cdot \mathbf{n}\, d\sigma + \int_{\partial V(t)} \dot{w}\mathbf{s} \cdot \mathbf{n}\, d\sigma,$$

represents a *generalized surface force* on a *microscopic length scale*.

The equivalent local form of (2.14) is

$$\partial_t e + \operatorname{div}_x(\mathbf{u}e) - \operatorname{div}_x(\mathbb{T} \cdot \mathbf{u}) - \operatorname{div}_x(\dot{c}\mathbf{t}) - div(\dot{w}\mathbf{s}) - \operatorname{div}_x(\mu \mathscr{F}) =: D \leq 0. \tag{2.15}$$

Now, (2.1) and (2.2) yield

$$\partial_t\left(\rho\frac{|\mathbf{u}|^2}{2}\right)+\mathrm{div}_x\left(\mathbf{u}\rho\frac{|\mathbf{u}|^2}{2}\right)=\mathrm{div}_x(\mathbb{T}\cdot\mathbf{u})-\mathbb{T}:\nabla_x\mathbf{u}., \tag{2.16}$$

which together with

$$\partial_t F+\mathrm{div}_x(F\mathbf{u})=\partial_t(\rho f)+\mathrm{div}_x(\rho f\mathbf{u})=\rho\partial_t f+\rho\mathbf{u}\nabla_x f=\rho\dot{f}. \tag{2.17}$$

Combining (2.16) and (2.17) yield

$$\partial_t\left(\rho\frac{|\mathbf{u}|^2}{2}+F\right)+\mathrm{div}_x\left(\left(\rho\frac{|\mathbf{u}|^2}{2}+F\right)\mathbf{u}\right)=\mathrm{div}_x(\mathbb{T}\cdot\mathbf{u})-\mathbb{T}:\mathbf{u}+\rho\dot{f}. \tag{2.18}$$

Now,

$$\mathrm{div}_x(\dot{c}\,\mathbf{t})+\mathrm{div}_x(\dot{w}\,\mathbf{s})+\mathrm{div}_x(\mu\mathscr{F})= \tag{2.19}$$

$$\dot{c}\,\mathrm{div}_x\mathbf{t}+\nabla_x\dot{c}\,\mathbf{t}+\dot{w}\,\mathrm{div}_x\mathbf{s}+\nabla_x\dot{w}\,\mathbf{s}+\mu\,\rho\,\dot{c}++m|\nabla_x\mu|^2=$$

$$(\rho\mu+\mathrm{div}_x\mathbf{t})\,\dot{c}+\mathrm{div}_x\mathbf{s}\,\dot{w}+\left[(\dot{\nabla_x c})\cdot\mathbf{t}+(\dot{\nabla_x w})\cdot\mathbf{s}\right]$$

$$+\nabla_x\mathbf{u}:[\mathbf{t}\otimes\nabla_x c]+\nabla_x\mathbf{u}:[\mathbf{s}\otimes\nabla_x w]+m|\nabla_x\mu|^2.$$

Note,

$$(\dot{\rho f})-(f\dot{\rho})=$$

$$\frac{\partial F}{\partial\rho}\dot{\rho}+\frac{\partial F}{\partial c}\dot{c}+\frac{\partial F}{\partial\nabla_x c}(\dot{\nabla_x c})+\frac{\partial F}{\partial w}\dot{w}+\frac{\partial F}{\partial\nabla_x w}(\dot{\nabla_x w})-f\dot{\rho}=$$

$$\rho\frac{\partial f}{\partial\rho}\dot{\rho}+\frac{\partial F}{\partial c}\dot{c}+\frac{\partial F}{\partial\nabla_x c}(\dot{\nabla_x c})+\frac{\partial F}{\partial w}\dot{w}+\frac{\partial F}{\partial\nabla_x w}(\dot{\nabla_x w}) \tag{2.20}$$

Combining now (2.15), (2.18), (2.19) and (2.20) we get

$$D:=\left(\frac{\partial F}{\partial c}-\mathrm{div}_x\mathbf{t}-\rho\mu\right)\dot{c} \tag{2.21}$$

$$+\left(\frac{\partial F}{\partial \nabla_x c}-\mathbf{t}\right)(\dot{\nabla_x c})+\left(\frac{\partial F}{\partial w}-\mathrm{div}_x\mathbf{s}\right)\dot{w}+\left(\frac{\partial F}{\partial \nabla_x w}-s\right)(\dot{\nabla_x w})$$

$$-\left(\mathbb{T}+\rho^2\frac{\partial(\rho^{-1}F)}{\partial\rho}\mathbb{I}+\mathbf{t}\otimes\nabla_x c+\mathbf{s}\otimes\nabla_x w\right):\nabla_x\mathbf{u}-m|\nabla_x\mu|^2.$$

Using (2.13) the last relation takes the form

$$D:=\left(\mathrm{div}_x\frac{\partial F}{\partial \nabla_x c}-\mathrm{div}_x\mathbf{t}\right)\dot{c}$$

$$+\left(\frac{\partial F}{\partial \nabla_x c}-\mathbf{t}\right)(\dot{\nabla_x c})+\left(\frac{\partial F}{\partial w}-\mathrm{div}_x\mathbf{s}\right)\dot{w}+\left(\frac{\partial F}{\partial \nabla_x w}-s\right)(\dot{\nabla_x w})$$

$$-\left(\mathbb{T}+\rho^2\frac{\partial(\rho^{-1}F)}{\partial\rho}\mathbb{I}+\mathbf{t}\otimes\nabla_x c+\mathbf{s}\otimes\nabla_x w\right):\nabla_x\mathbf{u}-m|\nabla_x\mu|^2.$$

So for example we want $\mathrm{div}_x\mathbf{s}=\frac{\partial F}{\partial w}=(c-M)=-\triangle w=\mathrm{div}_x(-\nabla w)$. Hence, making the following constitutive assumptions

$$\begin{cases}\mathbf{t} & =\nabla c=\frac{\partial F}{\partial \nabla_x c},\\[2ex] \mathbf{s} & =-\nabla w=\frac{\partial F}{\partial \nabla w},\\[2ex] \mathbb{S} & =\mathbb{T}+P(\rho,c,\nabla_x c,w,\nabla_x w)\mathbb{I}+\mathbf{t}\otimes\nabla_x c+\mathbf{s}\otimes\nabla_x w\\[2ex] & =2\nu(c)\mathbb{D}\mathbf{u}+\eta(c)\mathrm{div}_x\mathbf{u}\mathbb{I},\end{cases}\tag{2.22}$$

where,

$$P(\rho,c,\nabla_x c,w)=\rho^2\frac{\partial(\rho^{-1}F)}{\partial\rho}(\rho,c,\nabla_x c,w).\tag{2.23}$$

we conclude that

$$D=-2\nu(c)|\mathbb{D}\mathbf{u}|^2-\eta(c)|\mathrm{div}_x\mathbf{u}|^2-m|\nabla_x\mu|^2\leq 0.\tag{2.24}$$

Note that, the stress tensor $\mathbb{T}$ differs from the stress tensor for a single compressible Newtonian fluid by two extra stresses, namely the stress $\frac{\partial F}{\partial \nabla_x c} \otimes \nabla_x c$, which is often called *Ericksen's term* and the extra stress $\frac{\partial F}{\partial \nabla w} \otimes \nabla_x w$ which accounts for the nature of the joint *A & B subchain interactions* in the diblock copolymer macromolecule. Finally, if we specify F to be of the form

$$F(\rho, c, \nabla_x c, w) = \rho f(\rho, c) + \frac{1}{2}|\nabla c|^2 + (c - M)w - \frac{1}{2}|\nabla_x w|^2$$

we have

$$P(\rho) = \rho^2 \frac{\partial f}{\partial \rho}(\rho, c) - \frac{|\nabla_x c|^2}{2} - (c - M)w + \frac{1}{2}|\nabla_x w|^2 \tag{2.25}$$

Thus, (1.1)-(1.5) are obtained, due to (2.1)(2.2)(2.4) and (2.13), where an assumption is made for simplicity $m = 1$.

2.3 Preliminaries: A Priori Bounds and Compactness

In this section, a priori bounds for weak solutions of the system will be discussed.

2.3.1 Total mass conservation

By integration of (1.1) over Ω

$$\int_\Omega \rho(t)\, dx = \int_\Omega \rho_0\, dx \equiv M_0 \text{ for almost all } t \in (0, T). \tag{2.26}$$

And we do the same with (1.3),

$$\int_\Omega \rho(t) c(t)\, dx = \int_\Omega \rho_0 c_0\, dx \text{ for almost all } t \in (0, T) \tag{2.27}$$

2.3.2 Total energy balance

Integrating (2.15) with respect to Ω, using (1.9) and (2.24)

$$\frac{d}{dt}E(t)+\int_{\Omega}\left(2\nu(c,w)|\mathbb{D}(\mathbf{u})|^2+\eta(c,w)|\mathrm{div}_x\mathbf{u}|^2+m|\nabla_x\mu|^2\right)(t)dx=0 \tag{2.28}$$

for sufficiently smooth solutions, where $E(t)$ is as in (1.17). For weak solutions, this equality will turn into inequality as in (1.21). The total energy (1.21), together with (1.11), yields uniform estimates

$$\operatorname{ess\,sup}_{t\in(0,T)}\|\rho\|_{L^\gamma(\Omega)}\leq C(M_0,E_0) \tag{2.29}$$

$$\operatorname{ess\,sup}_{t\in(0,T)}\|\nabla_x c\|_{L^2(\Omega;\mathbb{R}^3)}\leq C(M_0,E_0) \tag{2.30}$$

$$\operatorname{ess\,sup}_{t\in(0,T)}\|\sqrt{\rho}\mathbf{u}\|_{L^2(\Omega;\mathbb{R}^3)}\leq C(M_0,E_0) \tag{2.31}$$

$$\operatorname{ess\,sup}_{t\in(0,T)}\|\nabla_x w\|_{L^2(\Omega;\mathbb{R}^3)}\leq C(M_0,E_0) \tag{2.32}$$

$$\int_0^T\|\nabla_x\mu\|^2_{L^2(\Omega;\mathbb{R}^3)}dt\leq C(M_0,E_0) \tag{2.33}$$

where E_0 denotes the initial energy defined in (1.18) and M_0 is the total mass as in (2.26). Moreover, by means of Korn's inequality and hypothesis (1.11),

$$\int_0^T\|\nabla_x\mathbf{u}\|^2_{L^2(\Omega;\mathbb{R}^3)}dt\leq C(M_0,E_0) \tag{2.34}$$

2.3.3 Cahn-Hilliard Type Equation

A weak formulation of (1.3)-(1.5), taking the boundary conditions for (c,w,μ) in (1.9) into account, reads

$$\int_{Q_T}(\rho c\partial_t\varphi+\rho c\mathbf{u}\cdot\nabla v)\,d(x,t)=\int_{Q_T}\nabla\mu\cdot\nabla\varphi d(x,t), \tag{2.35}$$

$$\int_{Q_T} \rho\mu\varphi d(x,t) = \int_{Q_T} \left(\rho \frac{\partial f(\rho,c)}{\partial c}\varphi + w\varphi + \nabla c \cdot \nabla \varphi\right) d(x,t) \tag{2.36}$$

$$\int_{Q_T} \nabla w \cdot \nabla \varphi \, d(x,t) = \int_{Q_T} (c-M)\varphi \, d(x,t) \tag{2.37}$$

for any test function $\varphi \in \mathscr{D}((0,T)\times\bar{\Omega})$. In order to estimate $\|c(t)\|_{L^2}$ and $\|\mathbf{u}(t)\|_{L^2}$, the following simple variant of Poincare's inequality (cf. Lemma 3.1 in [21]) is used:

Lemma 2.3.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Assume that ρ is a non-negative function such that*

$$0 < M = \int_\Omega \rho \, dx, \int_\Omega \rho^\gamma dx \leq K,$$

with $\gamma > \frac{6}{5}$. Then there exists a constant $C = C(\gamma, M, K)$ such that

$$\left\| w - \frac{1}{|\Omega|}\int_\Omega \rho w \, dx \right\|_{L^2(\Omega)} \leq C(\gamma, M, K)\|\nabla w\|_{L^2(\Omega;\mathbb{R}^3)}$$

for any $w \in W^{1,2}(\Omega)$.

Applying the lemma directly with estimates (2.29) (2.31) (2.34), boundary condition (1.9), and hypothesis on viscosity coefficients (1.11), following bounds are obtained:

$$\int_0^T \|\mathbf{u}\|^2_{W_0^{1,2}(\Omega;\mathbb{R}^3)} dt \leq C(M_0, E_0) \tag{2.38}$$

$$\int_0^T \|\mathbb{S}\|^2_{L^2(\Omega;\mathbb{R}^3)} dt \leq C(M_0, E_0) \tag{2.39}$$

Similarly, by estimates (2.27), (2.30), (1.16), and (2.32), the following estimates for c and w are obtained

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|c\|^2_{W^{1,2}(\Omega)} dt \leq C(c_0, M_0, E_0), \tag{2.40}$$

and

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|w\|^2_{W^{1,2}(\Omega)} dt \leq C(E_0). \tag{2.41}$$

Choosing $\varphi = \varphi(t)$, (2.36) becomes

$$\int_{Q_T} \rho(t)\mu(t)\varphi(t) d(x,t) = \int_{Q_T} \rho \frac{\partial f(\rho(t), c(t))}{\partial c(t)} \varphi \, dx \quad \text{for a.a. } t \in (0,T),$$

because of the estimate of w (1.16). The integral on the right-hand side

$$\left| \frac{\partial f}{\partial c} \right| \leq 1 + |c| \text{ for all c.}$$

Therefore, in accordance with (2.33), one have

$$\int_0^T \|\mu\|^2_{W^{1,2}(\Omega)} dt \leq C(c_0, M_0, E_0) \tag{2.42}$$

Weak formulation (2.37) and C^2 boundary condition enable us to apply improved regularity theorem. With the estimate for c (2.40) and estimate for w (2.41), we now obtain

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|w\|^2_{W^{3,2}(\Omega)} \leq C(c_0, M_0, E_0) \tag{2.43}$$

2.4 Strong compactness of gradients of concentration and interaction terms

This is one of the main ingredients of the proof. This result will be later used twice in the proof for existence theorem. Assume that $\rho_n \geq 0$,

$$\rho_n \to \rho \text{ in } C_{\text{weak}}\left([0,T]; L^\gamma(\Omega)\right), \tag{2.44}$$

$$c_n \to c \text{ weakly* in } L^\infty\left(0,T; W^{1,2}(\Omega)\right), \tag{2.45}$$

$$w_n \to w \text{ weakly* in } L^\infty\left(0,T; W^{3,2}(\Omega)\right), \tag{2.46}$$

$$\partial_t\left(\rho_n c_n\right) \text{ is bounded in } L^q\left(0,T; W^{-1,q}(\Omega)\right) \text{ for a certain } q > 1, \tag{2.47}$$

and, in addition,

$$\int_0^T \int_\Omega \nabla c_n \cdot \nabla \varphi \, dx dt = \int_0^T \int_\Omega \sqrt{\rho_n} f_n \varphi \, dx dt + \int_0^T \int_\Omega g_n \varphi \, dx dt + \int_0^T \int_\Omega w \varphi \, dx dt \quad (2.48)$$

$$\int_0^T \int_\Omega \nabla w_n \cdot \nabla \varphi \, dx dt = \int_0^T \int_\Omega (c_n - M_n) \varphi \, dx dt \quad (2.49)$$

for any $\varphi \in \mathscr{D}((0,T) \times \bar{\Omega})$, where

$$\left\{ \begin{array}{c} f_n \to f \text{ weakly in } L^2((0,T) \times \Omega), \\ g_n \to g \text{ (strongly) in } L^1 \left(0,T; L^{\frac{6}{5}}(\Omega)\right). \end{array} \right\} \quad (2.50)$$

Our first goal is to show that

$$\int_0^T \int_\Omega |\nabla c_n|^2 \mathrm{d}x \to \int_0^T \int_\Omega |\nabla c|^2 \mathrm{d}x \quad (2.51)$$

which yields with (2.45),

$$c_n \to c \text{ in } L^2(0,T; W^{1,2}(\Omega)).$$

To this end, we observe first that $\rho \geq 0$, and

$$c_n \to c \text{ a.a. on the set } \{\rho > 0\} \quad (2.52)$$

passing to a suitable subsequence as the case may be. Indeed it follows from (2.44), (2.45) that

$$\rho_n c_n \to \rho c \text{ weakly-(*) in } L^\infty(0,T; L^q(\Omega)) \text{ for a certain } q > \frac{6}{5},$$

which, together with (2.47), gives rise to

$$\rho_n c_n^2 \to \rho c^2 \text{ weakly-(*) in } L^\infty(0,T; L^r(\Omega)) \text{ for a certain } r > 1.$$

Since, by the same token,

$$(\rho_n - \rho)\, c_n^2 \to 0 \text{ weakly-(*) in } L^\infty(0,T;L^r(\Omega)) \text{ for a certain } r > 1,$$

we get

$$\int_0^T \int_\Omega \rho c_n^2 dxdt \to \int_0^T \int_\Omega \rho c^2 dxdt,$$

in particular, (2.52) follows.

We then want to show

$$w_n \to w \text{ a.a.} \tag{2.53}$$

passing to a suitable subsequence. Indeed it follows from (2.46) that

$$w_n \text{ is bounded in } L^\infty(0,T;W^{3,2}(\Omega))$$

which implies

$$w_n^2 \text{ is bounded in } L^\infty(0,T;L^r(\Omega)).$$

for some $r > 1$. The bound above give rise to

$$w_n^2 \to w^2 \text{ weakly* in } L^\infty(0,T;L^r(\Omega)),$$

for a subsequence of w_n, which yields,

$$\int_0^T \int_\Omega w_n^2\, dx\, dt \to \int_0^T \int_\Omega w^2\, dx\, dt.$$

In particular, (2.53) follows.

On the other hand, letting $n \to \infty$ in (2.48) yields

$$\int_0^T \int_\Omega \nabla c \cdot \nabla \varphi \, dx dt = \int_0^T \int_\Omega \overline{\sqrt{\rho} f} \varphi \, dx dt + \int_0^T \int_\Omega g \varphi \, dx dt + \int_0^T \int_\Omega w \varphi \, dx dt$$

for any $\varphi \in \mathscr{D}((0,T) \times \Omega)$, where

$$\sqrt{\rho_n} f_n \to \overline{\sqrt{\rho} f} \text{ weakly in } L^2\left(0,T;L^q(\Omega)\right) \text{ for a certain } q > \frac{6}{5}.$$

In particular, by means of a standard density argument,

$$\int_0^T \int_\Omega |\nabla c|^2 \, dx dt = \int_0^T \int_\Omega \overline{\sqrt{\rho} f} c \, dx dt + \int_0^T \int_\Omega g c \, dx dt + \int_0^T \int_\Omega w c \, dx dt \tag{2.54}$$

Finally, taking $\varphi = c_n$ and letting $n \to \infty$ in (2.48), we obtain

$$\lim_{n \to \infty} \int_\Omega |\nabla c_n|^2 \, dx dt = \int_0^T \int_\Omega \overline{\sqrt{\rho} f c} \, dx dt + \int_0^T \int_\Omega g c \, dx dt + \int_0^T \int_\Omega \overline{wc} \, dx dt$$

which, combined with (2.54), gives rise to the desired conclusion (2.51) as soon as we observe that

$$\overline{\sqrt{\rho} f c} = \overline{\sqrt{\rho} f} c \tag{2.55}$$

$$\overline{wc} = wc \tag{2.56}$$

where, according to the standard notation convention adopted in this paper, the bar stands for a weak limit in L^1.

In accordance with (2.52), relation (2.55) is satisfied on the set $\{\rho > 0\}$ where $c_n \to c$ strongly in $L^1(\Omega)$. Similarly, by (2.53), relation (2.56) is satisfied for any ρ. On the other hand, since ρ_n are non-negative,

$$\rho_n \to 0 \text{ (strongly) in } L^q(\{\rho = 0\}) \text{ for any } 1 \leq q < \gamma$$

whence (2.55) holds on the set $\{\rho = 0\}$ as well. The proof of (2.51) is now complete.

In the same spirit, our second goal

$$w_n \to w \text{ in } L^2(0,T;W^{1,2}(\Omega)).$$

can be obtain if

$$\overline{(c-M)w} = (c-M)w,$$

which follows from the strong convergence of c in $L^2(0,T;W^{1,2}(\Omega))$.

Chapter 3: Existence of the Weak Solution of System with Artificial Pressure

In this chapter, we will discuss the first part of the two-level approximation. In the first approximation level we add an artificial pressure term that ensures better integrability of ρ and ρc. This technique is well-known and can be found e.g. in [[20] , [41],[43]]. More precisely, we start with the approximate system with artificial pressure added

$$\begin{aligned}&\int_{Q_T}\left(\rho_\delta \mathbf{u}_\delta\cdot\partial_t\varphi+\rho_\delta\left(\mathbf{u}_\delta\otimes\mathbf{u}_\delta\right):\nabla\varphi+\left(p\left(\rho_\delta,c_\delta\right)+\delta\rho_\delta^\Gamma\right)\operatorname{div}\varphi\right)d(x,t)\\&=\int_0^T\int_\Omega\left(\mathbb{S}_\delta-\mathbb{P}_\delta\right):\nabla\varphi dxdt-\int_\Omega\rho_{0,\delta}\mathbf{u}_0\cdot\varphi(0)dx\end{aligned}\tag{3.1}$$

for any $\varphi\in\mathscr{D}\left([0,T)\times\Omega;\mathbb{R}^3\right)$,

$$\int_{Q_T}\left(\rho_\delta\partial_t\varphi+\rho_\delta\mathbf{u}_\delta\cdot\nabla\varphi\right)d(x,t)=-\int_\Omega\rho_{0,\delta}\varphi\Big|_{t=0}dx\tag{3.2}$$

$$\int_{Q_T}\left(\rho_\delta c_\delta\partial_t\varphi+\left(\rho_\delta c_\delta\mathbf{u}_\delta-\nabla\mu\right)\cdot\nabla\varphi\right)d(x,t)=-\int_\Omega\rho_{0,\delta}c_0\varphi\Big|_{t=0}dx\tag{3.3}$$

$$\int_{Q_T}\rho\mu\varphi\,d(x,t)=\int_{Q_T}\left(\rho\frac{\partial f}{\partial c}\varphi+w\varphi+\nabla_x c\cdot\nabla_x\varphi\right)d(x,t)\tag{3.4}$$

$$\int_{Q_T}\nabla w\cdot\nabla\varphi\,d(x,t)=\int_{Q_T}(C-M)\varphi\,d(x,t)\tag{3.5}$$

for any $\varphi \in \mathscr{D}\left([0,T)\times\bar{\Omega}\right)$, where

$$\mathbb{S}_\delta = 2\nu(c_\delta)\mathbb{D}(\mathbf{u}_\delta) + \eta(c_\delta)\mathrm{div}_x\mathbf{u}_\delta\mathbb{I}, \tag{3.6}$$

$$\mathbb{P}_\delta = \nabla_x c_\delta \otimes \nabla_x c_\delta - \frac{|\nabla_x c_\delta|^2}{2}\mathbb{I} - \nabla_x w_\delta \otimes \nabla_x w_\delta - (c_\delta - M_\delta)w_\delta + \frac{|\nabla_x w_\delta|^2}{2}\mathbb{I} \tag{3.7}$$

$$p_\delta = p(\rho) + \delta\rho^\Gamma + \delta\rho^2 c^2 \tag{3.8}$$

Here, $\rho_{0,\delta} \in L^\Gamma(\Omega)$ such that $\rho_{0,\delta} \geq 0$, and $\rho \to \rho_0$ in $L^\gamma(\Omega)$.....

$$E_\delta(t) + \int_{Q_{(s,t)}} (\mathbb{S}_\delta : \nabla_x u_\delta + |\nabla_x \mu_\delta|^2) \leq E_\delta(s) \tag{3.9}$$

for almost all $0 \leq s \leq t \leq T$ including $s = 0$, where

$$E_\delta = \int_\Omega \frac{\rho|u_\delta|^2}{2} + \rho_\delta f(\rho_\delta, c_\delta) + \frac{|\nabla_x c_\delta|^2}{2} + \frac{|\nabla_x w_\delta|^2}{2} + \frac{\delta}{\Gamma-1}\rho_\delta^\Gamma + \delta\rho^2 c^2 \tag{3.10}$$

The main goal of this section is to prove:

Theorem 3.0.1. *Let $\Gamma > 3$, $\delta > 0$, and let $0 < T < \infty$. Then for every non-negative $\rho_{0,\delta} \in L^\Gamma(\Omega)$, measurable $u_0 : \Omega \to \mathbb{R}^3$ with $\rho_{0,\delta}|u_0|^2 \in L^1(\Omega)$, $c_0 \in H^1(\Omega)$ and $w_0 \in H^1(\Omega)$ there are some $\rho_\delta \in L^\infty(0,T;L^\Gamma(\Omega)) \cap L^{\Gamma+1}(Q_T)$, $\delta \geq 0$, $u_\delta \in L^2(0,T;H^1(\Omega;\mathbb{R}^3))$, $c_\delta \in L^\infty(0,T;H^1(\Omega))$ $w_\delta \in L^\infty(0,T;H^1(\Omega))$ solving (3.1)- (3.8) and satisfying (3.9)*

In order to prove this theorem, we use a suitable time discretization to approximate (3.1)-(3.5). For simplicity we will drop the subscript δ in most quantities for the rest of this chapter.

3.1 Implicit Time Discretization

Let $h>0$. Given $(\mathbf{u}_k,\rho_k,c_k)\in L^2(\Omega)^3\times L^\Gamma(\Omega)\times H^1(\Omega)$ we determine $(\mathbf{u}_{k+1},\rho_{k+1},c_{k+1},\mu_{k+1})$ as a solution of the system

$$\begin{aligned}\frac{\rho\mathbf{u}-\rho_k\mathbf{u}_k}{h}&+\operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u})-\operatorname{div}\mathbb{S}\\&+\varepsilon(\nabla\rho\cdot\nabla)\mathbf{u}+\nabla p_\delta=\rho\mu\nabla c-\rho\frac{\partial f}{\partial c}(\rho,c)\nabla c\end{aligned}\tag{3.11}$$

$$\frac{\rho-\rho_k}{h}+\operatorname{div}(\rho\mathbf{u})=\varepsilon\Delta\rho\tag{3.12}$$

$$\rho_k\frac{c-c_k}{h}+\rho\mathbf{u}\cdot\nabla c=\Delta\mu\tag{3.13}$$

$$\rho_k\mu=\rho_k\frac{f(\rho_k,c)-f(\rho_k,c_k)}{c-c_k}+w-\Delta c\tag{3.14}$$

$$-\Delta w=c-M\tag{3.15}$$

where $p_\delta(\rho)=p(\rho)+\delta\rho^\Gamma+\delta\rho^2c^2$, $p(\rho)=\rho^2\frac{\partial f}{\partial\rho}=\rho^2\frac{\partial f_e}{\partial\rho}$ and

$$\mathbb{S}(c_k,\nabla\mathbf{u})=2\nu(c_k)\mathbb{D}(\mathbf{u})+\eta(c_k)\operatorname{div}_x\mathbf{u}\mathbb{I},\tag{3.16}$$

together with the boundary conditions

$$\mathbf{u}|_{\partial\Omega}=\partial_n\rho|_{\partial\Omega}=\partial_n c|_{\partial\Omega}=\partial_n w|_{\partial\Omega}=\partial_n\mu|_{\partial\Omega}=0\tag{3.17}$$

Here (3.11)-(3.15) will be understood in the sense of distributions and (3.17) in the sense of traces of Sobolev functions. Note that (3.12)-(3.13) implies that

$$\int_\Omega\rho_k\,dx=\int_\Omega\rho\,dx,\quad\int_\Omega\rho_kc_k\,dx=\int_\Omega\rho c\,dx$$

which is the time discrete version of (2.26)-(2.27). Moreover, corresponding to (1.16), the following equality holds

$$\int_\Omega w\,dx = 0. \tag{3.18}$$

Remark 3.1.1. *The right hand side of (3.11) is given by*

$$\begin{aligned}
&-\mathrm{div}_x\left[\nabla_x c\otimes\nabla_x c-\frac{1}{2}|\nabla_x c|^2\mathbb{I}\right]-\mathrm{div}_x\left[-\nabla_x w\otimes\nabla_x w-(c-M)w\mathbb{I}+\frac{1}{2}|\nabla_x w|^2\mathbb{I}\right]\\
&\quad=-\Delta c\nabla_x c+\Delta w\nabla_x w+\nabla_x c\,w+(c-M)\nabla_x w\\
&\quad=\rho\mu\nabla_x c-\frac{\partial f}{\partial c}\nabla_x c-w\nabla_x c-(c-M)\nabla_x w+\nabla_x c\,w+(c-M)\nabla_x w\\
&\quad=\rho\mu\nabla_x c-\rho\frac{\partial f}{\partial c}\nabla_x c
\end{aligned}\tag{3.19}$$

provided (1.4) (1.5) holds. In the discrete system above, although (1.4) does not hold, the form of (3.11)-(3.15) provide that a similar energy estimate holds.

Because we have assumption $p=\rho^2\frac{\partial f_e}{\partial\rho}$, we decompose $p=\tilde{p}_m+p_b$, where $p_b\in C^2([0,\infty))$, $p_b\le 0$ has a compact support, $p_m(0)=0$ and

$$\underline{\tilde{p}_m}(1+\rho^{\Gamma-1})\le\tilde{p}_m'(\rho)\le\overline{\tilde{p}_m}(1+\rho^{\Gamma-1}). \tag{3.20}$$

for some constants $\overline{p_m},\underline{p_m}>0$. Moreover, we can see that $\delta\rho^2c^2>0$. Therefore, by these assumptions we have that

$$p_\delta(\rho,c)=p_m(\rho,c)+p_b(\rho), \tag{3.21}$$

where $p_m(\rho,c)=\tilde{p}_m(\rho)+\delta\rho^2c^2\ge 0$ is again monotone with respect to ρ. The decomposition of p induces a decomposition of f

$$f(\rho,c)=f_m(\rho,c)+f_b(\rho)$$

where $f_m(\rho,c) = \int_0^\rho \frac{p_m(s,c)}{s^2}ds + f_0(c)$ and $\rho \mapsto \rho f_m(\rho,c)$ is convex and monotone. Moreover, we define the energy

$$E_m(\rho,u,c,w) = \int_\Omega \frac{\rho|u|^2}{2} + \rho f_m(\rho,c) + \frac{|\nabla_x c|^2}{2} + \frac{|\nabla_x w|^2}{2}\,dx$$

Lemma 3.1.1. *Let $(u_k,\rho_k,c_k,w_k) \in L^2(\Omega;\mathbb{R}^3) \times L^\gamma(\Omega) \times H^1(\Omega) \times H^1(\Omega)$, $\rho \geq 0$, and let $0 < \varepsilon \leq 1$. Then every $(u,\rho,c,w,\mu) \in H^1(\Omega;\mathbb{R}^3) \times H^2(\Omega)^4$, $\rho \geq 0$, solving (3.11)-(3.17) satisfies the discrete energy estimate*

$$\begin{aligned}
&E_m(\rho,\mathbf{u},c,w) + \varepsilon h \int_\Omega \frac{\partial_\rho p_m}{\rho}|\nabla\rho|^2 dx + \int_\Omega \frac{\rho_k\,|\mathbf{u}-\mathbf{u}_k|^2}{2}dx \\
&\quad + \frac{\|\nabla(c-c_k)\|_2^2}{2} + \frac{\|\nabla(w-w_k)\|_2^2}{2} + \alpha\,\|\rho-\rho_k\|_2^2 + h\int_\Omega \mathbb{S} : \nabla\mathbf{u}dx + h\|\nabla\mu\|_2^2 \\
&\quad \leq E_m(\rho_k,\mathbf{u}_k,c_k,w_k) + R_k
\end{aligned} \tag{3.22}$$

for some $\alpha > 0$ depending only on f_m. Here $p_m = p_m(\rho,c)$ and

$$R_k = h\int_\Omega p_b(\rho)\,\mathrm{div}\,\mathbf{u}\,dx - \varepsilon h \int_\Omega \nabla\rho\cdot\nabla c \frac{\partial^2(\rho f_m(\rho,c))}{\partial\rho\partial c}\,dx. \tag{3.23}$$

Moreover, there is some $h_0 > 0$ independent of (u_k,ρ_k,c_k,w_k) and $\varepsilon > 0$ such that any solution $(u,\rho,c,w,\mu) \in H^1(\Omega;\mathbb{R}^3) \times H^2(\Omega)^4$ with $\rho \geq 0$ satisfies

$$\begin{aligned}
&\left\|\left(\rho^{\frac{1}{2}}\mathbf{u},\nabla c,\nabla w\right)\right\|_2^2 + \|\rho\|_\Gamma^\Gamma + h\left\|\left(\mathbf{u},\nabla\mathbf{u},\mu,\nabla\mu,\varepsilon^{\frac{1}{2}}\nabla\rho,\varepsilon^{\frac{1}{2}}c\nabla\rho\right)\right\|_2^2 \\
&\quad \leq C\left(E_m(\rho_k,\mathbf{u}_k,c_k,w_k)+1\right)
\end{aligned} \tag{3.24}$$

where C is independent of h with $0 < h \leq h_0$, $0 < \varepsilon \leq 1$ and ρ_k, $\mathbf{u}_k$, c_k, but depends on $\int_\Omega \rho_k dx$ and $\int_\Omega \rho_k c_k dx$. Finally, for all $0 < h \leq h_0$ there is some $(u,\rho,c,w,\mu) \in H^1(\Omega;\mathbb{R}^3) \times H^2(\Omega)^4$ with $\rho \geq 0$ solving (3.11)-(3.17).

Proof. To prove the energy estimate, we multiply (3.12) with $\frac{1}{2}|\mathbf{u}|^2$ and integrating by parts

$$\begin{aligned}\int_\Omega \frac{\rho|\mathbf{u}|^2-\rho_k\mathbf{u}_k\cdot\mathbf{u}}{h}dx+\int_\Omega \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u})\cdot\mathbf{u}dx+\varepsilon\int_\Omega(\nabla\rho\cdot\nabla)\mathbf{u}\cdot\mathbf{u}dx\\ =\int_\Omega\rho\frac{|\mathbf{u}|^2}{2h}dx-\int_\Omega\rho_k\frac{|\mathbf{u}_k|^2}{2h}dx+\int_\Omega\rho_k\frac{|\mathbf{u}-\mathbf{u}_k|^2}{2h}dx\end{aligned}$$

Hence, testing (3.11) with $\mathbf{u}$ we obtain

$$\begin{aligned}\int_\Omega\rho\frac{|\mathbf{u}|^2}{2h}dx-\int_\Omega\rho_k\frac{|\mathbf{u}_k|^2}{2h}dx+\int_\Omega\rho_k\frac{|\mathbf{u}-\mathbf{u}_k|^2}{2h}dx+\int_\Omega\mathbb{S}:\nabla\mathbf{u}dx\\ =\int_\Omega p_\delta\operatorname{div}\mathbf{u}dx+\int_\Omega\rho\mu\nabla c\cdot\mathbf{u}dx-\int_\Omega\rho\frac{\partial f}{\partial c}(\rho,c)\nabla c\cdot\mathbf{u}dx.\end{aligned}\tag{3.25}$$

Moreover, multiplying (3.12) with $\partial_\rho F(\rho)$, where $F(\rho,c)=\rho f_m(\rho,c)$, we obtain

$$\begin{aligned}&\frac{\rho-\rho_k}{h}\partial_\rho F(\rho,c)+\operatorname{div}(F(\rho,c)\mathbf{u})\\ &\quad+p_m(\rho)\operatorname{div}\mathbf{u}=\varepsilon\Delta\rho\partial_\rho F(\rho,c)+\rho\partial_c f(\rho,c)\nabla c\cdot\mathbf{u}\end{aligned}$$

since $\rho\partial_\rho F(\rho,c)-F(\rho,c)=\rho^2\partial_\rho f_m(\rho,c)=p_m(\rho,c)$. Furthermore, since $\frac{\partial^2 F}{\partial\rho^2}(\rho,c)=\rho^{-1}\partial_\rho p_m(\rho,c)\geq\frac{\alpha}{2}>0$ for some $\alpha>0$ due to (3.20), we have

$$\frac{\partial F}{\partial c}(\rho,c)(\rho-\rho_k)\geq\rho f_m(\rho,c)-\rho_k f_m(\rho_k,c)+\alpha(\rho-\rho_k)^2.$$

Therefore

$$\begin{aligned}\frac{1}{h}\int_\Omega(\rho f_m-\rho_k f_m(\rho_k,c))dx+\alpha\|\rho-\rho_k\|_2^2\leq-\int_\Omega p_m\operatorname{div}\mathbf{u}dx\\ -\varepsilon\int_\Omega\frac{\partial_\rho p_m}{\rho}|\nabla\rho|^2dx-\varepsilon\int_\Omega\nabla\rho\cdot\nabla c\frac{\partial^2(\rho f)}{\partial\rho\partial c}dx+\int_\Omega\rho\frac{\partial f}{\partial c}\nabla c\cdot\mathbf{u}dx,\end{aligned}\tag{3.26}$$

where f_m,p_m and their derivatives depend on ρ,c if not stated differently. Moreover, multiplying (3.13) with μ and (3.14) with $\frac{c-c_k}{h}$, we obtain

$$\begin{aligned}
&\frac{1}{h}\int_\Omega \rho_k\left(f_m\left(\rho_k,c\right)-f_m\left(\rho_k,c_k\right)\right)dx+\frac{\|\nabla c\|_2^2}{2h}+\frac{\|\nabla\left(c-c_k\right)\|_2^2}{2h}\\
&+\frac{\|\nabla w\|_2^2}{2h}+\frac{\|\nabla\left(w-w_k\right)\|_2^2}{2h}+\int_\Omega|\nabla\mu|^2dx\\
&\leq\frac{\|\nabla c_k\|_2^2}{2h}+\frac{\|\nabla w_k\|_2^2}{2h}-\int_\Omega\rho\mu\nabla c\cdot\mathbf{u}+\frac{M-M_k}{h}w\,dx\\
&=\frac{\|\nabla c_k\|_2^2}{2h}+\frac{\|\nabla w_k\|_2^2}{2h}-\int_\Omega\rho\mu\nabla c\cdot\mathbf{u}\,dx
\end{aligned}\tag{3.27}$$

since equation (3.15) (3.18), $(a-b)\cdot a=\frac{|a|^2}{2}+\frac{|a-b|^2}{2}-\frac{|b|^2}{2}$ for all $a,b\in\mathbb{R}^3$ and $f\left(\rho_k,c_k\right)-f\left(\rho_k,c\right)=f_m\left(\rho_k,c_k\right)-f_m\left(\rho_k,c\right)$. Combining (3.25)-(3.27) we obtain (3.22).

In order to estimate R_k

$$\left|\int_\Omega\nabla\rho\cdot\nabla c\frac{\partial^2\left(\rho f_m(\rho,c)\right)}{\partial\rho\partial c}dx\right|=\int|\nabla\rho\cdot\nabla c\ (1+|c|)|dx\leq\|(1+|c|)\nabla_x\rho\|_2\|\nabla_x c\|_2$$

since $\frac{\partial^2(\rho f_m)}{\partial\rho\partial c}(\rho,c)=f_0'(c)$.Hence,

$$|R_k|\leq C(h\|\mathrm{div}_x\mathbf{u}\|_2+\varepsilon h\|(1+c)\nabla_x\rho\|_2\|\nabla_x c\|_2)\tag{3.28}$$

Moreover, (3.22) and $\rho^{-1}\partial p_m\geq 2c^2+\underline{\tilde{p_m}}(\rho^{-1}+\rho^{\Gamma-1})$ due to (3.20) imply that

$$\begin{aligned}
&\left\|\left(\rho^{\frac{1}{2}}\mathbf{u},\nabla c,\nabla w\right)\right\|_2^2+\|\rho\|_\Gamma^\Gamma+h\left\|\left(\mathbf{u},\nabla\mu,\varepsilon^{\frac{1}{2}}(1+\log\rho)\nabla\rho,\varepsilon^{\frac{1}{2}}c\nabla\rho\right)\right\|_2^2\\
&\leq C\left(E_m\left(\rho_k,\mathbf{u}_k,c_k\right)+|R_k(\rho,\mathbf{u},c)|\right)
\end{aligned}$$

By combining the last two estimates and using Young's inequalities,

$$\begin{aligned}
&\left\|\left(\rho^{\frac{1}{2}}\mathbf{u},\nabla c,\nabla w\right)\right\|_2^2+\|\rho\|_\Gamma^\Gamma+h\left\|\left(\mathbf{u},\nabla\mu,\varepsilon^{\frac{1}{2}}(1+\log\rho)\nabla\rho,\varepsilon^{\frac{1}{2}}c\nabla\rho\right)\right\|_2^2\\
&\leq C\left(E_m\left(\rho_k,\mathbf{u}_k,c_k,w_k\right)+1+\varepsilon h^{3/2}\|(1+c)\nabla_x\rho\|_2^2+h^{1/2}\|\nabla_x c\|_2^2\right)
\end{aligned}$$

where C is independent of ρ, $\mathbf{u}$, c, ρ_k, $\mathbf{u}_k$, ε, h. Then, there is some $h_0 > 0$ such that

$$\begin{aligned}&\left\|\left(\rho^{\frac{1}{2}}\mathbf{u}, \nabla c, \nabla w\right)\right\|_2^2 + \|\rho\|_\Gamma^\Gamma + h\left\|\left(\mathbf{u}, \nabla\mu, \varepsilon^{\frac{1}{2}}(1+\log\rho)\nabla\rho, \varepsilon^{\frac{1}{2}}c\nabla\rho\right)\right\|_2^2\\ &\quad \leq C\left(E_m(\rho_k, \mathbf{u}_k, c_k, w_k) + 1\right)\end{aligned}$$

for all $0 < h \leq h_0$. Finally, by the same estimates as in Section 2.3.3, Lemma 2.3.1 and (3.13) implies

$$\begin{aligned}\|c\|_2^2 + h\|\mu\|_2^2 + \|w\|_2^2 &\leq C\left(\|\nabla c\|_2^2 + h\|\nabla\mu\|_2^2 + \|\nabla w\|_2^2 + \left|\int_\Omega \rho c\, dx\right|^2 + h\left|\int_\Omega \rho_k \mu\, dx\right|^2\right)\\ &\leq C'\left(E_m(\rho_k, \mathbf{u}_k, c_k, w_k) + 1\right)\end{aligned}$$

where C, C' depend on $\int_\Omega \rho\, dx = \int_\Omega \rho_k dx$ and $\int_\Omega c\, dx = \int_\Omega \rho_k c_k dx$. This completes the proof of the uniform estimate (3.24).

Next we prove existence of solutions (for a fixed $0 < h \leq h_0$) with the aid of a homotopy argument and the Leray-Schauder degree. To this end we introduce operators $\mathscr{L}_k, \mathscr{F}_k : X \to Y$ with

$$\begin{aligned}&X = H_0^1(\Omega; \mathbb{R}^3) \times H_N^2(\Omega)^4, \quad H_N^2(W) = \{u \in H^2(W) : \partial_n u|_{\partial\Omega} = 0\},\\ &Y = H^{-1}(\Omega; \mathbb{R}^3) \times L^2(\Omega)^4,\end{aligned}$$

and

$$\mathscr{L}_k(\mathbf{u},\rho,c,w,\mu) = \begin{pmatrix} \operatorname{div} S(c_k,\nabla\mathbf{u}) \\ \lambda\rho + \operatorname{div}(\rho\mathbf{u}) - \varepsilon\Delta\rho \\ \Delta\mu + \int_\Omega \mu\, dx \\ -w + \Delta c + \int_\Omega c\, dx \\ \Delta w + \int_\Omega w\, dx \end{pmatrix}$$

$$\mathscr{F}_k(\mathbf{u},\rho,c,w,\mu) =$$

$$\begin{pmatrix} \frac{\rho\mathbf{u}-\rho_k\mathbf{u}_k}{h} + \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) + \varepsilon(\nabla\rho\cdot\nabla)\mathbf{u} + \nabla p_\delta - \rho\mu\nabla c + \rho\frac{\partial f}{\partial c}(\rho,c)\nabla c \\ \left(\lambda - \frac{1}{h}\right)[\rho]_+ + \frac{1}{h}\rho_k \\ \rho_k\frac{c-c_k}{h} + \rho\mathbf{u}\cdot\nabla c + \int_\Omega \mu\, dx \\ \rho_k\frac{f(\rho_k,c)-f(\rho_k,c_k)}{c-c_k} - \rho_k\mu + \int_\Omega c\, dx \\ \int_\Omega c\, dx - c + \int_\Omega c\, dx \end{pmatrix}$$

Here $\lambda \geq \max(\lambda_0, \frac{1}{h})$, where $\lambda_0 = \lambda_0(\varepsilon, K)$ is the constant in the statement of Lemma 3.1.2 below with K so large that $\|v\|_6 \leq K$ for any solution of (3.11)-(3.17). Then by Lemma 3.1.2 below and standard results on elliptic partial differential equations $\mathscr{L}_k : X \to Y$ is invertible. Moreover, if $\mathscr{L}_k(u,\rho,c,w,\mu) = \mathscr{F}_k(u,\rho,c,\mu)$ for some $(u,\rho,c,w,\mu) \in X$, then $\rho \geq 0$ by Lemma 3.1.2 and therefore $[\rho]_+ = \rho$. Hence $(u,\rho,c,w,\mu) \in X$ with $\rho \geq 0$ is a solution of (3.11)-(3.17) if and only if

$$\mathscr{L}_k(\mathbf{u},\rho,c,w,\mu) = \mathscr{F}_k(\mathbf{u},\rho,c,w,\mu) \Leftrightarrow (\mathbf{u},\rho,c,w,\mu) = \mathscr{L}_k^{-1}(\mathscr{F}_k(\mathbf{u},\rho,c,w,\mu))$$

i.e., $v = (u,\rho,c,\mu)$ solves $v - \mathscr{L}_k^{-1}(\mathscr{F}_k(v)) = 0$. . Moreover, the operator norms of $\mathscr{L}_k$ and $\mathscr{L}_k^{-1}$ can be bounded by a constant independent of c_k due to the bound for viscosity coefficients (1.11). Furthermore, it is easy to check that $\mathscr{L}_k^{-1}\mathscr{F}_k : X \to X$ is a continuous and compact mapping. Therefore, the Leray-Schauder degree of $I-\mathscr{L}_k^{-1}\mathscr{F}_k$ is well-defined, cf. e.g. []. In order to show that $\deg\left(I-\mathscr{L}_k^{-1}\mathscr{F}_k, B_R(0), 0\right)=1$ for sufficiently large $R>0$, let $\mathscr{F}_k^\tau(\mathbf{u},\rho,c,w,\mu), \tau\in[0,1]$, be the operator obtained by replacing for $\mathbf{u}_k,\rho_k,c_k,f$ in the definition of $\mathscr{F}_k(\mathbf{u},\rho,c,w,\mu)$ by $\mathbf{u}_k^\tau=(1-\tau)\mathbf{u}_k, \rho_k^\tau=(1-\tau)\rho_k+\tau, c_k^\tau=(1-\tau)c_k$ and

$$f^\tau(\rho,c)=\tau\left(\rho^{\Gamma-1}+1+c^2\right)+(1-\tau)f(\rho,c).$$

Then $v=(\mathbf{u},\rho,c,2,\mu)\in X$ solves $v-\mathscr{L}_k^{-1}\left(\mathscr{F}_k^\tau(v)\right)=0$ if and only if $(\mathbf{u},\rho,c,w,\mu)$ solve $(3.11)-(3.17)$ with $\mathbf{u}_k,\rho_k,c_k,f$ replaced by $\mathbf{u}_k^\tau,\rho_k^\tau,c_k^\tau,f^\tau$. Moreover, it is not difficult to check that for each fixed $\varepsilon>0$, $0<h\le h_0$, $\left\|\mathscr{F}_k^\tau(\mathbf{u},\rho,c,\mu)\right\|_Y$ can be estimated by the terms on the left-hand side of (3.24). Hence, if $v=(\mathbf{u},\rho,c,w,\mu)\in X$ solves $v-\mathscr{L}_k^{-1}\left(\mathscr{F}_k^\tau(v)\right)=0$, then

$$\|(\mathbf{u},\rho,c,\mu)\|_X\le C\,\|\mathscr{F}_k^\tau(\mathbf{u},\rho,c,\mu)\|_Y\le M\left(E_m\left(\rho_k,\mathbf{u}_k,c_k\right),\varepsilon,h\right)$$

for some continuous function M independent of $\tau\in[0,1]$. Hence there is some $R>1$ such that any solution of $w-\mathscr{L}_k^{-1}\mathscr{F}_k^\tau(w)=0$ with $0<h\le h_0$, and we got

$$\deg\left(I-\mathscr{L}_k^{-1}\mathscr{F}_k, B_R(0),0\right)=\deg\left(I-\mathscr{L}_k^{-1}\mathscr{F}_k^1, B_R(0),0\right)=1,$$

which concludes the proof of the lemma. $\square$

The following lemma is from Abel et al. [2, Lemma 3.4].

Lemma 3.1.2. *Let $K,\varepsilon>0$, and let $v\in H^1(\Omega;\mathbb{R}^3)$ with $\|v\|_6\le K$. Then there is some $\lambda_0=\lambda_0(\varepsilon,K)>0$ such that for any $\lambda\ge\lambda_0$ and any $f\in L^2(\Omega)$, there is a unique $\rho\in H^1(\Omega)$ solving*

$$\lambda(\rho,\varphi)_\Omega-(\mathbf{v}\rho,\nabla\varphi)_\Omega+\varepsilon(\nabla\rho,\nabla\varphi)_\Omega=(f,\varphi)_\Omega \tag{3.29}$$

for all $\varphi\in H^1(\Omega)$. Moreover, if $f\ge0$, then $\rho\ge0$.

Now, let $N \in \mathbb{N}$ be given and set $h = \frac{T}{N}$ and $\varepsilon = h$. If h_0 is the constant appear as in Lemma 3.1.1, then there is some N_0 such that $N \geq N_0$ implies $h \leq h_0$. Hence, if $N \geq N_0$, we can define $(\mathbf{u}_k, \rho_k, c_k, \mu_k, w_k)$, $k = 1, ..., N$, successively as solution of (3.11)-(3.17) with $(\mathbf{u}_0, \rho_0, c_0)$ as fixed initial values. Moreover, Define $g^N(t) : (-h, \infty)$ by $g^N(t) = g_k$ for $t \in ((k-1)h, kh]$, where $g \in \{\mathbf{u}, \rho, c, \mu\}$ (setting $\mu_0 = 0$) as well as $p_\delta^N = p\left(\rho^N, c^N\right) + \delta\left(\rho^N\right)^\Gamma + \delta(\rho^N)^2(c^N)^{2\Gamma}$. In what follows we denote

$$\begin{aligned}
\left(\Delta_h^+ f\right)(t) &= f(t+h) - f(t), \quad & \left(\Delta_h^- f\right)(t) &= f(t) - f(t-h), \\
\left(\tau_h g\right)(t) &= g(t-h), & \partial_{t,h}^{\pm} f &= \frac{1}{h}\Delta_h^{\pm} f.
\end{aligned}$$

Multiplication of (3.11) by $\int_{kh}^{k(h+1)} \varphi(x,t)dt$, integration in space, and summation over all $k \in \mathbb{N}_0$ gives

$$\begin{aligned}
&\left(\partial_{t,h}^{-}\left(\rho^N \mathbf{u}^N\right), \varphi\right)_{Q_T} - \left(\rho^N \mathbf{u}^N \otimes \mathbf{u}^N - \mathbb{S}^N + p_\delta^N \mathbb{I}, \nabla\varphi\right)_{Q_T} \\
&\quad + h\left(\nabla\rho^N \cdot \nabla\mathbf{u}^N, \varphi\right)_{Q_T} = \left(\rho^N\left(\mu^N - \partial_c f^N\right)\nabla c^N, \varphi\right)_{Q_T}
\end{aligned} \tag{3.30}$$

where $\varphi \in C_{(0)}^\infty\left(\Omega \times [0,T); \mathbb{R}^3\right)$ is arbitrary, $\partial_c f^N = \frac{\partial f}{\partial c}(\rho^N, c^N)$

$$\mathbb{S}^N = 2\nu\left(\tau_h c^N, \tau_h w^N\right)\mathbb{D}\left(\mathbf{u}^N\right) + \eta\left(\tau_h c^N, \tau_h w^N\right)\operatorname{div}\mathbf{u}^N \mathbb{I} \tag{3.31}$$

Moreover, using summation by parts, i.e.,

$$\left(\partial_{t,h}^{-}\left(\rho^N \mathbf{u}^N\right), \varphi\right)_{Q_T} = -\left(\rho^N \mathbf{u}^N, \partial_{t,h}^{+}\varphi\right)_{Q_T} - \left(\rho_0 \mathbf{u}_0, \varphi(0)\right)_\Omega,$$

we conclude

$$
\begin{aligned}
-\left(\rho^N \mathbf{u}^N, \partial_{t,h}^+ \varphi\right)_{Q_T} - (\rho_0 \mathbf{u}_0, \varphi|_{t=0})_\Omega - \left(\rho^N \mathbf{u}^N \otimes \mathbf{u}^N - \mathbb{S}^N + p_\delta^N \mathbb{I}, \nabla \varphi\right)_{Q_T} \\
+h\left(\nabla \rho^N \cdot \nabla \mathbf{u}^N, \varphi\right)_{Q_T} = \left(\rho^N \left(\mu^N - \partial_c f^N\right) \nabla c^N, \varphi\right)_{Q_T}
\end{aligned}
\tag{3.32}
$$

for all $\varphi \in C_{(0)}^\infty\left(\Omega \times [0,T); \mathbb{R}^3\right)$. In the same way, one obtains

$$
\left(\rho^N, \partial_{t,h}^+ \psi\right)_{Q_T} + (\rho_0, \psi|_{t=0})_\Omega + \left(\rho^N \mathbf{u}^N, \nabla \psi\right)_{Q_T} = h(\nabla \rho, \nabla \psi)_{Q_T}, \tag{3.33}
$$

$$
\left(\rho^N c^N, \partial_{t,h}^+ \psi\right)_{Q_T} + (\rho_0 c_0, \psi|_{t=0})_\Omega + \left(\rho^N c^N \mathbf{u}^N, \nabla \psi\right)_{Q_T} = \left(\nabla \mu^N, \nabla \psi\right)_{Q_T} \tag{3.34}
$$

for all $\psi \in C_{(0)}^\infty(\Omega \times [0,\infty))$, where we have used that (3.12)-(3.13) implies

$$
\frac{\rho_{k+1} c_{k+1} - \rho_k c_k}{h} + \operatorname{div}_x(\rho_k \mathbf{u}_k c_k) = \Delta \mu_{k+1} \tag{3.35}
$$

Moreover

$$
\tau_h \rho^N \mu^N = \tau_h \rho^N \frac{f(\tau_h \rho^N, c^N) - f(\tau_h \rho^N, \tau^N c^N)}{\Delta_h^- c^N} + w^N - \Delta c^N \tag{3.36}
$$

Finally, summation of (3.22) with respect to $k \in \mathbb{N}$ yields

$$
\begin{aligned}
&E_m(\rho^N(t), \mathbf{u}^N(t), c^N(t), w^N(t)) + h \int_{Q(s,t)} \frac{\partial_\rho p_m(\rho^N, c^N)}{\rho^N} |\nabla \rho^N|^2 \, d(x,\tau) \\
&+ \int_{Q(s,t)} \frac{\rho^N \left|\Delta_h^- \mathbf{u}^N\right|^2}{2} \, d(x,\tau) + \frac{1}{2h} \left\|\nabla \Delta_h^- c^N\right\|_{L^2(Q(s,t))}^2 + \frac{1}{2h} \left\|\nabla \Delta_h^- w^N\right\|_{L^2(Q(s,t))}^2 \\
&+ \alpha \left\|\Delta_h^- \rho\right\|_{L^2(Q(s,t))}^2 + \int_{Q(s,t)} \mathbb{S}^N : \nabla \mathbf{u}^N \, d(x,\tau) + \|\nabla \mu^N\|_{L^2(Q(s,t))}^2 \\
&\leq E_m\left(\rho^N(s), \mathbf{u}^N(s), c^N(s), w^N(s)\right) + R_{t,s}\left(\rho^N, \mathbf{u}^N, c^N\right)
\end{aligned}
\tag{3.37}
$$

for all $0 \le s \le t \le T$ with $s,t \in h\mathbb{N}_0$, where

$$R_{t,s}\left(\rho^N, \mathbf{u}^N, c^N\right) = \int_{Q(s,t)} \left(p_b(\rho^N)\operatorname{div}\mathbf{u}^N - h\nabla\rho^N \cdot \nabla c^N \frac{\partial^2(\rho f_m)}{\partial\rho\partial c}(\rho^N, c^N)\right) d(x,\tau) \tag{3.38}$$

Since $E_m(\rho^N(t), \mathbf{u}^N(t), c^N(t), w^N(t)) = E_m(\rho^N(t_k), \mathbf{u}^N(t_k), c^N(t_k), w^N(t_k))$ for all $t \in (t_k - h, t_k]$ if $t_k \in h\mathbb{N}_0 \cap (0,T)$, we conclude that (3.37) holds for all $0 \le s \le t \le T$ with

$$R_{t,s}\left(\rho^N, \mathbf{u}^N, c^N\right) = \int_{Q(s_k,t_k)} \left(p_b(\rho^N)\operatorname{div}\mathbf{u}^N - h\nabla\rho^N \cdot \nabla c^N \frac{\partial^2(\rho f_m)}{\partial\rho\partial c}(\rho^N, c^N)\right) d(x,\tau) \tag{3.39}$$

where $s_k, t_k \in h\mathbb{N}_0 \cap (0,T)$ are determined by the condition $t \in (t_k - h, t_k]$ and $s \in (s_k - h, s_k]$.

Lemma 3.1.3. *There is some $h_1 > 0$ independent of $\rho^N, \mathbf{u}^N, c^N$ and a constant $C(\rho_0, \mathbf{u}_0, c_0, w_0)$ depending only $\Omega, d, \rho_0, \mathbf{u}_0, c_0, w_0$ such that*

$$\begin{aligned} &\sup_{0\le t\le T}\left(\|\rho^N\|_\Gamma + \int_\Omega \rho^N|\mathbf{u}^N|^2dx + \|c^N\|_{H^1} + \|w^N\|_{H^1}\right) \\ &+h^{-\frac{1}{2}}\|\nabla\Delta_h^- c^N, \Delta_h^-\rho^N\|_{L^2(QT)} + \|\mathbf{u}^N, \mu^N, h^{\frac{1}{2}}\rho^N\log\rho^N, h^{\frac{1}{2}}c\nabla\rho\|_{L^2(0,T;H1)} \\ &\le C(\rho_0, \mathbf{u}_0, c_0) \end{aligned}$$

provided that $h = \frac{T}{N} \le h_1$

Proof. since $\frac{\partial^2(\rho f_m)}{\partial\rho\partial c} = f_0'(c)$ and since $p_b(\rho)$ is uniformly bounded, we have

$$|R_{0,T}(\rho^N, u^N, c^N)| \le C\left(\|\operatorname{div}\mathbf{u}^N\|_{L^2(QT)} + h^{3/2}\|(1+c)\nabla\rho\|^2_{L^2(QT)} + h^{\frac{1}{2}}\|\nabla c\|^2_{L2(QT)}\right).$$

On the other hand (3.37) and $\rho^{-1}\partial_\rho p(\rho) \geq 2|c|^2 + |1+\log\rho|^2$ due to (3.20) imply

$$\begin{aligned}
&\sup_{0\leq t\leq T}\left(\|\rho^N\|_\Gamma^\Gamma + \int_\Omega \rho^N|\mathbf{u}|^2 + \|\nabla c\|_2^2 + \|\nabla w\|_2^2\right) + h^{-1}\|\nabla\Delta_h^- c, \Delta_h^-\rho\|_{L^2(Q_T)}^2 \\
&\quad + \|\nabla\mathbf{u}^N, \nabla\mu^N, h^{\frac{1}{2}}\nabla(1+\log\rho), h^{\frac{1}{2}}c^2\nabla\rho\|_{L^2(Q_T)}^2 \\
&\quad \leq C\left(E_m(\rho_0, \mathbf{u}_0, c_0, w_0) + R_{0,T}(\rho^N, \mathbf{u}^N, c^N)\right)
\end{aligned}$$

Combining this with the previous estimate, choosing $0 < h \leq h_1$ sufficiently small, and using Young inequalities yields

$$\begin{aligned}
&\sup_{0\leq t\leq T}\left(\|\rho^N\|_\Gamma^\Gamma + \int_\Omega \rho^N|\mathbf{u}|^2 + \|\nabla c\|_2^2 + \|\nabla w\|_2^2\right) + h^{-1}\|\nabla\Delta_h^- c, \Delta_h^-\rho\|_{L^2(Q_T)}^2 \\
&\quad + \|\nabla\mathbf{u}^N, \nabla\mu^N, h^{\frac{1}{2}}\nabla(1+\log\rho), h^{\frac{1}{2}}c\nabla\rho\|^2 \\
&\quad \leq C\left(E_m(\rho_0, \mathbf{u}_0, c_0, w_0) + 1\right)
\end{aligned}$$

The remaining estimates of $\|c^N\|_{L^\infty(0,T;L^2)}$, $\|w^N\|_{L^\infty(0,T;L^2)}$ and $\|\mu^N\|_{L^2(QT)}$ are done in the same way as in the proof of Lemma 3.1.1 $\square$

3.2 Improved Density Estimate

In order to show that ρ^N, $N \geq N_0$, is uniformly bounded in $L^{\Gamma+1}(Q_T)$, we choose

$$\varphi = \psi(t)B[P_0\rho^N], \text{ where } P_0\rho^N = \rho^N - \frac{1}{|\Omega|}\int_\Omega \rho^N dx.$$

and $\psi \in C_0^\infty(0,T)$, in (3.32). Here B is the well-known Bogovskii operator, cf. Bogovskii [6] or Galdi [30, Chapter III.3]. In particular, $B : L^p_{(0)}(\Omega) \to W_0^{1,p}(\Omega;\mathbb{R}^3)$ is a bounded operator for all $1 < p < \infty$, where $L^p_{(0)}(\Omega) = P_0L^p(\Omega)$, provided that Ω is a Lipschitz domain.

Moreover, if $g \in L^p$, $g = \operatorname{div} \mathbf{v}$, $\mathbf{v} \in L^q(\Omega;\mathbb{R}^3)$ such that $\mathbf{v} \cdot n|_{\partial\Omega} = 0$, then

$$\|B[g]\|_{L^q(\Omega;\mathbb{R}^3)} \leq C(p,q)\|\mathbf{v}\|_{L^q(\Omega;\mathbb{R}^3)} \text{ for } 1 < p,q < \infty. \tag{3.40}$$

Because $B[P_0\rho^N] \in L^\infty(0,T;W^{1,\Gamma}(\Omega))$ (by improved regularity theorem) and $\Gamma > 3$,

$$B[P_0\rho^N] \in L^\infty(0,T;C^{r+\alpha}(\Omega;\mathbb{R}^3)) \text{ for } r+\alpha = 1 - \frac{3}{\Gamma}. \tag{3.41}$$

Therefore, since $\Gamma > 3$

$$\|B[P_0\rho^N]\|_{L^\infty(Q_T;\mathbb{R}^3)} \leq C(\rho_0) \tag{3.42}$$

where $P_0\rho = \operatorname{div} \tilde{\mathbf{v}}$.

The direct computation using (3.11) yields

$$\begin{aligned}
&\int_{Q_T} \psi(t) p_m\left(\rho^N, c^N\right) \rho^N d(x,t) - \int_0^T \psi(t) \int_\Omega p_m\left(\rho^N\right) dx \frac{1}{|\Omega|} \int_\Omega \rho^N dx dt \\
&= \int_{Q_T} \psi(t) \left(\mathbb{S}^N - \rho^N \mathbf{u}^N \otimes \mathbf{u}^N\right) : \nabla B\left[P_0\rho^N\right] d(x,t) \\
&\quad + \int_{Q_T} \psi(t) \left(\rho^N \left(\partial_c f^N - \mu^N\right) \nabla c^N + h \nabla \mathbf{u}^N \cdot \nabla \rho^N\right) \cdot B\left[P_0\rho^N\right] d(x,t) \\
&\quad + \int_{Q_T} \psi(t) \rho^N \mathbf{u}^N \tau_{-h} B\left[\operatorname{div}\left(\rho^N \mathbf{u}^N - h \nabla \rho^N\right)\right] d(x,t) \\
&\quad - \int_{Q_T} \rho^N \mathbf{u}^N \left(\partial_{t,h}^+ \psi\right) B\left[P_0\rho^N\right] d(x,t) \equiv \sum_{j=1}^4 I_j
\end{aligned}$$

where

$$
\begin{aligned}
|I_1| &\leq (\|S^N(\nabla u)\|_{L^2}+\|\rho^N\|_{L^\Gamma}\|\mathbf{u}\|_{L^6}^2)\left\|\nabla B\left[P_0\rho^N\right]\right\|_{L^2(Q_T)}\|\psi\|_\infty \\
&\leq C'\left(T,\rho_0,c_0,\mathbf{u}_0\right)\|\psi\|_\infty, \\
|I_2| &\leq \left\|\rho^N\right\|_{L^\infty\left(0,T;L^\Gamma\right)}(\|c\|_{L^\infty\left(0,T;L^6\right)}+\|\mu\|_{L^2(0,T;L^6)})\|\nabla c\|_{L\infty(0,T;L^2)}\|B[P_0\rho^N]\|_\infty\|\psi\|_\infty \\
&\leq C\left(T,\rho_0,c_0,\mathbf{u}_0\right)\|\psi\|_\infty, \\
|I_3| &\leq \left\|\rho^N\mathbf{u}^N\right\|_{L^2(Q_T)}\left\|B\left[\operatorname{div}\left(\rho^N\mathbf{u}^N-h\nabla\rho^N\right)\right]\right\|_{L^2(Q_T)}\|\psi\|_\infty \\
&\leq C\left(T,\rho_0,c_0,\mathbf{u}_0\right)\|\psi\|_\infty, \\
|I_4| &\leq C\left(T,\rho_0,c_0,\mathbf{u}_0\right)\|\partial_t\psi\|_{L^\infty(0,T)}
\end{aligned}
$$

since $\Gamma > 3$. Letting ψ to approach 1 we conclude

$$
\delta\int_{Q_T}(\rho^N)^{\Gamma+1}d(x,t) \leq C(T,\rho_0,\mathbf{u}_0,c_0) \tag{3.43}
$$

and

$$
\delta\int_{Q_T}(\rho^N)^3c^2d(x,t) \leq C(T,\rho_0,\mathbf{u}_0,c_0)
$$

uniformly in $N \geq N_0$. From the equation above, we get uniform bound in $N \geq N_0$

$$
\|\delta(\rho^N)^2(c^N)^2\|_{L^{\frac{9}{5}}} \leq \delta^{\frac{1}{3}}\|\delta^{\frac{2}{3}}\rho^2c^{\frac{4}{3}}\|_{L^{\frac{3}{2}}}\|c^{\frac{2}{3}}\|_{L^9} \leq \delta^{\frac{1}{3}}C(T,\rho_0,\mathbf{u}_0,c_0) \tag{3.44}
$$

3.3 Passing to the Limit

Using the a priori bounds given by Lemma 3.1.3, by (3.43) and by (3.44), we can pass to a subsequence again denoted by $(\rho^N, u^N, c^N, w^N, \mu^N)$ such that

$$\begin{aligned}
(\rho^N, c^N) &\xrightarrow[N\to\infty]{*} (\rho, c) && \text{in } L^\infty(0,T;L^\Gamma \times H^1(\Omega)) \\
\rho^N &\xrightharpoonup[N\to\infty]{} \rho && \text{in } L^{\Gamma+1}(Q_T) \\
p_\delta(\rho^N, c^N) &\xrightharpoonup[N\to\infty]{} \overline{p_\delta(\rho^N, c^N)} && \text{in } L^{(\Gamma+1)/\Gamma}(Q_T) \\
(\rho^N \mathbf{u}^N, \rho^N c^N, \mathbb{S}^N) &\xrightharpoonup[N\to\infty]{} (\overline{\rho\mathbf{u}}, \overline{\rho c}, \overline{\mathbb{S}}) && \text{in } L^2(Q_T;\mathbb{R}^4 \times \mathbb{R}^{3\times 3}) \\
(\mathbf{u}^N, \mu^N) &\xrightharpoonup[N\to\infty]{} (\mathbf{u}, \mu) && \text{in } L^2(0,T;H^1(\Omega;\mathbb{R}^4)) \\
w^N &\xrightharpoonup[N\to\infty]{} w && \text{in } L^2(0,T;H^3(\Omega;\mathbb{R}^3))
\end{aligned}$$

as well as

$$(h\nabla\rho^N, h\nabla(\rho^N \log\rho^N), hc\nabla\rho, \Delta_h^- c^N) \xrightarrow[N\to\infty]{} 0 \text{ in } L^2(Q_T;\mathbb{R}^7).$$

As a reminder, the bar stands for a weak limit in L^1.

Next we define $\tilde{\rho}^N$ and $\tilde{\rho c}^N$ as a piece-wise linear interpolation of $\rho^N(t_k)$, $\rho^N(t_k)c^N(tk)$, respectively, where $t_k = kh$, $k = 0,\ldots,N$. More precisely, $\tilde{\rho}^N = \frac{1}{h}\chi_{[0,h]} \star_t \rho^N$ and $\tilde{\rho c}^N = \frac{1}{h}\chi_{[0,h]} \star_t (\rho^N c^N)$, where the convolution is only taken with respect to the time variable t. Then

$$\partial_t \tilde{\rho}^N = \partial_{t,h}^- \rho^N \text{ and } \partial_t \tilde{\rho c}^N = \partial_{t,h}^-(\rho^N c^N) \text{ almost everywhere.}$$

Thus (3.33) yields that $\partial_t \tilde{\rho}^N$ is bounded in $L^2(0,T;H^{-1}(\Omega))$, which implies that $\tilde{\rho}^N \to_{N\to\infty} \rho$ in $L^r(0,T;H^{-\varepsilon}(\Omega))$ for all $1 \le r < \infty$, $\varepsilon > 0$

$$\tilde{\rho}^N \to_{N\to\infty} \rho \text{ in } L^r(0,T;H^{-\varepsilon}(\Omega)) \text{ for all } 1 \le r < \infty,\ \varepsilon > 0$$

by the Aubin-Lions Lemma. In particular, this implies

$$\tilde{\rho}^N \to \rho \text{ in } C_{\text{weak}}\left([0,T];L^\Gamma(\Omega)\right)$$

since $\left\|\tilde{\rho}^N\right\|_{L^\infty\left(0,T;L^\Gamma(\Omega)\right)}$ is uniformly bounded in $N \ge N_0$. Moreover,

$$\left\|\rho^N - \tilde{\rho}^N\right\|_{L^2(Q_T)} \le \left\|\Delta_h^- \rho^N\right\|_{L^2(Q_T)} \to_{N\to\infty} 0.$$

Hence $weak - \lim_{N\to\infty} \tilde{\rho}^N = weak - \lim_{N\to\infty} \rho^N = \rho$ and

$$\begin{aligned}
\overline{\rho u} &= \lim_{N\to\infty} \rho^N \mathbf{u}^N = \lim_{N\to\infty} \tilde{\rho}^N \mathbf{u}^N = \rho \mathbf{u} && \text{weakly in } L^2\left(Q_T\right)\\
\overline{\rho c} &= \lim_{N\to\infty} \rho^N c^N = \lim_{N\to\infty} \tilde{\rho}^N c^N = \rho c && \text{weakly in } L^2\left(Q_T\right)\\
& \lim_{N\to\infty} \rho^N \mu^N = \lim_{N\to\infty} \tilde{\rho}^N \mu^N = \rho \mu && \text{weakly in } L^2\left(Q_T\right)
\end{aligned}$$

since $\mathbf{u}^N$ and c^N converge weakly in $L^2\left(0,T;H^1(\Omega)\right)$. Moreover, we denote

$$\sqrt{\tau_h \rho^N} \frac{f\left(\tau_h \rho^N, c^N\right) - f\left(\tau_h \rho^N, \tau_h c^N\right)}{c^N - \tau_h c^N} = F\left(\tau_h \rho_N, c^N, \tau_h c^N\right)$$

using the convention $F(\rho,c,c) = \frac{\partial f}{\partial c}(\rho,c)$. Then $F\left(\rho, c_1, c_2\right)$ is a continuous function with respect to $(\rho, c_1, c_2) \in [0,\infty) \times \mathbb{R}^2$ satisfying

$$\left|F\left(\rho, c_1, c_2\right)\right| \le C\left(1 + \rho^{\frac{1}{2}} |\log \rho|\right)\left(1 + |c_1| + |c_2|\right).$$

Hence $\sqrt{\tau_h\rho^N \frac{f\left(\tau_h\rho^N,c^N\right)-f\left(\tau_h\rho^N,\tau_hc^N\right)}{c^N-\tau_hc^N}}$ is bounded in $L^2\left(Q_T\right)$, and we can apply the result of Section 2.4 to $\left(\tilde{\rho}^N, c^N\right)$ using (3.29) together with the fact that $\tau_h\rho^N - \tilde{\rho}^N$ converges strongly to zero in $L^\beta\left(Q_T\right)$ for all $1 \leq \beta < \Gamma+1$. We concludes that

$$c^N \to_{N\to\infty} c \quad \text{in } L^2\left(0,T;H^1(\Omega)\right).$$

$$w^N \to_{N\to\infty} w \quad \text{in } L^2\left(0,T;H^1(\Omega)\right).$$

In particular, $c^N \to_{N\to\infty} c$ almost everywhere and $w^N \to_{N\to\infty} w$ almost everywhere in Q_T and therefore

$$\begin{aligned}\overline{\mathbb{S}} &= \lim_{N\to\infty}\left(2\nu\left(\tau_hc^N,\tau_hw^N\right)\mathbb{D}\left(\mathbf{u}^N\right)+\eta\left(\tau_hc^N,\tau_hw^N\right)\operatorname{div}\mathbf{u}^{N}\mathbb{I}\right)\\ &= 2\nu(c,w)\mathbb{D}(\mathbf{u})+\eta(c,w)\operatorname{div}\mathbf{u}\mathbb{I}=\mathbb{S}.\end{aligned}$$

Furthermore, because of the growth estimate of F, we conclude that

$$\rho^N\frac{f\left(\tau_h\rho^N,c^N\right)-f\left(\tau_h\rho^N,\tau_hc^N\right)}{c^N-\tau_hc^N} \rightharpoonup^*_{N\to\infty} \overline{\rho\frac{\partial f}{\partial c}} \quad \text{in } L^\infty\left(0,T;L^{\frac{6}{5}}(\Omega)\right)$$

for a suitable subsequence.

Having all necessary results at hand, we see that $(\mathbf{u},\rho,c,w,\mu)$ solve

$$\begin{aligned}&-(\rho\mathbf{u},\partial_t\varphi)_{Q_T}+(\rho_0\mathbf{u}_0,\,\varphi|_{t=0})_\Omega-\frac{(\rho\mathbf{u}\otimes\mathbf{u}-\mathbb{S},\nabla\varphi)_{Q_T}}{\partial f}\\ &\quad=(\overline{p_\delta},\operatorname{div}\varphi)_{Q_T}+\left(\rho\mu\nabla c-\overline{\rho\frac{\partial f}{\partial c}}\nabla c,\varphi\right)_{Q_T}\end{aligned} \tag{3.45}$$

for all $\varphi \in C^\infty_{(0)}([0,T)\times\Omega;\mathbb{R}^3)$, as well as

$$
\begin{aligned}
(\rho,\partial_t\psi)_{Q_T}+(\rho_0,\,\psi|_{t=0})_\Omega+(\rho\mathbf{u},\nabla\psi)_{Q_T} &= 0 &(3.46)\\
(\rho c,\partial_t\psi)_{Q_T}+(\rho_0 c_0,\,\psi|_{t=0})_\Omega+(\rho c\mathbf{u},\nabla\psi)_{Q_T} &= (\nabla\mu,\nabla\psi)_{Q_T} &(3.47)\\
\left(\rho\mu-\overline{\rho\frac{\partial f}{\partial c}}-w,\psi\right)_{Q_T} &= (\nabla c,\nabla\psi)_{Q_T} &(3.48)\\
(c-M,\psi)_{Q_T} &= (\nabla w,\nabla\psi)_{Q_T} &(3.49)
\end{aligned}
$$

for all $\psi \in C^\infty_{(0)}([0,T)\times\bar{\Omega})$.

To show $\overline{\rho\frac{\partial f}{\partial c}}=\rho\frac{\partial f}{\partial c}(c)$, since we have convergence $\rho^N\to\rho$ in $C_{weak}((0,T],L^\Gamma(\Omega))$ and hypothesis (1.15), then

$$
\begin{aligned}
\int_0^T\int_\Omega\rho^N\frac{\partial f}{\partial c}(c^N)\varphi \mathrm{d}x\,\mathrm{d}t &= \int_0^T\left\langle\rho^N,\varphi\frac{\partial f}{\partial c}(c^N)\right\rangle_{W^{-1,2}(\Omega),W^{1,2}(\Omega)}\mathrm{d}t\\
&\to\int_0^T\langle\rho,\varphi\frac{\partial f}{\partial c}(c)\rangle_{W^{-1,2}(\Omega),W^{1,2}(\Omega)}\mathrm{d}t=\int_0^T\int_\Omega\rho\frac{\partial f}{\partial c}(c)\varphi\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
\tag{3.50}
$$

In order to pass the limit, the following result is still needed

$$\overline{p_\delta}=p_\delta(\rho),$$

The almost everywhere convergence of ρ^N to ρ can yield the above statement. It can be obtained by similar strategy as in [2, Section 3.3]. We will put it in the appendix A.

Finally, passing to the limit in (3.37) and (3.38), we obtain that

$$
\begin{aligned}
&E_m(\rho(t),\mathbf{u}(t),c(t),w(t))+\int_{Q_{(s,t)}}\left(\mathbb{S}:\nabla\mathbf{u}+|\nabla\mu|^2\right)d(x,\tau)\\
&\le E_m(\rho(s),\mathbf{u}(s),c(s),w(s))+\int_{Q_{s,t}}p_b(\rho)\operatorname{div}\mathbf{u}d(x,\tau)
\end{aligned}
$$

Now, using the renormalized transport equation (1.20) for $b(\rho)=p_b(\rho)$ and $\varphi=\chi_{[s,t]}$

(after a simple approximation), we conclude that

$$\int_{Q_{(s,t)}} p_b(\rho) \operatorname{div} \mathbf{u} d(x,\tau) = - \int_{\Omega} \rho(\tau) f_b(\rho(\tau)) dx \Big|_{\tau=s}^{\tau=t}$$

Summing up, we have proved (3.9), which completes the proof.

Chapter 4: Vanishing Artificial Pressure Limit

4.1 Uniform Bounds

By virtue of the coercivity of the functions f_e, f_0 postulated in (1.14), (1.15), the specific free energy E_δ is bounded from below, and, by the same arguments as in Sections 2.3.1-2.3.3, the energy inequality (3.9) yields the estimates (2.29)-(2.43) with $(\mathbf{u}, \rho, c, w, \mu)$ replaced by $(\mathbf{u}_\delta, \rho_\delta, c_\delta, w_\delta, \mu_\delta)$ uniformly in $\delta > 0$. Moreover, (3.9) and (3.10) imply that

$$\delta \operatorname{ess} \sup_{t \in (0,T)} \|\rho_\delta\|^{\Gamma}_{L^\Gamma(\Omega)} \leq C \tag{4.1}$$

4.2 Refined Pressure Estimates

To derive a uniform bound on the pressure in the reflexive Lebesgue space $L^p((0,T) \times \omega))$, $p > 1$, we follow the strategy as in [23]: show that the pressure family $\{p(\rho_\delta, c_\delta)\}_{\delta > 0}$ is equi-integrable in the following sense:

$$\int_0^T \int_\Omega p(\rho, c)\rho_\delta^\alpha + \delta \rho_\delta^{\Gamma+\alpha} + \delta \rho^{2+\alpha} c^2 \leq C. \tag{4.2}$$

Let B be the Bogovskii operator as introduced in Section 3.2. Pursuing the main idea of [23] we use quantities

$$\varphi(t,x) = \psi(t)B\left[\rho_\delta^\alpha - \frac{1}{|\Omega|}\int_\Omega \rho_\delta^\alpha \, dx\right], \quad \psi \in \mathscr{D}(0,T)$$

as test functions in the momentum balance (3.1).

$$\begin{aligned}
&\int_{Q_T} \psi(t)\left[p(\rho_\delta)\rho_\delta^\alpha + \delta\rho^{\Gamma+\alpha} + \delta\rho^{2+\alpha}c_\delta^2\right] d(x,t) \\
&\quad - \int_0^T \psi(t)\int_\Omega \left[p(\rho_\delta) + \delta\rho_\delta^\Gamma + \delta\rho_\delta^2 c_\delta^2\right] dx \frac{1}{|\Omega|}\int_\Omega \rho_\delta^\alpha dx dt \\
&= \int_{Q_T} \psi(t)\left(\mathbb{S}_\delta - \mathbb{P}_\delta - \mathbf{u}_\delta \otimes \mathbf{u}_\delta\right) : \nabla B\left[P_0\rho_\delta^\alpha\right] d(x,t) \\
&\quad + \int_{Q_T} \psi(t)\rho_\delta \mathbf{u}_\delta B\left[\operatorname{div}\left(\rho_\delta^\alpha \mathbf{u}_\delta\right)\right] d(x,t) \\
&\quad - \int_{Q_T} \rho_\delta \mathbf{u}_\delta \left(\partial_t \psi\right) B\left[P_0\rho_\delta^\alpha\right] d(x,t) \equiv \sum_{j=1}^{1} I_j
\end{aligned}$$

$$|I_2| \leq \|\rho_\delta^{\frac{1}{2}}\mathbf{u}_\delta\|_{L^\infty(0,T;L^2)}\|\rho_\delta^{\frac{1}{2}}\|_{L^\infty(0,T;L^{2\gamma})}\|\rho_\delta^\alpha\|_{L^\infty\left(0,T;L^{\frac{6\gamma}{2\gamma-3}}\right)}\|\mathbf{u}_\delta\|_{L^2(0,T;L^6)}$$

$$|I_3| \leq \|\rho_\delta^{\frac{1}{2}}\mathbf{u}_\delta\|_{L^\infty(0,T;L^2)}\|\rho_\delta^{\frac{1}{2}}\|_{L^\infty(0,T;L^{2\gamma})}\|\rho_\delta^\alpha\|_{L^\infty\left(0,T;L^{\frac{2\gamma}{\gamma-1}}\right)}$$

If we pick $0 < \alpha < \frac{2\gamma-3}{6}$, $|I_2|$, $|I_3| \leq C(\rho_0, c_0, \mathbf{u}_0)$. In order to bound $|I_1|$, a uniform bound

$$\|\mathbb{P}_\delta\|_{L^p(0,T;L^p(\Omega;\mathbb{R}^3))} \text{ for a certain } p > 1 \tag{4.3}$$

is to be obtained. The constitutive relation (3.4), hypothesis (1.15), and estimates (2.29), (2.40), (2.42) imply

$$\|\Delta c_\delta\|_{L^\infty(0,T;L^q)} \leq C \text{ for } q > \frac{6}{5}$$

then with standard elliptic estimates

$$\|\nabla c_\delta\|_{L^\infty(0,T;L^r)} \leq C \text{ for } r > 2. \tag{4.4}$$

The constitutive relation (3.5) and estimates (2.43) yields

$$\|\nabla w_\delta\|_{L^\infty(0,T;L^r)} \leq C \text{ for } r > 2. \tag{4.5}$$

Moreover,

$$\|(c-M)w\|_{L^\infty(0,T;L^3)} \leq \|w\|_{L^\infty(0,T;L^6)}\|c\|_{L^\infty(0,T;L^6)} \tag{4.6}$$

Thus, (4.3) follows from (4.4), (4.5), and (4.6)

4.3 Strong Compactness of the Concentration Gradients

We follow the arguments of Section 2.4 to obtain that

$$c_\delta \to c \text{ in } L^2(0,T;W^{1,2}(\Omega)). \tag{4.7}$$

$$w_\delta \to w \text{ in } L^2(0,T;W^{1,2}(\Omega)). \tag{4.8}$$

4.4 Asymptotic Limit for $\delta \to 0$

First we observe, in accordance with (2.38),

$$\mathbf{u}_\delta \to \mathbf{u} \text{ in } L^2(0,T;W_0^{1,2}(\Omega;\mathbb{R}^3)) \tag{4.9}$$

Then with relation (3.2) and estimate (2.29), we have

$$\frac{d}{dt}\int_{\Omega}\rho_{\delta}\varphi\,dx = \int_{\Omega}\rho_{\delta}\mathbf{u}_{\delta}\cdot\nabla\varphi\mathrm{d}x.$$

where the right hand side is bounded in $L^2(0,T)$. Thus,

$$\left\|\int_{\Omega}\rho_{\delta}\varphi\,dx\right\|_{C^{0,\frac{1}{2}}(0,T)} \leq \left\|\int_{\Omega}\rho_{\delta}\varphi\,dx\right\|_{W^{1,2}(0,T)} \leq C$$

which implies equi-continuity of $\int_{\Omega}\rho_{\delta}\varphi\,dx(t)$. Then, with this fact, we verity

$$\rho_{\delta}\to\rho \text{ in } C_{weak}([0,T];L^{\gamma}(\Omega)) \tag{4.10}$$

for a suitable subsequence of $\delta\to 0$ by applying [19, Corollary 2.1]. This fact together with the momentum equation (3.1) imply

$$\rho_{\delta}\mathbf{u}_{\delta}\to\rho\mathbf{u} \text{ in } C_{weak}([0,T];L^{q}(\Omega)),\ q=\frac{2\gamma}{1+\gamma}; \tag{4.11}$$

whence

$$\rho_{\delta}\mathbf{u}_{\delta}\otimes\mathbf{u}_{\delta}\to\rho\mathbf{u}\otimes\mathbf{u} \text{ in } L^{2}([0,T];L^{q}(\Omega)),\ q=\frac{6\gamma}{3+4\gamma}. \tag{4.12}$$

Similarly, by virtue of (1.15) (2.40),(4.7),

$$\rho_{\delta}c_{\delta}\to\rho c \quad \text{in } C_{\text{weak}}\left(0,T;L^{q}\left(\Omega;\mathbb{R}^{3}\right)\right) \text{ for } q=\frac{6\gamma}{6+\gamma}, \tag{4.13}$$

$$\rho_{\delta}c_{\delta}\mathbf{u}_{\delta}\to^{*}\rho c\mathbf{u} \text{ in } L^{\infty}\left(0,T;L^{q}\left(\Omega;\mathbb{R}^{3}\right)\right) \quad \text{for } q=\frac{3\gamma}{3+\gamma}, \tag{4.14}$$

$$\rho_{\delta}\frac{\partial f}{\partial c}(c_{\delta})\to\rho\frac{\partial f}{\partial c}(c) \quad \text{in } C_{\text{weak}}\left(0,T;L^{q}\left(\Omega;\mathbb{R}^{3}\right)\right) \text{ for } q=\frac{6\gamma}{6+\gamma}. \tag{4.15}$$

and, in view of (2.42) and (2.43),

$$\mu_\delta \to \mu \text{ weakly in } L^2\left(0,T;W^{1,2}(\Omega)\right). \tag{4.16}$$

$$w_\delta \to w \text{ weakly in } L^2\left(0,T;W^{3,2}(\Omega)\right). \tag{4.17}$$

Finally, it follows from the refined pressure estimates established in (4.2) that

$$\|\delta\rho_\delta^\Gamma\|_{\frac{\Gamma+\alpha}{\Gamma}}^{\frac{\Gamma+\alpha}{\Gamma}} \leq C\delta^{\frac{\alpha}{\Gamma}}$$

$$\|\delta\rho_\delta^2 c_\delta^2\|_{\frac{4+2\alpha}{4+\alpha}} \leq \|\delta\rho_\delta^2 c_\delta^{\frac{2}{2+\alpha}}\|_{\frac{2+\alpha}{2}} \|c^{\frac{2\alpha}{2+\alpha}}\|_{\frac{4+2\alpha}{\alpha}} \leq C\delta^{\frac{\alpha}{2+\alpha}}$$

Then, we conclude the convergence of the artificial pressure,

$$\left.\begin{aligned} p(\rho_\delta, c_\delta) &\rightharpoonup \overline{p(\rho,c)}, \\ \delta\rho_\delta^\Gamma &\rightharpoonup 0 \\ \delta\rho_\delta^2 c_\delta^2 &\rightharpoonup 0 \end{aligned}\right\} \text{ in } L^q((0,T)\times\Omega) \text{ for a certain } q>1 \tag{4.18}$$

At this stage, it is easy to let $\delta \to 0$ in (3.1) - (3.7) in order to obtain

$$\int_0^T \int_\Omega (\rho\partial_t\varphi + \rho\mathbf{u}\cdot\nabla\varphi)\,dxdt + \int_\Omega \rho_0\varphi\Big|_{t=0} dx = 0 \tag{4.19}$$

for any test function $\varphi \in \mathscr{D}([0,T)\times\bar{\Omega})$,

$$\begin{aligned} &\int_0^T \int_\Omega \left(\rho\mathbf{u}\cdot\partial_t\varphi + \rho\mathbf{u}\otimes\mathbf{u} : \nabla\varphi + \overline{p(\rho,c)}\,\mathrm{div}\,\varphi\right) dxdt \\ &\quad = \int_0^T \int_\Omega (\mathbb{S}-\mathbb{P}) : \nabla\varphi dxdt + \int_\Omega \mathbf{m}_0\cdot\varphi\Big|_{t=0} dxdt \end{aligned} \tag{4.20}$$

for any $\varphi \in \mathscr{D}\left([0,T)\times\Omega;\mathbb{R}^3\right)$,

$$\int_0^T \int_\Omega (\rho c\partial_t\varphi + \rho c\mathbf{u}\cdot\nabla\varphi - \nabla\mu\cdot\nabla\varphi)\,dxdt - \int_\Omega \rho_0 c_0\varphi\Big|_{t=0} dxdt = 0 \tag{4.21}$$

for any $\varphi \in \mathscr{D}([0,T)\times\bar{\Omega})$,

$$\int_0^T\int_\Omega \nabla w\cdot\nabla\varphi = \int_0^T\int_\Omega (c-M)w\,dx\,dt \tag{4.22}$$

for any $\varphi \in \mathscr{D}([0,T)\times\bar{\Omega})$, where $\mathbb{S}$ satisfies (1.6),

$$\mathbb{P} = \nabla c\otimes\nabla c - \tfrac{|\nabla c|^2}{2}\mathbb{I} - \nabla w\otimes\nabla w - (c-M)w\mathbb{I} + \tfrac{|\nabla w|^2}{2}\mathbb{I} \tag{4.23}$$

$$\mu = \rho\tfrac{\partial f}{\partial c} + w - \Delta c \tag{4.24}$$

provided the family of initial data $\{\rho_{0,\delta},(\rho\mathbf{u})_{0,\delta},(\rho c)_{0,\delta}\}_{\delta>0}$ converges at least weakly in L^1.

To complete the prove, we need to remove the bar in (4.20), or equivalently, verifying

$$\rho_\delta \to \rho \text{ (strongly) in } L^1((0,T)\times\Omega),$$

which will be show in the last section.

4.5 Strong Convergence of the Approximate Densities

To prove strong L^1 convergence of ρ_δ, we apply the method based on certain fine properties of the effective viscous flux established by P.-L.Lions[41], further investigated in [20] for the case of non-constant viscosity coefficients.

First, we notice that $\rho_\delta,\mathbf{u}_\delta$ satisfy (3.2) in the sense of renormalized solutions introduced by DiPerna and P.-L.Lions[16], cf. (1.20):

$$\begin{aligned}\int_{Q_T}\big(\tilde{b}(\rho_\delta)\partial_t\varphi + \tilde{b}(\rho_\delta)\mathbf{u}\cdot\nabla\varphi - (\tilde{b}(\rho_\delta) - \rho_\delta\tilde{b}'(\rho_\delta))\operatorname{div}\mathbf{u}\varphi\big)\,d(x,t)\\ = -\int_\Omega \tilde{b}(\rho_{0,\delta})\varphi(0)\,dx\end{aligned} \tag{4.25}$$

Above integral identity holds for any $\varphi \in \mathscr{D}([0,T) \times \Omega)$, and $\tilde{b}(\rho) = \rho B(\rho) \in C[0,\infty)$ any such that $\tilde{b}'(\rho) \equiv 0$ for $\rho > M_b$ large enough.

To deduce relation (4.25) from (3.2), we need to apply the regularization technique developed by Diperna and P.-L.Lions[16] or [43, Lemma 6.9], a step that requires $\rho_\delta \in L^2((0,T) \times \Omega$ and $\mathbf{u}_\delta \in L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3)$. The former condition holds as a result of of artificial pressure term $\frac{\delta}{\Gamma-1}\rho^\delta$.

The next thing we want to show is

$$\overline{\Big(p(\rho,c) - \mathscr{R} : \mathbb{S}\Big)\tilde{b}(\rho)} = \overline{\Big(p(\rho,c) - \mathscr{R} : \mathbb{S}\Big)}\,\overline{\tilde{b}(\rho)} \tag{4.26}$$

where $\mathscr{R} = (\partial_{x_i}\partial_{x_j}\delta^{-1})_{i,j}$. This relation is the core of the existence theory for barotropic Navier-Stokes system developed by P.-L.Lions [41].

To show (4.26), we take the quantity

$$\varphi = \psi(t)v(x)\nabla\Delta^{-1}[\chi_\Omega \tilde{b}(\rho_\delta)]$$

as test function in (3.1), and

$$\varphi = \psi(t)v(x)\nabla\Delta^{-1}[\chi_\Omega \overline{\tilde{b}(\rho)}]$$

in (4.20). And we obtain the following two estimates,

$$
\begin{aligned}
&\int_{Q_T} \psi v\left((p(\rho_\delta, c_\delta)+\delta\rho_\delta^\Gamma+\delta\rho^2c^2)\mathbb{I}-\mathbb{S}_\delta\right):\mathscr{R}\left[\tilde{b}(\rho_\delta)\right]d(x,t)\\
&=\int_{Q_T}\psi v\mathbf{u}_\delta\left(\tilde{b}(\rho_\delta)\nabla\operatorname{div}\Delta^{-1}(\rho_\delta\mathbf{u}_\delta)-\rho_\delta\mathbf{u}_\delta\cdot\mathscr{R}\left[\tilde{b}(\rho_\delta)\right]\right)d(x,t)\\
&\quad+\int_{Q_T}\psi v\rho_\delta\mathbf{u}_\delta\cdot\nabla\Delta^{-1}\left(\left(\tilde{b}'(\rho_\delta)\rho_\delta-\tilde{b}(\rho_\delta)\right)\operatorname{div}\mathbf{u}\right)\\
&\quad+\int_{Q_T}\psi v\mathbb{P}_\delta:\mathscr{R}[\tilde{b}(\rho_\delta)]d(x,t)\\
&\quad+\int_{Q_T}\left(-\rho_\delta\mathbf{u}_\delta(\partial_t\psi)v\left(\nabla\Delta^{-1}\tilde{b}(\rho_\delta)\right)+g\cdot\nabla\Delta^{-1}\tilde{b}(\rho_\delta)\right)d(x,t)
\end{aligned}
\tag{4.27}
$$

where $\mathscr{R}=\nabla^2\Delta^{-1}$ and

$$
g=-\left((p+\delta\rho_\delta^\Gamma+\delta\rho^2c^2)\mathbb{I}-\mathbb{S}_\delta+\mathbb{P}_\delta+\rho_\delta\mathbf{u}_\delta\otimes\mathbf{u}_\delta\right)\cdot\psi\nabla v \quad ;
$$

and

$$
\begin{aligned}
&\int_{Q_T}\psi(\overline{p}\mathbb{I}-\mathbb{S}):\mathscr{R}\left[\overline{\tilde{b}(\rho)}\right]d(x,t)\\
&=\int_{Q_T}\psi v\mathbf{u}\left(\overline{\tilde{b}(\rho)}\nabla\operatorname{div}\Delta^{-1}(\rho\mathbf{u})-\rho\mathbf{u}\cdot\mathscr{R}\left[\overline{\tilde{b}(\rho)}\right]\right)d(x,t)\\
&\quad+\int_{Q_T}\psi v\rho\mathbf{u}\cdot\nabla\Delta^{-1}\left(\left(\tilde{b}'(\rho)\rho-\tilde{b}(\rho)\right)\operatorname{div}\mathbf{u}\right)\\
&\quad+\int_{Q_T}\psi v\mathbb{P}:\mathscr{R}[\overline{\tilde{b}(\rho)}]\\
&\quad+\int_{Q_T}\left(-\rho\mathbf{u}(\partial_t\psi)v\left(\nabla\Delta^{-1}\overline{\tilde{b}(\rho)}\right)+g\cdot\nabla\Delta^{-1}\overline{\tilde{b}(\rho)}\right)d(x,t)
\end{aligned}
\tag{4.28}
$$

where

$$
g=-(\bar{p}\mathbb{I}-\mathbb{S}+\mathbb{P}+\rho\mathbf{u}\otimes\mathbf{u}d)\cdot\psi\nabla v.
$$

From Div-Curl lemma [20, Lemma 4.2],

$$
\tilde{b}(\rho_\delta)\nabla\operatorname{div}\Delta^{-1}(\rho_\delta\mathbf{u}_\delta)-\rho_\delta\mathbf{u}_\delta\cdot\mathscr{R}\left[\tilde{b}(\rho_\delta)\right]\to\overline{\tilde{b}(\rho)}\nabla\operatorname{div}\Delta^{-1}(\rho\mathbf{u})-\rho\mathbf{u}\cdot\mathscr{R}\left[\overline{\tilde{b}(\rho)}\right]
$$

As a consequence of (4.7), the "pressure" term $\mathbb{P}_\delta$ satisfies

$$\mathbb{P}_\delta : \mathscr{R}[\tilde{b}(\rho_\delta)] \to \mathbb{P} : \mathscr{R}[\overline{\tilde{b}(\rho)}] \text{ weakly in } L^1((0,T)\times\Omega)$$

Then, the convergence results (4.10)-(4.18) with the results above yields (4.26).

Our next step is to use (4.26) to deduce the following

$$\overline{p(\rho,c)b(\rho)} - \overline{p(\rho,c)}\,\overline{b(\rho)} = \left(\frac{4}{3}\nu(c,w)+\eta(c,w)\right)\left(\overline{\operatorname{div} b(\rho)\mathbf{u}} - \overline{b(\rho)}\operatorname{div}\mathbf{u}\right) \tag{4.29}$$

where the quantity $p-\left(\frac{4}{3}\nu+\eta\right)$ div $\mathbf{u}$ is usually termed the effective viscous flux.

In order to get (4.29), we apply Lemma A.0.1 as in Appendix A and obtain

$$\begin{aligned} &\mathscr{R} : \left[\nu\left(c_\delta,w_\delta\right)\left(\nabla\mathbf{u}_\delta+\nabla\mathbf{u}_\delta^T\right)\right] - 2\nu\left(c_\delta,v_\delta\right)\operatorname{div}\mathbf{u}_\delta \\ &\overrightarrow{\mathscr{R}} : \left[\nu(c,w)\left(\nabla\mathbf{u}+\nabla\mathbf{u}^T\right)\right] - 2\nu(c,w)\operatorname{div}\mathbf{u} \end{aligned} \tag{4.30}$$

weakly in $L^2\left(0,T;W^{\omega,q}(\Omega)\right)$ for a certain $\omega>0$. On the other hand, as ρ_δ satisfies the renormalized equation (4.25),

$$b\left(\rho_\delta\right) \to \overline{b(\rho)} \text{ in } C_{\text{weak}}\left([0,T];L^q(\Omega)\right) \text{ for any finite } q\geq 1 \tag{4.31}$$

as soon as b is uniformly bounded. Combining relation (4.26) with (4.30), (4.31) we obtain (4.29) (see [19, Chapter 6] and [20] for details).

Equation (4.29) can give us the strong convergence of densities:

$$\rho_\delta \to \rho \text{ in } L^1((0,T)\times\Omega),$$

following the same strategy as in [2, Section 4.5]. I will put the detail in Appendix B.

The proof for the main theorem is now complete.

Chapter 5: Large-Time Behavior

In this chapter, we analyze the large time behavior under assumption that the limit of c and μ as $t \to \infty$ is Lipschitz continuous.

Theorem 5.0.1 (Large-Time Asymptotic)**.** *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a boundary of class $C^{2+\nu}$, $\nu > 0$ and $\{\rho, \mathbf{u}, c\}$ be a weak solution of the model (1.1)-(1.5) with the initial data $\{\rho_0, \mathbf{u}_0, c_0\}$ satisfying all of the hypotheses in Theorem 1.2.1. Then either (i) $E(t) \to \infty$ as $t \to \infty$,*
or (ii) there exist positive measurable functions $\rho_s = \rho_s(x)$, measurable functions c_s and

$$\begin{cases} \rho(t) \to \rho_s \text{ in } L^1(\Omega), \\ c(t) \to c_s \text{ weakly in } L^2((0,T);W^{1,2}(\Omega)) \\ \mu(t) \to \mu_s \text{ weakly in } L^2((0,T);W^{1,2}(\Omega)) \end{cases} \tag{5.1}$$

as $t \to \infty$.

Moreover, if c_s and μ_s are Lipschitz continuous, then ρ_s is positive, and there exists a scalar potential $F \in C^1(\Omega)$ such that $\mu_s \nabla_x c_s - \frac{\partial f}{\partial c} \nabla_x c_s = \nabla_x F$ in Ω and (ρ_s, c_s, μ_s) solves the state problem

$$\begin{cases} \nabla_x p(\rho_s) = \rho_s \nabla_x F = \rho_s (\mu_s \nabla_x c - \frac{\partial f}{\partial c} \nabla_x c_s) \text{ in } \Omega, \end{cases} \tag{5.2}$$

in the sense of distribution.

Let the sequence $\{t_n\}_{n=1}^{\infty}$ be such that $t_n \to \infty$ and define the sequences $\mathbf{u}_n$, ρ_n, c_n, f_n by

$$\mathbf{u}_n(t,x) = \mathbf{u}(t+t_n,x), \rho_n(t,x) = \rho(t+t_n,x),$$

and

$$c_n(t,x) = c(t+t_n,x), f_n(t,x) = f(c_n(t,x))$$

for each $k = 1,2,...,N$.

Lemma 5.0.1. *Under the hypotheses of Theorem 5.0.1, we have*

$$\lim_{n\to\infty}\int_0^{\varepsilon} \|\nabla \mathbf{u}_n\|^2_{L^{p_1}(\Omega)} + \left\|\rho_n|\mathbf{u}_n|^2\right\|_{L^{p_2}(\Omega)} + \|\rho_n \mid \mathbf{u}_n|\|^2_{L^{p_3}(\Omega)} + \|\nabla_x \mu_n\|^2_{L^{p_1}(\Omega)}\, \mathrm{d}t = 0,$$

where

$$p_1 = 2, p_2 = \frac{3\gamma}{\gamma+3}, p_3 = \frac{6\gamma}{\gamma+6}.$$

The proof of above lemma can be done in similar way as in [26, Lemma 3.1.], using energy dissipation (2.28).

Observe that, for each fixed n, the triple $(\rho_n, \mathbf{u}_n, c_n)$ is a weak solution to the particle interaction model in the sense as in the theorem 1.2.1.

As we discussed in section 2.3, we recall some estimates. In accordance with (2.29), (2.38), (2.31), (2.39), (2.40), and (2.42)

$$\begin{cases} \rho_n \in L^\infty([0,\varepsilon];L^\gamma(\Omega)), \\ \mathbf{u}_n \in L^2([0,\varepsilon];W_0^{1,2}(\Omega)), \\ \sqrt{\rho_n}\mathbf{u}_n \in L^\infty([0,\varepsilon];L^2(\Omega)), \\ \rho\mathbf{u}_n \in L^\infty([0,\varepsilon];L^{\frac{2\gamma}{1+\gamma}}(\Omega)), \\ \mathbb{S}_n \in L^2([0,\varepsilon]\times\Omega), \\ c_n \in L^2([0,\varepsilon];W_0^{1,2}(\Omega)) \\ \mu_n \in L^2([0,\varepsilon];W^{1,2}(\Omega)), \end{cases} \tag{5.3}$$

where ε is a positive constant.

Therefore, up to a subsequence, (5.3) yields

$$\begin{cases} \rho_n \rightharpoonup \rho_s \text{ in } L^\infty([0,\varepsilon];L^\gamma(\Omega)), \\ \mathbf{u}_n \rightharpoonup \mathbf{u}_s \text{ in } L^2([0,\varepsilon];W_0^{1,2}(\Omega)), \\ c_n \rightharpoonup c_s \text{ in } L^2([0,\varepsilon];W_0^{1,2}(\Omega)), \\ \mu_n \rightharpoonup \mu_s \text{ in } L^2([0,\varepsilon];W^{1,2}(\Omega)). \end{cases} \tag{5.4}$$

Moreover, Lemma 5.0.1 and the estimates $\mathbf{u}_n \in L^2([0,\varepsilon];W_0^{1,2}(\Omega))$ in (5.3) yield

$$\lim_{n\to\infty}\int_0^\varepsilon \|\mathbf{u}_n\|^2_{L_2(\Omega)}dt = 0, \tag{5.5}$$

which yields

$$\mathbf{u}_s = 0, \text{ a.e. in } (0,\varepsilon)\times\Omega, \tag{5.6}$$

thanks to the compactness of $H^1 \hookrightarrow L_6$.

We notice that, in accordance with section 4.5

$$\begin{aligned} \rho_n \to \rho_s \text{ in } L^1(0,\varepsilon;\Omega) \\ c_n \to c_s \quad \text{in } L^2\left(0,\varepsilon;H^1(\Omega)\right) \\ w_n \to w_s \quad \text{in } L^2\left(0,\varepsilon;H^1(\Omega)\right). \end{aligned} \tag{5.7}$$

From equation for concentration (1.3), we observe that

$$\partial_t(\rho c) = -\mathrm{div}_x(\rho c\mathbf{u}) + \mathrm{div}_x(m\nabla_x\mu)$$

in the sense of of distributions on $(0,\varepsilon)\times\Omega$. Hence,

$$\int_0^\varepsilon \|\rho_n c_n\|_{W^{-1,1}(\Omega)} \leq \|c_n\|_{L^2(0,\varepsilon;L^6(\Omega))}\|\rho_n\mathbf{u}_n\|_{L^2(0,\varepsilon;L^{\frac{6\gamma}{\gamma+6}}(\Omega))} + \|\nabla_x\mu_n\|_{L^2(0,\varepsilon;L^2(\Omega))}$$

The right hand side converges to 0, due to Lemma 5.0.1.By Aubin-Lions lemma, thus, we show $\rho_s c_s$ is independent of t.

By taking the limit $n\to\infty$ in the continuity equation (1.1), we can show

$$\int_0^\varepsilon \int_\Omega \rho_s\phi_t\,dx\,dt = 0\;\forall\phi\in C_0^\infty((0,\varepsilon)\times\Omega)$$

Thus, ρ_s is independent of t.

Our next goal is to show if μ_s and c_s is Lipschitz continuous, the following holds

- ρ_s is strictly positive on Ω,
- (ρ_s, c_s, μ_s) solves the state problem (5.2)

Because $\lim_{n\to\infty}\int_0^\varepsilon\|\nabla_x\mu_n\|_{L^2}\mathrm{d}x = 0$ and the assumption on μ_s , we have μ_s is constant everywhere.

Together with (5.3) (5.4) (5.7), we can pass to the limit in the equation of balance of momentum, we get

$$\nabla_x p(\rho_s) = \rho_s(\mu_s \nabla_x c - \frac{\partial f}{\partial c}\nabla_x c_s) \text{ in } \Omega. \tag{5.8}$$

The right hand side of (5.8)can be expressed as $\rho_s \nabla_x F$, where

$$F = \mu_s c_s - f(c_s) \tag{5.9}$$

The assumption on c_s makes F Lipschitz continuous. Note that we have the following property:

$$\frac{1}{\rho_s}\nabla_x p(\rho_s) = \frac{1}{\rho_s} p'(\rho_s)\nabla_x \rho_s =: \nabla_x \Phi(\rho_s), \tag{5.10}$$

where $\Phi(\rho) = \int_0^\rho Z^{-1} p'(Z) dZ$. We have that Φ is a strict increasing function for variable ρ, thanks to $\Phi'(\rho) = \frac{p'(\rho)}{\rho} > 0$ and thus F is also a strict increasing function. Using the state equation in (5.8) with the above property we just show, we achieve the positivity of ρ_s, thanks to [25, Theorem 2.1].

The proof of Theorem 5.0.1 is now complete.

Chapter 6: Conclusions and open problems

This dissertation is the first one that treats the interaction of diblock copolymer melt with compressible fluid. The works that have been done focus on static problem without the influence of time, or on the incompressible case which is too idealized. The existence of weak solutions to the model of compressible viscous diblock copolyer fluid has been proved here.

The limitation of our results is that this model does not cover the case for free energy $f_0(\rho,c)$ depending also on the density ρ. In physics perspective, it is natural for the free energy to be related with density in a way that $f_0(\rho,c)=H(c)\log\rho+G(c)$. The result can be improved by using a more complicated form of free energy density $\rho f(\rho,c)$ involving logarithm as in [2]:

$$\begin{aligned}\rho f(\rho,c)=&\alpha_1\rho\frac{1-c}{2}\ln\left(\rho\frac{1-c}{2}\right)+\alpha_2\rho\frac{1+c}{2}\ln\left(\rho\frac{1+c}{2}\right)-\beta c^2\\=&\rho\left(\alpha_1\frac{1-c}{2}\ln\frac{1-c}{2}+\alpha_2\frac{1+c}{2}\ln\frac{1+c}{2}\right)\\&+\rho\ln\rho\left(\alpha_1\frac{1-c}{2}+\alpha_2\frac{1+c}{2}\right)-\beta c^2,\end{aligned}\tag{6.1}$$

where $\alpha_1,\alpha_2,\beta>0$. Moreover, this form of free energy density come from Cahn and Hilliard [9]: $\alpha_1\frac{1-c}{2}\ln\frac{1-c}{2}+\alpha_2\frac{1+c}{2}\ln\frac{1+c}{2}$. It is also called Flory–Huggins logarithmic potential. This $f(\rho,c)$ has a corresponding pressure $p(\rho,c)$ which depends on both density ρ and concentration difference c. The pressure depending on concentration difference is more physical. Dealing with this type free energy and pressure requires more complicated work in the proof.

This dissertation gives an insight for large time behavior assuming the regularity on the limiting system. The large time behavior problem for compressible Navier-Stokes-Cahn-Hilliard equations still remains open, due to the complicated right hand side of the equation for conservation of momentum. The existing literature (e.g.[25],[26],[24]) on large-time behavior for Navier-Stokes system and Navier-Stokes-Fourier system relies heavily on the regularity of the forcing term. The forcing term is in the form of $\rho f(x)$ where $f(x)$ satisfies confinement hypothesis: bounded and Lipschitz continuous in $\bar{\Omega}$ and the sub-level sets $[f(x) < k]$ are connected in Ω for any $k > 0$. There are also works on large-time behavior of fluid–particle interaction model[10] and multicomponent reactive flows[39]. In these works, they also follows the idea by Feireisl and Petzetolva [25],[26],[24] and has restriction on the forcing term.

Talking about possible future work, it might be possible to work on dynamic boundary condition problem for this model, if we follow the strategy as in [12] to address the quantities on boundaries. The idea is to assume the boundary Γ sufficiently smooth, and to impose the generalized Navier boundary conditions

$$\left.\begin{aligned} \mathbf{u}\cdot\mathbf{n} &= 0 \\ \left(\mathbb{S}\left(c, \nabla_x \mathbf{u}\right)\mathbf{n}\right)_\tau + \beta \mathbf{u}_\tau &= \mathscr{L}(c)\nabla_\tau c \end{aligned}\right\} \text{ on } \Gamma,$$

where $\beta > 0$, together with the Neumann boundary condition for the chemical potential μ and w

$$\nabla_x \mu \cdot \mathbf{n} = \nabla_x w \cdot \mathbf{n} = 0, \text{ on } \Gamma.$$

And for c the dynamic boundary conditions is imposed as follow:

$$\left.\begin{aligned} \partial_t c + \mathbf{u}_\tau \nabla_\tau c &= -\mathscr{L}(c) \\ \mathscr{L}(c) &= -\triangle_\tau c + \xi c + k(c) + \partial_{\mathbf{n}} c \end{aligned}\right\} \text{ on } \Gamma$$

where $\xi > 0$ is a constant and k a suitable nonlinear function to be specified later. Such boundary condition can be interpreted as a parabolic equation on Γ.

The incompressible case for diblock copolymer model is another aspect we can look into. The proof will be completely different and based on spaces of fractional time and interpolation argument. It is likely doable if we follow the strategies as in [1] which address the incompressible case for NSCH equation.

There is no existing numerical work done for the interaction of compressible fluid made of diblock copolymer. But there are simulation results for static case [13][14][15] and incompressible case[11][40]. Our strategy on implicit-time discretization may be utilized in proving the convergence of numerical approximations. Because this dissertation focuses on the existence theory and the convergence of solutions in large time, the coefficient for average of concentration difference over space is denoted by M, and the coefficient for intrinsic length scale for minimizer is set to be 1. So in my work, the relation to Fig 1 is not shown. But our existence theory allows people to do numerical investigation with any kind of coefficients, so any kind of mesoscopic domains can be considered.

Appendix A:

In this section, we will show the omitted detail in section 3.3 for the proof of the following,

$$\overline{p_\delta} = p_\delta(\rho), \tag{A.1}$$

as $\delta \to 0$.

Since $\rho \in L^2(Q_T), \mathbf{u} \in L^2\left(0,T;H^1(\Omega)\right)$, we can use the regularizing procedure of DiPerna and Lions [16] or [43, Lemma 6.9], to conclude that ρ is a renormalized solution of the transport equation (1.2) as in (1.20) for all $B(\rho)$ such that $\tilde{b}(\rho) = \rho B(\rho) \in C^0([0,\infty)) \cap C^1(0,\infty)$ and

$$\left|\tilde{b}'(\rho)\right| \leq \begin{cases} Ct^{-\lambda_0} & \text{if } t \in (0,1] \\ Ct^{\lambda_1} & \text{if } t > 1 \end{cases}$$

for some $\lambda_0 < 1$ and $\lambda_1 \leq 1$. In particular, we can choose $B(\rho) = \log\rho$, which implics that

$$\partial_t(\rho\log\rho) + \operatorname{div}(\rho\log(\rho)\mathbf{u}) + \rho\operatorname{div}\mathbf{u} = 0 \quad \text{in } \mathscr{D}'\left(\mathbb{R}^3 \times (0,T)\right) \tag{A.2}$$

We choose $\psi = \Psi'\left(\rho^N\right)\chi_{[0,t]}$ in (3.33), where $\Psi : \mathbb{R} \to \mathbb{R}_+$is a smooth and convex function, then we get

$$
\begin{aligned}
&\int_\Omega \Psi\left(\rho^N(t)\right) dx - \int_\Omega \Psi\left(\rho_0\right) dx \\
&\leq \frac{1}{h}\int_{t-h}^t \int_\Omega \Psi\left(\rho^N(\tau)\right) dx d\tau - \int_\Omega \Psi\left(\rho_0\right) dx \\
&= \int_{Q_t} \partial_{\tau,h}^- \Psi\left(\rho^N(\tau)\right) d(x,\tau) \leq \int_{Q_t} \Psi'\left(\rho^N\right) \partial_{\tau,h}^- \rho^N(\tau) d(x,\tau) \\
&= -\int_{Q_t} \Psi'\left(\rho^N\right) \operatorname{div}\left(\rho^N \mathbf{u}^N\right) d(x,\tau) + h\int_{Q_t} \Delta\rho^N \Psi'\left(\rho^N\right) d(x,\tau) \\
&= -\int_{Q_t} \left(\left(\Psi'\left(\rho^N\right)\rho^N - \Psi\left(\rho^N\right)\right)\operatorname{div}\mathbf{u}^N + h\Psi''\left(\rho^N\right)\left|\nabla\rho^N\right|^2\right) d(x,\tau) \\
&\leq -\int_{Q_t} \left(\Psi'\left(\rho^N\right)\rho^N - \Psi\left(\rho^N\right)\right)\operatorname{div}\mathbf{u}^N d(x,\tau)
\end{aligned}
$$

because of Jensen's inequality and $\tilde{\rho}^N = \frac{1}{h}\chi_{[0,h]} *_t \rho^N$. After a simple approximation we can replace $\Psi(s)$ by $s\log s$. Hence, passing to the limit $N \to \infty$ and using (A.2), we have the following

$$
\int_\Omega (\overline{\rho\log\rho} - \rho\log\rho)(t) dx \leq \int_{Q_t} (\rho \operatorname{div}\mathbf{u} - \overline{\rho\operatorname{div}\mathbf{u}}) d(x,\tau) \tag{A.3}
$$

for almost all $t \in (0,T)$. In what follows, the symbol $\Delta^{-1} f = K * f$ denotes the convolution of f with the fundamental solution of the Laplacean on $\mathbb{R}^3$, where functions defined on Ω are extended by zero to functions on $\mathbb{R}^3$. We choose $\varphi = \psi\nabla\Delta^{-1}\left[\rho^N\right], \psi \in C_0^\infty(Q_T)$ in (3.32) and obtain

$$
\begin{aligned}
&\int_{Q_T} \psi\left(p_\delta^N I - \mathbb{S}^N\right) \mathscr{R}\left[\rho^N\right] d(x,t) \\
&\quad = \int_{Q_T} \psi\mathbf{u}^N \left(\rho^N \nabla\operatorname{div}\Delta^{-1}\left(\rho^N\mathbf{u}^N\right) - \rho^N\mathbf{u}^N \cdot \mathscr{R}\left[\rho^N\right]\right) d(x,t) \\
&\qquad + \int_{Q_T} \psi\rho^N\mathbf{u}^N \tau_{-h} \nabla\operatorname{div}\Delta^{-1}\left(\rho^N\mathbf{u}^N - h\nabla\rho^N\right) d(x,t) \\
&\qquad + \int_{Q_T} \left(-\rho^N\mathbf{u}^N\left(\partial_{t,h}^+\psi\right)\tau_{-h}\left(\psi\nabla\Delta^{-1}\rho^N\right) + g^N \cdot \nabla\Delta^{-1}\rho^N\right) d(x,t)
\end{aligned}
$$

where $\mathscr{R} = \nabla^2\Delta^{-1}$ and

$$g^N = -\left(p_\delta^N I - \mathbb{S}^N + \rho^N \mathbf{u}^N \otimes \mathbf{u}^N\right) \cdot \nabla\psi$$
$$+ h\psi\nabla\rho^N \cdot \nabla\mathbf{u}^N + \rho^N\left(\partial_c f^N - \mu^N\right)\nabla c^N \psi$$

With the help of corollary 6.1 in [19], we conclude

$$\begin{aligned}
&\lim_{N\to\infty}\int_{Q_T} \psi\mathbf{u}^N\left(\rho^N\Delta^{-1}\nabla\operatorname{div}\left(\rho^N\mathbf{u}^N\right) - \rho^N\mathbf{u}^N\cdot\mathscr{R}\left[\rho^N\right]\right)d(x,t)\\
&= \int_{Q_T} \psi u\left(\rho\Delta^{-1}\nabla\operatorname{div}(\rho\mathbf{u}) - \rho u\cdot\mathscr{R}[\rho]\right)d(x,t).
\end{aligned}$$

Moreover, using the previous results on strong and weak convergence, it is easy to pass to the limit in all remaining terms to conclude that

$$\begin{aligned}
&\lim_{N\to\infty}\int_{Q_T} \psi\left(p_\delta^N I - \mathbb{S}^N\right)\mathscr{R}\left[\rho^N\right]d(x,t)\\
&\quad= \int_{Q_T} \psi\mathbf{u}\left(\rho\nabla\operatorname{div}\Delta^{-1}(\rho\mathbf{u}) - \rho u\cdot\mathscr{R}[\rho]\right)d(x,t)\\
&\qquad+ \int_{Q_T}\left(-\rho\mathbf{u}\left(\partial_t\psi\right)\left(\nabla\Delta^{-1}\rho\right) + g\cdot\nabla\Delta^{-1}\rho\right)d(x,t)
\end{aligned}$$

where

$$g = -\left(p_\delta I - \mathbb{S} + \rho\mathbf{u}\otimes\mathbf{u}\right)\cdot\nabla\psi + \overline{\rho\frac{\partial f}{\partial c}}\nabla c - \rho\mu\nabla c.$$

On the other hand, choosing $\varphi = \psi\nabla\Delta^{-1}\rho$ in (3.45) and comparing it with the latter identity, we obtain

$$\begin{aligned}
&\int_{Q_T}\psi\left(\overline{p_\delta}I - \mathbb{S}\right)\mathscr{R}[\rho]d(x,t)\\
&\quad= \lim_{N\to\infty}\int_{Q_T}\psi\left(p_\delta^N I - \mathbb{S}^N\right)\mathscr{R}\left[\rho^N\right]d(x,t)
\end{aligned} \tag{A.4}$$

for all $\psi \in C_0^\infty(Q_T)$. Our next goal is to show that

$$\begin{aligned}\lim_{N\to\infty}&\left(\mathscr{R}:\left[\psi\nu\left(c^N,w^N\right)\left(\nabla\mathbf{u}^N+\left(\nabla\mathbf{u}^N\right)^T\right)\right]-\psi 2\nu\left(c^N,w^N\right)\operatorname{div}\mathbf{u}^N\right)\\&=\mathscr{R}:\left[\psi\nu(c,w)\left(\nabla\mathbf{u}+\nabla\mathbf{u}^T\right)\right]-\psi 2\nu(c,w)\operatorname{div}\mathbf{u}\end{aligned}\tag{A.5}$$

weakly in $L^2(0,T;W^{\omega,q}(\Omega))$ for some $\omega>0, q>1$. In order to see (A.5), we adapt the technique of [20]. In particular, we report the following lemma [20, Lemma 4.2].

Lemma A.0.1. *Let $w \in W^{1,r}\left(\mathbb{R}^d\right)$ and $\mathbf{V}\in L^2\left(\mathbb{R}^d;\mathbb{R}^d\right)$ be given, where $r>\frac{2d}{d+2}$. Then there exists $\omega=\omega(r)>0$ and $q=q(r)>1$ such that*

$$\|\mathscr{R}[w\mathbf{V}]-w\mathscr{R}[\mathbf{V}]\|_{W^{\omega,q}\left(\mathbb{R}^d;\mathbb{R}^d\right)}\leq C(r)\|w\|_{W^{1,r}\left(\mathbb{R}^d\right)}\|\mathbf{V}\|_{L^2\left(\mathbb{R}^d;\mathbb{R}^d\right)}$$

Extending $c^N, \partial_{x_j}\mathbf{u}^N$ to be zero outside Ω we intend to apply Lemma A.0.1 to

$$w=\nu\left(c_\delta\right), \mathbf{V}=\left[V_1,V_2,V_3\right], V_i=\partial_{x_i}u_{\delta,j}+\partial_{x_j}u_{\delta,i}, i=1,2,3,$$

where $j=1,2,3$ is fixed. Indeed as the shear viscosity coefficient ν is (globally) Lipschitz in c and w, the uniform estimate stated in Lemma 3.1.3 allows us to apply Lemma A.0.1, with $r=2$.

Following step by step the arguments of Section 2.4 we obtain that

$$c^N\to c \text{ in } L^2(0,T;W^{1,2}(\Omega)).\tag{A.6}$$

$$w^N\to w \text{ in } L^2(0,T;W^{1,2}(\Omega)).\tag{A.7}$$

Also, we observe, in accordance with (2.38),

$$\mathbf{u}^N\to\mathbf{u} \text{ in } L^2(0,T;W_0^{1,2}(\Omega;\mathbb{R}^3))\tag{A.8}$$

Consequently, in accordance with (A.6), (A.7), (A.8), we get (A.5). Combining (A.5) with (A.4), we obtain the essential relation

$$\int_{Q_t} \psi\,(\overline{p_\delta \rho} - \overline{p_\delta}\rho)\,d(x,\tau) = \int_{Q_t} \psi \left(\frac{4}{3}\nu(c) + \eta(c)\right)(\overline{\rho \operatorname{div}\mathbf{u}} - \rho \operatorname{div}\mathbf{u})d(x,\tau).$$

Choosing $\psi = \left(\frac{4}{3}\nu(c) + \eta(c)\right)^{-1}$ above and using (A.3), we obtain

$$\int_\Omega (\overline{\rho \log \rho} - \rho \log \rho)dx \le \int_{Q_t} \left(\frac{4}{3}\nu(c) + \eta(c)\right)^{-1} (\overline{p_\delta \rho} - \overline{p_\delta}\rho)\,d(x,\tau)$$

for some $\Lambda > 0$, where, because of the decomposition (3.21),

$$\int_{Q_t} \left(\frac{4}{3}\nu(c) + \eta(c)\right)^{-1} (\overline{p_\delta \rho} - \overline{p_\delta}\rho)\,d(x,\tau) \le \Lambda \int_{Q_t} (\overline{\rho \log \rho} - \rho \log \rho)$$

by the same arguments as in [19, Section 6.6.3]. Hence

$$\int_\Omega (\overline{\rho \log \rho} - \rho \log \rho)(t)dx \le \Lambda \int_{Q_t} (\overline{\rho \log \rho} - \rho \log \rho)d(x,t)$$

which implies

$$\int_\Omega (\overline{\rho \log \rho} - \rho \log \rho)(t)dx \equiv 0$$

for all $t \in (0,T)$ because of Gronwall's lemma. Thus ρ^N converges almost everywhere to ρ and

$$\overline{p_\delta} = p_\delta(\rho), \quad \overline{\rho \frac{\partial f}{\partial c}} = \rho \frac{\partial f}{\partial c}(\rho, c).$$

Appendix B:

In this section, we want to show strong convergence of densities $\rho_\delta \to \rho$. We use the renormalized equation (4.25) for $b(\rho) = \rho L_k(\rho)$, where

$$L_k(\rho) = \int_1^\rho \frac{T_k(z)}{z^2} dz,$$

$$T_k(\rho) = \min\{\rho, k\}, \rho \geq 0.$$

Accordingly, we obtain

$$\int_\Omega \rho_\delta L_k(\rho_\delta)(\tau) dx + \int_{Q_T} T_k(\rho_\delta) \operatorname{div} \mathbf{u}_\delta d(x,t) = \int_\Omega \rho_{0,\delta} L_k(\rho_{0,\delta}) dx. \tag{B.1}$$

At this stage, we have to show that the limit quantities ρ, $\mathbf{u}$ represent a renormalized solution of (4.19). Following the approach of [22] we introduce the concept of oscillations defect measure associated to the family $\{\rho_\delta\}_{\delta>0}$:

$$\operatorname{osc}_p[\rho_\delta \to \rho](O) = \sup_{k\geq 1} \left(\limsup_{\delta \to 0} \int_O |T_k(\rho_\delta) - T_k(\rho)|^p \, dxdt \right).$$

We report the following result [19, Chapter 6, Proposition 6.3].

Lemma B.0.1. *Let*

$$\operatorname{osc}_p[\rho_\delta \to \rho]((0,T) \times \Omega) < \infty \textit{ for a certain } p > 2. \tag{B.2}$$

Then $\rho, \mathbf{u}$ represent a renormalized solution of (4.19).

In order to show (B.2), we make use of relation (4.29) for $b = T_k$. To begin with, as the pressure p is given through the constitutive relation (1.13) and $\{c_\delta\}_\delta$ converges strongly, we observe that

$$\overline{p(\rho,c)T_k(\rho)} = \overline{p(\rho,\cdot)T_k(\rho)}, \overline{p(\rho,c)} = \overline{p(\rho,\cdot)},$$

where

$$\overline{p(\rho,\cdot)T_k(\rho)} = \text{weak}_{L^1} \lim_{\delta\to 0} p\left(\rho_\delta, c\right) T_k\left(\rho_\delta\right),$$

and, similarly,

$$\overline{p(\rho,\cdot)} = \text{weak}_{L^1} \lim_{\delta\to 0} p\left(\rho_\delta, c\right).$$

On the other hand, in accordance with hypotheses (1.14), (1.15), the pressure can be written in the form

$$p(\rho,c) = a\rho^\gamma + p_m(\rho,c) + p_b(\rho), a > 0, \tag{B.3}$$

where p_m is non-decreasing in ρ and $p_b \in C^2[0,\infty)$ has compact support in $[0,\infty)$.

As p_m is non-decreasing in ρ and $0 \leq \overline{T_k(\rho)} \leq k$, it is easy to check that

$$\left(p_m\left(\rho_n, c\right) - p_m\left(\overline{T_k(\rho)}, c\right)\right)\left(T_k\left(\rho_n\right) - \overline{T_k(\rho)}\right) \geq 0;$$

whence letting $n \to \infty$ we get

$$\overline{p_m(\rho,\cdot)T_k(\rho)} - \overline{p_m(\rho,\cdot)}\,\overline{T_k(\rho)} \geq 0 \tag{B.4}$$

while, exactly as in [19, Proposition 6.2], we can show that

$$\begin{aligned}
&\int_0^T \int_\Omega \left(\overline{\rho^\gamma T_k(\rho)} - \overline{\rho^\gamma}\,\overline{T_k(\rho)}\right) dxdt \\
&\quad \geq \limsup_{\delta \to 0} \int_0^T \int_\Omega \left|T_k(\rho_\delta) - T_k(\rho)\right|^{\gamma+1} dxdt.
\end{aligned} \tag{B.5}$$

Equation (B.4), (B.5), together with (4.29) and Young's inequality yields

$$\operatorname{osc}_{\gamma+1}[\rho_\delta \to \rho]((0,T)\times\Omega) < \infty. \tag{B.6}$$

In particular, with Lemma B.0.1 , the limit functions ρ, u represent a renormalized solution of (3.2). Thus we get

$$\int_\Omega \rho_\delta L_k(\rho)(\tau) dx + \int_0^\tau \int_\Omega T_k(\rho) \operatorname{div} \mathbf{u} dxdt = \int_\Omega \rho_0 L_k(\rho_0) dx$$

which, together with (B.1), gives rise to

$$\begin{aligned}
&\int_\Omega \left(\overline{L_k(\rho)} - L_k(\rho)\right)(\tau) dx + \int_0^\tau \int_\Omega \left(\overline{T_k(\rho)\operatorname{div}\mathbf{u}} - \overline{T_k(\rho)}\operatorname{div}\mathbf{u}\right) dxdt \\
&\quad = \int_0^\tau \int_\Omega \left(T_k(\rho) - \overline{T_k(\rho)}\right) \operatorname{div}\mathbf{u}\, \mathrm{d}x\mathrm{d}t
\end{aligned} \tag{B.7}$$

for any $\tau \in [0,T]$ since $\rho_{0,\delta} \to \rho_0$ in $L^1(\Omega)$. Finally, as a consequence of (B.6),

$$\int_0^\tau \int_\Omega \left(T_k(\rho) - \overline{T_k(\rho)}\right) \operatorname{div}\mathbf{u} dxdt \to 0 \text{ as } k \to \infty$$

whence, by virtue of (4.29),(B.3)-(B.5), we can let $k \to \infty$ in (B.7) in order to obtain

$$\int_\Omega (\overline{\rho \log(\rho)} - \rho \log(\rho))(\tau) dx \leq \Lambda \int_0^\tau \int_\Omega (\overline{\rho \log(\rho)} - \rho \log(\rho))(\tau) dxdt$$

for a certain $\Lambda > 0$ (see Section 6.6 in Chapter 6 in [19] for details). Thus, by means of

Gronwall's lemma,

$$\overline{\rho \log(\rho)} = \rho \log(\rho) \text{ a.a. in } (0,T) \times \Omega,$$

in particular

$$\rho_\delta \to \rho \text{ in } L^1((0,T) \times \Omega).$$

We have shown the strong convergence of densities.

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