ABSTRACT

Title of Thesis: A Modern Overview of Local Sections of Flows

Name of degree candidate: Helen Marie Colston

Degree and Year: Master of Arts, 1990

Thesis directed by: Dr. Nelson Markley, Professor,
Department of Mathematics

This paper examines local cross sections of a continuous flow on a locally compact metric space. Some of the history of the study of local cross sections is reviewed, with particular attention given to H. Whitney's work. The paper presents a modern proof that local cross sections always exist at non-critical points of a flow. Whitney is the primary source for the key idea in the existence proof; he also gave characterizations of local cross sections on 2- and 3-dimensional manifolds. We show various topological properties of local cross sections, the most important one being that local cross sections on the same orbit are locally homeomorphic. A new elementary proof using the Jordan Curve Theorem shows that when a flow is given on a 2-manifold, a local cross section will be an arc. Whitney is cited for a similar result on 3-manifolds. Finally, the so-called "dog-bone" space of R. Bing is used to construct a flow on a 4-manifold with a point at which every local cross section is not homeomorphic to a 3-dimensional disk.

A MODERN OVERVIEW OF

LOCAL SECTIONS

OF FLOWS

by

Helen Marie Colston

Thesis submitted to the Faculty of the Graduate School
of The University of Maryland in partial fulfillment
of the requirements for the degree of
Master of Arts
1990

Advisory Committee:

Professor Nelson Markley, Chairman/Advisor
Associate Professor Kenneth Berg
Professor Guydo Lehner

DEDICATION

For

Ken and Thalia

ACKNOWLEDGEMENT

I would like to express my profound gratitude to
Professor Markley for his encouragement,
advice, and guidance.

TABLE OF CONTENTS

Section	Page
Introduction	1
Chapter 1 Fundamental Ideas	3
Definition of Local Cross Section	4
Fundamental Construction	5
Proof of Existence	7
Chapter 2 Historical Review	10
H. Whitney	11
Nemitsky and Stepanov	15
Ordinary Differential Equations	17
R. Bing	20
Chapter 3 Properties of Local Cross Sections	22
Closure of Interior	23
Crossing of Section by Trajectory	25
Locally Homeomorphic Sections	26
Bending a Section by a Continuous Function	28
Arbitrarily Long Local Sections	30
Chapter 4 Cross Sections of Flows on Manifolds	32
Local Sections on Surfaces are Arcs	32
Local Sections on 3-Manifolds	35
Counterexample on a 4-Manifold	35
References	38

INTRODUCTION

The existence of a local cross section to a solution of an n-dimensional system of autonomous differential equations is a standard theorem in the theory of ordinary differential equations. These local cross sections exist at non-fixed points of the system, and they have certain useful properties. In particular, these local cross sections can be constructed as closed (n-1)-dimensional disks, and the trajectories of the solutions in a neighborhood of the section are homeomorphic to parallel line segments in a neighborhood of $\mathbb{R}^n$. Construction of local cross sections to solutions to a system of autonomous differential equations is straightforward, since a non-zero tangent exists at each non-critical point; then a small disk in the hyperplane perpendicular to this tangent will be a local cross section. In the more general case of a continuous flow, it is also possible to show that local cross sections exist at non-critical points. Our goal is to present an overview of important results regarding the existence of local cross sections to a continuous flow and to exhibit topological properties of these local cross sections.

This paper will be organized into four chapters. The first will cover fundamental notions related to local cross sections and prove that local cross sections exist for continuous flows. In the second chapter we give an historical review of the origins of the main ideas on cross sections. The most important past results will be examined rather carefully. The third chapter contains theorems describing the topological properties of local cross sections. Finally, the fourth chapter will be devoted to the question of when local cross sections for a flow on an n-dimensional manifold are homeomorphic to the (n-1)-dimensional closed cell. It will include an unpublished elementary proof of this fact for surfaces and a counterexample for n = 4.

Chapter 1: FUNDAMENTAL IDEAS

Let X be a locally compact metric space and let $\mathbb{R}$ denote the real numbers. Let ρ denote the metric on X. A flow on X is a mapping $f: X \times \mathbb{R} \to X$ such that:

i) $f(x,\emptyset) = x$,

ii) f is continuous on $X \times \mathbb{R}$,

iii) $f(f(x,s),t) = f(x,s+t)$.

For ease of notation, we may abbreviate this flow by suppressing the f, so we have $f(x,t) = xt$. Note that for each t, $x \to xt$ is a homeomorphism of X onto X. This one-parameter group of transformations of the space X onto itself possessing these properties is also called a dynamical system. Henceforth, we suppose f is a flow on a locally compact metric space, X. This includes $\mathbb{R}^n$, topological manifolds (metric spaces such that every point has a neighborhood homeomorphic to $\{v \in \mathbb{R}^n : \|v\| < 1\}$), and compact metric spaces.

The set of fixed points of a continuous flow is defined by

$$C = \{x: xt = x \text{ for all } t \text{ in } \mathbb{R}\}.$$

A point not in C which for some $T > \emptyset$ satisfies the condition $x(t + T) = xt$ for all t is called periodic. Since this equation also admits solutions of nT ($n = \pm 1, \pm 2, \ldots$), the smallest positive T satisfying the condition above is called the <u>period</u> of the flow xt.

The set of points $\{f(p,t), -\infty < t < +\infty\}$ for fixed p is called the <u>trajectory</u> or <u>orbit</u> of the point p and can be denoted $f(p, -\infty, +\infty)$ or $\mathcal{O}(p)$. The set of points

$$\{f(p,t), T' \leq t \leq T''\},$$

where p is fixed and $-\infty < T' < T'' < +\infty$, is called a <u>finite arc</u> of the trajectory and will be written $f(p,[T',T''])$ or simply $p[T',T'']$. The number $T'' - T'$ is called the <u>time length</u> of this arc of the trajectory.

<u>Definition</u>. Let $x_0 \notin C$. A closed subset S of X is a <u>local cross section</u> of length 2λ at $x_0 \in X$ if the map

$$h: S \times \{t: |t| \leq \lambda\} \to X \text{ given by } h(x,t) = xt \text{ is a}$$

homeomorphism of $S \times \{|t| \leq \lambda\}$ onto a neighborhood of x_0, that is, x_0 is an interior point of the image. The image of this homeomorphism is called a <u>rectangular neighborhood</u> of x_0.

We would like to show that local cross sections exist. To this end, we now construct a special function, G(xt), which will be used to prove the existence of local cross sections for a flow f. This construction and some intermediate lemmas will lead to the main theorem on existence, Theorem 1. Theorem 2 and its corollary will give some properties of local cross sections that can be derived from our construction.

We will use the notation Cl and Int for the operations of taking the closure and interior, respectively, of a subset of X.

Let x_0 be a non-fixed point of f. Since X is metric, it satisfies the Hausdorff condition, and since it is locally compact, it is completely regular. Hence there exists a continuous, real-valued function P on X such that

$$P(x_o) = \emptyset < P(x_o t_o) = 1,$$

where $t_o > \emptyset$, $x_o t_o \neq x_o$. Since we are assuming X is metric, then $P = \rho(x_o, x)/\rho(x_o, x_o t_o)$ will work.

Set $G(x) = \int_0^{t_o} P(xs)\,ds$. It follows that

$$G(xt) = \int_0^{t_o} P[(xt)s]\,ds = \int_0^{t_o} P[x(t+s)]\,ds$$

$$= \int_t^{t+t_o} P(xs)\,ds, \text{ and}$$

$$\partial G(xt)/\partial t = P[x(t+t_o)] - P(xt).$$

Clearly $G(xt)$ and $\partial G(xt)/\partial t$ are both continuous functions of (x,t).

From this construction it follows in particular that if $t = \emptyset$ and $x = x_o$, then $\partial G(xt)/\partial t = 1$ and there exists a neighborhood U_1 of x_o and $\lambda > \emptyset$ such that

$$\partial G(xt)/\partial t > \emptyset \text{ for } x \in U_1, |t| \leq 4\lambda.$$

So $G(x_o\lambda) > G(x_o) > G(x_o(-\lambda))$, and thus there is a neighborhood U_2 of x_o such that

$$G(x\lambda) > G(x_o) > G(x(-\lambda)) \text{ for all } x \text{ in } U_2.$$

Let U be any neighborhood of x_o with $Cl(U) \subset U_1 \cap U_2$. Set

$$S = \{x: G(x) = G(x_o)\} \cap Cl(U)[-\lambda, \lambda],$$

$$F = S[-\lambda, \lambda].$$

We will now consider some properties of these objects which will enable us to prove that S is a local cross section. As we begin to study S and F, let us note that clearly S and F are closed and $x_0 \in S$.

Lemma 1. $Cl(U) \subset F$.

Proof. Take any x in Cl(U). Because $Cl(U) \subset U_2$, we know that $G(x\lambda) > G(x_0) > G(x(-\lambda))$, where x_0 is the non-fixed point we chose to construct U_1 and U_2. Therefore, we can find $t \in (-\lambda,\lambda)$ such that $G(xt) = G(x_0)$. But this implies that xt is an element of S. Therefore, x is an element of F. Hence $Cl(U) \subset F$. //

Lemma 2. If $x \in S$, $xt \in S$, and $|t| \leq 2\lambda$, then $t = 0$.

Proof. Suppose there exists $x \in S$ and $t \in (0,2\lambda]$ such that $xt \in S$. By definition, S is a subset of $Cl(U)[-\lambda,\lambda]$, which implies that $x = yt'$, where $y \in Cl(U)$ and $|t'| \leq \lambda$. Now consider the set $y[-3\lambda,3\lambda]$. Both x and xt are in $y[-3\lambda,3\lambda]$. Since $Cl(U) \subset U_1$ by construction, we know $y \in U_1$ and so $\partial G(yt)/\partial t > 0$ for $t \in [-3\lambda,3\lambda]$. Also t+t' and t' are in $[-3\lambda,3\lambda]$, and so we get

$$G(yt') = G(x) = G(x_0) = G(xt) = G(y(t+t')).$$

But this is an impossibility, since G(ys) is increasing on $[-3\lambda,3\lambda]$. //

Lemma 3. For each $x \in F$ there exists a unique $y \in S$ and t, $|t| \leq \lambda$, such that $x = yt$.

Proof. Suppose we have y_1 and y_2 in S, and t_1 and t_2 with $|t_i| \leq \lambda$, i = 1, 2, such that $x = y_1t_1$ and $x = y_2t_2$. This implies that $y_1 t_1 = y_2 t_2$. By the group property of the flow this implies $y_1 = y_2(t_2 - t_1)$. But $|t_2 - t_1| \leq 2\lambda$. So $y_1 \in S$ implies $t_2 - t_1 = \emptyset$ by Lemma 2. Hence $t_1 = t_2$. Then $y_1 t_1 = y_2 t_2 = y_2 t_1$ which implies $y_1 = y_2$. //

Now we are ready to show the main result of this chapter.

Theorem 1. Local cross sections exist at non-fixed points of the flow f.

Proof. Let S and λ be constructed as above. Consider the map $h: S \times [-\lambda, \lambda] \to X$ given by $(x,t) \to xt$.

Since X is locally compact we can assume $Cl(U_1)$ is compact. It follows that $Cl(U)$, $Cl(U)[-\lambda, \lambda]$, and S are all compact. The map h is continuous because f is. By Lemma 3, h is one-to-one and onto F. Hence, h is a homeomorphism, because $S \times [-\lambda, \lambda]$ is compact. Finally, Lemma 1 implies that F is a neighborhood of x_0 since U is open and this lemma showed that $Cl(U) \subset F$. //

Let us define the map $p: F \to S$ by $p(x) = y$ if $x = yt$, where $y \in S$ and $|t| \leq \lambda$.

Lemma 4. The map p is continuous, closed, open, and $p(Cl(U)) = S$.

Proof. By Lemma 3, the map p is well-defined. The map p will be continuous if for every convergent sequence (x_n) in F converging to x, say, the sequence $(p(x_n)) \to p(x)$. Let (x_n) be a sequence in F converging to $x \in F$, $p(x) = y = xt$, $|t| \leq \lambda$, and $p(x_n) = y_n = x_n t_n$, $|t_n| \leq \lambda$. If t_n does not converge to t, there exists a subsequence x_{n_k} such that t_{n_k} converges to $t' \neq t$. Clearly $t' \in [-\lambda, \lambda]$ since $|t_n| \leq \lambda$. But

$p(x_{n_k}) = x_{n_k}t_{n_k} \rightarrow xt'$, and $xt' \in Cl(S) = S$ because $x_{n_k}t_{n_k} \in S$. Therefore, $t' = t$, $p(x_n) \rightarrow p(x)$, and so p is continuous.

Now to show that p is closed let $p(x_n) = x_n t_n$ be a sequence in S converging to $y \in S$. Because $[-\lambda, \lambda]$ is compact, there exists a subsequence (x_{n_k}) such that (t_{n_k}) converges to t' in $[-\lambda, \lambda]$. We know that $(x_{n_k}) = ((x_{n_k}t_{n_k})(-t_{n_k})) \longrightarrow (y(-t')) \in F$ since F is closed. Clearly, $p(y(-t')) = y$. It follows that p is closed.

To show that p is open, let V be open in $S[-\lambda, \lambda]$. Let $x \in V$, so $p(x) = y$, where $y \in S$. We need to show that $p(V)$ is a neighborhood of y in S. Since V is open and h is a homeomorphism, $V \supset h(U \times I)$ where U is open in S and I is open in $[-\lambda, \lambda]$. Clearly $U = p(h(U \times I)) \subset p(V)$, so p is open.

To complete the proof, we show that $p(Cl(U)) = S$. Since $Cl(U) \subset F$ and $S \subset Cl(U)[-\lambda, \lambda]$, the map p is onto. Hence $S \subset p(Cl(U))$. Obviously, $p(Cl(U)) \subset S$. So $p(Cl(U)) = S$. //

Given the mapping p above and the properties we showed it has, we observe that if Cl(U) has a property, say $\mathcal{P}$, and property $\mathcal{P}$ is preserved by closed or open continuous maps, then S has property $\mathcal{P}$, by Lemma 4. (This will be important later.) In particular, we have the next theorem.

<u>Theorem</u> <u>2</u>. The following hold:

(a) If Cl(U) is connected, then S is connected.

(b) If Cl(U) is locally connected, then S is locally connected.

<u>Proof</u> <u>of</u> <u>(a)</u>. The image of a connected space under a continuous map is connected.

Proof of (b). Lemma 4 showed that p is a closed map, and also by Lemma 4 we have p(Cl(U)) = S. A standard result from topology (see, for example, Hocking and Young, [6], p. 125) is that the image of a locally connected space under a closed map is locally connected. Hence p(Cl(U)) = S is locally connected. //

Theorem 2 has a corollary which gives as a consequence an important property of local cross sections when X is a manifold.

Corollary 1. If Cl(U) can be chosen compact, connected, and locally connected, then S is arcwise connected. In particular, if X is a manifold, there exist arcwise connected local cross sections.

Proof. Theorem 2 proved that if Cl(U) is connected and locally connected, then S will be connected and locally connected. S will also be compact since it is a closed subset of Cl(U), which is compact by assumption. By a standard theorem in topology (see [6], p. 116), each two points of a compact, connected, and locally connected metric space can be joined by an arc in the space. //

Chapter 2: HISTORICAL REVIEW

The proof for the existence of a local cross section presented in the previous section is a modification of that given by Hajek in his 1965 paper [4], and is very similar to but less cumbersome than Nemitsky and Stepanov's proof, page 333 of [7]. Hajek refers to the work of Whitney [8], whose three papers ([8], [9], [1Ø]) appear to be the sources for most of the important ideas on local cross sections of flows. This will be more fully explained later in this section. In addition to his existence proof, which is in the slightly more general context of separated uniformisable spaces, Hajek [4] presents a proof of Whitney's result that if there is a flow on a 2-manifold with a section S which is a locally connected continuum, then the section is either a simple arc or a simple closed curve. His proof relies on showing that S is a local dendrite, and later in our paper we will give a more direct route to this result. Nemistsky and Stepanov [7] cite Bebutov [1] as the source for their existence proof, but do not mention Whitney.

Let us look carefully at Whitney's results. In [8] Whitney's primary purpose is to study families of curves which satisfy certain regularity conditions. To understand his definition of a regular family of curves, we will need to use the notion of the span between two arcs. Whitney considers a family of curves whose points form a separable metric space with metric ρ. He gives the following definition, first introduced by Frechet.

Definition. Given two arcs, pq and p'q', homeomorphic, with p corresponding to p' and q to q'. Let d(H) be the upper bound of all numbers $\rho(r,r')$ for corresponding points r, r'. Then the span between the arcs pq and p'q', written $\sigma(pq,p'q')$ is defined to be the lower bound of the numbers d(H) for all such homeomorphisms between the two arcs.

Now, we turn to Whitney's object of study.

Definition. A set of curves forms a regular family, F, if no two intersect, and if given any arc pq of a curve C of the family and any $\varepsilon > 0$ there is a $\delta > 0$ such that if p' lies on C' and $\rho(p,p') < \delta$, then there is an arc p'q' of C' such that $\sigma(pq,p'q') < \varepsilon$.

These regularity conditions are satisfied by solution curves of autonomous systems of differential equations and also by trajectories of a general flow owing to the continuity in initial conditions. In fact, Whitney's main purpose in [8] is to show that a flow can be defined over any regular family of paths. The bulk of his paper is devoted to proving the next theorem, which we quote from [8], p. 269.

Theorem. A function f(p,t) can be defined over any (regular) family of paths, with the following properties:

(1) For each point p of X and any number t, $-\infty < t < +\infty$,

there is a unique point $q = f(p,t)$ which lies on the curve C through p, or coincides with p if p is an invariant point. Further, for each point q of C there is a t such that $q = f(p,t)$.

(2) $f(p,t)$ is continuous in both variables.

(3) If p is not a critical point, $f(p,t)$ moves in the positive (negative) direction along the curve C through p as t increases (decreases).

(4) For each point p of X and any two numbers t' and t'', $f[f(p,t'),t''] = f(p,t' + t'')$; also $f(p,0) = p$.

Any such function of f we say defines a flow in X.

Without presenting a proof of this theorem, we will give a brief sketch to try to show the flavor of Whitney's construction. Whitney covers a regular family of curves with what he calls tubes and pseudotubes consisting of sets of arcs. These arcs are non-intersecting and self-compact, that is, any sequence chosen from these arcs contains a subsequence approaching an arc pq of the set: $\lim_{n \to \infty} \sigma(pq, p_n q_n) = 0$. For tubes, in addition, any set of arcs in the tube must contain inner points. He shows that the ends of these tubes are "cross sections." On the tubes he defines a continuous mapping which associates a unique number t to each point on an arc. He patches tubes and pseudotubes together at fixed points so that the mapping will be continuous over the whole arc. By modifying some of his earlier constructions of continuous functions over families of regular curves, he is able to show that a continuous flow - a function satisfying all the conditions of his theorem - can be constructed over any regular family of paths.

In the course of Whitney's analysis of families of regular curves there are two important sidelights. They are the key idea used to prove the existence of cross sections for flows and the remark that on surfaces local cross sections are arcs. We will look at these two subtopics separately and in some detail.

We begin by examining the fundamental idea for the local cross section existence proof. Early in this paper on regular curves, [8], Whitney gives the following definition, which we quote.

Definition. Take an arc pq of the regular family F and a $\lambda > \emptyset$. The λ-neighborhood of pq, $N_\lambda(pq)$, consists of all arcs p'q' of F such that $\sigma(pq, p'q') < \lambda$.

He uses this idea of neighborhood as a basis for his definition of a "cross section," which we quote from [8] p. 256.

Definition. Let p be a point interior to a curve C of the regular family F. A set of points S forms a cross section through p if the following hold:

(1) Each point p' of S lies within an arc $q_0'q_1'$ of F such that for some $\lambda' > \emptyset$ each arc of $N_{\lambda'}(q_0'q_1')$ contains at most one point of S.

(2) p lies within an arbitrarily small arc q_0q_1 of F such that for some $\lambda > \emptyset$, each arc of $N_\lambda(q_0q_1)$ contains exactly one point of S (hence p is in S).

(3) S is closed.

Whitney concludes [8] with a sketch of an argument intended to show that cross sections exist through non-fixed points of a flow. He sets $q_t = f(q,t)$ and defines a continuous function

$$\theta(q) = \int_0^1 \rho(q_t,p)\,dt.$$

By the group property of a flow he knows $(q_\varepsilon)_t = q_{\varepsilon+t}$. Whitney concludes ([8], section 29):

"Hence, given a point q_ε on the curve through q, we find by a change of variable

$$\theta(q_\varepsilon) - \theta(q) = \int_1^{1+\varepsilon} \rho(q_t,p)\,dt - \int_0^{\varepsilon} \rho(q_t,p)\,dt$$

and thus θ has a partial derivative along each curve:

$$\theta'(q) = \partial\theta(q)/\partial t = \rho(q_1,p) - \rho(q,p).$$

θ' is continuous and positive near p: $\theta'(p) = \rho(p_1,p)$. Hence on each curve passing near p we can find a point q for which $\theta(q) = \theta(p)$; these points are easily seen to form a cross section."

Whitney does not offer more in the way of proof for his claim, but this function θ is exactly the function φ used by Nemitsky and Stepanov when they rigorously prove the existence of cross sections in [7]. This function is also the key construction in our proof of existence; we called the function G. Our construction followed Hajek [4], who offered a mild generalization of Whitney's idea by doing his proof for the case where X is a separated, uniformisable space. More importantly, in the last sentence in the above quote Whitney gives the strategy used in all the existence proofs for local cross sections.

Now we turn to the second important subtopic in Whitney's 1933 paper [8]. This is his remark that on surfaces local cross sections are arcs. Whitney gives a short proof that if the space X is locally connected, then any cross section is locally connected at all inner points ([8], p. 259). Then he asks, if F is a regular family of curves filling a region in Euclidean n-space, is there a cross section through any inner point of a curve which is a closed (n-1)-cell? His full reply, in the case of surfaces, is as follows:

"If n = 2, the answer is yes. For any connected cross section S contains two points p and q such that dropping out any other point of S disconnects these points; hence S is an arc." ([8], p. 260)

Moreover, in the study of flows on surfaces, Whitney's paper of regular curves has been the standard reference for the existence of local cross sectional arcs. He also says in the same paper that the answer is in the affirmative for n = 3. We discuss this later.

Now let us look at the results of another source. Nemitsky and Stepanov prove existence of local cross sections for both systems of autonomous differential equations and general flows.

First, consider a non-fixed point of an autonomous system of differential equations. If N is a neighborhood of this point, Nemitsky and Stepanov define a section of N to be a closed set which has one and only one point in common with every trajectory arc in N. They prove such sections exist by constructing one through an arbitrary non-critical point of a differentiable system ([7], pp. 30 - 32).

With more effort, Nemitsky and Stepanov flesh out the proof of the existence of cross sections for the more general flows introduced in Whitney's paper. Given a finite tube or flow box Y = $f(E,[-T,T])$ on any set $E \subset X$, they call a set $F \subset Y$, closed in Y, a local section of Y if to each point $q \in Y$ there corresponds a unique number t_q such that $f(q,t_q) \in F$ and $|t_q| < 2T$. Then they prove this theorem.

<u>Theorem</u>. Given a non-fixed point, p, of a flow on a metric space, for a sufficiently small $t' > 0$ there can be found a number $\delta > 0$ so that the tube constructed on $S(p,\delta) = \{y \in X: \rho(p,y) < \delta\}$ of time length $2t'$ has a local section.

For their proof, they define a function akin to our $G(xt)$ as Whitney does, and they show that the section for the tube Y = $f(ClS(p,\delta),[-t', t'])$ is the set Q of points $q \in Y$ for which $G(q0) = G(p0)$. Even though E can be any subset of X in their definition of cross section, we note that they prove the existence of a cross section in the special case where $E = S(p,\delta)$.

We mention that the local cross section constructed by Nemitsky and Stepanov is not equivalent to ours, so their proof of existence is not quite adequate for us. The most that can be said if we are given a cross section constructed by Nemitsky and Stepanov is that a subset of it can be found which is a local cross section by our definition. However, a local cross section by our definition will also be one according to theirs.

Nemitsky and Stepanov also prove that a non-critical point of an autonomous differential system which satisfies existence and uniqueness conditions (such as a Lipschitz condition) has a neighborhood, N, in which the trajectories of the system are homeomorphic to parallel straight lines in $E \subset \mathbb{R}^n$ or $\mathbb{R}^{n+1}$. ([7], pp. 30-32.) For a general flow, their main result regarding the topological properties of the flow in a metric space in a neighborhood of a non-critical point is the following theorem.

<u>Theorem</u>. If a finite tube Y of time length 2T, constructed on a set E, has a local section S, then Y is homeomorphic to a system of parallel segments of a Hilbert space. ([7], pp. 236-238]).

Of course, local cross sections or transversals have been used in ordinary differential equations since the time of Poincaré. A modern construction of a transversal is given by Hale, for example. Suppose we are given an autonomous differential equation

$$\dot{x} = f(x) \qquad (*)$$

where $f: X \to \mathbb{R}^n$ is continuously differentiable and X is open in $\mathbb{R}^n$. Let γ_p be the orbit through a non-critical point p in X. Suppose a function $\psi: B_\alpha \to \mathbb{R}^n$ is continuously differentiable, where $B_\alpha = \{u \text{ in } \mathbb{R}^{n-1}, |u| < \alpha\}$. Let

$\Psi(\emptyset) = p$, and suppose the rank of the Jacobian matrix $[\partial \Psi(u)/\partial u]$ is $n-1$ for all u in $B\alpha$. Then the set

$$E_p^{n-1} = \{x \text{ in } \mathbb{R}^n,\ x = \Psi(u),\ u \text{ in } B\alpha\}$$

is called a differentiable (n-1)-cell through p. E^{n-1} is said to be a <u>transversal</u> to γ_p at p if for each $p' \in E_p^{n-1}$, the path $\gamma_{p'}$ through p' is not tangent to E^{n-1} at p'. This is equivalent to the condition that the tangent vector to $\gamma_{p'}$ at p' is linearly independent of those vectors tangent to E^{n-1} at p'. This means

$$D(x,u) = \det [\partial \Psi(u)/\partial u,\ f(x)] \neq \emptyset \text{ for } u < \alpha \text{ for some } \alpha > \emptyset.$$

Assume $D(p,\emptyset) \neq \emptyset$. Hence there is an α sufficiently small to ensure that $D(x,u) \neq \emptyset$ for $|x-p| < \alpha$, $|u| < \alpha$, since $D(x,u)$ is continuous. So when p is a non-critical point of (*) a transversal to γ_p will always exist at p, and with α chosen small enough, E^{n-1} will be transverse to all the paths that cross it.

Now let us specify that $\varphi(t,p')$ such that $\varphi(\emptyset,p') = p'$ is the solution of (*) which describes the orbit $\gamma_{p'}$ through p'. There is an interval I having zero in its interior such that each $\varphi(t,p')$, $p' \in E^{n-1}$ is defined for $t \in I$. Therefore, the function $\varphi(t,p') = \varphi(t, \Psi(u))$ can be thought of as a mapping H of $I \times B\alpha \to \mathbb{R}^n$. H is continuously differentiable in $I \times B\alpha$. If we define

$$F(x,t,u) = -x + \Psi(u) + \int_0^t f(\varphi(s, \Psi(u)))\,ds$$

then $F(\varphi(t, \Psi(u)),t,u) = \emptyset$. For convenience, let $y = (t,u)$ and $H(y) = \varphi(t, \Psi(u))$. So we have $F(H(y),y) = \emptyset$.

Finally, by the chain rule we know that $\partial H/\partial y$ will be invertible whenever $\partial F(x,y)/\partial y$ and $\partial F/\partial x$ are invertible at $(H(y),y)$. Hence, $\partial H/\partial y$ will be invertible at $(\emptyset,\emptyset)$. So the inverse function theorem implies that H^{-1} exists on a neighborhood of p. That is, there exists $\alpha > \emptyset$, $\bar{t} > \emptyset$ such that H is a continuously differentiable homeomorphism of $I(\bar{t}) \times B_\alpha$ onto the open set $\varphi(I(\bar{t}), \Psi(B_\alpha(\emptyset)))$ in $\mathbb{R}^n$, where $I(\bar{t}) = \{t: -\bar{t} < t < \bar{t}\}$. Because of this homeomorphism, E^{n-1} corresponds exactly to our local cross section. Also, for fixed u, $\{H(t,u): |t| < \bar{t}\}$ coincides with the arc γ' given by the solution $\varphi(t,p')$, $-\bar{t} < t < \bar{t}$. The range of the map H is often called an <u>open path cylinder</u>. For details, see Hale [5], pages 43-44.

The central ideas here — the existence of a transversal through any non-critical point, p, and the existence of an open path cylinder having p in its interior -- are the ideas we sought to generalize with our definition of local cross section and the proof of its existence.

Earlier in this chapter we mentioned that Whitney also claimed in [8] that a local cross section through a point in 3-space will be a closed 2-cell. This claim is based on work he published in his papers [9] and [10]. In [9] Whitney gives a characterization of a closed 2-cell based on the following theorems that we quote:

<u>Theorem 1</u>: Let R be a continuous curve containing the simple closed curve J, such that

(1) J is irreducibly homologous to zero in R, and

(2) If g is an arc with just its two end points a and b on J, then R - g is not connected.

Let R' and J' be defined similarly. Then R and R' are homeomorphic, with J corresponding with J'.

Then, the fact that R is a closed 2-cell follows immediately from his next theorem, which states:

Theorem 2: If I is a circle in the plane and S is I with its interior, then S and I satisfy the conditions prescribed for R and J in Theorem 1.

He notes that J in Theorem 1 corresponds with the circle I, that is, J is the boundary of R.

Whitney proves in [10] that the hypotheses of Theorem 1 hold when R = Q where Q is a cross section through any non-fixed point in a regular family of curves in $\mathbb{R}^3$. Thus Q, a local cross section, is a closed 2-cell.

A natural conjecture might be that this result of Whitney's holds for all n. However, the work of Bing ([2], [3]) provides an interesting counterexample in $\mathbb{R}^4$ to the fact that cross sections of flows in 2 or 3 dimensions are homeomorphic to closed 1- or 2-cells, respectively. In his 1957 paper [2], Bing describes a decomposition space of $\mathbb{R}^3$ which is topologically different from $\mathbb{R}^3$, and it is not a manifold at at least one point. The second paper we cite [3] proves the surprising result that the cartesian product of this decomposition space with $\mathbb{R}$ is homeomorphic to $\mathbb{R}^4$.

Now, given any space Y, we can always define a flow on $X = Y \times \mathbb{R}$ by setting

$$f((y,s),t) = (y,(s + t)).$$

Now, if U is an open set of Y containing $y \in Y$, then $\mathrm{Cl}U \times \{\emptyset\}$ is a local section at $x_0 = (y_0,\emptyset) \in X$ of arbitrary length. In fact, $Y \times \{\emptyset\}$ is a local cross section. We now apply these remarks to Bing's space. Let y_0 be a point in Y at which Y fails to be a manifold. Then for any neighborhood U of y_0 in Y, $S = \mathrm{Cl}(U) \times \{\emptyset\}$ is a local cross section at $x_0 = (y_0,\emptyset)$ which is not homeomorphic to a closed 3-disk because Y is not a manifold at y_0. Moreover, no section at any other point on the trajectory of x_0 will be homeomorphic to a closed 3-disk. We will show this later when we examine the topological properties of local cross sections in Chapter 3. Using Bing's dog bone space, as his example is called, to construct a flow on a four manifold for which local cross sections cannot be homeomorphic to a 3-dimensional disk has been part of the folklore of dynamical systems for at least twenty-five years.

Chapter 3: PROPERTIES OF LOCAL CROSS SECTIONS

In this chapter, we will examine the properties of a local cross section which can be derived from its definition. It will be assumed throughout the chapter that the local cross section, S, is a compact subset of the locally compact metric space, X. The main results are:

1) The closure of the "interior" of a local cross section is a local cross section;

2) In a finite length of time a trajectory can cross a local cross section only finitely many times;

3) The topology of a local cross section is unique in that local cross sections through distinct points of an orbit always contain neighborhoods of these points which are homeomorphic;

4) A local cross section containing no periodic points can be distorted by means of a continuous function to obtain another local cross section; and

5) A local cross section at a non-periodic point always has a subset which is a local cross section of any specified length.

We will present our results as a series of propositions.

We first describe a construction for modifying a local cross section so that the rectangular neighborhood is the closure of its interior. To do this, we will need to identify those points in S which are "interior" to S.

As usual, let S be a local cross section at $x_o \notin C$. Let

$$S^i = \{x \in S: x \in \mathrm{Int}(S[-\lambda,\lambda])\}.$$

We know S^i is not empty since x_o is in S^i. Now set

$$S^* = \mathrm{Cl}(S^i) = \mathrm{Cl}\{x \in S: x \in \mathrm{Int}(S[-\lambda,\lambda])\}.$$

Proposition 1. (a) S^* is a local cross section.

(b) $(S^*)^* = S^*$.

(c) $S^i(-\lambda,\lambda)$ is open.

(d) $S^i(-\lambda,\lambda) = \mathrm{Int}(S^*[-\lambda,\lambda])$.

(e) $S^*[-\lambda,\lambda] = \mathrm{Cl}(S^i(-\lambda,\lambda))$.

(f) $S^*[-\lambda,\lambda] = \mathrm{Cl}[\mathrm{Int}(S^*[-\lambda,\lambda])]$.

Proof of (a). We know that the mapping $h: S \times [-\lambda,\lambda] \to X$ given by $h(x,t) = xt$ is a homeomorphism, so h restricted to $S^* \times [-\lambda,\lambda]$ will also be a homeomorphism because $S^* \subset S$. S^* is closed by construction. So, it remains only to show that $S^*[-\lambda,\lambda]$ has some points of S^* in its interior. Let $x \in S^i \subset S^*$. Then there exists V open in X such that $x \in V \subset S[-\lambda,\lambda]$, by definition of S^i. This implies $h^{-1}(V)$ is open, since h is a homeomorphism. Hence there exists a set U, an open neighborhood of x in S, and a $\delta > \emptyset$ such that $U \times (-\delta,\delta)$ is a subset of $h^{-1}(V)$. And, again since h is a homeomorphism, we see that $h(U,(-\delta,\delta))$ is an open subset of X containing U and contained in $S[-\lambda,\lambda]$. This implies $U \subset S^i \subset S^*$, and the open set $h(U,(-\delta,\delta))$ is a subset of $S^*[-\lambda,\lambda]$.

We have shown S^* is a local section, but we have also shown more. In fact, this argument proves $(S^*)^i = S^i$. It is clear that $(S^*)^i \subset S^i$ because $S^* \subset S$. Since x in the above argument was an arbitrary point in S^i, we have shown the reverse inclusion, namely $S^i \subset (S^*)^i$. So, $S^i = (S^*)^i$.

Proof of (b). This follows immediately from (a) because $S^i = (S^*)^i$ implies $S^* = \mathrm{Cl}(S^i) = \mathrm{Cl}[(S^*)^i] = (S^*)^*$.

Proof of (c). Take $y \in S^i(-\lambda, \lambda)$. Then $y = h(x,t^*) = xt^*$, where $x \in S^i$, $|t^*| < \lambda$. Because $x \in S^i$ we can construct U and δ as in the proof of (a) with the added condition that $\delta < \lambda - |t^*|$. Since $V = h(U,(-\delta,\delta))$ is an open neighborhood of x, Vt^* is an open neighborhood of y. Moreover, we are guaranteed that $Vt^* \subset S^i(-\lambda,\lambda)$ by our choice of δ. Hence $S^i(-\lambda,\lambda)$ is open.

Proof of (d). First, it is clear from (c) that $S^i(-\lambda,\lambda)$ is a subset of $\mathrm{Int}(S^*[-\lambda,\lambda])$. To show the opposite inclusion, let $y \in \mathrm{Int}(S^*[-\lambda,\lambda])$. So $y = xt^*$, where $x \in S^*$, $|t^*| \leq \lambda$. Let us first show that $|t^*| \neq \lambda$. Suppose $t^* = \lambda$. Since $\lim_{n\to\infty} \lambda + 1/n = \lambda$, this means $\lim_{n\to\infty} x(t^* + 1/n) = y$. So every neighborhood of y contains points which are not elements of $S^*[-\lambda,\lambda]$, since it can be shown that $x(t^* + 1/n) = x(\lambda + 1/n)$ is not in $S^*[-\lambda,\lambda]$ for large n. Similarly, it can be shown that $t^* \neq -\lambda$.

Now, using the fact that h is a homeomorphism as before, there exists U open in S^* and $0 < \delta < \lambda - |t^*|$ such that $h(U \times (t^* - \delta, t^* + \delta)) = V$ is an open set in $S^*[-\lambda,\lambda]$ containing y. It follows that $V(-t^*)$ is an open set containing x and $V(-t^*) \subset S^*[-\lambda,\lambda]$. Hence, $x \in (S^*)^i$ and we are done because we have already shown that $(S^*)^i = S^i$.

<u>Proof of (e)</u>. It is clear that $Cl(S^i(-\lambda,\lambda))$ is a subset of $S^*[-\lambda,\lambda]$, so we need to show the opposite inclusion. Let $y \in S^*[-\lambda,\lambda]$ so $y = xt^*$ where $x \in S^*$ and $|t^*| \leq \lambda$. If $x \in S^i$ and $|t^*| < \lambda$ there is nothing to prove. We will consider two cases.

First, if $x \in S^i$ and $t^* = \lambda$, then $y = \lim_{n\to\infty} x[\lambda(1 - 1/n)]$, that is, y is the limit point of a sequence of points in $S^i(-\lambda,\lambda)$. Therefore, $y \in Cl(S^i(-\lambda,\lambda))$. The argument is similar for $t^* = -\lambda$.

In the second case, $y = xt^*$ but $x \notin S^i$. Hence, there exists (x_n) in S^i such that $x_n \to x$. Assume for convenience $t^* > 0$. As before $t^*(1 - 1/n) \to t^*$ and $t^*(1 - 1/n) \in (-\lambda,\lambda)$ for large n. By the continuity of the flow we have

$$\lim_{n\to\infty} x_n[t^*(1 - 1/n)] = (\lim_{n\to\infty} x_n)(\lim_{n\to\infty} t^*(1 - 1/n)) = xt^* = y.$$

So y is the limit of a sequence of points in $S^i(-\lambda,\lambda)$, and $y \in Cl(S^i(-\lambda,\lambda))$. Again, similar proofs work for $t^* < 0$, and $t^* = 0$.

We have shown in all cases $S^*[-\lambda,\lambda] \subset Cl(S^i(-\lambda,\lambda))$ and hence $S^*[-\lambda,\lambda] = Cl(S^i(-\lambda,\lambda))$.

<u>Proof of (f)</u>. This follows immediately from (e) using the result in (d). //

<u>Proposition 2</u>. Given $x \in X$ and $T > 0$, and S a local cross section of length 2λ.

(a) If $T \leq 2\lambda$ then $x[-T,T]$ can intersect the local cross section S at most one time.

(b) If $T > 2\lambda$ then $x[-T,T]$ intersects S at most finitely many times. The number of intersections is at most $T/2\lambda$.

(c) An orbit $x\mathbb{R}$ can intersect S at most a countable number of times.

Proof. (a) Since S has length 2λ, this is clear.

(b) By (a), $\mathcal{O}(x)$ can intersect S at most once for every 2λ units of time.

(c) $\mathbb{R} = \bigcup_{n=-\infty}^{\infty}[n2\lambda,(n+1)2\lambda]$, and $xt \in S$ at most once for $t \in [n2\lambda,(n+1)2\lambda]$. //

The next result is that the topology of a local cross section is unique. This idea is stated precisely in Proposition 3.

Proposition 3. Let S_1 and S_2 be local cross sections and let x_1 and x_2 lie in S_1^i and S_2^i, respectively. If x_1 and x_2 are on the same orbit, then there exist neighborhoods U_1 and U_2 of x_1 and x_2 in S_1 and S_2, respectively, such that U_1 and U_2 are homeomorphic.

Proof. Let $2\lambda_i$ be the length of S_i, $i = 1, 2$. Without loss of generality, assume $\lambda_2 \leq \lambda_1$. Since $\mathcal{O}(x_1) = \mathcal{O}(x_2)$, we must have $x_2 = x_1 t^*$ for some t^*. Since $x_2 \in S_2^i$, there exists an $\varepsilon > \emptyset$ such that the ε-ball at x_2, $B_\varepsilon(x_2) = \{x \in X: \rho(x_2,x) < \varepsilon\}$, is a subset of $S_2[-\lambda_2/2, \lambda_2/2]$. By continuity, choose δ so that $B_\delta(x_1)t^* \subset B_\varepsilon(x_2)$ and $B_\delta(x_1) \subset S_1[-\lambda_1, \lambda_1]$. Now, using the usual homeomorphism from $S_1 \times [-\lambda_1, \lambda_1] \to S_1[-\lambda_1, \lambda_1]$, we can find U_1, an open subset of S_1, with $x_1 \in U_1$, and $\eta > \emptyset$, such that $h(\mathrm{Cl}U_1 \times [-\eta,\eta]) \subset$

$B_\delta(x_1)$. Let V be the open set $h(U_1 \times (-\eta,\eta)) = V$.

Now consider the map $r\colon \mathrm{Cl}U_1 \to S_2$ given by

$$r(x) = p(f(x,t^*)) = y,$$

where the map p was defined after Theorem 1 in Chapter 1 for S_2. Specifically, $p(xt^*) = y$, provided $xt^* = yt$ for $y \in S_2$ and $|t| \leq \lambda_2$.

Since r is the composition of two continuous functions, it is continuous. We now show r is one-to-one on $\mathrm{Cl}U_1$.

If $r(x') = r(x'')$, $x', x'' \in \mathrm{Cl}U_1$, then $p(x't^*) = p(x''t^*) = y$, where $y \in S_2$. That is, $yt' = x't^*$ and $yt'' = x''t^*$, where $|t'|, |t''| < \lambda_2$. Solving for y gives $x' = x''(t' - t'')$. If $x' \neq x''$, we have $2\lambda_2 \geq |t'-t''| > 2\lambda_1$, but by construction $2\lambda_1 \geq 2\lambda_2$. This contradiction implies $|t'-t''| = \emptyset$ and hence $x' = x''$.

Since r is one-to-one, r is a homeomorphism on $\mathrm{Cl}U_1$ by compactness. This implies that r restricted to U_1 is also a homeomorphism.

In order to show that $r(U_1) = U_2$ is open in S_2 and complete the proof, consider again $V = h(U_1,(-\eta,\eta))$ constructed above. Since V is open, Vt^* is open, and by construction $Vt^* \subset B_\varepsilon(x_2) \subset S_2[-\lambda_2/2, \lambda_2/2]$. Since p is an open map by Lemma 4, Chapter 1, $p(Vt^*)$ is open in S_2 and $x_2 \in p(Vt^*)$. Now $U_1 \subset V$, so $U_2 = p(U_1t^*) \subset p(Vt^*)$. Also, every point in Vt^* can be expressed as $x(t+t^*)$ for some $x \in U_1$ and t with $|t| < \eta < \lambda_2$. In fact, since xt^* and $x(t+t^*)$ are both in $B_\varepsilon(x_2) \subset S_2[-\lambda_2/2, \lambda_2/2]$, we know $yt' = xt^*$ for some $y \in S_2$ and $|t'| \leq \lambda_2/2$, and similarly

$x(t+t^*) = y't''$. Solving for y' we have $y' = y(t'+t-t'')$, where $|t'+t-t''| \leq |t'| + |t-t''| < \lambda_2/2 + \lambda_2 + \lambda_2/2 = 2\lambda_2$. So Proposition 2 above implies $|t'+t-t''|$ must be $\emptyset$. Therefore, $y = y'$ and $p(x(t+t^*)) = p(xt^*)$, and we see that $p(Vt^*) \subset p(U_1 t^*)$. Therefore, $p(Vt^*) = p(U_1 t^*) = U_2$ and $U_2 = p(U_1 t^*)$ is open. //

Proposition 3 is important because it guarantees that the local topological properties of a local cross section are essentially the same at any non-fixed point of a trajectory. We use this fact in Chapter 4 when we give a local cross section at a point in $\mathbb{R}^4$, and this local cross section turns out not to be homeomorphic to $\mathbb{R}^3$. The local cross section is Bing's dog bone space. Then we can use Proposition 3 to claim that there will not be <u>any</u> point on the trajectory at which the local cross section is a 3-dimensional manifold.

The fourth result of this chapter is that under certain conditions, a local cross section containing no periodic points can be distorted by a continuous function to obtain another local cross section. If S is a local cross section of length 2λ, let

$$\gamma = \min\{ |t| : xt \in S \text{ for some } x \in S \text{ and } t \neq \emptyset\}, \text{ and set}$$

$$\gamma = \infty \text{ if } xt \notin S \text{ for all } t \neq \emptyset \text{ and } x \in S.$$

It is clear that $\gamma > 2\lambda$ (by Lemma 2, Chapter 1). We have the following result.

<u>Proposition 4</u>. Let S be a local cross section of length 2λ containing no periodic points and let $g: S \to \mathbb{R}$ be continuous. Let γ be as above. If $|g(x) - g(xt)| < \gamma - 2\lambda$ whenever x and xt, $t \neq \emptyset$, lie in S, then $S' = \{xg(x): x \in S\}$ is also a local

cross section of length 2λ.

Proof. S is a closed set and S' is the image of S under the continuous map $x \to xg(x)$, so S' is compact and hence closed.

Now consider the map $h^*: S' \times [-\lambda, \lambda] \to X$ where $h^*(y,t) = yt = f(xg(x),t) = x(g(x)+t)$ when $y = xg(x)$, $x \in S$. h^* is clearly continuous, since the flow, f, is. Since S is compact, we will have shown h^* is a homeomorphism if we can show h^* is one-to-one.

Now suppose $h^*(y_1,t_1) = h^*(y_2,t_2)$. This implies

$[x_1 g(x_1)]t_1 = [x_2 g(x_2)]t_2$ for some $x_1, x_2 \in S$, and so

$[x_1 g(x_1)](t_1 - t_2) = [x_2 g(x_2)]$, or equivalently,

$x_1(g(x_1) + t_1 - t_2) = x_2 g(x_2)$, and so

$x_1[t_1 - t_2 + g(x_1) - g(x_2)] = x_2$, where $|t_1 - t_2| < 2\lambda$.

This means x_1 and x_2 are on the same orbit.

If $x_1 = x_2$, then $g(x_1) = g(x_2)$, which means $x_1(t_1 - t_2) = x_2$ and $t_1 = t_2$, because there are no periodic points in S.

If $x_1 \neq x_2$, then $|t_1 - t_2 + g(x_1) - g(x_2)|$ must be greater than γ. However,

$$\begin{aligned} |t_1 - t_2 + g(x_1) - g(x_2)| &\leq |t_1 - t_2| + |g(x_1) - g(x_2)| \\ &< 2\lambda + \gamma - 2\lambda = \gamma. \end{aligned}$$

This contradiction shows that in all cases $(y_1,t_1) = (y_2,t_2)$ when $h^*(y_1,t_1) = h^*(y_2,t_2)$, so h^* is one-to-one.

To finish the proof, we also have to show that $\mathrm{Int}(S'[-\lambda,\lambda]) \neq \phi$. Let $x_0 \in S^i$. Choose $\delta = \lambda/2$. Since h is a homeomorphism, we can construct an open set $U(-\delta, \delta)$ in $S[-\lambda,\lambda]$ where U is open in S and $x_0 \in U$ and, by continuity, we can also guarantee that $|g(x) - g(x')| < \lambda/2$ for all x, x' in

U. Then $x_0 g(x_0) \in [U(-\delta,\delta)]g(x_0)$, which is open. It suffices to show that $[U(-\delta,\delta)]g(x_0) \subset S'[-\lambda,\lambda]$.

Now, for any y in $[U(-\delta,\delta)]g(x_0)$,

$$\begin{aligned} y &= (xt)g(x_0) \quad (\text{where } x \in S,\ |t| < \lambda/2) \\ &= x(t+g(x_0)). \end{aligned}$$

Also, for this x, $xg(x) \in S'$. Since $xg(x)$ and $(xt)g(x_0)$ are on the same orbit, there is a time $\bar{t}$ such that

$$xg(x)\bar{t} = x(t + g(x_0)).$$

So, since there are no periodic points, $\bar{t} = t + g(x_0) - g(x)$, and

$$\begin{aligned} |\bar{t}| &\leq |t| + |g(x_0) - g(x)| \\ &< \lambda/2 + \lambda/2 = \lambda. \end{aligned}$$

Therefore, $y \in [U(-\delta,\delta)]g(x_0)$ implies $y \in S'[-\lambda,\lambda]$. Thus $[U(-\delta,\delta)]g(x_0) \subset S'[-\lambda,\lambda]$. Hence $\mathrm{Int}(S'[-\lambda,\lambda]) \neq \phi$. Therefore, S' is a local cross section of length 2λ. //

The two obvious extreme cases to which this proposition applies are given in the following corollaries.

<u>Corollary 1</u>. If S is a local cross section and $g \equiv$ constant, then $S' = \{xg(x): x \in S\}$ is a local cross section.

<u>Corollary 2</u>. If S is a local cross section such that $x \in S$ implies $xt \notin S$ for all $t \neq 0$, and if g is any continuous function from $S \to \mathbb{R}$, then $S' = \{xg(x): x \in S\}$ is a local cross section.

<u>Proposition 5</u>. If q is any non-fixed, non-periodic point of the flow f on X and S is a local cross section at q of length 2λ, then there exists a local cross section $S' \subset S$ at q of any specified length.

<u>Proof</u>. Without loss of generality, assume $S = S^*$. Fix

$T > 0$. Let $k = \min\{\rho(q,qt): \lambda \leq |t| \leq 2T\}$. By Proposition 2 above, k is positive. By continuity in initial conditions, there exists $\delta > 0$ such that $\rho(q,x) < \delta$ implies $\rho(qt,xt) < k/2$ for $-2T \leq t \leq 2T$. Without loss of generality, choose $\delta < k/2$.

Now, the homeomorphism h defined by the flow on $S \times [-\lambda,\lambda] \rightarrow S[-\lambda,\lambda]$ can be used to construct an open set $V = h(U,(-\eta,\eta))$ where U is an open set in S containing q such that $h(Cl(U),[-\eta,\eta]) \subset B_\delta(q)$. Let $S' = Cl(U)$. Then S' is a local cross section at q of length 2T.

S' is closed and, by construction, q is an interior point of $S'[-T,T]$. The map h of $S' \times [-T,T] \rightarrow S'[-T,T]$ taking $(x,t) \rightarrow xt$ is continuous and onto.

It remains to show h is one-to-one. Suppose $x_1t_1 = x_2t_2$, $|t_1|, |t_2| \leq T$. Then $x_1 = x_2(t_2 - t_1)$, where $|t_2 - t_1| \leq 2T$. If $|t_2 - t_1| < \lambda$, then $|t_2 - t_1| = 0$. So assume $\lambda \leq |t_2 - t_1| \leq 2T$. By construction, $\rho(q,x_2) < \delta < k/2$ implies $\rho(q(t_2 - t_1),x_2(t_2 - t_1)) < k/2$. Using the triangle inequality, it follows that

$$\begin{aligned}\rho(q,q(t_2 - t_1)) &\leq \rho(q,x_1) + \rho(x_1,q(t_2 - t_1))\\ &= \rho(q,x_1) + \rho(x_2(t_2 - t_1),q(t_2 - t_1))\\ &< k/2 + k/2 = k.\end{aligned}$$

This contradiction implies $|t_2 - t_1| = 0$. So h is one-to-one. //

Chapter 4: CROSS SECTIONS OF FLOWS ON MANIFOLDS

In this chapter we consider flows on manifolds. We define an n-dimensional manifold as a compact metric space, each point of which has a neighborhood homeomorphic to an open ball in Euclidean n-space. It has long been known that flows on 2- and 3-dimensional manifolds have local cross sections which are, respectively, arcs and closed disks. We will present a new elementary proof of this fact for the 2-dimensional case and complete the discussion begun in Chapter 2 of a counterexample to show that this result does not hold when $n = 4$.

Our first result shows that a local cross section of a flow on a surface will be an arc.

Theorem 1. Let X be a 2-manifold on which a flow f is defined. If $x_0 \notin C$, then there exists a local cross section at x_0 homeomorphic to $[0,1]$.

Before proving the theorem, we will prove the following lemma.

Lemma 1. Let X be a manifold with a flow, f, and S a local cross section. If $x \in S^i$ and W is a neighborhood of x in S, then there exists an arcwise connected neighborhood W' of x in S with $W' \subset W$.

Proof of Lemma. Let $2\lambda > 0$ be the length of S. Without loss of generality, W is open in S. Hence, $W(-\lambda, \lambda)$ is open so there is a closed ball $B_\eta(x) \subset W(-\lambda, \lambda)$. Consider the map $p: Cl(B_\eta(x)) \to S$, defined in Chapter 1, taking $y \to x$, where $y = xt$, $x \in S'$, $|t| \leq \lambda$. Since p is a closed, continuous map and $Cl(B_\eta(x))$ is compact, connected, and locally connected, then $p(Cl(B_\eta(x))) = W'$ is arcwise connected as in the proof of Corollary 1 in Chapter 1. Finally, W' is a neighborhood of x because the map p is open. //

<u>Proof of Theorem</u>. Construct the set $Cl(U)$ as in Theorem 1, Chapter 1, such that $Cl(U)$ is a closed disk and with the added condition that $x(\pm 2\lambda) \notin U$ for all x in U. Let S be the local cross section at x_0 of length 2λ constructed as in Chapter 1. By continuity of the flow and the fact that the homeomorphism h maps x_0 into a rectangular neighborhood of x_0, we can find $U' \subset Cl(U)$ and $\delta > \emptyset$ so that $x_0 \in U'$ open, and $U'(-\delta, \delta) \subset U$. Then $W = U' \cap S$ is a neighborhood of x_0 in S. By the lemma above, there exists $W' \subset W$ such that W' is an arcwise connected neighborhood of x_0. Consider $V = W'(-\delta, \delta)$, which is open. By our choice of δ we know $V \subset U$, since $W' \subset U'$. Therefore, V is a neighborhood of x_0 contained in U. Consider the trajectory of x_0 in $Cl(U)$. There is a first negative time, say α, and a first positive time, say β, at which the trajectory of x_0 leaves $Cl(U)$. The Jordan Curve Theorem implies that $Cl(U)$ is divided into two pieces by $x_0(\alpha, \beta)$, and $x_0(\alpha, \beta)$ is on the boundary between these disjoint pieces. Let A be the piece of $Cl(U)$ "above" $x_0(\alpha, \beta)$ and B the region "below" $x_0(\alpha, \beta)$. Since

$V \subset U \subset Cl(U)$ and V is also a neighborhood of x_0, $V \cap A \neq \phi$. This implies $W' \cap A \neq \phi$, and, similarly, $W' \cap B \neq \phi$. So there exists $p \in W' \cap A$ and $q \in W' \cap B$.

By Lemma 1 above there exists an arc, γ, in $W' \subset S$ from p to q. This arc must intersect $x_0(\alpha, \beta)$ since it stays inside U', but $x_0(\alpha, \beta)$ intersects S at exactly one point: x_0. Hence the arc γ goes through x_0. Since $\gamma \subset S$, this arc will be the desired local cross section at x_0 if x_0 is in $Int(\gamma[-\lambda, \lambda])$.

Consider a continuous function $g: \gamma \rightarrow \mathbb{R}$, where

$$g(p) = g(q) = \emptyset, \text{ and } \emptyset < g(x) < \delta < \lambda \text{ for } x \neq p, q.$$

Let $\gamma g = \{xt: x \in \gamma, t = g(x)\}$. Then the set $\gamma g \cup \gamma(-g) = J$ will be a simple closed curve containing p and q. We know that γg and $\gamma(-g)$ do not intersect except at p and q because S, $S(\emptyset, \lambda)$, and $S(-\lambda, \emptyset)$ are disjoint. The Jordan Curve Theorem implies that J divides U into two open regions: a bounded interior, J_I, and an exterior, J_E. Clearly $x_0 \in J_I \subset Int(\gamma[-\lambda, \lambda])$ and the proof is complete. //

Corollary 1. If X is a 2-manifold and S is a local cross section, then S^i is a 1-dimensional manifold.

Proof. Through any point x in S^i we can construct a local cross section which is homeomorphic to $[\emptyset, 1]$, by the above theorem. Therefore, x has a neighborhood in S which is homeomorphic to the open 1-dimensional ball, $(\emptyset, 1)$. But S^i is a local cross section at each $x \in S^i$. Hence, by Proposition 3, Chapter 3, each x in S^i must have a neighborhood in S^i which is homeomorphic to $(\emptyset, 1)$. This means S^i is a 1-dimensional manifold. //

Corollary 2. If A is any space such that A x $\mathbb{R}$ is homeomorphic to $\mathbb{R}^2$, then A is homeomorphic to $\mathbb{R}$.

Proof. Using the projection map of A x $\mathbb{R}$ onto A it follows that A is connected and separable since $\mathbb{R}^2$ is connected and separable. Define the usual flow on the space A x $\mathbb{R}$. Then S = A x {Ø} is a local cross section and S^i = A, since A x (-1,1) is open in A x $\mathbb{R}$. By Corollary 1, A is a 1-dimensional manifold. So either A = $\mathbb{R}$ or A = $\mathbb{T}$. But $\mathbb{T}$ x $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^2$ since $\mathbb{T}$ x $\mathbb{R}$ is not simply connected. Therefore, A is homeomorphic to $\mathbb{R}$. //

Turning to the three dimensional case, we said in Chapter 2 that H. Whitney gave a proof in [1Ø] that if a regular family of curves fills a region of Euclidean 3-space, then any "cross-section" contains a "cross-section" which is a 2-cell. For his proof of this, Whitney used a characterization of the closed 2-cell which he established in [9]. In his later paper, [1Ø], Whitney then showed that the cross-section through any non-fixed point of a regular family in $\mathbb{R}^3$ satisfies all the conditions of this characterization. Therefore he concludes that the cross-section must be a closed 2-cell.

If we ask how far we can generalize these two results to higher dimensions, we see right away that the analogous situation does not hold when n = 4.

Theorem 2. There exists a flow on a 4-manifold with a point x_0 at which every local cross section is not homeomorphic to a 3-

dimensional disk.

Proof. Let Y be Bing's dog bone space, introduced in Chapter 2. Define a flow on $X = Y \times \mathbb{R}$ by

$$f((y,s),t) = (y,(s+t)).$$

Bing's paper [3] proves that X is a 4-manifold, in fact $\mathbb{R}^4$.

Let y_0 be a point in Y at which Y fails to be a 3-manifold. Such a point exists by Bing's paper, [2], which proves that Y is not homeomorphic to a 3-manifold. Then if U is an open set of Y containing $y_0 \in Y$, U is not a 3-manifold either at y_0 and $Cl(U) \times \{\emptyset\}$ is a local cross section at $x_0 = (y_0, \emptyset) \in X$ of any specified length. In fact, $Y \times \{\emptyset\}$ is a local cross section at x_0.

Let S' be a local cross section at x_0. By Proposition 3 of Chapter 3 there exists U' open in S' with $x_0 \in U'$ such that U' is homeomorphic to an open set V of $Y \times \{\emptyset\}$. Clearly $V = U \times \{\emptyset\}$ for some open neighborhood of y_0 in Y. Therefore, S' is not homeomorphic to a 3-disk. //

In the particular case in which a flow on a differentiable n-manifold is determined by solutions to differential equations, then this flow will always have a local cross section homeomorphic to an (n-1)-disk at any non-critical point. The reason for this is that the solution to the differential equation is differentiable with respect to x to the same degree that f is. So the construction of a transversal, which we outlined in Chapter 2, can always be accomplished at a non-critical point. It should be emphasized that the differentiability of the flow is the key here. For a general flow, we used the function $G(x)$ defined in Chapter 1 to construct the local cross section, $S = \{x : G(x) = G(x_0)\} \cap Cl(U)[-\lambda, \lambda]$. Since we could not assume that G was differentiable in x, we could not use the Implicit and Inverse Function Theorems to study S and we had to use different methods to prove that S is analogous to the transversal of ordinary differential equations and shares its useful properties.

REFERENCES

[1] Bebutoff, M., "Sur la représentation des trajectoires d'un système dynamique sur un système de droites parallèles," Bull. Math. Univ. Moscou. 2, no. 3, 1939.

[2] Bing, R. H., "A decomposition of E^3 into points and tame arcs such that the decomposition space is topologically different from E^3," Ann of Math. 65 (1957), pp. 484-500.

[3] Bing, R. H., "The cartesian product of a certain nonmanifold and a line is E^4," Ann of Math. 70, No. 3, (1959), pp. 399-412.

[4] Hajek, O., "Sections of Dynamical Systems," The Czechoslovak Math. J. 15(90) (1965), pp. 205-211.

[5] Hale, J., "Ordinary Differential Equations," Pure and Applied Math. Series, XXI, John Wiley & Sons, Inc., New York, 1969.

[6] Hocking, J. G. and G. S. Young, "Topology," Addison-Wesley Publishing Co., Inc., Reading, Mass.-London, 1961.

[7] Nemitsky, V. V. and V. V. Stepanov, "Qualitative Theory of Differential Equations," Princeton Mathematical Series, 22, Princeton University Press, Princeton, 1960.

[8] Whitney, H., "Regular families of curves," Ann of Math. (2)34, 1933, pp. 244-270.

[9] Whitney, H., "A characterization of the closed 2-cell," Am. Math. Society Transactions, V.35, 1933, pp. 261-273.

[10] Whitney, H., "Cross-sections of curves in 3-space," Duke Math. Jour., vol. 4 (1938), pp. 222-226.