ABSTRACT

Title of dissertation: BRANCHING DIFFUSION PROCESSES IN PERIODIC MEDIA

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In the first part of this manuscript, we investigate the asymptotic behavior of solutions to parabolic partial differential equations (PDEs) in $\mathbb{R}^d$ with space-periodic diffusion matrix, drift, and potential. The asymptotics is obtained up to linear in time distances from the support of the initial function. Using this asymptotics, we describe the behavior of branching diffusion processes in periodic media. For a super-critical branching process, we distinguish two types of behavior for the normalized number of particles in a bounded domain, depending on the distance of the domain from the region where the bulk of the particles is located. At distances that grow linearly in time, we observe intermittency (i.e., the $k-$th moment dominates the $k-$th power of the first moment for some k), while, at distances that grow sub-linearly in time, we show that all the moments converge.

In the second part of the manuscript, we obtain asymptotic expansions for the distribution functions of continuous time stochastic processes with weakly dependent increments in the domain of large deviations. As a key example, we show that

additive functionals of solutions of stochastic differential equations (SDEs) satisfying Hörmander condition on a d–dimensional compact manifold admit asymptotic expansions of all orders in the domain of large deviations.

BRANCHING DIFFUSION PROCESSES IN PERIODIC MEDIA

by

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Dedication

I dedicate this to my mother and father.

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Chapter 1: Introduction

This thesis considers asymptotic problems in the domain of large deviations, arising in the study of branching diffusion processes in periodic media and in the study of stochastic processes with weakly dependent increments.

Branching processes are used to model evolution of various populations such as bacteria, cancer cells, sub-atomic particles, etc., where each member of the population may die (be annihilated) or produce offspring independently of the rest. The individuals involved in the process are referred to as particles. Important real-life applications of branching processes are found in chemistry and physics (certain chemical and nuclear chain reactions), in life sciences (population dynamics), etc. Branching diffusion processes, in addition to modeling the branching phenomena, also take into account spatial movement of particles. That is, particles undergo branching and diffusion, with each particle behaving independently of the rest. For a variety of applications of branching diffusion processes in biology, see, for example, Sawyer [1] and the book of Bansaye, Méléard [2] and references therein.

In Chapter 2 we study branching diffusions in $\mathbb{R}^d$ with space-periodic media. In these processes, the particle density and higher order moments satisfy a system of parabolic partial differential equations (PDEs). Therefore, asymptotic behavior of such processes is closely linked to solutions to parabolic PDEs in $\mathbb{R}^d$. In Section 2.2 we derive the precise asymptotics of solutions of parabolic PDEs using probabilistic tools - namely the theory of large deviations and homogenization theory. The asymptotics is obtained up to linear in time distances from the support of the initial function i.e., in the domain of large deviations. We then use this asymptotics to study the higher order correlation functions recursively, extending the results of Koralov, Molchanov [3], where processes with constant and compactly supported branching rates were considered, and where the expression for the heat kernel was explicit.

More precisely, we study the moments of the random variables $n^y(t,x)$, that denote the number of particles in a $d-$dimensional unit cube containing $y \in \mathbb{R}^d$ at time t, assuming that at time zero there was a single particle located at $x \in [0,1)^d$. For a super critical branching process, using the asymptotics of the heat kernel obtained in Section 2.2, we show two different limiting behaviors of the moments, depending on the distance $y(t)$ from the bulk of the particles. In Section 2.3, we show that, when the distance of $y(t)$ from the bulk of the particles is of order t, intermittency occurs, that is, $k-$th moment of $n^{y(t)}(t,x)$ dominates the $k-$th power of its first moment for some k. In section 2.4, we identify a sequence of periodic continuous functions, $f_k(x)$ which, as we show in section 2.5, serve as limits of the $k-$th moments of $n^{y(t)}(t,x)/\mathbb{E}(n^{y(t)}(t,x))$ whenever the distance of $y(t)$ from the bulk of the particles is sub-linear.

In Chapter 3, we show that asymptotic expansions for the distribution functions of continuous time stochastic processes in the domain of large deviations can be obtained under a set of weak dependence conditions on their increments. These expansions are commonly referred to as strong large deviation results. They are in the spirit of Edgeworth expansions for the Central Limit Theorem, but in the domain of large deviations. In our earlier paper [4], we established natural conditions (in the context of dynamical systems and Markov chains) that guarantee the existence of asymptotic expansions for the distribution functions of sums of weakly dependent random variables in the domain of large deviations. The goal of this chapter is to extend the discrete time result from [4] to continuous time stochastic processes under additional assumptions and to describe a key example, where we show that the distribution functions of additive functionals of diffusion processes on a compact manifold admit expansions of all orders in the domain of large deviations. The motivation for focusing on this example comes form the work on branching diffusions in periodic media, and from the large deviation problems for coupled stochastic differential equations (SDEs) studied in Veretennikov [5] and Liptser [6]. Note that each co-ordinate of the location of a particle undergoing a diffusion process in $\mathbb{Z}^d$ periodic media can be considered as an additive functional of a diffusion process on a $d-$dimensional torus. The conditions imposed on our additive functional in Section 3.4 are somewhat more stringent (they include a Wiener process independent of the Wiener process in the underlying diffusion process on the compact manifold), and are not immediately satisfied by the processes considered in Section 2.2. On the other hand, in Chapter 3, we go beyond the first term of the asymptotics studied in Section 2.2.

Chapter 2: Branching diffusion processes in periodic media

2.1 Introduction

In this chapter, we consider a collection of particles $Y_1(t), Y_2(t), \dots$ in $\mathbb{R}^d$ that move diffusively and independently according to

$$dY_k(t) = b(Y_k(t))\,dt + \sigma(Y_k(t))\,dW_k(t), \tag{2.1}$$

where W_k denote independent Brownian motions in $\mathbb{R}^d$. Each particle branches independently into two particles at rate $\alpha(y) \geq 0$, and each particle is annihilated independently at rate $\beta(y) \geq 0$. The newly created particles starting at the location of their parent then repeat this process independently of each other. This process is referred to as a $d-$ dimensional branching diffusion process. We suppose that the drift $b(y)$, the non-degenerate diffusion matrix $\sigma(y)$, and the rates $\alpha(y)$ and $\beta(y)$ are all Lipschitz continuous and $\mathbb{Z}^d$ periodic (and thus bounded). That is, $b(y+k) = b(y)$ for all $y \in \mathbb{R}^d$ and $k \in \mathbb{Z}^d$, and similarly for σ, α and β.

The main topic of interest here is the limiting behavior of branching diffusion processes in periodic media in the supercritical regime. The study of branching diffusions can be traced as far back as the work of Ikeda, Nagasawa, Watanabe [7], followed by important contributions of Bramson [8], [9], etc. For an overview of the literature on these topics, see the book of Bovier [10] and the notes of Berestycki [11].

Our main goal is to study the distribution of the number of particles in regions whose spatial location depends on time. With probability that tends to one, the entire population is confined to a region that grows linearly in time (see Chapter 7.3 in the book of Freidlin [12]). The effective drift of a branching process can be understood heuristically as the speed at which the bulk of the particles is traveling in space. We will rigorously define the notion of effective drift in Section 2.2. For a bounded region at a fixed location, assuming that the effective drift is zero, the structure of the population is similar to that in the compact setting. See, for example, Engländer, Harris, Kyprianou [13] and references therein. For a time dependent region inside the linearly growing front, the normalized number of particles converges almost surely (see, for example, Uchiyama [14] in the case of constant coefficients). The nature of this convergence, however, depends on how distant the region is from the location of the initial particle (assuming for simplicity that the effective drift is zero). At linear in time distances, we will show intermittency (i.e., the $k-$th moment dominates the $k-$th power of the first moment for some k), while, at distances that grow sub-linearly in time, we will prove that all the moments converge. For the case of homogeneous media and for the case of compactly supported branching term, this question has been studied in the work of Koralov [15] as well as Koralov, Molchanov [3].

Given a single particle initially at $x \in \mathbb{R}^d$, the transition kernel $u(t, x, y)$ is defined by

$$\int_{\mathbb{R}^d} u(t,x,y) f(y)\, dy = \mathbb{E}_x \left[\sum_k f(Y_k(t)) \right],$$

where $f \in C_b(\mathbb{R})$ and the sum is over all particles alive at time $t \geq 0$. The function $(t, y) \mapsto u(t, x, y)$ satisfies

$$\partial_t u = \mathcal{L}^* u, \quad y \in \mathbb{R}^d, \;\; t > 0, \tag{2.2}$$

with initial condition

$$u(0, x, \cdot) = \delta_x(\cdot),$$

where $\mathcal{L}^*$ is the adjoint of

$$\mathcal{L}u = \frac{1}{2} \sum_{ij=1}^{d} a_{ij}(y) \frac{\partial^2 u}{\partial y_i \partial y_j} + \sum_{i=1}^{d} b_i(y) \frac{\partial u}{\partial y_i} + r(y) u, \tag{2.3}$$

$a(y) = \sigma(y)\sigma^*(y)$, and $r(y) = \alpha(y) - \beta(y)$. The operator $\mathcal{L} - r(y)$ is the generator of the process (2.1). As a first step, we give a precise asymptotic description of $u(t, x, y)$, valid up to the large deviation scale, that is, for $\|x - y\| = O(t)$.

There are two main parts in the asymptotic analysis of $u(t, x, y)$. First, we transform the operator in order to alter the effective drift of the process, while simultaneously turning the branching rate into a constant. Thus, the problem reduces to studying the transition kernel of an altered diffusion process near the diagonal, where $\|x - y\| = O(\sqrt{t})$. The next part is to prove a local limit theorem for the new transformed kernel at this diffusive scale.

The ingredients we use - exponential change of measure, homogenization and local limit theorems for the resulting diffusion process are fairly standard. In spite of this, the precise asymptotics of the transition kernel that holds up to linear in time distances has not been published (in 2007, Agmon gave a talk [16], where this result was announced). Here, we provide a simple probabilistic proof that establishes uniform asymptotics of the transition kernel for $d-$dimensional second-order parabolic operators with periodic coefficients. The precise asymptotics in the $1-$ dimensional case has been obtained previously by Tsuchida in [17].

Prior results in this direction, in $d-$dimensions, give estimates of the heat kernel, as opposed to precise asymptotics. The seminal work of Aronson [18] gives global estimates on the heat kernel, while in [19] Norris proves a generalization of Aronson's Gaussian bounds in the case of periodic coefficients and identifies an effective drift of the heat flow. The upper and lower bounds of Norris [19] have different constants in the Gaussian term. We provide a stronger result that correctly identifies the main term of the asymptotic expansion of the transition kernel, which is precise up to the domain of large deviations (up to distances in space that are linear in time). The asymptotics of Green's function for the corresponding elliptic problem for different values of the spectral parameter has been studied extensively (see, e.g., Murata, Tsuchida [20], Kuchment, Raich [21]).

The asymptotics proved in Section 2.2 plays a crucial role in analyzing the behavior of the branching diffusion process in periodic media, in Sections 2.3-2.5. The bulk of the particles will be seen to be located around of $\bar{\mathbf{v}}t$ where $\bar{\mathbf{v}}$ denotes the effective drift of the process (to be defined rigorously later). Let $n^y(t,x)$ denote the number of particles located in a unit $d-$dimensional cube containing $y \in \mathbb{R}^d$, assuming that, initially, there is one particle located at $x \in \mathbb{R}^d$. In Section 2.3, for a super-critical branching process, we study the asymptotic behavior of $n^y(t,x)$

in the domain of large deviations, that is when $\|y - \bar{\mathbf{v}}t\| = O(t)$. We observe the effect of intermittency, that is, for each vector $\mathbf{v} \in \mathbb{R}^d$, $\mathbf{v} \neq \bar{\mathbf{v}}$, there exists $k \geq 2$ such that the $k-$th moment of $n^{\mathbf{v}t}(t,x)$ grows exponentially faster than the $k-$th power of the first moment. This result was first proved in [3] in the case of a super-critical branching diffusion process in $\mathbb{R}^d$ with identity diffusion matrix, zero drift, and a positive constant potential. Here, in contrast to [3], we do not have explicit expressions for the transition kernel, but only have asymptotic formulas. This makes the analysis of the higher order moments much more involved.

In Section 2.4, we define a sequence of periodic functions $f_k(x)$ that serve as limits for the $k-$th moments of $N(t,x)/\mathbb{E}(N(t,x))$, where $N(t,x)$ denotes the total number of particles in $\mathbb{R}^d$, assuming that, initially, there is one particle located at $x \in \mathbb{R}^d$.

In Section 2.5, we again study $n^y(t,x)$, but here we assume that $\|y - \bar{\mathbf{v}}t\| = o(t)$. That is, we study the distribution of particles near the region where the bulk of the particles is located (i.e, near $\bar{\mathbf{v}}t$). In this region, we show that the $k-$th moment of $n^y(t,x)/\mathbb{E}(n^y(t,x))$ converges to the periodic function $f_k(x)$ identified in Section 2.4.

2.2 Asymptotics of solutions of parabolic PDEs

Given a positive function $h : \mathbb{R}^d \to \mathbb{R}$ that is sufficiently smooth, the $h-$transform of the operator $\mathcal{L}$ (given in (2.3)) is defined as

$$\mathcal{L}_h u = \frac{1}{h}\mathcal{L}(hu).$$

For each $t \geq 0$ and $x, y \in \mathbb{R}^d$, the transition kernel $u^h(t,x,y)$ corresponding to $\mathcal{L}_h$ satisfies:

$$u^h(t,x,y) = \frac{1}{h(x)} u(t,x,y) h(y), \tag{2.4}$$

where $u(t,x,y)$, satisfying (2.2), is the transition kernel corresponding to $\mathcal{L}$. We choose h from among a special family of eigenfunctions of $\mathcal{L}$ having exponential growth in a given direction. For $v \in \mathbb{R}^d$, let φ_v be the principal positive periodic eigenfunction of the operator $e^{-v\cdot y}\mathcal{L}(e^{v\cdot y}\cdot)$. That is φ_v satisfies

$$e^{-v\cdot y}\mathcal{L}(e^{v\cdot y}\varphi_v) = \mu(v)\varphi_v, \tag{2.5}$$

with eigenvalue $\mu(v) \in \mathbb{R}$. Let φ_v^* denote the solution of the adjoint problem, that is,

$$e^{v\cdot y}\mathcal{L}^*(e^{-v\cdot y}\varphi_v^*) = \mu^*(v)\varphi_v^*$$

where $\mu^*(v)$ is the principal eigenvalue of the adjoint operator, and hence $\mu^*(v) = \mu(v)$. We normalize φ_v and φ_v^* by

$$\int_{[0,1)^d} \varphi_v(y)\varphi_v^*(y)\, dy = 1 = \int_{[0,1)^d} \varphi_v^*(y)\, dy. \tag{2.6}$$

Now we define h_v by

$$h_v(y) = e^{v\cdot y}\varphi_v(y), \quad \text{that is,} \quad \mathcal{L}h_v = \mu(v)h_v.$$

With this choice of $h = h_v$, (2.4) can be written as

$$\begin{aligned} u(t,x,y) &= \frac{h_v(x)}{h_v(y)} u^{h_v}(t,x,y) \\ &= e^{-v\cdot(y-x)} \frac{\varphi_v(x)}{\varphi_v(y)} u^{h_v}(t,x,y) \end{aligned}$$

$$= e^{-t\left(v\cdot\frac{(y-x)}{t}-\mu(v)\right)}\frac{\varphi_v(x)}{\varphi_v(y)}e^{-t\mu(v)}u^{h_v}(t,x,y), \tag{2.7}$$

Let us define $p^v(t,x,y) := e^{-t\mu(v)}u^{h_v}(t,x,y)$. The function $p^v(t,x,y)$ is the transition kernel for the operator

$$\begin{aligned}
\mathcal{K}_v w &:= (\mathcal{L}_{h_v} - \mu(v))w \\
&= \frac{1}{e^{v\cdot y}\varphi_v(y)}\mathcal{L}(e^{v\cdot y}\varphi_v(y)w(y)) - \mu(v)w \\
&= \frac{1}{2}\sum_{ij} a_{ij}w_{x_ix_j} + \sum_i \left(b_i + \sum_j a_{ij}(v_j + \partial_{x_j}\log\varphi_v)\right)w_{x_i}.
\end{aligned} \tag{2.8}$$

Compared to $\mathcal{L}$, this operators $\mathcal{K}_v$ has an additional periodic drift $a\nabla\log h_v = av + a\nabla\log\varphi_v$, but no branching term $r(y)$. Let ψ_v and ψ_v^* denote the top eigenfunctions corresponding to the top eigenvalue (which is equal to zero) of the operator $\mathcal{K}_v$ and $\mathcal{K}_v^*$ on the torus, respectively, and suppose that

$$\int_{[0,1)^d}\psi_v(y)\psi_v^*(y)\,dy = 1 = \int_{[0,1)^d}\psi_v^*(y)\,dy.$$

It is easy to see that

$$\psi(y) \equiv 1 \quad \text{and} \quad \psi_v^*(y) \equiv \varphi_v^*(y)\varphi_v(y).$$

Now we choose the direction $v \in \mathbb{R}^d$ in an optimal way. Let Φ denote the Legendre transform of $\mu(v)$:

$$\Phi(c) = \sup_{v\in\mathbb{R}^d}\left(v\cdot c - \mu(v)\right). \tag{2.9}$$

The properties of μ, from Theorem 2.10 in Chapter 8 of the book of Pinsky [22], guarantee that $\Phi \in C^2$ is well-defined. For each $c \in \mathbb{R}^d$, the supremum in (2.9) is attained at a unique point which will be denoted by $\hat{v} = \hat{v}(c)$, that is

$$\Phi(c) = \hat{v}\cdot c - \mu(\hat{v}).$$

Thus, $c = \nabla\mu(\hat{v})$. In addition, for each $c \in \mathbb{R}^d$, we have $\nabla\Phi(c) = \hat{v}(c)$. Now, given $(t,x,y) \in \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d$, let $c = c(t,x,y) = (y-x)/t$. Therefore substituting $v = \hat{v}(c)$ in to (2.7), we get

$$u(t,x,y) = e^{-t\Phi(\frac{y-x}{t})} \frac{\varphi_{\hat{v}}(x)}{\varphi_{\hat{v}}(y)} p^{\hat{v}}(t,x,y). \tag{2.10}$$

Therefore, to obtain the exact asymptotics of $u(t,x,y)$ in the domain of large deviations, we need to choose $\hat{v}$ appropriately, and provide an exact asymptotics of the transition density $p^{\hat{v}}(t,x,y)$. The reason for introducing this transformed kernel is that, momentarily assuming $y = y(t) = x + ct$, the effective drift of the process corresponding to $p^{\hat{v}}(t,x,y)$ is c. And therefore, the problem reduces to estimating the density of the transition kernel of the operator $\mathcal{K}_{\hat{v}}$ at a diffusive scale. The following proposition, which will be proved later, gives the exact asymptotics of the transition density $p^{\hat{v}}(t,x,y)$.

Proposition 2.2.1. *Fix $L_0 > 0$. For $(t,x,y) \in \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d$, define $c = c(t,x,y) = (y-x)/t$ and $\hat{v} = \hat{v}(t,x,y) = \nabla\Phi(\frac{y-x}{t})$. Then*

$$\lim_{t\to\infty} \sup_{\|x-y\|\le tL_0} \left| \frac{1}{\varphi_{\hat{v}}(y)\varphi^*_{\hat{v}}(y)} \det[D^2\Phi(\frac{y-x}{t})]^{-1/2} (2\pi t)^{d/2} p^{\hat{v}}(t,x,y) - 1 \right| = 0. \tag{2.11}$$

The following theorem gives the exact asymptotics of $u(t,x,y)$.

Theorem 2.2.2. *Fix $L > 0$. The following asymptotic relation holds as $t \to \infty$ for all $x, y \in \mathbb{R}^d$ such that $\|y - x\| \le Lt$:*

$$u(t,x,y) = (2\pi t)^{-d/2} \det[D^2\Phi(\frac{y-x}{t})]^{1/2} e^{-t\Phi(\frac{y-x}{t})} \varphi_{\hat{v}}(x)\varphi^*_{\hat{v}}(y) \left[1 + o_L(1)\right], \tag{2.12}$$

where $\hat{v} = \hat{v}(t,x,y) = \nabla\Phi(\frac{y-x}{t})$.

Proof. Fix $L > 0$. From Proposition 2.2.1 and (2.10), for all $(t, x, y) \in \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d$ with $\frac{\|y-x\|}{t} \leq L$, we obtain

$$\begin{aligned} u(t,x,y) &= e^{-t\Phi(\frac{y-x}{t})} \frac{\varphi_{\hat{v}}(x)}{\varphi_{\hat{v}}(y)} p^{\hat{v}}(t,x,y) \\ &= e^{-t\Phi(\frac{y-x}{t})} \frac{\varphi_{\hat{v}}(x)}{\varphi_{\hat{v}}(y)} \Big(\varphi_{\hat{v}}(y) \varphi^*_{\hat{v}}(y) [\det D^2\Phi(\frac{y-x}{t})]^{1/2} (\sqrt{2\pi t})^{-d} (1 + o(1)) \Big) \\ &= (\sqrt{2\pi t})^{-d} [\det D^2\Phi(\frac{y-x}{t})]^{1/2} e^{-t\Phi(\frac{y-x}{t})} \varphi_{\hat{v}}(x) \varphi^*_{\hat{v}}(y)(1 + o(1)), \end{aligned}$$

uniformly for $\|y - x\| \leq Lt$. This concludes the proof of Theorem 2.2.2. □

Proof of Proposition 2.2.1. Let X_t be the diffusion process with generator $\mathcal{K}_v$,

$$dX_t = V(X_t)dt + \sigma(X_t)\, dW_t, \quad X_0 = x, \tag{2.13}$$

with

$$V_i(x) = b_i(x) + \sum_j a_{ij}(x)(v_j + \partial_{x_j} \log \varphi_v(x)).$$

From homogenization theory (see Freidlin [23] and Theorem 2.6 in the book of Jikov, Kozlov, Oleinik [24]), it is well known that the following result holds for diffusion processes with periodic coefficients: There exists a vector $\ell(v) \in \mathbb{R}^d$ (called the effective drift of X_t) and a positive definite matrix Ξ_v (called the effective diffusivity of X_t) such that

$$\frac{X_t - \ell(v)t}{\sqrt{t}} \to \mathcal{N}(0, \Xi_v) \quad \text{as} \quad t \to \infty,$$

in distribution, where $\mathcal{N}(0, \Xi_v)$ denotes the normal random vector with mean zero and covariance matrix Ξ_v. These quantities are given by the formulas:

$$\ell(v) = \int_{[0,1)^d} V(y)\psi^*_v(y)dy = \int_{[0,1)^d} V(y)\varphi_v(y)\varphi^*_v(y)\, dy, \tag{2.14}$$

$$\Xi_v = \int_{[0,1)^d} (\nabla\eta_v + I)a(y)(\nabla\eta_v + I)\varphi_v(y)\varphi_v^*(y)\,dy, \tag{2.15}$$

where $\eta_v(y)$ is a periodic (vector-valued) solution to

$$\mathcal{K}_v\eta_v = \ell(v) - V(y),$$

which is determined uniquely up to an additive constant.

We also refer to $\ell(v)$ and Ξ_v as the effective drift and the effective diffusivity of the operator $\mathcal{K}_v$ and hence, of the operator $\mathcal{L}_{h_v}$ since it only differs from $\mathcal{K}_v$ by a constant potential term. For the operator $\mathcal{L}$, whose potential term is not necessarily a constant, the effective drift cannot be defined by simply removing the potential term. Instead, by the effective drift of $\mathcal{L}$, we mean $\ell(0)$ (which we also denote by $\bar{\mathbf{v}}$), and by the effective diffusivity of $\mathcal{L}$, we mean Ξ_0.

We now state the following lemma about properties of the principal eigenvalue $\mu(v)$. The proof of this lemma can be found in the book of Pinsky [22] (Chapter 8, Theorem 2.10).

Lemma 2.2.3. *The function $\mu : \mathbb{R}^d \to \mathbb{R}$ is twice continuously differentiable and strictly convex. In addition, for each $v \in \mathbb{R}^d$,*

$$\nabla\mu(v) = \ell(v), \tag{2.16}$$

and,

$$D^2\mu(v) = \Xi_v. \tag{2.17}$$

Since Φ is the Legendre transform of the function μ, we have the relation

$$D^2\mu(v) = \Big[D^2\Phi(\nabla\mu(v))\Big]^{-1}.$$

Therefore, for each $v \in \mathbb{R}^d$,

$$[\det D^2\Phi\big(\ell(v)\big)\big]^{-1/2} = \big[\det(\Xi_v)\big]^{1/2}. \tag{2.18}$$

Let $\mathcal{B}$ denote the Banach space of $\mathbb{Z}^d$ periodic continuous functions $f : \mathbb{R}^d \to \mathbb{C}$, equipped with the supremum norm. In order to prove a local limit theorem for the process X_t corresponding to the operator $\mathcal{K}_v$, we first need to estimate $(\sqrt{2\pi t})^d \mathbb{E}_x[f(X_t)g([X_t])]$ for $f \in \mathcal{B}$ and $g : \mathbb{Z}^d \to \mathbb{R}$ having bounded support.

For $g : \mathbb{Z}^d \to \mathbb{R}$, for $\theta \in [0, 2\pi)^d$, $z \in \mathbb{Z}^d$, use the following definitions of Fourier Transform and Inverse Fourier Transform:

$$\hat{g}(\theta) = \frac{1}{(\sqrt{2\pi})^d} \sum_{z \in \mathbb{Z}^d} g(z) e^{i\theta z},$$

$$g(z) = \frac{1}{(\sqrt{2\pi})^d} \int_{[0,2\pi)^d} \hat{g}(\theta) e^{-i\theta z} d\theta.$$

Letting $\hat{g}(-\theta) = \tilde{g}(\theta)$, we have

$$\begin{aligned}(\sqrt{2\pi t})^d \mathbb{E}_x[f(X_t)g([X_t])] &= t^{d/2}\mathbb{E}_x[f(X_t) \int_{[0,2\pi)^d} \tilde{g}(\theta) e^{i\theta[X_t]}\, d\theta] \\ &= t^{d/2} \int_{[0,2\pi)^d} \tilde{g}(\theta) \mathbb{E}_x[f(X_t) e^{i\theta[X_t]}]\, d\theta. \end{aligned} \tag{2.19}$$

For $\theta \in \mathbb{R}^d$, $t \geq 0$, let is define the *Fourier Kernels* $Q(\theta, t)$, acting on $\mathcal{B}$, by

$$Q(\theta, t) f(x) = \mathbb{E}_x(f(X_t) e^{i\theta([X_t]-[x]-\ell(v)t)}). \tag{2.20}$$

Observe that $\{Q(\theta, \cdot)\}_{t \geq 0}$ is a family of compact operators on $\mathcal{B}$ and $e^{i\theta\ell(v)t}Q(\theta, t)$ is $2\pi\mathbb{Z}^d$ periodic in the parameter θ. Next, we show that, for a fixed $\theta \in [0, 2\pi)^d$, the family $\{Q(\theta, \cdot)\}_{t \geq 0}$ forms a semigroup. That is, for each $x \in \mathbb{R}^d$, $t, s \geq 0$,

$$Q(\theta, t) \circ Q(\theta, s) f(x) = Q(\theta, t + s) f(x). \tag{2.21}$$

To show (2.21), recall that $p^v(t,x,y)$ is the transition density of the operator $\mathcal{K}_v$, which is the generator of the diffusion process X_t. That is,

$$Q(\theta,s)f(x) = \sum_{m\in\mathbb{Z}^d}\int_{[0,1)^d} f(y)e^{i\theta(m-[x]-\ell(v)s)}p^v(s,x,y+m)dy.$$

Therefore,

$$Q(\theta,t)\big(Q(\theta,s)f\big)(x) = \sum_{n\in\mathbb{Z}^d}\int_{[0,1)^d} Q(\theta,s)f(z)e^{i\theta(n-[x]-\ell(v)t)}p^v(t,x,z+n)dz$$

$$= \sum_{n\in\mathbb{Z}^d}\int_{[0,1)^d}\Big(\sum_{m\in\mathbb{Z}^d}\int_{[0,1)^d} f(y)e^{i\theta(m-\ell(v)s)}p^v(s,z,y+m)dy\Big)e^{i\theta(n-[x]-\ell(v)t)}p^v(t,x,z+n)dz$$

$$= \sum_{n\in\mathbb{Z}^d}\sum_{m\in\mathbb{Z}^d}\int_{[0,1)^d} f(y)e^{i\theta(m+n-[x]-\ell(v)(t+s))}\int_{[0,1)^d} p^v(s,z,y+m)p^v(t,x,z+n)dzdy.$$

Since all the coefficients of the operator $\mathcal{K}_v$ are periodic, we observe that, for each $n\in\mathbb{N}$, $p^v(s,z,y+m) = p^v(s,z+n,y+m+n)$. Thus,

$$Q(\theta,t)Q(\theta,s)f(x) =$$

$$= \sum_{n\in\mathbb{Z}^d}\sum_{m\in\mathbb{Z}^d}\int_{[0,1)^d} f(y)e^{i\theta(m+n-[x]-\ell(v)(t+s))}\int_{[0,1)^d} p^v(s,z+n,y+m+n)p^v(t,x,z+n)dzdy$$

$$= \sum_{k\in\mathbb{Z}^d}\int_{[0,1)^d} f(y)e^{i\theta(k-[x]-\ell(v)(t+s))}\sum_{n\in\mathbb{Z}^d}\int_{[0,1)^d} p^v(s,z+n,y+k)p^v(t,x,z+n)dzdy$$

$$= \sum_{k\in\mathbb{Z}^d}\int_{[0,1)^d} f(y)e^{i\theta(k-[x]-\ell(v)(t+s))}p^v(t+s,x,y+k)dy = Q(\theta,t+s)f(x).$$

This concludes the proof of (2.21).

Let us denote the principal eigenvalue of the operator $Q(\theta,1)$ by $e^{\lambda(v,\theta)}$. Thus, from the semigroup property and the time homogeneity of the coefficients of the partial differential operator $\mathcal{K}_v$, we conclude that the principal eigenvalue of the operator $Q(\theta,t)$ is $e^{\lambda(v,\theta)t}$ for each $t\geq 0$.

Since the effective drift of the process X_t is $\ell(v)$, we observe that the asymptotic mean of the process $\{[X_t] - [x] - \ell(v)t\}_{t\geq 0}$ is zero. That is

$$\lim_{t\to\infty} \frac{\mathbb{E}([X_t] - [x] - \ell(v)t)}{t} = 0.$$

Similarly, the asymptotic covariance matrix of the process $\{[X_t] - [x] - \ell(v)t\}_{t\geq 0}$ is the effective diffusivity of the process X_t, that is, for each $1 \leq i, j \leq d$,

$$\Xi_v^{ij} = \lim_{t\to\infty} \frac{\mathbb{E}(([X_t^i] - [x^i] - \ell(v)_i t)([X_t^j] - [x^j] - \ell(v)_j t)}{t}.$$

Observe that, for $\theta = 0$, the operator $Q(0,t)$ is the Markov operator corresponding to the process X_t, which is generated by $\mathcal{K}_v$. Therefore, since the principal eigenvalue of the operator $\mathcal{K}_v$ is zero, we have, $e^{\lambda(v,0)t} = 1$.

The following lemma provides a spectral decomposition of the operator $Q(\theta, t)$, which will be useful in the proof of Proposition 2.2.1.

Lemma 2.2.4. *For a fixed $L > 0$, there exist $\theta_0 > 0$, $q > 0$ and $\eta > 0$ such that, for each $t > 1$, $\|\theta\| < \theta_0$, $f \in \mathcal{B}$, we have*

$$Q(\theta,t)f(x) = e^{\lambda(v,\theta)t}\big[\langle \varphi_v \varphi_v^*, f\rangle + (M(\theta,t)f)(x)\big] + (N(\theta,t)f)(x), \tag{2.22}$$

where the following bounds for the operator $M(\theta,t)$ and $N(\theta,t)$ hold:

$$\|M(\theta,t)f\|_{L^\infty} \leq q\|f\|_\infty \|\theta\|, \qquad \|N(\theta,t)f\|_{L^\infty} \leq qe^{-\eta t}\|f\|_\infty, \tag{2.23}$$

uniformly for all $\|v\| \leq L$. Moreover, there exists a constant C_1 such that for each $\theta \in \mathbb{R}^d$ satisfying $\|\frac{\theta}{\sqrt{t}}\| \leq \theta_0$ we have

$$\|e^{\left(\lambda(v,\frac{\theta}{\sqrt{t}})\right)t} - e^{-\frac{\theta^T \Xi_v \cdot \theta}{2}}\| \leq \frac{C_1}{\sqrt{t}}\|\theta\|^3 e^{\frac{-\theta^T \Xi_v \cdot \theta}{4}}, \tag{2.24}$$

uniformly for all $\|v\| \leq L$.

Remark 2.2.1. *The proof of the above lemma can be found in Hennion, Hervé [25] (Proposition VI.2), in the discrete time one dimensional setting. The assumptions of this Proposition, denoted bt $H''[2]$ in [25] (assumptions on the Banach space being sufficiently big, $Q(0,1)$ having 1 as its simple eigenvalue corresponding to the eigenfunction $f \equiv 1$, and the operators $Q(\theta,1)$ being sufficiently regular in the variable θ in a small neighborhood around $\theta = 0$) are all satisfied in our setting, uniformly in $\|v\| \leq L$. The proof of (2.24) (or, rather, its analog in [25]) relies on the fact that $\nabla_\theta \lambda(v,\theta)\big|_{\theta=0} = 0$ and $D^2_\theta \lambda(v,\theta)\big|_{\theta=0} = -\Xi_v$, which follows from arguments similar to those used in proving (2.18). The arguments in the proof of Proposition VI.2 of [25] also go through in the continuous time $d-$dimensional setting.*

We now show that the all the eigenvalues of $Q(\theta,t)$ are strictly less than 1 for all $\theta \in (0,2\pi)^d$, $t \geq 0$, that is,

$$\lambda(v,\theta) < 0 \quad \text{for each} \quad \theta \in (0,2\pi)^d. \tag{2.25}$$

(It is clear that $\lambda(v,\theta) \leq 0$ for all $\theta \in [0,2\pi)^d$.) This fact will be useful in the proof of Lemma 2.2.5. Suppose that, on the contrary, for some $\theta \in (0,2\pi)^d$ there exists an eigenfunction $f \in \mathcal{B}$ of the operator $Q(\theta,t)$ with $\|f\| = 1$ such that $\lambda(\theta,v) = 0$. That is, for each $x \in [0,1)^d$,

$$\mathbb{E}_x(f(X_t)e^{i\theta([X_t]-[x]-\ell(v)t)}) = f(x). \tag{2.26}$$

We know that 1 is the top eigenvalue of the operator $Q(0,t)$. Thus, there exists an eigenfunction $g \in \mathcal{B}$ of $Q(0,t)$ such that, for all $x \in [0,1)^d$,

$$\mathbb{E}_x g(X_t) = g(x). \tag{2.27}$$

Note that we can choose $g \in \mathcal{B}$ such that, for all $x \in [0,1)^d$, $g(x) > 0$, and that $|f(x)| \leq g(x)$. In addition, we can assume that there exists a point $x_0 \in [0,1)^d$ such that $|f(x_0)| = g(x_0)$. Now,

$$\mathbb{E}_{x_0}\left|f(X_t)e^{i\theta([X_t]-[x]-\ell(v)t)}\right| \geq |\mathbb{E}_{x_0}(f(X_t)e^{i\theta([X_t]-[x]-\ell(v)t)})|$$

$$= |f(x_0)| = g(x_0) = \mathbb{E}_{x_0} g(X_t).$$

This implies that,

$$\mathbb{E}_{x_0}(|f(X_t)| - g(X_t)) = Q(0,t)(|f| - g)(x_0) \geq 0.$$

Since $|f| \leq g$ and $Q(0,t)$ is a positive operator, we conclude that

$$\mathbb{E}_{x_0}(|f(X_t)| - g(X_t)) = 0.$$

That is,

$$\int_{\mathbb{R}^d} (|f(y)| - g(y)) p^v(t, x_0, y) dy = 0.$$

Since X_t is a non-degenerate diffusion, for a fixed $x_0 \in [0,1)^d$, $p^v(t, x_0, y) > 0$ for all $y \in \mathbb{R}^d$, $t \geq 0$. Thus, there exists a continuous $\mathbb{Z}^d$ periodic function h such that $f(y) = e^{ih(y)} g(y)$ for all $y \in \mathbb{R}^d$. Therefore,

$$\mathbb{E}_x(e^{ih(X_t)} g(X_t) e^{i\theta([X_t]-[x]-\ell(v)t)}) = e^{ih(x)} g(x) = e^{ih(x)} \mathbb{E}_x(g(X_t)).$$

Thus,

$$\mathbb{E}_x\Big(g(X_t)\Big[e^{i\big(\theta([X_t]-[x]-\ell(v)t)+h(X_t)-h(x)\big)} - 1\Big]\Big) = 0,$$

that is, $\theta([y]-[x]-\ell(v)t)+h(y)-h(x)\in 2\pi\mathbb{Z}$, for all $x,y\in\mathbb{R}^d$, $t\geq 0$. This is a contradiction since, taking $y=x+m$ with $m\in\mathbb{Z}^d$, we get $\theta(m-\ell(v)t)\in 2\pi\mathbb{Z}$ for all $m\in\mathbb{Z}^d$, which is impossible.

Let $\mathcal{B}_{+,r}$ be defined as follows:

$$\mathcal{B}_{+,r}=\{f : f\in\mathcal{B}, f\geq 0, \|f\|_\infty<r\}.$$

Let $B_0(L)=\{v\in\mathbb{R}^d\Big|\|v\|\leq L\}$ denote the ball of radius L centered at 0 in $\mathbb{R}^d$. For $\chi=(z,x,f,v)\in(\mathbb{Z}^d,\mathbb{R}^d,\mathcal{B}_{+,r},B_0(L))$, we define two families of measures on $\mathbb{Z}^d$:

$$m_t^\chi(k)=\det(\Xi_v)^{1/2}(\sqrt{2\pi t})^d\mathbb{E}_x(f(X_t)\mathbf{1}_{\{k\}}([X_t]-z)),$$

$$m_t^{'\chi}(k)=e^{-\frac{(z-\ell(v)t-[x])^T\Xi_v^{-1}(z-\ell(v)t-[x])}{2t}}\langle\varphi_v\varphi_v^*,f\rangle.$$

Note that neither of these needs to be a probability measure. In fact, the latter one is a measure assigning the same weight to each $k\in\mathbb{Z}^d$, but both can be applied to functions defined on $\mathbb{Z}^d$.

Lemma 2.2.5. *Let $g:\mathbb{Z}^d\to\mathbb{R}$ be a function with bounded support and $\chi=(z,x,f,v)\in(\mathbb{Z}^d,\mathbb{R}^d,\mathcal{B}_{+,r},B_0(L))$. Then*

$$\lim_{t\to\infty}\sup_\chi\|m_t^\chi(g)-m_t^{'\chi}(g)\|=0.$$

Remark 2.2.2. *A similar lemma and its proof can be found in Hennion, Hervé [25] (Lemma VI.4), in the discrete time one dimensional setting. For a bounded linear operator Q on a Banach Space $\mathcal{B}$, let $r(Q)$ denote its spectral radius. The difference between [25] and our setting is that, in Lemma VI.4, the set $\{\theta\in\mathbb{R}^d\Big|r(Q(\theta,1))\geq e^{\lambda(v,0)}=1\}$ was required to be $\{0\}$. This is clearly not the case in our setting since* *the operators* $Q(\theta,1)e^{i\theta\ell(v)}$ *are* $2\pi\mathbb{Z}^d$ *periodic in* $\theta\in\mathbb{R}^d$*. Instead, we have shown in* (2.25) *that* $\{\theta\in[0,2\pi)^d \big| r(Q(\theta,1))\geq 1\}=\{0\}$.

Proof of Lemma 2.2.5. Using the Fourier inversion formula and Fubini's theorem,

$$m_t^\chi(g)=\det(\Xi_v)^{1/2}(\sqrt{2\pi t})^d\mathbb{E}_x(f(X_t)g([X_t]-z))$$

$$=\det(\Xi_v)^{1/2}t^{d/2}\int_{[0,2\pi)^d}\tilde{g}(\theta)e^{-i\theta(z-\ell(v)t-[x])}Q(\theta,t)f(x)d\theta,$$

where $\tilde{g}(\theta):=\hat{g}(-\theta)$. From Lemma 2.2.4, we know that there exists a $\theta_0>0$ such that, for all $\|\theta\|\leq\theta_0$ the decomposition (2.22) holds. Therefore, we can write

$$m_t^\chi(g)=J_t^1(\chi)+J_t^2(\chi)+J_t^3(\chi),$$

where

$$J_t^1(\chi):=$$

$$=\det(\Xi_v)^{1/2}t^{d/2}\int_{[0,2\pi)\cap(\|\theta\|<\theta_0)}\tilde{g}(\theta)e^{-i\theta(z-\ell(v)t-[x])}e^{\lambda(v,\theta)t}\big[\langle\varphi_v\varphi_v^*,f\rangle+M(\theta,t)f(x)\big]d\theta$$

and $J_t^2(\chi)$ and $J_t^3(\chi)$, are defined as follows:

$$J_t^2(\chi):=\det(\Xi_v)^{1/2}t^{d/2}\int_{[0,2\pi)^d\cap(\|\theta\|<\theta_0)}\tilde{g}(\theta)e^{-i\theta(z-\ell(v)t-[x])}N(\theta,t)f(x)d\theta,$$

$$J_t^3(\chi):=\det(\Xi_v)^{1/2}t^{d/2}\int_{[0,2\pi)^d\cap(\|\theta\|\geq\theta_0)}\tilde{g}(\theta)e^{-i\theta(z-\ell(v)t-[x])}Q(\theta,t)f(x)d\theta.$$

The change of variable $\theta=\dfrac{s}{\sqrt{t}}$ gives

$$J_t^1(\chi)=\int_{\mathbb{R}^d}k_t(s)e^{-i(\frac{s(z-\ell(v)t-[x])}{\sqrt{t}})}\big[\langle\varphi\varphi_v^*,f\rangle+M(\frac{s}{\sqrt{t}},t)f(x)\big]ds,$$

where

$$k_t(s)=\det(\Xi_v)^{1/2}\mathbf{1}_{[0,2\pi)^d\cap(\|\theta\|<\theta_0)}(\frac{s}{\sqrt{t}})\tilde{g}(\frac{s}{\sqrt{t}})e^{\lambda(v,\frac{s}{\sqrt{t}})t}.$$

On the other hand, we have

$$m_t^{'\chi}(g) = \int_{\mathbb{R}^d} e^{-i\frac{s(z-\ell(v)t-[x])}{\sqrt{t}}} k(s)\langle \varphi_v\varphi_v^*, f\rangle ds,$$

where,

$$k(s) := \det(\Xi_v)^{1/2}\ \tilde{g}(0)e^{-\frac{s^T\Xi_v s}{2}}.$$

For each $s \in [0,2\pi)^d$ such that $\|\frac{s}{\sqrt{t}}\| < \theta_0$, from Lemma 2.2.4, we have that

$$\|M(\frac{s}{\sqrt{t}},t)f(x)\| \leq q\|f\|\frac{\|s\|}{\sqrt{t}}.$$

Hence,

$$\|J_t^1(\chi) - m_t^{'\chi}(g)\| \leq |\langle\varphi_v\varphi_v^*, f\rangle| \int_{\mathbb{R}^d} |k_t(s)-k(s)|ds + q\|f\| \int_{\mathbb{R}^d} |k_t(s)|\frac{\|s\|}{\sqrt{t}}ds.$$

We observe from (2.24) that the sequence $\{k_t\}_{t\geq 1}$ converges point-wise to k. Since the function g has bounded support in $\mathbb{Z}^d$, $\|\tilde{g}\|_\infty < \infty$. Thus, setting $c_g := \|\tilde{g}\|_\infty$, we have

$$\|k_t(s)\| \leq \det(\Xi_v)c_g e^{-\frac{s^T\Xi_v s}{4}}.$$

By defining

$$\epsilon_t^1 := \|\varphi_v\varphi_v^*\| \int_{\mathbb{R}^d} |k_t(s)-k(s)|ds, \qquad \epsilon_t^2 := q\int_{\mathbb{R}^d} |k_t(s)|\frac{\|s\|}{\sqrt{t}}ds,$$

we get,

$$\|J_t^1(\chi) - m_t^{'\chi}(g)\| \leq (\epsilon_t^1+\epsilon_t^2)\|f\|.$$

Using the Lebesgue dominated convergence theorem, $\lim_{t\to\infty}\epsilon_t^1 = \lim_{t\to\infty}\epsilon_t^2 = 0$. Now it remains to consider the terms $J_t^2(\chi)$ and $J_t^3(\chi)$. For $\|\theta\| \leq \theta_0$, we have from Lemma 2.2.4 that $\|N(\theta,t)\| \leq qe^{-\eta t}$, and therefore

$$J_t^2(\chi) \leq \det(\Xi_v)^{1/2} t^{d/2} q e^{-\eta t}\|f\| \int_{[0,2\pi)^d\cap(\|\theta\|<\theta_0)} |\tilde{g}(\theta)|d\theta =: \epsilon_t^3\|f\|,$$

where

$$\epsilon_t^3 := \det(\Xi_v)^{1/2} t^{d/2} q e^{-\eta t} \int_{[0,2\pi)^d \cap (\|\theta\| < \theta_0)} |\tilde{g}(\theta)| d\theta.$$

It is clear that $\lim_{t\to\infty} \epsilon_t^3 = 0$. Let $\beta_t = \sup\{\|Q(\theta,t)\| : \theta \in (\|\theta\| \geq \theta_0) \cap [0,2\pi)^d, \|v\| \leq L\}$. From (2.25), by choosing

$$\delta = \sup\{r(Q(\theta,1)) \Big| \theta \in [0,2\pi)^d, \|\theta\| \geq \theta_0, \|v\| \leq L\} < 1,$$

we have $\beta_t \leq \delta^t \to 0$ exponentially fast, as $t \to \infty$. Now,

$$\|J_t^3(\chi)\| \leq \det(\Xi_v)^{1/2} t^{d/2} \|f\| \beta_t \int_{[0,2\pi)^d} \tilde{g}(\theta) d\theta =: \epsilon_t^4 \|f\|,$$

where

$$\epsilon_t^4 := \det(\Xi_v)^{1/2} t^{d/2} \beta_t \int_{[0,2\pi)^d} \tilde{g}(\theta) d\theta.$$

It is clear to see that $\lim_{t\to\infty} \epsilon_t^4 = 0$. Therefore,

$$\lim_{t\to\infty} \sup_{\chi} \|m_t^\chi(g) - m_t^{'\chi}(g)\| = 0.$$

□

From the above lemma, for the function $g(k) = \mathbf{1}_0(k)$, we have

$$\lim_{t\to\infty} \sup_{\substack{f\in C([0,1)^d)\\ \|f\|<r}} \sup_{\substack{x\in\mathbb{R}^d, z\in\mathbb{Z}^d\\ \|v\|\leq L}} \Big| \det(\Xi_v)^{1/2} (\sqrt{2\pi t})^d \int_{[0,1)^d} f(y) p^v(t,x,y+z) dy$$

$$- \exp\Big(- \frac{(z-[x]-\ell(v)t)^T \Xi_v^{-1} (z-[x]-\ell(v)t)}{2t} \Big) \langle \varphi_v \varphi_v^*, f \rangle \Big| = 0. \qquad (2.28)$$

The formula above gives the asymptotics of p^v in a slightly weaker form, than the one we claimed. Namely, we would like to be able to replace f by a delta function at $y \in [0,1)^d$. This is easily justified if we have an appropriate bound on the derivative of $p^v(t,x,y)$ in the y variable. In this case, the weighted average of p^v over a small domain approximates the value of p^v at any point inside the domain. To get such bounds on the derivative of p^v, we observe that $p^v(t,x,y) \leq c/t^{d/2}$ for all $x,y \in \mathbb{R}^d$, since $p^v(t,x,y)$ is the fundamental solution of the PDE with periodic coefficients, with no potential term (see, for example, arguments in the proof of Lemma 2.3.2). From the Schauder estimate (see, Friedman [26], Theorem 1), it then follows that $\|\nabla_y p^v(t,x,y)\| \leq \sup\{p^v(s,x',y')\Big| s \in (t-1,t), x', y' \in \mathbb{R}^d\} \leq c/(t-1)^{d/2} \leq \tilde{c}/t^{d/2}$. These bounds on the gradient of $p^v(t,x,y)$ along with (2.28) imply that

$$\lim_{t\to\infty} \sup_{\substack{x\in\mathbb{R}^d, y\in[0,1)^d, z\in\mathbb{Z}^d \\ \|v\|\leq L}} \Big| \det(\Xi_v)^{1/2}(\sqrt{2\pi t})^d p^v(t,x,y+z) \tag{2.29}$$

$$- \exp\Big(- \frac{(z-[x]-\ell(v)t)^T \Xi_v^{-1}(z-[x]-\ell(v)t)}{2t}\Big)\varphi_v(y)\varphi_v^*(y)\Big| = 0.$$

Writing $y \in \mathbb{R}^d$ instead of $y+z$ with $y \in [0,1)^d, z \in \mathbb{Z}^d$, we obtain

$$\lim_{t\to\infty} \sup_{\substack{x,y\in\mathbb{R}^d \\ \|v\|\leq L}} \Big| \det(\Xi_v)^{1/2}(\sqrt{2\pi t})^d p^v(t,x,y) - e^{-\frac{t}{2}\left(\frac{y-x}{t}-\ell(v)\right)^T \Xi_v^{-1}\left(\frac{y-x}{t}-\ell(v)\right)}\varphi_v(y)\varphi_v^*(y)\Big| = 0.$$

Note that the exponent in the above formula is slightly different. But the difference is negligible in the limit.

Now suppose that $L_0 > 0$ is fixed, and $\|y-x\|/t \leq L_0$, for all $x,y \in \mathbb{R}^d$ and $t > 0$. Then, if we choose $c = (y-x)/t$, we have a corresponding $\hat{v}$ such that, $\nabla\Phi(c) = \hat{v}$ and there exists a L such that $\hat{v} \leq L$ for all $t > 0$. Thus the above result can be applied to those c, and $\hat{v}$ uniformly to obtain

$$\lim_{t\to\infty} \sup_{\|x-y\|\leq tL_0} \Big| \det(\Xi_v)^{1/2}(\sqrt{2\pi t})^d p^{\hat{v}}(t,x,y) - \varphi_{\hat{v}}(y)\varphi_{\hat{v}}^*(y)\Big| = 0.$$

Substituting $[\det D^2\Phi\big(\frac{y-x}{t}\big)]^{-1/2} = \big[\det(\Xi_{\hat{v}})\big]^{1/2}$ from (2.18), we get

$$\lim_{t\to\infty} \sup_{\|x-y\|\le tL_0} \Big| \frac{1}{\varphi_{\hat{v}}(y)\varphi^*_{\hat{v}}(y)} \big[\det D^2\Phi\big(\frac{y-x}{t}\big)\big]^{-1/2} (\sqrt{2\pi t})^d p^{\hat{v}}(t,x,y) - 1 \Big| = 0.$$

□

2.3 Intermittency of a supercritical branching process

Here, we assume that our branching diffusion process is super-critical, that is, the principle eigen-value of the operator $\mathcal{L}$ when considered as an operator on the torus $\mathbb{T}^d$ is positive (i.e, $\mu(0) > 0$). We know that Φ is a twice continuously differentiable, strictly convex function such that its minimum value is achieved at $\bar{\mathbf{v}}$, and $\Phi(\bar{\mathbf{v}}) = -\mu(0) < 0$.

For $y = (y_1, y_2, \cdots, y_d)$, let $\mathbb{T}^d_y$ denote the $d-$dimensional cube:

$$\mathbb{T}^d_y = [y_1, y_1 + 1) \times [y_2, y_2 + 1) \times \cdots \times [y_d, y_d + 1).$$

Let $n^y(t, x)$ denote the number of particles in $\mathbb{T}^d_y$ at time t, assuming that at time zero there was a single particle located at $x \in \mathbb{R}^d$.

Theorem 2.3.1. *For each $k \in \mathbb{N}\cup\{0\}$, and each $x \in [0, 1)^d$, the following statements hold:*

(a) For each $\mathbf{v} \in \mathbb{R}^d$, there exists the limit,

$$\gamma_k(\mathbf{v}) = \lim_{t\to\infty} \frac{\ln \mathbb{E}(n^{t\mathbf{v}}(t, x)^k)}{t}.$$

(b) Define $G_k = \{\mathbf{v} \in \mathbb{R}^d : \gamma_1(\mathbf{v}) \ge 0, \gamma_k(\mathbf{v}) = k\gamma_1(\mathbf{v})\}$ for each $k \in \mathbb{N}$. Then G_k's are closed subsets of $\mathbb{R}^d$ and $G_{k+1} \subseteq G_k$.

(c) There exists a sequence of constants $\alpha_k > 0$ *such that* $B_{\alpha_k}(\bar{\mathbf{v}}) \subseteq G_k$*, and* $\cap_{k\in\mathbb{N}} G_k = \{\bar{\mathbf{v}}\}$.

Remark 2.3.1. *We first make a few elementary observations regarding the statements in the above theorem.*

1. *Since* $n^{t\mathbf{v}}(t,x) \in \mathbb{N} \cup \{0\}$*, using the Jensen's inequality, it is clear that for each* $\mathbf{v} \in \mathbb{R}^d$, $\mathbb{E}(n^{t\mathbf{v}}(t,x)^k) \geq (\mathbb{E}(n^{t\mathbf{v}}(t,x)))^k$ *for each* $k \in \mathbb{N}$*, and therefore, as long as the limits in* (a) *exist, we have,* $\gamma_k(\mathbf{v}) \geq k\gamma_1(\mathbf{v})$ *for each* $k \in \mathbb{N}$*. Thus,* $G_1 \setminus G_k = \{\mathbf{v} \in G_1 : \gamma_k(\mathbf{v}) > k\gamma_1(\mathbf{v})\}$.
2. *Using Hölder's inequality, it is easily seen that* $\ln \mathbb{E}(n^{t\mathbf{v}}(t,x)^k)$ *is a convex function of* k *for each fixed* $t \in \mathbb{R}^+, \mathbf{v} \in \mathbb{R}^d$*. In addition,* $\gamma_0 \equiv 0$ *and therefore* $\gamma_k(\mathbf{v})/k$ *is a non-decreasing function of* k*, which implies that, if* $\gamma_k(\mathbf{v}) > k\gamma_1(\mathbf{v})$*, then* $\gamma_{k+1}(\mathbf{v}) > (k+1)\gamma_1(\mathbf{v})$*. Therefore,* $G_{k+1} \subseteq G_k$ *for each* $k \in \mathbb{N}$.

The main idea of the proof is to look at the higher order correlation functions and the corresponding PDEs they solve and then use the asymptotics of the density function obtained in Theorem 2.2.2 and techniques developed in [3] to obtain logarithmic asymptotics of the moments $\mathbb{E}(n^{t\mathbf{v}}(t,x)^k)$.

Remark 2.3.2. *Note that* $\bar{\mathbf{v}} = \ell(0) = \nabla\mu(0)$ *is the effective drift of the branching process detailed in Section 2.2 (also see Lemma 2.2.3). From the properties of the function* μ*, it is clear that* $\Phi \in C^2$ *is strictly convex (see Lemma 2.2.3), and the minimum of* Φ *is achieved at a point* $\bar{\mathbf{v}} \in \mathbb{R}^d$*. The minimum value of the function* Φ *is* $\Phi(\bar{\mathbf{v}}) = -\mu(0) < 0$.

For simplicity of notation, we assume that $\bar{\mathbf{v}} = 0$.

Let $B_\delta(y)$ denote a ball of radius $\delta > 0$ centered at $y \in \mathbb{R}^d$. For $t > 0$ and $x, y_1, y_2, ... \in \mathbb{R}^d$ with all y_i distinct, define the particle density $\rho_1(t,x,y)$ and the higher order correlation functions $\rho_n(t,x,y_1,....,y_n)$ as the limits of probabilities of finding n distinct particles in $B_\delta(y_1),...B_\delta(y_n)$, respectively, divided by the n-th power of the volume of $B_\delta(0) \subset \mathbb{R}^d$. For a fixed y_1, the density satisfies

$$\partial_t \rho_1(t,x,y_1) = \mathcal{L}_x \rho_1(t,x,y_1), \qquad \rho_1(0,x,y_1) = \delta_{y_1}(x). \tag{2.30}$$

The equations on ρ_n , $n > 1$, are as follows

$$\partial_t \rho_n(t,x,y_1,y_2,...,y_n) = \mathcal{L}_x \rho_n(t,x,y_1,y_2,...,y_n) + \alpha(x) H_n(t,x,y_1,y_2,...,y_n), \tag{2.31}$$

$$\rho_n(0,x,y_1,y_2,...,y_n) \equiv 0,$$

where

$$H_n(t,x,y_1,y_2,...,y_n) = \sum_{U \subset Y, U \neq \emptyset} \rho_{|U|}(t,x,U)\rho_{n-|U|}(t,x,Y \setminus U),$$

where $Y = (y_1,...,y_n)$, U is a proper non-empty subsequence of Y , and $|U|$ is the number of elements in this subsequence.

Define $m_k^y(t,x) = \int_{\mathbb{T}_y^d} .... \int_{\mathbb{T}_y^d} \rho_k(t,x,y_1,y_2,...,y_k) dy_1...dy_k$. It follows that

$$\partial_t m_1^y(t,x) = \mathcal{L}_x m_1^y(t,x), \qquad m_1^y(0,x) = \chi_{\mathbb{T}_y^d}(x), \tag{2.32}$$

while for $k \geq 2$,

$$\partial_t m_k^y(t,x) = \mathcal{L}_x m_k^y(t,x) + \alpha(x) \sum_{i=1}^{k-1} \beta_i^k m_i^y(t,x) m_{k-i}^y(t,x), \qquad m_k^y(0,x) \equiv 0, \tag{2.33}$$

where $\beta_i^k = k!/(i!(k-i)!)$. Notice that

$$\mathbb{E}(n^{t\mathbf{v}}(t,x)^k) = \sum_{i=1}^{k} S(k,i) \int_{\mathbb{T}_{t\mathbf{v}}^d} .. \int_{\mathbb{T}_{t\mathbf{v}}^d} \rho_i(t,x,y_1,y_2,..,y_i) dy_1...dy_i$$

$$= \sum_{i=1}^{k} S(k,i) m_i^{t\mathbf{v}}(t,x), \tag{2.34}$$

where $S(k,i)$ is the Stirling number of the second kind (the number of ways to partition k elements into i nonempty subsets).

To see this, let $n^y(t,x,\Delta(z))$ denote the number of particles at time t in a small set $\Delta(z) \subseteq \mathbb{T}_y^d$ containing z, assuming that at time zero there was a single particle located at $x \in \mathbb{R}^d$. Then, we can write

$$n^y(t,x) = \sum_j n^y(t,x,\Delta(z_j)),$$

by choosing the disjoint subsets $\Delta(z_j)$, $j \geq 1$, such that $\bigcup_j \Delta(z_j) = \mathbb{T}_y^d$. We now show that (2.34) holds for $k=2$. Observe that

$$\mathbb{E}(n^y(t,x)^2) = \sum_{i,j} \mathbb{E}(n^y(t,x,\Delta(z_i)) n^y(t,x,\Delta(z_j))).$$

By taking the limit as $\max_j\{\mathrm{diam}(\Delta(z_j))\} \to 0$, when $i \neq j$, we get

$$\mathbb{E}(n^y(t,x,\Delta(z_i)) n^y(t,x,\Delta(z_j))) \approx \rho_2(t,x,z_i,z_j) Vol(\Delta(z_i)) Vol(\Delta(z_j)).$$

On the other hand, when the diameter of the set $\Delta(z_i)$ is small enough, with overwhelming probability $n^y(t,x,\Delta(z_i)) \in \{0,1\}$. Thus $\mathbb{E}(n^y(t,x,\Delta(z_i))^2) \approx \mathbb{E}(n(t,x,\Delta(z_i))) \approx \rho_1(t,x,z_i)\mathrm{Vol}(\Delta(z_i))$. Thus, we get

$$\sum_{i,j} \mathbb{E}(n^y(t,x,\Delta(z_i)) n^y(t,x,\Delta(z_j))) \approx$$
$$\approx \sum_{i \neq j} \rho_2(t,x,z_i,z_j)\mathrm{Vol}(\Delta(z_i))\mathrm{Vol}(\Delta(z_j)) + \sum_i \rho_1(t,x,z_i)\mathrm{Vol}(\Delta(z_i))$$
$$\to \int_{\mathbb{T}_y^d}\int_{\mathbb{T}_y^d} \rho_2(t,x,y_1,y_2) dy_1 dy_2 + \int_{\mathbb{T}_y^d} \rho_1(t,x,y_1) dy_1 = m_2^y(t,x) + m_1^y(t,x).$$

This proves (2.34) for $k=2$. We do not detail the arguments for all $k>2$ since they are similar to those for $k=2$.

Proof of Theorem 2.3.1. In Part 1 of the proof, we will use induction to show the following:

(i) For each $k \geq 1$, there exists a constant $a_k > 0$ such that

$$m_k^y(t,x) \leq a_k \exp\left(a_k t - \frac{\|y-x\|^2}{a_k(t+1)}\right) \tag{2.35}$$

for all $(t,x,y) \in \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d$.

(ii) For each $k \geq 1$, for each $L > 0$, the following two limits exist uniformly for $\mathbf{v} \in \mathbb{R}^d$, with $\|\mathbf{v}\| \leq L$ and for $x \in [0,1)^d$, and satisfy

$$\gamma_k(\mathbf{v}) = \lim_{t\to\infty} \frac{\ln \mathbb{E}(n^{t\mathbf{v}}(t,x)^k)}{t} = \lim_{t\to\infty} \frac{\ln m_k^{t\mathbf{v}}(t,x)}{t}. \tag{2.36}$$

Moreover, $\gamma_k : \mathbb{R}^d \to \mathbb{R}$ is continuous for all $k \in \mathbb{N}$.

(iii) For each $L > 0$, there exists $M_0 = M_0(L,k)$ such that, for all $M \geq M_0$,

$$\gamma_k(\mathbf{v}) = \sup_{\|w-\mathbf{v}\|\leq M, u\in(0,1)} \left[u\gamma_{k-1}\left(\frac{\mathbf{v}-w}{u}\right) + u\gamma_1\left(\frac{\mathbf{v}-w}{u}\right) + (1-u)\gamma_1\left(\frac{w}{1-u}\right)\right], \tag{2.37}$$

when $\|\mathbf{v}\| \leq L$, $k \geq 2$. In addition, $\gamma_k(\mathbf{v}) \geq \gamma_{k-1}(\mathbf{v})$ for $k \geq 2$.

We will conclude the proof by showing, in Part 2, that there exists a sequence of constants $\alpha_k > 0$ such that $B_{\alpha_k}(0) \subseteq G_k$ and that $\bigcap_{k\in\mathbb{N}} G_k = \{0\}$.

Before we start the rigorous proof of the above statements (including (2.37)), we make a few remarks about the intuition behind formula (2.37). From (2.33) and the Duhamel's Formula, we see that

$$m_k^y(t,x) = \int_0^t \int_{\mathbb{R}^d} \alpha(z) \sum_{i=1}^{k-1} \beta_i^k m_i^y(s,z) m_{k-i}^y(s,z) \rho_1(t-s,x,z)\,dz\,ds$$

$$\geq C \int_0^t \int_{\mathbb{R}^d} \alpha(z) m_{k-1}^y(s,z) m_1^y(s,z) \rho_1(t-s,x,z) dz ds. \tag{2.38}$$

Let us, for the time being, assume that the asymptotics of $m_k^{t\mathbf{v}}(t,x)$ is captured by the integral in (2.38). It is clear to see from the precise asymptotics of the transition density obtained in Section 2.2 and from the definition of $m_1^y(t,x)$ that the logarithmic asymptotics of $m_1^{t\mathbf{v}}(t,x)$ and $\rho_1(t,x,t\mathbf{v})$ is given by $\gamma_1(\mathbf{v}) = -\Phi(\mathbf{v})$.

It will be seen that the main contribution to the logarithmic asymptotics of the integral in (2.38) comes from a small neighborhood of the point (z,s) where the maximum of the integrand is achieved. Thus, we take the logarithm of the integrand in (2.38). Moreover, formally applying (2.36), replacing $\ln m_{k-1}^{t\mathbf{v}}(s,z)$ by $s\gamma_{k-1}(\frac{t\mathbf{v}-z}{s})$, $\ln m_1^{t\mathbf{v}}(s,z)$ by $s\gamma_1(\frac{t\mathbf{v}-z}{s})$ and $\ln \rho_1(t-s,x,z)$ by $(t-s)\gamma_1(\frac{z}{t-s})$, substitute $z/t = w$ and $s/t = u$, and maximizing the integrand in u and w, we get

$$\gamma_k(\mathbf{v}) = \sup_{w \in \mathbb{R}^d u \in (0,1)} \Big[u\gamma_{k-1}\Big(\frac{\mathbf{v}-w}{u}\Big) + u\gamma_1\Big(\frac{\mathbf{v}-w}{u}\Big) + (1-u)\gamma_1\Big(\frac{w}{1-u}\Big)\Big].$$

However, observe that the asymptotics of ρ_1 obtained in Section 2.2 is only valid in the regions of space that grow linearly in time. In addition, the relation $\ln m_k^y(t,x) = t\gamma_k(\frac{y-x}{t})$ is also meaningful only when $\|y-x\| = O(t)$. Thus, we need additional analysis to prove that the formula holds when the supremum is taken over the set where $\|w-\mathbf{v}\| \leq M$, and that the contribution from the rest of the space is negligible.

Part 1 of the Proof:

For $k=1$, from Theorem 2.2.2, for a fixed $L > 0$, for all $x, y \in \mathbb{R}^d$ with $\|x-y\| \leq Lt$, we have

$$m_1^y(t,x) = \int_{\mathbb{T}_y^d} \rho_1(t,x,z)\, dz =$$

$$= (\sqrt{2\pi t})^{-d}\varphi_0(x)\left(\int_{\mathbb{T}^d_y} \det[D^2\Phi(\frac{z-x}{t})]^{1/2} e^{-t\Phi(\frac{z-x}{t})}\varphi_0^*(z)\,dz\right)[1+o_L(1)]\,, \quad (2.39)$$

where Φ, φ_0 and φ_0^* are defined before Theorem 2.2.2.

Lemma 2.3.2. *If f is the fundamental solution to a linear parabolic PDE with $\mathbb{Z}^d$ periodic coefficients, then there exists a constant $c > 0$ such that f satisfies the following Aronson-type estimate. For all $(t, x, y) \in \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d$,*

$$f(t,x,y) \le ct^{-d/2}\exp\left(ct - \frac{\|x-y\|^2}{ct}\right). \quad (2.40)$$

Proof. Suppose that $\mathcal{M}$ is an elliptic operator with $\mathbb{Z}^d$ periodic coefficients, and suppose f is the fundamental solution to the linear parabolic PDE driven by this operator. Denote the top eigenvalue of the operator $\mathcal{M}$ on the torus by λ, the top eigenfunction by ψ, and the effective drift (defined by (2.14)) of the operator $\mathcal{M}$ by $\mathbf{v} \in \mathbb{R}^d$. Since the operator is elliptic, the function $\psi : \mathbb{T}^d \to \mathbb{R}$ is strictly positive.

Following the calculations involving the $h-$ transform done in (2.7) and (2.8), we know that $e^{-t\lambda}f(t,x,y)\psi(y)/\psi(x)$ is the fundamental solution to the linear parabolic PDE driven by an elliptic operator without a potential term. Applying Theorem 1.1 from Norris [19] to this operator, there exists $C > 0$ such that for all $x, y \in \mathbb{R}^d$ and $t > 0$,

$$C^{-1}t^{-d/2}e^{\frac{-C\|y-x\|^2}{t}} \le e^{-t\lambda}f(t,x,y+\mathbf{v}t)\psi(y)/\psi(x) \le Ct^{-d/2}e^{\frac{-\|y-x\|^2}{Ct}}.$$

Therefore,

$$f(t,x,y) \le \tilde{C}t^{-d/2}\exp\left(\lambda t - \frac{\|y-\mathbf{v}t-x\|^2}{Ct}\right).$$

Now, it is a simple exercise in algebra to show that, for a fixed $\mathbf{v} \in \mathbb{R}^d$, there exist a $a > 1$, $b > 0$ such that, for all $t \geq 0$ and for all $x \in \mathbb{R}^d$,

$$\frac{\|x - \mathbf{v}t\|^2}{t} \geq \frac{\|x\|^2}{at} - bt.$$

For simplicity, we show the above statement for real numbers ($x, \mathbf{v} \in \mathbb{R}$) instead of vectors ($x, \mathbf{v} \in \mathbb{R}^d$). Therefore, we need to show that

$$\inf_{t \geq 0} \left\{ x^2 + \mathbf{v}^2 t^2 - 2x\mathbf{v}t - \frac{x^2}{a} + bt^2 \right\} \geq 0.$$

That is,

$$\inf_{t \geq 0} \left\{ t^2 \big(\mathbf{v}^2 + b \big) - 2x\mathbf{v}t \right\} \geq x^2 \left[\frac{1}{a} - 1 \right].$$

If $\mathbf{v} < 0$, the infimum is achieved at $t = 0$. and therefore, choosing any $a > 1$, we get the required inequality. If $\mathbf{v} > 0$, the infimum is achieved at $t = \dfrac{x\mathbf{v}}{\mathbf{v}^2 + b}$, and therefore, we need

$$-\frac{\mathbf{v}^2}{(\mathbf{v}^2 + b)} \geq \left[\frac{1}{a} - 1 \right].$$

This is true if we choose $a > 1$ and $b > 0$ such that

$$a \geq 1 + \frac{\mathbf{v}^2}{b}.$$

Thus,

$$f(t, v, y) \leq \tilde{C} t^{-d/2} \exp\left(\lambda t - \frac{\|y - \mathbf{v}t - x\|^2}{Ct} \right) \leq \tilde{C} t^{-d/2} \exp\left((\lambda + \frac{d}{C})t - \frac{\|y - x\|^2}{aCt} \right).$$

Choosing $c = \max\{\tilde{C}, (\lambda + \dfrac{d}{C}), aC\}$ we get the required bound (2.40). □

Since ρ_1 is the fundamental solution to a linear parabolic PDE with periodic coefficients, the conclusion of Lemma 2.3.2 applies. In addition, since the effective drift of the process, $\bar{\mathbf{v}}$, is assumed to be zero, we have

$$\rho_1(t,x,y) \le ct^{-d/2} \exp\left(\mu(0)t - \frac{\|x-y\|^2}{ct}\right). \tag{2.41}$$

By the first equality in (2.39), there exists $a_1 > 0$ such that

$$m_1^y(t,x) \le a_1 \exp\left(a_1 t - \frac{\|y-x\|^2}{a_1(t+1)}\right). \tag{2.42}$$

This proves (i) for $k = 1$. Now suppose that (i) holds up-to $k-1$. From (2.33) and the Duhamel's Formula, we see that

$$m_k^y(t,x) = \int_0^t \int_{\mathbb{R}^d} \alpha(z) \sum_{i=1}^{k-1} \beta_i^k m_i^y(s,z) m_{k-i}^y(s,z) \rho_1(t-s,x,z) dz ds. \tag{2.43}$$

Note that, since (2.35) holds up-to $k-1$, it also holds for $\sum_{i=1}^{k-1} \beta_i^k m_i^y(s,z) m_{k-i}^y(s,z)$ (with a different constant $\tilde{a}_{k-1}$). Thus there exists a constant $a_k > 0$ such that $m_k^y(t,x) \le a_k \exp\left(a_k t - \frac{\|y-x\|^2}{a_k(t+1)}\right)$, since the convolution of two functions satisfying the estimate (2.35), with two different constants also satisfies (2.35). That is, (i) holds for k.

We next show that (ii) holds for $k = 1$ and $k = 2$, and (iii) holds for $k = 2$. From (2.39), we obtain

$$\gamma_1(\mathbf{v}) = \lim_{t\to\infty} \frac{\ln m_1^{t\mathbf{v}}(t,x)}{t} = -\Phi(\mathbf{v}), \tag{2.44}$$

and γ_1 is continuous since Φ is continuous. In addition, from (2.34), for each $t > 0$, $\mathbb{E}(n^{t\mathbf{v}}(t,x)) = m_1^{t\mathbf{v}}(t,x)$. Thus (ii) holds for $k = 1$.

Next we show that, for $k = 2$, the second limit on the right hand side of (2.36) exists and satisfies formula (2.37). In the arguments below, we treat x and $\mathbf{v}$ as fixed, but all the estimates are easily seen to be uniform in $\|\mathbf{v}\| \leq L$ and $x \in [0,1)^d$.

Let us recall that

$$m_2^y(t,x) = \int_0^t \int_{\mathbb{R}^d} 2\alpha(z)(m_1^y(s,z))^2 \rho_1(t-s,x,z)dzds.$$

Let $0 < \varepsilon < \alpha < 1$. Let us define the following

$$A_1(t,x,t\mathbf{v}) := \int_0^{\varepsilon t} \int_{\|z-t\mathbf{v}\| \geq \varepsilon^{1/4} t} 2\alpha(z)(m_1^{\mathbf{v}t}(s,z))^2 \rho_1(t-s,x,z)dzds,$$

$$B_1(t,x,t\mathbf{v}) := \int_0^{\varepsilon t} \int_{\|z-t\mathbf{v}\| \leq \varepsilon^{1/4} t} 2\alpha(z)(m_1^{\mathbf{v}t}(s,z))^2 \rho_1(t-s,x,z)dzds,$$

$$C_1(t,x,t\mathbf{v}) := \int_{\varepsilon t}^{\alpha t} \int_{\|z-\mathbf{v}t\| \geq tM} 2\alpha(z)(m_1^{\mathbf{v}t}(s,z))^2 \rho_1(t-s,x,z)dzds,$$

$$D_1(t,x,t\mathbf{v}) := \int_{\varepsilon t}^{\alpha t} \int_{\|z-\mathbf{v}t\| \leq tM} 2\alpha(z)(m_1^{\mathbf{v}t}(s,z))^2 \rho_1(t-s,x,z)dzds,$$

$$E_1(t,x,t\mathbf{v}) := \int_{\alpha t}^{t} \int_{\|z\| \geq (1-\alpha)^{1/4} t} 2\alpha(z)(m_1^{\mathbf{v}t}(s,z))^2 \rho_1(t-s,x,z)dzds,$$

$$F_1(t,x,t\mathbf{v}) := \int_{\alpha t}^{t} \int_{\|z\| \leq (1-\alpha)^{1/4} t} 2\alpha(z)(m_1^{\mathbf{v}t}(s,z))^2 \rho_1(t-s,x,z)dzds.$$

Note that $m_2^{t\mathbf{v}}(t,x) = (A_1 + B_1 + C_1 + D_1 + E_1 + F_1)(t,x,t\mathbf{v})$.

From (2.42), in the region where $s < \varepsilon t$, $\|\mathbf{v}t - z\| > \varepsilon^{1/4} t$,

$$(m_1^{\mathbf{v}t}(s,z))^2 \leq a_1^2 \exp(2(a_1\varepsilon - \frac{\sqrt{\varepsilon}}{a_1(\varepsilon + \frac{1}{t})})t),$$

which can be made exponentially small (as $t \to \infty$), with an arbitrarily large negative exponent, by choosing ε small enough. Using estimate on ρ_1 from (2.41), for each $r > 0$, for all sufficiently small $\varepsilon > 0$,

$$\limsup_{t\to\infty} \frac{\ln A_1(t,x,t\mathbf{v})}{t} \leq -r.$$

Similarly, by exchanging the roles of $(m_1^{t\mathbf{v}}(s,z))^2$ and $\rho_1(t-s,z,x)$, we obtain for each $r>0$, for all $\alpha \in (0,1)$ sufficiently close to 1,

$$\limsup_{t\to\infty} \frac{\ln E_1(t,x,t\mathbf{v})}{t} \le -r.$$

For each $s < \varepsilon t, \|\mathbf{v}t - z\| < \varepsilon^{1/4} t$, using (2.42), we conclude that there exists a $C_1 > 0$ such that

$$(m_1^{\mathbf{v}t}(s,z))^2 \le C_1 e^{C_1 \varepsilon t}.$$

By Theorem 2.2.2, in the region $s < \varepsilon t, \|\mathbf{v}t - z\| < \varepsilon^{1/4} t$, there exists $C_2 > 0$ such that

$$\rho_1(t-s,x,z) \le C_2 (t-s)^{-d/2} e^{-(t-s)\Phi(\frac{z-x}{t-s})}.$$

By choosing $\varepsilon > 0$ small enough, and choosing sufficiently large t, the value of $-\Phi(\dfrac{z-x}{t-s})$ in this region can be made arbitrarily close to $\gamma_1(\mathbf{v})$. Thus, for each $\delta > 0$, for all sufficiently small $\varepsilon > 0$,

$$\limsup_{t\to\infty} \frac{\ln B_1(t,x,t\mathbf{v})}{t} \le \gamma_1(\mathbf{v}) + \delta.$$

Similarly, by exchanging the roles of $(m_1^{t\mathbf{v}}(s,z))^2$ and $\rho_1(t-s,z,x)$, we obtain for each $\delta > 0$, for all $\alpha \in (0,1)$ sufficiently close to 1,

$$\limsup_{t\to\infty} \frac{\ln F_1(t,x,t\mathbf{v})}{t} \le 2\gamma_1(\mathbf{v}) + \delta.$$

Now let us assume $1 > \alpha > \varepsilon > 0$ are fixed. From (2.41), (2.42), it follows that, given $r > 0$, we can choose M large enough such that

$$\limsup_{t\to\infty} \frac{\ln C_1(t,x,t\mathbf{v})}{t} < -r.$$

Let us now examnine the asymptotics of $D_1(t,x,t\mathbf{v})$. Observe that the volume of the region $(\varepsilon t, \alpha t)\times B_{Mt}(t\mathbf{v})\subset\mathbb{R}^{d+1}$ grows polynomially in t. The uniform asymptotics of the logarithm of the integrand in $D_1(t,x,t\mathbf{v})$ is available in (2.44) and Theorem 2.2.2. Observe that $\tilde{c}$ is periodic, non-negative and not identically 0. Therefore, substituting $z/t=w$ and $s/t=u$, we get

$$\lim_{t\to\infty}\frac{\ln D_1(t,x,t\mathbf{v})}{t}=\sup_{\|w-\mathbf{v}\|\le M, u\in(\varepsilon,\alpha)}\Big[2u\gamma_1\Big(\frac{\mathbf{v}-w}{u}\Big)+(1-u)\gamma_1\Big(\frac{w}{1-u}\Big)\Big].$$

Combining this with the estimates on $\ln A_1, \ln B_1, \ln C_1, \ln E_1, \ln F_1$, since $\delta>0$ was arbitrarily small, and since ε and α are arbitrarily close to 0 and 1 respectively, we conclude that the expression on the right hand side of (2.37)(with $k=2$) captures the logarithmic asymptotics of $m_2^y(t,x)$. Thus, we obtain that the second limit on the right hand side of (2.36) exists and that formula (2.37) holds for $k=2$, that is,

$$\lim_{t\to\infty}\frac{\ln(m_2^{t\mathbf{v}}(t,x))}{t}=\sup_{\|w-\mathbf{v}\|\le M, u\in(0,1)}\Big[2u\gamma_1\Big(\frac{\mathbf{v}-w}{u}\Big)+(1-u)\gamma_1\Big(\frac{w}{1-u}\Big)\Big].$$

We define

$$\gamma_2(\mathbf{v}):=\lim_{t\to\infty}\frac{\ln(m_2^{t\mathbf{v}}(t,x))}{t} \tag{2.45}$$

Next we show that $\gamma_2(\mathbf{v})\ge\gamma_1(\mathbf{v})$ for all $\mathbf{v}\in\mathbb{R}^d$.

Recall that $m_1^y(t,x)$ and $m_2^y(t,x)$ solve the following PDEs:

$$\partial_t m_1^y(t,x)=\mathcal{L}_x m_1^y(t,x),\qquad m_1^y(0,x)=\chi_{\mathbb{T}_y^d}(x), \tag{2.46}$$

$$\partial_t m_2^y(t,x)=\mathcal{L}_x m_2^y(t,x)+\alpha(x)(m_1^y(t,x))^2,\qquad m_2^y(0,x)\equiv 0. \tag{2.47}$$

We will show that there exists a $C_L>0$ such that, for each $t\ge 1$ and $x,y\in\mathbb{R}^d$ with $\|x-y\|\le Lt$, we have

$$m_2^y(t,x)\ge C_L m_1^y(t,x).$$

Fix $R > 0$ such that $[0,1)^d \in B_R(0)$. Observe that, since $m_1^y(0,x) = \chi_{\mathbb{T}_y^d}(x)$, there exists a $\delta_1 > 0$ such that

$$m_1^y(t,x) \geq \delta_1 \chi_{B_R(y)}(x) \qquad \text{for all} \qquad t \in [1/8, 1/4].$$

Also observe that there exists a $\delta_2 > 0$ such that, for all $x, y \in \mathbb{R}^d$ with $\|x-y\| \leq 2R$ and $t \in [1/4, 1/2]$,

$$\rho_1(t,x,y) \geq \delta_2.$$

Now, observe that $\tilde{c}$ is periodic, non-negative and not identically 0. Thus, from (2.47), using Duhamel's Formula, for $x \in \mathbb{T}_y^d$,

$$\begin{aligned} m_2^y(1/2,x) &= \int_0^{1/2} \int_{\mathbb{R}^d} 2\alpha(z)(m_1^y(s,z))^2 \rho_1(\frac{1}{2} - s, x, z) dz ds \\ &\geq \int_{1/8}^{1/4} \int_{B_R(y)} 2\alpha(z)\delta_1^2 \delta_2 dz ds \\ &\geq \frac{1}{4}\delta_1^2\delta_2 \int_{[0,1)^d} \alpha(z) dz := \delta_3 > 0, \end{aligned}$$

that is,

$$m_2^y(1/2,x) \geq \delta_3 \chi_{\mathbb{T}_y^d}(x). \tag{2.48}$$

Now, comparing the PDEs (2.46) and (2.47), and taking into account (2.48), we see that for all $t \geq 0$, $x, y \in \mathbb{R}^d$,

$$m_2^y(t + 1/2, x) \geq \delta_3 m_1^y(t,x). \tag{2.49}$$

For a fixed $L > 0$, for all $x, y \in \mathbb{R}^d$ with $\dfrac{\|x-y\|}{t} \leq L$, $t \geq 1/2$, from Theorem 2.2.2, there exists $c > 0$ such that

$$m_1^y(t,x) \geq c m_1^y(t + 1/2, x). \tag{2.50}$$

From (2.50) and (2.49), we conclude that there exists a constant $C_L > 0$ such that

$$m_2^y(t,x) \geq C_L m_1^y(t,x), \tag{2.51}$$

for all $x, y \in \mathbb{R}^d$ with $\dfrac{\|x-y\|}{t} \leq L$ and $t \geq 1$. $\square$

In particular, for each $\mathbf{v} \in \mathbb{R}^d$, we have

$$\begin{aligned}
\gamma_1(\mathbf{v}) &= \lim_{t\to\infty} \frac{\ln \mathbb{E}(n^{t\mathbf{v}}(t,x))}{t} = \lim_{t\to\infty} \frac{\ln m_1^{t\mathbf{v}}(t,x)}{t} \\
&\leq \lim_{t\to\infty} \frac{\ln m_2^{\mathbf{v}}(t,x)}{t} = \lim_{t\to\infty} \frac{\ln(m_1^{t\mathbf{v}}(t,x) + 2m_2^{t\mathbf{v}}(t,x))}{t} \\
&= \lim_{t\to\infty} \frac{\ln \mathbb{E}(n^{t\mathbf{v}}(t,x)^2)}{t} = \gamma_2(\mathbf{v}),
\end{aligned}$$

where the last equality follows from the definition fo γ_2 (formula (2.45)). Thus (ii) and (iii) hold for $k = 2$. This completes the basis for induction.

Now suppose that (ii) and (iii) hold up to $k-1$ with $k \geq 2$. From (2.43), there exists a constant $C_1 > 0$ such that

$$m_k^y(t,x) \geq C_1 \int_0^t \int_{\mathbb{R}^d} \alpha(z) m_1^y(s,z) m_{k-1}^y(s,z) \rho_1(t-s,x,z)\,dz\,ds =: C_1 I_1(t,x,y).$$

Since $\mathbb{E}(n^{t\mathbf{v}}(t,x)^k)$ is a convex function of k, for each $1 \leq i \leq k-1$,

$$\mathbb{E}(n^{t\mathbf{v}}(t,x)^{k-1})\mathbb{E}(n^{t\mathbf{v}}(t,x)) \geq \mathbb{E}(n^{t\mathbf{v}}(t,x)^{k-i})\mathbb{E}(n^{t\mathbf{v}}(t,x)^i).$$

Thus, using (2.34), there exists a constant $C_2 > 0$ such that,

$$\begin{aligned}
m_k^y(t,x) &\leq C_2 \int_0^t \int_{\mathbb{R}^d} \alpha(z) \mathbb{E}(n^{t\mathbf{v}}(t,x)^{k-1})\mathbb{E}(n^{t\mathbf{v}}(t,x)) \rho_1(t-s,x,z)\,dz\,ds \\
&=: C_2 I_2(t,x,y).
\end{aligned}$$

In order to prove that the second limit on the right hand side of (2.36) exists, we need to show that,

$$\lim_{t\to\infty}\frac{\ln I_1(t,x,t\mathbf{v})}{t}=\lim_{t\to\infty}\frac{\ln I_2(t,x,t\mathbf{v})}{t}. \tag{2.52}$$

We claim that, for all sufficiently large $M>0$,

$$\gamma_k(\mathbf{v}):=\lim_{t\to\infty}\frac{\ln(I_1(t,x,t\mathbf{v}))}{t}=\lim_{t\to\infty}\frac{\ln(I_2(t,x,t\mathbf{v}))}{t}=\lim_{t\to\infty}\frac{m_k^{t\mathbf{v}}(t,x)}{t}$$
$$=\sup_{\|w-\mathbf{v}\|\le M,u\in(0,1)}\left[u\gamma_{k-1}\Big(\frac{\mathbf{v}-w}{u}\Big)+u\gamma_1\Big(\frac{\mathbf{v}-w}{u}\Big)+(1-u)\gamma_1\Big(\frac{w}{1-u}\Big)\right]. \tag{2.53}$$

As before, let $0<\varepsilon<\alpha<1$. Let us define the following

$$A(t,x,t\mathbf{v}):=\int_0^{\varepsilon t}\int_{\|z-t\mathbf{v}\|\ge\varepsilon^{1/4}t}\alpha(z)m_1^{\mathbf{v}t}(s,z)m_{k-1}^{\mathbf{v}t}(s,z)\rho_1(t-s,x,z)dzds,$$

$$B(t,x,t\mathbf{v}):=\int_0^{\varepsilon t}\int_{\|z-t\mathbf{v}\|\le\varepsilon^{1/4}t}\alpha(z)m_1^{\mathbf{v}t}(s,z)m_{k-1}^{\mathbf{v}t}(s,z)\rho_1(t-s,x,z)dzds,$$

$$C(t,x,t\mathbf{v}):=\int_{\varepsilon t}^{\alpha t}\int_{\|z-\mathbf{v}t\|\ge tM}\alpha(z)m_1^{\mathbf{v}t}(s,z)m_{k-1}^{\mathbf{v}t}(s,z)\rho_1(t-s,x,z)dzds,$$

$$D(t,x,t\mathbf{v}):=\int_{\varepsilon t}^{\alpha t}\int_{\|z-\mathbf{v}t\|\le tM}\alpha(z)m_1^{\mathbf{v}t}(s,z)m_{k-1}^{\mathbf{v}t}(s,z)\rho_1(t-s,x,z)dzds,$$

$$E(t,x,t\mathbf{v}):=\int_{\alpha t}^{t}\int_{\|z\|\ge(1-\alpha)^{1/4}t}\alpha(z)m_1^{\mathbf{v}t}(s,z)m_{k-1}^{\mathbf{v}t}(s,z)\rho_1(t-s,x,z)dzds,$$

$$F(t,x,t\mathbf{v}):=\int_{\alpha t}^{t}\int_{\|z\|\le(1-\alpha)^{1/4}t}\alpha(z)m_1^{\mathbf{v}t}(s,z)m_{k-1}^{\mathbf{v}t}(s,z)\rho_1(t-s,x,z)dzds.$$

Note that $I_1(t,x,t\mathbf{v})=(A+B+C+D+E+F)(t,x,t\mathbf{v})$.

Using the same arguments as above, it is not difficult to show that, for each $r>0$, for each $\delta>0$, for all sufficiently small $\varepsilon>0$, for all $\alpha\in(0,1)$ sufficiently close to 1, for all sufficiently large M,

$$\limsup_{t\to\infty}\frac{\ln A(t,x,t\mathbf{v})}{t}\le -r,$$

$$\limsup_{t\to\infty} \frac{\ln E(t,x,t\mathbf{v})}{t} \le -r,$$

$$\limsup_{t\to\infty} \frac{\ln B(t,x,t\mathbf{v})}{t} \le \gamma_1(\mathbf{v}) + \delta,$$

$$\limsup_{t\to\infty} \frac{\ln F(t,x,t\mathbf{v})}{t} \le \gamma_1(\mathbf{v}) + \gamma_{k-1}(\mathbf{v}) + \delta,$$

$$\limsup_{t\to\infty} \frac{\ln C(t,x,t\mathbf{v})}{t} < -r,$$

and

$$\lim_{t\to\infty} \frac{\ln D(t,x,t\mathbf{v})}{t} = \sup_{\|w-\mathbf{v}\|\le M, u\in(\varepsilon,\alpha)} \Big[u\gamma_1\Big(\frac{\mathbf{v}-w}{u}\Big)+u\gamma_{k-1}\Big(\frac{\mathbf{v}-w}{u}\Big)+(1-u)\gamma_1\Big(\frac{w}{1-u}\Big)\Big].$$

Thus, we have proved that

$$\lim_{t\to\infty} \frac{\ln(I_1(t,0,t\mathbf{v}))}{t} =$$

$$= \sup_{\|w-\mathbf{v}\|\le M, u\in(0,1)} \Big[u\gamma_{k-1}\Big(\frac{\mathbf{v}-w}{u}\Big) + u\gamma_1\Big(\frac{\mathbf{v}-w}{u}\Big) + (1-u)\gamma_1\Big(\frac{w}{1-u}\Big)\Big]. \quad (2.54)$$

Now, we justify (2.52), that is, the logarithmic asymptotics of the integrals I_1 and I_2 are equal. The difference between I_1 and I_2 is that $\mathbb{E}(n^{t\mathbf{v}}(t,x)^i)$ in I_2 replaces $m_i^{t\mathbf{v}}(t,x)$ in I_1. The properties of $m_i^{t\mathbf{v}}(t,x)$ that were used to derive the asymptotics of I_1 included estimate (2.35) and the uniform asymptotics of the logarithm (formula (2.36)). By the inductive assumption, the same uniform asymptotics holds for $\mathbb{E}(n^{t\mathbf{v}}(t,x)^i)$ for $i \le k-1$. Moreover, by formula (2.34), the analogue of (2.35) holds for $\mathbb{E}(n^{t\mathbf{v}}(t,x)^i)$. That is, there exist constants $d_i > 0$ such that

$$\mathbb{E}(n^{t\mathbf{v}}(t,x)^i) \le d_i \exp\left(d_i t - \frac{\|y-x\|^2}{d_i(t+1)}\right) \quad (2.55)$$

for all $(t,x,y) \in \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d$, for all $1 \le i \le k-1$. Therefore, the logarithmic asymptotics of I_2 are the same as that of I_1 i.e., (2.52) holds. From (2.34),

$$\liminf_{t\to\infty} \frac{\ln \mathbb{E}(n^{t\mathbf{v}}(t,x)^k)}{t} \ge \lim_{t\to\infty} \frac{\ln m_k^{t\mathbf{v}}(t,x)}{t}.$$

From the formula (2.53) which now holds for k and $k-1$ and the inductive hypothesis that $\gamma_{k-1}(v) \ge \gamma_{k-2}(v)$ for each $v \in \mathbb{R}^d$, we observe that that

$$\gamma_k(\mathbf{v}) = \sup_{\|w-\mathbf{v}\|\le M, u\in(0,1)} \Big[u\gamma_{k-1}\Big(\frac{\mathbf{v}-w}{u}\Big) + u\gamma_1\Big(\frac{\mathbf{v}-w}{u}\Big) + (1-u)\gamma_1\Big(\frac{w}{1-u}\Big)\Big]$$

$$\ge \sup_{\|w-\mathbf{v}\|\le M, u\in(0,1)} \Big[u\gamma_{k-2}\Big(\frac{\mathbf{v}-w}{u}\Big) + u\gamma_1\Big(\frac{\mathbf{v}-w}{u}\Big) + (1-u)\gamma_1\Big(\frac{w}{1-u}\Big)\Big] = \gamma_{k-1}(\mathbf{v}).$$

This, along with the inductive hypothesis that $\gamma_{i-1}(\mathbf{v}) \le \gamma_i(\mathbf{v})$ for each $2 \le i \le k-1$, by (2.34), implies that

$$\limsup_{t\to\infty} \frac{\ln \mathbb{E}(n^{t\mathbf{v}}(t,x)^k)}{t} \le \lim_{t\to\infty} \frac{\ln m_k^{t\mathbf{v}}(t,x)}{t}.$$

Therefore both the limits in (2.36) exist and are equal.

Form the inductive assumption that γ_i is a continuous function for $1 \le i \le k-1$, using formula (2.54), we conclude that γ_k is continuous. This concludes the proof of (i)-(iii) through induction.

Part 2 of the Proof:

In Part 1 of the proof, we have shown that, for all sufficiently large $M > 0$,

$$\gamma_k(\mathbf{v}) = \sup_{\|w-\mathbf{v}\|\le M, u\in(0,1)} \Big[u\gamma_{k-1}\Big(\frac{\mathbf{v}-w}{u}\Big) + u\gamma_1\Big(\frac{\mathbf{v}-w}{u}\Big) + (1-u)\gamma_1\Big(\frac{w}{1-u}\Big)\Big].$$

Therefore, in the rest of the proof, we will use the following formula for γ_k, $k \in \mathbb{N}$:

$$\gamma_k(\mathbf{v}) = \sup_{w\in\mathbb{R}^d, u\in(0,1)} \Big[u\gamma_{k-1}\Big(\frac{\mathbf{v}-w}{u}\Big) + u\gamma_1\Big(\frac{\mathbf{v}-w}{u}\Big) + (1-u)\gamma_1\Big(\frac{w}{1-u}\Big)\Big].$$

Observe that, for each $k \in \mathbb{N}$, $\mathbf{v} \in \mathbb{R}^d$, $\gamma_k(\mathbf{v}) \leq k\gamma_1(0)$. To justify this, we use induction. For $k = 1$, the statement is obvious since γ_1 achieves its maximum at 0. Now suppose the statement holds up to $k-1$. Then, from the definition of γ_k,

$$\sup_{w \in \mathbb{R}^d, u \in (0,1)} \Big[u\gamma_{k-1}\Big(\frac{\mathbf{v}-w}{u}\Big) + u\gamma_1\Big(\frac{\mathbf{v}-w}{u}\Big) + (1-u)\gamma_1\Big(\frac{w}{1-u}\Big)\Big] \leq \tag{2.56}$$

$$\leq \Big[u(k-1)\gamma_1(0) + u\gamma_1(0) + (1-u)\gamma_1(0)\Big] = k\gamma_1(0).$$

We know that $-\Phi(0) = \gamma_1(0) = \mu(0) > 0$, and Φ is continuous, therefore, the region G_1 is non-empty. As a consequence of Remark 2.3.1, Part (2), we have that $G_{k+1} \subseteq G_k$ for each $k \geq 1$. We know that γ_k is continuous for each $k \geq 1$. Therefore, from the definition of G_k, these are closed subsets of $\mathbb{R}^d$.

Next let us show that each set G_k contains a small ball centered at the origin. As a first step, the following lemma establishes an important property of the functions γ_k.

Lemma 2.3.3. *For each $k \geq 1$, $\mathbf{v} \in \mathbb{R}^d$ and $\alpha \in [0,1]$, $\gamma_k(\mathbf{v}) \leq \gamma_k(\alpha\mathbf{v})$.*

Proof. We use induction for this proof. For $k = 1$, the statement of the lemma holds since $\gamma_1(\mathbf{v})$ is a twice differentiable strictly concave function and $\bar{\mathbf{v}} = 0$ is its maximizer.

Suppose the statement of the lemma holds for each $1 \leq i \leq k-1$. To show this for k, we have, for $0 \leq \alpha \leq 1$,

$$\gamma_k(\alpha\mathbf{v}) = \sup_{w \in \mathbb{R}^d, u \in (0,1)} \Big[u\gamma_{k-1}\Big(\frac{\alpha\mathbf{v}-w}{u}\Big) + u\gamma_1\Big(\frac{\alpha\mathbf{v}-w}{u}\Big) + (1-u)\gamma_1\Big(\frac{w}{1-u}\Big)\Big].$$

Now, substituting $w = \alpha z$, we have

$$\gamma_k(\alpha\mathbf{v}) = \sup_{z \in \mathbb{R}^d, u \in (0,1)} \Big[u\gamma_{k-1}\Big(\frac{\alpha\mathbf{v}-\alpha z}{u}\Big) + u\gamma_1\Big(\frac{\alpha\mathbf{v}-\alpha z}{u}\Big) + (1-u)\gamma_1\Big(\frac{\alpha z}{1-u}\Big)\Big]$$

$$\geq \sup_{z\in\mathbb{R}^d, u\in(0,1)} \Big[u\gamma_{k-1}\Big(\frac{\mathbf{v}-z}{u}\Big) + u\gamma_1\Big(\frac{\mathbf{v}-z}{u}\Big) + (1-u)\gamma_1\Big(\frac{z}{1-u}\Big)\Big] = \gamma_k(\mathbf{v}).$$

□

Now, in order to prove that each set $G_k = \{\mathbf{v} \in \mathbb{R}^d \big| \gamma_k(\mathbf{v}) = k\gamma_1(\mathbf{v}), \gamma_1(\mathbf{v}) \geq 0\}$ contains a small ball centered at the origin, we introduce functions f_k defined below. For each $k \geq 2$, we will first show that there is a small ball centered around the origin on which $f_k(\mathbf{v}) = k\gamma_1(\mathbf{v}) \geq 0$. Then we will use induction to show that there is a (smaller) ball centered around the origin on which $f_k(\mathbf{v}) = \gamma_k(\mathbf{v})$.

Let us define, for $k \geq 2$,

$$g_k^{\mathbf{v}}(w,u) := \Big[ku\gamma_1(\frac{\mathbf{v}-w}{u}) + (1-u)\gamma_1(\frac{w}{1-u})\Big], \quad w \in \mathbb{R}^d, \quad u \in (0,1),$$

and

$$f_k(\mathbf{v}) := \sup_{w\in\mathbb{R}^d, u\in(0,1)} g_k^{\mathbf{v}}(w,u). \tag{2.57}$$

Observe that $f_2 = \gamma_2$. For $k > 2$, the formula for function f_k is similar to the formula of γ_k, but with $(\gamma_{k-1} + \gamma_1)$ replaced by $k\gamma_1$. For $w = \mathbf{v}(1-u)$, we have

$$g_k^{\mathbf{v}}(\mathbf{v}(1-u),u) = ku\gamma_1(\mathbf{v}) + (1-u)\gamma_1(\mathbf{v}) = (1+(k-1)u)\gamma_1(\mathbf{v}) \to k\gamma_1(\mathbf{v}) \quad \text{as} \quad u \uparrow 1.$$

Therefore, $f_k(\mathbf{v}) \geq k\gamma_1(\mathbf{v})$ for each $\mathbf{v} \in \mathbb{R}^d$.

The analysis of $g_k^{\mathbf{v}}(w,u)$ is detailed in the following three lemmas. They show that, for each $k \geq 2$, there is a small ball centered around the origin $B_{\beta_k}(0)$, such that, for $\mathbf{v} \in B_{\beta_k}(0)$, the value of the supremum of $g_k^{\mathbf{v}}(w,u)$ on $\mathbb{R}^d \times (0,1)$ is $k\gamma_1(\mathbf{v})$ which, as shown above, can be nearly achieved when w is close to 0 and u is close to 1.

The first of the three lemmas, Lemma 2.3.4, shows that the value of the supremum of $g_k^{\mathbf{v}}(w,u)$ over the region where w is bounded and u is close to 1 is $k\gamma_1(\mathbf{v})$.

Lemma 2.3.4. *There exist constants $L > 0$, $\delta, \varepsilon_0 > 0$ such that for all $\mathbf{v} \in B_\delta(0)$,*

$$\sup\{g_k^{\mathbf{v}}(w,u)\Big|\|w\| \leq L, u \in (1-\varepsilon_0, 1)\} = k\gamma_1(\mathbf{v}).$$

Proof. We prove the above lemma in 2 steps. In Step I, we show that there exist $\delta_1 > 0$, $M > 0$ and $\varepsilon_1 > 0$ such that, for each $\mathbf{v} \in B_{\delta_1}(0)$, for each $(w,u) = (\ell\varepsilon, 1-\varepsilon)$, with $L/\varepsilon > \|\ell\| > M$, and $\varepsilon \in (0, \varepsilon_1)$, we have $g_k^{\mathbf{v}}(\ell\varepsilon, 1-\varepsilon) < k\gamma_1(\mathbf{v})$.

In Step II, we show that there exist constants $\delta < \delta_1, \varepsilon_0 \in (0, \varepsilon_1)$ such that for all $\mathbf{v} \in B_\delta(0)$, for all $\|\ell\| \leq M$,

$$\frac{d}{d\varepsilon}\Big[g_k^{\mathbf{v}}(\ell\varepsilon, 1-\varepsilon)\Big] \leq 0$$

for all $\varepsilon < \varepsilon_0$.

Step I: Note that from Lemma 2.3.3,

$$g_k^{\mathbf{v}}(\ell\varepsilon, 1-\varepsilon) = k(1-\varepsilon)\gamma_1\big(\frac{\mathbf{v}-\ell\varepsilon}{1-\varepsilon}\big) + \varepsilon\gamma_1(\ell) \leq k\gamma_1\big(\mathbf{v}-\ell\varepsilon\big) + \varepsilon\gamma_1(\ell),$$

where $L > 0$ and $\delta_1 > 0$ are such that $\gamma_1(\mathbf{v} - \ell\varepsilon) > 0$ for all $\mathbf{v} \in B_{\delta_1}(0)$, and $\|\ell\varepsilon\| \leq L$. Thus, in order to prove that $g_k^{\mathbf{v}}(\ell\varepsilon, 1-\varepsilon) < k\gamma_1(\mathbf{v})$, it is enough to show that

$$\frac{\varepsilon}{k}\gamma_1(\ell) \leq \gamma_1(\mathbf{v}) - \gamma_1\big(\mathbf{v} - \ell\varepsilon\big).$$

From (2.42), we know that, for all $v \in \mathbb{R}^d$,

$$\gamma_1(v) \leq a_1 - \frac{\|v\|^2}{a_1}.$$

Therefore, we only need to show that

$$\frac{\varepsilon}{k}(a_1 - \frac{\|\ell\|^2}{a_1}) \leq \gamma_1(\mathbf{v}) - \gamma_1\big(\mathbf{v} - \ell\varepsilon\big).$$

Using Taylor's formula, we have

$$\gamma_1(\mathbf{v}) - \gamma_1\big(\mathbf{v} - \ell\varepsilon\big) = \varepsilon\langle\nabla\gamma_1(\mathbf{v}), \ell\rangle - \frac{\varepsilon^2}{2}\langle D^2\gamma_1(\mathbf{v} - q\ell\varepsilon)\ell, \ell\rangle$$

for some $q \in (0,1)$. Thus, we need to show that

$$\frac{1}{k}(a_1 - \frac{\|\ell\|^2}{a_1}) \leq \langle\nabla\gamma_1(\mathbf{v}), \ell\rangle - \frac{\varepsilon}{2}\langle D^2\gamma_1(\mathbf{v} - q\ell\varepsilon)\ell, \ell\rangle.$$

That is, we need to show that

$$\frac{1}{k}(a_1 - \frac{\|\ell\|^2}{a_1}) + \frac{\varepsilon}{2}\langle D^2\gamma_1(\mathbf{v} - q\ell\varepsilon)\ell, \ell\rangle - \langle\nabla\gamma_1(\mathbf{v}), \ell\rangle \leq 0. \tag{2.58}$$

Let $\mathbf{v} \in B_{\delta_1}(0)$. Let $C = \sup\{\|\partial_i\gamma_1(\mathbf{v})\| \Big| \mathbf{v} \in B_{\delta_1}(0)\}$. Then we have the following lower bound,

$$\langle\nabla\gamma_1(\mathbf{v}), \ell\rangle \geq -C\|\ell\|.$$

Let us fix $M \geq 1$ such that the following quadratic expression is positive, that is,

$$\frac{x^2}{2ka_1} - Cx - \frac{a_1}{k} \geq 0 \quad \text{for all } \ \|x\| \geq M.$$

For each $\mathbf{v} \in B_{\delta_1}(0)$, $\|\ell\varepsilon\| \leq L$, $q \in (0,1)$ we have $(\mathbf{v} - \varepsilon q\ell) \in B_{\delta_1+L}(0)$. Let

$$R = \sup\Big\{\|\partial_{i,j}\gamma_1(v)\| \Big| v \in B_{(\delta_1+L)}(0), 1 \leq i, j \leq d\Big\}.$$

This is a finite constant since the function γ_1 is twice continuously differentiable. Choose $\varepsilon_1 > 0$ such that $\varepsilon_1 R < \frac{1}{2a_1k}$. Then, for all $L/\varepsilon \geq \|\ell\| \geq M$ and $\varepsilon < \varepsilon_1$, $\mathbf{v} \in B_{\delta_1}(0)$,

$$\frac{\|\ell\|^2}{a_1k} + \langle\nabla\gamma_1(\mathbf{v}), \ell\rangle - \frac{a_1}{k} - \frac{\varepsilon}{2}\langle D^2\gamma_1(\mathbf{v} - q\ell\varepsilon)\ell, \ell\rangle \geq$$

$$\geq \frac{\|\ell\|^2}{2a_1k} - \frac{\varepsilon}{2}\langle D^2\gamma_1(\mathbf{v} - q\ell\varepsilon)\ell, \ell\rangle \geq \varepsilon_1 R\|\ell\|^2 - \frac{\varepsilon}{2}\langle D^2\gamma_1(\mathbf{v} - q\ell\varepsilon)\ell, \ell\rangle \geq 0,$$

which proves (2.58).

Step II: Recall that

$$g_k^{\mathbf{v}}(\ell\varepsilon, 1-\varepsilon) = k(1-\varepsilon)\gamma_1\big(\frac{\mathbf{v} - \ell\varepsilon}{1-\varepsilon}\big) + \varepsilon\gamma_1(\ell).$$

Differentiating with respect to ε we obtain,

$$\frac{d}{d\varepsilon}\Big[g_k^{\mathbf{v}}(\ell\varepsilon, 1-\varepsilon)\Big] = -k\gamma_1\big(\frac{\mathbf{v} - \ell\varepsilon}{1-\varepsilon}\big) + \frac{k(\mathbf{v} - \ell)}{1-\varepsilon}\nabla\gamma_1\big(\frac{\mathbf{v} - \ell\varepsilon}{1-\varepsilon}\big) + \gamma_1(\ell).$$

Using the fact that the maximum of the function $\gamma_1(\mathbf{v})$ is achieved at $\mathbf{v} = 0$ and the fact that $\gamma_1(\mathbf{v})$ is strictly concave, choose $\delta_2 \in (0, \delta_1)$ be such that

$$\min\{\gamma_1(\mathbf{v})\Big|\mathbf{v} \in B_{\delta_2}(0)\} > \frac{7}{4k}\gamma_1(0).$$

Let $\varepsilon_2 \in (0, \varepsilon_1)$ be such that for each $\mathbf{v} \in B_{\frac{\delta_2}{2}}(0)$, for all $\varepsilon < \varepsilon_2$, and $\|\ell\| \leq M$, the vector $\big(\frac{\mathbf{v} - \ell\varepsilon}{1-\varepsilon}\big)$ belongs to the $B_{\delta_2}(0)$.

Now choose $\delta \in (0, \delta_2/2)$ such that, for each $\mathbf{v} \in B_\delta(0)$, we have

$$k\langle(\mathbf{v} - l), \nabla\gamma_1(\mathbf{v})\rangle < \frac{1}{8}\gamma_1(0),$$

for all $\|\ell\| \leq M$. This is possible since γ_1 achieves its maximum at 0, that is $\nabla\gamma_1(0) = 0$. Choose $\varepsilon_0 > 0$ with $\varepsilon_0 < (0, \varepsilon_2)$ such that for all $\mathbf{v} \in B_\delta(0)$ and for all $\|\ell\| \leq M$, we have

$$k\langle\frac{(\mathbf{v} - \ell)}{1-\varepsilon_0}, \nabla\gamma_1\big(\frac{\mathbf{v} - \ell\varepsilon_0}{1-\varepsilon_0}\big)\rangle < \frac{1}{4}\gamma_1(0).$$

Thus, for all $\mathbf{v} \in B_\delta(0)$, for all $\varepsilon < \varepsilon_0$ and for all $\|\ell\| \leq M$,

$$\frac{d}{d\varepsilon}\Big[g_k^{\mathbf{v}}(\ell\varepsilon, 1-\varepsilon)\Big] < -\frac{7}{4}\gamma_1(0) + \frac{1}{4}\gamma_1(0) + \gamma_1(0) = -\frac{1}{2}\gamma_1(0) < 0.$$

Thus, we conclude that, for all $\mathbf{v} \in B_\delta(0)$,

$$\sup\{g_k^{\mathbf{v}}(w,u)\Big| \|w\| \le L, u \in (1-\varepsilon_0, 1)\} \le k\gamma_1(\mathbf{v}).$$

But we know that, if $(w,u) = (\mathbf{v}(1-u), u)$ and u approaches 1, the value of $g_k^{\mathbf{v}}(w,u)$ approaches $k\gamma_1(\mathbf{v})$. Therefore,

$$\sup\{g_k^{\mathbf{v}}(w,u)\Big| \|w\| \le L, u \in (1-\varepsilon_0, 1)\} = k\gamma_1(\mathbf{v}).$$

□

The next lemma shows that the supremum of g cannot be achieved if u is close to 1, and w is separated from the origin.

Lemma 2.3.5. *For each $L > 0$, there exist $\delta > 0$ and $\varepsilon_0 > 0$ such that, for all $\mathbf{v} \in B_\delta(0)$,*

$$\sup\{g_k^{\mathbf{v}}(w,u)\Big| 1-\varepsilon_0 < u \le 1, \|w\| \ge L\} < k\gamma_1(\mathbf{v}).$$

Proof. Note that,

$$\|\frac{\mathbf{v}-w}{u}\| \ge \frac{L}{2} > 0 \ \text{ for each } \ \|w\| \ge L, \quad \|\mathbf{v}\| \le L/2, \quad u \in [1/2, 1).$$

Take $\alpha \in (0,1)$ such that $\gamma_1(\ell) \le \alpha\gamma_1(0)$ for all $\|\ell\| \ge L/2$. Here, we used the fact that the maximum of the function γ_1 is achieved at $\mathbf{v} = 0$ and $\gamma_1(\mathbf{v})$ is continuous.

Choose an $\varepsilon_0 < 1/2$ such that $\alpha + \dfrac{\varepsilon_0}{k} < 1$. Thus,

$$g_k^{\mathbf{v}}(w,u) = ku\gamma_1\big(\frac{\mathbf{v}-w}{u}\big) + (1-u)\gamma_1\big(\frac{w}{1-u}\big) \le (k\alpha + \varepsilon_0)\gamma_1(0),$$

for all $\mathbf{v} \in B_{L/2}(0)$, $\|w\| > L, u \in [1-\varepsilon_0, 1)$. Now we choose a $\delta > 0$ with $\delta < L/2$ such that, for all $\mathbf{v} \in B_\delta(0)$, we have

$$\gamma_1(\mathbf{v}) > (\alpha + \frac{\varepsilon_0}{k})\gamma_1(0).$$

We can choose such a $\delta > 0$ since $1 > (\alpha + \frac{\varepsilon_0}{2}) > 0$, the maximum of the function $\gamma_1(\mathbf{v})$ is achieved at $\mathbf{v} = 0$, and $\gamma_1(\mathbf{v})$ is continuous.

Thus, for all $\mathbf{v} \in B_\delta(0)$, we have

$$\sup\{g_k^{\mathbf{v}}(w,u)\Big| 1 - \varepsilon_0 < u \leq 1, \|w\| \geq L\} < k\gamma_1(\mathbf{v}).$$

□

The last of the three lemmas, Lemma 2.3.6, shows that there is a small ball centered around the origin, on which the value of the supremum of $g_k^{\mathbf{v}}(w,u)$ in the region where $w \in \mathbb{R}^d$ and u is away from 1 is strictly less than $k\gamma_1(\mathbf{v})$.

Lemma 2.3.6. *For each $\varepsilon > 0$, there exists a $\delta > 0$ such that, for all $\mathbf{v} \in B_\delta(0)$,*

$$\sup\{g_k^{\mathbf{v}}(w,u)\Big| 1 - \varepsilon > u \geq 0, w \in \mathbb{R}^d\} < k\gamma_1(\mathbf{v}).$$

Proof. Choose $\delta > 0$ such that for all $\mathbf{v} \in B_\delta(0)$, $\gamma_1(\mathbf{v}) > (1 - \frac{(k-1)\varepsilon}{k})\gamma_1(0)$. We can choose such a $\delta > 0$ since $1 > (1 - \frac{(k-1)\varepsilon}{k}) > 0$, the maximum of the function γ_1 is achieved at $\mathbf{v} = 0$ and $\gamma_1(\mathbf{v})$ is continuous. Then, for all $\mathbf{v} \in B_\delta(0)$, for all $w \in \mathbb{R}^d$ and $u \in [0, 1-\varepsilon)$,

$$\begin{aligned}
g_k^{\mathbf{v}}(w,s) &= ku\gamma_1(\frac{\mathbf{v}-w}{u}) + (1-u)\gamma_1(\frac{w}{1-u}) \\
&\leq ku\gamma_1(0) + (1-u)\gamma_1(0) = (1 + (k-1)u)\gamma_1(0) \\
&\leq (k - (k-1)\varepsilon)\gamma_1(0) < k\gamma_1(\mathbf{v}).
\end{aligned}$$

Therefore, for all $\mathbf{v} \in B_\delta(0)$,

$$\sup\{g_k^{\mathbf{v}}(w,u)\Big| 1 - \varepsilon > u \geq 0, w \in \mathbb{R}^d\} < k\gamma_1(\mathbf{v}).$$

□

Thus, by the above three lemmas, there exists a sequence of positive constants $\{\beta_k\}_{k\geq 1}$ such that, for all $\mathbf{v} \in B_{\beta_k}(0)$,

$$f_k(\mathbf{v}) = \lim_{u\uparrow 1} g_k^{\mathbf{v}}(\mathbf{v}(1-u), u) = k\gamma_1(\mathbf{v}). \tag{2.59}$$

Now let us show that there exists a sequence of positive constants $\{\alpha_k\}_{k\geq 1}$ such that, for all $\mathbf{v} \in B_{\alpha_k}(0)$, $f_k(\mathbf{v}) = \gamma_k(\mathbf{v})$. This will be proved by induction.

For $k = 2$, by the definition of γ_2, we have that, $f_2(\mathbf{v}) = \gamma_2(\mathbf{v})$ for each $\mathbf{v} \in \mathbb{R}^d$. Now suppose there exists constants α_i for $1 \leq i \leq k-1$ with $\alpha_i \in (0, \beta_i]$ such that $\gamma_i(\mathbf{v}) = f_i(\mathbf{v}) = i\gamma_1(\mathbf{v})$ for all $\mathbf{v} \in B_{\alpha_i}(0)$. We need to show that there exists $\alpha_k \in (0, \beta_k]$ such that, for all $\mathbf{v} \in B_{\alpha_k}(0)$, we have

$$\begin{aligned}
\gamma_k(\mathbf{v}) &:= \sup_{w\in\mathbb{R}^d, u\in(0,1)} \Big[u\gamma_{k-1}\Big(\frac{\mathbf{v}-w}{u}\Big) + u\gamma_1\Big(\frac{\mathbf{v}-w}{u}\Big) + (1-u)\gamma_1\Big(\frac{w}{1-u}\Big)\Big] \\
&= \sup_{w\in\mathbb{R}^d, u\in(0,1)} \Big[ku\gamma_1\Big(\frac{\mathbf{v}-w}{u}\Big) + (1-u)\gamma_1\Big(\frac{w}{1-u}\Big)\Big] =: f_k(\mathbf{v}).
\end{aligned}$$

To show this, it is enough to show that the supremum in the definition of γ_k is achieved in the part of the space where the values of γ_{k-1} and $(k-1)\gamma_1$ coincide. Let us define the cone $\Gamma_k(\mathbf{v}) = \{(w, u) \in \mathbb{R}^d \times (0, 1) : \dfrac{|\mathbf{v} - w|}{u} \leq \alpha_{k-1}\} \subseteq \mathbb{R}^d \times (0, 1)$. It remains to show that $\Big[u\gamma_{k-1}\Big(\dfrac{\mathbf{v}-w}{u}\Big) + u\gamma_1\Big(\dfrac{\mathbf{v}-w}{u}\Big) + (1-u)\gamma_1\Big(\dfrac{w}{1-u}\Big)\Big]$ on the set $\Gamma_k(\mathbf{v})^c := \mathbb{R}^d \times (0, 1) \setminus \Gamma_k(\mathbf{v})$ is dominated by the supremum of the same expression over the set $\Gamma_k(\mathbf{v})$. We will show that there exists $\alpha_k > 0$ such that, for all $\mathbf{v} \in B_{\alpha_k}(0)$, we have

$$\begin{aligned}
&\sup_{(w,u)\in\Gamma_k(\mathbf{v})^c} \Big[u\gamma_{k-1}\Big(\frac{\mathbf{v}-w}{u}\Big) + u\gamma_1\Big(\frac{\mathbf{v}-w}{u}\Big) + (1-u)\gamma_1\Big(\frac{w}{1-u}\Big)\Big] \\
&\leq \sup_{(w,u)\in\Gamma_k(\mathbf{v})} \Big[u\gamma_{k-1}\Big(\frac{\mathbf{v}-w}{u}\Big) + u\gamma_1\Big(\frac{\mathbf{v}-w}{u}\Big) + (1-u)\gamma_1\Big(\frac{w}{1-u}\Big)\Big].
\end{aligned} \tag{2.60}$$

Note that, for each $(w,u) \in \Gamma_k(\mathbf{v})$, the expression on the RHS is

$$\sup_{(w,u)\in\Gamma_k(\mathbf{v})} \Big[ku\gamma_1\Big(\frac{\mathbf{v}-w}{u}\Big) + (1-u)\gamma_1\Big(\frac{w}{1-u}\Big)\Big].$$

This expression, as follows from (2.59), is equal to $f_k(\mathbf{v}) = k\gamma_1(\mathbf{v})$, as long as $(w,u) = (\mathbf{v}(1-u),u) \in \Gamma_k(\mathbf{v})$ and $\|\mathbf{v}\| \le \beta_k$. That is, $\|\mathbf{v}\| \le \min\{\alpha_{k-1},\beta_k\}$. The inequality (2.60) is justified by the following lemma.

Lemma 2.3.7. *There exists* $0 < \alpha_k < \min\{\alpha_{k-1},\beta_k\}$ *such that, for each* $\mathbf{v} \in B_{\alpha_k}(0)$,

$$u\gamma_{k-1}\Big(\frac{\mathbf{v}-w}{u}\Big) + u\gamma_1\Big(\frac{\mathbf{v}-w}{u}\Big) + (1-u)\gamma_1\Big(\frac{w}{1-u}\Big) < k\gamma_1(\mathbf{v}), \tag{2.61}$$

for all $(w,u) \in \Gamma_k(\mathbf{v})^c$.

Proof. The lemma will be proved in 2 steps. In Step I, the part of set $\Gamma_K(\mathbf{v})^c$ where u is close to 1 is considered. In this part of the set, we make use of the fact that w is bounded from below.

In Step II, the part of set $\Gamma_k(\mathbf{v})^c$ where u is away from 1 is considered. In this part of the set, the left hand side of (2.61) can be made strictly smaller than $k\gamma_1(0)$, while the right hand side can be made arbitrarily close to $k\gamma_1(0)$ by choosing $\mathbf{v}$ in a small enough ball around the origin.

Step I: Let $\delta_1 = \min\{\alpha_{k-1},\beta_k\}/4$. For all $(w,u) \in \Gamma_k(\mathbf{v})^c$, $\|\mathbf{v}\| \le \delta_1$ and $u \in [3/4, 1)$, we have,

$$4\delta_1 \le \alpha_{k-1} < \|\frac{\mathbf{v}-w}{u}\| \le \frac{4}{3}\|\mathbf{v}-w\| \le \frac{4}{3}\big(\|\mathbf{v}\| + \|w\|\big) \le \frac{4}{3}\big(\delta_1 + \|w\|\big).$$

Therefore, $\|w\| \ge 2\delta_1$. Using (2.42), there exist $a > 0$ and $M > 0$ such that, for all $\|\ell\| \ge M$,

$$\gamma_1(\ell) < -a\|\ell\|^2. \tag{2.62}$$

In addition, we choose $M > 0$ large enough such that $\delta_1/M < 1/4$. Observe that, from (2.62),

$$(1-u)\gamma_1\Big(\frac{w}{1-u}\Big) \le -a\frac{\|w\|^2}{1-u} \le -\frac{4a\delta_1^2}{1-u},$$

for all $u \in [1-\dfrac{\delta_1}{M}, 1)$, $(w,u) \in \Gamma_k(\mathbf{v})^c$ if $\|\mathbf{v}\| \le \delta_1$. From (2.56), for each $(\mathbf{v}, w, u) \in \mathbb{R}^d \times \mathbb{R}^d \times (0,1)$, we know that

$$u\gamma_{k-1}\Big(\frac{\mathbf{v}-w}{u}\Big) + u\gamma_1\Big(\frac{\mathbf{v}-w}{u}\Big) \le uk\gamma_1(0).$$

Choosing $\varepsilon \in (0, \delta_1/M)$ such that $k\gamma_1(0) < 4a\delta_1^2/2\varepsilon$, we obtain that, for each $(w,u) \in \mathbb{R}^d \times (1-\varepsilon, 1) \setminus \Gamma_k(\mathbf{v})$ for $\|\mathbf{v}\| \le \delta_1$, the left-hand side of equation (2.61) is negative.

We now choose $\delta_2 \in (0,\delta)$ such that, for all $\|\mathbf{v}\| < \delta_2$, we have $k\gamma_1(\mathbf{v}) > 0$. Thus the inequality (2.61) holds for all $\|\mathbf{v}\| < \delta_2$, for each $(w,u) \in \mathbb{R}^d \times (1-\varepsilon, 1) \setminus \Gamma_k(\mathbf{v})$.

Step II: Let $\varepsilon > 0$ be fixed. Choose $\alpha_k \in (0, \delta_2)$ such that $\gamma_1(\mathbf{v}) > (1-\varepsilon+\dfrac{\varepsilon}{k})\gamma_1(0)$ for all $|\mathbf{v}| < \alpha_k$. Using Lemma 2.3.3, for each $\ell \in \mathbb{R}^d$, $u \in (0,1), i \in \mathbb{N}$, we have $\gamma_i(\ell) < \gamma_i(u\ell)$. Therefore,

$$u\gamma_{k-1}\Big(\frac{\mathbf{v}-w}{u}\Big)+u\gamma_1\Big(\frac{\mathbf{v}-w}{u}\Big) < u\gamma_{k-1}(\mathbf{v}-w)+u\gamma_1(\mathbf{v}-w) \le (k-1)u\gamma_1(0)+u\gamma_1(0),$$

where the last inequality follows from the trivial observation that $\gamma_i(\ell) \le i\gamma_1(0)$, for all $i \in \mathbb{N}, \ell \in \mathbb{R}^d$. Therefore, for $u \in (0, 1-\varepsilon)$, the left hand side of (2.61) can be bounded above as follows,

$$u\gamma_{k-1}\Big(\frac{\mathbf{v}-w}{u}\Big)+u\gamma_1\Big(\frac{\mathbf{v}-w}{u}\Big)+(1-u)\gamma_1\Big(\frac{w}{1-u}\Big) \le (k-1)u\gamma_1(0)+u\gamma_1(0)+(1-u)\gamma_1(0)$$

$$\le k(1-\varepsilon+\frac{\varepsilon}{k})\gamma_1(0).$$

From the definition of α_k, for all $|\mathbf{v}| < \alpha_k$, $k\gamma_1(\mathbf{v}) > k(1 - \varepsilon + \dfrac{\varepsilon}{k})\gamma_1(0)$. Thus, we have shown that inequality (2.61) holds for all $|\mathbf{v}| < \alpha_k$, for each $(w, u) \in \mathbb{R}^d \times (0, 1 - \varepsilon) \setminus \Gamma_k(\mathbf{v})$.

$\square$

Now we prove that $\displaystyle\bigcap_{k\geq 1} G_k = \{0\}$. Let $\mathbf{v} \in G_1$ be fixed, with $\|\mathbf{v}\| > 0$. Now, we show that there exists $k \in \mathbb{N}$, large enough, such that $\gamma_k(\mathbf{v}) > k\gamma_1(\mathbf{v})$. That is, there exists a pair $(w, u) \in \mathbb{R}^d \times (0, 1)$ such that

$$u\gamma_{k-1}\Big(\frac{\mathbf{v} - w}{u}\Big) + u\gamma_1\Big(\frac{\mathbf{v} - w}{u}\Big) + (1 - u)\gamma_1\Big(\frac{w}{1 - u}\Big) > k\gamma_1(\mathbf{v}).$$

We first pick $w = \mathbf{v}$. Then we need to show that there exist $u \in (0, 1)$ and $k \in \mathbb{N}$, such that

$$u\gamma_1(0) + \frac{1 - u}{k}\gamma_1\Big(\frac{\mathbf{v}}{1 - u}\Big) > \gamma_1(\mathbf{v}). \tag{2.63}$$

Let $u = 1 - \varepsilon$ where $\varepsilon > 0$ small enough such that $(1 - \varepsilon)\gamma_1(0) > \gamma_1(\mathbf{v})$. This is possible because $\|\mathbf{v}\| > 0$ and $\gamma_1(\mathbf{v})$ achieves its maximum value at $\mathbf{v} = 0$. Define $\eta = (1 - \varepsilon)\gamma_1(0) - \gamma_1(\mathbf{v}) > 0$. Keeping $\mathbf{v}$ and ε fixed, we pick $k \in \mathbb{N}$ large enough such that,

$$\Big|\frac{\varepsilon}{k}\gamma_1\Big(\frac{\mathbf{v}}{\varepsilon}\Big)\Big| < \eta/2.$$

Therefore,

$$(1 - \varepsilon)\gamma_1(0) + \frac{\varepsilon}{k}\gamma_1\Big(\frac{\mathbf{v}}{\varepsilon}\Big) = \eta + \gamma_1(\mathbf{v}) + \frac{\varepsilon}{k}\gamma_1\Big(\frac{\mathbf{v}}{\varepsilon}\Big) > \gamma_1(\mathbf{v}) + \frac{\eta}{2} > \gamma_1(\mathbf{v}).$$

Thus $\displaystyle\bigcap_{k\geq 1} G_k = \{0\}$. This concludes the proof of Theorem 2.3.1. $\square$

2.4 Distribution of total number of particles

Here, we again assume that our branching diffusion process is super-critical and that the effective drift of the process is zero. Following notation introduced in Section 2.2, recall that φ_0 is the principal periodic eigenfunction of the operator $\mathcal{L}$. It satisfies

$$\mathcal{L}(\varphi_0) = \mu(0)\varphi_0, \tag{2.64}$$

with eigenvalue $\mu(0) \in \mathbb{R}$. The function φ_0^* will denote the solution of the adjoint eigenvalue problem:

$$\mathcal{L}^*(\varphi_0^*) = \mu^*(0)\varphi_0^*,$$

where $\mu^*(0)$ is the principal eigenvalue of the adjoint operator, and hence $\mu^*(0) = \mu(0)$. We normalize φ_0 and φ_0^* by

$$\int_{[0,1)^d} \varphi_0(y)\varphi_0^*(y)\, dy = 1 = \int_{[0,1)^d} \varphi_0^*(y)\, dy. \tag{2.65}$$

In this section, to simplify notation, we will denote φ_0, φ_0^* and $\mu(0)$ by φ, φ^* and μ. For $t > 0$, $x, y \in [0,1)^d$, let $\varrho(t, x, y)$ denote the fundamental solution of the following PDE on the torus:

$$\partial_t \varrho(t, x, y) = \mathcal{L}_x \varrho(t, x, y), \qquad \varrho(0, x, y) = \delta_y(x).$$

Observe that there exists $C_0 > 0$ such that, for every $t > 0$,

$$\int_{[0,1)^d} \varrho(t, x, z) dz \leq C_0 e^{t\mu}. \tag{2.66}$$

Let $N(t, x)$ denote the total number of particles in $\mathbb{R}^d$ at time t, assuming that, at time $t = 0$, there is one particle at $x \in [0,1)^d$. In the following theorem, all the moments of the normalized total number of particles are shown to converge.

Theorem 2.4.1. *For each $k \in \mathbb{N}$, the following limit exists uniformly in $x \in [0,1)^d$:*

$$\lim_{t\to\infty} \frac{\mathbb{E}(N(t,x)^k)}{e^{k\mu t}} = f_k(x), \tag{2.67}$$

where the functions f_k are defined recursively as follows,

$$f_1(x) = \varphi(x),$$

and, for $k \geq 2$,

$$f_k(x) = \sum_{i=1}^{k-1} \beta_i^k \int_0^\infty \int_{[0,1)^d} e^{-k\mu t} \alpha(z) f_i(z) f_{k-i}(z) \varrho(t,x,z) dz dt, \tag{2.68}$$

where $\beta_i^k = k!/(i!(k-i)!)$. In addition, there exists a real valued random variable ξ_x, whose distribution is determined uniquely, such that $\mathbb{E}(\xi_x^k) = f_k(x)$ for each $k \in \mathbb{N}$.

Remark 2.4.1. *The functions $f_k(x)$ defined recursively the formulas (2.68) will be shown to be well defined, that is, the integrals in (2.68) will be shown to be convergent.*

Remark 2.4.2. *The above theorem implies that the total number of particles $N(t,x)$, normalized by its expected value behaves "regularly". That is, the $k-$th moment of $N(t,x)$ is commensurate with the $k-$th power of the first moment. In the next section, we show that $n^y(t,x)$ also exhibits the same "regular" behavior when $\|y - t\bar{\mathbf{v}}\| = o(t)$. In contrast, in Section 2.3 we show that $n^{t\mathbf{v}}(t,x)$ exhibits intermittent behavior when $\mathbf{v} \neq \bar{\mathbf{v}}$, i.e., the k-th moment of $n^{t\mathbf{v}}(t,x)$ grows much faster than the k-th power of the first moment for some $k \in \mathbb{N}$.*

Proof of Theorem 2.4.1. Since the functions $f_1(z) = \varphi(z)$, and $\varrho(t, x, z)$ are strictly positive and continuous for $x, z \in [0, 1)^d$, $t > 0$ and the function $\alpha(z)$ is non negative and continuous for $z \in [0, 1)^d$, and $\tilde{r}$ is not identically zero, the functions f_k are clearly positive and continuous on $[0, 1)^d$, as follows from their recursive definition. As in (2.34), for each $k \geq 1$,

$$\mathbb{E}(N(t, x)^k) = \sum_{i=1}^{k} S(k, i)\bar{m}_i(t, x), \tag{2.69}$$

where

$$\bar{m}_i(t, x) = \int_{\mathbb{R}^d} .... \int_{\mathbb{R}^d} \rho_k(t, x, y_1, y_2, ..., y_k) dy_1 ... dy_k,$$

where ρ_i's are the particle density and higher order correlation functions, as defined in (2.30) and (2.31). Thus, we observe that $\bar{m}_i(t, x)$ satisfy the following PDEs on $\mathbb{T}^d$:

$$\partial_t \bar{m}_1(t, x) = \mathcal{L}_x \bar{m}_1(t, x), \qquad \bar{m}_1(0, x) \equiv 1, \tag{2.70}$$

while, for $k \geq 2$,

$$\partial_t \bar{m}_k(t, x) = \mathcal{L}_x \bar{m}_k(t, x) + \tilde{c}(x) \sum_{i=1}^{k-1} \beta_i^k \bar{m}_i(t, x)\bar{m}_{k-i}(t, x), \qquad \bar{m}_k(0, x) \equiv 0, \tag{2.71}$$

where $\beta_i^k = k!/(i!(k-i)!)$.

We will prove the following lemma after completing the proof of the theorem.

Lemma 2.4.2. *For each $k \in \mathbb{N}$, $x \in [0, 1)^d$,*

$$\bar{m}_k(t, x) = e^{k\mu t}\Big[f_k(x) + q_k(t, x)\Big], \tag{2.72}$$

where $\lim_{t\to\infty} q_k(t, x) = 0$ uniformly in $x \in [0, 1)^d$, and f_k have been defined in (2.68).

Using formula (2.72) in (2.69), we get

$$\frac{\mathbb{E}(N(t,x)^k)}{e^{k\mu t}} = \sum_{i=1}^{k} S(k,i)e^{-(k-i)\mu t}\Big[f_i(x) + q_i(t,x)\Big].$$

Therefore,

$$\begin{aligned}\lim_{t\to\infty}\frac{\mathbb{E}(N(t,x)^k)}{e^{k\mu t}} &= f_k(x) + \lim_{t\to\infty}\Big(q_k(t,x) + \sum_{i=1}^{k-1} S(k,i)e^{-(k-i)\mu t}\Big[f_i(x)+q_i(t,x)\Big]\Big) \\ &= f_k(x).\end{aligned}$$

Now, we use induction to show that there exists a constant $A > 0$ such that, for every $x \in [0,1)^d$, $f_k(x) \le A^k k!$. For $k = 1$, we know that the eigenfunction $\varphi(x)$ corresponding to the top eigenvalue μ of the operator $\mathcal{L}$ on the $d-$dimensional torus $\mathbb{T}^d$ is a positive and continuous function. Therefore, there exists a constant $A_1 > 1$ such that, for every $x \in (0,1]^d$, $\varphi(x) \le A_1$.

Suppose that for all $1 \le j \le k-1$, $x \in [0,1)^d$, $f_j(x) \le A_1^j j!$. Then, from the definition of the function f_k, we get

$$\begin{aligned} f_k(x) &= \sum_{i=1}^{k-1}\beta_i^k \int_0^\infty \int_{[0,1)^d} e^{-k\mu t}\alpha(z) f_i(z) f_{k-i}(z)\varrho(t,x,z)dzdt \\ &\le A_1^k (k-1)k! \int_0^\infty \int_{[0,1)^d} e^{-k\mu t}\alpha(z)\varrho(t,x,z)dzdt.\end{aligned}$$

Recall that the operator $\mathcal{L}-\mu$ has top eigenvalue zero, while the top eigenfunction of the adjoint operator $(\mathcal{L}-\mu)^*$ is φ^* (with $\int_{[0,1)^d}\varphi^*(z)\,dz = 1$). Therefore, there exists a constant $C > 0$ such that, for every $x \in [0,1)^d$, $t > 0$,

$$\int_{[0,1)^d} e^{-t\mu}\alpha(z)\varrho(t,x,z)dz \le \big(\sup_{x\in[0,1)^d}\alpha(x)\big)\int_{[0,1)^d} e^{-t\mu}\varrho(t,x,z)dz \le C.$$

Therefore,

$$f_k(x) \le CA_1^k(k-1)k!\int_0^\infty e^{-(k-1)\mu t}dt \le k! A_1^k C/\mu.$$

If $C/\mu \le 1$, we pick $A = A_1$, and if $C/\mu > 1$, choose $A = A_1 C/\mu$. With this choice of A we obtain that, for every $x \in [0,1)^d$, $f_k(x) \le A^k k!$. From the convergence of all the moments of $N(t,x)/e^{\mu t}$, it follows that, there exists a random variable ξ_x with the moments $f_k(x)$ (see [27]). The uniqueness of the distribution of ξ_x follows from the bound on f_k by the Carleman theorem. This concludes the proof of Theorem 2.4.1. □

Proof of Lemma 2.4.2. We use induction to prove this lemma. The top eigenvalue of the operator $\mathcal{L}$ is $\mu > 0$, and the corresponding eigenfunction $\varphi(x) > 0$. Thus, from the theory of elliptic operators on $\mathbb{T}^d$, from (2.70), there exists a function $q_1(t,x)$ such that

$$\bar{m}_1(t,x) = e^{\mu t}\big[\varphi(x) + q_1(t,x)\big],$$

where

$$\lim_{t\to\infty} q_1(t,x) = 0$$

uniformly in $x \in [0,1)^d$. This gives (2.72) for $k = 1$ with $f_1(x) = \varphi(x)$. Suppose that the conclusion of the lemma holds up to $k-1$, where $k \ge 2$. From (2.71), using Duhamel's formula, we get

$$\bar{m}_k(t,x) = \int_0^t \int_{[0,1)^d} \alpha(z) \sum_{i=1}^{k-1} \beta_i^k \bar{m}_i(s,z)\bar{m}_{k-i}(s,z)\varrho(t-s,x,z)dzds.$$

By the inductive assumption,

$$\begin{aligned}\bar{m}_k(t,x) = &\sum_{i=1}^{k-1} \beta_i^k \int_0^t \int_{[0,1)^d} e^{k\mu s}\alpha(z) f_i(z) f_{k-i}(z)\varrho(t-s,x,z)dzds\\ &+ \sum_{i=1}^{k-1} \beta_i^k \int_0^t \int_{[0,1)^d} e^{k\mu s}\alpha(z) h_i(s,z)\varrho(t-s,x,z)dzds,\end{aligned}$$

where

$$h_i(s,z) := q_i(s,z)q_{k-i}(s,z) + q_i(s,z)f_{k-i}(z) + q_{k-i}(s,z)f_i(z).$$

After the change of variables $u = t - s$, we get

$$\begin{aligned}
\bar{m}_k(t,x) &= e^{k\mu t}\sum_{i=1}^{k-1}\beta_i^k \int_0^t \int_{[0,1)^d} e^{-uk\mu}\alpha(z)f_i(z)f_{k-i}(z)\varrho(u,x,z)dzdu \\
&+ e^{k\mu t}\sum_{i=1}^{k-1}\beta_i^k \int_0^t \int_{[0,1)^d} e^{-uk\mu}\alpha(z)h_i(t-u,z)\varrho(u,x,z)dzdu \\
&= e^{k\mu t}f_k(x) - e^{k\mu t}\sum_{i=1}^{k-1}\beta_i^k \int_t^\infty \int_{[0,1)^d} e^{-uk\mu}\alpha(z)f_i(z)f_{k-i}(z)\varrho(u,x,z)dzdu \\
&+ e^{k\mu t}\sum_{i=1}^{k-1}\beta_i^k \int_0^t \int_{[0,1)^d} e^{-uk\mu}\alpha(z)h_i(t-u,z)\varrho(u,x,z)dzdu.
\end{aligned}$$

Define

$$\begin{aligned}
q_k(t,x) := \sum_{i=1}^{k-1}\beta_i^k\Big(&\int_0^t \int_{[0,1)^d} e^{-uk\mu}\alpha(z)h_i(t-u,z)\varrho(u,x,z)dzdu \\
&- \int_t^\infty \int_{[0,1)^d} e^{-uk\mu}\alpha(z)f_i(z)f_{k-i}(z)\varrho(u,x,z)dzdu\Big).
\end{aligned}$$

Thus, we have

$$\bar{m}_k(t,x) = e^{k\mu t}\Big[f_k(x) + q_k(t,x)\Big].$$

It remains to show that $\lim_{t\to\infty} q_k(t,x) = 0$ uniformly in $x \in [0,1)^d$. Since the functions $\tilde{c}, f_i, f_{k-i}$ are non-negative and continuous on $[0,1)^d$, there exists a constant $C_i > 0$ such that $0 \le \alpha(z)f_i(z)f_{k-i}(z) < C_i$ for all $z \in [0,1)^d$. Therefore,

$$\int_t^\infty \int_{[0,1)^d} e^{-uk\mu}\alpha(z)f_i(z)f_{k-i}(z)\varrho(u,x,z)dzdu \le C_i \int_t^\infty \int_{[0,1)^d} e^{-uk\mu}\varrho(u,x,z)dzdu. \tag{2.73}$$

Therefore, from (2.66), the right hand side of the (2.73) goes to zero uniformly in $x \in [0,1)^d$. To deal with the sum in the definition of $q_k(t,x)$, we break up the integral in two parts as follows,

$$
\begin{aligned}
&\Big|\int_0^t \int_{[0,1)^d} e^{-uk\mu}\alpha(z)h_i(t-u,z)\varrho(u,x,z)dzdu\Big| \\
&\le \Big|\int_0^{t/2} \int_{[0,1)^d} e^{-uk\mu}\alpha(z)h_i(t-u,z)\varrho(u,x,z)dzdu\Big| \\
&\quad + \Big|\int_{t/2}^{t} \int_{[0,1)^d} e^{-uk\mu}\alpha(z)h_i(t-u,z)\varrho(u,x,z)dzdu\Big| \\
&\le \sup_{s\in(t/2,t),x\in[0,1)^d} |\alpha(x)h_i(s,x)| \int_0^{t/2} \int_{[0,1)^d} e^{-uk\mu}\varrho(u,x,z)dzdu \\
&\quad + \sup_{s\in(0,t/2),x\in[0,1)^d} |\alpha(x)h_i(s,x)| \int_{t/2}^{t} \int_{[0,1)^d} e^{-uk\mu}\varrho(u,x,z)dzdu
\end{aligned}
$$

From (2.66), the integral in the first term is bounded and from the inductive hypothesis,

$$
\lim_{t\to\infty} \sup_{s\in(t/2,t),x\in[0,1)^d} |\alpha(x)h_i(s,x)| = 0,
$$

and therefore, the first term converges to zero uniformly in $x \in [0,1)^d$.

Similarly, from the inductive hypothesis, the supremum in the second term is bounded, while, from (2.66), the integral in the second term converges to zero uniformly in $x \in [0,1)^d$. Thus, we conclude that

$$
\lim_{t\to\infty} \sup_{x\in[0,1)^d} q_k(t,x) = 0,
$$

which completes the proof of Lemma 2.4.2. □

2.5 Distribution of the number of particles near the region where the bulk of the particles is located

Here, we again assume that our branching diffusion process is super-critical and that the effective drift of the process is zero (i.e., $\bar{\mathbf{v}} = 0$). Let $n^y(t,x)$ denote the number of particles in $\mathbb{T}_y^d$ at time $t \in \mathbb{R}^+$, given that there was one particle at $x \in [0,1)^d$ at time $t = 0$. Define

$$g(t,y) = (\sqrt{2\pi t})^{-d} \det[D^2\Phi(0)]^{1/2} e^{-t\Phi(\frac{y}{t})}.$$

From this formula of $g(t,y)$, since the minimum of the twice continuously differentiable function $\Phi(v)$ is achieved at $v = 0$, for each $\alpha \in (0,1)$, we get

$$g(t,\alpha y) \geq g(t,y). \tag{2.74}$$

Theorem 2.5.1. *Let $r(t) = o(t)$. For each $k \in \mathbb{N}$,*

$$\lim_{t\to\infty} \frac{\mathbb{E}(n^{y(t)}(t,x)^k)}{g(t,y(t))^k} = f_k(x)$$

uniformly in $x \in [0,1)^d$ and $\|y(t)\| \leq r(t)$.

Proof. Recall from (2.34),

$$\mathbb{E}(n^y(t,x)^k) = \sum_{i=1}^{k} S(k,i) m_i^y(t,x). \tag{2.75}$$

We will show the following two statements by induction:

(i) For each $r(t) = o(t)$, there exists the limit

$$\lim_{t\to\infty} \frac{m_k^{y(t)}(t,x)}{g(t,y(t))^k} = f_k(x)$$

uniformly in $x \in [0,1)^d$ and $\|y(t)\| \leq r(t)$.

(ii) Let $\bar{r}(t) = o(t)$ be a function satisfying $r(t) = o(\bar{r}(t))$, with $\sqrt{t}/\bar{r}(t) \to 0$. Then

$$\lim_{t\to\infty} \frac{m_k^{\bar{y}(t)}(t,x)}{g(t,y(t))^k} = 0$$

uniformly in $x \in [0,1)^d$, $\|y(t)\| \le r(t)$, and $\|\bar{y}(t)\| \ge \bar{r}(t)$.

The theorem will then immediately follow from (i) since $g(t,y) \to \infty$ as $t \to \infty$ for $\|y\| \le r(t)$ and therefore, the term with $i = k$ dominates in the sum in formula (2.75).

For $k = 1$, using the asymptotic formula for $\rho_1(t,x,y)$ that was given in Theorem 2.2.2, we get

$$m_1^y(t,x) = \int_{\mathbb{T}_y^d} \rho_1(t,x,z) dz =$$

$$= (\sqrt{2\pi t})^{-d} \varphi(x) \left(\int_{\mathbb{T}_y^d} \det[D^2\Phi(\frac{z-x}{t})]^{1/2} e^{-t\Phi(\frac{z-x}{t})} \varphi^*(z)\, dz \right) [1 + o_L(1)], \quad (2.76)$$

for all $x, y \in \mathbb{R}^d$ with $\|x - y\| \le Lt$. Observe that the following limits exit uniformly in $x \in [0,1)^d$, $z \in \mathbb{T}_y^d$, and $\|y(t)\| \le r(t)$,

$$\lim_{t\to\infty} \det[D^2\Phi(\frac{z-x}{t})] = \det[D^2\Phi(0)], \qquad \lim_{t\to\infty} e^{-t\Phi(\frac{z-x}{t}) + t\Phi(\frac{y}{t})} = 1,$$

while $\int_{\mathbb{T}_y^d} \varphi^*(z) dz = 1$. Therefore, (i) holds for $k = 1$. To prove (ii) for $k = 1$, it enough to show that

$$\lim_{t\to\infty} e^{t\Phi(\frac{y(t)}{t}) - t\Phi(\frac{\bar{y}(t)}{t})} = 0 \quad (2.77)$$

uniformly in $\|y(t)\| \le r(t)$, $\|\bar{y}(t)\| \ge \bar{r}(t)$.

First observe that, given a small $c > 0$, there exists a constant $m > 0$ such that $-\mu - \Phi(v) \le -c$ for all $\|v\| \ge m$. In addition, since $\frac{y(t)}{t} \to 0$ as $t \to \infty$, there exists $T_1 > 0$ such that $|\Phi\Big(\frac{y(t)}{t}\Big) + \mu| \le c/2$ for all $t \ge T_1$. Therefore, whenever $\|\frac{\bar{y}(t)}{t}\| \ge m$, we have $\Phi\Big(\frac{y(t)}{t}\Big) - \Phi\Big(\frac{\bar{y}(t)}{t}\Big) \le -c/2$. for all $t \ge T_1$. That is, if $\|\frac{\bar{y}(t)}{t}\| \ge m$, then

$$e^{t\Phi(\frac{y(t)}{t}) - t\Phi(\frac{\bar{y}(t)}{t})} \le e^{-tc/2} \quad (2.78)$$

for all $t \geq T_1$.

We choose $T_2 > T_1$ such that, for all $t \geq T_2$, $\|\frac{y(t)}{t}\| \leq m$. Observe that there exist constants $c_1, c_2 > 0$ such that

$$c_1\|\mathbf{v}\|^2 \leq |\langle D^2\Phi\,(v)\,\mathbf{v}, \mathbf{v}\rangle| \leq c_2\|\mathbf{v}\|^2 \tag{2.79}$$

for all $v \in \mathbb{R}^d$ with $\|v\| \leq m$. Whenever $\|\frac{\bar{y}(t)}{t}\| \leq m$, using Taylor's formula, for all $t \geq T_2$, there exist $\alpha_1, \alpha_2 \in (0,1)$ such that

$$\begin{aligned} t\Big(\Phi\big(\frac{y(t)}{t}\big) - \Phi\big(\frac{\bar{y}(t)}{t}\big)\Big) &= t\Big[\langle\frac{y(t)}{t}, D^2\Phi\left(\frac{\alpha_1 y(t)}{t}\right)\frac{y(t)}{t}\rangle - \langle\frac{\bar{y}(t)}{t}, D^2\Phi\left(\frac{\alpha_2\bar{y}(t)}{t}\right)\frac{\bar{y}(t)}{t}\rangle\Big] \\ &\leq c_2\frac{\|\bar{r}(t)\|^2}{t}\Big[\Big\|\frac{r(t)}{\bar{r}(t)}\Big\|^2 - \frac{c_1}{c_2}\Big]. \end{aligned} \tag{2.80}$$

Since $\sqrt{t}/\bar{r}(t) \to 0$, and $r(t) = o(\bar{r}(t))$, (2.80) and (2.78) imply (2.77). This concludes the proof of (i) and (ii) for $k = 1$.

Now, let us assume that (i) and (ii) hold up to $k-1$, where $k \geq 2$. We first prove (i) for k. Let $\bar{r}(t) = o(t)$ be a function satisfying $r(t) = o(\bar{r}(t))$, with $\sqrt{t}/\bar{r}(t) \to 0$. Recall from (2.43),

$$m_k^y(t,x) = \int_0^t \int_{\mathbb{R}^d} \alpha(z) \sum_{i=1}^{k-1} \beta_i^k m_i^y(s,z) m_{k-i}^y(s,z) \rho_1(t-s,x,z)\,dz\,ds.$$

Let $\varepsilon \in (0,1)$, to be selected later. Let us define the following

$$A_k(t,x,y(t)) := \int_0^{\varepsilon t} \int_{\mathbb{R}^d} \alpha(z) \sum_{i=1}^{k-1} \beta_i^k m_i^{y(t)}(s,z) m_{k-i}^{y(t)}(s,z) \rho_1(t-s,x,z)\,dz\,ds,$$

$$B_k(t,x,y(t)) := \int_{\varepsilon t}^t \int_{\|z-y(t)\| \geq \bar{r}(t)} \alpha(z) \sum_{i=1}^{k-1} \beta_i^k m_i^{y(t)}(s,z) m_{k-i}^{y(t)}(s,z) \rho_1(t-s,x,z)\,dz\,ds,$$

$$C_k(t,x,y(t)) := \int_{\varepsilon t}^t \int_{\|z-y(t)\| \leq \bar{r}(t)} \alpha(z) \sum_{i=1}^{k-1} \beta_i^k m_i^{y(t)}(s,z) m_{k-i}^{y(t)}(s,z) \rho_1(t-s,x,z)\,dz\,ds.$$

By (2.35), we can choose $\varepsilon > 0$ small enough so that, for each $1 \le i \le k-1$,

$$m_i^y(s,z) m_{k-i}^y(s,z) \le c e^{\mu t/2}$$

for all $0 \le s \le \varepsilon t$ and $z, y \in \mathbb{R}^d$. For this fixed $\varepsilon > 0$, choosing a sufficiently large $L > 0$, we use the asymptotic formula for $\rho_1(t,x,z)$ that was given in Theorem 2.2.2 in the region $\|z-x\| \le Lt$ and the estimate (2.41) elsewhere, to obtain that $A_k(t,x,y) \le c_1 e^{3\mu t/2}$, for all $x, y \in \mathbb{R}^d$. Therefore, there exists a constant $C > 0$ such that,

$$\lim_{t\to\infty} \frac{A_k(t,x,y(t))}{g(t,y(t))^k} \le \lim_{t\to\infty} C(2\pi t)^{d/2} e^{(3/2-k)\mu t} = 0,$$

uniformly in $x \in [0,1)^d$ and $\|y(t)\| \le r(t)$, since $k \ge 2$. Next we show that

$$\lim_{t\to\infty} \frac{B_k(t,x,y(t))}{g(t,y(t))^k} = 0.$$

Since the operator $\mathcal{L}$ is periodic, we first observe that $m_k^y(t,x) = m_k^{y-[x]}(t,\{x\})$ for all $k \in \mathbb{N}$, $x, y \in \mathbb{R}^d$, and $t \ge 0$. For all $1 \le i \le k-1$, from (ii) we have

$$\lim_{t\to\infty} \frac{m_i^y(s(t),z)}{g(s(t),y(t))^i} = \lim_{t\to\infty} \frac{m_i^{y-[z]}(s(t),\{z\})}{g(s(t),y(t))^i} = 0$$

uniformly in $s(t) \in (\varepsilon t, t)$, $\|y(t)\| \le r(t)$ and $\|z - y(t)\| \ge \bar{r}(t)$ where $[\cdot]$ denotes the greatest integer function in $d-$dimensions, and $\{z\} = z - [z]$. Thus, it is enough to show that there exists a $C > 0$ such that

$$\lim_{t\to\infty} \frac{\sup\left\{g(s,y(t))^k \middle| s \in (\varepsilon t, t)\right\}}{g(t,y(t))^k} \int_{\varepsilon t}^{t} \int_{\|z-y(t)\| \ge \bar{r}(t)} \alpha(z) \rho_1(t-s,x,z) dz ds \le C.$$

Choosing a sufficiently large $L > 0$, we use the asymptotic formula for $\rho_1(t,x,z)$ that was given in Theorem 2.2.2 in the region $\|z-x\| \le Lt$ and the estimate (2.41) elsewhere, to obtain that

$$\int_{\mathbb{R}^d} \rho_1(t-s,x,z)dz \le ae^{\mu(t-s)}. \tag{2.81}$$

Thus, it is enough to show that

$$\lim_{t\to\infty} \sup\left\{e^{kt\left[\Phi\left(\frac{y(t)}{t}\right)-\frac{s}{t}\Phi\left(\frac{y(t)}{s}\right)\right]}e^{\mu(t-s)}\Big|s\in(\varepsilon t,t)\right\}=1. \tag{2.82}$$

Note that $e^{kt\left[\Phi\left(\frac{y(t)}{t}\right)-\frac{s}{t}\Phi\left(\frac{y(t)}{s}\right)\right]}e^{\mu(t-s)}=1$ when $s=t$. We show that, for sufficiently large t, the supremum in the above expression is achieved when $s=t$. To show the claim, for $s=t-\delta$, we will show that

$$kt\Big[\Phi\big(\frac{y(t)}{t}\big)-\Phi\big(\frac{y(t)}{t-\delta}\big)\Big]+k\delta\Phi\big(\frac{y(t)}{t}\big)+\mu\delta<0.$$

Recall that Φ is continuous and the minimum value of the function Φ is achieved at 0, which is $\Phi(0)=-\mu<0$. In addition, recall that $r(t)=o(t)$. Thus, since $k\ge 2$, we conclude that there exists $\eta>0$ such that, for all sufficiently large t,

$$k\delta\Phi\big(\frac{y(t)}{t}\big)+\mu\delta<-\eta.$$

Thus, it is enough to show that, for all sufficiently large t,

$$\Big|kt\Big(\Phi\big(\frac{y(t)}{t}\big)-\Phi\big(\frac{y(t)}{t-\delta}\big)\Big)\Big|<\eta/2. \tag{2.83}$$

Indeed, for large t, the value of $\|y(t)/t\|$ is close to 0, while $\|y(t)/(t-\delta)\| = \|y(t)/s\| \le \frac{1}{\varepsilon}\|y(t)/t\|$ is also close to 0. Thus, using the fact that $\nabla\Phi(0)=0$ and Taylor's formula, we obtain that (2.83) holds. Thus, we have shown that the supremum in (2.82) is achieved when $s=t$, when t is large enough. This completes the proof of (2.82). Next we show that

$$\lim_{t\to\infty}\frac{C_k(t,x,y(t))}{g(t,y(t))^k}=f_k(x).$$

In the region $\|z-y(t)\|\le \bar{r}(t)$, by the inductive assumption, we can replace

$$\sum_{i=1}^{k-1}\beta_i^k m_i^{y(t)}(s,z)m_{k-i}^{y(t)}(s,z) \quad \text{by} \quad g(s,y(t)-[z])^k\sum_{i=1}^{k-1}\beta_i^k f_i(z)f_{k-i}(z)$$

in the integral. Therefore, we obtain

$$\lim_{t\to\infty}\frac{C_k(t,x,y(t))}{g(t,y(t))^k}=$$

$$=\lim_{t\to\infty}\sum_{i=1}^{k-1}\beta_i^k\int_{\varepsilon t}^{t}\int_{\|z-y(t)\|\le\bar{r}(t)}\Big(\frac{g(s,y(t)-[z])}{g(t,y(t))}\Big)^k\alpha(z)f_i(z)f_{k-i}(z)\rho_1(t-s,x,z)dzds.$$

Therefore, using a change of variable, it remains to show that, for each $1\le i\le k-1$,

$$\lim_{t\to\infty}\int_0^{(1-\varepsilon)t}\int_{\|z-y(t)\|\le\bar{r}(t)}\Big(\frac{g(t-s,y(t)-[z])}{g(t,y(t))}\Big)^k\alpha(z)f_i(z)f_{k-i}(z)\rho_1(s,x,z)dzds= \tag{2.84}$$

$$=\int_0^{\infty}\int_{[0,1)^d}e^{-k\mu s}\alpha(z)f_i(z)f_{k-i}(z)\varrho(s,x,z)dzds.$$

Let $\eta>0$ be fixed. From (ii), if $\|y(t)-[z]\|/\|y(t)\|\to\infty$, then,

$$\Big(\frac{g(t,y(t)-[z])}{g(t,y(t))}\Big)^k\to\infty.$$

Note that,

$$\frac{g(t-s,y(t)-[z])}{g(t,y(t))}=$$

$$=\Big(\frac{t}{t-s}\Big)^{(d/2)}\exp\Big[t\Big(\Phi\Big(\frac{y(t)}{t}\Big)-\Phi\Big(\frac{y(t)-[z]}{t-s}\Big)\Big)+s\Phi\Big(\frac{y(t)-[z]}{t-s}\Big)\Big].$$

The term $(t/t-s)^{(d/2)}$ is bounded when $0\le s\le(1-\varepsilon)t$. Given $\delta>0$ small, using the fact that $\|z-y(t)\|\le\bar{r}(t)=o(t), \|y(t)\|\le r(t)=o(t)$ and $\nabla\Phi(0)=0$, from Taylor's formula, there exists $\alpha\in(0,1)$ and $\ell(t)=\frac{y(t)-[z]}{t-s}+\alpha\Big(\frac{y(t)}{t}-\frac{y(t)-[z]}{t-s}\Big)$ such that, for all sufficiently large t, for all $0\le s\le(1-\varepsilon)t$,

$$t\Big(\Phi\Big(\frac{y(t)}{t}\Big)-\Phi\Big(\frac{y(t)-[z]}{t-s}\Big)\Big)+s\Phi\Big(\frac{y(t)-[z]}{t-s}\Big)$$

$$\leq t\|\nabla\Phi\Big(\ell(t)\Big)\|\Big\|\frac{t[z]-sy(t)}{t(t-s)}\Big\|+(-\mu+\delta)s$$

$$\leq \delta\|z\|+(-\mu+2\delta)s. \tag{2.85}$$

Therefore, using (2.41), we conclude that

$$\int_{\|z-y(t)\|\leq\bar r(t)}\Big(\frac{g(t-s,y(t)-[z])}{g(t,y(t))}\Big)^k\alpha(z)f_i(z)f_{k-i}(z)\rho_1(s,x,z)dz\leq$$

$$\leq C\int_{\|z-y(t)\|\leq\bar r(t)}\exp\Big[k(-\mu+2\delta)s+k\delta\|z\|\Big]\rho_1(s,x,z)dz\leq$$

$$\leq \tilde C s^{-d/2}\exp\Big[k(-\mu+2\delta)s+\mu s\Big]\int_{\|z-y(t)\|\leq\bar r(t)}\exp\Big[k\delta\|z\|-\frac{\|z\|^2}{cs}\Big]dz\leq$$

$$\leq \tilde C s^{-d/2}\exp\Big[-(k-1)\mu s+k2\delta s+cs\delta^2k^2/4\Big]\int_{\mathbb{R}^d}\exp\Big[-\Big(\frac{\|z\|}{\sqrt{cs}}-\frac{k\delta\sqrt{cs}}{2}\Big)^2\Big]dz.$$

Now, by choosing δ small enough so that $k\delta<\mu/8$, and $c\delta^2k^2<\mu$, we have, for all $k\geq 2$,

$$-(k-1)\mu s+k\delta s+cs\delta^2k^2/4\leq-\mu s/2.$$

Therefore, there exists $m>0$ such that, for all t sufficiently large,

$$\int_m^{(1-\varepsilon)t}\int_{\|z-y(t)\|\leq\bar r(t)}\Big(\frac{g(t-s,y(t)-[z])}{g(t,y(t))}\Big)^k\alpha(z)f_i(z)f_{k-i}(z)\rho_1(s,x,z)dzds<\eta/10 \tag{2.86}$$

uniformly in $x\in[0,1)^d$ and $\|y(t)\|\leq r(t)$. From (2.66), we can show that

$$\int_m^\infty\int_{[0,1)^d}e^{-k\mu s}\alpha(z)f_i(z)f_{k-i}(z)\varrho(s,x,z)dzds<\eta \tag{2.87}$$

uniformly in $x\in[0,1)^d$. Now it remains to show that, for each $0\leq s\leq m$,

$$\int_0^m\int_{\|z-y(t)\|\leq\bar r(t)}\Big(\frac{g(t-s,y(t)-[z])e^{\mu s}}{g(t,y(t))}\Big)^k\alpha(z)f_i(z)f_{k-i}(z)\rho_1(s,x,z)dzds$$

$$\to \int_0^m \int_{[0,1)^d} \alpha(z) f_i(z) f_{k-i}(z) \varrho(s,x,z) dz ds.$$

Observe that ρ_1 is the fundamental solution of the operator $\mathcal{L}$ on $\mathbb{R}^d$, while ϱ is the fundamental solution of the same operator on $\mathbb{T}^d$. Therefore, for each $s \geq 0$, $x \in [0,1)^d$, and each continuous $\mathbb{Z}^d-$ periodic function $h : \mathbb{R}^d \to \mathbb{R}$, we have the relation

$$\int_{\mathbb{R}^d} h(z)\rho_1(s,x,z)dz = \int_{[0,1)^d} h(z)\varrho(s,x,z)dz.$$

Also, if $\|z - y(t)\| \geq \bar{r}(t)$, and $\|y\| \leq r(t)$ where $r(t) = o(\bar{r}(t))$, we conclude that, for sufficiently large t, $\|z - x\| \leq c\bar{r}(t)$, for some $c > 0$. Therefore, for $0 \leq s \leq m$, $1 \leq i \leq k-1$, for sufficiently large t, using (2.41), we obtain

$$\int_0^m \int_{|z-x| \geq c\bar{r}(t)} \alpha(z) f_i(z) f_{k-i}(z) \rho_1(s,x,z) dz ds < \eta. \tag{2.88}$$

Thus, it remains to show that, $1 \leq i \leq k-1$,

$$\int_{\|z-y(t)\| \leq \bar{r}(t)} \left| \left(\frac{g(t-s, y(t) - [z]) e^{\mu s}}{g(t, y(t))} \right)^k - 1 \right| \alpha(z) f_i(z) f_{k-i}(z) \rho_1(s,x,z) dz \to 0$$

uniformly in $s \in [0,m]$. Let us first prove that there exists $R > 0$ such that, for $0 \leq s \leq m$, $1 \leq i \leq k-1$,

$$\int_{\substack{\|z-y(t)\| \leq \bar{r}(t) \\ \|z\| \geq R}} \left| \left(\frac{g(t-s, y(t) - [z]) e^{\mu s}}{g(t, y(t))} \right)^k - 1 \right| \alpha(z) f_i(z) f_{k-i}(z) \rho_1(s,x,z) dz < \eta. \tag{2.89}$$

As in (2.85), and using (2.41), given $\delta > 0$ small, for all sufficiently large t, and for $0 \leq s \leq m$, we get,

$$\int_{\substack{\|z-y(t)\| \leq \bar{r}(t) \\ \|z\| \geq R}} \left(\frac{g(t-s, y(t) - [z]) e^{\mu s}}{g(t, y(t))} \right)^k \alpha(z) f_i(z) f_{k-i}(z) \rho_1(s,x,z) dz \leq$$

$$\le \tilde{C} s^{-d/2} \exp\Big[\mu s + 2k\delta s + cs\delta^2 k^2/4\Big] \int_{\|z\|\ge R} \exp\Big[-\Big(\frac{\|z\|}{\sqrt{cs}} - \frac{k\delta\sqrt{cs}}{2}\Big)^2\Big] dz.$$

By choosing $R > 0$ large enough, the right can be made arbitrarily small uniformly for all $0 \le s \le m$. Thus, (2.89) holds. Now it remains to show that for this positive constant $R > 0$, we have

$$\lim_{t\to\infty} \sup\Big\{\Big|\frac{g(t-s, y(t)-[z])e^{\mu s}}{g(t, y(t))} - 1\Big|\Big| s \in [0,m], \|y(t)\| \le r(t), \|z\| \le R\Big\} = 0. \tag{2.90}$$

To see this, as before, we observe that,

$$\frac{g(t-s, y(t)-[z])e^{\mu s}}{g(t, y(t))} = \Big(\frac{t}{t-s}\Big)^{(d/2)} e^{\Big[t\Big(\Phi\Big(\frac{y(t)}{t}\Big) - \Phi\Big(\frac{y(t)-[z]}{t-s}\Big)\Big) + s\Big(\mu + \Phi\Big(\frac{y(t)-[z]}{t-s}\Big)\Big)\Big]}.$$

Given $\delta > 0$ small, for each sufficiently large t,

$$t\Big|\Phi\Big(\frac{y(t)}{t}\Big) - \Phi\Big(\frac{y(t)-[z]}{t-s}\Big)\Big| < \delta/2, \quad \text{and} \quad s\Big|\mu + \Phi\Big(\frac{y(t)-[z]}{t-s}\Big)\Big| < \delta/2,$$

for all $\|z\| \le R$, $\|y(t)\| \le r(t)$ $\|y(t)-z\| \le \bar{r}(t)$ and $0 \le s \le m$. Therefore, we can choose $\delta > 0$ small enough, such that, for all sufficiently large t,

$$\int_{\substack{\|z-y(t)\|\le\bar{r}(t)\\ \|z\|\le R}} \Big|\Big(\frac{g(t-s, y(t)-[z])e^{\mu s}}{g(t, y(t))}\Big)^k - 1\Big| \alpha(z) f_i(z) f_{k-i}(z) \rho_1(s,x,z) dz < \eta. \tag{2.91}$$

Since $\eta > 0$ was arbitrary, (2.86), (2.87), (2.88), (2.89) and (2.91) complete the proof of (i) for k.

We now prove (ii) for k. For fixed $r(t)$ and $\bar{r}(t)$ as in (ii), choose $p(t)$ such that $r(t) \ll p(t) \ll \bar{r}(t)$. Again, divide the integral in the definition of $m_k^{\bar{y}(t)}(t,x)$ into the following three integrals:

$$\bar{A}_k(t,x,\bar{y}(t)) := \int_0^{\varepsilon t} \int_{\mathbb{R}^d} \alpha(z) \sum_{i=1}^{k-1} \beta_i^k m_i^{\bar{y}(t)}(s,z) m_{k-i}^{\bar{y}(t)}(s,z) \rho_1(t-s,x,z) dz ds,$$

$$\bar{B}_k(t,x,\bar{y}(t)) := \int_{\varepsilon t}^{t}\int_{|z-\bar{y}(t)|\geq p(t)} \alpha(z)\sum_{i=1}^{k-1}\beta_i^k m_i^{\bar{y}(t)}(s,z)m_{k-i}^{\bar{y}(t)}(s,z)\rho_1(t-s,x,z)dzds,$$

$$\bar{C}_k(t,x,\bar{y}(t)) := \int_{\varepsilon t}^{t}\int_{|z-\bar{y}(t)|\leq p(t)} \alpha(z)\sum_{i=1}^{k-1}\beta_i^k m_i^{\bar{y}(t)}(s,z)m_{k-i}^{\bar{y}(t)}(s,z)\rho_1(t-s,x,z)dzds.$$

From the proof of (i), following the arguments used to show that

$$A_k(t,x,y(t))/g(t,y(t))^k \to 0 \quad \text{and} \quad B_k(t,x,y(t))/g(t,y(t))^k \to 0,$$

we can also show that

$$\bar{A}_k(t,x,\bar{y}(t))/g(t,y(t))^k \to 0 \quad \text{and} \quad \bar{B}_k(t,x,\bar{y}(t))/g(t,y(t))^k \to 0,$$

uniformly in $\|\bar{y}(t)\| \geq \bar{r}(t), \|y(t)\| \leq r(t)$. Next we show that, for $1 \leq i \leq k-1$, the following limit holds uniformly in $\|\bar{y}(t)\| \geq \bar{r}(t), \|y(t)\| \leq r(t)$

$$\lim_{t\to\infty}\int_0^{(1-\varepsilon)t}\int_{\|z-\bar{y}(t)\|\leq\bar{p}(t)}\Big(\frac{g(t-s,\bar{y}(t)-[z])}{g(t,y(t))}\Big)^k\alpha(z)f_i(z)f_{k-i}(z)\rho_1(s,x,z)dzds = 0.$$

Following the same arguments that are detailed before (2.86), it is enough to show that, for all $1 \leq i \leq k-1$,

$$\lim_{t\to\infty}\int_0^{m}\int_{\|z-\bar{y}(t)\|\leq\bar{p}(t)}\Big(\frac{g(t-s,\bar{y}(t)-[z])}{g(t,y(t))}\Big)^k\alpha(z)f_i(z)f_{k-i}(z)\rho_1(s,x,z)dzds = 0.$$

uniformly in $\|\bar{y}(t)\| \geq \bar{r}(t), \|y(t)\| \leq r(t)$. The idea here is that, $\|\bar{y}\|$ as well as $\|y(t)\|$ can be bounded from above by $2\|z\|$ on the domain of integration. Therefore, repeating the arguments from (2.85), using Taylor's formula, given $\delta > 0$ small. since $\|z-\bar{y}(t)\| \leq p(t) = o(t), \|y(t)\| \leq r(t) = o(t)$ and $\nabla\Phi(0) = 0$, along with the estimate (2.41), there exists $C > 0$ such that, for all sufficiently large t and all $1 \leq i \leq k-1$,

$$\int_0^{m}\int_{\|z-\bar{y}(t)\|\leq\bar{p}(t)}\Big(\frac{g(t-s,\bar{y}(t)-[z])}{g(t,y(t))}\Big)^k\alpha(z)f_i(z)f_{k-i}(z)\rho_1(s,x,z)dzds$$

$$\leq C \int_0^m e^{-(k-1)\mu s + 2\delta s} \frac{1}{s^{d/2}} \int_{\|z-\bar{y}(t)\| \leq \bar{p}(t)} e^{\delta\|z\| - \frac{\|z-x\|^2}{cs}} dz ds.$$

Since $\|\bar{y}(t)\| \geq \bar{r}(t)$ and $\|z - \bar{y}(t)\| \leq \bar{p}(t) = o(\bar{r}(t))$, we know, for sufficiently large t, $\|z - x\| \geq \bar{r}(t)/2$. Thus, there exists $a > 0$ such that, for sufficiently large t,

$$\delta\|z\| - \frac{\|z-x\|^2}{cs} \leq \delta\|x\| + \Big(\delta\|z-x\| - \frac{\|z-x\|^2}{cs}\Big) \leq \delta\|x\| - \frac{\|z-x\|^2}{as}.$$

Therefore, there exists a constant $\tilde{C} > 0$ such that, for all $1 \leq i \leq k-1$,

$$\lim_{t\to\infty} \int_0^m \int_{\|z-\bar{y}(t)\| \leq p(t)} \Big(\frac{g(t-s, \bar{y}(t) - [z])}{g(t, y(t))}\Big)^k \alpha(z) f_i(z) f_{k-i}(z) \rho_1(s, x, z) dz ds \leq$$

$$\leq \tilde{C} \lim_{t\to\infty} \int_0^m e^{-(k-1)\mu s + \delta s} \Big(\frac{1}{s^{d/2}} \int_{\|z-x\| \geq \bar{r}(t)} e^{-\frac{\|z-x\|^2}{as}} dz\Big) ds = 0,$$

if δ is sufficiently small. This concludes the proof of (ii) for k.

□

Chapter 3: Asymptotic expansions for Large Deviation Principles

3.1 Introduction

Suppose $\{X_n\}_{n\geq 1}$ is a sequence of centered random variables and $S_n = \sum_{i=1}^{n} X_i$. In the case when $\{X_n\}_{n\geq 1}$ is a independent, identically distributed (iid) sequence of random variables with exponential moments, Cramér's Large Deviation Principle ([28, Chapter 1]) states that the tail probabilities of S_n/n decay exponentially fast. It is natural to ask if this could be made more precise by finding the exact asymptotics. The first rigorous treatment of exact large deviation asymptotics for S_n in the case when $\{X_n\}_{n\geq 1}$ is a iid sequence of random variables, was done by Cramér in [29] assuming the existence of an absolutely continuous component in the distribution of X_1. In the a non-iid setting, in [30], they obtain the exact pre–exponential factor under a decay condition on the Fourier–Laplace transform of the distribution of X_1. In our earlier paper [4], we show that, under a set of natural conditions, which we refer to as weak dependence conditions, distribution functions of sums of random variables (in the discrete time setting) admit higher order asymptotic expansions in the domain of large deviations.

Asymptotic expansions for the Central Limit Theorem (CLT) (called Edgeworth expansions), were first discussed rigorously in [29], later in [31–38], and more recently in [25, 39–42]. In [42, 43], Edgeworth expansions for sums of weakly dependent lattice random variables are obtained. The expansions we obtain in [4], and in this chapter, are in the spirit of Edgeworth expansions, but in the domain of large deviations.

In this chapter, we extend the results obtained in [4], to show that distribution functions of continuous time stochastic processes under additional assumptions (which are detailed in Section 3.2), admit higher order asymptotic expansions in the domain of large deviations (defined rigorously below). The processes that satisfy the conditions detailed in Section 3.2 will be referred to as stochastic processes with weakly dependent increments.

Definition 1 (Strong Expansions for LDP). *Let $\{S_t\}_{t\geq 0}$ be a stochastic process with asymptotic mean zero. Suppose that, for some $r \in \mathbb{N}$, for each $a \in (0,\delta)$, the asymptotic expansion for the distribution function of a stochastic process S_t of the form:*

$$\mathbb{P}(S_t \geq at)e^{I(a)t} = \sum_{k=0}^{[r/2]} \frac{D_k(a)}{t^{k+1/2}} + o_{r,a}\left(\frac{1}{t^{\frac{r+1}{2}}}\right) \qquad as \qquad t \to \infty, \tag{3.1}$$

where, the $I(a)$ denotes the rate function, and $D_k(a)$ are constants. Then, we refer to (3.1) as the strong expansion for LDP of order r in the range $(0,\delta)$.

Here, in Section 3.3, by proving a key proposition (Proposition 3.2.1), we show that the proofs in the discrete time can be adapted to obtain the strong expansions for LDPs for stochastic processes with weakly dependent increments.

We then apply our continuous time results to study additive functionals of diffusion processes satisfying Hörmander's condition on a d–dimensional compact manifold. In Section 3.4 we show that the additive functionals of such diffusion processes have weakly dependent increments. That is, they satisfy the conditions detailed in Section 3.2 that guarantee the existence of strong expansions for LDPs. For related work on large deviation problems for coupled SDEs, see [5, 6].

Now we make a few remarks about the relationship between the setting in Chapter 2, Section 2.2 and the setting here. First observe that each coordinate of the location of a particle undergoing a diffusion process in $\mathbb{Z}^d$ periodic media, Y_t^i (described in (2.1), setting the branching term equal to zero) can be viewed as an additive functional of a diffusion process on a $d-$dimensional torus. That is, suppose $X_t \in \mathbb{T}^d$ is the diffusion process generated by the following partial differential operator on $\mathbb{T}^d$,

$$\mathcal{L} = \frac{1}{2}\sum_{ij=1}^{d} a_{ij}(y)\frac{\partial^2}{\partial y_i \partial y_j} + \sum_{i=1}^{d} b_i(y)\frac{\partial}{\partial y_i}.$$

Then viewing $X_t \in \mathbb{T}^d$ as taking values in $[0,1)^d \subset \mathbb{R}^d$, we can write $Y_t \in \mathbb{R}^d$ as

$$dY_t^i = dX_t^i, \qquad Y_0 = 0, \tag{3.2}$$

for each $1 \leq i \leq d$. Therefore, the analysis of diffusion processes in periodic media in the large deviation domain, done in Chapter 2, Section 2.2 to obtain the exact asymptotics for LDPs, is closely related to the question we pose in this Chapter. In the setting detailed in Section 3.4 of this Chapter, we assume that X_t denotes the solution of a SDE (driven by a $k-$dimensional Wiener process W_t) that satisfies Hörmander's Hypoellipticity condition (as opposed to ellipticity condition, satisfied in Section 2.2) on a arbitrary $d-$dimensional smooth compact manifold, and we assume that $Y_t \in \mathbb{R}^d$ is an additive functional of X_t such that

$$dY_t = h(X_t)d\widetilde{W}_t + c(X_t)dt, \tag{3.3}$$

where the Wiener process $\widetilde{W}_t$ is independent of W_t, $h(x)$ is non-degenerate for each $x \in M$, and h, c are Lipschitz continuous. The difference between (3.2) and (3.3) is that in (3.3) the Wiener process $\widetilde{W}_t$ is independent of W_t, while in (3.2) the process Y_t and X_t have the same underlying $d-$dimensional Wiener process W_t (in X_t it is viewed as a Wiener process on the $d-$dimensional torus while in Y_t it is viewed as a Wiener process on $\mathbb{R}^d$). However, in this chapter, under this stronger requirement of independence of the Weiner processes, we obtain higher order terms of the asymptotic expansion, as opposed to just the first term that was obtained in Section 2.2.

3.2 Overview and main results.

Let $\{S_t\}_{t\geq 0}$ be a stochastic process with asymptotic mean zero, i.e.,

$$\lim_{t\to\infty} \frac{1}{t}\mathbb{E}(S_t) = 0.$$

Suppose that there exists a Banach space $\mathcal{B}$, a family of bounded linear operators $\mathcal{L}(z,t) : \mathcal{B} \to \mathcal{B}$, and vectors $v \in \mathcal{B}, \ell \in \mathcal{B}'$ such that

$$\mathbb{E}(e^{zS_t}) = \ell(\mathcal{L}(z,t)v),\ t > 0,$$

for $z \in \mathbb{C}$ for which the conditions $(D1)$ and $(D2)$ and $(D3)$ (which are detailed below) are satisfied and the family of operators $\mathcal{L}(z,\cdot)$ forms a C^0–semigroup on the

Banach space $\mathcal{B}$. That is

$$\mathcal{L}(z, t_1 + t_2) = \mathcal{L}(z, t_1) \circ \mathcal{L}(z, t_2), \quad \text{for each} \;\; t_1, t_2 \geq 0, \quad \mathcal{L}(z, 0) = \mathrm{Id},$$

and

$$\lim_{t \to 0} \mathcal{L}(z, t) = \mathcal{L}(z, 0) = \mathrm{Id},$$

where the above limit is with respect to the operator norm.

Condition (D1) The family of operators $\mathcal{L}(z, 1 + \eta)$ satisfies the condition $[B]$ (from [4]), uniformly in $\eta \in [0, 1]$. That is,

1. There exists $\delta > 0$ such that the following conditions hold for all $\eta \in [0, 1]$:
   - (B1) $z \mapsto \mathcal{L}(z, 1 + \eta)$ is continuous on the strip $|\mathrm{Re}(z)| < \delta$ and holomorphic on the disc $|z| < \delta$.
   - (B2) For each $\theta \in (-\delta, \delta)$, the operator $\mathcal{L}(\theta, 1 + \eta)$ has an isolated and simple eigenvalue $\lambda(\theta, 1+\eta) > 0$ and the rest of its spectrum is contained inside the disk of radius smaller than $\lambda(\theta, 1 + \eta)$ (spectral gap). In addition, $\lambda(0, 1 + \eta) = 1$.
   - (B3) For each $\theta \in (-\delta, \delta)$, for all real numbers $s \neq 0$, the spectrum of the operator $\mathcal{L}(\theta + is, 1+\eta)$, denoted by $\mathrm{sp}(\mathcal{L}(\theta+is, 1+\eta))$, satisfies: $\mathrm{sp}(\mathcal{L}(\theta+is, 1+\eta)) \subseteq \{z \in \mathbb{C} \mid |z| < \lambda(\theta, 1 + \eta)\}$.
2. For each $\theta \in (-\delta, \delta)$, there exist positive numbers r_1, r_2, K and N_0 such that

$$\|\mathcal{L}(\theta + is, t)\| \leq \frac{\lambda(\theta)^t}{t^{r_2}} \tag{3.4}$$

   for all $t > N_0$, for all $K < |s| < t^{r_1}$.

Condition (D2) Suppose $z \in \mathbb{C}$ is such that, for all $\eta \in [0, 1]$, $\mathcal{L}(z, 1 + \eta)$ has an isolated simple eigenvalue $\lambda(z, 1+\eta)$. Then the projection to the top eigenspace, $\Pi(z, 1+\eta)$, satisfies $\Pi(z, 1+\eta) = \Pi(z, 1)$ for all $\eta \in [0, 1]$.

We denote $\Pi(\theta, 1)$ by Π_θ. Using the above condition, along with the semigroup property, we conclude that for each $t > 0$, the top eigenvalue of the operator $\mathcal{L}(z, t)$ (whenever it exists) is equal to $\lambda(z, 1)^t$.

Due to (D1), the operators $\mathcal{L}(\theta, 1+\eta)$ with $\theta \in (-\delta, \delta)$ and $\eta \in [1, 2]$ take the form

$$\mathcal{L}(\theta, 1+\eta) = \lambda(\theta)^{1+\eta}\Pi(\theta, 1+\eta) + \Lambda(\theta, 1+\eta), \tag{3.5}$$

where $\Pi(\theta, 1+\eta)$ is the eigenprojection corresponding to the eigenvalue $\lambda(\theta)^{1+\eta}$ of the operator $\mathcal{L}(\theta, 1+\eta)$ and $\Pi(\theta, 1+\eta)\Lambda(\theta, 1+\eta) = \Lambda(\theta, 1+\eta)\Pi(\theta, 1+\eta) = 0$. Due to (D1) we can use the perturbation theory of linear operators (see [44, Chapter 7]) to conclude that $\lambda(\cdot)$, $\Pi(\cdot, 1+\eta)$ and $\Lambda(\cdot, 1+\eta)$ are analytic.

As a consequence of (3.5) and condition (D2), the family of operators $\Lambda(\theta, t)$ defined as $\mathcal{L}(\theta, t) - \lambda(\theta)^t\Pi_\theta$ also forms a semigroup, and the spectral radius of the operator $\Lambda(\theta, 1)$ is less than $\lambda(\theta)$ for all $\theta \in (-\delta, \delta)$.

Condition (D3) For all $\theta \in (-\delta, \delta)$, $\ell(\Pi_\theta v) > 0$ and for all $\eta \in [0, 1]$,

$$\frac{\partial^2}{\partial\theta^2} \log \lambda(\theta, 1+\eta) > 0.$$

The following proposition, which will be proved in Section 3.3, is the key idea in adapting the proofs of discrete time results from [4] to continuous time.

Proposition 3.2.1. *Suppose that the conditions* (*D*1) *and* (*D*2) *hold. Then, for a fixed* $\theta \in (-\delta, \delta)$*, there exists* $\tilde{\delta} > 0$ *such that, for each* $s \in (-\tilde{\delta}, \tilde{\delta})$*, for each* $t \geq 1$*,* *the operator* $\mathcal{L}(\theta + is, t)$ *has a simple top eigenvalue* $\lambda(\theta + is)^t$ *and*

$$\mathcal{L}(\theta + is, t) = \lambda(\theta + is)^t \Pi_{\theta+is} + \Lambda(\theta + is, t), \tag{3.6}$$

where $\Pi_{\theta+is} \equiv \Pi(\theta + is, t)$ *is the eigenprojection corresponding to the eigenvalue* $\lambda(\theta + is)^t$ *and* $\Pi(\theta + is, t)\Lambda(\theta + is, t) = \Lambda(\theta + is, t)\Pi(\theta + is, t) = 0$*. In addition, the family of operators* $\{\Lambda(\theta + is, t)\}_{t\geq 1}$ *satisfies* $\Lambda(\theta + is, tN) = \Lambda(\theta + is, t)^N$ *for all* $t \geq 1, N \in \mathbb{N}$ *and the spectral radius of the operator* $\Lambda(\theta + is, 1)$ *is less than* $|\lambda(\theta + is)|$.

The following theorems are the continuous time analogues of Theorem 2.6 and Theorem 2.7 from [4], respectively. We do not repeat the proofs of Theorems 3.2.2 and 3.2.3 in our current continuous time setting, since the proofs are completely analogous to those in [4]. The crucial point, however, is that the continuous time results require the use of Proposition 3.2.2 which we prove in the next section.

Theorem 3.2.2. *Let* $r \in \mathbb{N}$, $r \geq 2$*. Suppose that conditions* $(D1), (D2)$ *and* $(D3)$ *hold with* $r_1 > r/2$*. Then, for each* $a \in \left(0, \dfrac{\log \lambda(\delta)}{\delta}\right)$*, there exist constants* $D_k(a)$ *such that*

$$\mathbb{P}(S_t \geq at)e^{I(a)t} = \sum_{k=0}^{[r/2]} \frac{D_k(a)}{t^{k+1/2}} + o_{r,a}\left(\frac{1}{t^{\frac{r+1}{2}}}\right) \qquad as \qquad t \to \infty,$$

where, the rate functional $I(a)$ *is defined as*

$$I(a) := \sup_{\theta \in (0,\delta)} [a\theta - \log \lambda(\theta, 1)] = a\theta_a - \log \lambda(\theta_a, 1).$$

Theorem 3.2.3. *Suppose that* $(D1)-(1),(D2)$ *and* $(D3)$ *hold. Then, for each* $a \in \Big(0, \dfrac{\log \lambda(\delta)}{\delta}\Big)$,

$$\mathbb{P}(S_t \geq at)e^{I(a)t} = \frac{\ell(\Pi_{\theta_a} v)\sqrt{I''(a)}}{\theta_a \sqrt{2\pi t}}\Big(1+o(1)\Big) \quad \textit{as } t \to \infty.$$

3.3 Proofs of Proposition 3.2.1 and Theorems 3.2.2 and 3.2.3

Proof of Proposition 3.2.1. Let $\theta \in (-\delta, \delta)$ and $\eta \in [0,1]$ be fixed. Consider the two parameter perturbation of the operator $\mathcal{L}(\theta, 1+\eta)$ of the form $\mathcal{L}(\theta + is, 1+\eta+\varepsilon)$. From condition (D1), for a fixed η, $z \mapsto \mathcal{L}(z, 1+\eta)$ is holomorphic on the disc $|z| < \delta$ and for each fixed z, the family of operators $\mathcal{L}(z,t)$ forms a C^0–semigroup. In addition, the two parameter operator $\mathcal{L}(z,t)$ is uniformly bounded on the region $\{(z,t) : |z| < \delta, t \in [1,2]\}$. From here, using the Cauchy integral formula for analytic functions it is clear to see that this two parameter perturbation is continuous. Hence, by perturbation theory, for each $\eta \in [0,1]$, there exists $\delta_\eta > 0$ such that, on the set $\{(s,\varepsilon) : |s| < \delta_\eta, \varepsilon < \delta_\eta\}$,

$$\mathcal{L}(\theta + is, 1+\eta+\varepsilon) = \lambda(\theta + is, 1+\eta+\varepsilon)\Pi(\theta+is, 1+\eta+\varepsilon) + \Lambda(\theta+is, 1+\eta+\varepsilon),$$

where $\Pi(\theta+is, 1+\eta+\varepsilon)$ is the projection on the top eigenfunction of the operator $\mathcal{L}(\theta+is, 1+\eta+\varepsilon)$ corresponding to the simple top eigenvalue $\lambda(\theta+is, 1+\eta+\varepsilon)$ and

$$\Pi(\theta+is,1+\eta+\varepsilon)\Lambda(\theta+is,1+\eta+\varepsilon) = \Lambda(\theta+is,1+\eta+\varepsilon)\Pi(\theta+is,1+\eta+\varepsilon) = 0.$$

In addition, the spectral radius of $\Lambda(\theta+is,1+\eta+\varepsilon)$ is less than $|\lambda(\theta+is,1+\eta+\varepsilon)|$.

Since the interval $[0,1]$ is compact, we can choose $\eta_1, \eta_2, \cdots, \eta_k$ such that the set $\{\eta : |\eta - \eta_i| < \delta_{\eta_i}, i = 1, 2, \cdots k\}$ contains the interval $[0,1]$. Put $\tilde{\delta} = \min_{i=1,2,\cdots k} \delta_{\eta_i}$. Thus, for all $\eta \in [0,1]$ and s such that $|s| < \tilde{\delta}$,

$$\mathcal{L}(\theta + is, 1 + \eta) = \lambda(\theta + is, 1 + \eta)\Pi(\theta + is, 1 + \eta) + \Lambda(\theta + is, 1 + \eta),$$

and the spectral radius of $\Lambda(\theta + is, 1 + \eta)$ is less than $|\lambda(\theta + is, 1 + \eta)|$.

Put $\Pi_{\theta+is} = \Pi(\theta + is, 1)$. From (D2) we know that $\Pi(\theta + is, 1 + \eta) = \Pi_{\theta+is}$ for all $\eta \in [0,1]$ and $|s| < \tilde{\delta}$. This, along with the semigroup property of the operators $\mathcal{L}(\theta + is, t)$, implies that $\lambda(\theta + is, 1 + \eta) = \lambda(\theta + is)^{1+\eta}$ for all for all $\eta \in [0,1]$, $|s| < \tilde{\delta}$. To see this, first note that we do not assume that the top eigen-value for the operator $\mathcal{L}(\theta + is, \eta)$ exists for $\eta \in [0,1)$. Now, if η is rational, we have $\eta = p/q$ for some $p, q \in \mathbb{N}, q \neq 0$. Let $v(\theta + is) \in \mathcal{B}$ be a non-zero vector be such that $\Pi(\theta + is, 1 + \eta)v(\theta + is) = \Pi_{\theta+is}v(\theta + is) = v(\theta + is)$ for all $\eta \in [1,2]$. Then we have,

$$\begin{aligned}
\lambda(\theta + is)^{q+p}v(\theta + is) &= \mathcal{L}(\theta + is, 1)^{q+p}v(\theta + is) \\
&= \mathcal{L}(\theta + is, q + p)v(\theta + is) \\
&= \mathcal{L}(\theta + is, 1 + p/q)^q v(\theta + is) \\
&= \lambda(\theta + is, 1 + p/q)^q v(\theta + is).
\end{aligned}$$

Therefore, $\lambda(\theta + is)^{1+\eta} = \lambda(\theta + is, 1 + \eta)$ for all rational $\eta \in [0,1]$. Since, the semigroup $\mathcal{L}(\theta+is, t)$ is continuous in t, we have that the top eigenvalue $\lambda(\theta+is, 1+\eta)$ is continuous in η, and therefore, the relation $\lambda(\theta + is)^{1+\eta} = \lambda(\theta + is, 1 + \eta)$ holds for all $\eta \in [0,1]$.

For $t \geq 1$, define the new family of operators $\Lambda(\theta + is, t) = \mathcal{L}(\theta + is, t) - \lambda(\theta + is)^t\Pi_{\theta+is}$. It is clear to see from this definition that $\Lambda(\theta + is, tN) = \Lambda(\theta + is, t)^N$ for all $t \geq 1, N \in \mathbb{N}$. Then, using the fact that $\frac{t}{[t]} \in [1,2]$, we have

$$\mathcal{L}(\theta + is, t) = \mathcal{L}\Big(\theta + is, \frac{t}{[t]}\Big)^{[t]} = \Big(\lambda(\theta + is)^{\frac{t}{[t]}}\Pi_{\theta+is} + \Lambda\Big(\theta + is, \frac{t}{[t]}\Big)\Big)^{[t]}$$
$$= \lambda(\theta + is)^t \Pi_{\theta+is} + \Lambda(\theta + is, t).$$

Here, the spectral radius of the operator $\Lambda(\theta + is, 1)$ is less than $|\lambda(\theta + is)|$. This concludes the proof of Proposition 3.2.1. □

Remark 3.3.1. *Our equation* (3.6) *is the continuous time analogue of equation* (2.2) *from [4]. This, along with assumption* ($D3$)*, allows us to obtain proofs of Theorems 3.2.2 and 3.2.3 by replacing the discrete time steps n by $t \in \mathbb{R}^+$ and replacing $\overline{\mathcal{L}}_s^n$ by $\overline{\mathcal{L}}(s,t) = \frac{e^{-iast}}{\lambda(\theta)^t}\mathcal{L}(\theta_a + is, t)$ in the proofs of the corresponding discrete time results from [4].*

3.4 SDEs satisfying Hörmander Hypoellipticity condition

Let M be a compact $d-$ dimensional smooth manifold and $\{V_0, \ldots, V_k\}$ be a collection of smooth vector fields of M such that $D = \{V_1, \ldots V_k\}$ satisfies the Hörmander Hypoellipticity condition, i.e., the Lie algebra generated by D evaluated at x spans the tangent space T_xM at each $x \in M$.

Let W_t be the $k-$dimensional Wiener process with components W_t^i for $1 \leq i \leq k$. Let X_t be the process on M, and Y_t be the process on $\mathbb{R}$ satisfying the coupled SDEs,

$$dX_t = \sum_{i=1}^{k} V_i(X_t) \circ dW_t^i + V_0(X_t)\, dt, \qquad X_0 = x, \tag{3.7}$$

$$dY_t = \sigma(X_t) \circ d\widetilde{W}_t + b(X_t)\, dt, \qquad Y_0 = y, \tag{3.8}$$

where the real valued function $b : M \to \mathbb{R}$ and the real valued function $\sigma : M \to \mathbb{R}$ are smooth and $\widetilde{W}_t$ is a $1-$dimensional Wiener process independent of the $k-$dimensional Wiener process W_t. We also assume that σ is non-degenerate, i.e, $\sigma^2(x) > 0$ for each $x \in M$. The right hand sides of (3.7) and (3.8) are interpreted in the Stratonovich sense. Observe that, in (3.8), it is equivalent to consider the Itô or the Stratonovich sense, since the coefficient $\sigma(X_t)$ of the Wiener process $\widetilde{W}_t$ is independent of Y_t. Note that the distribution of X_t for each $t > 0$ is absolutely continuous by Hörmander's theorem.

Theorem 3.4.1. *If the above assumptions hold, then for all $r \in \mathbb{N} \cup \{0\}$, Y_t admits the strong expansion for LDP of order r in the range $(0, \infty)$.*

Proof. The infinitesimal generator of the joint Markov process (X_t, Y_t) is a partial differential operator $\mathcal{M}$ acting on functions u defined on $M \times \mathbb{R}$ given by

$$\mathcal{M}u = \frac{1}{2}\nabla_x[(V(x)V^T(x))\nabla_x u] + \frac{1}{2}(\sigma^2(x))\Delta_y u + V_0(x)\nabla_x u + b(x)\nabla_y u, \tag{3.9}$$

where $V(x)$ is the $d \times k$ matrix formed by the vectors $\{V_1, \ldots V_k\}$ as columns.

Let $\bar{\rho}(x)$ be the invariant density of the process X_t on M, that is, $\bar{\rho}(x)$ is the density of a measure defined on M, satisfying

$$\mathcal{M}^* \bar{\rho} = 0, \qquad \int_M \bar{\rho} = 1.$$

We assume that

$$\int_M b(x)\, d\bar{\rho}(x) = 0.$$

The above condition guarantees that the asymptotic mean of the random process

Y_t is zero, since

$$\bar{Y} = \lim_{t\to\infty} \frac{1}{t}\mathbb{E}(Y_t) = \lim_{t\to\infty} \frac{1}{t}\mathbb{E}\Big(\int_0^t b(X_s)\, ds\Big) = \int_M b(x)\, d\bar{\rho}(x)$$

We also observe that, from the Kolmogorov Forward Equation, the transition density for the Markov process $(X_t, Y_t)_{t\geq 0}$ is given by $p(t, (x_0, y_0), (x, y))$, and it satisfies the PDE

$$\begin{aligned} \partial_t p &= \mathcal{M}^*_{(x,y)} p, \\ p(0, (x_0, y_0), (x, y)) &= \delta_{(x_0,y_0)}(x, y). \end{aligned} \tag{3.10}$$

Let $\mathcal{B}$ be the Banach space of complex valued continuous functions defined on M equipped with the supremum norm. Define, for each $z \in \mathbb{C}$, $t \geq 0$, the bounded linear operator $\mathcal{L}(z, t) : \mathcal{B} \to \mathcal{B}$ given by

$$\mathcal{L}(z, t)f(x) = \mathbb{E}_{(x,y)}(f(X_t)e^{z(Y_t-y)}),$$

where the right hand side clearly does not depend on y. That is, for the constant function $v = \mathbf{1} \in \mathcal{B}$, and the measure $\ell = \delta_x \in \mathcal{B}'$ (the space of bounded linear functionals on $\mathcal{B}$) we have

$$\mathbb{E}_{(x,0)}\left(e^{zY_t}\right) = \ell(\mathcal{L}(z, t)v), \tag{3.11}$$

The family of operators $\{\mathcal{L}(z, t)\}_{\{t\geq 0\}}$ forms a semigroup since

$$\begin{aligned} \mathcal{L}(z, t) \circ \mathcal{L}(z, s)f(x) &= \mathbb{E}_{(x,y)}((\mathcal{L}(z, s)f)(X_t)e^{z(Y_t-y)}) \\ &= \mathbb{E}_{(x,y)}(\mathbb{E}_{(X_t,Y_t)}(f(X_s)e^{z(Y_s-Y_t)})e^{z(Y_t-y)}) \\ &= \mathbb{E}_{(x,y)}(\mathbb{E}_{(X_t,Y_t)}(f(X_s)e^{z(Y_s-y)})) \\ &= \mathbb{E}_{(x,y)}(f(X_{s+t})e^{z(Y_{s+t}-y)}) \end{aligned}$$

$$= \mathcal{L}(z, t+s)f(x).$$

Now we will verify conditions $(D1)$, $(D2)$ and $(D3)$ from Section 3.2 for the family of operators $\mathcal{L}(z,t)$. To verify condition $(D1)$, we will show that $(B1)-(B3)$ hold uniformly on $t \in [1,2]$ and show that (3.4) holds.

Condition (B1) We first observe that the map $z \mapsto \mathcal{L}(z,t)$ is infinitely differentiable in z for all $z \in \mathbb{C}$. Indeed, for each $f \in \mathcal{B}$, $\alpha \in \mathbb{Z}_+$, and $z \in \mathbb{C}$, $D_z^\alpha(\mathcal{L}(z,t)f)(x_0) = \mathbb{E}_{(x_0,0)}(Y_t^\alpha f(X_t)e^{zY_t})$. We know that Y_t is a stochastic process on $\mathbb{R}$ with bounded diffusion and drift coefficients, which implies that Y_t has all exponential moments. Hence, $D_z^\alpha \mathcal{L}(z,t)$ is a well defined bounded linear operator on $\mathcal{B}$ for all $\alpha \in \mathbb{Z}_+$ and $z \in \mathbb{C}$.

Note that $\mathcal{L}(0,t)$ is a compact operator on $\mathcal{B}$ since, if we define

$$q_{0,t}(x_0, x) = \int_{\mathbb{R}} p(t, (x_0, 0), (x, y))dy,$$

then, for any $f \in \mathcal{B}$, $\mathcal{L}(0,t)f(x_0) = \displaystyle\int_M f(x)q_{0,t}(x_0,x)dx$, where $q_{0,t}$ is positive and continuous in $(x_0, x) \in M \times M$. We note that 1 is the top eigenvalue of $\mathcal{L}(0,t)$ with constant functions forming the eigenspace. All the other eigenvalues of $\mathcal{L}(0,t)$ have absolute values less than 1, by the Perron–Frobenius theorem.

We note that if $\theta \in \mathbb{R}$, then $q_{\theta,t}(x_0,x) = \displaystyle\int_{\mathbb{R}} e^{\theta y} p(t,(x_0,0),(x,y))dy > 0$ for all $x_0, x \in M$. This kernel is continuous in $(x_0, x) \in M \times M$. That is, $\mathcal{L}(\theta, t)$ is a positive, compact operator for all $\theta \in \mathbb{R}$.

Condition (D2): We observe that the coefficients of the operator $\mathcal{M}$ are independent of the time variable t, and therefore the Markov process (X_t, Y_t) is time homogeneous. Thus, the top eigenspace of the operators $\mathcal{L}(\theta,t)$ is the same for all

$t > 0$. Thus, $\Pi(\theta, t) = \Pi(\theta, 1)$ for all $t > 0$, in particular, condition (D2) holds.

Condition (B2) Using (D2) and the semigroup property, condition (B2) is satisfied since there exists a $\lambda(\theta) > 0$ for all θ, the top eigenvalue $\lambda(\theta)^t$ of the operator $\mathcal{L}(\theta, t)$ exists, and other eigenvalues of $\mathcal{L}(\theta, t)$ have absolute values less than $\lambda(\theta)^t$.

Condition (B3) We need to show that $\mathrm{sp}(\mathcal{L}(\theta + is, t)) \subseteq \{|z| < \lambda(\theta)^t\}$. We first note that

$$|\mathcal{L}(\theta + is, t)f(x)| = |\mathbb{E}_{(x,y)}(f(X_t)e^{(\theta+is)(Y_t-y)})| \leq \mathbb{E}_{(x,y)}(|f(X_t)e^{(\theta+is)(Y_t-y)}|)$$

$$= \mathbb{E}_{(x,y)}(|f(X_t)|e^{\theta(Y_t-y)}) = \mathcal{L}(\theta, t)|f|(x).$$

Thus $\mathrm{sp}(\mathcal{L}(\theta + is, t)) \subseteq \{|z| \leq \lambda(\theta)^t\}$. To prove that there is inclusion with strict inequality, using the fact that the top eigenvalue of the operator $\mathcal{L}(\theta, t)$ is $\lambda(\theta)^t$, it is enough to show that $\mathrm{sp}(\mathcal{L}(\theta + is, 1)) \subseteq \{|z| < \lambda(\theta)\}$. We suppose, on the contrary, that there exists an eigenfunction $f \in \mathcal{B}$ of the operator $\mathcal{L}(\theta + is, 1)$, with $\|f\| = 1$ corresponding to the eigenvalue $\lambda(\theta + is)$ such that $|\lambda(\theta + is)| = \lambda(\theta)$. That is, for all $x \in M$,

$$\mathbb{E}_{(x,0)}(f(X_1)e^{(\theta+is)Y_1}) = \lambda(\theta + is)f(x). \tag{3.12}$$

We know $\lambda(\theta)$ is the top eigenvalue of the operator $\mathcal{L}(\theta, 1)$. Thus, there exists an eigenfunction $g \in \mathcal{B}$ of $\mathcal{L}(\theta, 1)$, corresponding to the eigenvalue $\lambda(\theta)$, which implies that for all $x \in M$,

$$\mathbb{E}_{(x,0)}(g(X_1)e^{\theta Y_1}) = \lambda(\theta)g(x). \tag{3.13}$$

Without loss of generality, we can assume that, for all $x \in M$, $g(x) > 0$, and that $|f(x)| \leq g(x)$. In addition, we can assume that there exists a point $x_0 \in M$ such that $|f(x_0)| = g(x_0)$. Now,

$$|\mathbb{E}_{(x_0,0)}(f(X_1)e^{(\theta+it)Y_1})| = |\lambda(\theta)f(x_0)| = \lambda(\theta)g(x_0) = \mathbb{E}_{(x_0,0)}(g(X_1)e^{\theta Y_1}).$$

Thus,

$$\mathbb{E}_{(x_0,0)}(|f(X_1)e^{(\theta+it)Y_1}|) \geq \mathbb{E}_{(x_0,0)}(g(X_1)e^{\theta Y_1}).$$

This implies that

$$\mathbb{E}_{(x_0,0)}(e^{\theta Y_1}(|f(X_1)e^{itY_1}| - g(X_1))) \geq 0,$$

and therefore,

$$\mathbb{E}_{(x_0,0)}(e^{\theta Y_1}(|f(X_1)| - g(X_1))) = \mathcal{L}(\theta,1)(|f| - g)(x_0) \geq 0.$$

We have from our assumption that $|f| \leq g$, and we know that $\mathcal{L}(\theta,1)$ is a positive operator. We conclude that,

$$\mathcal{L}(\theta,1)(|f| - g)(x_0) = \mathbb{E}_{(x_0,0)}(e^{\theta Y_1}(|f(X_1)| - g(X_1))) = 0.$$

Now,

$$\mathbb{E}_{(x_0,0)}(e^{\theta Y_1}(|f(X_1)| - g(X_1))) = \int_M (|f(x)| - g(x))q_{\theta,1}(x_0,x)dx.$$

From the definition of $q_{\theta,1}$, we know that, for a fixed $x_0 \in M$, $q_{\theta,1}(x_0,x) > 0$, $x \in M$. Therefore, for all $x \in M$, $|f(x)| = g(x)$. Thus, there exists a continuous function ϕ defined on M such that $f(x) = e^{i\phi(x)}g(x)$ for all $x \in M$. Substituting this in (3.12), we get

$$\mathbb{E}_{(x,0)}(e^{i\phi(X_1)}g(X_1)e^{(\theta+is)Y_1}) = \lambda(\theta+is)e^{i\phi(x)}g(x)$$

$$= e^{i\phi(x)}\mathbb{E}_{(x,0)}(g(X_1)e^{\theta Y_1})\frac{\lambda(\theta+is)}{\lambda(\theta)},$$

where the last equality follows from equation (3.13). In addition, since $|\lambda(\theta+is)| = \lambda(\theta)$, there exists a constant c such that $\dfrac{\lambda(\theta+is)}{\lambda(\theta)} = e^{ic}$. Therefore,

$$\mathbb{E}_{(x,0)}(e^{i\phi(x)}e^{\theta Y_1}e^{ic}g(X_1)(e^{isY_1+i\phi(X_1)-i\phi(x)-ic}-1)) = 0.$$

This implies that whenever $p(1,(x,0),(\tilde{x},\tilde{y})) > 0$,

$$s\tilde{y} + \phi(\tilde{x}) - \phi(x) - ic = 0 \pmod{2\pi}.$$

This is not possible since the Brownian motion $\widetilde{W}$ (in the definition of Y_1) is independent of W (in the definition of X_1). Thus, $\mathrm{sp}(\mathcal{L}(\theta+is,1)) \subseteq \{|z| < \lambda(\theta)\}$, which implies $\mathrm{sp}(\mathcal{L}(\theta+is,t)) \subseteq \{|z| < \lambda(\theta)^t\}$.

Condition ((3.4)) Let $\theta \in \mathbb{R}$ be fixed. Let $g_\theta(x)$ be such that $\|g_\theta\| = 1$ and $\mathcal{L}(\theta,1)g_\theta(x) = \lambda(\theta)g_\theta(x)$ for all $x \in M$. Then we also have $\mathcal{L}(\theta,t)g_\theta(x) = \lambda(\theta)^t g_\theta(x)$ for all $x \in M$, since condition (D2) holds. In addition, since $\mathcal{L}(\theta,1)$ is a positive operator, the eigenfunction g_θ is positive. We observe that g_θ satisfies the PDE $e^{-\theta y}\mathcal{M}(e^{\theta y}g_\theta(x)) = \mu(\theta)g_\theta(x)$ for all $x \in M$, $y \in \mathbb{R}$, where $\mu(\theta) = \log\lambda(\theta)$. Since the coefficients of the operator $e^{-\theta y}\mathcal{M}(e^{\theta y}\cdot)$ are differentiable in θ, the function g_θ is differentiable in θ.

We first consider a new family of operators $\tilde{\mathcal{L}}(z,t) : \mathcal{B} \to \mathcal{B}$ defined by

$$\tilde{\mathcal{L}}(z,t)f(x_0) = \int_M f(x)\tilde{q}_{z,t}(x_0,x)\,dx,$$

where $\tilde{q}_{z,t}(x_0,x) = \displaystyle\int_{\mathbb{R}} e^{zy}p_\theta(t,(x_0,0),(x,y))\,dy$ and

$$p_\theta(t,(x_0,0),(x,y)) := \frac{e^{\theta y}g_\theta(x)p(t,(x_0,0),(x,y))}{\lambda(\theta)^t g_\theta(x_0)}.$$

Let $\mathbf{1}$ denote the function that takes the value 1 for all $x_0 \in M$. Note that

$$\tilde{\mathcal{L}}(0,t)\mathbf{1}(x_0) = \int_M 1 \cdot \tilde{q}_{0,t}(x_0,x)\,dx$$

$$
\begin{aligned}
&= \int_M \int_{\mathbb{R}} p_\theta(t,(x_0,0),(x,y))\,dy\,dx \\
&= \int_M \int_{\mathbb{R}} \frac{e^{\theta y} g_\theta(x) p(t,(x_0,0),(x,y))}{\lambda(\theta)^t g_\theta(x_0)}\,dy\,dx \\
&= \frac{1}{\lambda(\theta)^t g_\theta(x_0)} \int_M \int_{\mathbb{R}} e^{\theta y} g_\theta(x) p(t,(x_0,0),(x,y))\,dy\,dx \\
&= \frac{1}{\lambda(\theta)^t g_\theta(x_0)} \mathcal{L}(\theta,t) g_\theta(x_0) = 1.
\end{aligned}
$$

Hence, $\mathbf{1}$ is an eigenfunction for the operator $\tilde{\mathcal{L}}(0,t)$ corresponding to the top eigenvalue 1.

Observe that the operators $\tilde{\mathcal{L}}$ and $\mathcal{L}$ satisfy, for all $f \in \mathcal{B}$,

$$
\tilde{\mathcal{L}}(z,t) f(x_0) = \frac{1}{\lambda(\theta)^t g_\theta(x_0)} \mathcal{L}(\theta+z,t)(f g_\theta)(x_0).
$$

It is easy to see that the new family of operators $\{\tilde{\mathcal{L}}(z,t)\}_{t\geq 0}$ also forms a C_0 semigroup. Thus, in order to prove (3.4), we need to show that there exist positive numbers r_1, r_2, K and N_0 such that

$$
\|\tilde{\mathcal{L}}(is,t)\| \leq \frac{1}{t^{r_2}}
$$

for all $t > N_0$, for all $K < |s| < t^{r_1}$. In fact, it will be enough to show that there exists an $\epsilon \in (0,1)$ such that, for all $t \in [1,2]$ and for all $|s| > K$,

$$
\|\tilde{\mathcal{L}}(is,t)\| < 1 - \epsilon, \tag{3.14}
$$

since the above relation would imply that, for all $t > 2$,

$$
\|\tilde{\mathcal{L}}(is,t)\| = \left\| \tilde{\mathcal{L}}\Big(is, \frac{t}{[t]}\Big)^{[t]} \right\| \leq \left\| \tilde{\mathcal{L}}\Big(is, \frac{t}{[t]}\Big) \right\|^{[t]} \leq (1-\epsilon)^{[t]}.
$$

showing exponential decay.

We observe that for any $f \in \mathcal{B}$, and $x_o \in M$,

$$\tilde{\mathcal{L}}(is,t)f(x_0) = \int_M f(x)\tilde{q}_{is,t}(x_0,x)\,dx$$

where,

$$\tilde{q}_{is,t}(x_0,x) = \int_{\mathbb{R}} \frac{e^{(\theta+is)y}g_\theta(x)p(t,(x_0,0),(x,y))}{\lambda(\theta)^t g_\theta(x_0)}\,dy,$$

ans therefore, it is enough to show that there exists an $\epsilon \in (0,1)$ and $K > 0$ such that for all $|s| > K$, and for all $t \in [1,2]$,

$$|\tilde{q}_{is,t}(x_0,x)| \leq 1 - \epsilon. \tag{3.15}$$

Let $\mathcal{F}_t$ denote the sigma algebra generated by the process $\{W_u\}_{u\in[0,t]}$. Note that the following equality holds,

$$\begin{aligned}\tilde{q}_{is,t}(x_0,x) &= \frac{g_\theta(x)}{\lambda(\theta)^t g_\theta(x_0)}\mathbb{E}_{(x_0,0)}\left(e^{(\theta+is)Y_t}\middle|X_t = x\right)\\ &= \frac{g_\theta(x)}{\lambda(\theta)^t g_\theta(x_0)}\mathbb{E}_{(x_0,0)}\left(\mathbb{E}(e^{(\theta+is)\left(\int_0^t \sigma(X_u)\,d\widetilde{W}_u + \int_0^t + b(X_u)\,du\right)}\middle|\mathcal{F}_t)\middle|X_t = x\right)\right)\end{aligned}$$

We know that $\left\{e^{\int_0^t(\theta+is)\sigma(X_u)\,d\widetilde{W}_u - \frac{1}{2}\int_0^t(\theta+is)^2\sigma^2(X_u)\,du}\middle|\mathcal{F}_t\right\}$ forms a martingale for all $t > 0$. Therefore,

$$\begin{aligned}&\mathbb{E}(e^{(\theta+is)Y_t}|\mathcal{F}_t) = \mathbb{E}(e^{(\int_0^t(\theta+is)^2\sigma^2(X_u)\,du + (\theta+is)\int_0^t b(X_u)\,du)}|\mathcal{F}_t)\\ &= \mathbb{E}\left(e^{(\theta^2\int_0^t\sigma^2(X_u)\,du - s^2\int_0^t\sigma^2(X_u)\,du + 2is\theta\int_0^t\sigma^2(X_u)\,du + (\theta+is)\int_0^t b(X_u)\,du)}\middle|\mathcal{F}_t\right).\end{aligned}$$

Let $\epsilon \in (0,1)$. Since $\sigma(x)$, $b(x)$ are smooth on the compact manifold M, and $\sigma(x) > 0$ for all $x \in M$, for a fixed $\theta > 0$, we can choose $K > 0$ such that for all $t \in [1,2]$, $|s| > K$,

$$\Big|\mathbb{E}(\exp\Big((\theta^2\int_0^t\sigma^2(X_u)\,du - s^2\int_0^t\sigma^2(X_u)\,du +$$

$$+ 2is\theta \int_0^t \sigma^2(X_u)\, du + (\theta + is) \int_0^t b(X_u)\, du)\Big)|\mathcal{F}_t)\Big|$$

$$< (1-\epsilon)\frac{\|g_\theta\| \sup\{\lambda(\theta)^t | t \in [1,2]\}}{\inf\{g_\theta(x)|x \in M\}}.$$

Note that the quantities $\sup\{\lambda(\theta)^t \mid t \in [1,2]\}$ and $\inf\{g_\theta(x) \mid x \in M\}$ are strictly positive and finite due to condition (B2) and the fact that eigenfunction g_θ is strictly positive on M. Therefore,

$$\Big|\mathbb{E}_{(x_0,0)}(e^{(\theta+is)Y_t}|X_t = x)\Big| = \Big|\mathbb{E}_{(x_0,0)}\Big(e^{(\theta+is)\left(\int_0^t \sigma(X_u)\, d\widetilde{W}_u + \int_0^t b(X_u)\, du\right)}\Big|\mathcal{F}_t\Big)\Big|X_t = x\Big)\Big|$$

$$\le \mathbb{E}_{(x_0,0)}\Big(\Big|\Big(\mathbb{E}(e^{(\theta+is)(\int_0^t \sigma(X_u)\, d\widetilde{W}_u + \int_0^t b(X_u)\, du)}\Big|\mathcal{F}_t\Big)\Big|\ \Big|X_t = x\Big)$$

$$\le (1-\epsilon)\frac{\|g_\theta\| \sup\{\lambda(\theta)^t | t \in [1,2]\}}{\inf\{g_\theta(x)|x \in M\}},$$

As a result $|\tilde{q}_{is,t}(x_0, x)| \le (1-\epsilon)$. This implies that for all $t \in [1,2]$, $|s| > K$, $\|\tilde{\mathcal{L}}(is, t)\| < 1 - \epsilon$, which concludes the proof of condition (D1).

Condition (D3): First, observe that $\ell(\Pi_\theta v) = \delta_x(\Pi_{g_\theta}\mathbf{1}) = g_\theta(x)\int_M g_\theta > 0$. Now, that the top eigenvalue of operators $\mathcal{L}(z, 1+\eta)$ is $\lambda(\theta)^{1+\eta}$. Thus, it is enough to show that $\log\lambda(\theta)$ is twice continuously differentiable and the second derivative is positive for all $\theta \in \mathbb{R}$. Let $\mu(\theta) = \log\lambda(\theta)$.

Let $\theta > 0$ be fixed. We know that the function g_θ is such that

$$\mathcal{L}(\theta, t) g_\theta = e^{t\mu(\theta)} g_\theta. \tag{3.16}$$

Let ψ_θ be a linear functional in $\mathcal{B}'$ satisfying $\langle \psi_\theta, \mathcal{L}(\theta, t) f\rangle = e^{t\mu(\theta)}\langle\psi_\theta, f\rangle$ for all $f \in \mathcal{B}$, and $\langle\psi_\theta, g_\theta\rangle = 1$. Let us define a new operator $\mathcal{L}'(\theta, t)$, which is the derivative of the operator $\mathcal{L}(\theta, t)$ with respect to θ. Thus,

$$\Big(\mathcal{L}'(\theta, t) f\Big)(x_0) = \mathbb{E}_{(x_0,0)}(f(X_t) Y_t e^{\theta Y_t}).$$

We differentiate equation (3.16) on both sides with respect to θ to obtain

$$\begin{aligned}\mathcal{L}'(\theta,t)g_\theta(x_0)+\mathcal{L}(\theta,t)g'_\theta(x_0)&=\mathbb{E}_{(x_0,0)}(g_\theta(X_t)Y_te^{\theta Y_t})+\mathcal{L}(\theta,t)g'_\theta(x_0)\\&=t\mu'(\theta)e^{t\mu(\theta)}g_\theta(x_0)+e^{t\mu(\theta)}g'_\theta(x_0).\end{aligned}\tag{3.17}$$

Therefore, applying the linear functional ψ_θ on both sides, we obtain,

$$\langle\psi_\theta,\mathbb{E}_{(x,0)}(g_\theta(X_t)Y_te^{\theta Y_t})\rangle+\langle\psi_\theta,\mathcal{L}(\theta,t)g'_\theta\rangle=t\mu'(\theta)e^{t\mu(\theta)}\langle\psi_\theta,g_\theta\rangle+e^{t\mu(\theta)}\langle\psi_\theta,g'_\theta\rangle,$$

which simplifies to

$$\langle\psi_\theta,\mathbb{E}_{(x,0)}(g_\theta(X_t)Y_te^{\theta Y_t})\rangle+e^{t\mu(\theta)}\langle\psi_\theta,g'_\theta\rangle=t\mu'(\theta)e^{t\mu(\theta)}+e^{t\mu(\theta)}\langle\psi_\theta,g'_\theta\rangle.$$

Thus, we obtain the following formula for $\mu'(\theta)$.

$$\mu'(\theta)=\frac{\langle\psi_\theta,\mathbb{E}_{(x,0)}(g_\theta(X_t)Y_te^{\theta Y_t})\rangle}{te^{t\mu(\theta)}}.\tag{3.18}$$

Differentiating the equation (3.17) again with respect to θ and taking the action of the linear functional ψ_θ on both sides, we obtain,

$$\begin{aligned}&\langle\psi_\theta,\mathbb{E}_{(x,0)}(g_\theta(X_t)Y_t^2e^{\theta Y_t})\rangle+2\langle\psi_\theta,\mathbb{E}_{(x,0)}(g'_\theta(X_t)Y_te^{\theta Y_t})\rangle+e^{t\mu(\theta)}\langle\psi_\theta,g''_\theta\rangle\\&=t\mu''(\theta)e^{t\mu(\theta)}+t^2(\mu'(\theta))^2e^{t\mu(\theta)}+2t\mu'(\theta)e^{t\mu(\theta)}\langle\psi_\theta,g'_\theta\rangle+e^{t\mu(\theta)}\langle\psi_\theta,g''_\theta\rangle.\end{aligned}$$

Thus, rearranging the terms, we obtain the following formula for $\mu''(\theta)$:

$$\begin{aligned}\mu''(\theta)=&\frac{\langle\psi_\theta,\mathbb{E}_{(x,0)}(g_\theta(X_t)Y_t^2e^{\theta Y_t})\rangle-t^2(\mu'(\theta))^2e^{t\mu(\theta)}}{te^{t\mu(\theta)}}\\&+2\frac{\langle\psi_\theta,\mathbb{E}_{(x,0)}(g'_\theta(X_t)Y_te^{\theta Y_t})\rangle-t\mu'(\theta)e^{t\mu(\theta)}\langle\psi_\theta,g'_\theta\rangle}{te^{t\mu(\theta)}}.\end{aligned}$$

Using the formula for $\mu'(\theta)$ in the above expression we obtain

$$\mu''(\theta)=\frac{\langle\psi_\theta,\mathbb{E}_{(x,0)}(g_\theta(X_t)Y_t^2e^{\theta Y_t-t\mu(\theta)})\rangle-(\langle\psi_\theta,\mathbb{E}_{(x,0)}(g_\theta(X_t)Y_te^{\theta Y_t-t\mu(\theta)})\rangle)^2}{t}\tag{3.19}$$

$$+ 2\frac{\langle\psi_\theta, \mathbb{E}_{(x,0)}(g'_\theta(X_t)Y_t e^{\theta Y_t - t\mu(\theta)})\rangle - \langle\psi_\theta, \mathbb{E}_{(x,0)}(g_\theta(X_t)Y_t e^{\theta Y_t - t\mu(\theta)})\rangle\langle\psi_\theta, g'_\theta\rangle}{t}.$$

Let $\tilde{\mathcal{B}}$ be the Banach space of bounded continuous functions defined on $M \times \mathbb{R}$ equipped with the supremum norm. We define a new family of bounded linear operators $N(\theta, t) : \tilde{\mathcal{B}} \to \tilde{\mathcal{B}}$, $t \geq 0$ by

$$N(\theta, t)f(x_0, y_0) := \mathbb{E}_{(x_0,y_0)}\Big(f(X_t, Y_t)e^{\theta(Y_t - y_0) - t\mu(\theta)}\frac{g_\theta(X_t)}{g_\theta(x_0)}\Big) \tag{3.20}$$

for each $f \in \tilde{\mathcal{B}}$. Note that the family $\{N(\theta, t)\}_{t\geq 0}$ forms a C^0 semigroup.

We first observe that the operators $\{N(\theta, t)\}_{t\geq 0}$ are positive, and $N(\theta, t)\mathbf{1} = \mathbf{1}$, where $\mathbf{1}$ denotes the constant function taking the value 1 on $M \times \mathbb{R}$.

The operator $N(\theta, t)$ is also an operator on $\mathcal{B}$ because, for $f \in \mathcal{B}$,

$$\begin{aligned} N(\theta, t)f(x_0) &= \mathbb{E}_{(x_0,y_0)}\Big(f(X_t)e^{\theta(Y_t - y_0) - t\mu(\theta)}\frac{g_\theta(X_t)}{g_\theta(x_0)}\Big) \\ &= \Big[\frac{e^{-t\mu(\theta)}}{g_\theta}\mathcal{L}(\theta, t)(g_\theta f)\Big](x_0) \in \mathcal{B}. \end{aligned}$$

Now, corresponding to this family of operators, we have a new Markov process $(\tilde{X}_t, \tilde{Y}_t)$on $M \times \mathbb{R}$, such that, $N(\theta, t)f(x_0, y_0) = \mathbb{E}_{(x_0,y_0)}(f(\tilde{X}_t, \tilde{Y}_t))$. In addition, we observe that $\langle\psi_\theta g_\theta, N(\theta, t)f\rangle = \langle\psi_\theta g_\theta, f\rangle$ for all $f \in \mathcal{B}$. That is, $\psi_\theta g_\theta$ is the invariant measure for the process $\widetilde{X}_t$ on the manifold M for all $t \geq 0$.

Let us define the function $h \in \tilde{\mathcal{B}}$ by $h(x, y) = y$ for all $(x, y) \in M \times \mathbb{R}$. Now, we re-write the formula (3.19) for $\mu''(\theta)$ as

$$\begin{aligned} \mu''(\theta) &= \frac{1}{t}\Big(\langle\psi_\theta(x), N(\theta, t)(h^2)(x, 0)g_\theta(x)\rangle_x - (\langle\psi_\theta(x), N(\theta, t)(h)(x, 0)g_\theta(x)\rangle_x)^2\Big) \\ &+ \frac{2}{t}\Big[\langle\psi_\theta(x), N(\theta, t)\Big(\frac{hg'_\theta}{g_\theta}\Big)(x, 0)g_\theta(x)\rangle_x \\ &- \langle\psi_\theta(x), N(\theta, t)(h)(x, 0)g_\theta(x)\rangle_x\langle\psi_\theta(x), g'_\theta(x)\rangle_x\Big] \end{aligned}$$

$$= \frac{1}{t}\Big(\langle \psi_\theta g_\theta, N(\theta,t)(h^2)\rangle - (\langle \psi_\theta g_\theta, N(\theta,t)(h)\rangle)^2 \Big)$$
$$+ \frac{2}{t}\Big(\langle \psi_\theta g_\theta, N(\theta,t)\Big(\frac{h g'_\theta}{g_\theta}\Big)\rangle - \langle \psi_\theta g_\theta, N(\theta,t)(h)\rangle \langle \psi_\theta g_\theta, \frac{g'_\theta}{g_\theta}\rangle \Big).$$

Therefore, we have,

$$\mu''(\theta) = \frac{1}{t}\Big(\langle \psi_\theta g_\theta, \mathbb{E}_{(x,0)}(\widetilde{Y}_t^2)\rangle - (\langle \psi_\theta g_\theta, \mathbb{E}_{(x,0)}(\widetilde{Y}_t)\rangle)^2 \Big)$$
$$+ \frac{2}{t}\Big(\langle \psi_\theta g_\theta, \mathbb{E}_{(x,0)}\Big(\frac{\widetilde{Y}_t g'_\theta(\widetilde{X}_t)}{g_\theta(\widetilde{X}_t)}\Big)\rangle - \langle \psi_\theta g_\theta, \mathbb{E}_{(x,0)}(\widetilde{Y}_t)\rangle \langle \psi_\theta g_\theta, \frac{g'_\theta}{g_\theta}\rangle \Big).$$

Denoting $\langle \psi_\theta g_\theta, \mathbb{E}_{(x,0)}(f(\widetilde{X}_t, \widetilde{Y}_t))\rangle$ by $\mathbb{E}_{\psi_\theta g_\theta}(f(\widetilde{X}_t, \widetilde{Y}_t))$, the above formula can be written as

$$\mu''(\theta) = \frac{1}{t}\Big(\mathbb{E}_{\psi_\theta g_\theta}(\widetilde{Y}_t^2) - (\mathbb{E}_{\psi_\theta g_\theta}(\widetilde{Y}_t))^2 \Big) +$$
$$+ \frac{2}{t}\Big(\mathbb{E}_{\psi_\theta g_\theta}\Big(\frac{\widetilde{Y}_t g'_\theta(\widetilde{X}_t)}{g_\theta(\widetilde{X}_t)}\Big) - \mathbb{E}_{\psi_\theta g_\theta}(\widetilde{Y}_t)\langle \psi_\theta g_\theta, \frac{g'_\theta}{g_\theta}\rangle \Big). \tag{3.21}$$

Now, in order to prove that $\mu''(\theta) > 0$, we first show that the first term in (3.21) is the effective diffusivity of the process $\tilde{Y}_t$, which is strictly positive.Then we prove that that the second term in (3.21) goes to zero as t goes to infinity, since the processes $\tilde{X}_t$ and $\tilde{Y}_t$ de-correlate as as t goes to infinity.

In order to analyze the process $(\widetilde{X}_t, \widetilde{Y}_t)$, we first study the transition kernel of the associated Markov Operator $N(\theta, t)$. For $f \in \tilde{\mathcal{B}}$,

$$N(\theta,t)f(x_0,y_0) = \int_M \int_{\mathbb{R}} f(x,y) k(t,(x_0,y_0),(x,y))\, dy\, dx,$$

where

$$k(t,(x_0,y_0),(x,y)) := e^{-t\mu(\theta)} \frac{e^{\theta y} g_\theta(x)}{e^{\theta y_0} g_\theta(x_0)} p(t,(x_0,y_0),(x,y)).$$

From (3.10), we see that $k(t,(x_0,y_0),(x,y))$ solves the PDE

$$\partial_t k = g_\theta(x) e^{\theta y} \mathcal{M}^*_{(x,y)}\Big(\frac{k}{g_\theta(x)e^{\theta y}}\Big) - \mu(\theta)k =: \widetilde{\mathcal{M}}^* k,$$

$$k(0,(x_0,y_0),(x,y)) = \delta_{(x_0,y_0)}(x,y),$$

where, we have a new differential operator $\widetilde{\mathcal{M}}$ acting on functions $u: \ M \times \mathbb{R} \to \mathbb{R}$ given by

$$\widetilde{\mathcal{M}}^* u = g_\theta(x) e^{\theta y} \mathcal{M}^*_{(x,y)} \Big(\frac{u}{g_\theta(x) e^{\theta y}} \Big) - \mu(\theta) u.$$

Observe that

$$\begin{aligned}\widetilde{\mathcal{M}}^* k = \mathcal{M}^* k &- \frac{\nabla_x g_\theta}{g_\theta} (V(x)V^T(x)) \nabla_x k - \theta \sigma^2(x) \nabla_y k \\ &+ \Big[\frac{V_0(x) \nabla_x g_\theta(x)}{g_\theta(x)} + \frac{1}{2} \theta^2 \sigma^2(x) - \frac{\nabla_x((V(x)V^T(x)) \nabla_x g_\theta(x))}{2 g_\theta(x)} \\ &\frac{(\nabla_x g_\theta)^2}{g_\theta^2} (V(x)V^T(x)) + b(x)\theta - \mu(\theta) \Big] k.\end{aligned}$$

From the choice of g_θ, we know that $e^{-\theta y} \mathcal{M}(e^{\theta y} g_\theta(x)) = \mu(\theta) g_\theta(x)$. That is,

$$\frac{1}{2} \nabla_x [(V(x)V^T(x)) \nabla_x g_\theta] + V_\theta \nabla_x g_\theta + b(x)\theta g_\theta + \frac{1}{2} (\sigma^2(x)) \theta^2 g_\theta = \mu(\theta) g_\theta.$$

Therefore, the above expression simplifies to

$$\begin{aligned}\widetilde{\mathcal{M}}^* k = \mathcal{M}^* k &- \frac{\nabla_x g_\theta}{g_\theta} (V(x)V^T(x)) \nabla_x k - \theta \sigma^2(x) \nabla_y k \\ &+ \Big(\frac{(\nabla_x g_\theta)^2 (V(x)V^T(x))}{g_\theta^2} - \frac{\nabla_x [(V(x)V^T(x)) \nabla_x g_\theta]}{g_\theta} \Big) k.\end{aligned}$$

Thus, the operator $\widetilde{\mathcal{M}}$ simplifies to

$$\widetilde{\mathcal{M}} k = \mathcal{M} k + \frac{\nabla_x g_\theta}{g_\theta} (V(x)V^T(x)) \nabla_x k + \theta \sigma^2(x) \nabla_y k.$$

From the above expression of the generator of the new process $(\widetilde{X}_t, \widetilde{Y}_t)$, we conclude that the process $(\widetilde{X}_t, \widetilde{Y}_t)$ differ from the process (X_t, Y_t) only by the additional drift terms in x and y. The asymptotic variance (also referred to as Effective Diffusivity) of the process $\widetilde{Y}_t$ is given by

$$\Xi = \lim_{t \to \infty} \frac{\mathbb{E}_{\psi_\theta g_\theta} \Big(\big(\widetilde{Y}_t - \mathbb{E}_{\psi_\theta g_\theta} \tilde{Y}_t \big)^2 \Big)}{t}.$$

Let $c_\theta \in \mathbb{R}$ be given by,

$$c_\theta = \int_M (b + \theta\sigma^2)\psi_\theta g_\theta$$

Choose a function $f : M \to \mathbb{R}$ such that $\widetilde{\mathcal{M}} f + b + \sigma^2\theta = c_\theta$ on M. The existence of such a function f is guaranteed because $\int_M (b+\theta\sigma^2 - c_\theta)\psi_\theta g_\theta = 0$. The process $\widetilde{Y}_t + f(\widetilde{X}_t) - c_\theta t$ forms a martingale, and therefore,

$$\begin{aligned}
&\widetilde{Y}_t + f(\widetilde{X}_t) - c_\theta t - \widetilde{Y}_0 - f(\widetilde{X}_0) \\
&= \int_0^t V(\widetilde{X}_u)\nabla_x f(\widetilde{X}_u)\, dW_u + \int_0^t \sigma(\widetilde{X}_u)\, d\widetilde{W}_u + \\
&+ \int_0^t (\widetilde{\mathcal{M}} f(\widetilde{X}_u) + b(\widetilde{X}_u) + \theta\sigma^2(\widetilde{X}_u) - c_\theta)\, du \\
&= \int_0^t V(\widetilde{X}_u)\nabla_x f(\widetilde{X}_u)\, dW_u + \int_0^t \sigma(\widetilde{X}_u)\, d\widetilde{W}_u.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\mathbb{E}_{\psi_\theta g_\theta}\big(\widetilde{Y}_t - \mathbb{E}_{\psi_\theta g_\theta}\widetilde{Y}_t\big)^2 \\
&= \mathbb{E}_{\psi_\theta g_\theta}\Big(\int_0^t V(\widetilde{X}_u)\nabla_x f(\widetilde{X}_u)\, dW_u + \int_0^t \sigma(\widetilde{X}_u)\, d\widetilde{W}_u \\
&\qquad\qquad - \big(f(\widetilde{X}_t) - \mathbb{E}\psi_\theta g_\theta(f(\widetilde{X}_t))\big)\Big)^2 \\
&= \mathbb{E}_{\psi_\theta g_\theta}\Big(\frac{1}{2}\int_0^t (V(\widetilde{X}_u)\nabla_x f(\widetilde{X}_u))(V(\widetilde{X}_u)\nabla_x f(\widetilde{X}_u))^*\, du\Big) \\
&\quad + \mathbb{E}_{\psi_\theta g_\theta}\Big(\frac{1}{2}\int_0^t \sigma^2(\widetilde{X}_u)\, du\Big) + \mathbb{E}_{\psi_\theta g_\theta}(f(\widetilde{X}_t)^2) - \mathbb{E}_{\psi_\theta g_\theta}(f(\widetilde{X}_t))^2 \\
&\quad - 2\mathbb{E}_{\psi_\theta g_\theta}(f(\widetilde{X}_t))\Big(\int_0^t V(\widetilde{X}_u)\nabla_x f(\widetilde{X}_u)\, dW_u + \int_0^t \sigma(\widetilde{X}_u)\, d\widetilde{W}_u\Big)
\end{aligned}$$

Also, note that

$$\lim_{t\to\infty} \frac{\mathbb{E}_{\psi_\theta g_\theta}(f(\widetilde{X}_t))\Big(\int_0^t V(\widetilde{X}_u)\nabla_x f(\widetilde{X}_u)\, dW_u + \int_0^t \sigma(\widetilde{X}_u)\, d\widetilde{W}_u\Big)}{t} = 0.$$

Therefore, using the fact that $\psi_\theta g_\theta$ is the invariant measure of the process $\widetilde{X}_t$ on M, we have,

$$\Xi = \frac{1}{2}\int_M \big((V(x)\nabla_x f(x))(V(x)\nabla_x f(x))^* + \sigma^2(x)\big)\psi_\theta g_\theta\, dx$$

$$+ \lim_{t\to\infty} \frac{\mathbb{E}_{\psi_\theta g_\theta}(f(\widetilde{X}_t)^2) - \mathbb{E}_{\psi_\theta g_\theta}(f(\widetilde{X}_t))^2}{t}.$$

Since $\sigma > 0$ for all $x \in M$, we have $\Xi > 0$. Thus we have shown that the first term in (3.21) is positive. Now it remains to show that the limit of the second term in (3.21) is zero as t approaches infinity. That is,

$$\lim_{t\to\infty} \frac{\mathbb{E}_{\psi_\theta g_\theta}\big(\frac{\widetilde{Y}_t g'_\theta(\widetilde{X}_t)}{g_\theta(\widetilde{X}_t)}\big) - \mathbb{E}_{\psi_\theta g_\theta}(\widetilde{Y}_t)\langle \psi_\theta g_\theta, \frac{g'_\theta}{g_\theta}\rangle}{t} = 0.$$

First, we observe that

$$\lim_{t\to\infty} \frac{1}{t}\mathbb{E}_{\psi_\theta g_\theta}(\widetilde{Y}_t) - c_\theta = \tag{3.22}$$

$$= \lim_{t\to\infty} \frac{1}{t}\mathbb{E}_{\psi_\theta g_\theta}\Big(\widetilde{Y}_0 + f(\widetilde{X}_0) - f(\widetilde{X}_t) + \int_0^t V(\widetilde{X}_u)\nabla_x f(\widetilde{X}_u)\, dW_u + \int_0^t \sigma(\widetilde{X}_u)\, d\widetilde{W}_u\Big) = 0.$$

Therefore,

$$\lim_{t\to\infty} \frac{\mathbb{E}_{\psi_\theta g_\theta}(\widetilde{Y}_t)\langle \psi_\theta g_\theta, \frac{g'_\theta}{g_\theta}\rangle}{t} = c_\theta\langle \psi_\theta g_\theta, \frac{g'_\theta}{g_\theta}\rangle.$$

Thus, we only need to show that

$$\lim_{t\to\infty} \frac{1}{t}\mathbb{E}_{\psi_\theta g_\theta}\left(\frac{\widetilde{Y}_t g'_\theta(\widetilde{X}_t)}{g_\theta(\widetilde{X}_t)}\right) = c_\theta\langle \psi_\theta g_\theta, \frac{g'_\theta}{g_\theta}\rangle,$$

that is, to show that

$$\lim_{t\to\infty} \frac{1}{t}\mathbb{E}_{\psi_\theta g_\theta}\left(\frac{(\widetilde{Y}_t - c_\theta t)g'_\theta(\widetilde{X}_t)}{g_\theta(\widetilde{X}_t)}\right) = 0$$

Since $0 < \Xi < \infty$, there exists a constant $K > 0$ such that

$$\mathbb{E}_{\psi_\theta g_\theta}\frac{(\widetilde{Y}_t - c_\theta t)^2}{t} \le K$$

Using Cauchy- Schwartz inequality, and the upper bound on $\mathbb{E}_{\psi_\theta g_\theta}((\widetilde{Y}_t - c_\theta t)^2)$, stated above, we have

$$\mathbb{E}_{\psi_\theta g_\theta}\left(\left|\frac{(\widetilde{Y}_t - c_\theta t)g'_\theta(\widetilde{X}_t)}{g_\theta(\widetilde{X}_t)}\right|\right) \leq \mathbb{E}_{\psi_\theta g_\theta}\big((\widetilde{Y}_t - c_\theta t)^2\big)^{1/2}\mathbb{E}_{\psi_\theta g_\theta}\left(\frac{(g'_\theta(\widetilde{X}_t))^2}{g^2_\theta(\widetilde{X}_t)}\right)^{1/2}$$
$$\leq \sqrt{K}\sqrt{t}\sup_{x\in M}\left|\frac{(g'_\theta(x))^2}{g^2_\theta(x)}\right|$$

Therefore, we have,

$$\lim_{t\to\infty}\frac{1}{t}\mathbb{E}_{\psi_\theta g_\theta}\left(\left|\frac{(\widetilde{Y}_t - c_\theta t)g'_\theta(\widetilde{X}_t)}{g_\theta(\widetilde{X}_t)}\right|\right) \leq \lim_{t\to\infty}\frac{1}{t}\sqrt{K}\sqrt{t}\sup_{x\in M}\left|\frac{(g'_\theta(x))^2}{g^2_\theta(x)}\right| = 0.$$

We have shown that the conditions (D1), (D2) and (D3) hold with r_1 arbitrarily large. As a result, for all r, Y_t admits the strong expansion for LDP of order r in the range $(0, \infty)$. $\square$

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