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geometric random graphs with non-vanishing densities

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# A strong zero-one law for connectivity in one-dimensional geometric random graphs with non-vanishing densities

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## Abstract

We consider the geometric random graph where  $n$  points are distributed independently on the unit interval  $[0, 1]$  according to some probability distribution function  $F$ . Two nodes communicate with each other if their distance is less than some transmission range. When  $F$  admits a continuous density  $f$  which is strictly positive on  $[0, 1]$ , we show that the property of graph connectivity exhibits a strong critical threshold and we identify it. This is achieved by generalizing a limit result on maximal spacings due to Lévy for the uniform distribution.

**Keywords:** Geometric random graphs, Critical threshold functions, Zero-one laws, Non-vanishing densities.

## 1 Introduction

Starting with a recent paper by Gupta and Kumar [10], there has been renewed interest in geometric random graphs [19] as models for wireless net-

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works. Although much of the subsequent work has been carried out in dimensions two and three, some attention has been given to one-dimensional case, e.g., see [3, 5, 6, 7, 8, 9, 13, 15, 18, 17, 22, 23, 24] and references therein.

Most of these references deal with the following situation: The network comprises  $n$  nodes which are distributed independently and *uniformly* on the interval  $[0, 1]$ . Two nodes are then said to communicate with each other if their distance is less than some transmission range  $\tau > 0$ . In this setting it is well known that the property of network connectivity admits strong zero-one laws which are associated with a sharp phase transition [3, 5, 6, 8, 13, 15, 17].

In this paper, we consider the case when the nodes are placed independently on the interval  $[0, 1]$  according to an arbitrary distribution  $F$ . We only assume that  $F$  admits a continuous density  $f$  which is strictly positive on  $[0, 1]$ . Under this assumption we show (Theorem 2.1) that the property of network connectivity also obeys a strong zero-one law and we identify the corresponding critical threshold. This answers an open problem stated in [17].

We approach this problem through the asymptotic properties of maximal spacings from the univariate distribution  $F$ . The main technical contribution of the paper, summarized in Proposition 4.3, represents a generalization of a well-known result obtained by Lévy for the maximal spacings under the uniform distribution [4, 16]. The limiting result obtained here is related to earlier results of Deheuvels [2, Thm. 4, p. 1183], and is also compatible with a multi-dimensional version of a result by Penrose [20].

The paper is organized as follows: The model and main result (Theorem 2.1) are presented in Section 2. Section 4 organizes the proof of Theorem 2.1 into two key technical steps, given as Propositions 4.1 and 4.2. In Section 5, a representation of spacings associated with the uniform distribution is given in terms of i.i.d. exponentially distributed rvs. This representation is key to the approach used in establishing Propositions 4.1 and 4.2 given in Sections 6 and 7, respectively. We give some concluding remarks and an open problem in Section 8

## 2 Model and main results

First a word on notation and conventions: We assume that the rvs under consideration are *all* defined on the *same* probability triple  $(\Omega, \mathcal{F}, \mathbb{P})$ . All probabilistic statements are made with respect to this probability measure  $\mathbb{P}$ .

The notation  $\xrightarrow{P}_n$  (resp.  $\implies_n$ ) is used to signify convergence in probability (resp. convergence in distribution) with  $n$  going to infinity. Also, we use the notation  $=_{st}$  to indicate distributional equality.

Let  $\{X_i, i = 1, 2, \dots\}$  denote a sequence of i.i.d. rvs which are distributed on the unit interval  $[0, 1]$  according to some common probability distribution function  $F$ . We assume that  $F$  admits a density function  $f : [0, 1] \rightarrow \mathbb{R}_+$  which is continuous on the interval  $[0, 1]$ , and write

$$f_\star = \inf (f(x), x \in [0, 1]). \quad (1)$$

The continuity of  $f$  on the compact  $[0, 1]$  guarantees that this infimum is achieved by at least one element  $x_\star$  in  $[0, 1]$ . Throughout we make the key assumption that

$$f_\star = f(x_\star) > 0. \quad (2)$$

Of course such minimizers are not necessarily unique.

For each  $n = 2, 3, \dots$ , we think of  $X_1, \dots, X_n$  as the locations of  $n$  nodes, labelled  $1, \dots, n$ , in the interval  $[0, 1]$ . Given a fixed transmission range  $\tau > 0$ , two nodes are said to be connected or adjacent if their distance is at most  $\tau$ , i.e., nodes  $i$  and  $j$  are connected if  $|X_i - X_j| \leq \tau$ , in which case an undirected edge is said to exist between these two nodes. This notion of connectivity gives rise to the undirected geometric random graph  $\mathbb{G}_F(n; \tau)$ . We write

$$P(n; \tau) := \mathbb{P} [\mathbb{G}_F(n; \tau) \text{ is connected}]$$

where as usual,  $\mathbb{G}_F(n; \tau)$  is said to be connected if every pair of nodes can be linked by at least one path over the edges of the graph. We refer to the quantity  $P(n; \tau)$  as the probability of graph connectivity. We shall find it convenient to set  $P(n; \tau) = 1$  for  $\tau \geq 1$ .

A range function  $\tau$  is defined as any mapping  $\tau : \mathbb{N}_0 \rightarrow \mathbb{R}_+$ . The range function  $\tau^\star : \mathbb{N}_0 \rightarrow \mathbb{R}_+$  defined by

$$\tau_n^\star = \frac{\log n}{n}, \quad n = 1, 2, \dots \quad (3)$$

occupies a special place with respect to zero-one laws for graph connectivity in  $\mathbb{G}_F(n; \tau)$ .

**Theorem 2.1** *Consider a range function  $\tau : \mathbb{N}_0 \rightarrow \mathbb{R}_+$  such that*

$$\lim_{n \rightarrow \infty} \frac{\tau_n}{\tau_n^\star} = \frac{c}{f_\star} \quad (4)$$

for some  $c > 0$ . Under the enforced assumptions, it holds that

$$\lim_{n \rightarrow \infty} P(n; \tau_n) = \begin{cases} 0 & \text{if } 0 < c < 1 \\ 1 & \text{if } 1 < c. \end{cases} \quad (5)$$

Theorem 2.1 is the main result of the paper; its proof is outlined in Section 4 with the technical details presented in Sections 5, 6 and 7. Theorem 2.1 identifies the range function  $\tau_F^* : \mathbb{N}_0 \rightarrow \mathbb{R}_+$  given by

$$\tau_{F,n}^* = \frac{1}{f_\star} \cdot \frac{\log n}{n} = \frac{1}{f_\star} \cdot \tau_n^*, \quad n = 1, 2, \dots \quad (6)$$

as the *critical* scaling for graph connectivity. Roughly speaking, for  $n$  large, a communication range  $\tau_n$  suitably larger (resp. smaller) than  $\tau_{F,n}^*$  ensures that the graph  $\mathbb{G}_F(n; \tau_n)$  is connected (resp. disconnected) with very high probability if  $\tau_n \sim c\tau_{F,n}^*$  if  $c > 1$  (resp.  $0 < c < 1$ ).

It is customary [18, p. 376] to summarize (5) by stating that the range function  $\tau_F^* : \mathbb{N}_0 \rightarrow \mathbb{R}_+$  is a *strong* threshold. This is to be contrasted with the statement, readily implied by Theorem 2.1, to the effect that

$$\lim_{n \rightarrow \infty} P(n; \tau_n) = \begin{cases} 0 & \text{if } \lim_{n \rightarrow \infty} \frac{\tau_n}{\tau_{F,n}^*} = 0 \\ 1 & \text{if } \lim_{n \rightarrow \infty} \frac{\tau_n}{\tau_{F,n}^*} = \infty \end{cases} \quad (7)$$

with range function  $\tau : \mathbb{N}_0 \rightarrow \mathbb{R}_+$ . According to (7), the one law (resp. zero law) emerges when considering range functions  $\tau : \mathbb{N}_0 \rightarrow \mathbb{R}_+$  which are *at least* an order of magnitude larger (resp. smaller) than  $\tau_F^*$ . Contrast this with (5) where the one law (resp. zero law) does hold with range functions  $\tau : \mathbb{N}_0 \rightarrow \mathbb{R}_+$  which are not only larger (resp. smaller) than  $\tau_F^*$  but of the *same* order of magnitude as  $\tau_F^*$ ! It is therefore natural to refer to the situation (7) as one where the range function  $\tau_F^*$  is a *weak* threshold [18, p. 376].

Note that  $\tau^*$  is also a weak threshold for connectivity under *any* distribution  $F$  satisfying the assumptions of Theorem 2.1, a somewhat robust, albeit weak, conclusion. When  $f_\star = 0$ , a blind application of (6) yields  $\tau_{F,n}^* = \infty$  for all  $n = 1, 2, \dots$ . This begs the question as to what is the appropriate analog of Theorem 2.1 when the density  $f$  vanishes. In [14] the authors show through a counterexample that only weak critical thresholds exist when  $f_\star = 0$ .

### 3 Preliminaries

Fix  $n = 2, 3, \dots$  and  $\tau$  in  $(0, 1)$ . With the node locations  $X_1, \dots, X_n$ , we associate the rvs  $X_{n,1}, \dots, X_{n,n}$  which are the location of the  $n$  users arranged in increasing order, i.e.,  $X_{n,1} \leq \dots \leq X_{n,n}$  with the convention  $X_{n,0} = 0$  and  $X_{n,n+1} = 1$ . The rvs  $X_{n,1}, \dots, X_{n,n}$  are known as the order statistics associated with the rvs  $X_1, \dots, X_n$ . We also define the spacings

$$L_{n,k} := X_{n,k} - X_{n,k-1}, \quad k = 1, \dots, n+1. \quad (8)$$

The graph  $\mathbb{G}_F(n; \tau)$  is connected if and only if  $L_{n,k} \leq \tau$  for all  $k = 2, \dots, n$ , so that

$$P(n; \tau) = \mathbb{P}[M_n \leq \tau] \quad (9)$$

where

$$M_n := \max(L_{n,k}, k = 2, \dots, n). \quad (10)$$

The first step in establishing Theorem 2.1 lies in the following equivalence.

**Lemma 3.1** *Under the enforced assumptions, the convergence (5) under (4) is equivalent to*

$$f_\star \frac{M_n}{\tau_n^\star} \xrightarrow{P} n \ 1. \quad (11)$$

In other words, the zero-one law of Theorem 2.1 is an expression of a limiting property of the maximal spacings  $\{M_n, n = 2, \dots\}$ . Establishing (11) is the main technical contribution of this paper and we summarize it in Proposition 4.3 for easy reference.

**Proof.** First, we note that (11) is equivalent to

$$f_\star \frac{M_n}{\tau_n^\star} \Longrightarrow_n 1 \quad (12)$$

since the modes of convergence in distribution and in probability are equivalent when the limit is a constant. However, the convergence (12) amounts to

$$\lim_{n \rightarrow \infty} P(n; \frac{c}{f_\star} \tau_n^\star) = \begin{cases} 0 & \text{if } 0 < c < 1 \\ 1 & \text{if } 1 < c. \end{cases} \quad (13)$$

We conclude by noting that (5) under (4) is equivalent to (13). This conclusion is a simple consequence of the fact that the function  $\tau \rightarrow P(n; \tau)$  is monotone increasing on  $[0, 1]$ . Details are left to the interested reader. ■

We close this section with some easy facts concerning  $F$  and  $f$ : By virtue of (2), the mapping  $F : [0, 1] \rightarrow [0, 1]$  is strictly increasing, hence invertible. Let  $F^{-1} : [0, 1] \rightarrow [0, 1]$  denote the inverse mapping of  $F$ . This inverse mapping is strictly increasing and continuous since  $F$  is itself strictly increasing and continuous. Also the differentiability of  $F$  implies that of  $F^{-1}$ . Differentiating both sides of the identity  $F^{-1}(F(t)) = t$  on  $[0, 1]$  and making use of the chain rule, we get

$$\begin{aligned} \frac{d}{dt}F^{-1}(t) &= \frac{1}{f(F^{-1}(t))} \\ &= \frac{1}{g(t)}, \quad 0 \leq t \leq 1 \end{aligned} \tag{14}$$

where the mapping  $g : [0, 1] \rightarrow \mathbb{R}_+$  is defined by

$$g(t) = f(F^{-1}(t)), \quad 0 \leq t \leq 1.$$

As a result, we can write

$$F^{-1}(x) = \int_0^x \frac{1}{g(t)} dt, \quad 0 \leq x \leq 1$$

since  $F(0) = 0$ .

Consider any  $x_*$  in  $[0, 1]$  which achieves the minimum of  $f$ . By the strict monotonicity of  $F$  under (2), there exists a unique  $t_*$  in  $[0, 1]$  such that  $F^{-1}(t_*) = x_*$ , namely  $F(x_*) = t_*$ . Note that  $x_* = 0$  (resp.  $0 < x_* < 1$ ,  $x_* = 1$ ) if and only if  $t_* = 0$  (resp.  $0 < t_* < 1$ ,  $t_* = 1$ ). Moreover, as the composition of two continuous mappings, the mapping  $g$  is also continuous and (2) yields the bound

$$g(t) \geq g(t_*) = f(x_*) = f_*, \quad 0 \leq t \leq 1. \tag{15}$$

## 4 An outline of the proof

In addition to the i.i.d.  $[0, 1]$ -valued rvs  $\{X_i, i = 1, 2, \dots\}$ , consider a second collection of i.i.d., rvs  $\{U_i, i = 1, 2, \dots\}$  which are all uniformly distributed on

$[0, 1]$ . In analogy with the notation introduced earlier, for each  $n = 2, 3, \dots$ , we introduce the order statistics  $U_{n,1}, \dots, U_{n,n}$  associated with the  $n$  i.i.d. rvs  $U_1, \dots, U_n$  and we again use the convention  $U_{n,0} = 0$  and  $U_{n,n+1} = 1$ .

Key to our approach is the well-known stochastic equivalence

$$(X_1, \dots, X_n) =_{st} (F^{-1}(U_1), \dots, F^{-1}(U_n)) \quad (16)$$

which leads to the representation

$$\begin{aligned} & (X_{n,1}, \dots, X_{n,n}) \\ &=_{st} (F^{-1}(U_{n,1}), \dots, F^{-1}(U_{n,n})). \end{aligned} \quad (17)$$

It is now plain that

$$\begin{aligned} M_n &= \max(L_{n,k}, k = 2, \dots, n) \\ &=_{st} \max\left(\int_{U_{n,k-1}}^{U_{n,k}} \frac{1}{g(t)} dt, k = 2, \dots, n\right) \end{aligned}$$

as we note that

$$F^{-1}(U_{n,k}) - F^{-1}(U_{n,k-1}) = \int_{U_{n,k-1}}^{U_{n,k}} \frac{1}{g(t)} dt$$

for each  $k = 1, \dots, n + 1$ .

These observations suggest that the convergence (11) is likely to emerge as a consequence of limiting properties of the rvs  $\{U_{n,k}, k = 0, \dots, n + 1\}$  and of properties of the function  $f$  (via  $g$ ). As we shall see shortly, this is indeed the case. We shall find it convenient to write

$$M_n^u := \max(L_{n,k}^u, k = 2, \dots, n) \quad (18)$$

with

$$L_{n,k}^u := U_{n,k} - U_{n,k-1}, \quad k = 1, \dots, n + 1. \quad (19)$$

The quantities defined at (19) and (18) coincide with the quantities defined at (8) and (10), respectively, when  $F$  is the uniform distribution on  $[0, 1]$ .

For each  $n = 1, 2, \dots$ , define the rv  $\widetilde{M}_n$  by

$$\widetilde{M}_n := \max\left(\frac{L_{n,k}^u}{g(U_{n,k-1})}, k = 2, \dots, n\right).$$

The next result shows that when establishing (11) we can replace  $M_n$  by the simpler quantity  $\widetilde{M}_n$ .

**Proposition 4.1** *Under the enforced assumptions, it holds that*

$$\frac{M_n - \widetilde{M}_n}{\tau_n^*} \xrightarrow{P} {}_n 0. \quad (20)$$

Proposition 4.1 is established in Section 6. We next show that the convergence (11) indeed holds when  $M_n$  is replaced by  $\widetilde{M}_n$ .

**Proposition 4.2** *Under the enforced assumptions, it holds that*

$$f_\star \frac{\widetilde{M}_n}{\tau_n^*} \xrightarrow{P} {}_n 1. \quad (21)$$

We give a proof of Proposition 4.2 in Section 7. Combining Proposition 4.1 and Proposition 4.2 readily leads to the desired result.

**Proposition 4.3** *Under the enforced assumptions, the convergence statement (11) holds.*

If we specialize either Proposition 4.2 or Proposition 4.3 to the case when  $F$  is the uniform distribution, we get

$$\frac{M_n^u}{\tau_n^*} \xrightarrow{P} {}_n 1 \quad (22)$$

since  $f_\star = 1$ . This result was already obtained by Lévy [4, 16], and yields Theorem 2.1 when  $F$  is the uniform distribution. Theorem 2.1 is now within easy reach: Just combine Lemma 3.1, and Proposition 4.3.

## 5 A useful representation

The starting point in proving Propositions 4.1 and 4.2 resides in the representation (17). We shall leverage it by relying on a useful representation of the order statistics  $\{U_{n,k}, k = 0, 1, \dots, n+1\}$  via i.i.d. exponential rvs: Thus, consider a collection of  $\{\xi_j, j = 1, 2, \dots\}$  of i.i.d. rvs which are exponentially distributed with unit parameter, and set

$$T_0 = 0, \quad T_k = \xi_1 + \dots + \xi_k, \quad k = 1, 2, \dots$$

For all  $n = 1, 2, \dots$ , the stochastic equivalence

$$(U_{n,1}, \dots, U_{n,n}) =_{st} \left( \frac{T_1}{T_{n+1}}, \dots, \frac{T_n}{T_{n+1}} \right) \quad (23)$$

is known to hold [21, p. 403] (and references therein).

This representation makes it possible to provide an elementary proof for a technical fact used repeatedly in what follows. For each  $n = 1, \dots$ , let  $K_n$  denote a non-empty subset of  $\{1, \dots, n+1\}$ , and let  $|K_n|$  denote its cardinality. Also set

$$M(K_n) := \max(\xi_k, k \in K_n).$$

**Lemma 5.1** *The convergence*

$$\frac{M(K_n)}{\log n} \xrightarrow{P}_n 1 \quad (24)$$

takes place whenever there exists some  $\theta$  in  $(0, 1]$  such that

$$\lim_{n \rightarrow \infty} \frac{|K_n|}{n} = \theta. \quad (25)$$

**Proof.** Fix  $n = 1, 2, \dots$  and  $t \geq 0$ . By independence, we get

$$\begin{aligned} \mathbb{P}[M(K_n) \leq t] &= \mathbb{P}[\xi_k \leq t, k \in K_n] \\ &= (1 - e^{-t})^{|K_n|} \end{aligned}$$

so that

$$\begin{aligned} \mathbb{P}\left[\frac{M(K_n)}{\log n} \leq t\right] &= (1 - e^{-t \log n})^{|K_n|} \\ &= \left(1 - \frac{n^{1-t}}{n}\right)^{|K_n|}. \end{aligned}$$

With the help of (25) it is straightforward to check that

$$\lim_{n \rightarrow \infty} \mathbb{P}\left[\frac{M(K_n)}{\log n} \leq t\right] = \begin{cases} 0 & \text{if } 0 \leq t < 1 \\ 1 & \text{if } 1 < t. \end{cases}$$

As this last convergence implies

$$\frac{M(K_n)}{\log n} \xrightarrow[n]{P} 1,$$

the convergence (24) follows from the fact that convergence in distribution is equivalent to convergence in probability when the limit is a constant. ■

**Lemma 5.2** *Under the assumptions of Lemma 5.1 we also have*

$$\frac{1}{\tau_n^*} \left( \max_{k \in K_n} (L_{n,k}^u) \right) \xrightarrow[n]{P} 1. \quad (26)$$

**Proof.** By virtue of (19) and the stochastic identity (23), we need only show that

$$\frac{1}{\tau_n^*} \left( \max_{k \in K_n} \left( \frac{\xi_k}{T_{n+1}} \right) \right) \xrightarrow[n]{P} 1, \quad (27)$$

a convergence statement which is equivalent to

$$\frac{n}{T_{n+1}} \frac{M(K_n)}{\log n} \xrightarrow[n]{P} 1. \quad (28)$$

The validity of this convergence statement follows from Lemma 5.1 and from the fact that

$$\lim_{n \rightarrow \infty} \frac{T_{n+1}}{n} = 1 \quad a.s. \quad (29)$$

by the Strong Law of Large Numbers. ■

Specializing this last result to  $K_n = \{2, \dots, n\}$ , we get the convergence (22) originally obtained by Lévy [4, 16].

## 6 A proof of Proposition 4.1

Fix  $n = 2, 3, \dots$  and pick  $k = 2, \dots, n$ . Upon writing

$$\begin{aligned} \Delta_{n,k} &:= \int_{U_{n,k-1}}^{U_{n,k}} \frac{1}{g(t)} dt - \frac{U_{n,k} - U_{n,k-1}}{g(U_{n,k-1})} \\ &= \int_{U_{n,k-1}}^{U_{n,k}} \left( \frac{1}{g(t)} - \frac{1}{g(U_{n,k-1})} \right) dt, \end{aligned}$$

we find

$$|\Delta_{n,k}| \leq f_{\star}^{-2} \int_{U_{n,k-1}}^{U_{n,k}} |g(t) - g(U_{n,k-1})| dt.$$

Recalling the definition (19), we then get

$$|\Delta_{n,k}| \leq f_{\star}^{-2} G_{n,k} \cdot L_{n,k}^u$$

where we have set

$$G_{n,k} := \max(|g(t) - g(U_{n,k-1})|, U_{n,k-1} \leq t \leq U_{n,k}).$$

These facts lead to

$$\begin{aligned} |\widetilde{M}_n - M_n| &\leq \max(|\Delta_{n,k}|, k = 2, \dots, n) \\ &\leq f_{\star}^{-2} \max(G_{n,k} \cdot L_{n,k}^u, k = 2, \dots, n) \\ &\leq f_{\star}^{-2} G_n \cdot M_n^u \end{aligned}$$

where  $M_n^u$  is defined at (18) and

$$G_n := \max(G_{n,k}, k = 2, \dots, n).$$

The bound

$$\frac{|\widetilde{M}_n - M_n|}{\tau_n^{\star}} \leq f_{\star}^{-2} G_n \cdot \frac{M_n^u}{\tau_n^{\star}}$$

is now immediate. Thus, from (22) we see that (20) holds if we show that  $G_n \xrightarrow{P} 0$ . In other words, for arbitrary  $\varepsilon > 0$ , we need to show that

$$\lim_{n \rightarrow \infty} \mathbb{P}[G_n > \varepsilon] = 0. \quad (30)$$

To do so, we recall that the mapping  $g$  is continuous on the compact  $[0, 1]$ , hence uniformly continuous on  $[0, 1]$ . Thus, for every  $\varepsilon > 0$ , there exists  $\delta = \delta(\varepsilon) > 0$  such that with  $x$  and  $y$  in  $[0, 1]$ ,

$$|g(x) - g(y)| \leq \varepsilon \quad (31)$$

whenever  $|x - y| \leq \delta$ .

Fix  $\varepsilon > 0$  and consider an arbitrary integer  $n = 2, 3, \dots$ . Obviously,  $G_n \leq \varepsilon$  if and only if  $G_{n,k} \leq \varepsilon$  for all  $k = 2, \dots, n$ . In view of the comments at (31), this will occur if  $L_{n,k}^u \leq \delta$  for all  $k = 2, \dots, n$ , a condition equivalent to  $M_n^u \leq \delta$ . Consequently,

$$\mathbb{P}[G_n \leq \varepsilon] \geq \mathbb{P}[M_n^u \leq \delta],$$

or equivalently,

$$\mathbb{P}[G_n > \varepsilon] \leq \mathbb{P}[M_n^u > \delta]. \quad (32)$$

But we have  $M_n^u \xrightarrow{P} 0$  by virtue of (22) since  $\lim_{n \rightarrow \infty} \tau_n^* = 0$ . Therefore,  $\lim_{n \rightarrow \infty} \mathbb{P}[M_n^u > \delta] = 0$  and we readily get (30) upon letting  $n$  go to infinity in the inequality (32). This completes the proof of Proposition 4.1.  $\blacksquare$

## 7 A proof of Proposition 4.2

Fix  $n = 2, 3, \dots$ . By virtue of (23), the representation

$$\widetilde{M}_n =_{st} \max \left( \frac{\xi_k}{T_{n+1} \cdot g\left(\frac{T_{k-1}}{T_{n+1}}\right)}, k = 2, \dots, n \right)$$

holds so that

$$\frac{\widetilde{M}_n}{\tau_n^*} =_{st} \frac{n}{T_{n+1}} \cdot \widehat{M}_n$$

where we have used the notation

$$\widehat{M}_n := \max \left( \frac{\xi_k}{g\left(\frac{T_{k-1}}{T_{n+1}}\right)}, k = 2, \dots, n \right).$$

By the Strong Law of Large Numbers (29), the convergence (21) will be established if we show that

$$f_\star \frac{\widehat{M}_n}{\log n} \xrightarrow{P} 1. \quad (33)$$

Thus, we need to show that for every  $\varepsilon > 0$ , we have

$$\lim_{n \rightarrow \infty} \mathbb{P} \left[ \left| f_\star \frac{\widehat{M}_n}{\log n} - 1 \right| \geq \varepsilon \right] = 0 \quad (34)$$

and this is equivalent to establishing the simultaneous validity of the two convergence statements

$$\lim_{n \rightarrow \infty} \mathbb{P} \left[ 1 + \varepsilon \leq f_* \frac{\widehat{M}_n}{\log n} \right] = 0 \quad (35)$$

and

$$\lim_{n \rightarrow \infty} \mathbb{P} \left[ f_* \frac{\widehat{M}_n}{\log n} \leq 1 - \varepsilon \right] = 0. \quad (36)$$

To do so, we start with the easy upper bound

$$f_* \frac{\widehat{M}_n}{\log n} \leq \frac{M(\{2, \dots, n\})}{\log n}, \quad n = 2, 3, \dots \quad (37)$$

so that the convergence (35) now follows readily from (24) (specialized to  $K_n = \{2, \dots, n\}$ ).

The proof of (36), given next, is somewhat more involved. It will require the introduction of a family of lower bounds (in contrast with the proof of (35) which relied on the single upper bound (37)): Pick any element  $x_*$  in  $[0, 1]$  which achieves the minimum of  $f$ . We structure the forthcoming discussion according to whether  $x_* = 0$ ,  $0 < x_* < 1$  and  $x_* = 1$ . Below we give a complete discussion for the case  $0 < x_* < 1$ , as the two other cases can be handled *mutatis mutandi*.

Thus, with  $0 < x_* < 1$ , we have  $0 < t_* < 1$  where  $t_* = F(x_*)$ . Now pick  $\theta$  such that

$$0 < \theta < \min(t_*, 1 - t_*). \quad (38)$$

For each  $n = 2, 3, \dots$ , we introduce  $K_n(\theta)$  as the subset of  $\{1, \dots, n + 1\}$  defined by

$$K_n(\theta) := \{\lceil n(t_* - \theta) \rceil, \dots, \lceil n(t_* + \theta) \rceil\}.$$

Since we are interested in limiting results, we need only consider  $n \geq n^*(\theta)$  with  $n^*(\theta) = 2(t_* - \theta)^{-1}$  (as we do from now on), in which case  $\lceil n(t_* - \theta) \rceil \geq 2$  and  $K_n(\theta) \subseteq \{2, \dots, n\}$ . The lower bound

$$\widehat{M}_n(\theta) \leq \widehat{M}_n \quad (39)$$

is then immediate where we have set

$$\widehat{M}_n(\theta) := \max \left( \frac{\xi_k}{g\left(\frac{T_{k-1}}{T_{n+1}}\right)}, k \in K_n(\theta) \right).$$

To proceed, we observe the following elementary facts: For each  $a = 0, \pm 1$ , it is plain that

$$\lim_{n \rightarrow \infty} \frac{\lceil n(t_\star + a\theta) \rceil}{n} = t_\star + a\theta,$$

so that

$$\lim_{n \rightarrow \infty} \frac{T_{\lceil n(t_\star + a\theta) \rceil - 1}}{T_{n+1}} = t_\star + a\theta \quad a.s. \quad (40)$$

by the Strong Law of Large Numbers. Building on this observation, with  $\eta > 0$ , we introduce for each  $n \geq n^\star(\theta)$ , the events

$$\Omega_n^a(\theta; \eta) := \left[ \left| \frac{T_{\lceil n(t_\star + a\theta) \rceil - 1}}{T_{n+1}} - (t_\star + a\theta) \right| \leq \eta \right]$$

where  $a = 0, \pm 1$ . If we set

$$\Omega_n(\theta; \eta) := \cap_{a=0, \pm 1} \Omega_n^a(\theta; \eta),$$

then the convergence (40) implies

$$\lim_{n \rightarrow \infty} \mathbb{P}[\Omega_n(\theta; \eta)] = 1, \quad \eta > 0. \quad (41)$$

Fix  $n \geq n^\star(\theta)$  and pick  $\eta > 0$  such that  $\theta + \eta < t_\star < 1 - (\theta + \eta)$ . Such a choice of  $\eta$  is possible under (38), in which case on the event  $\Omega_n(\theta; \eta)$ , the inequalities

$$\left| \frac{T_{k-1}}{T_{n+1}} - t_\star \right| \leq (\theta + \eta), \quad k \in K_n(\theta) \quad (42)$$

hold.

We are now in a position to complete the proof: Fix  $\zeta > 0$  and set  $\delta = \delta(\zeta)$  where  $\delta(\zeta)$  insures (31) (with  $\varepsilon$  replaced by  $\zeta$ ) as a result of the uniform continuity of  $g$ . Pick  $\theta$  in  $(0, 1)$  and  $\eta > 0$  such that  $\theta + \eta \leq \delta$ . By selecting  $\theta$  and  $\eta$  sufficiently small, one can ensure that the constraints (38) and  $\theta + \eta < t_\star < 1 - (\theta + \eta)$  can also be satisfied simultaneously. With this choice, it follows from (42) that the inequalities

$$\left| g\left(\frac{T_{k-1}}{T_{n+1}}\right) - g(t_\star) \right| \leq \zeta, \quad k \in K_n(\theta)$$

all hold on the event  $\Omega_n(\theta; \eta)$ . Therefore,

$$f_\star \leq g\left(\frac{T_{k-1}}{T_{n+1}}\right) \leq f_\star + \zeta, \quad k \in K_n(\theta)$$

since  $g(t_\star) = f(x_\star) = f_\star$ , and we obtain the inequality

$$(f_\star + \zeta)^{-1} \cdot M(K_n(\theta)) \leq \widehat{M}_n(\theta). \quad (43)$$

We now return to the lower bound (39). On the event  $\Omega_n(\theta; \eta)$ , for a given  $\varepsilon > 0$ , the inequality  $f_\star \frac{\widehat{M}_n}{\log n} \leq 1 - \varepsilon$ , when coupled with (43), readily implies

$$\frac{M(K_n(\theta))}{\log n} \leq a(\varepsilon; \zeta) \quad (44)$$

with

$$a(\varepsilon; \zeta) := (1 - \varepsilon) \cdot \frac{f_\star + \zeta}{f_\star}.$$

As a result, by standard bounding and decomposition arguments, we get

$$\begin{aligned} & \mathbb{P} \left[ f_\star \frac{\widehat{M}_n}{\log n} \leq 1 - \varepsilon \right] \\ & \leq \mathbb{P} \left[ \left[ \frac{M(K_n(\theta))}{\log n} \leq a(\varepsilon; \zeta) \right] \cap \Omega_n(\theta; \eta) \right] + \mathbb{P} [\Omega_n(\theta; \eta)^c] \\ & \leq \mathbb{P} \left[ \frac{M(K_n(\theta))}{\log n} \leq a(\varepsilon; \zeta) \right] + 1 - \mathbb{P} [\Omega_n(\theta; \eta)]. \end{aligned} \quad (45)$$

Note that (36) needs to be established only for  $0 < \varepsilon < 1$  for otherwise the convergence is trivially true. Thus, pick  $0 < \varepsilon < 1$  and note that  $\zeta > 0$  can be selected sufficiently small such that  $a(\varepsilon; \zeta) < 1$ . Indeed this last condition is equivalent to

$$\zeta < \frac{\varepsilon}{1 - \varepsilon} \cdot f_\star.$$

With such a selection of  $\zeta$ , Lemma 5.1 (with  $K_n = K_n(\theta)$ ) implies

$$\lim_{n \rightarrow \infty} \mathbb{P} \left[ \frac{M(K_n(\theta))}{\log n} \leq a(\varepsilon; \zeta) \right] = 0. \quad (46)$$

Let  $n$  go to infinity in (45). The desired result (36) follows from (41) and (46).

The cases  $x_\star = 0$  and  $x_\star = 1$  can be analyzed in a similar way: Now, still with  $t_\star = F(x_\star)$ , we have  $t_\star = 0$  and  $t_\star = 1$ , respectively. As a result we need only change the definition of  $K_n(\theta)$  to read  $\{2, \dots, \lceil n(t_\star + \theta) \rceil\}$  and  $\{\lceil n(t_\star - \theta) \rceil, \dots, n\}$ , respectively, for  $n$  large enough in order to ensure  $K_n(\theta) \subset \{2, \dots, n\}$ . This completes the proof of Proposition 4.2.  $\blacksquare$

## 8 Concluding remarks

**Strong versions of Lévy's result** Slud has shown [25, Thm. 2.1, p. 343] that

$$nM_n^u - \log n = O(\log \log n) \quad a.s. \quad (47)$$

so that the convergence (22) also holds in the a.s. sense.

The convergence (11) is compatible with a multi-dimensional result obtained by Penrose [20]: Formally setting  $d = 1$  in Theorem 1.1 [20, p. 247] (discussed under the dimensional assumption  $d \geq 2$ ), we obtain (11) in its a.s. form.

**Connections with earlier results** In principle, Proposition 4.3 would follow from results by Deheuvels [2, Thm. 4, p. 1183]. However, these earlier results are given under stronger conditions than the one used here: (i) The minimizer  $x_*$  appearing in (2) is assumed to be an *isolated* minimizer; (ii) For some finite constant  $r > 0$ , we have  $0 < d_r \leq D_r < \infty$  where<sup>1</sup>

$$d_r := \liminf_{h \rightarrow 0} \frac{f(x_* + h) - f(x_*)}{|h|^r}$$

and

$$D_r := \limsup_{h \rightarrow 0} \frac{f(x_* + h) - f(x_*)}{|h|^r}.$$

Neither conditions (i) nor (ii) are needed here, but the result (11) is not as sharp as the earlier results in [2]. As a result of this trade-off, we are able to give a simple and direct proof of the convergence (11).

**Zero-one laws and critical transmission ranges** For each  $n = 2, 3, \dots$ , the *critical transmission range* for the  $n$  node network is defined as the rv  $R_n$  given by

$$R_n := \min(\tau > 0 : \mathbb{G}(n; \tau) \text{ is connected}).$$

In short,  $R_n$  is the smallest transmission range that ensures that the node set  $X_1, \dots, X_n$  forms a connected network. The obvious identity

$$R_n = M_n$$

---

<sup>1</sup>This is the form that the conditions take when  $x_*$  is an interior point of the interval  $[0, 1]$ . Obvious modifications need to be made when either  $x_* = 0$  or  $x_* = 1$ .

leads to an operational interpretation of critical thresholds: By Lemma 3.1, the range function  $\tau_F^* : \mathbb{N}_0 \rightarrow \mathbb{R}_+$  is a strong critical threshold if and only if  $R_n \sim \tau_{F,n}^*$  for  $n$  large in some appropriate distributional sense (formalized at (11)). Thus, the critical threshold serves as a proxy or estimate of the critical transmission range for the many node networks.

**Refinements** As pointed out earlier, (11) was already obtained by Lévy [4, 16] when  $F$  is the uniform distribution. In fact, this result (in the form (22)) was given a direct proof at the end of Section 5. However, it is also known [4, 16] that

$$nM_n^u - \log n \Longrightarrow_n \Lambda \quad (48)$$

where  $\Lambda$  is a Gumbel variable. To the best of the authors's knowledge, it is not known whether the analog of (48) also holds more generally for an arbitrary probability distribution  $F$  satisfying (2), say in the form

$$nf_*M_n - \log n \Longrightarrow_n \Lambda' \quad (49)$$

for some rv  $\Lambda'$ . Of course, the validity of (49) may require additional conditions on the continuous density function  $f$ . Results such as (48) and (49) imply (22) and (11), respectively, and can be viewed as complementing these results.

Interest in such questions arises from the following observation: In the uniform case, the convergence (48) implies exact asymptotics on the transition width for the phase transition associated with the property of graph connectivity [12, 15]. The validity of (49) would immediately yield a similar result under more general node placement distributions.

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