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Tracking and Stabilization for Control Systems on Matrix Lie Groups

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Tracking and Stabilization for Control Systems on Matrix Lie Groups *

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Abstract

A wide range of dynamical systems from fields as diverse as mechanics, electrical networks and molecular chemistry can be modeled by invariant systems on matrix Lie groups. This paper introduces control systems on matrix Lie groups and studies open-loop tracking and feedback stabilization for these systems in the presence of nonholonomic constraints. Using the concept of approximate inversion, results for drift-free, left-invariant systems on specific matrix Lie groups are presented.

1 Introduction

Invariant systems on matrix Lie groups arise from a wide range of fields encompassing problems as diverse as motion planning and control for autonomous vehicles, power conversion with switching circuits and coherent control of molecular dynamics. For instance the kinematics of the orientation of an under-actuated satellite or underwater vehicle can be expressed as a system evolving on the special orthogonal group $SO(3)$ [13], while the kinematics of a tractor with n trailers can be described locally as a system on a certain subgroup of the unipotent matrices (see [16] and our result in Section 3.1). Some electrical networks used for power conversion can be modeled as evolving on the higher-dimensional groups $SO(k)$ and $SE(k)$, where the dimension k depends on the complexity of the network [25]. Moreover, so called multilevel systems used to model molecular bonds in the context of coherent control of quantum dynamics can naturally be represented as invariant systems evolving on the complex unitary group $U(n)$ [8].

Apart from the fact that invariant systems on matrix Lie groups arise naturally from numerous applications, our study of these systems is also motivated by more theoretical

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aspects. In the analysis of nonholonomic systems the Lie algebra of the vector fields involved has turned out to be the crucial characteristic of these systems. Systems on matrix Lie groups make this Lie algebra structure naturally explicit and categorizable. Moreover, they form an important subclass of nonholonomic systems since the invariance of the involved vector fields on the group manifold implies that the corresponding Lie algebras are finite dimensional. There are numerous mathematical tools and results available for Lie groups and algebras, which can provide valuable geometric insight and sometimes elegant shortcuts. For example the existence of a smooth, static feedback globally, asymptotically stabilizing the origin of a system on $SO(n)$ can be precluded immediately since it is shown in [24] that the domain of attraction of an asymptotically stable equilibrium point is diffeomorphic to $\Re^n$, while it is well known that $SO(n)$ is not. This global geometric fact might not be apparent in a local coordinate representation of the system. Of course we have to venture into local coordinates when we do our computations but we have a range of local representations at our disposal and can work with the one most suitable for a certain problem at hand.

In contrast with linear systems where the controllability Gramian can be used directly to construct stabilizing or tracking controls, to date no methods exist to extract such controls for general nonlinear systems from the controllability Lie algebra. Focusing on the class of drift-free affine systém some progress has been made in this direction starting in the late 80's in the context of nonholonomic motion planning [5, 16]. The underlying principle here was to make use of oscillatory controls which systematically "excite" higher order brackets of the system to steer the system in the desired direction. Leonard and Krishnaprasad considered the problem in a Lie group setting and, applying averaging theory, obtained in [14] results on approximate point-to-point constructive controls for left-invariant systems on arbitrary finite-dimensional Lie groups. Sussmann and Liu gave a very general result for path constructive controllability of drift-free, nilpotent systems on $\Re^n$ [20]. In parallel to these developments in motion planning, Coron [6] showed the existence of time-varying stabilizing controls for a general class of drift-free nonlinear systems while Pomet presented in [18] a method to explicitly construct these control laws for a more restrictive class of systems.

Brockett, achieved in [2, 3, 4] an appealing synthesis of the problems of open-loop tracking and feedback stabilization via the concept of *approximate inversion* of a system as captured in a simple, three dimensional nilpotent system. After constructing controls for asymptotic tracking in the high frequency limit, the concatenation of the approximate inverse with the original system yields approximately the identity operator in the path space of the system. While this is already the solution for the motion planning problem, the problem of finding a stabilizing controller is reduced to the stabilization of a perturbed identity operator which can be tackled via methods of robust control. Here we investigate Brockett's notion of approximate inversion in the setting of systems on Lie groups.

After demonstrating our tools in Section 2, we present in Section 3 approximate tracking controls for a class of nilpotent systems and outline a method to obtain approximate tracking controls for non-nilpotent system using the example of a system on $SE(2)$. Section 4 relates approximate inversion to feedback stabilization and presents a control law exponentially stabilizing systems on three-dimensional real matrix Lie groups.

Finally, we will summarize our results in Section 5 and discuss possible extensions and generalizations.

2 Preliminaries

2.1 Lie Groups and Lie Algebras

For general definitions and properties pertaining to Lie groups, Lie algebras and the exponential map we refer the reader to the standard literature on this topic such as [22, 7]. In this context we will only present the mathematical tools which relate specifically to the problem of approximate inversion of systems on matrix Lie groups.

2.2 Control Systems on Matrix Lie Groups

In this section we will define the class of systems we are concerned with, focusing for now on controllable, drift-free, left–invariant systems on matrix Lie groups. Generalization to systems with drift and on generic Lie groups will be discussed at a later stage of this work.

Given a real matrix Lie group G of dimension n and the Lie algebra $\mathcal{G}$ associated with G having a basis $\{A_1, \ldots, A_n\}$ of constant matrices in $\mathcal{G}$, let X denote a curve in G. Let $U(t),\ t \geq 0$ denote a curve in an m-dimensional subspace of $\mathcal{G}$ which can be written without loss of generality as $U(t) = \sum_{i=1}^{m} u_i(t)A_i,\ m \leq n$, where the scalar functions $u_i(t), t \geq 0, i = 1, \ldots, m$ are interpreted as controls. A drift-free, left–invariant system on a matrix Lie group G can then be written as

$$\begin{aligned} \dot{X}(t) &= X(t)U(t), \qquad \forall t \geq 0 \\ X(0) &= e, \end{aligned} \tag{1}$$

where

$$U(t) = \sum_{i=1}^{m} u_i(t)A_i,\ m \leq n.$$

and e is the identity in G.

Definitions and characterizations associated with systems on matrix Lie groups, in particular controllability and the notion of a depth-k system, can be found for instance in [13].

2.3 Exponential Representations

The main tools in our study will be the so called single exponential representation and the product of exponential representation for the solution of (1). Both are in general of local nature, but have distinctive properties which make them more or less appropriate for specific problems. After introducing both representations and the local equivalents of (1) in these representations, we end this section with a short comparison by pointing out their virtues and weaknesses.

2.3.1 Single Exponential Representation

It is well known (see e.g. [22]) that the exponential map $\exp : \mathcal{G} \to G$ is a local diffeomorphism for finite-dimensional Lie groups G on a neighborhood $U \subset \mathcal{G}$ of the origin.

Moreover, a result by Lazard and Tits [12] shows that U can be chosen reasonably large for the three-dimensional matrix Lie groups presented above. In particular, for $SE(2), SO(3)$, and $SL(2)$ we can choose U to be an open ball of radius π, where we assume a standard Euclidean norm on $\mathcal{G}$. For $H(3)$ and its higher-dimensional nilpotent generalizations to be presented below the exponential map is actually a *global* diffeomorphism.

This suggests that we locally represent the solution of (1) as

$$X(t) = \exp(Z(t)) = e^{Z(t)},\ X(t) \in \exp(U) \subset G,\ t \geq 0$$

with

$$Z(t) = \sum_{i=1}^{n} z_i(t) A_i,$$

and call the $z_i,\ i = 1, 2, \ldots, n$ the *canonical coordinates of the first kind* for G. In a similar vein we write $Z = \log(X)$. This representation of the solution to (1) was characterized by Magnus in [15]. Adhering to our convention we will present a *left*–invariant version of Magnus' result without proof.

Theorem 1 (Magnus) *Consider the left-invariant, drift-free system (1) on a matrix Lie group G and the single exponential representation $Z(t)$, $t \geq 0$ of its solution. Then, if certain unspecified conditions of convergence are satisfied, $Z(t)$ can be written in the form*

$$\begin{aligned} \dot{Z}(t) &= \frac{ad_{Z(t)}}{1 - \exp(-ad_{Z(t)})} U(t) \\ &= (I + \frac{1}{2} ad_{Z(t)} + \sum_{p=1}^{\infty} \frac{\beta_{2p}}{(2p)!} ad^{2p}_{Z(t)}) U(t) \\ &= U(t) + \frac{1}{2}[Z(t), U(t)] + \frac{1}{12}[Z(t), [Z(t), U(t)]] - \frac{1}{720}[Z(t), [Z(t), [Z(t), U(t)]]] \pm \ldots, \end{aligned} \tag{2}$$

where the β_{2p} are Bernoulli numbers.

Note that given a k-step nilpotent system (all Lie brackets of depth higher than k are zero) the in general infinite sum terminates and we need only consider the first $k+1$ terms on the right hand side. Moreover, in the general situation the rapidly decreasing coefficients on the right hand side of (2) could be of importance for approximate inversion of truncated versions of the original system, which would then solve the original problem up to a perturbation of small magnitude.

Example 1 Using the Lie algebra structure of $SE(2)$ as defined in the Appendix and the anti–symmetry of the Lie bracket, we obtain for the Magnus equation of the left-invariant system on $SE(2)$ with input $U(t) = u_1(t)A_1 + u_2(t)A_2$

$$
\begin{aligned}
\dot{Z}(t) &= U(t) + \frac{1}{2}[Z(t), U(t)] + \frac{1}{12}[Z(t), [Z(t), U(t)]] \pm \dots \\
&= u_1(t)A_1 + u_2(t)A_2 + \frac{1}{2}\left[z_1(t)A_1 + z_2(t)A_2 + z_3(t)A_3,\ u_1(t)A_1 + u_2(t)A_2\right] \\
&\quad + \frac{1}{12}\left[z_1(t)A_1 + z_2(t)A_2 + z_3(t)A_3,\ [z_1(t)A_1 + z_2(t)A_2 + z_3(t)A_3,\ u_1(t)A_1 + u_2(t)A_2]\right] \pm \dots \\
&= u_1(t)A_1 + u_2(t)A_2 + \frac{1}{2}\Big(z_1(t)u_2(t)[A_1, A_2] + z_2(t)u_1(t)[A_2, A_1]\Big) \\
&\quad + \frac{1}{12}\Big(z_1(t)(z_1(t)u_2(t) - z_2(t)u_1(t))\Big)[A_1, A_3] \pm \dots \\
&= u_1(t)A_1 + u_2(t)A_2 + \frac{1}{2}\Big(z_1(t)u_2(t) - z_2(t)u_1(t)\Big)A_3 \\
&\quad \frac{1}{12}\Big(z_1(t)z_2(t)u_1(t) - z_1^2(t)u_2(t)\Big)A_2 \pm \dots .
\end{aligned}
$$

Due to the linear independence of the basis vectors $A_i,\ i = 1, 2, 3$ this can also be written in the standard state space form

$$
\begin{aligned}
\dot{z}_1 &= u_1 \\
\dot{z}_2 &= u_2 + \frac{1}{12}\Big(z_1 z_2 u_1 - z_1^2 u_2\Big) \pm \dots \\
\dot{z}_3 &= \frac{1}{2}\Big(z_1 u_2 - z_2 u_1\Big) \pm \dots .
\end{aligned}
$$

Another single exponential representation relevant for approximate inversion but not used in this paper is the Fomenko-Chakon recursive expansion [9]. It can be understood as a continuous version of the Baker–Campbell-Hausdorff formula and describes the solution to (2) as an in general infinite series of quadratures involving only $U(t)$.

2.3.2 Product of Exponentials Representation

Another canonical construction of coordinates for a finite-dimensional Lie group G involves expanding $X \in G$ in a neighborhood of the identity as a product of elements of one parameter subgroups corresponding to basis vectors $A_1, \dots, A_n$ of the Lie algebra $\mathcal{G}$

$$
X = \prod_{i=1}^{n} e^{x_i A_i} = e^{x_1 A_1} e^{x_2 A_2} \cdots e^{x_n A_n}. \tag{3}
$$

The $x_i,\ i = 1, \dots, n$ are called the *canonical coordinates of the second kind.* The product of exponentials representation of (1) is characterized in [23] by the following theorem where we use an extended control vector $u = (u_1, \dots, u_n)^T$ with $u_i = 0, i = m+1, \dots, n$.

Theorem 2 (Wei–Norman) *Consider the system (1) and its solution $X(t),\ t \geq 0$. Then, in a neighborhood of $t = 0$ the solution may be expressed in the form*

$$
X(t) = e^{x_1(t)A_1} e^{x_2(t)A_2} \cdots e^{x_n(t)A_n}. \tag{4}
$$

The coordinate functions $x_i(t)$ evolve according to

$$\begin{pmatrix} \dot{x}_1(t) \\ \vdots \\ \dot{x}_n(t) \end{pmatrix} = M(x_1, \ldots, x_n) \begin{pmatrix} u_1(t) \\ \vdots \\ u_n(t) \end{pmatrix}, \tag{5}$$

where M is analytic in the coordinates x_i and depends only on the structure of the Lie algebra $\mathcal{G}$.

Moreover, if $\mathcal{G}$ is solvable, then there exists a basis and an ordering of this basis for which the representation (4) is global and the x_i can be computed by quadratures.

2.3.3 Comparison of Exponential Representations

Since both exponential representations hold on neighborhoods of the identity e of G, there exists a neighborhood U of e on which both representations hold. Thus we are free to choose whichever representation is more appropriate for the problem at hand and are able to locally transform our results from one to the other. Moreover, since a metric on the canonical coordinates induces a metric on a neighborhood of the identity of the group G our results could be expressed locally directly for the system (1).

The strength of the single exponential representation is that the right hand side of the Magnus equation is structured as a sum of Lie brackets of increasing order and therefore nicely reflects the Lie algebra structure of the system at hand. While no conditions for the domain of convergence of (2) are specified in [15], a conservative estimate is given by Fomenko and Chakon in [9].

The advantage of the product of exponentials representation with the Wei-Norman equation (5) lies in the fact, that it often leads to more compact representations and that we are guaranteed a global quadrature solution for solvable Lie algebras $\mathcal{G}$.

3 Approximate Inversion

Our approach to open-loop tracking for nonholonomic systems on Lie groups is inspired by [2, 3] and is based on the well-known fact that periodic controls with appropriate phase and frequency relation create a motion in the direction of a certain higher order Lie bracket of the system. If we want to move the system in several higher order Lie bracket directions simultaneously we have to additively superimpose several of these oscillatory control components, namely some carrier wave modulated by a function specifying the desired velocity in the corresponding Lie bracket direction. These kinds of control laws are evocatively described in [3] as "multiplexing" the desired motion, since they accommodate the motions pertaining to different subspaces of the Lie algebra in distinct frequency bands. In this way we can solve the problem of having more states to track (receivers) than controls (channels). It turns out that if we let the carrier frequencies go to infinity these motions are increasingly independent of each other and even though the underlying system is nonlinear an *approximate high-frequency superposition principle* [20] emerges.

We are free to use arbitrarily shaped periodic signals as carrier waves, e.g. harmonic functions, square waves, etc., as long as they have a certain phase relation suitable to "create area" in the phase space of $\int u_1$ and $\int u_2$. In this sense the approximate tracking problem does not have a unique solution unless we impose additional (optimality) conditions to make the problem well posed.

Modeling these open-loop control laws as time-varying nonlinear systems with the desired trajectories as inputs we obtain the so called approximate inverse of the original system. This terminology is motivated by the fact, that concatenating the approximate inverse and the original system results in an approximate identity operator on the space of trajectories. We will see in Section 4 that this idea has interesting consequences for feedback stabilization of nonholonomic systems.

3.1 Approximate Inversion on a Subgroup of Unipotent Matrices

In this section we will introduce a subgroup of unipotent matrices which is of particular interest due to its simple Lie algebra structure. A two-input system of the form (1) on this matrix group will turn out to be equivalent to a so called *chained-form system.* The practical relevance of the two-input chained-form system (10) stems from the fact that the kinematic model of a car or a tractor and trailer can locally be transformed into (10), as is shown in [16].

Consider a k–dimensional subgroup G of the unipotent matrices consisting of elements X of the form

$$X = \begin{pmatrix} 1 & x_2 & x_3 & x_4 & x_5 & \cdots & x_k \\ 0 & 1 & x_1 & \frac{1}{2}x_1^2 & \frac{1}{6}x_1^3 & \cdots & \frac{1}{(k-2)!}x_1^{k-2} \\ & & 1 & x_1 & \frac{1}{2}x_1^2 & \ddots & \vdots \\ & & \ddots & 1 & x_1 & \ddots & \frac{1}{6}x_1^3 \\ \vdots & & & & 1 & \ddots & \frac{1}{2}x_1^2 \\ & & & & & \ddots & x_1 \\ 0 & & \cdots & & & 0 & 1 \end{pmatrix}, \qquad x = (x_1, \ldots, x_k)^T \in \Re^k, \tag{6}$$

which we call $SUP(k)$ for future reference. Note, that for $k = 3$ $SUP(k)$ is isomorphic to the real Heisenberg group $H(3)$.

Fix the following basis for the Lie algebra of $SUP(k)$:

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & & & \ddots & 0 \\ & & & & 1 \\ 0 & & \cdots & & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & & \cdots & & 0 \\ \vdots & & \ddots & & \vdots \\ & & & & \\ 0 & & \cdots & & 0 \end{pmatrix}, \tag{7}$$

$$A_3 = \begin{pmatrix} 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & & \cdots & & & 0 \\ \vdots & & \ddots & & & \vdots \\ 0 & & \cdots & & & 0 \end{pmatrix}, \ldots, A_k = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ & & & 0 \\ \vdots & \ddots & & \\ & & & \vdots \\ 0 & \cdots & & 0 \end{pmatrix}.$$

This choice of a basis results in the following non-zero brackets

$$\begin{aligned} ad_{A_1}A_2 &= -A_3 \\ ad^2_{A_1}A_2 &= A_4 \\ &\vdots \\ ad^{k-2}_{A_1}A_2 &= (-1)^{k-2}A_k. \end{aligned} \tag{8}$$

Thus $SUP(k)$ is nilpotent of order $k-2$.

We would like to solve the approximate tracking problem for the two-input drift-free system

$$\dot{X} = X(u_1A_1 + u_2A_2) \tag{9}$$

where $X \in SUP(k)$ and A_1, A_2 are defined as above. It follows from (8) that (9) is controllable, more specifically, that (9) is a depth-$(k-2)$ system (see e.g. [13]).

Plugging (4) into (9), differentiating on both sides and using the Lie algebra structure of $SUP(k)$ yields the product of exponentials representation of (9):

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= u_1x_2 \\ \dot{x}_4 &= u_1x_3 \\ &\vdots \\ \dot{x}_k &= u_1x_{k-1} \end{aligned} \tag{10}$$

This representation and the following result hold globally by Theorem 2 since $SUP(k)$ is solvable. Note that we could have derived (10) also by directly writing (9) in its natural coordinates $x_1, \ldots, x_k$ taken from (6).

Due to its lower triangular structure we can integrate (10) by quadratures and the solution for the n^{th} state can be written as

$$x_n(t) = \int_0^t u_1(\tau_1) \int_0^{\tau_1} u_1(\tau_2) \ldots \int_0^{\tau_{n-3}} u_1(\tau_{n-2}) \int_0^{\tau_{n-2}} u_2(\tau_{n-1}) d\tau_{n-1} \cdots d\tau_1, \tag{11}$$

assuming $x(0) = 0$. The succession of integrations and multiplications with u_1 in (11) suggests to use oscillating control components in u_2 with integrally distributed frequencies which are "down-modulated" by multiplications with the fundamental harmonic in u_1, so that this oscillatory component in x_{i-1} "resonates" at state x_i with the cosine in u_1.

Theorem 3 *Given a (n+2)-dimensional chained-form system (10), let* $\bar{x}(t) = (\bar{x}_1(t), \bar{x}_2(t), \ldots, \bar{x}_{n+2}(t))^T$, $t \in [0,T]$ *denote the desired trajectory. Assume that* $\bar{x}$ *is twice differentiable on* $[0,T]$ *and that* $x(0) = \bar{x}(0) = \dot{\bar{x}}(0) = 0$. *Define a sequence* $\{u^{(\omega)}\}_{\omega=1}^{\infty}$ *of controls*

$$\begin{aligned} u_1^{(\omega)} &= \dot{\bar{x}}_1(t) + 2\,\omega^{\frac{n}{n+1}}\cos(\omega t) \\ u_2^{(\omega)} &= \dot{\bar{x}}_2(t) + \sum_{m=1}^{n} \alpha_m(t)\,\omega^{-\frac{mn}{n+1}}\,\frac{m!}{m^m}\,\frac{d^m}{dt^m}(\cos(m\omega t)), \end{aligned} \tag{12}$$

where

$$\alpha_m(t) = \dot{\bar{x}}_{m+2}(t) - \dot{\bar{x}}_1(t)\bar{x}_{m+1}(t).$$

Let $x^{(\omega)}(t)$, $0 \le t \le T$ *be the solution of (10) with* $u^{(\omega)}$ *as input. Then*

$$\lim_{\omega\to\infty} x^{(\omega)}(t) = \bar{x}(t), \qquad \forall t \in [0,T] \tag{13}$$

where the convergence is uniform with respect to t.

Proof: We start by proving a lemma which enables us to discard terms in the solution of (10) which vanish in the high-frequency limit.

Lemma 1 *Let* f *be of bounded variation on* $[a,b]$ *and let* $\phi \in [0, 2\pi]$. *Then as* $\omega \to \infty$

$$\int_a^b f(t)\cos(\omega t + \phi)dt = o(1/\omega). \tag{14}$$

Proof: (Lemma 1) By the Jordan Decomposition Theorem we can write a function f of bounded variation as the difference of two non-decreasing functions and therefore it suffices to show the lemma for non-decreasing functions.

Now, assuming f to be non-negative and non-decreasing and g continuous, it follows from a Bonnet form of the Second Mean Value Theorem that there exists a $\xi \in [a,b]$ such that

$$\int_a^b f(t)g(t)dt = f(b)\int_\xi^b g(t)dt$$

Hence, there exists a $\xi \in [a,b]$ such that

$$\begin{aligned} \left|\int_a^b f(t)\cos(\omega t+\phi)dt\right| &= \left|f(b)\int_\xi^b \cos(\omega t+\phi)dt\right| \\ &\le \frac{2f(b)}{\omega}, \end{aligned}$$

and (14) follows from the boundedness of f on $[a,b]$. □

Note that we can readily apply Lemma 1 in our context since the smoothness assumption on $\bar{x}$ implies that the $\bar{x}_i$, $i = 1, 2, \ldots, n+2$ are of bounded variation.

We proceed to show convergence $x_i^{(\omega)}(t) \to \bar{x}_i(t),\ 1 \leq i \leq n+2$ for $\omega \to \infty$, which implies the convergence $x^{(\omega)}(t) \to \bar{x}(t)$ with respect to the standard norm on $\Re^{n+2}$. Writing out the solution for the first state

$$\begin{aligned}
x_1^{(\omega)}(t) &= x_1(0) + \int_0^t u_1^{(\omega)}(\tau) d\tau \\
&= \int_0^t \dot{\bar{x}}_1(\tau) + 2\,\omega^{\frac{n}{n+1}} \cos(\omega\tau) d\tau \\
&= \bar{x}_1(t) + 2\,\omega^{-\frac{1}{n+1}} \sin(\omega t),
\end{aligned}$$

it follows that

$$\lim_{\omega\to\infty} x_1^{(\omega)}(t) = \bar{x}_1(t), \qquad \forall t \in [0,T], \tag{15}$$

where the convergence is uniform with respect to t.

Using integration by parts we have for the second state

$$\begin{aligned}
x_2^{(\omega)}(t) &= x_2(0) + \int_0^t u_2^{(\omega)}(\tau) d\tau \\
&= \int_0^t \dot{\bar{x}}_2(\tau) + \sum_{m=1}^{n} \alpha_m(\tau)\,\omega^{-\frac{mn}{n+1}} \frac{m!}{m^m} \frac{d^m}{d\tau^m}(\cos(m\omega\tau)) d\tau \\
&= \bar{x}_2(t) + \sum_{m=1}^{n} \alpha_m(t)\,\omega^{-\frac{mn}{n+1}} \frac{m!}{m^m} \frac{d^{m-1}}{dt^{m-1}}(\cos(m\omega t)) \\
&\quad - \sum_{m=1}^{n} \int_0^t \dot{\alpha}_m(t)\,\omega^{-\frac{mn}{n+1}} \frac{m!}{m^m} \frac{d^{m-1}}{dt^{m-1}}(\cos(m\omega t)),
\end{aligned}$$

where the first sum is of order $\omega^{-\frac{1}{n+1}}$ and the second sum is of order $\omega^{-(1+\frac{1}{n+1})}$ by Lemma 1. Thus

$$\lim_{\omega\to\infty} x_2^{(\omega)}(t) = \bar{x}_2(t), \qquad \forall t \in [0,T], \tag{16}$$

where the convergence is uniform with respect to t.

In writing down the solution for the third state it will become clear how the successive multiplications with u_1 and subsequent integrations required to solve (10) affect the limit behavior of the involved terms. Plugging in for $\alpha_1(\tau)$, using the identities

$$\cos(\omega t)\cos(n\omega t) = \frac{1}{2}\Big(\cos((n-1)\omega t) + \cos((n+1)\omega t)\Big), \tag{17}$$

$$2\cos(\omega t) \sum_m \frac{m!}{m^m} \frac{d^{m-1}}{d\tau^{m-1}}(\cos(m\omega\tau)) = \sum_m \frac{(m-1)!}{(m-1)^{m-1}} \frac{d^{m-1}}{d\tau^{m-1}}\Big(\cos((m-1)\omega\tau)+\cos((m+1)\omega\tau)\Big)$$

and integrating by parts we obtain:

$$\begin{aligned}
&x_3^{(\omega)}(t) \\
&= x_3(0) + \int_0^t u_1^{(\omega)}(\tau)\, x_2^{(\omega)}(\tau) d\tau
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^t \Big(\dot{\bar{x}}_1(\tau) + 2\,\omega^{\frac{n}{n+1}}\cos(\omega\tau)\Big)\ \Big(\bar{x}_2(\tau) + \sum_{m=1}^{n} \alpha_m(\tau)\,\omega^{-\frac{mn}{n+1}}\,\frac{m!}{m^m}\,\frac{d^{m-1}}{d\tau^{m-1}}(\cos(m\omega\tau)) \\
&\quad + o(\omega^{-(1+\frac{1}{n+1})})\Big)\,d\tau \\
&= \int_0^t \Big\{\dot{\bar{x}}_1(\tau)\,\bar{x}_2(\tau) + \dot{\bar{x}}_1(\tau)\alpha_1(\tau)\,\omega^{-\frac{n}{n+1}}\cos(\omega\tau) \\
&\quad + 2\,\omega^{\frac{n}{n+1}}\bar{x}_2(\tau)\cos(\omega\tau) + \alpha_1(\tau) + \alpha_1(\tau)\cos(2\omega\tau) \\
&\quad + \Big(\dot{\bar{x}}_1(\tau) + 2\,\omega^{\frac{n}{n+1}}\cos(\omega\tau)\Big)\ \Big(\sum_{m=2}^{n} \alpha_m(\tau)\,\omega^{-\frac{mn}{n+1}}\,\frac{m!}{m^m}\,\frac{d^{m-1}}{d\tau^{m-1}}(\cos(m\omega\tau)) \\
&\quad + o(\omega^{-(1+\frac{1}{n+1})})\Big)\Big\}d\tau \\
&= \int_0^t \Big\{\dot{\bar{x}}_3(\tau) + \dot{\bar{x}}_1(\tau)\alpha_1(\tau)\,\omega^{-\frac{n}{n+1}}\cos(\omega\tau) + 2\,\omega^{\frac{n}{n+1}}\bar{x}_2(\tau)\cos(\omega\tau) + \alpha_1(\tau)\cos(2\omega\tau) \\
&\quad + \sum_{m=2}^{n} \alpha_m(\tau)\,\omega^{-\frac{mn}{n+1}}\,\frac{(m-1)!}{(m-1)^{m-1}}\,\frac{d^{m-1}}{d\tau^{m-1}}(\cos((m-1)\omega\tau) + \cos((m+1)\omega\tau)) \\
&\quad + 2\,\omega^{\frac{n}{n+1}}\cos(\omega\tau)\,o(\omega^{-(1+\frac{1}{n+1})}) \\
&\quad + \dot{\bar{x}}_1(\tau)\Big(\sum_{m=1}^{n} \alpha_m(\tau)\,\omega^{-\frac{mn}{n+1}}\,\frac{m!}{m^m}\,\frac{d^{m-1}}{d\tau^{m-1}}(\cos(m\omega\tau)) + o(\omega^{-(1+\frac{1}{n+1})})\Big)\Big\}d\tau \\
&= \bar{x}_3(t) + 2\,\omega^{-\frac{1}{n+1}}\,\bar{x}_2(t)\,\sin(\omega t) \\
&\quad + \sum_{m=2}^{n} \alpha_m(t)\,\omega^{-\frac{mn}{n+1}}\,\frac{(m-1)!}{(m-1)^{m-1}}\,\frac{d^{m-2}}{dt^{m-2}}(\cos((m-1)\omega t) + \cos((m+1)\omega t)) \\
&\quad - \int_0^t \Big\{\sum_{m=2}^{n} \dot{\alpha}_m(\tau)\,\omega^{-\frac{mn}{n+1}}\,\frac{(m-1)!}{(m-1)^{m-1}}\,\frac{d^{m-2}}{d\tau^{m-2}}(\cos((m-1)\omega\tau) + \cos((m+1)\omega\tau))\Big\}d\tau \\
&\quad + o(\omega^{-1}) \\
&= \bar{x}_3(t) + o(\omega^{-\frac{1}{n+1}})
\end{aligned}
$$

Again as a consequence of Lemma 1 all terms in the expression for $x_3(t)$ except for $\bar{x}_3(t)$ are of negative order with respect to ω and therefore

$$
\lim_{\omega\to\infty} x_3^{(\omega)}(t) = \bar{x}_3(t), \qquad \forall t \in [0,T], \tag{18}
$$

where convergence is again uniform with respect to t.

To establish an induction argument assume for the states $x_j, j = 4, \ldots, n$ recall that for $i = 1, \ldots, n-2$

$$
x_{i+2}(t) = \int_0^t u_1(\tau_1) \int_0^{\tau_1} u_1(\tau_2) \ldots \int_0^{\tau_{i-1}} u_1(\tau_i) \int_0^{\tau_i} u_2(\tau_{i+1})d\tau_{i+1} \cdots d\tau_1, \tag{19}
$$

i.e. x_{i+2} is obtained by applying an iteration to u_2 consisting of integration and multiplication with u_1. According to (17) the terms of u_2 are iteratively frequency shifted by $\pm\omega$ for each multiplication with the $2\,\omega^{\frac{n}{n+1}}\cos(\omega t)$-terms of u_1. We write

$$
\begin{aligned}
x_{i+2}^{(\omega)}(t) &= \bar{x}_{i+2} + \sum_{m=i+1}^{n} \{\alpha_m(t)\,\omega^{\frac{-(m-i)n}{n+1}}\,\frac{(m-i)!}{(m-i)^{m-i}}\,\frac{d^{m-i+1}}{dt^{m-i+1}}(\cos((m-i)\omega t))\} \\
&\quad + 2\,\omega^{-\frac{1}{n+1}}\,\bar{x}_{i+1}(t)\,\sin(\omega t) + \rho_{i+2} + o(\omega^{-1}),
\end{aligned} \tag{20}
$$

where the summation in (20) is comprised of the terms whose frequencies have been shifted by $-\omega$ for all previous multiplications with $2\,\omega^{\frac{n}{n+1}}\cos(\omega t)$. The terms whose frequencies have been shifted at least once by $+\omega$ due to multiplication with $2\,\omega^{\frac{n}{n+1}}\cos(\omega t)$ are subsumed under ρ_{i+2}. The terms in ρ_{i+2} are of lower order in ω as compared to terms in the summation with the same frequency. It can be verified that therefore the contributions of ρ_{i+2} to the states $x_j, j = i+3, \ldots, n$ vanishes in the high-frequency limit. We assume further that $\rho_{i+2} = o(\omega^{\frac{-1}{n+1}})$ such that $\lim_{\omega\to\infty} x_{i+2}^{(\omega)}(t) = \bar{x}_{i+2}(t)$, $\forall t \in [0,T]$. Note that $x_3(t)$ is of the form (20).

The following state can be written as

$$
\begin{aligned}
&x_{i+3}^{(\omega)}(t)\\
&= \int_0^t u_1(\tau)x_{i+2}(\tau)d\tau\\
&= \int_0^t \left(\dot{\bar{x}}_1(\tau) + 2\,\omega^{\frac{n}{n+1}}\cos(\omega\tau)\right)\left(\bar{x}_{i+2}(\tau) + 2\,\omega^{-\frac{1}{n+1}}\,\bar{x}_{i+1}(\tau)\,\sin(\omega\tau)\right.\\
&\quad \left. + \sum_{m=i+1}^{n} \alpha_m(\tau)\,\omega^{\frac{-(m-i)n}{n+1}}\,\frac{(m-i)!}{(m-i)^{m-i}}\,\frac{d^{m-i+1}}{d\tau^{m-i+1}}(\cos((m-i)\omega\tau)) + \rho_{i+2} + o(\omega^{-1})\right) d\tau\\
&= \int_0^t \Big\{\dot{\bar{x}}_1(\tau)\bar{x}_{i+2}(\tau) + \dot{\bar{x}}_1(\tau)(x_{i+2}(\tau) - \bar{x}_{i+2}(\tau))\\
&\quad +2\,\omega^{\frac{n}{n+1}}\bar{x}_{i+2}(\tau)\cos(\omega\tau) + 2\,\omega^{\frac{n-1}{n+1}}\bar{x}_{i+1}(\tau)sin(\omega\tau) + \alpha_{i+1}(\tau) + \alpha_{i+1}(\tau)\cos(2\omega\tau)\\
&\quad + \sum_{m=i+2}^{n} \alpha_m(\tau)\,\omega^{\frac{-(m-i-1)n}{n+1}}\,\frac{(m-i-1)!}{(m-i-1)^{m-i-1}}\,\frac{d^{m-i+1}}{d\tau^{m-i+1}}\left(\cos((m-i-1)\omega\tau) + \cos((m-i+1)\omega\tau)\right)\\
&\quad +2\,\omega^{\frac{n}{n+1}}\cos(\omega\tau)(\rho_{i+2} + o(\omega^{-1}))\Big\}d\tau\\
&= \bar{x}_{i+3}(t) + 2\,\omega^{\frac{-1}{n+1}}\bar{x}_{i+2}(\tau)\sin(\omega\tau)\\
&\quad + \sum_{m=i+2}^{n} \alpha_m(t)\,\omega^{\frac{-(m-i-1)n}{n+1}}\,\frac{(m-i-1)!}{(m-i-1)^{m-i-1}}\,\frac{d^{m-i}}{dt^{m-i}}(\cos((m-i-1)\omega t) + \cos((m-i+1)\omega t)\\
&\quad + \int_0^t \Big\{\dot{\bar{x}}_1(\tau)(x_{i+2}(\tau) - \bar{x}_{i+2}(\tau))\\
&\quad - \sum_{m=i+2}^{n} \dot{\alpha}_m(\tau)\,\omega^{\frac{-(m-i-1)n}{n+1}}\,\frac{(m-i-1)!}{(m-i-1)^{m-i-1}}\,\frac{d^{m-i}}{d\tau^{m-i}}(\cos((m-i-1)\omega\tau) + \cos((m-i+1)\omega\tau)\\
&\quad +2\,\omega^{\frac{n}{n+1}}\cos(\omega\tau))(\rho_{i+2} + o(\omega^{-1}))\Big\}d\tau\\
&= \bar{x}_{i+3}(t) + \sum_{m=i+2}^{n} \alpha_m(t)\,\omega^{\frac{-(m-i-1)n}{n+1}}\,\frac{(m-i-1)!}{(m-i-1)^{m-i-1}}\,\frac{d^{m-i}}{dt^{m-i}}(\cos((m-i-1)\omega t))\\
&\quad +2\,\omega^{\frac{-1}{n+1}}\bar{x}_{i+2}(\tau)\sin(\omega\tau) + \rho_{i+3} + o(\omega^{-1})
\end{aligned}
$$

Note that also here the contributions of ρ_{i+3} and the $o(\omega^{-1})$ to $x_j, j = i+4, n$ vanish in the high-frequency limit. Again, we have

$$
\begin{aligned}
\lim_{\omega \to \infty} x_{i+3}^{(\omega)}(t) &= \lim_{\omega \to \infty} \left(\bar{x}_{i+3}(t) + o(\omega^{-\frac{1}{n+1}}) \right) \\
&= \bar{x}_{i+3}(t), \qquad \forall t \in [0, T]
\end{aligned}
$$

where the convergence is uniform in t. Our claim follows by induction on i.

□

Remarks:

- Since the above convergence is uniform in t, also the tracking error, for instance defined as $E = \int_0^T \|\bar{x}(\tau) - x^{(\omega)}(\tau)\| \, d\tau$, goes to zero with $\omega \to \infty$.
- The residual terms are of order $-\frac{1}{n+1}$ in ω. Hence the convergence properties w.r.t. the frequency parameter ω worsen with increasing dimension $n+2$ of the chained form system.
- We have assumed $x(0) = \bar{x}(0) = \dot{\bar{x}}(0) = 0$ so that we can discard the initial conditions and evaluations of the lower limit of any definite integral in the proof. Nevertheless, Theorem 3 holds whenever $x(0) = \bar{x}(0)$ and $\dot{\bar{x}}(0) = 0$. If we relax $\dot{\bar{x}}(0) = 0$, we have to take care that $x(0)$ cancels the evaluations of the lower limits of integrals in order to avoid off-sets of the modulated *cosine* functions that are added to the state.
- The approximate tracking controls u_1 and u_2 given in the theorem above are simpler than if they had been derived directly via the more general method of [20]. Here, we need only one frequency $m\omega$ for each Lie bracket rather than a whole set of frequencies. This is possible because of the exceptionally simple Lie algebra structure encountered here, specifically since only brackets involving the A_1 direction are non-zero.

3.2 Approximate Tracking on $SE(2)$

Below we will present approximate tracking controls for the system (1) on $G = SE(2)$ which is remarkable since $SE(2)$ is not nilpotent as are $H(3)$ or the matrix groups mentioned in Section 3.1. Since the controls involve feedback they do not directly define an approximate inverse for the given system (see remark below). We will state the theorem for the Wei-Norman representation of (1), where it follows from the solvability of $SE(2)$ and Theorem 2 that this result is globally valid.

Theorem 4 *Consider the drift-free controllable system (1) on $G = SE(2)$ with associated Wei-Norman representation*

$$
\begin{aligned}
\dot{x}_1(t) &= u_1(t) \\
\dot{x}_2(t) &= u_2(t) + u_1(t) x_3(t) \\
\dot{x}_3(t) &= -u_1(t) x_2(t)
\end{aligned}
\tag{21}
$$

and a twice differentiable desired trajectory $\bar{x}(t) = (\bar{x}_1(t), \bar{x}_2(t), \bar{x}_3(t))^T$, $t \in [0,T]$, such that $\bar{x}(0) = \dot{\bar{x}}(0) = x(0) = 0$. Define the parameterized family of controls

$$
\begin{aligned}
u_1^{(\omega)}(t) &= \dot{\bar{x}}_1(t) + 2\omega^{\frac{1}{2}}\cos(\omega t) \\
u_2^{(\omega)}(t) &= \dot{\bar{x}}_2(t) - \omega^{\frac{1}{2}}\alpha(t)\sin(\omega t) - u_1^{(\omega)}(t)x_3(t)
\end{aligned} \tag{22}
$$

with $\alpha(t) = -\dot{\bar{x}}_3(t) - \dot{\bar{x}}_1(t)\bar{x}_2(t))$ and let $x^{(\omega)}(t), t \geq 0$ be the solution of (21) with controls $u_1^{(\omega)}(t), u_2^{(\omega)}(t)$ as input. Then

$$\lim_{\omega\to\infty} x^{(\omega)}(t) = \bar{x}(t), \quad \forall t \in [0,T],$$

uniformly in t.

Proof: Rewrite the controls $u(t) = (u_1(t)u_2(t))^T$ as $u(t) = H(x(t))v(t)$ with

$$
\begin{aligned}
H(x(t)) &= \begin{pmatrix} 1 & 0 \\ -x_3(t) & 1 \end{pmatrix} \\
\begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix} &= \begin{pmatrix} \dot{\bar{x}}_1(t) + 2\omega^{\frac{1}{2}}\cos(\omega t) \\ \dot{\bar{x}}_2(t) - \omega^{\frac{1}{2}}\alpha(t)\sin(\omega t) \end{pmatrix}
\end{aligned}
$$

and express equation (21) accordingly as

$$\dot{x}(t) = F(x(t))u(t) = F(x(t))H(x(t))v(t) = \tilde{F}(x(t))v(t) \tag{23}$$

System (21) with controls (22) thus assumes the form

$$
\begin{aligned}
\dot{x}_1(t) &= v_1(t) \\
\dot{x}_2(t) &= v_2(t) \\
\dot{x}_3(t) &= -v_1(t)x_2(t)
\end{aligned} \tag{24}
$$

i.e. it is of chained form. It follows that system (21) with controls (22) can be reduced to the situation of Theorem 3 using the input transformation $H(x)$ which is along with its inverse globally defined. The difference in sign for $\dot{x}_3$ is accounted for by the difference in sign in the definition of α as compared with α_i of Theorem 3.

□

The above result is an application of the concept of nilpotentization since the input transformation $H(x)$ is introduced to make the distribution spanned by the control vector fields of (21) nilpotent. In [19] we how show this approach can be applied to other non-nilpotent matrix Lie groups and how the state feedback can be replaced by a state estimate, resulting in *open-loop* control laws defining an approximate inverse for the systems considered.

The effect of the choice of ω on the quality of the approximation is demonstrated in Figure 3.2.

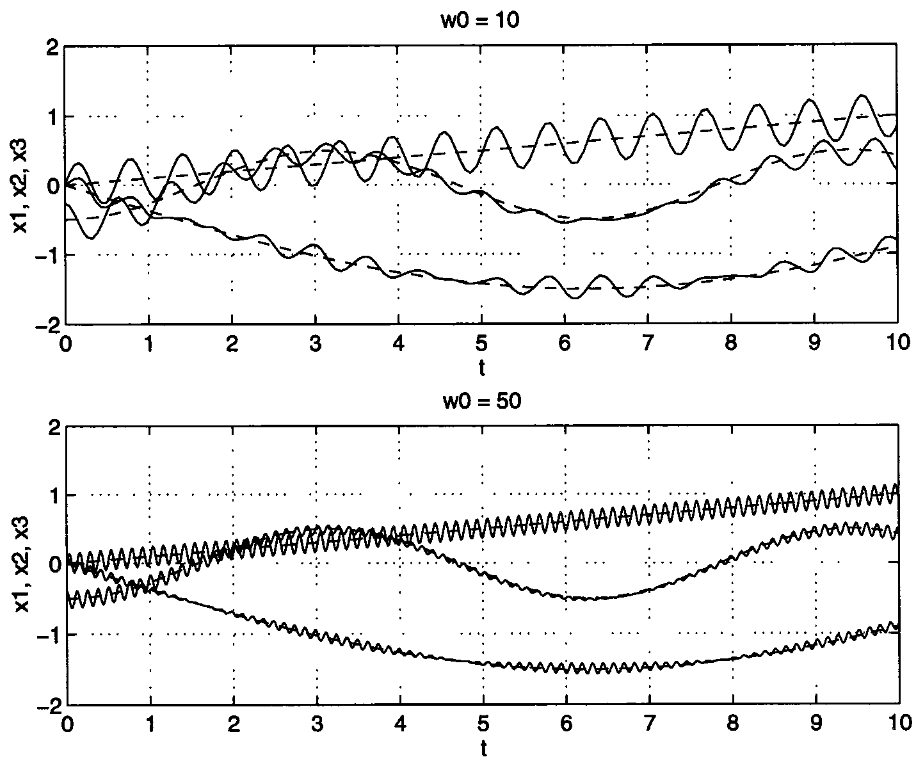

Figure 1: Approximate tracking for Wei-Norman representation of system on $SE(2)$ with two carrier frequencies

4 Feedback Stabilization

Ever since the existence of smooth time–*invariant* feedback laws for the asymptotic stabilization of drift-free nonholonomic systems was ruled out [1], a good deal of attention has been devoted to proving existence [6] and constructing [18] time–*varying*, periodic feedback laws for these systems. In these works the problem was tackled via Lyapunov's direct method and the approach of Jurdjevic and Quinn [11] was used to explicitly construct time-varying controls and the corresponding Lyapunov function. Following Brockett [2, 3] we take a different route in emphasizing the importance of open-loop steering for nonholonomic stabilization: if it is possible to steer a system along a general class of trajectories the only task left for the feedback is specifying appropriate trajectories leading the state to the origin. As described in Section 3 composing the approximate inverse with the original system yields an approximate identity operator, which can be stabilized by standard methods of robust control, e.g feeding back the negative integrals of the state to the approximate inverse. As a consequence of this different approach our control law achieves exponential rather than just uniform stability of the origin.

The control law we present here is essentially the same as in [2, 3] and is based on the approximate inverse of the Magnus representation

$$\begin{aligned}
\dot{z}_1 &= u_1 \\
\dot{z}_2 &= u_2 \\
\dot{z}_3 &= u_1 z_2 - u_2 z_1
\end{aligned}$$

of a system on $H(3)$. We will give a modified proof and show furthermore that the same control law also asymptotically stabilizes the left–invariant systems on $SE(2)$, $SO(3)$, and $SL(2)$ (see the Appendix for more information on these three-dimensional matrix Lie groups).

We use the Magnus equation representation of (1) in this context, since it allows us to interpret the $SE(2)$, $SO(3)$, and $SL(2)$ systems as a system on $H(3)$ with additional higher order perturbation terms (see Example 1). The feedback law derived for $H(3)$ turns out to be robust enough to locally stabilize the perturbed systems as well.

Note also, that since the Magnus equation is globally valid for the system on $H(3)$ we obtain *global* exponential stability for $H(3)$.

Theorem 5 *Consider the left-invariant drift-free systems on the three-dimensional real matrix Lie groups $H(3)$, $SE(2)$, $SO(3)$, and $SL(2)$ and their corresponding Magnus equations*

$$\dot{Z} = U + \frac{1}{2}[Z,U] + \frac{1}{12}[Z,[Z,U]] \mp \dots, \tag{25}$$

where $Z = z_1 A_1 + z_2 A_2 + z_3 A_3$.
Then the control $U = u_1 A_1 + u_2 A_2$ with

$$\begin{aligned}
u_1 &= -z_1 + \sqrt{2\omega|z_3|}\, sign(z_3) \sin(\omega t) \\
u_2 &= -z_2 + \sqrt{2\omega|z_3|} \cos(\omega t)
\end{aligned} \tag{26}$$

makes the zero solution of the systems (25) exponentially stable for ω sufficiently high.

Proof: We will show that after partitioning the state space and applying a coordinate transform the linearizations of the resulting systems are exponentially stable. Thus, we can conclude that also the zero solutions of the original systems are exponentially stable.

First, note that with $z_3 = 0$ we have for the closed loop

$$\begin{aligned}
[Z,U] &= (z_1 u_2 - z_2 u_1)[A_1, A_2] + z_3 u_1 [A_3, A_1] + z_3 u2 [A_3, A_2] \\
&= (z_1 \sqrt{2\omega|z_3|} \cos(\omega t) - z_2 \sqrt{2\omega|z_3|} \mathrm{sign}(z_3) \sin(\omega t))[A_1, A_2] \\
&\quad + z_3 u_1 [A_3, A_1] + z_3 u_2 [A_3, A_2] \\
&= 0
\end{aligned}$$

such that

$$\dot{Z} = U, \qquad \forall Z \in S_0 = \{Z | z_3 = 0\}. \tag{27}$$

Since U has only components in the direction of A_1 and A_2 it follows that $\dot{Z}$ has no component in the direction of A_3 for $Z \in S_0$ and hence S_0 is an invariant set of the closed loop system (25). Moreover, trajectories starting outside of S_0 do not enter S_0 in finite time and hence we have a partition of the state space into three invariant sets S_0, $S_1 = \{Z|z_3 > 0\}$, and $S_2 = \{Z|z_3 < 0\}$. Thus, we can carry out the stability analysis on each of the sets separately and study if trajectories starting in a neighborhood of the origin in the augmented sets S_0, $S_1 \cup \{0\}$, and $S_2 \cup \{0\}$ converge to it exponentially.

For $Z(0) \in S_0$ the closed loop system system (25) written in the coordinates of the Lie algebras $\mathcal{G}$ simplifies to

$$\begin{aligned} \dot{z}_1 &= -z_1 \\ \dot{z}_2 &= -z_2 \\ \dot{z}_3 &= 0 \end{aligned}$$

and is clearly exponentially stable on S_0.

Now, for $Z(0)$ in S_1 or S_2 write (25) as

$$\begin{aligned} \dot{z}_1 &= -z_1 + \sqrt{2\omega|z_3|}\,\mathrm{sign}(z_3)\sin(\omega t) + p_1(z,t) \\ \dot{z}_2 &= -z_2 + \sqrt{2\omega|z_3|}\cos(\omega t) + p_2(z,t) \\ \dot{z}_3 &= z_1\sqrt{2\omega|z_3|}\cos(\omega t) - z_2\sqrt{2\omega|z_3|}\,\mathrm{sign}(z_3)\sin(\omega t) + p_3(t,z) \end{aligned} \tag{28}$$

where p_1 and p_2 represent the contributions of the second and higher order terms on the right hand side of (25), while p_3 represents the contribution of the third and higher order terms in (25). Define a coordinate map $x_1 = z_1$, $x_2 = z_2$, and $x_3 = \mathrm{sign}(z_3)\sqrt{2|z_3|}$, which is a diffeomorphism away from S_0 and apply it to (28) resulting in

$$\begin{aligned} \dot{x}_1 &= -x_1 + \sqrt{\omega}\,x_3\sin(\omega t) + \tilde{p}_1(x,t) \\ \dot{x}_2 &= -x_2 + \sqrt{\omega}\,\mathrm{sign}(x_3)x_3\cos(\omega t) + \tilde{p}_2(x,t) \\ \dot{x}_3 &= \frac{1}{2}(x_1\sqrt{\omega}\cos(\omega t) - x_2\sqrt{\omega}\,\mathrm{sign}(x_3)\sin(\omega t)) + \tilde{p}_3(x,t). \end{aligned} \tag{29}$$

Linearizing (29) at the origin and assuming $Z(0) \in S_1$ yields

$$\begin{aligned} \dot{x} &= \begin{pmatrix} -1 & 0 & \sqrt{\omega}\sin(\omega t) \\ 0 & -1 & \sqrt{\omega}\cos(\omega t) \\ \frac{1}{2}\sqrt{\omega}\cos(\omega t) & -\frac{1}{2}\sqrt{\omega}\sin(\omega t) & 0 \end{pmatrix} x \\ &= A_1(t)x, \end{aligned} \tag{30}$$

a time-varying system system with periodic coefficients of period $T = 2\pi/\omega$.

By a standard Floquet argument the origin of (30) will be exponentially stable if the Floquet multipliers, i.e. the eigenvalues of the transition matrix $\Phi_{A_1}(T,0)$ all lie in the unit circle. Writing out the Peano-Baker-Series for $\Phi_{A_1}(T,0)$ and using $T = 2\pi/\omega$ we obtain

$$\begin{aligned}
\Phi_{A_1}(T,0) &= I_3 + \int_0^T A_1(\sigma_1)d\sigma_1 + \int_0^T A_1(\sigma_1)\left(\int_0^{\sigma_1} A_1(\sigma_2)d\sigma_2\right)d\sigma_1 + \ldots \\
&= I_3 + \begin{pmatrix} -T & 0 & 0 \\ 0 & -T & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} \frac{T}{4}+\frac{T^2}{2} & 0 & -\frac{T^{3/2}}{\sqrt{2\pi}} \\ 0 & \frac{T}{4}+\frac{T^2}{2} & 0 \\ 0 & -\frac{T^{3/2}}{2\sqrt{2\pi}} & -\frac{T}{2} \end{pmatrix} + o(T^2) \\
&= \begin{pmatrix} 1-\frac{3T}{4} & 0 & 0 \\ 0 & 1-\frac{3T}{4} & 0 \\ 0 & 0 & 1-\frac{T}{2} \end{pmatrix} + o(T^{3/2}).
\end{aligned}$$

Since the off-diagonal elements of $\Phi_{A_1}(T,0)$ are at least of order $3/2$ in T, also the radii r_i, $i = 1,2,3$ of the corresponding Geršgorin circles are of that order. Thus choosing T sufficiently small or ω sufficiently large the r_i are small compared to the diagonal elements and the eigenvalues of $\Phi_{A_1}(T,0)$ are therefore guaranteed to be within the unit circle. This proves that the linearization (30) is exponentially stable and consequently also trajectories starting in a neighborhood of the origin intersected with S_1 converge to the origin exponentially.

The argument for $Z(0)$ in an intersection of a neighborhood of the origin and S_2 is analogous and the details are left to the reader. It should suffice to mention that we obtain

$$\begin{aligned}
\dot{x} &= \begin{pmatrix} -1 & 0 & \sqrt{\omega}\sin(\omega t) \\ 0 & -1 & -\sqrt{\omega}\cos(\omega t) \\ \frac{1}{2}\sqrt{\omega}\cos(\omega t) & \frac{1}{2}\sqrt{\omega}\sin(\omega t) & 0 \end{pmatrix} x \\
&= A_2(t)x,
\end{aligned}$$

for the linearization in this case, which differs from $A_1(t)$ only in a change of sign for two matrix elements. This leads, as in the S_1 case, to

$$\Phi_{A_2}(T,0) = \begin{pmatrix} 1-\frac{3T}{4} & 0 & 0 \\ 0 & 1-\frac{3T}{4} & 0 \\ 0 & 0 & 1-\frac{T}{2} \end{pmatrix} + o(T^{3/2})$$

since the change of sign in $A_2(t)$ affects only the higher order terms of $\Phi_{A_2}(T,0)$. Thus, given a sufficiently large ω there exists a neighborhood U of the origin such that trajectories starting in each intersection of U with S_i, $i = 1,2,3$ converge to the origin exponentially. This proves our result.

□

The exponential decay of the states achieved by the feedback law above is shown in Fig. 2 for a system on $SO(3)$.

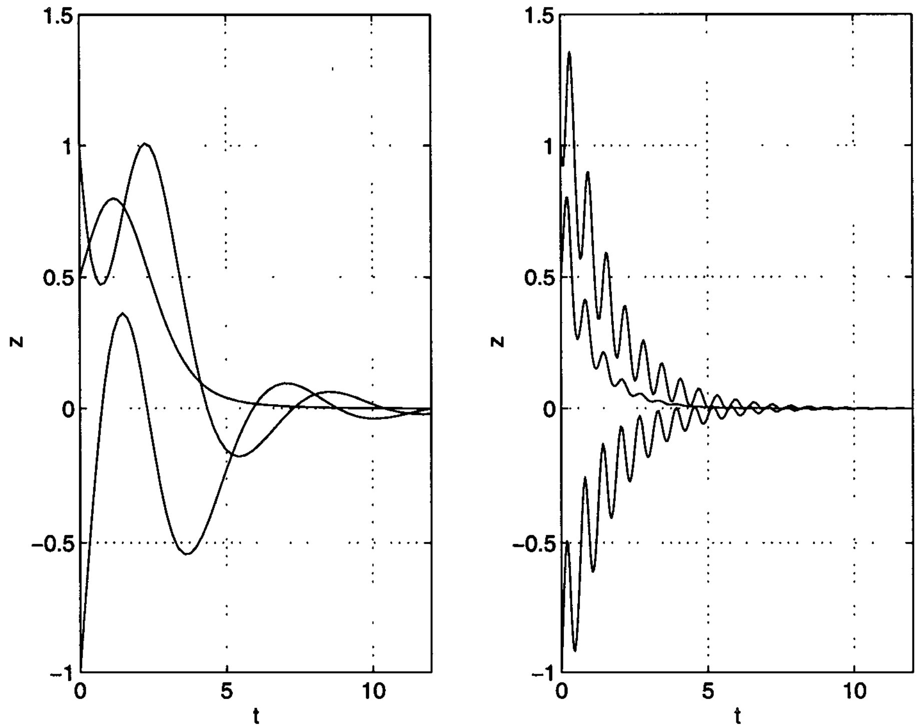

Figure 2: Feedback stabilization of Magnus equation of system on $SO(3)$ with $\omega = 1$ (left) and $\omega = 10$ (right) and initial conditions $z_1(0) = 1$, $z_2(0) = -1$, $z_3(0) = 0.5$

5 Discussion

We have presented approximate inversion controls for left-invariant drift-free systems on a nilpotent subgroup of the unipotent matrices, which are equivalent to chained form systems. Further we have derived approximate tracking controls involving feedback for a non-nilpotent system on $SE(2)$, outlining a more general method for approximated inversion of non-nilpotent systems. Finally a feedback control law based on the idea of approximate inversion has been shown to exponentially stabilize the origin for systems on $H(3)$, $SE(2)$, $SO(3)$, and $SL(2)$. These solutions for specific matrix Lie groups should serve as examples to point out the significance of the approximate inversion approach to open-loop tracking and feedback stabilization of nonholonomic systems.

We plan to generalize these results, heading toward a method to construct an approximate inverse for a system on a matrix Lie group based solely on the structure of the Lie algebra and the nature and number of the inputs. Numerical solutions might have to be taken into account where closed form results are not available.

A rigorous characterization of the approximate inverse together with the original system as a time–varying perturbation of an identity operator needs to be developed for a more unified approach to feedback stabilization of nonholonomic systems. Robust stability methods could then be applied drawing only on the characterization of the approximate identity operator rather than the properties of the original system. Eventually we would like to extend our results to nonholonomic systems with drift.

Appendix

A Three-dimensional Matrix Lie Groups

For reference we present here a canonical list of non-Abelian three-dimensional matrix Lie groups and Lie algebras, which are characterized by a choice of bases A_1, A_2, A_3 of their Lie algebras satisfying $[A_1, A_2] = \alpha A_3,\ \alpha \in \Re$. This list "nearly" represents a complete classification of non-Abelian three-dimensional Lie algebras neglecting only the solvable Lie algebras not isomorphic to $se(2)$ (for a complete classification see for example [21]). Our focus on three-dimensional real Lie groups is motivated by their simple structure and their immediate relevance for applications in mechanics and nonholonomic motion planning which holds especially for $SE(2)$ and $SO(3)$.

Let $H(3)$ denote the Heisenberg group of real 3×3 upper triangular matrices and fix a basis for the corresponding Lie algebra $h(3)$

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},\ A_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

yielding the following bracket structure:

$$[A_1, A_2] = A_3,\ [A_1, A_3] = 0,\ [A_2, A_3] = 0.$$

Let $SE(2)$ denote the Special Euclidean group representing rigid motions in the plane and fix a basis for the corresponding Lie algebra $se(2)$

$$A_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ A_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

yielding the following bracket structure:

$$[A_1, A_2] = A_3,\ [A_1, A_3] = -A_2,\ [A_2, A_3] = 0.$$

Let $SO(3)$ denote the Special Orthogonal group representing rotations in three-dimensional Euclidean space and fix a basis for the corresponding Lie algebra $so(3)$

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},\ A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},\ A_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

yielding the following bracket structure:

$$[A_1, A_2] = A_3,\ [A_1, A_3] = -A_2,\ [A_2, A_3] = A_1.$$

Let $SL(2)$ denote the Special Linear group of 2×2 matrices with determinant one and fix a basis for the corresponding Lie algebra $sl(2)$

$$A_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; A_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \; A_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

yielding the following bracket structure:

$$[A_1, A_2] = 2\,A_3, \;\; [A_1, A_3] = -A_1, \;\; [A_2, A_3] = A_2.$$

The following table summarizes some properties of the listed matrix Lie groups.

$H(3)$	$SE(2)$	$SO(3)$	$SL(2)$
nilpotent	solvable	simple	simple
not compact	not compact	compact	not compact
connected	connected	connected	connected

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