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**Design of Robust Digital
Controllers and Sampling-Time
Selection for SISO Systems**

by

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Design of robust digital controllers and sampling-time selection for SISO systems

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The stability of a digital control system and its performance in terms of the continuous plant output are studied. A two-step controller design is proposed. In the first step, the assumption of no modelling error is made and a controller that combines properties of the algorithm that minimizes the sum of squared errors and a deadbeat-type algorithm is designed so that no intersample rippling appears. In the second step, a filter is designed so that appropriate conditions which guarantee robust stability and performance in the presence of model-plant mismatch are satisfied. The effect of the sampling time on the achievable performance and the robustness properties of the system is examined and the results are incorporated in a complete procedure for sampling-time selection and robust controller design. Finally, the procedure and some theoretical implications are illustrated with examples.

1. Introduction

The importance of obtaining control designs which are robust with respect to model-plant mismatch has been well emphasized in the literature in the last few years. For sampled-data systems, although information on the plant output is available only at the sample points and the manipulated variable is discrete, it is important that robust performance is guaranteed in terms of the continuous plant output. The internal model control (IMC) structure will be used to make some qualitative aspects of the problem clear and to derive quantitative robustness conditions. A synthesis method that makes use of these conditions will also be developed.

The selection of the sampling time is an integral and very important part of any control system design. The sampling time directly affects the achievable performance and the robustness properties of the control system. A clear qualitative understanding and a quantification of these relations will be attempted, which will then lead to a criterion for sampling-time selection built into the controller synthesis method.

Finally, the theoretical results will be incorporated into a complete step by step procedure for robust controller design and sampling-time selection.

2. System description and design goals

2.1. Structure

The classical feedback structure is shown in Fig. 1(a). Wave lines are used to represent paths along which the signals are digital. The transfer function of the zero-order hold is

$$H(s) = \frac{1 - \exp(-sT)}{s} \quad (1)$$

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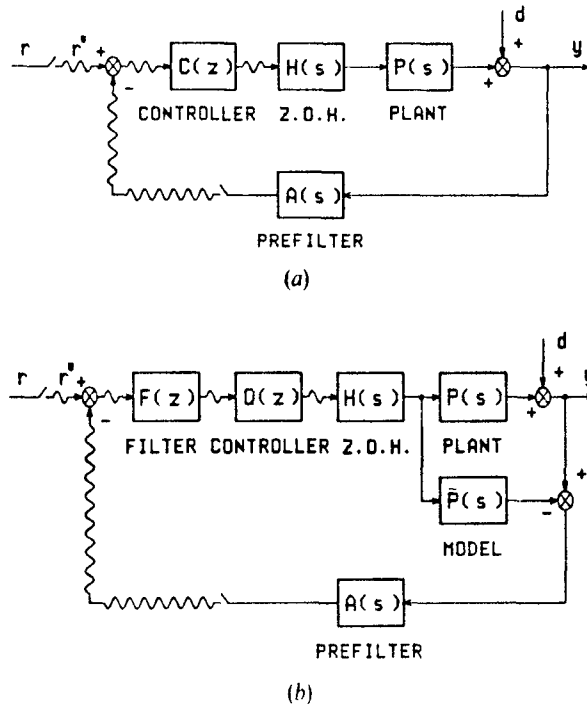


Figure 1. (a) Classical feedback structure. (b) Internal model control structure.

where T is the sampling time. $A(s)$ is an analogue anti-aliasing prefilter. A detailed explanation of the problem of aliasing can be found in digital control books (see, for example, Åström and Wittenmark 1984, and Franklin and Powell 1980). Briefly one can see the problem by looking at (2) which relates a continuous signal $a(s)$ to its z -transform $a^*(z)$:

$$a^*(\exp i\omega T) = \frac{1}{T} \sum_{k=-\infty}^{\infty} a(i\omega + ik \frac{2\pi}{T}) \quad (2)$$

Equation (2) shows that the value of a^* at a frequency ω is the sum of the values of the continuous signal a at the frequencies $\omega + k \frac{2\pi}{T}$ divided by T . The result is that after sampling, a high-frequency disturbance or measurement noise cannot be distinguished from an equivalent low-frequency one. The prefilter serves the function of cutting off high-frequency components from the analogue signals before sampling, when that is necessary. It is clear from (2) that $a^*(\exp i\omega T)$ is periodic in ω with period $2\pi/T$. It is also important to note that for a rational function $a^*(z)$ we have $\overline{a^*(z)} = a^*(\bar{z})$, where the overbar indicates the complex conjugate, and therefore for $\pi/T < \omega < 2\pi/T$ we have

$$a^*(\exp i\omega T) = \overline{a^*(\exp -i\omega T)} = \overline{a^*(\exp i(2\pi/T - \omega)T)} \quad (3)$$

Hence in addition to the periodicity, a rational z -transform $a^*(z)$ has the property that its values for frequencies greater than π/T are uniquely determined by those for $0 \leq \omega \leq \pi/T$.

The IMC structure was introduced by Garcia and Morari (1982). This structure is

a theoretical tool that simplifies the design of a controller which is robust to plant-model mismatch. This will become clear in the following sections of this paper. In Fig. 1 (b) an IMC structure which includes both digital and analogue signals is given. $\tilde{P}(s)$ is the process model and $P(s)$ the actual process. Note that though the block $\tilde{P}(s)$ appears in the structure, one will not have to implement an analogue block for the model. The structure that will be implemented is that of Fig. 1 (a) and the feedback controller $C(z)$ can be obtained from the IMC controller $Q(z)$ and filter $F(z)$ by

$$C(z) = \frac{F(z)Q(z)}{1 - F(z)Q(z)\tilde{P}_A^*(z)} \quad (4)$$

where

$$\tilde{P}_A^*(z) = \mathcal{Z} \mathcal{L}^{-1} \{H(s)\tilde{P}(s)A(s)\} \quad (5)$$

When (4) holds the mappings between the inputs r, d and the output y are the same for the two structures.

Let

$$\tilde{P}^*(z) = \mathcal{Z} \mathcal{L}^{-1} \{H(s)\tilde{P}(s)\} \quad (6)$$

$$P^*(z) = \mathcal{Z} \mathcal{L}^{-1} \{H(s)P(s)\} \quad (7)$$

$$P_A^*(z) = \mathcal{Z} \mathcal{L}^{-1} \{H(s)P(s)A(s)\} \quad (8)$$

$$d^*(z) = \mathcal{Z} \mathcal{L}^{-1} \{d(s)\} \quad (9)$$

$$d_A^*(z) = \mathcal{Z} \mathcal{L}^{-1} \{A(s)d(s)\} \quad (10)$$

$$y^*(z) = \mathcal{Z} \mathcal{L}^{-1} \{y(s)\} \quad (11)$$

Then the continuous plant output $y(s)$ is given by

$$y(s) = \frac{F(\exp sT)Q(\exp sT)H(s)P(s)}{1 + F(\exp sT)Q(\exp sT)[P_A^*(\exp sT) - \tilde{P}_A^*(\exp sT)]} \times [r^*(\exp sT) - d_A^*(\exp sT)] + d(s) \quad (12)$$

In the above the transformation $z = \exp sT$ is used. Sampling of (12) yields

$$y^*(z) = \frac{F(z)Q(z)P^*(z)}{1 + F(z)Q(z)[P_A^*(z) - \tilde{P}_A^*(z)]} [r^*(z) - d_A^*(z)] + d^*(z) \quad (13)$$

The digital controller output $u(z)$ is given by

$$u(z) = \frac{F(z)Q(z)}{1 + F(z)Q(z)[P_A^*(z) - \tilde{P}_A^*(z)]} [r^*(z) - d_A^*(z)] \quad (14)$$

2.2. Plant uncertainty description

In order to be able to design a control system which is robust with respect to model-plant mismatch one should have some bounds on this mismatch, in other words one should know how 'far' the actual process can be from the model.

The most commonly used descriptions of plant uncertainty for control purposes is the additive and multiplicative uncertainty (Doyle and Stein 1981). This kind of description can be obtained from bounds on the values of the estimated parameters of

the process model, either analytically, as illustrated in the examples of § 6, or numerically.

We can write

$$P(s) = \tilde{P}(s) + E_a(s) \quad (15)$$

$$P(s) = \tilde{P}(s)[1 + E_m(s)] \quad (16)$$

where for the additive and the multiplicative uncertainty, E_a and E_m , we have

$$|E_a(i\omega)| \leq l_a(\omega) \quad \forall \omega \quad (17)$$

$$|E_m(i\omega)| \leq l_m(\omega) \quad \forall \omega \quad (18)$$

and the bounds $l_a(\omega)$ and $l_m(\omega)$ are known. Note that

$$l_a(\omega) = |\tilde{P}(i\omega)| \cdot l_m(\omega) \quad (19)$$

Typically, $l_m(\omega)$ becomes equal to 1 or greater for high frequencies where nothing is known about the phase characteristics.

Equation (12) indicates that we need to obtain a bound for $P_\lambda^*(\exp sT) - \tilde{P}_\lambda^*(\exp sT)$. We have from (5), (8), and (15)

$$P_\lambda^*(z) - \tilde{P}_\lambda^*(z) = \mathcal{Z} \mathcal{L}^{-1} \{H(s)[P(s) - \tilde{P}(s)]A(s)\} = \mathcal{Z} \mathcal{L}^{-1} \{H(s)E_a(s)A(s)\}$$

and from (2) it follows

$$P_\lambda^*(\exp i\omega T) - \tilde{P}_\lambda^*(\exp i\omega T) = \frac{1}{T} \sum_{k=-\infty}^{\infty} H E_a A(i\omega + ik 2\pi/T) \quad (20)$$

Equations (17) and (20) can now be used to obtain the following bound

$$|P_\lambda^*(\exp i\omega T) - \tilde{P}_\lambda^*(\exp i\omega T)| \leq \frac{1}{T} \sum_{k=-\infty}^{\infty} |H A(i\omega + ik 2\pi/T)| l_a(\omega + k 2\pi/T) \triangleq l_a^*(\omega) \quad (21)$$

Since the plant is a physical system, $P(s)$ and $\tilde{P}(s)$ are strictly proper and so $|E_a(i\omega)| \rightarrow 0$ at least as fast as $1/\omega$, as $\omega \rightarrow \infty$. Hence we can always obtain a bound $l_a(\omega)$ in (17) such that $l_a(\omega) \rightarrow 0$, as $\omega \rightarrow \infty$. Also $|H A(i\omega)| \rightarrow 0$ at least as fast as $1/\omega$ as $\omega \rightarrow \infty$ even if $A(s) = 1$ and therefore $|H A(i\omega)| l_a(\omega) \rightarrow 0$ faster than $1/\omega$ as $\omega \rightarrow \infty$, which implies that the sum in (21) converges. Still, note that if a prefilter $A(s)$ is used, the property $l_a(\omega) \rightarrow 0$ as $\omega \rightarrow \infty$ is not needed for convergence. Finally, note that for computational purposes only a few terms in (21) need be considered. The reason for this is that $A(s)$ is small for ω larger than π/T in order to cut off the high-frequency components. Also from (1) it follows that $H(i\omega)/T$ is small for $\omega \geq \pi/T$. Hence for computational purposes one need only consider two or three terms in (21). Actually there is one dominant term in (20) and (21), which is the one for which $-\pi/T \leq \omega + k 2\pi/T \leq \pi/T$. Hence for $0 \leq \omega \leq \pi/T$, the dominant term corresponds to $k = 0$.

2.3. Design goals and procedure

2.3.1. Zero offset

The property of zero steady-state offset for some class of external inputs is an essential property of the control system. The conditions that have to be satisfied in order for this to happen impose certain requirements on the controller $Q(z)$, the filter $F(z)$ and the anti-aliasing prefilter $A(s)$, described by the following theorem.

Theorem 1

For an open-loop stable plant and provided that the closed-loop system is stable, the necessary and sufficient conditions for no offset for the class of external inputs $r(s)$ and $d(s)$ with all poles in the open left-half plane except l poles at $s = 0$ where $l \leq m$ and m is specified, are the following:

$$F(1)Q(1)\tilde{P}^*(1) = 1 \quad (22)$$

$$\left. \frac{d^k}{dz^k} (F(z)Q(z)\tilde{P}^*(z)) \right|_{z=1} = 0, \quad k = 1, \dots, m-1 \quad (23)$$

$$A(0) = 1 \quad (24)$$

$$\left. \frac{d^k}{ds^k} A(s) \right|_{s=0} = 0, \quad k = 1, \dots, m-1 \quad (25)$$

For $m = 1$, only (22) and (24) apply.

Proof

See Appendix A.

The implications of the above relations on the design of Q and F will be considered in subsequent sections. Let us discuss only briefly the design of the prefilter $A(s)$, whose performance specification is quite simple, namely to cut off high-frequency components. Most digital control books (Åström and Wittenmark 1984, Franklin and Powell 1980) discuss different types of anti-aliasing prefilters, which satisfy (24). In the case of $m > 1$, a simple modification can be used; let us write

$$A(s) = A_1(s)A_m(s) \quad (26)$$

where

$$A_m(s) = \frac{c_{m-1}s^{m-1} + \dots + c_1s + 1}{(\tau s + 1)^{m-1}}, \quad m \geq 2 \quad (27)$$

and $A_1(s)$ is an appropriate prefilter for $m = 1$. Then for a specified τ , (25) can be used to compute the coefficients c_1, \dots, c_{m-1} . Qualitatively it is clear that the use of $A_m(s)$ to satisfy (25) should not significantly change the behaviour of $A_1(s)$. The reason is that (25) simply adds some properties at $\omega = 0$ and this can be done without affecting the high-frequency properties of $A_1(s)$. A large τ should be used to push the effect of $A_m(s)$ towards $\omega = 0$. Indeed for a usual second-order $A_1(s) = \omega_0^2/(s^2 + 2\omega_0\zeta s + \omega_0^2)$ and for $m = 2$ (ramp inputs), (25) yields $c_1 = \tau + 2\zeta/\omega_0$ and therefore for a sufficiently large τ , $A_m(s)$ does not significantly affect the high-frequency performance of $A(s)$.

2.3.2. IMC design procedure

The purpose of the control system is to guarantee stability and good performance not only when the model is exact but also in the presence of model-plant mismatch. The IMC structure gives rise naturally to a two-step design procedure. From Fig. 1 (b) it is clear that when no modelling error is present, the design of the IMC controller $Q(z)$ reduces to the design of an open-loop controller. Indeed for $P_A^* = \tilde{P}_A^*$ and $P = \tilde{P}$, (12) becomes

$$y(s) = Q(\exp sT)H(s)\tilde{P}(s)(r^*(\exp sT) - d_A^*(\exp sT)) + d(s) \quad (28)$$

where the filter F is assumed to be the identity. Hence in the first step, $Q(z)$ can be designed so that some desired response is achieved. Inherent performance limitations exist, imposed by non-minimum-phase elements and potential intersample rippling, but the simple form of (28) simplifies the design considerably. For example, stability is not an issue if $P = \tilde{P}$, since then for an open-loop stable plant, a stable Q is all that is required for overall stability. Section 3 deals with the design of Q and the effect of sampling time on the achievable performance.

A mismatch between the model and the plant will generate a feedback signal which may cause performance deterioration or instability. The IMC filter $F(z)$ is used to take care of this problem by acting on this signal before it is fed to $Q(z)$. The filter should be designed so that stability and acceptable performance are guaranteed for a given set of possible plants. Section 4 of the paper deals with the derivation of the robustness conditions that have to be satisfied, and the filter design.

The fact that in the first step of the procedure Q is designed so that no offset is produced for a given class of inputs when $F(z) = 1$, means that according to Theorem 1, $Q(z)$ has to satisfy

$$Q(1)\tilde{P}^*(1) = 1 \quad (29)$$

$$\left. \frac{d^k}{dz^k} (Q(z)\tilde{P}^*(z)) \right|_{z=1} = 0, \quad k = 1, \dots, m-1 \quad (30)$$

Then clearly a filter $F(z)$ will satisfy (22) and (23) for a $Q(z)$ that satisfies (29) and (30) if and only if

$$F(1) = 1 \quad (31)$$

$$\left. \frac{d^k}{dz^k} F(z) \right|_{z=1} = 0, \quad k = 1, \dots, m-1 \quad (32)$$

3. Controller design for no modelling error

3.1. Effect of sampling on performance

In § 1, it was mentioned that sampling puts a limitation on the achievable performance. We shall now demonstrate this fact quantitatively. Consider (28)

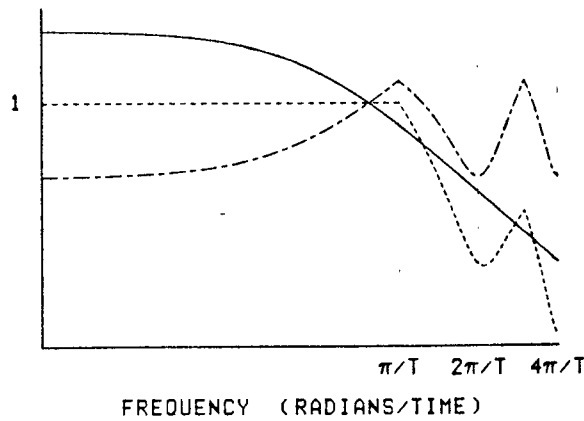


Figure 2. Effect of sampling on performance (logarithmic plot): ——— $|\tilde{P}(i\omega)|$; — · — $|Q(\exp i\omega T)|$; - - - $|Q(\exp i\omega T)\tilde{P}(i\omega)|$.

obtained from (12) for no modelling error. $H(s)r^*(\exp sT)$ and $H(s)d^*(\exp sT)$ are zero-order-hold reconstructions of the set-point $r(s)$ and output disturbance $d(s)$. Though these are not exact, we shall assume that they are, in order to demonstrate the limitation that comes from the periodicity of Q . Then (28) implies that for good performance $\tilde{P}(s)Q(\exp sT)$ should be close to one.

In Fig. 2 a typical Bode plot of $\tilde{P}(s)$ is shown. For perfect performance we need a Q equal to the inverse of $\tilde{P}(s)$. However, as described by (3), Q is periodic in ω with period $2\pi/T$ and its values for frequencies greater than π/T are uniquely determined by those for $\omega \leq \pi/T$. In Fig. 2, an ideal Q is plotted which inverts $\tilde{P}(s)$ for ω up to π/T . In order for this to be accomplished, Q has actually to be of infinite order. However even for this Q , it is clear in Fig. 2 that the closed-loop transfer function $\tilde{P}(s)Q(\exp sT)$ cannot have a bandwidth larger than π/T .

3.2. The controller Q

We have seen that sampling limits the achievable performance. The question that arises is how to design Q so that for a given sampling time T we obtain the 'best' possible performance. In addition, the design method should be simple enough so that designing Q for more than one sampling time is not time-consuming. The necessity for repeating the design for more than one T will become apparent in § 5.

A detailed study of the advantages and disadvantages and the theoretical reasons behind them, for a number of well-known digital control algorithms led Zafriou and Morari (1985) to a simple method for designing Q from the model $\tilde{P}^*(z)$. To do so one should first obtain the controller Q_{SE} which minimizes the sum of squared errors between the sampled system output y^* and a specified external input. Then to obtain $Q(z)$ one should substitute the poles of $Q_{SE}(z)$ which have a negative real part with poles at the origin while preserving the property of zero steady-state offset. The reason for the substitution is that poles with a negative real part produce undesirable intersample rippling in the continuous plant output, which does not reveal itself in the sum of squared errors, computed only at discrete points in time. The introduction of poles at the origin, aims at incorporating in the design some of the advantages of a deadbeat-type response, while at the same time avoiding known problems of deadbeat controllers like overshoot and undershoot (Zafriou and Morari 1985).

We can always write $\tilde{P}^*(z)$ as

$$\tilde{P}^*(z) = K \frac{(z - a_1) \dots (z - a_{n-1})}{(z - p_1) \dots (z - p_n)} z^{-N} \quad (33)$$

where N is the largest integer such that NT is less than or equal to the dead time. All the poles are assumed to be stable.

Let $v(s)$ be the external input which we want our system to follow (for v a setpoint) or reject (for v an output disturbance). All poles of $v(s)$ are assumed to be in the open left-half plane, except possibly some at $s = 0$. It is important however to note that in order for the control system to yield zero steady-state offset for all inputs with m or less poles at $s = 0$, the controller $Q(z)$ must have been designed for an input $v(s)$ which has m such poles. Let

$$v^*(z) = \mathcal{Z} \mathcal{L}^{-1} \{v(s)\} \quad (34)$$

Write

$$v^*(z) = z^{-N_v}(v_0 + v_1 z^{-1} + v_2 z^{-2} + \dots) \quad (35)$$

where v_l , $l = 0, 1, \dots$, are the values of $\mathcal{L}^{-1}\{v(s)\}$ at time $t = (N_v + l)T$, $v_0 \neq 0$. Define

$$\begin{aligned} v_N^*(z) &= [z^N v^*(z) - (v_0 + v_1 z^{-1} + \dots + v_N z^{-N})] z^{N+1} \\ &= v_{N+1} + v_{N+2} z^{-1} + v_{N+3} z^{-2} + \dots \end{aligned} \quad (36)$$

This represents the part of $\mathcal{L}^{-1}\{v^*(z)\}$ for time points greater than the dead-time of the plant, moved by $(N+1)T$ to the left on the time axis. Another way to compute $v_N^*(z)$ is to find from $\mathcal{L}^{-1}\{v(s)\}$ the Laplace transform of the continuous time function that corresponds to these points and then take the z -transform from tables.

Without loss of generality, assume that a_1, \dots, a_f and $a_{v,1}, \dots, a_{v,h}$ are the zeros of $\tilde{P}^*(z)$ and $v^*(z)$ respectively, which are outside the unit circle. Define

$$\tilde{P}_+^*(z) = \prod_{j=1}^f \frac{(1 - a_j^{-1})(z - a_j)}{(1 - a_j)(z - a_j^{-1})} \quad (37)$$

$$v_+^*(z) = \prod_{j=1}^h \frac{(1 - a_{v,j}^{-1})(z - a_{v,j})}{(1 - a_{v,j})(z - a_{v,j}^{-1})} \quad (38)$$

and

$$\tilde{P}_-^*(z) = (\tilde{P}_+^*(z))^{-1} \tilde{P}^*(z) z^N \quad (39)$$

$$v_-^*(z) = (v_+^*(z))^{-1} v^*(z) z^N \quad (40)$$

Then $Q_{SE}(z)$ is given by

$$Q_{SE}(z) = (\tilde{P}_-^*(z) v_-^*(z))^{-1} \{ (\tilde{P}_+^*(z) v_+^*(z))^{-1} v_N^*(z) z^{-1} \}_- \quad (41)$$

where $\{\cdot\}_-$ is obtained by taking a partial fraction expansion of $\{\cdot\}$ and discarding the terms with poles *outside* the unit circle. Terms with poles at $z = 1$ are retained. The constant term is zero because $\{\cdot\}$ is strictly proper. The steps used to arrive at (41) are given in Appendix B.

In the case of set-point following, one often has available and supplies to the controller future values of the set-point, which one wants the system output to follow after some time steps. By doing so, better servo behaviour is accomplished. In this case, $Q_{SE}(z)$ can be obtained by using $v_{N_0}^*(z)$ instead of $v_N^*(z)$ in (41), where $v_{N_0}^*$ is defined as follows:

$$v_{N_0}^*(z) = [z^{N_0} v^*(z) - (v_0 + v_1 z^{-1} + \dots + v_{N_0} z^{-N_0})] z^{N_0+1} \quad (42)$$

with

$$N_0 = \max \{N - N_p, 0\} \quad (43)$$

and N_p the number of time steps ahead for which the set-point is supplied. How this conclusion is reached, is discussed in Appendix B.

$Q(z)$ can now be obtained from $Q_{SE}(z)$ as

$$Q(z) = q(z) Q_{SE}(z) B(z) \quad (44)$$

where

$$q(z) = z^{-\kappa} \prod_{j=1}^{\kappa} \frac{(z - \pi_j)}{(1 - \pi_j)} \quad (45)$$

$$B(z) = \sum_{j=0}^{m-1} b_j z^{-j} \quad (46)$$

and π_j , $j = 1, \dots, \kappa$, are the poles of $Q_{SE}(z)$ with a negative real part. The coefficients b_j , $j = 0, \dots, m-1$, will be determined so that the controller produces zero steady-state

offset for all inputs $v(s)$ with m or less poles at $s=0$. By its construction, $Q_{se}(z)$ produces no offset and therefore it satisfies (29) and (30). Then clearly $Q(z)$ will satisfy (29) and (30) if and only if

$$q(1)B(1) = 1 \quad (47)$$

$$\left. \frac{d^k}{dz^k} (q(z)B(z)) \right|_{z=1} = 0, \quad k = 1, \dots, m-1 \quad (48)$$

Equation (47) yields

$$b_0 = 1 - (b_1 + \dots + b_{m-1}) \quad (49)$$

Note that for $m=1$, only (47) need be considered and then (46) and (49) yield $B(z) = 1$. Equation (48) is equivalent to

$$\left. \frac{d^k}{d\lambda^k} (q(\lambda^{-1})B(\lambda^{-1})) \right|_{\lambda=1} = 0, \quad k = 1, \dots, m-1 \quad (50)$$

Note that $q(\lambda^{-1})$ is a polynomial in λ and therefore its derivatives with respect to λ can be computed easily. Hence simple successive substitution will reduce (50) to

$$\left. \frac{d^k}{d\lambda^k} B(\lambda^{-1}) \right|_{\lambda=1} = \gamma_k, \quad k = 1, \dots, m-1 \quad (51)$$

where the γ 's are known. Equation (51) can be written in terms of the b s as

$$N_{m-1} \begin{bmatrix} b_1 \\ \vdots \\ b_{m-1} \end{bmatrix} = \begin{bmatrix} \gamma_1 \\ \vdots \\ \gamma_{m-1} \end{bmatrix} \quad (52)$$

where the matrix N_k is defined for some k as a matrix of dimension $(m-1) \times k$ whose elements v_{ij} are

$$v_{ij} = \begin{cases} 0 & \text{for } i > j \\ \frac{j!}{(j-i)!} & \text{for } i \leq j \end{cases} \quad (53)$$

Equation (52) can be solved by successive substitution. Note that for the simple case of $m=2$ we get

$$b_1 = \gamma_1 = \sum_{j=1}^{\infty} \frac{\pi_j}{(1-\pi_j)}, \quad \text{for } m=2 \quad (54)$$

The case of $v(s) = 1/s$ is of special interest because this is the most commonly considered input. In this case $v^*(z) = v_N^*(z) = v_-^*(z) = z/(z-1)$ and $v_+^*(z) = 1$. Then (41) yields $Q_{se}(z) = (z\tilde{P}_-^*(z))^{-1}$ and therefore the $Q(z)$ obtained from (44) for $B(z) = 1$ can actually be constructed by using the following rules (Zafiriou and Morari 1985).

- (i) Use as zeros of $Q(z)$ the poles of $\tilde{P}^*(z)$.
- (ii) Use as poles of $Q(z)$ the zeros of $\tilde{P}^*(z)$ with a positive real part that are inside the unit circle, also the inverses of those with a positive real part which are outside the unit circle, and as many at the origin as there are zeros with a negative real part.

- (iii) An additional pole of $Q(z)$ at the origin must be present because of the inherent time delay of a sampled-data system.
- (iv) The steady-state gain of $Q(z)$ should be $Q(1) = 1/\tilde{P}^*(1)$.

This controller designed for step inputs combines the advantages of the algorithm that minimizes the sum of squared errors and of deadbeat-type algorithms. In the case where all the unstable zeros of $\tilde{P}^*(z)$ have a negative real part, it yields a deadbeat controller which drives the discrete output of the system to the set-point in a finite number of time steps. When $\tilde{P}^*(z)$ has unstable zeros with a positive real part, the controller drives the output to the set-point asymptotically in order to avoid large overshoot or undershoot. When all the zeros, stable or unstable, have a positive real part, it minimizes the sum of the squared errors of the output. The same properties are maintained for a controller designed for inputs other than steps according to (41), (44), (45) and (46), when the minimum number of coefficients b_i necessary to satisfy (47) and (48) is used.

4. Filter design for model-plant mismatch

In this section conditions for stability and good performance in the presence of a modelling error are derived and a method for designing a filter so that these conditions are satisfied is proposed. Also the effect of sampling time on robustness is discussed.

4.1. Robust stability

As mentioned earlier, the plant is assumed to be open-loop stable and therefore, when $P = \tilde{P}$, all that is required for overall stability is that Q and F are also stable. Application of the 'small gain' theorem to (13) and (14) and use of (21) will then yield the following stability condition for $P \neq \tilde{P}$ (see Doyle and Stein 1981).

Theorem 2

Let $P(s)$, $Q(z)$ and $F(z)$ be stable. Then the system in Fig. 1 (b) is stable

$$\forall P(s) \text{ s.t. } |P_A^*(\exp i\omega T) - \tilde{P}_A^*(\exp i\omega T)| \leq l_s^*(\omega) \quad \text{for } 0 \leq \omega \leq \pi/T$$

if and only if

$$|Q(\exp i\omega T)F(\exp i\omega T)| \cdot l_s^*(\omega) < 1 \quad \text{for } 0 \leq \omega \leq \pi/T \quad (55)$$

Note that the periodicity and (3) imply that if (55) holds for $0 \leq \omega \leq \pi/T$, then it holds for all ω . Note also that stability of the system in Fig. 1 (b) is equivalent to stability of the classical feedback structure in Fig. 1 (a), provided that $C(z)$ is related to Q and F through (4).

The above condition is both necessary and sufficient if stability for all the plants in the set described by (21) is required. Any conservativeness comes only from the fact that this set is, in general, larger than that of the plants which one actually needs to consider. For a specified sampling time, $l_s^*(\omega)$ is obtained from (21) and $Q(z)$ constructed according to § 3.2. If one then selects $F(z) = 1$, the condition (55) may or may not be satisfied. If not, one should use an $F(z) \neq 1$ to achieve it. The simplest filter $F(z)$ is a first-order one:

$$F_1(z) = \frac{(1 - \alpha)z}{z - \alpha} \quad (56)$$

This filter satisfies (31) but not (32) and therefore it can be used only for external inputs $r(s)$ with one or less poles at $s = 0$ ($m = 1$) when no offset is required. We shall assume here that this is the case. The structure of the filter for $m \geq 2$ will be discussed in § 4.3.1.

It is clear from (56) that by changing α we can affect the value of $|F(\exp i\omega T)|$ at every frequency except $\omega = 0$. Hence it is important to examine (55) at $\omega = 0$. From (21) we get $l_a^*(0) = l_a(0)$ since $H(i(2\pi k)/T) = 0$ for $k = \pm 1, \pm 2, \dots$, and $H(0)T = A(0) = 1$. Then (29), (19) and the fact that $\tilde{P}^*(1) = \tilde{P}(0)$ imply that (55) will be satisfied for $\omega = 0$ and an $F(z)$ given by (56) if and only if $l_m(0) < 1$. All that this means is that the error between the steady-state gain of the actual plant and that of the model should not be more than 100% of the gain of the model, which is a rather easily satisfied condition. For example, if all the possible plants have steady-state gains with the same sign, then one can always choose an appropriate gain for the model so that $l_m(0)$ is less than one.

Hence if (55) is not satisfied we can always increase the time constant of the filter until it does. Clearly such an α will always exist provided that $|Q(\exp i\omega T)|/l_a^*(\omega)$ is finite for all $0 \leq \omega \leq \pi/T$ and of course that $l_m(0) < 1$. However increasing the filter time-constant means that we are simply reducing the closed-loop bandwidth of the nominal system (i.e. no modelling error) and in § 3.1 we saw that this is equivalent to using a larger sampling time T . This becomes clearer if we write (55) as

$$|\tilde{P}(i\omega)Q(\exp i\omega T)F(\exp i\omega T)| \leq |\tilde{P}(i\omega)|/l_a^*(\omega) \quad (57)$$

One can see that the bandwidth of the left-hand-side term can be reduced by either increasing α in $F(z)$ or leaving $F(z) = 1$ and increasing T . A graphical depiction of the above is given in Fig. 3. Note that in Fig. 3 the right-hand-side term of (56) is assumed to be independent of T by using the approximation $l_a^*(\omega) \approx l_a(\omega)$. For illustrative purposes, this is a reasonable approximation for $0 \leq \omega \leq \pi/T$ but it should not be used to check (55); $l_a^*(\omega)$ should be computed from (21).

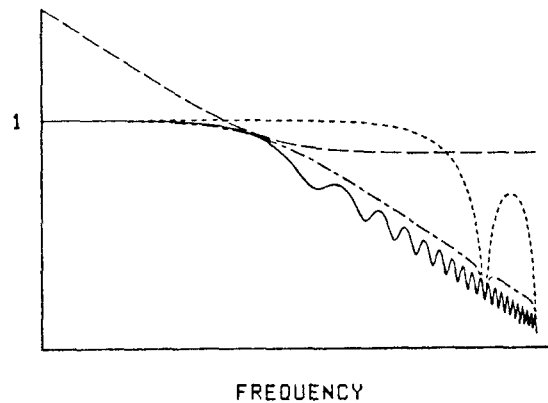


Figure 3. Effect of sampling on robust stability (logarithmic plot): ——— $1/l_m(\omega)$; ——— $|Q(\exp i\omega T)\tilde{P}(i\omega)|$, $T = T_1$; - - - - $|Q(\exp i\omega T)\tilde{P}(i\omega)|$, $T = T_2 < T_1$; - · - · $|F_1(\exp i\omega T)Q(\exp i\omega T)\tilde{P}(i\omega)|$, $T = T_2$.

4.2. Robust performance

In § 3.1 it was demonstrated that since the values of $Q(z)$ for frequencies larger than π/T are uniquely determined from its values for $0 \leq \omega \leq \pi/T$, by using a digital controller one can only try to guarantee good performance for frequencies less than

π/T . The same holds for $F(z)$, of course, since it is a rational function in z . Therefore, we shall now proceed to obtain a condition for acceptable performance in the presence of model-plant mismatch by considering all frequencies such that $0 \leq \omega \leq \pi/T$. According to § 2.2, there is one dominant term in the infinite sum in (20) which for $0 \leq \omega \leq \pi/T$ is the term corresponding to $k=0$. Hence (20), yields the following approximation:

$$P_A^*(\exp i\omega T) - \tilde{P}_A^*(\exp i\omega T) \approx \frac{1}{T} H(i\omega) E_a(i\omega) A(i\omega) \quad \text{for } 0 \leq \omega \leq \pi/T \quad (58)$$

Similarly, from (10) we obtain

$$d_A^*(\exp i\omega T) \approx \frac{1}{T} A(i\omega) d(i\omega) \quad \text{for } 0 \leq \omega \leq \pi/T \quad (59)$$

Note that if no prefilter is used, i.e. if $A(s) = 1$, then in order to obtain (59), T must be such that $d(s)$ is small for $\omega > \pi/T$.

Substitution of (15), (58) and (59) into (12) yields for the response of the system $y(s)$ to a disturbance $d(s)$:

$$y(i\omega) \approx \frac{1 - \tilde{P}(i\omega)Q(\exp i\omega T)F(\exp i\omega T)H(i\omega)A(i\omega)/T}{1 + E_a(i\omega)Q(\exp i\omega T)F(\exp i\omega T)H(i\omega)A(i\omega)/T} d(i\omega), \quad \text{for } 0 \leq \omega \leq \frac{\pi}{T} \quad (60)$$

Note that the transfer function between the error $(y(s) - r(s))$ and the setpoint $r(s)$ is the same as in (60) with A in the numerator substituted by the identity.

A limitation on the performance deterioration caused by model-plant mismatch can now be set by requiring that the magnitude of the function connecting y and d in (60) is bounded by a designer-specified function of ω . This transfer function is similar to the sensitivity function defined between the error $y(s) - r(s)$ and $r(s)$ or $d(s)$ for continuous control systems. For sampled-data systems however one cannot obtain an equation in the form of (60) without making the approximations in (58) and (59), because (12) describes a time-varying relation between $y(s)$ and $d(s)$. Let us use the notation

$$K(s) = Q(\exp sT)F(\exp sT)H(s)A(s)/T \quad (61)$$

For robust performance we require

$$\left| \frac{1 - \tilde{P}(i\omega)K(i\omega)}{1 + E_a(i\omega)K(i\omega)} \right| \leq S(\omega) \quad \forall E_a \text{ s.t. } |E_a(i\omega)| \leq l_a(\omega) \quad \text{for } 0 \leq \omega \leq \pi/T \quad (62)$$

where $S(\omega)$ is designer-specified. Note however that $S(\omega)$ cannot be chosen arbitrarily small because even for $E_a = 0$, the left-hand side of (62) may be non-zero. The selection of $S(\omega)$ will be discussed in § 4.3.2.

We shall now proceed to write (62) in a different form without making any conservative steps. The idea behind the following steps is based on the concept of the structured singular value, introduced by Doyle (1982).

We can write (62) \Leftrightarrow

$$\begin{aligned} & 1 + \frac{1 - \tilde{P}(i\omega)K(i\omega)}{1 + E_a(i\omega)K(i\omega)} \frac{1}{S(\omega)} \Delta(\omega) \neq 0, \\ & \forall E_a \text{ s.t. } |E_a(i\omega)| \leq l_a(\omega) \quad \text{and} \quad \forall \Delta \text{ s.t. } |\Delta(\omega)| < 1 \quad \text{for } 0 \leq \omega \leq \pi/T \\ & \Leftrightarrow 1 + K(i\omega)l_a(\omega) \frac{E_a(i\omega)}{l_a(\omega)} + \frac{1 - \tilde{P}(i\omega)K(i\omega)}{S(\omega)} \Delta(\omega) \neq 0 \end{aligned}$$

$$\begin{aligned} & \forall E_a \text{ s.t. } |E_a(i\omega)/l_a(\omega)| \leq 1 \quad \text{and} \quad \forall \Delta \text{ s.t. } |\Delta(\omega)| < 1 \quad \text{for } 0 \leq \omega \leq \pi/T \\ & \Leftrightarrow L(\omega) \triangleq |K(i\omega)|l_a(\omega) + \frac{|1 - \tilde{P}(i\omega)K(i\omega)|}{S(\omega)} \leq 1 \quad \text{for } 0 \leq \omega \leq \pi/T \end{aligned} \quad (63)$$

Hence the above condition is non-conservative. It may however be somewhat optimistic because of the approximations in (58) and (59). Any such optimism though is related only to performance. Robust stability is guaranteed from (55). Finally, note that (63) is different from (55) in the sense that increasing the sampling time does not lead to satisfaction of (63). An optimization over the filter time-constant in an effort to satisfy (63) is necessary at each T and the result is improved as T is reduced.

4.3. Filter design

This section deals with the design of a filter so that the robustness conditions derived in §§ 4.1 and 4.2 are satisfied.

4.3.1. Filter structure

In the case where $m \geq 2$ a filter described by (56) is not sufficient. However we can write

$$F(z) = (\beta_0 + \beta_1 z^{-1} + \dots + \beta_w z^{-w}) \frac{(1 - \alpha)z}{z - \alpha} \quad (64)$$

where the coefficients β_0, \dots, β_w are such that $F(z)$ satisfies (31) and (32), for some specified α . Equation (31) implies that we must have

$$\beta_0 = 1 - (\beta_1 + \dots + \beta_w) \quad (65)$$

The following theorem allows the computation of β_1, \dots, β_w .

Theorem 3

Equation (32) is satisfied for an $F(z)$ given by (64) and (65) if and only if the coefficients β_1, \dots, β_w ($w \geq m - 1$) satisfy

$$N_w \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_w \end{bmatrix} = \begin{bmatrix} -\alpha/(1 - \alpha) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \left. \vphantom{\begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_w \end{bmatrix}} \right\} (m - 1) \quad (66)$$

where the elements of the matrix N_w of dimension $(m - 1) \times w$ are defined in (53).

Proof

See Appendix C.

For a choice of $w > m - 1$, there are more than one solution to (66) and then one can obtain β_1, \dots, β_w as the minimum-norm solution (see Stewart 1973). Then, as $w \rightarrow \infty$ the norm of this solution goes to zero and from (64) and (65) it follows that the properties of $F(z)$ are not significantly different from those of $F_1(z)$. Finally, note that for $m = 2$, one should choose a $w \geq 2$ in order to avoid the trivial solution $F(z) = 1$.

Then the minimum-norm solution for $m = 2$ and $w \geq 2$, is

$$\beta_k = -\frac{6k\alpha}{(1-\alpha)w(w+1)(2w+1)}, \quad k = 1, \dots, w \quad (67)$$

4.3.2. Selection of the weight $S(\omega)$

The choice of $S(\omega)$ depends on the performance requirements set by the designer. However, one can use as a guide a function $S_0(\omega)$ determined by the model $\tilde{P}(s)$. Let ζ_1, \dots, ζ_n be the right half-plane zeros of $\tilde{P}(s)$ and τ_D its time delay. Define

$$\tilde{P}_+(s) = \prod_{j=1}^n \frac{(-s + \zeta_j)}{(s + \zeta_j)} \exp(-\tau_D s) \quad (68)$$

Then the optimal sensitivity function in terms of minimizing the integral squared error for a step input is $1 - \tilde{P}_+(s)$ (Frank 1974; see also Kwakernaak and Sivan 1972). Hence a reasonable choice is

$$S(\omega) \geq S_0(\omega) = |1 - \tilde{P}_+(i\omega)| \quad (69)$$

The above sensitivity function however is achieved only by a non-proper controller. The properness requirement simply adds to (69) the condition that $S(\infty) \geq 1$, even if there are no right half-plane zeros and time delays. Also note that though $S_0(0) = 0$, there is no need to choose $S(0) = 0$, since $Q(z)$ and $F(z)$ have been designed so that the conditions (22)–(25) are satisfied, which guarantee no steady-state offset under modelling error, provided that stability is maintained.

4.3.3. Computation of α

The filter parameter α has to be adjusted in an effort to satisfy (55) and (63). Equation (55) is equivalent to placing a lower bound α^* on α . This can be obtained from a Bode plot of $(|Q(\exp i\omega T)|/l_s^*(\omega))^{-1}$. If this quantity is never less than one, then $\alpha^* = 0$. If it attains values less than one, then α^* can be found from the Bode plot of a first-order filter so that (55) is satisfied. For example, if ω_l is the frequency at which the above quantity becomes equal to 0.7, then α should be larger than approximately $\exp(-T\omega_l)$, i.e.

$$\alpha^* = \exp(-T\omega_l) \quad (70)$$

Note that as explained in § 4.3.1, the properties of $F(z)$ in (64) are practically the same as those of $F_1(z)$ in (56) for a sufficiently large w . Subsequently one should obtain

$$\psi(T) = \min_{\alpha^* \leq \alpha < 1} \left(\max_{0 \leq \omega \leq \pi/T} L(\omega) \right) \quad (71)$$

where $L(\omega)$ is defined in (63).

The above minimization can be done by simply computing $L(\omega)$ for a number of values for α . The computational effort is very small. It is advisable however that one write $\alpha = \exp(-T/\tau)$ where τ is in $[\tau^*, \infty)$ with $\alpha^* = \exp(-T/\tau^*)$ and minimize over τ . Then it is, in general, sufficient to only examine τ s such that $1/\tau$ is in the frequency range where any significant changes in the value of $S(\omega)$ occur. The optimal τ will be denoted by τ_{opt} and the corresponding α by α_{opt} . Both τ_{opt} and α_{opt} are functions of T .

5. Sampling-time selection

5.1. Sampling-time bounds

These are imposed from the following:

- (a) *Open-loop bandwidth*: Let ω_B be the frequency at which $|\tilde{P}(i\omega_B)/\tilde{P}(0)| = 0.7$. Then letting π/T be less than ω_B , clearly makes no engineering sense. For example, it may not be possible to take care of undesirable open-loop response characteristics like overshoot described by a peak at a frequency less than ω_B , since the controller can only guarantee good performance up to π/T . Therefore one should choose

$$\pi/T \geq \omega_B \quad (72)$$

- (b) *Expected disturbances*: Let the frequency content of any expected disturbances be negligible for frequencies larger than ω_d . Then Shannon's sampling theorem (Åström and Wittenmark 1984) implies that if one wants to reconstruct those disturbances then one has to use a π/T at least as large as ω_d . If not then the aliasing problem will appear unless an anti-aliasing analogue prefilter is used.
- (c) *Prefilter*: Since this is an analogue device, hardware and cost considerations put a limit ω_A on how small the prefilter cut-off frequency can be. Hence one should choose

$$\pi/T \geq \min \{\omega_d, \omega_A\} \quad (73)$$

- (d) *Digital computer*: It is clear that, depending on the particular machine to be used and the total load that it should accommodate, there exists a lower bound T_{comp} on the possible sampling times. Combination with (72) and (73) yields:

$$T_{\text{comp}} \leq T \leq \pi / \max \{\omega_B, \min \{\omega_d, \omega_A\}\} \quad (74)$$

5.2. Initial choice for T

The discussion in § 4.1 on (57) which is illustrated in Fig. 3, indicates that a reasonable starting point would be a T_{init} such that

$$T_{\text{init}} = \pi/\omega_c \quad (75)$$

where ω_c is the smallest frequency at which $|\tilde{P}(i\omega)|/|l_a(\omega)| = 0.7 (\Leftrightarrow 1/l_m(\omega) = 0.7)$, since such a T would tend to satisfy the robust stability condition (57) for $F(z) = 1$. If there is no ω at which this happens, then one can choose as ω_c the corner frequency at which $1/l_m(\omega)$ settles to its value for $\omega \rightarrow \infty$. T_{init} of course should be kept within the limits specified by (74).

5.3. Iteration on the sampling time.

The first step in each iteration is to design $Q(z)$ and $F(z)$ for a given T . Then depending on the value of the quantity $\psi(T)$ defined in (71), either a new value for T is determined or the decision is taken to terminate the iteration. Two cases are possible:

- (a) $\psi(T) > 1$: Then the performance condition (63) is not satisfied; stability however is guaranteed through satisfaction of (55). A smaller T should be used next. If there has been no trial at a T where $\psi \leq 1$, then a $T_{\text{next}} = T/10$ (moving π/T to the right by one decade) is a reasonable choice. If the bounds of (74) are violated, T should be set equal to the corresponding bound.

- (b) $\psi(T) \leq 1$: Both the robust stability and performance requirements are satisfied and therefore the design is an acceptable one. However, it may be that the same requirements are also satisfied for a larger sampling time. To find the largest T where the conditions are satisfied, defined as

$$T_0 = \max \{T: \psi(T) \leq 1\} \quad (76)$$

one should increase T . If no larger T has been tried, then a reasonable choice is to move π/T by one or one-half decade to the left. If a larger T where $\psi > 1$ is known, then T_{next} can be chosen as the geometric mean of the two values.

Finally if no T can be found within the limits of (74), for which $\psi(T) \leq 1$, that means that the performance requirements set by the designer through the choice of $S(\omega)$ are too strict to be satisfied. The only course of action is to choose a different $S(\omega)$ and repeat the procedure. A plot of $L(\omega)$ as a function of ω for $\alpha = \alpha_{\text{opt}}$ at the smallest T that was used can help locate the frequency range where $S(\omega)$ was too strict.

6. Illustrations

The controller design for two systems will be presented. The first example will serve as an illustration of the design procedure. In the second, the procedure will be applied on a system that is difficult to control and a high-frequency external input will be considered in order to demonstrate that fast sampling does not necessarily help to achieve good performance.

6.1. Example 1

Let

$$\tilde{P}(s) = \frac{3}{(s+1)(s+3)} \quad (77)$$

A delay-type uncertainty is assumed, i.e.

$$P(s) = \tilde{P}(s) \exp(-\tau_D s) \quad (78)$$

where

$$0 \leq \tau_D \leq 0.05 \quad (79)$$

then (16) and (78) $\Rightarrow |E_m(i\omega)| = \sqrt{2} (1 - \cos(\omega\tau_D))^{1/2} \Rightarrow$ (from (79))

$$l_m(\omega) = \begin{cases} 2 & \text{for } \omega \geq 20\pi \\ \sqrt{2} (1 - \cos(0.05\omega))^{1/2} & \text{for } 0 \leq \omega \leq 20\pi \end{cases} \quad (80)$$

and $l_s(\omega)$ can be obtained through (19).

Bounds on T can now be obtained from (74). For the system of (77), $\omega_B = 0.92$. The assumption will be made that no high-frequency disturbances are expected ($\omega_d = \omega_B$) and that $T_{\text{comp}} \rightarrow 0$. Finally, no prefilter will be used ($\omega_A = \infty$). Then (74) yields

$$0 < T \leq \pi/\omega_B = 3.4 \quad (81)$$

Since $\tilde{P}_+(s) = 1$, the only restriction on $S(\omega)$ is that it is larger than 1 at $\omega = \infty$. Its shape depends on how strict a performance requirement one wishes to set. A Bode

plot of $|\tilde{P}(i\omega)|$ is helpful in this respect. For this design, a choice of

$$S(\omega) = 0.4 \left[\frac{(\omega^2/2^2 + 1)}{(\omega^2/10^2 + 1)} \right]^{1/2} \quad (82)$$

is made based on the observation that at $\omega = 2$, $|\tilde{P}(i\omega)|$ is small enough (≈ 0.35) to justify a relaxation of the performance requirement. Also $S(\infty) = 2 > 1$. It should be noted that the above choice is a rather strict performance requirement, but it is justified because the system is not inherently difficult to control and the uncertainty is small.

Equation (80) yields a value $\omega_c = 31$ and then from (75) we get $T_{\text{init}} = 0.101$. An iteration on T according to the outline in § 5.3 yields the values shown in Table 1.

T	$\psi(T)$	$1/\tau_{\text{opt}}$	ω_l
0.1013	1.22	7.713	($z^* = 0$)
0.0101	0.90	6.581	31.9
0.0320	0.98	7.124	41.0

Table 1.

At the final choice of $T = 0.032$, the controller $Q(z)$ designed according to the rules in § 3.2 is given by

$$Q(z) = \frac{345.9(z^2 - 1.877z + 0.8797)}{z^2} \quad (83)$$

and the filter for $\alpha = \alpha_{\text{opt}} = \exp(-T/\tau_{\text{opt}})$ is

$$F(z) = \frac{0.2041z}{z - 0.7959} \quad (84)$$

The controller for the classical feedback structure of Fig. 1 (a) can be obtained through (4).

In Fig. 4 (a) the response to a step set-point input is given for $P(s) = \tilde{P}(s)$. In Fig. 4 (b) a model-plant mismatch is assumed and $P(s) = \tilde{P}(s) \exp(-0.05s)$ is used. For comparison, simulations are also given for a continuous controller

$$QF_{\text{cont}}(s) = \frac{1(s+1)(s+3)}{3(0.0737s+1)^2} \quad (85)$$

designed with the same performance condition and bound $S(\omega)$. Also the controller obtained by a Tustin approximation at $T = 0.032$ of the classical feedback controller corresponding to $QF_{\text{cont}}(s)$ is simulated. It is clear that the design obtained with the proposed design procedure performs robustly under modelling error. On the other hand, the discrete approximation of the continuous controller tends not to be robust and a smaller sampling time would be required to improve it. It should, however, be repeated that this is a system which is rather easy to control and which was chosen solely to illustrate the design procedure.

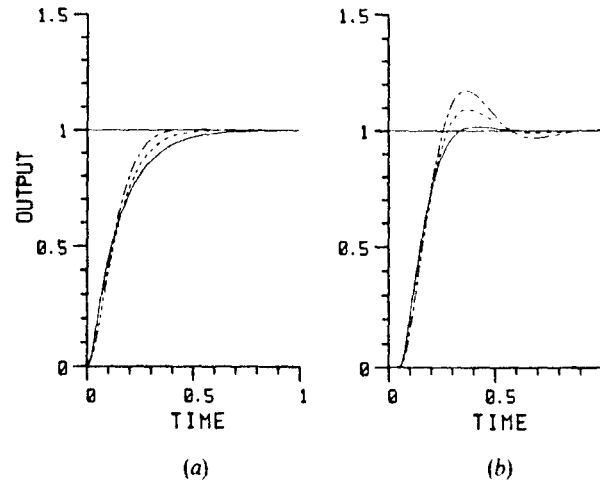


Figure 4. Example 1, step set-point response: (a) $P = \tilde{P}$, (b) $P \neq \tilde{P}$, $\tau_D = 0.05$; ——— $Q(z)F(z)$, $T = 0.032$; - - - $QF_{\text{cont}}(s)$; - · - Tustin approximation of $QF_{\text{cont}}(s)$ for $T = 0.032$.

6.2. Example 2

Let

$$P(s) = K \frac{-0.5s + 1}{(s + 1)(0.25s + 1)} \exp(-\tau_D s) \quad (86)$$

where

$$0.95 \leq K \leq 1.05 \quad (87)$$

$$0.35 \leq \tau_D \leq 0.45 \quad (88)$$

and the nominal values $K = 1$, $\tau_D = 0.4$ are used for the model $\tilde{P}(s)$.

The $|E_m(i\omega)| = |K \exp[i(0.4 - \tau_D)\omega] - 1|$ and after some algebra

$$l_m(s) = \begin{cases} 2.05 & \text{for } \omega > 20\pi \\ [2.1025 - 2.1 \cos(0.05\omega)]^{1/2} & \text{for } 10\pi \leq \omega \leq 20\pi \\ [2.1025 - 1.9 \cos(0.05\omega)]^{1/2} & \text{for } 0 \leq \omega \leq 10\pi \end{cases} \quad (89)$$

For this system $\omega_B = 1.2$. High-frequency disturbances requiring an $\omega_d = 63$ ($\pi/\omega_d = 0.05$) are considered possible and T_{comp} is assumed to be practically zero. Two cases will be distinguished with respect to the use of a prefilter:

Case I. No limitations on the use of a prefilter, i.e. $\omega_A = 0$. Then (74) yields

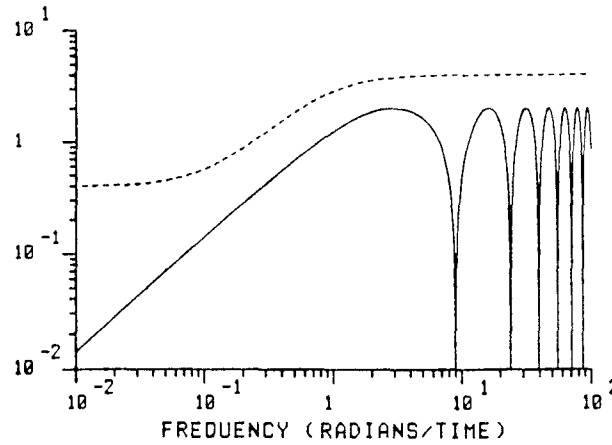
$$0 < T \leq \pi/\omega_B = 2.6 \quad (90)$$

Case II. No prefilter can be used. Hence $\omega_A = \infty$ and from (74)

$$0 < T \leq \pi/\omega_d = 0.05 \quad (91)$$

In Fig. 5 a plot of $S_0(\omega)$ is shown. $S(\omega)$ is chosen as

$$S(\omega) = 0.4 \left[\frac{(\omega^2/0.1^2 + 1)}{(\omega^2 + 1)} \right]^{1/2} \quad (92)$$

Figure 5. Example 2: ——— $S_0(\omega)$; ---- $S(\omega)$.

so that (69) is satisfied. It is clear that the non-minimum-phase elements in $\tilde{P}(s)$ limit the achievable performance even for no modelling error (Holt and Morari 1985 a, b).

Equation (89) yields $\omega_c = 30$ and then from (75) we obtain $T_{\text{init}} = 0.105$. This T_{init} is outside the bound in (91) and therefore in Case II a $T_{\text{init}} = 0.05$ will be used.

Case I: A second-order Butterworth prefilter with a cutoff frequency of $\pi/2T$ is selected for each sampling time. The iteration on T yields the values shown in Table 2.

T	$\psi(T)$	$1/\tau_{\text{opt}}$	α^*
0.105	0.963	2.157	0
0.331	1.027	2.603	0
0.186	0.986	2.317	0
0.248	1.000	3.070	0

Table 2.

For the final choice of $T = 0.248$, the controller $Q(z)$ and the filter $F(z)$ are given by

$$QF_I(z) = 1.458 \frac{z^2 - 1.150z + 0.2889}{z^2 - 1.088z + 0.2900} \quad (93)$$

The anti-aliasing prefilter is

$$A(s) = \frac{1}{0.02499s^2 + 0.2236s + 1} \quad (94)$$

Case II: For $T = T_{\text{init}} = 0.05$, $\psi(T) = 0.936 < 1$ and therefore this is the final choice since (91) allows no larger T . We have $1/\tau_{\text{opt}} = 2.062 < \omega_l = 27$ which yields

$$QF_{II}(z) = 1.0585 \frac{z^2 - 1.770z + 0.7788}{z^2 - 1.8065z + 0.8159} \quad (95)$$

The response of the two control systems to a high-frequency disturbance

$$d(s) = \frac{1}{s} \cdot \frac{1}{(0.001003s^2 + 0.006334s + 1)} \quad (96)$$

justifying an $\omega_d = 63$, is shown in Fig. 6 (a) for $P(s) = \tilde{P}(s)$ ($K = 1$, $\tau_D = 0.4$) and in Fig. 6 (b) for $K = 1.05$ and $\tau_D = 0.45$. The two designs are robust to model-plant mismatch and behave quite similarly. This was to be expected because the faster sampling in Case II does not aim at a faster response, but at avoiding the aliasing problem that would appear if no prefilter were used. The speed of response is determined by the robustness requirements. To demonstrate this, we shall proceed to design the controller $Q(z)$ for the particular input $v(s) = d(s)$ for $T = 0.05$ and ignore any robustness requirements. Equations (33)–(41) are used to obtain $Q_{SE}(z)$ and then (44) yields

$$Q_d(z) = 6.247 \frac{0.8000z^4 - 1.250z^3 + 1.093z^2 - 1.222z + 0.5947}{z^3(z - 0.9045)} \quad (97)$$

One can see in Fig. 6 (a) that the response is faster, though not much, since as mentioned earlier the non-minimum-phase elements limit the achievable performance. However, even this small improvement for the nominal case ($P = \tilde{P}$) is paid for by instability in the presence of modelling error as Fig. 6 (b) shows. Note that for $Q = Q_d$ and $\alpha = 0$, (55) is not satisfied, therefore indicating potential instability.

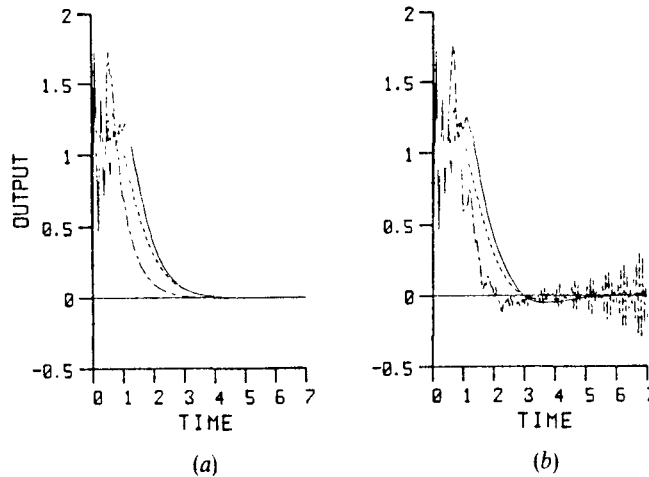


Figure 6. Example 2. response to $d(s)$: (a) $P = \tilde{P}$, (b) $P \neq \tilde{P}$, $K = 1.05$, $\tau_D = 0.45$; ——— $QF_d(z)$; $T = 0.248$; ---- $QF_{II}(z)$, $T = 0.05$; - - - $Q_d(z)$, $T = 0.05$.

7. Conclusions

Two main goals were accomplished in this paper. The first was the derivation of conditions that guarantee robust stability and performance for sampled-data systems and the development of a controller synthesis method. The conditions that were obtained can easily be checked and the computation effort required for the design is small. Any particular external input (set-point or disturbance) can be considered and the performance requirements are defined through a designer-specified frequency weight for the selection of which guidelines are given.

The second goal was the illustration of the effect of the sampling time on the achievable performance and the robustness properties of the control system. These relations were quantified in an iterative procedure for robust controller design and sampling-time selection. This design procedure has the advantage that it can easily be programmed on the computer in an interactive form.

Appendix A. Proof of Theorem 1

The disturbance $d(s)$ is fed through $A(s)$ before it is sampled and therefore for no offset it is clear that we need

$$\lim_{\text{time} \rightarrow \infty} \mathcal{L}^{-1}\{d(s) - A(s)d(s)\} = 0 \quad (\text{A } 1)$$

Application of the final-value theorem on (A 1) yields

$$\lim_{s \rightarrow 0} (s[1 - A(s)]d(s)) = 0 \quad (\text{A } 2)$$

Since (A 2) must be satisfied for all $d(s)$ with m or less poles at $s = 0$, we need $[1 - A(s)]$ to have m zeros at $s = 0$, which will be the case if and only if (24) and (25) hold.

Equation (A 1) implies that $\lim_{\text{time} \rightarrow \infty} \mathcal{Z}^{-1}\{d^*(z) - d_\lambda^*(z)\} = 0$ or $\lim_{z \rightarrow 1} ((1 - z^{-1}) \times [d^*(z) - d_\lambda^*(z)]) = 0$. Hence for offset considerations, $d_\lambda^*(z)$ can be substituted for $d^*(z)$ in (13). Consider an external input $v(s)$ and let

$$v^*(z) = \mathcal{Z} \mathcal{L}^{-1}\{v(s)\} \quad (\text{A } 3)$$

Then for both cases: (i) $v^*(z) = -r^*(z)$, $d_\lambda^*(z) = 0$; (ii) $v^*(z) = d_\lambda^*(z)$, $r^*(z) = 0$, (13) yields after substitution of d_λ^* for d^* :

$$y^*(z) - r^*(z) = \frac{1 - F(z)Q(z)\tilde{P}_\lambda^*(z) + F(z)Q(z)[P_\lambda^*(z) - P^*(z)]}{1 + F(z)Q(z)[P_\lambda^*(z) - \tilde{P}_\lambda^*(z)]} v^*(z) \quad (\text{A } 4)$$

Assume $P = \tilde{P}$; then (A 4) becomes

$$y^*(z) - r^*(z) = [1 - F(z)Q(z)\tilde{P}^*(z)]v^*(z) \quad (\text{A } 5)$$

The final-value theorem implies that for no offset we need

$$\lim_{z \rightarrow 1} ((1 - z^{-1})[1 - F(z)Q(z)\tilde{P}^*(z)]v^*(z)) = 0 \quad (\text{A } 6)$$

If $v(s)$ has l poles at $s = 0$, then from (A 3) it follows that $v^*(z)$ has l poles at $z = 1$. Hence (A 6) will be satisfied for all $l \leq m$ if and only if $(1 - F(z)Q(z)\tilde{P}^*(z))$ has m zeros at $z = 1$, i.e. if and only if (22) and (23) hold. Note that (A 1) means that the steady-state value of a signal in the class of inputs considered, going through $[1 - A(s)]$ is zero. This will remain zero even after passing through some other stable systems, say P or \tilde{P} . Hence

$$\lim_{z \rightarrow 1} ((1 - z^{-1})[P_\lambda^*(z) - P^*(z)]v^*(z)) = \lim_{z \rightarrow 1} ((1 - z^{-1})[\tilde{P}_\lambda^*(z) - \tilde{P}^*(z)]v^*(z)) = 0$$

and therefore

$$\begin{aligned} \lim_{z \rightarrow 1} ((1 - z^{-1})[1 - F(z)Q(z)\tilde{P}_\lambda^*(z) + F(z)Q(z)[P_\lambda^*(z) - P^*(z)]]v^*(z)) \\ = \lim_{z \rightarrow 1} ((1 - z^{-1})[1 - F(z)Q(z)\tilde{P}^*(z)]v^*(z)) = 0 \end{aligned} \quad (\text{A } 7)$$

Then from (A 4) and (A 7) it follows that the offset is zero even when $P \neq \tilde{P}$. Hence conditions (22), (23), (24) and (25) are all that is needed.

Appendix B. Proof of (41)

Sampling of (28) yields

$$y^*(z) = \tilde{P}^*(z)Q(z)r^*(z) + [1 - \tilde{P}^*(z)Q(z)]d^*(z) \quad (\text{B } 1)$$

Note that to obtain (B 1) from (28) we have to assume $A(s) = 1$. If this is not the case, then we should use $d_A^*(z)$ instead of $d^*(z)$ in (B 1), which yields

$$y^*(z) = \tilde{P}^*(z)Q(z)r^*(z) + [1 - \tilde{P}^*(z)Q(z)]d_A^*(z) \quad (\text{B } 2)$$

This equation cannot be obtained from (28) by sampling. However it can be used for control purposes because using an $A(s) \neq 1$ means that we opted for the rejection of $A(s)d(s)$ instead of $d(s)$.

$Q_{SE}(z)$ is the stable proper rational function which minimizes Φ :

$$\Phi = \sum_{k=0}^{\infty} (\mathcal{Z}^{-1}\{r^*(z) - y^*(z)\})^2 \quad (\text{B } 3)$$

Then by substituting (B 1) or (B 2) into (B 3) we obtain

$$\Phi = \sum_{k=0}^{\infty} (\mathcal{Z}^{-1}\{[1 - \tilde{P}^*(z)Q(z)]r^*(z)\})^2 \quad (\text{B } 4)$$

whether r is a set-point ($r = r, d = 0$) or a disturbance ($r = 0, v = d$ or $v = Ad$).

We now have to perform the following to assure that we shall obtain a proper Q . From (33) we see that because of the time delay, Q does not affect the first $N + 1$ terms in (B 4). If $r^*(z)$ also contains a delay N_r , i.e. $z^{N_r}r^*(z)$ is semi-proper, then this introduces an additional number of N_r terms in (B 4) which are not affected by Q . By using (35) we obtain

$$\begin{aligned} [1 - \tilde{P}^*(z)Q(z)]r^*(z) &= z^{-N_r}[z^{N_r}r^*(z) - \tilde{P}^*(z)Q(z)z^{N_r}r^*(z)] \\ &= z^{-N_r}(r_0 + r_1z^{-1} + \dots + r_Nz^{-N}) + z^{-N_r}z^{-N-1} \\ &\quad \times [r_{N+1} + r_{N+2}z^{-1} + \dots - z^{N+1}\tilde{P}^*(z)Q(z)z^{N_r}r^*(z)] \end{aligned} \quad (\text{B } 5)$$

It is clear that in (B 5) the first term of the right-hand side involves only the first $N_r + N + 1$ terms in (B 4) while the second, which contains Q , involves only the remainder of the terms. Hence minimizing Φ is equivalent to minimizing $\hat{\Phi}$:

$$\begin{aligned} \hat{\Phi} &= \sum_{k=N_r+N+1}^{\infty} (\mathcal{Z}^{-1}\{z^{-N_r-N-1}[r_N^*(z) - z^{N+1}\tilde{P}^*(z)Q(z)z^{N_r}r^*(z)]\})^2 \\ &= \sum_{k=0}^{\infty} (\mathcal{Z}^{-1}\{r_N^*(z) - z^{N_r+N+1}\tilde{P}^*(z)Q(z)r^*(z)\})^2 \end{aligned} \quad (\text{B } 6)$$

where $r_N^*(z)$ is defined in (36).

By applying Parseval's theorem on the right-hand side of (B 6) we obtain

$$\hat{\Phi} = \frac{1}{2\pi} \int_{-\pi}^{\pi} |r_N^*(\exp i\theta) - \exp[i(N_r + N + 1)\theta]\tilde{P}^*(\exp i\theta)Q(\exp i\theta)r^*(\exp i\theta)|^2 d\theta \quad (\text{B } 7)$$

For the $\tilde{P}_+^*(z)$ and $r_+^*(z)$ defined by (37) and (38) we can easily check that

$$|\tilde{P}_+^*(\exp i\theta)| = 1 \quad \text{for } -\pi \leq \theta \leq \pi \quad (\text{B } 8)$$

$$|r_+^*(\exp i\theta)| = 1 \quad \text{for } -\pi \leq \theta \leq \pi \quad (\text{B } 9)$$

since any complex zeros of $\tilde{P}^*(z)$ and $v^*(z)$ come in complex conjugate pairs.

Use of (39), (40), (B 8) and (B 9) in (B 7) yields

$$\hat{\Phi} = \frac{1}{2\pi} \int_{-\pi}^{\pi} |[\tilde{P}_+^*(\exp i\theta)v_+^*(\exp i\theta)]^{-1}v_-^*(\exp i\theta)\exp(-i\theta) - \tilde{P}_-^*(\exp i\theta)v_-^*(\exp i\theta)Q(\exp i\theta)|^2 d\theta \quad (\text{B } 10)$$

Define

$$f_1(z) = (\tilde{P}_+^*(z)v_+^*(z))^{-1}v_-^*(z)z^{-1} \quad (\text{B } 11)$$

$$f_2(z) = \tilde{P}_-^*(z)v_-^*(z) \quad (\text{B } 12)$$

Then $f_2(z)Q(z)$ is strictly proper and stable (poles strictly inside the unit circle) except possible for some poles of $f_2(z)$ at $z = 1$. $f_1(z)$ is strictly proper but not stable. Write

$$f_1(z) = \{f_1(z)\}_+ + \{f_1(z)\}_- \quad (\text{B } 13)$$

where $\{f_1\}_+$ contains only the unstable poles (strictly outside the unit circle), and $\{f_1\}_-$ only the stable poles. Any poles at $z = 1$ are included in $\{f_1\}_-$. The reason is that we shall assume at this point that the optimal $Q(z)$ is such that these poles are cancelled in both $f_1 - f_2Q$ and $\{f_1\}_- - f_2Q$. It should, however, be verified that the optimal Q has this property. We can obtain $\{f_1\}_+$ and $\{f_1\}_-$ from f_1 by partial-fraction expansion. The constant term is zero since $f_1(z)$ is strictly proper.

Let $L_2(-\pi, \pi)$ be the space of functions $f(\exp i\theta)$ which are square-integrable with respect to θ , i.e. for which

$$\int_{-\pi}^{\pi} |f(\exp i\theta)|^2 d\theta < \infty \quad (\text{B } 14)$$

The inner product in this space is defined by

$$\langle f_a, f_b \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{f_a(\exp i\theta)} f_b(\exp i\theta) d\theta \quad (\text{B } 15)$$

where the overbar indicates complex conjugate. Then by using (B 10) to (B 15) we obtain

$$\begin{aligned} \hat{\Phi} &= \langle \{f_1\}_+ + \{f_1\}_- - f_2Q, \{f_1\}_+ + \{f_1\}_- - f_2Q \rangle \\ &= \langle \{f_1\}_+, \{f_1\}_+ \rangle + \langle \{f_1\}_- - f_2Q, \{f_1\}_- - f_2Q \rangle \\ &\quad + \langle \{f_1\}_+, \{f_1\}_- - f_2Q \rangle + \langle \{f_1\}_- - f_2Q, \{f_1\}_+ \rangle \end{aligned} \quad (\text{B } 16)$$

Note that the first of the four terms in the right-hand side of (B 16) is independent of Q . As for the last two, they are both zero because they represent the inner products between a strictly proper stable and a strictly proper but totally unstable function and these two subspaces of $L_2(-\pi, \pi)$ are orthogonal (Francis and Zames 1983).

Hence our problem reduces to minimizing $\langle \{f_1\}_- - f_2Q, \{f_1\}_- - f_2Q \rangle$. The obvious solution to this is

$$Q_{SE}(z) = f_2^{-1}(z) \{f_1(z)\}_- \quad (\text{B } 17)$$

Note that the above $Q_{SE}(z)$ is stable and proper. Also we have $\{f_1\}_- - f_2Q_{SE} = 0$ and $f_1 - f_2Q_{SE} = \{f_1\}_+$, which has no poles at $z = 1$ and therefore the assumption that Q is such that the poles at $z = 1$ cancel out in both $f_1 - f_2Q$ and $\{f_1\}_- - f_2Q$ holds.

Hence the above $Q_{SE}(z)$ is acceptable and therefore it is the solution we were seeking.

In the case of set-point-following where future values of the set-point are supplied to the controller, which we want our system output to follow after N_p time steps, the objective function Φ in (B 3) should be written

$$\Phi = \sum_{k=0}^{\infty} (\mathcal{Z}^{-1}\{z^{-N_p}r^*(z) - y^*(z)\})^2 \quad (\text{B } 18)$$

By following the same steps used to find $Q_{SE}(z)$, we can easily see that (B 17) is obtained, but with $v_N^*(z)$ substituted with $v_{N_0}^*(z)$ in (B 11), where $v_{N_0}^*(z)$ is defined by (42).

Appendix C. Proof of Theorem 3

The following lemma will be used.

Lemma

Let $h(\lambda) = (1 - \alpha)/(1 - \alpha\lambda)$. Then

$$h^{(k)}(\lambda) = (1 - \alpha)k!\alpha^k(1 - \alpha\lambda)^{-(k+1)} \quad (\text{C } 1)$$

where the superscript (k) denotes the k th derivative.

Proof (by induction)

$$\begin{aligned} k=1 \quad & \frac{d}{d\lambda} h(\lambda) = (1 - \alpha)\alpha(1 - \alpha\lambda)^{-2} \\ k=n \quad & \text{Let } h^{(n)}(\lambda) = (1 - \alpha)n!\alpha^n(1 - \alpha\lambda)^{-(n+1)} \\ k=n+1 \quad & \text{From (C 2) we get} \end{aligned} \quad (\text{C } 2)$$

$$h^{(n+1)}(\lambda) = (1 - \alpha)n!\alpha^n \frac{d}{d\lambda} (1 - \alpha\lambda)^{-(n+1)} = (1 - \alpha)(n+1)!\alpha^{n+1}(1 - \alpha\lambda)^{-(n+2)} \quad \square$$

Proof of Theorem 3

Equation (32) is equivalent to

$$\left. \frac{d^k}{d\lambda^k} F(\lambda^{-1}) \right|_{\lambda=1} = 0, \quad k=1, \dots, m-1 \quad (\text{C } 3)$$

From (57) we get

$$F(\lambda^{-1}) = \Gamma(\lambda)h(\lambda) \quad (\text{C } 4)$$

where

$$\Gamma(\lambda) = \beta_0 + \beta_1\lambda + \dots + \beta_w\lambda^w \quad (\text{C } 5)$$

From the Lemma we get

$$h^{(k)}(1) = k!\alpha^k(1 - \alpha)^{-k} \quad (\text{C } 6)$$

then (C 3) for $k=1$ yields

$$\Gamma^{(1)}(1)h(1) + \Gamma(1)h^{(1)}(1) = 0$$

or

$$\Gamma^{(1)}(1) = -h^{(1)}(1) = -\alpha/(1-\alpha) \quad (\text{C } 7)$$

We shall now show that (C 3) for $k = 2, \dots, m-1$ yields $\Gamma^{(k)}(1) = 0$ for $k = 2, \dots, m-1$. The proof will be by induction.

$k = 2$

Equation (C 3) for $k = 2$ yields

$$\Gamma^{(2)}(1)h(1) + 2\Gamma^{(1)}(1)h^{(1)}(1) + \Gamma(1)h^{(2)}(1) = 0$$

or by using (C 6) and (C 7)

$$\Gamma^{(2)}(1) = 0$$

$k \leq n$ for $2 \leq n < m-1$

$$\text{Let } \Gamma^{(k)}(1) = 0 \quad (\text{C } 8)$$

$k = n+1$

Equation (C 3) for $k = n+1$ yields, because of (C 8)

$$\Gamma^{(n+1)}(1)h(1) + (n+1)\Gamma^{(1)}(1)h^{(n)}(1) + \Gamma(1)h^{(n+1)}(1) = 0$$

or by using (C 6) and (C 7)

$$\Gamma^{(n+1)}(1) = 0$$

Hence by induction

$$\Gamma^{(k)}(1) = 0, \quad k = 2, \dots, m-1 \quad (\text{C } 9)$$

But one can easily see that

$$\begin{bmatrix} \Gamma^{(1)}(1) \\ \vdots \\ \Gamma^{(m-1)}(1) \end{bmatrix} = N_w \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_w \end{bmatrix} \quad (\text{C } 10)$$

and therefore Theorem 3 follows from (C 7), (C 9) and (C 10). \square

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REFERENCES

- ÅSTRÖM, K. J., and WITTENMARK, B., 1984, *Computer Controlled Systems* (Englewood Cliffs, NJ: Prentice-Hall).
- DOYLE, J. C., 1982, *Proc. Instn electron. Engrs*, Pt D, **129**, 242.
- DOYLE, J. C., and STEIN, G., 1981, *I.E.E.E. Trans. autom. Control*, **26**, 4.
- FRANCIS, B. A., and ZAMES, G., 1983, *Proc. 22nd I.E.E.E. Conf. on Decision and Control*, San Antonio, TX, p. 103.
- FRANK, P. M., 1974, *Entwurf von Regelkreisen mit Vorgeschrieben Verhalten* (Karlsruhe: G. Braun Verlag).
- FRANKLIN, G. F., and POWELL, J. D., 1980, *Digital Control of Dynamic Systems* (New York: Addison-Wesley).
- GARCIA, C. E., and MORARI, M., 1982, *Ind. Engng Chem. Proc. Des. Dev.*, **21**, 308.
- HOLT, B. R., and MORARI, M., 1985 a, *Chem. Engng Sci.*, **40**, 9; 1985 b, *Ibid.*, **40**, 1229.
- KWAKERNAK, H., and SIVAN, R., 1972, *Linear Optimal Control Systems* (New York: Wiley-Interscience).
- STEWART, G. W., 1973, *Introduction to Matrix Computations* (New York: Academic Press).
- ZAFIRIOU, E., and MORARI, M., 1985, *Int. J. Control*, **42**, 855.