ABSTRACT

Title of dissertation: COMPUTING LOCAL L-FACTORS FOR THE UNRAMIFIED PRINCIPAL SERIES OF $\mathrm{Sp}_2(F)$ AND ITS METAPLECTIC COVER

Christian A. Zorn, Doctor of Philosophy, 2007

Dissertation directed by: Professor Stephen S. Kudla
Department of Mathematics

One of the central goals of this thesis is to verify the local Langlands correspondence for the rank two symplectic group $\mathrm{Sp}_2(F)$, where F is a p-adic local field with $p \neq 2$. This correspondence seeks to parameterize admissible representations of various matrix groups over F with representations of the Weil-Deligne group of F (denoted $\mathcal{W}'_F$). This correspondence should include an equality of certain local factors, one being the local L-factors attached to both representations of both the matrix group and the Weil group.

We will restrict our attention to constituents of the unramified principal series of $\mathrm{Sp}_2(F)$. In particular, we employ some criteria of Lusztig to assign these representations Weil-Deligne data. While computing the L-factor for representations of the Weil-Deligne group is well known and understood, we require a method for defining the local L-factor for representations of the matrix group.

Our method for defining L-factors for representations of $\mathrm{Sp}_2(F)$ is a modification of the doubling integral of Piatetski-Shapiro and Rallis [8]. While Piatetski-

Shapiro and Rallis formulate a definition of L-factor via this doubling method, we seek to realize the Weil-Deligne L-factor as an application of our modified integral evaluated on certain "good test vectors". Such choices will rely on a wide range of machinery, including intertwining operators, the Weil representation and studying local densities of quadratic form. We tie this wide range of material together, in great detail, through the course of the thesis.

Finally, this method of defining L-factors can be extended in a natural way to representations of the metaplectic cover of $\mathrm{Sp}_2(F)$. While the Local Langlands correspondence does not apply to this group, we are still able to produce Weil-Deligne data and L-factors for these representations by using Lusztig's criteria on constituents of the unramified principal series of $\mathrm{SO}_5(F)$. In particular, we demonstrate a bijection between constituents of the genuine unramified principal series of $\widetilde{\mathrm{Sp}_2}(F)$ and the unramified principal series of $\mathrm{SO}_5(F)$ in such a way that the doubling L-factor for a representation on the metaplectic group matches the Weil-Deligne L-factor for the corresponding representation on the special orthogonal group.

COMPUTING LOCAL L-FACTORS FOR THE UNRAMIFIED PRINCIPAL SERIES OF $\mathrm{Sp}_2(F)$ AND ITS METAPLECTIC COVER

by

Christian Alexander Zorn

Dissertation submitted to the Faculty of the Graduate School of the
University of Maryland, College Park in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
2007

Advisory Committee:
Professor Stephen S. Kudla, Chair/Advisor
Professor Jeffrey D. Adams
Professor Thomas J. Haines
Professor Lawrence C. Washington
Professor David Mount, Dean's Representative

Acknowledgments

I would like to take this opportunity to acknowledge some of the outstanding people that have left such a lasting impression on me over the course of my graduate school career. All of these people helped make my experience at the University of Maryland one of the most worthwhile periods of my life.

First, and foremost, I would like to thank my advisor, Professor Stephen Kudla. He has excelled as a teacher and as a mentor. I have appreciated the opportunity to work on such an outstanding problem with him over this past few years. Moreover, I am very grateful for the various opportunities that Steve helped open for me. For instance, Steve has always been very generous with his financial support so that I could travel to various meetings and conferences that I feel were essential for my development as a mathematician.

Second, I would like to thank the rest of the faculty in the Mathematics Department at the University of Maryland. In particular, I owe a special mention to Professors Larry Washington, Tom Haines and Jeff Adams. They were always generous with their time and expertise. I truly appreciate them letting me discuss both problems related to my thesis as well as issues related to the job search. Finally, I would like to thank my final committee member, Professor David Mount in the Computer Science Department for agreeing to be my Dean's representative.

Third, I would like to acknowledge Professor Jim Cogdell at the Ohio State University Mathematics Department. I appreciate both your comments on the rough draft of my thesis as well as your assistance in my being offered employment in Columbus.

Fourth, I should mention my family's support. I would like to express my endless gratitude to my future bride, Colleen. She has shown a great deal of patience when my commitment to graduate school takes precedent over our spending time together. I am also deeply grateful for her willingness to move away from an area where she is already established so that I can further pursue my career. Next, my mother, father and stepmother have always been a great source of strength for me throughout this endeavor. I especially appreciate my parents being receptive to helping me navigate frustrations and placing my successes in proper context. Finally, my brothers, their families and my best friend, Bryan, have been a constant source of encouragement through these years.

Finally, I would like to mention all of my friends at the University of Maryland, especially those from the Mathematics Department. I owe a particularly special acknowledgment to my friend, Eric Errthum. Eric and I have been friends and officemates since we arrived in the program. We have had several shared triumphs and frustrations; I think that the last three semesters would have been especially unbearable without his help. Thank you much Eric and wherever I end up, I hope it is an easy trip to Winona.

It has been an eventful five years and those that know me, know that I can be very forgetful. As such, I would also like to say apologize to anyone that I have overlooked and extend my most genuine appreciation.

I offer a very sincere thank you.

Table of Contents

List of Tables

Chapter 1

Introduction

1.1 The Thesis Problem

One goal of the Langlands program is generalizing the results of local class field theory. For instance, let F be a p-adic field and let

$$\mathcal{W}_F \subset \mathrm{Gal}(\bar{F}/F),$$

be its Weil group. It is a well known fact from local class field theory that we have a reciprocity isomorphism

$$\mathrm{Art}_F : F^\times \to \mathcal{W}_F^{ab}.$$

So characters of the group $F^\times$ are identified with characters on $\mathcal{W}_F$. Furthermore, there exists a uniformizer $\varpi \in F$ and a Frobenius element $\mathrm{Fr}_q \in \mathcal{W}_F$ such that

$$\mathrm{Art}_F(\varpi) = \overline{\mathrm{Fr}}_q,$$

where $\overline{\mathrm{Fr}}_q$ is the image of Fr_q in $\mathcal{W}_F^{ab}$. Consequently, if

$$\chi_v : F^\times \to \mathbb{C}^\times$$

and

$$\sigma_q : \mathcal{W}_F^{ab} \to \mathbb{C}^\times$$

are corresponding characters, then we have an equality of local Tate factors

$$(1 - \chi_v(\varpi)q^{-s})^{-1} = (1 - \sigma_q(\overline{\mathrm{Fr}}_q)q^{-s})^{-1}.$$

In this thesis, we seek to demonstrate a generalization of these results to certain representations of $G = \mathrm{Sp}_2(F)$, the rank two symplectic group. In particular, we will be working with constituents of the unramified principal series. While we will eventually discuss the local Langlands correspondence in some greater generality, we ultimately seek to prove a particular case of that very general conjecture. In particular, for π a constituent of the unramified principal series of $\mathrm{Sp}_2(F)$, we will define an L-factor $L(s, \pi, \mathbf{r}_{st})$ as the result of applying a modified doubling integral (see Chapter 4, [22] and [8]) on a particular set of "good test vectors" (see Tables B.7-B.8). We then compare this local factor to one arising from a representation of the Weil-Deligne group $\mathcal{W}'_F$ (see Chapter 2). In particular, one might state the first goal of this thesis as follows.

Theorem 1.1.1. *Let π be a constituent of the unramified principal series of $G = \mathrm{Sp}_2(F)$ (see Table B.1), then there exists an admissible representation*

$$\rho' = (\rho, N) : \mathcal{W}'_F \to {}^L G^0 = \mathrm{SO}_5(\mathbb{C}),$$

determined via some criteria of Lusztig (see Chapter 2 and [25]), satisfying the following property: Let $\mathbf{r}_{st} : \mathrm{SO}_5(F) \to \mathrm{GL}_5(\mathbb{C})$ be the obvious inclusion, so that

$$\mathbf{r}_{st} \circ \rho' : \mathcal{W}'_F \to \mathrm{GL}_5(\mathbb{C})$$

is a Weil-Deligne representation, then

$$L(s, \pi, \mathbf{r}_{st}) = L(s, \mathbf{r}_{st} \circ \rho').$$

Note that Tables B.5 and B.6, give the description of the Weil-Deligne representations that we associate to the various π by Lusztig's criteria. While we will explain this in greater detail in Section 2.3.3, it can be summarized as following (with π and ρ' as in the theorem above):

1. Let π' be the spherical representation parameterized by ρ. Then π and π' are constituents of the same induced representation.

2. If π is spherical, then $N = 0$. Otherwise, Lusztig's method determines the N according to whether π is a tempered representation or induced from a tempered representation on the Levi factor of a proper parabolic subgroup.

Further, we will occasionally refer to the set of representations mapping to a fixed $\rho' = (\rho, N)$ under Lusztig's criteria as the **L-packet** defined by ρ'; the equality of L-factors from our theorem provides some evidence to support this definition of an L-packet.

As we will eventually show, computing the factor $L(s, \mathbf{r}_{st} \circ \rho')$ can be done in a completely general way and offers us a set of L-values that conjecturally match the L-factors we compute for constituents of the unramified principal series of G. While computing L-factors on the "Galois" side of the correspondence can be done in a very general setting, there is no known general method for defining $L(s, \pi, \mathbf{r}_{st})$ for arbitrary matrix groups. There are various techniques that require specific conditions. Some require conditions on the actual matrix group G, others place conditions on the representation π.

For the purpose of this thesis, we employ a variant of the doubling integral of Piatetski-Shapiro and Rallis to define the various the L-factor for an admissible representation of G. This method uses a global Rankin-Selberg integral that can be unwound as a product of local integrals. It puts no extra conditions on the representation π, but it requires a group such as $\mathrm{Sp}_2(F)$ that is defined as preserving an inner product.

Ultimately, one goal of this thesis is to realize the Weil-Deligne L-factors by computing a slight variant of the doubling integral on some "good test vectors". At the various nonarchimedean local places F, Piatetski-Shapiro and Rallis generate an ideal of the ring $\mathbb{C}[q^{-s}]$, where q is the cardinality of the residue field of F, by computing the doubling integral over several large families of functions. The normalized generator of this ideal defines the L-factor for the representation. Our method seeks to find sufficient "good test vectors" that produce an L-factor in the same fractional ideal that matches L-factor predicted by the local Langlands correspondence, where we use Lusztig's method for assigning L-homomorphisms to unramified principal series representations of $\mathrm{Sp}_2(F)$.

The other results of this thesis concern genuine principal series representations of $\widetilde{\mathrm{Sp}_2}(F)$, the metaplectic cover of $\mathrm{Sp}_2(F)$. With only the most minor adjustments, we will produce local L-factors for certain constituents of the genuine principal series of the metaplectic group. Although there is no local Langlands correspondence for metaplectic group, we would still like to make sense of the L-factors. In this case, we should be able to relate them to L-factors of certain representations of $\mathrm{SO}_5(F)$.

Before we continue, it is worth noting which covering group $\widetilde{\mathrm{Sp}_n}(F)$ we are dealing with. Much of the literature is devoted to the 2-fold cover of $\mathrm{Sp}_n(F)$ (which we will denote $\widetilde{\mathrm{Sp}_n}^{(2)}(F)$); however, our cover will be infinite. In fact, our covering group satisfies the following exact sequence

$$1 \to \mathbb{C}^1 \to \widetilde{\mathrm{Sp}_n}(F) \xrightarrow{\tau} \mathrm{Sp}_n(F) \to 1$$

and realized as

$$\widetilde{\mathrm{Sp}_2}(F) = \mathrm{Sp}_2(F) \times \mathbb{C}^1$$

with

$$[g_1, z_1]_L \cdot [g_2, z_2]_L = [g_1 g_2, c_L(g_1, g_2) z_1 z_2]_L.$$

Note that the cocycle we consider is not the typical Rao cocycle [31] that is used to define the double cover. Instead we will employ the Leray cocycle (also defined in [31]) which is valued in the eighth roots of unity. The advantage is that larger cover offers more splitting of subgroups of $\mathrm{Sp}_n(F)$ than the double cover. The Leray cocycle also allows us to define parabolic induction on $\widetilde{\mathrm{Sp}_n}(F)$ in a manner more analogous to parabolic induction on $\mathrm{Sp}_n(F)$. For parabolic subgroups $P = MN$ contained in the Siegel parabolic, we show that there exists a splitting

$$P \to \widetilde{\mathrm{Sp}_n}(F) \qquad p \mapsto [p, 1]_L.$$

In particular, this splitting applies to the Borel subgroup $P_\emptyset \subset P$. So for a representation (σ, V) of the Levi factor M, we can inflate to P and then extend this to a homomorphism σ' on $\tilde{P} = \tau^{-1}P$ via

$$\sigma'([p, z]_L)v = z\sigma(p)v.$$

We can then induce this representation up to $\widetilde{\mathrm{Sp}_2}(F)$. Note that this is not quite the same process used to define induced representations for the double cover of $\mathrm{Sp}_2(F)$

defined by the Rao cocycle. However, we will show that our construction relates induced representations on the double cover in a natural way.

Let $\tilde{\pi}$ be a constituent of the genuine unramified principal series of $\widetilde{\mathrm{Sp}_2}(F)$. As in the case of $\mathrm{Sp}_2(F)$, we will define an L-factor $L(s, \tilde{\pi}, \mathrm{r}_{st})$ as the result of our doubling integral applied to certain "good test vectors." However, the local Langlands conjecture does not apply to covering groups like $\widetilde{\mathrm{Sp}_n}(F)$. Instead, we will show demonstrate a bijection between constituents $\tilde{\pi}$ of the genuine unramified principal series of $\widetilde{\mathrm{Sp}_2}(F)$ and constituents π of the unramified principal series of the split group $\mathrm{SO}_5(F)$. Under this bijection, the doubling L-factor $L(s, \tilde{\pi}, \mathbf{r}_{st})$ matches the Weil-Deligne L-factor $L(s, \mathbf{r}_{st} \circ \rho')$ attached to the Weil-Deligne data ρ' to which π maps by Lusztig's criteria. In particular, we would like to show the following result.

Theorem 1.1.2. *There exists a bijection between constituents $\tilde{\pi}$ of $\mathrm{Ind}_{\tilde{P}_\emptyset}^{\tilde{G}}((\chi_1 \otimes \chi_2)')$ with*

$$\chi_i : F^\times \to \mathbb{C}^\times$$

unramified quasi-characters (see Table B.9) and constituents π of

$$\mathrm{Ind}_{P'_\emptyset}^{\mathrm{SO}_5(F)}(\chi_1 \otimes \chi_2)$$

(see Table B.10; note $P'_\emptyset$ is a fixed Borel subgroup) with the following property. For

$$\pi \mapsto \rho' = (\rho, N)$$

given by Lusztig's criteria [25], then

$$L(s, \tilde{\pi}, \mathbf{r}_{st}) = L(s, \mathbf{r}_{st} \circ \rho')$$

(see Tables B.15-B.18).

As a future project, one might ask if this bijection could also be realized using the theta correspondence on the dual reductive pair $(\widetilde{\mathrm{Sp}_2}(F), \mathrm{SO}_5(F))$.

Now having described the main results, we will outline the content of the thesis. Also note that while much of this thesis can be applied to extensions of $\mathbb{Q}_p$ for arbitrary p, there are also large portions that require $p \neq 2$. Consequently, we will just enforce the condition that $p \neq 2$ throughout the thesis. This is especially necessary in much of Chapters 4 and 5, which contain the essential results building toward our main theorems.

It is also worth mentioning that the doubling method introduced in [22] does include the $p = 2$ case. In fact, computing the $p = 2$ largely motivated this modified doubling integral of Kudla, Rapoport and Yang. However, the $p = 2$ case is more complicated than the $p \neq 2$ in [22]. Thus, one could reasonably assume that extending the results of this thesis to $p = 2$ is possible, but likely more complicated than the $p \neq 2$ case.

1.2 An Outline of the Thesis

The next chapter will discuss some representation theory of general p-adic groups as well as some representation theory specific to $\mathrm{Sp}_n(F)$. While we won't mention metaplectic covers in this chapter, much of the machinery we introduce will be applicable to metaplectic groups with little or no modification. In particular, we will describe parabolic induction, Jacquet modules, Frobenius reciprocity as well as results particular to unitarizable representations. This material will be crucial later when we develop various methods for ascertaining the explicit values of Iwahori-fixed vectors in principal series representations. The next section of this chapter deals with reducibility points of principal series representations of $\mathrm{Sp}_2(F)$. Much of this material can also be found in [37] and [33]. In particular, Sally and Tadić determine the reducibility points of both the principal series of $\mathrm{GSp}_2(F)$ and $\mathrm{Sp}_2(F)$. The final section of the next chapter will explain the local Langlands conjecture as it applies to split p-adic groups. In particular, we will discuss Lusztig's criteria for assigning principal series representations to L-packets and how to compute the L-factor associated to each packet.

The subsequent chapter will discuss the metaplectic cover of $\mathrm{Sp}_n(F)$. We will define these covering groups and discuss genuine representations for these groups. In particular, we will discuss the relationship of our definition for parabolic induction to the one used for the double cover defined by the Rao cocycle. This is important because of a conjectured correspondence between representations of $\widetilde{\mathrm{Sp}_n}^{(2)}(F)$ and $\mathrm{SO}_{2n+1}(F)$. We would like to use Lusztig's criteria for finding conjectural L-values for representations applied to $\mathrm{SO}_5(F)$ and then prove that such L-factors are identical those on defined by doubling on the corresponding representations of $\widetilde{\mathrm{Sp}_2}(F)$.

Another section of this chapter explains the Weil representation $(\omega_V, S(V^n))$ on $\widetilde{\mathrm{Sp}_n}(F)$. Aside from being an important tool for producing liftings between representations of various groups, we also employ this representation in our construction of the doubling integral. In particular, this representation allows us to recast the question of choosing functions in a certain induced representation to that of selecting appropriate smooth functions on a quadratic vector space.

Chapter 4 explains the method used to compute local L-factors on $\mathrm{Sp}_2(F)$ and its metaplectic cover. We explain the classical doubling method of Piatetski-Shapiro and Rallis from [8] and then give a detailed explanation of the variant of Kudla, Rapoport and Yang [22]. Ultimately, our method integrates a function from our constituent on $\mathrm{Sp}_2(F)$ against a test vector from an induced representation on $\mathrm{Sp}_4(F)$. Notice that this construction computes L-factors for representations on $\mathrm{Sp}_2(F)$ by using the symplectic group of double rank; hence the name of the method. The variant in [22] actually defines an operator on the given representation. So for π an irreducible constituent of the unramified principal series of $\mathrm{Sp}_2(F)$, we can compute $L(s, \pi, \mathbf{r}_{st})$, as defined in [8], via the following steps.

1. Find a parahoric subgroup J_G with

$$\dim_{\mathbb{C}}(\pi^{J_G}) = 1.$$

2. Choose a family of function $\{\Phi_s\}_{\mathrm{Re}(s)>>0}$ on $\mathrm{Sp}_4(F)$ that has a related invariance property.

3. For the correct choice of $\{\Phi_s\}$, the doubling integral

$$Z(s, \Phi, f) \in \pi^{J_G}.$$

4. In particular, we should have

$$Z(s, \Phi, f) = \frac{L(s, \pi, \mathbf{r}_{st})}{d_\Phi(s)} f$$

where $d_\Phi(s)^{-1} \in \mathbb{C}[q^{-s}]$.

As was mentioned above, we will employ a interpolation method to transfer the problem of selecting a Φ_s to that of selecting a Schwartz function on a quadratic vector space.

Because our computation relies on the explicit values for various Iwahori-fixed vectors, we devote one section of this chapter to several methods for ascertaining such values. One method relies on the vanishing of standard intertwining operators on the full induced representations. In particular, we will derive relations that are satisfied by the Iwahori invariants of a given representation. Ideally, we should find enough relations to determine the explicit values on the set of Iwahori invariants. This technique works very well for regular inducing data. For irregular data, we derive some techniques involving exactness of parabolic induction as well as inner products on unitarizable subquotients in order to derive similar relations as above. In both cases, we leverage explicit information regarding the Iwahori invariance of the inducing data in order to determine information about the Iwahori invariance of the constituents on the larger group.

The last section of this chapter deals with the topic of local densities of quadratic forms. In particular, we explore the work of Tonghai Yang [42] that computes local densities of arbitrary quadratic forms in low rank cases. In order to compute the local doubling integral, we spend a great deal of effort reducing the doubling integral to a linear combination of Whittaker functions. These Whittaker functions are closely related to local densities of quadratic forms. Unfortunately, the results of Yang are not quite general enough to apply to these Whittaker functions, so we also spend some effort reducing arbitrary Whittaker functions to terms that can be computed using Yang [42].

The final chapter computes some local L-factors using the doubling integral and the other results that we develop throughout the thesis. This section is very detailed and explicitly escorts the reader through the entire calculation for several representations. In particular, we compute the local L-factor in the spherical case on both the linear and metaplectic group. For these representations, the calculations are still sufficiently simple to be computed by hand. We also demonstrate a calculation for one ramified representation of the linear group. The ramified cases are much more computationally intensive and require some help from Mathematica.

This thesis also contains two appendices. The first is simply a compilation of material needed to compute the various Weil indexes that are ubiquitous in the study of Weil representations. The second appendix is a table that summarizes all the results outlined above. There are also several tables that contain data about the unramified principal series of $\mathrm{Sp}_2(F)$ as well as similar data for the relevant representations on $\widetilde{\mathrm{Sp}_2}(F)$ and $\mathrm{SO}_5(F)$.

So in order to establish our theorem, we have the following steps:

For $\mathrm{Sp}_2(F)$, we must:

1. Determine all admissible representations π having an Iwahori-fixed vector.

2. Associate such representations to L-packets $\rho' = (\rho, N)$.

3. Compare the doubling L-factor $L(s, \pi, \mathbf{r}_{st})$ with the Galois L-factor

$L(s, \mathbf{r}_{st} \circ \rho')$.

For $\widetilde{\mathrm{Sp}_2}(F)$, we must:

1. Determine all admissible representations $\tilde{\pi}$ having an Iwahori-fixed vector.

2. Associate such a representation to an admissible π of $\mathrm{SO}_5(F)$ having an Iwahori-fixed vector.

3. Associate the various π to L-packets $\rho' = (\rho, N)$.

4. Compare the doubling L-factor $L(s, \tilde{\pi}, \mathbf{r}_{st})$ with the Galois L-factor $L(s, \mathbf{r}_{st} \circ \rho')$.

Notation

The following notation holds throughout the thesis:

$$N_1 = \left\{ n_1(c) = \begin{pmatrix} 1 & c & & \\ & 1 & & \\ & & 1 & \\ & & -c & 1 \end{pmatrix} \mid c \in F \right\},$$

$$N_2 = \left\{ n_2(b) = \begin{pmatrix} I & b \\ & I \end{pmatrix} \mid b \in \mathrm{Sym}_2(F) \right\}$$

$$A = \left\{ m(a) = \begin{pmatrix} a & \\ & {}^t a^{-1} \end{pmatrix} \mid a \in \mathrm{GL}_2(F) \right\}$$

and

$$\mathrm{diag}(a_1, a_2, \ldots, a_n) = \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{pmatrix}.$$

Finally bold letters denote the image under the splitting map

$$P_\alpha \hookrightarrow \widetilde{\mathrm{Sp}_2}(F).$$

For instance, $\mathbf{m}(a) = [m(a), 1]_L$.

Let k be any field with $\mathrm{char}(k) \neq 2$. For our purposes, the standard parabolic subgroups of $G = \mathrm{Sp}_2(k)$ will be subgroups $P \supset P_\emptyset$. Moreover, $P_\emptyset$ is a Borel subgroup with

$$P_\emptyset = M_\emptyset N_\emptyset$$

where

$$M_\emptyset \simeq (k^\times)^2$$

is the diagonal torus and

$$N_1, N_2 \subset N_\emptyset.$$

The standard parabolic subgroups of interest are:

- (Siegel Parabolic) $P_\alpha = M_\alpha N_\alpha$ with

$$M_\alpha \simeq \mathrm{GL}_2(k) \quad \text{and} \quad N_\alpha \simeq \mathrm{Sym}_2(k).$$

- (Long Root Parabolic) $P_\beta = M_\beta N_\beta$ with

$$M_\beta \simeq k^\times \times \mathrm{Sp}_1(k) \quad \text{and} \quad M_\beta \simeq \mathbb{H}(k^2)$$

  where $\mathbb{H}(k^2)$ is the Heisenberg group associated to the quadratic space

$$(k^2, Q(x, y) = 2xy).$$

Let F and $\mathcal{O}$ be as in the introduction. Further, set $K = \mathrm{Sp}_2(\mathcal{O})$, then a standard parahoric subgroup will be either the subgroup K or the full universe image of one of the standard parabolic subgroups under the map

$$K \to \mathrm{GL}(\mathcal{O}/\mathcal{P}).$$

Chapter 2

General Representation/Representation Theory of $\mathrm{Sp}_2(F)$

2.1 Representation Theory of p-adic Groups

In this first chapter, we will explore some general representation theory that applies to all manner of p-adic topological group. We will then discuss results that are more applicable to arbitrary symplectic groups $\mathrm{Sp}_n(F)$ and finally we will specialize all our results to the group $\mathrm{Sp}_2(F)$. While some of the results discussed in this section are well known, we will apply them toward the following goals.

1. Determine the reducibility points of the unramified principal series of $\mathrm{Sp}_2(F)$. The results are contained in Table B.1.

2. Begin building techniques for computing the explicit values of Iwahori-fixed vectors and the dimension of the Iwahori invariants of various representations. The three main constructs applied here are the standard intertwining operators, inner products for unitarizable representations and the exactness of parabolic induction. These techniques should apply to fairly general p-adic groups.

3. Describe the local Langlands conjecture for split p-adic groups. Applied to $\mathrm{Sp}_2(F)$ this allows us to produce conjecture L-factors that we will later compute on representations of $\mathrm{Sp}_2(F)$.

Finally, we begin with some very basic representation theory for general p-adic groups.

2.1.1 Induction, Jacquet functors and Frobenius reciprocity

Let us recall some of the basic facts from the representation theory of p-adic groups and give some context as to how they will be used in this thesis. For instance, let $G = \mathbb{G}(F)$ be the p-adic points of an algebraic group and $P \subset G$ be any parabolic subgroup with Levi decomposition MN. Then for any admissible representation (σ, W) of the group M, we define the induced representation

$$\mathrm{Ind}_P^G(\sigma) = \left\{ f : g \to W \mid \begin{array}{r} f(mng) = \delta_P(m)^{\frac{1}{2}}\sigma(m)f(g)\ \forall m \in M, n \in N, g \in G \\ \exists K \text{ compact open, } f(gk) = f(g)\ \forall k \in K, g \in G \end{array} \right\},$$

where $\delta_P : P \to \mathbb{C}^\times$ is the modulus character of P. G acts on this space via right translation; such a construction is called (normalized) smooth induction. So for the purpose of this thesis, all induction is assumed to be normalized. This has the benefit that inducing from unitary data results in a unitary representation. Furthermore, for a representation (π, V) of the group G, we consider the set

$$V(N) = \mathrm{span}_{\mathbb{C}}\{\pi(n)v - v \mid v \in V, n \in N\}$$

and notice the quotient $V_N = V/V(N)$ is a representation of the group M via the action

$$m \cdot (v + V(N)) = \delta_P(m)^{-\frac{1}{2}}\pi(m)v + V(N).$$

We call r_P^G the (normalized) Jacquet functor with respect to P and $r_P^G(\pi)$ the (normalized) Jacquet modules for π with respect to P. It is known that both constructions respect admissibility. Thus, let $\mathfrak{R}(G)$ is the Grothendieck group of the category of smooth finite-length representations of G. This is essentially the free abelian group of the set of finite-length representations of G. We notice that the functors for parabolic induction Ind_P^G and the Jacquet functor r_P^G lift to obvious maps on the Grothendieck groups $\mathfrak{R}(G)$ and $\mathfrak{R}(M)$. Further, $\mathfrak{R}(G)$ has a natural partial ordering $\pi_1 \leq \pi_2$ if $m(\tau, \pi_1) \leq m(\tau, \pi_2)$ for all smooth irreducible τ, where $m(\tau, \pi)$ is the multiplicity of τ in π. Additionally, parabolic induction and the Jacquet functor satisfy the following adjointness property.

Theorem 2.1.1 (Frobenius Reciprocity). *Let $\pi \in \mathfrak{R}(G)$ and $\sigma \in \mathfrak{R}(M)$, then we have*

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_P^G(\sigma)) = \mathrm{Hom}_M(r_P^G(\pi), \sigma).$$

Proof. See [2]. □

While this appears to be a categorical statement regarding two adjoint functors, we will use it in some explicit computations. Moreover, let $G = \mathrm{Sp}_2(F)$ and $P = P_\emptyset$, a Borel subgroup with $M_\emptyset = (F^\times)^2$. The irreducible representations of $M_\emptyset$ are parameterized by pairs of quasicharacters $\chi_i : F^\times \to \mathbb{C}^\times$. In fact, we have

$$\chi_1 \otimes \chi_2 : M_\emptyset \to \mathbb{C}^\times \qquad \chi_1 \otimes \chi_2(\begin{pmatrix} a_1 & \\ & a_2 \end{pmatrix}) = \chi_1(a_1)\chi_2(a_2).$$

Furthermore, we define an action of the Weyl group $W_G = W_{\mathrm{Sp}_2}$ on these pairs by

$$\chi_1 \otimes \chi_2 \circ w(\begin{pmatrix} a_1 & \\ & a_2 \end{pmatrix}) = \chi_1 \otimes \chi_2(w \begin{pmatrix} a_1 & \\ & a_2 \end{pmatrix} w^{-1}).$$

In the case of $\mathrm{Sp}_2(F)$, $W_G =< w_\alpha, w_\beta >$ where w_α (resp. w_β) corresponds to the short (resp. long) root in the Dynkin diagram for $\mathrm{Sp}_2(F)$. On $M_\emptyset$ we have that

$$\begin{aligned} \chi_1 \otimes \chi_2 \circ w_\alpha &= \chi_2 \otimes \chi_1 \\ \chi_1 \otimes \chi_2 \circ w_\beta &= \chi_1 \otimes \chi_2^{-1} \end{aligned}$$

It is elementary to show that

$$r_P^G(\mathrm{Ind}_{P_\emptyset}^G(\chi_1 \otimes \chi_2)) = \sum_{w \in W_G} \chi_1 \otimes \chi_2 \circ w \in \mathfrak{R}(M_\emptyset).$$

So for any irreducible constituent π of $\mathrm{Ind}_{P_\emptyset}^G(\chi_1 \otimes \chi_2)$, Frobenius Reciprocity determines for which $\mathrm{Ind}_{P_\emptyset}^G(\chi_1 \otimes \chi_2 \circ w)$ our data π appears as a submodule. In fact, if $\chi_1 \otimes \chi_2$ is regular (i.e., $\chi_1 \otimes \chi_2 \circ w \neq \chi_1 \otimes \chi_2$ for any $w \in W_G$), then

$$\begin{aligned} \dim_{\mathbb{C}}(\mathrm{Hom}_G[\pi, \mathrm{Ind}_{P_\emptyset}^G(\chi_1 \otimes \chi_2 \circ w)]) &= \dim_{\mathbb{C}}(\mathrm{Hom}_{M_\emptyset}[r_{P_\emptyset}^G(\pi), \chi_1 \otimes \chi_2 \circ w]) \\ &= \begin{cases} 1 \text{ if } \chi_1 \otimes \chi_2 \circ w \leq r_{P_\emptyset}^G(\pi) \\ 0 \text{ if } \chi_1 \otimes \chi_2 \circ w \not\leq r_{P_\emptyset}^G(\pi). \end{cases} \end{aligned}$$

Moreover, since π was taken to be irreducible, we know that any non-zero element of $\mathrm{Hom}[\pi, \mathrm{Ind}_{P_\emptyset}^G(\chi_1 \otimes \chi_2 \circ w)]$ is an embedding. In fact, Frobenius Reciprocity tells us that

$$\mathrm{Hom}_G(\mathrm{Ind}_{P_\emptyset}^G(\chi_1 \otimes \chi_2), \mathrm{Ind}_{P_\emptyset}^G(\chi_1 \otimes \chi_2 \circ w))$$

is one dimensional for regular $\chi_1 \otimes \chi_2$. In particular, we will show that this one dimensional space is spanned by a **standard intertwining integral** over a certain subgroup of the unipotent radical $N_\emptyset$. We will define this useful operator in a later section.

2.1.2 Some results regarding contragradient and unitarizability

Most of the machinery that we have discussed works in a great deal of generality, but is particularly effective when our inducing data is regular. We now discuss some additional tools that are quite useful in several of the cases that the inducing data is irregular. For $\mathrm{Sp}_2(F)$, most reducible representations coming from irregular data will have certain unitarizable constituents. We begin with a brief sketch of results pertaining to unitarizability.

Let (π, V) be an admissible representation of $G = \mathbb{G}(F)$ and let (π^*, V^*) be the (smooth) contragradient. Thus these are the smooth vectors in the space of linear functionals on V. One can show that for (π, V) admissible, that (π^*, V^*) is admissible as well [2], [6]. Moreover, we have the following properties of contragradients [6].

1. The functor $\pi \mapsto \pi^*$ is a exact and contravariant.

2. $(\pi^*)^* = \pi$.

3. For $P \subset G$ a parabolic subgroup with $P = MN$ and (σ, V_σ) an admissible representation of M, we have

$$\mathrm{Ind}_P^G(\sigma^*) = \mathrm{Ind}_P^G(\sigma)^*.$$

The third property actually follows from the following result in [4] and [38].

Lemma 2.1.1. *Let G be a p-adic group and $P = MN$ a parabolic subgroup. Further, let $C^\infty(P\backslash G; \delta_P)$ be the set of smooth, complex valued functions on G such that*

$$f(pg) = \delta_P(p) f(g)$$

where δ_P is the modulus character of P. Then there exists a G-invariant linear functional on $C^\infty(P\backslash G; \delta_P)$ given by

$$f \mapsto \int_K f(k)dk.$$

So we see that for $f \in \mathrm{Ind}_P^G(\sigma)$ and $f^* \in \mathrm{Ind}_P^G(\sigma^*)$ that the function

$$\Psi_{f,f^*}(g) =< f(g), f^*(g) >$$

belongs to $C^\infty(P\backslash G; \delta_P)$ and

$$<< f, f^* >>= \int_K < f(k), f^*(k) > dk$$

is non-degenerate pairing on $\mathrm{Ind}_P^G(\sigma) \times \mathrm{Ind}_P^G(\sigma^*)$. Now that we have considered some general theory regarding contragradients, let us examine the unitarizability of two kinds of representation that are useful in this thesis.

First, let us suppose that $P = MN$ is a parabolic subgroup of G and that (σ, V_σ) is a unitary representation of M with non-degenerate inner product $< \cdot, \cdot >_\sigma$. This inner product gives us a natural isomorphism between (σ, V_σ) and (σ^*, V_σ^*) in the usual way. Moreover, for $f, f' \in \mathrm{Ind}_P^G(\sigma)$

$$\Psi_{f,f'}(g) =< f(g), f'(g) >_\sigma \in C^\infty(P\backslash G; \delta_p).$$

Therefore,

$$<< f, f' >>= \int_K < f(k), f'(k) >_\sigma dk$$

is a non-degenerate Hermitian form on $\mathrm{Ind}_P^G(\sigma)$. In particular, if $P = B$ is a Borel for G and $\{\chi_i\}_{i=1}^n$ are unitary characters, then

$$\mathrm{Ind}_B^G(\chi_1, \chi_2, \ldots, \chi_n)$$

is unitary with inner product

$$<< f, f' >>= \int_K f(k)\overline{f'(k)}dk.$$

Such a representation is completely reducible. For instance, let $\xi : F^\times \to \mathbb{C}^1$ be the unique unramified character of order 2. For $G = \mathrm{Sp}_1(F) \simeq \mathrm{SL}_2(F)$ and B the upper triangular Borel,

$$\mathrm{Ind}_B^G(\xi) = T_\xi^1 \oplus T_\xi^2$$

with T_ξ^1 the spherical constituent. So we see that for unitary inducing data (σ, V_σ), the resulting induced representation is also unitary and we have an explicit formula for an inner product on this space that is derived from the inducing data.

The second type of representation of interest in this section are the **square-integrable** representations. Let (π, V) be a representation and (π^*, V^*) its contragradient. A **matrix coefficient** is a function of the form

$$c_{v,v^*}(g) :=< \pi(g)v, v^* >$$

for $v \in V$ and $v^* \in V^*$. An irreducible representation (π, V) is called square-integrable if the following two criteria hold:

- $\pi|_{Z(G)}$ acts via a unitary character, where $Z(G)$ is the center of G (i.e., has a unitary central character).
- All the matrix coefficients belong to $L^2(Z\backslash G)$ (all matrix coefficients are square-integrable mod center).

An important representation of this type is the **Steinberg representation** of a group G. For the groups we are interested in, this representation is the unique square-integrable subquotient of $\mathrm{Ind}_B^G(\delta_B^{\frac{1}{2}})$, where B is a Borel subgroup of G.

In general, irreducible square-integrable representations are **unitarizable**. In particular, they admit the following G-invariant inner product. Let (π, V) be an irreducible square-integrable representation and (π^*, V^*) its contragredient. Further, fix a $v_0^* \in V^*$, then we get the following non-degenerate Hermitian inner product on V

$$<< u, v >>:= \int_{Z\backslash G} < \pi(g)u, v_0^* > \overline{< \pi(g)v, v_0^* >} dg.$$

The square-integrability ensures that this definition makes sense.

While the examples above do not exhaust the unitarizable representations of G, they do include the cases of unitarizable representations that are needed for us to compute the explicit values of our Iwahori invariants for representations induced from irregular data. In nearly all these cases, our representations will have unitarizable inducing data. Thus the inducing data will have a non-degenerate inner product that we can extend to the induced representation as described above.

2.1.3 Iwahori factorization and the dimension of Iwahori invariants

As was suggested in the introduction, we also require a method of computing the dimesion of vectors fixed by various standard parahoric subgroups. Finding an upper bound for these dimensions is trivial. Let $G = \mathbb{G}(F)$ be an arbitrary p-adic group as above, $P \subset G$ a parabolic subgroup with Levi decomposition $P = MN$

and $J \subset G$ a parahoric subgroup. Then for a representation (σ, V) on M,

$$\dim_{\mathbb{C}}(\mathrm{Ind}_P^G(\sigma)^J) \leq \#(P\backslash G/J).$$

So for $\pi \subset \mathrm{Ind}_P^G(\sigma)$, then $\dim_{\mathbb{C}}(\pi^J) \leq \#(P\backslash G/J)$. Now we will study the various Jacquet modules will determine a lower bound on these dimensions. In the case of $\mathrm{Sp}_2(F)$, we will show that this suffices to determine the exact dimensions. For this section, we follow the notes of Casselman [5].

Let (π, V) be a representation of a p-adic group G. We begin by defining an Iwahori factorization for a compact open subgroup of G with respect to a parabolic subgroup $P = MN$. While Casselman has a more precise definition, for our purposes a compact open subgroup K_0 of G has an **Iwahori factorization** with respect to the parabolic subgroup $P = MN$ if

$$K_0 = (K_0 \cap \bar{N})(K_0 \cap M)(K_0 \cap N)$$

along with a condition pertaining to conjugating $K_0 \cap \bar{N}$ and $K_0 \cap N$ by a subset of the center of M (see [5]). If K_0 has an Iwahori factorization with respect to the parabolic P, then Casselman shows that

$$\pi^{K_0} \to r_P^G(\pi)^{(K_0 \cap M)}$$

is surjective under the natural projection of V onto its Jacquet module V_N. For $G = \mathrm{Sp}_2(F)$, one can easily verify that each of our standard parahoric subgroups $I_* \subset K$ ($* \in \{\emptyset, \alpha, \beta\}$) has an Iwahori factorization with respect to the corresponding parabolic subgroup $P_* = M_* N_*$. Consequently, we find that

$$\mathrm{Ind}_{P_\emptyset}^G(\chi_1 \otimes \chi_2)^{I_*} \to r_{P_*}^G(\mathrm{Ind}_{P_\emptyset}^G(\chi_1 \otimes \chi_2))^{M_*(\mathcal{O})}.$$

is surjective. So $\mathrm{Ind}_{P_\emptyset}^G(\chi_1 \otimes \chi_2)$ has at least as many I_*-fixed vectors as there are spherical representation in $r_{P_*}^G(\mathrm{Ind}_{P_\emptyset}^G(\chi_1 \otimes \chi_2))$. Moreover, we have computed all the applicable Jacquet modules and find that there are $\#(P_\emptyset \backslash G/I_*)$ such vectors. Thus our map

$$\mathrm{Ind}_{P_\emptyset}^G(\chi_1 \otimes \chi_2)^{I_*} \to r_{P_*}^G(\mathrm{Ind}_{P_\emptyset}^G(\chi_1 \otimes \chi_2))^{M_*(\mathcal{O})}$$

is a vector space isomorphism. Therefore for any constituent π of the unramified principal series of $\mathrm{Sp}_2(F)$,

$$\dim_{\mathbb{C}}(\pi^{I_*}) = \dim_{\mathbb{C}}(r_{P_*}^G(\pi)^{M_*(\mathcal{O})}).$$

Thus knowing all the Jacquet modules with respect to the standard parabolic subgroups is sufficient for determining the dimensions of parahoric invariants. It is interesting to note that for $\mathrm{Sp}_2(F)$, every constituent π of the unramified principal series has a parahoric I_* with $\dim_{\mathbb{C}}(\pi^{I_*}) = 1$.

2.2 Reducibility of Principal Series for $\mathrm{Sp}_2(F)$

In the previous sections, we developed some of the basic tools in the study of p-adic groups. However, we have yet to discuss the reducibility points of the their induced representations. Consequently, this section will be devoted to the reducibility points for principal series representations of p-adic groups. First, we will discuss some general results that offer a partial solution to finding such reducibility points [37]. Moreover, the reducibility points for the unramified principal series of $\mathrm{Sp}_2(F)$ are known. So we will summarize the work of Paul Sally, Jr. and Marko Tadić [33].

2.2.1 Some general reducibility results of Tadić

We now aim to describe some general reducibility and irreducibility criteria determined by Marko Tadić [37]. Also note that we will follow Tadić's notation in this section as well. In particular, let $G = \mathbb{G}(F)$ be the p-adic points of an algebraic group. As in the previous sections, $\mathfrak{R}(G)$ will denote the Grothendieck group of G. Tadić has the following reducibility criteria for representations of p-adic groups G :

Suppose $P_0 = M_0N_0$ and $P = MN$ are standard parabolic subgroups of G. Further, suppose σ is a smooth representation of M_0 and π, Π smooth finite-length representations of G. Finally, suppose that

1. $\mathrm{Ind}_{P_0}^G(\sigma) \leq \Pi, \pi \leq \Pi$.

2. $r_P^G(\mathrm{Ind}_{P_0}^G(\sigma)) + r_P^G(\pi) \neq r_P^G(\Pi)$.

3. $r_P^G(\mathrm{Ind}_{P_0}^G(\sigma)) \not\leq r_P^G(\pi)$.

Then, $\mathrm{Ind}_{P_0}^G(\sigma)$ is reducible.

As a concrete application of this criteria, let $G = \mathrm{Sp}_2(F)$ and let us consider the representations $\mathrm{Ind}_{P_\alpha}^G(\nu^{\frac{3}{2}} St_{\mathrm{GL}_2})$ and $\mathrm{Ind}_{P_\beta}^G(\nu^2 \otimes St_{\mathrm{Sp}_1})$. Here ν denotes the quasicharacter $\nu(a) = |a|$ for $a \in F^\times$. Finally, let $\Pi = \mathrm{Ind}_{P_\emptyset}^G(\nu^2 \otimes \nu)$. First, by exactness of induction, we see that

$$\mathrm{Ind}_{P_\beta}^G(\nu^2 \otimes St_{\mathrm{Sp}_1}) \subset \mathrm{Ind}_{P_\beta}^G(\nu^2 \otimes \mathrm{Ind}_B^{\mathrm{Sp}_1(F)}(\nu)) = \mathrm{Ind}_{P_\emptyset}^G(\nu^2 \otimes \nu)$$

and

$$\mathrm{Ind}_{P_\alpha}^G(\nu^{\frac{3}{2}} St_{\mathrm{GL}_2}) \subset \mathrm{Ind}_{P_\alpha}^G(\nu^{\frac{3}{2}} \mathrm{Ind}_B^{\mathrm{GL}_2(F)}(\nu^{\frac{1}{2}} \otimes \nu^{-\frac{1}{2}})) = \mathrm{Ind}_{P_\emptyset}^G(\nu^2 \otimes \nu).$$

Thus criterion (1) holds. Next we see that

$$\begin{aligned}
r^G_{P_\emptyset}(\mathrm{Ind}^G_{P_\emptyset}(\nu^2 \otimes \nu)) &= \nu^2 \otimes \nu + \nu^2 \otimes \nu^{-1} + \nu^{-1} \otimes \nu^2 + \nu^{-1} \otimes \nu^{-2} \\
&\quad + \nu^{-2} \otimes \nu^{-1} + \nu^{-2} \otimes \nu + \nu \otimes \nu^{-2} + \nu \otimes \nu^2 \\
r^G_{P_\emptyset}(\mathrm{Ind}^G_{P_\alpha}(\nu^{\frac{3}{2}} St_{\mathrm{GL}_2})) &= \nu^2 \otimes \nu + \nu^2 \otimes \nu^{-1} + \nu^{-1} \otimes \nu^2 + \nu^{-1} \otimes \nu^{-2} \\
r^G_{P_\emptyset}(\mathrm{Ind}^G_{P_\beta}(\nu^2 \otimes St_{\mathrm{Sp}_1})) &= \nu^2 \otimes \nu + \nu \otimes \nu^2 + \nu \otimes \nu^{-2} + \nu^{-2} \otimes \nu.
\end{aligned}$$

So verifying criteria (2) and (3) are routine. Further notice that for our choices above, either representation induced from the maximal parabolic subgroups P_α or P_β can function as the π or $\mathrm{Ind}^G_{P_0}(\sigma)$ in Tadić's criteria. Thus we conclude that both representations are reducible. In particular, St_{Sp_2} is a submodule for both and their quotients are two inequivalent Langlands quotients.

Tadić also has a methodology for ascertaining the irreducibility of various representations through the use of Jacquet modules. In particular, Tadić shows that given σ, an irreducible representation of M_0, one can derive criteria regarding the various Jacquet modules for $\mathrm{Ind}^G_{P_0}(\sigma)$. Let $\mathrm{Ind}^G_{P_0}(\sigma) = \pi_1 + \pi_2$ with $\pi_i > 0$ in $\mathfrak{R}(G)$. For any standard parabolic subgroup $P = MN$, let

$$T_{i,P} = r^G_P(\pi_i) \in \mathfrak{R}(M),$$

then the following must hold (see [37]):

1. $T_{i,P} \geq 0$ and $T_{1,P} \neq 0$ if and only if $T_{2,P} \neq 0$.

2. $T_{1,P} + T_{2,P} = r^G_P(\mathrm{Ind}^G_{P_0}(\sigma))$.

3. $r^{M_1}_{P_2 \cap M_1}(T_{i,P_1}) = T_{i,P_2}$ for $P_1 \supset P_2$.

So the representation $\mathrm{Ind}_{P_0}^G(\sigma)$ is irreducible if there is no parabolic P such that we write $r_P^G(\mathrm{Ind}_{P_O}^G(\sigma))$ as a sum $S_{1,P} + S_{2,P}$ with the $S_{i,P}$ conforming to the above criteria.

For a concrete example of this, let us consider the representation $\mathrm{Ind}_{P_\beta}^G(\chi \otimes St_{\mathrm{Sp}_1})$ where $\chi \notin \{\xi, \nu^{\pm 1}, \nu^{\pm 2} \mid \xi^2 = 1\}$. Here ν and ξ are quasicharacters on $F^\times$ with

$$\nu(a) = |a| \quad \text{and} \quad \xi(a) = |a|^{\frac{\pi i}{\log q}}.$$

The Jacquet modules with respect to the standard parabolic subgroups are listed below. In particular, the center column contains the Jacquet module for our example with respect to the Borel subgroup $P_\emptyset$. The lines in the diagram match a representation on a larger Levi component τ with $r_{M \cap P_\emptyset}^M(\tau)$. We choose to draw these lines to help illustrate how criterion (3) of Tadić's irreducibility criteria creates some rigid requirements that will actually force our representation to be irreducible.

$$\begin{tikzcd}
\underline{r_{P_\alpha}^G} & \underline{r_{P_\emptyset}^G} & \underline{r_{P_\beta}^G} \\
\mathrm{Ind}_B^{\mathrm{GL}_2(F)}(\chi \otimes \nu) \arrow[r, dash] \arrow[dr, dash] & \chi \otimes \nu \arrow[r, dash] & \chi \otimes St_{\mathrm{Sp}_1} \\
 & \nu \otimes \chi \arrow[r, dash] & \nu \otimes \mathrm{Ind}_B^{\mathrm{Sp}_1(F)}(\chi) \\
\mathrm{Ind}_B^{\mathrm{GL}_2(F)}(\chi^{-1} \otimes \nu) \arrow[r, dash] \arrow[dr, dash] & \nu \otimes \chi^{-1} \arrow[ur, dash] & \\
 & \chi^{-1} \otimes \nu \arrow[r, dash] & \chi^{-1} \otimes St_{\mathrm{Sp}_1}
\end{tikzcd}$$

So let us suppose that $\mathrm{Ind}_{P_\beta}^G(\chi \otimes St_{\mathrm{Sp}_1}) = \pi_1 + \pi_2$. We will show that either π_1 or π_2 must be the whole representation and so our original representation is irreducible. Now suppose that $r_{P_\emptyset}^G(\pi_1) > 0$ in $\mathfrak{R}(G)$. Without loss of generality, let

$r^G_{P_\emptyset}(\pi_1) \geq \chi \otimes \nu$. Since $r^G_{P_\emptyset}(\pi_1) \geq \chi \otimes \nu$, then

$$r^G_{P_\alpha}(\pi_1) \geq \mathrm{Ind}_B^{\mathrm{GL}_2(F)}(\chi \otimes \nu).$$

However, criterion (3) tells us that

$$r^G_{P_\emptyset}(\pi_1) \geq \chi \otimes \nu + \nu \otimes \chi.$$

Similarly, we see that

$$r^G_{P_\beta}(\pi_1) \geq \chi \otimes St_{\mathrm{Sp}_1} + \nu \otimes \mathrm{Ind}_B^{\mathrm{Sp}_1(F)}(\chi) \quad \text{and} \quad r^G_{P_\emptyset}(\pi_1) \geq \chi \otimes \nu + \nu \otimes \chi + \nu \otimes \chi^{-1}.$$

and

$$r^G_{P_\alpha}(\pi_1) \geq \mathrm{Ind}_B^{\mathrm{GL}_2(F)}(\chi \otimes \nu) + \mathrm{Ind}_B^{\mathrm{GL}_2(F)}(\chi^{-1} \otimes \nu).$$

Thus we know that $r^G_{P_\alpha}(\pi_1) \geq r^G_{P_\alpha}(\mathrm{Ind}^G_{P_\beta}(\chi \otimes St_{\mathrm{Sp}_1}))$. So π_1 is our entire representation. Notice that criterion (3) forces π_1 to contain all the available submodules of $r^G_P(\mathrm{Ind}^G_{P_\beta}(\chi \otimes St_{\mathrm{Sp}_1}))$. This ensures that

$$\pi_1 = \mathrm{Ind}^G_{P_\beta}(\chi \otimes St_{\mathrm{Sp}_1})$$

is irreducible.

It is also worth noting that if $\chi \in \{\nu^{\pm 2}, \xi \mid \xi^2 = 1\}$, $\mathrm{Ind}^G_{P_\beta}(\chi \otimes St_{\mathrm{Sp}_1})$ is reducible (thus we wouldn't expect Tadić's irreducibility criteria to hold). When $\chi = \nu^{\pm 1}$ the representation is still irreducible, however we need more sophisticated machinery to establish this. Notice that in such a case, the data is irregular. As with material in previous sections, Tadić's various criteria are most useful when the inducing data is regular. To resolve the remainder of the cases, Sally and Tadić derive the reducibility points for the unramified principal series of $\mathrm{GSp}_2(F)$ and deduce the reducibility points for $\mathrm{Sp}_2(F)$ from these.

2.2.2 Some results of Sally and Tadić

Next we mention some of results of Sally and Tadić in [33]. In this paper, Sally and Tadić determine the reducibility points for representations of the principal series of $\mathrm{GSp}_2(F)$ and $\mathrm{Sp}_2(F)$. Much of the paper proves reducibility results for $\mathrm{GSp}_2(F)$ and then leverages them to derive similar results for $\mathrm{Sp}_2(F)$. In particular, they use the following defintions and results for $\mathrm{GSp}_n(F)$ and $\mathrm{Sp}_n(F)$.

Definition 2.2.1. *Let $(F^\times)^\sim$ denote the set of quasi-characters of $F^\times$. For $\pi \in \mathfrak{R}(\mathrm{GSp}_n(F))$ define*

$$X_{\mathrm{Sp}_n}(\pi) = \{\chi \in (F^\times)^\sim \mid \chi\pi = \pi \,\mathrm{in}\, \mathfrak{R}(\mathrm{GSp}_n(F))\}.$$

Here $\chi\pi$ represents the twist of π by a quasicharacter χ composed with the similitude character $\mathrm{GSp}_n(F) \to F^\times$. Sally and Tadić then state that for $\pi \in \mathfrak{R}(\mathrm{GSp}_n(F))$,

$$\dim_{\mathbb{C}}(\mathrm{End}_{\mathrm{Sp}_n(F)}(\pi|_{\mathrm{Sp}_n(F)})) = \#X_{\mathrm{Sp}_n}(\pi). \tag{2.1}$$

To see an example of the utility of these statements, let us consider the case when $n = 1$. Thus $\mathrm{GSp}_1(F) \simeq \mathrm{GL}_2(F)$ and $\mathrm{Sp}_1(F) \simeq \mathrm{SL}_2(F)$. Let B (resp. B') be the upper triangular Borel subgroup of $\mathrm{GL}_2(F)$ (resp. $\mathrm{SL}_2(F)$). Then we have that

$$\mathrm{Ind}_B^{\mathrm{GL}_2(F)}(\chi_1 \otimes \chi_2)|_{\mathrm{SL}_2(F)} \simeq \mathrm{Ind}_{B'}^{\mathrm{SL}_2(F)}(\chi_1\chi_2^{-1})$$

as representations of $\mathrm{SL}_2(F)$. One direction of this isomorphism is obvious. In particular, we have

$$f \longrightarrow f|_{\mathrm{SL}_2(F)}.$$

Furthermore, for any $f' \in \mathrm{Ind}_{B'}^{\mathrm{SL}_2(F)}(\chi_1\chi_2^{-1})$, we define

$$f(g) := \chi_2(\det(g))|\det(g)|^{-\frac{1}{2}} f'(\begin{pmatrix} 1 & \\ & \det(g)^{-1} \end{pmatrix} g).$$

One can verify that $f \in \mathrm{Ind}_B^{\mathrm{GL}_2(F)}(\chi_1 \otimes \chi_2)$ and its restriction to $\mathrm{SL}_2(F)$ is obviously f'. Finally, we note that both maps are $\mathrm{SL}_2(F)$ intertwining maps and are inverse bijections of each other.

So now, let us consider the representation $\pi = \mathrm{Ind}_B^{\mathrm{GL}_2(F)}(\xi \otimes \mathbb{1}_{F^\times})$, where $\xi^2 = 1$ and $\xi \neq 1$. It is known that this representation is irreducible. Moreover, by the previous paragraph, we see that

$$\mathrm{Ind}_B^{\mathrm{GL}_2(F)}(\xi \otimes \mathbb{1}_{F^\times})|_{\mathrm{SL}_2(F)} \simeq \mathrm{Ind}_{B'}^{\mathrm{SL}_2(F)}(\xi).$$

Notice that

$$X_{\mathrm{SL}_2}(\pi) = \{\mathbb{1}_{F^\times}, \xi\}$$

so

$$\dim_{\mathbb{C}}(\mathrm{End}_{\mathrm{SL}_2(F)}(\pi|_{\mathrm{SL}_2})) = 2.$$

This fact, along with some elementary arguments, brings us to the conclusion $\mathrm{Ind}_{B'}^{\mathrm{SL}_2(F)}(\xi)$ is completely reducible (see [38]). In fact,

$$\mathrm{Ind}_{B'}^{\mathrm{SL}_2(F)}(\xi) = T_\xi^1 \oplus T_\xi^2$$

where T_ξ^1 is spherical.

Using similar results, as well as the previously mentioned reducibility criteria, Sally and Tadić give a complete list of reducibility points for the principal series of $\mathrm{GSp}_2(F)$ and $\mathrm{Sp}_2(F)$. The results for the unramified principal series of $\mathrm{Sp}_2(F)$ is included in Tables B.1-B.4 in Appendix B.

2.3 Representations of the Weil-Deligne Group

The last section of this chapter seeks to explain a conjectured correspondence between representations on p-adic algebraic groups and related Galois representations. For $G = F^\times$, such results are a well-understood part of local class field theory. One goal of the Langlands Program is to generalize this to the non-abelian setting. There are several formulations that depend on various properties of the p-adic group in question. In our case, we are working with $\mathrm{Sp}_2(F)$, a split p-adic group. Thus we shall use the formulation of the local Langlands correspondence that is commensurate with the split p-adic case. In particular, a more general case involves an action of $\mathrm{Gal}(\bar{F}/F)$ that is trivial for split groups.

2.3.1 Basic definitions

Let us begin with some of the basic definitions. Consider the exact sequence,

$$1 \to \mathcal{I}_F \to \mathrm{Gal}(\bar{F}/F) \to \mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q) \to 1,$$

where $\mathbb{F}_q$ is identified with $\mathcal{O}/\mathcal{P}$ and $\mathbb{Z} \subset \mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q) \simeq \hat{\mathbb{Z}}$. We let $\mathcal{W}_F$ be the inverse image of $\mathbb{Z}$ in $\mathrm{Gal}(\bar{F}/F)$ and we let $\mathrm{Fr}_q \in \mathcal{W}_F$ be an element that maps to $1 \in \mathbb{Z}$. We call $\mathcal{W}_F$ the **Weil group** of F and Fr_q is referred to as a **Frobenius element** of $\mathcal{W}_F$.

There is a more general group $\mathcal{W}'_F$ called the Weil-Deligne group associated to F; however for our purposes, we don't need much of the extra structure. Futher details can be found in the Corvallis proceedings [39] and [3]. For our applications, one could largely treat $\mathcal{W}'_F$ as the aforementioned $\mathcal{W}_F$.

2.3.2 Generalizing local class field theory

According to local class field theory, there is a reciprocity isomorphism

$$F^\times \simeq \mathcal{W}_F^{ab} \qquad \text{with} \qquad \varpi \leftrightarrow \mathrm{Fr}_q.$$

Quasicharacters χ_v of $F^\times$ correspond to homomorphisms

$$\sigma_q : \mathcal{W}_F^{ab} \to \mathbb{C}^\times$$

by composition with the reciprocity map. Moreover, there is an equality of local factors

$$(1-\chi_v(\varpi)q^{-s})^{-1} = (1-\sigma_q(\mathrm{Fr}_q)q^{-s})^{-1}.$$

It is these results that one would hope to generalize to arbitrary p-adic groups. In particular, we will now describe the local Langlands correspondence for split p-adic groups.

For such a group G, a complex Weil-Deligne representation of $\mathcal{W}_F'$ is a pair $\rho' = (\rho, N)$ where ρ is a continuous homomorphism

$$\rho : \mathcal{W}_F' \to \mathrm{GL}(V)$$

(here V is a $\mathbb{C}$-vector space) along with a nilpotent endomorphism $N \in \mathrm{End}(V)$ such that

$$\rho(\mathrm{Fr}_q)N\rho(\mathrm{Fr}_q)^{-1} = |\varpi|N. \tag{2.2}$$

Such a representation (ρ, N) is called **admissible** if ρ is semi-simple as a representation on $\mathcal{W}_F'$. Two such representations (ρ_1, N_1) and (ρ_2, N_2) are called equivalent if there exists a $g_0 \in \mathrm{GL}(V)$ with

$$\mathrm{Inn}(g_0)\circ\rho_1 = \rho_2 \quad \text{and} \quad N_2 = g_0 N_1 g_0^{-1}$$

where $\mathrm{Inn}(g_0)(x) = g_0 x g_0^{-1}$ is the inner automorphism defined by g_0.

As with representations of G, Weil-Deligne representations have L-factors attached to them. We can then define the following functions

$$L(s, \rho') = L(s, \rho, N) := \det(1 - q^{-s}\rho(\mathrm{Fr}_q)|_{V_N^{\mathcal{I}_F}})^{-1}$$

where $V_N = \ker(N)$ and $V^{\mathcal{I}_F}$ are the $\rho(\mathcal{I}_F)$ invariant vectors in V.

Furthermore, for a split p-adic group G, there exists a complex group ${}^LG^0$, called the **Langlands dual group**, associated to G. In the case of $G = \mathrm{Sp}_2(F)$, ${}^LG^0 = \mathrm{SO}_5(\mathbb{C})$. Let us suppose we have a pair $\rho' = (\rho, N)$ as above such that

$$\rho : \mathcal{W}'_F \to {}^LG^0$$

and $N \in \mathrm{Lie}({}^LG^0)$ is a nilpotent operator satisfying Equation 2.2. Then any representation

$$\mathbf{r} : {}^LG^0 \to \mathrm{GL}(V),$$

with V a $\mathbb{C}$ vector space, give us a Weil-Deligne representation

$$\mathbf{r} \circ \rho' : \mathcal{W}'_F \to \mathrm{GL}(V).$$

For instance, $\mathrm{Sp}_2(F)$ has the map

$$\mathbf{r}_{st} : \mathrm{SO}_5(\mathbb{C}) \to \mathrm{GL}_5(\mathbb{C})$$

as the obvious inclusion. So given any pair $\rho' = (\rho, N)$ with

$$(\rho, N) : \mathcal{W}'_F \to {}^LG^0$$

we can compose this map with $\mathbf{r}_{st}$ to yield a Weil-Deligne representation $\mathbf{r}_{st} \circ \rho'$.

Let $\mathcal{A}_F(G)$ denote the isomorphism classes of admissible representations of G and let $\mathcal{G}_F(G)$ denote the equivalence classes of admissible representations of $\mathcal{W}'_F \to {}^LG^0$. The Local Langlands Conjecture seeks to partition the set $\mathcal{A}_F(G)$ via elements of $\mathcal{G}_F(G)$. In particular, let $\rho' = (\rho, N) \in \mathcal{G}_F(G)$, then it is conjectured that there exists $\Pi_{\rho'} \subset \mathcal{A}_F(G)$ with several properties that include the following:

- $\Pi_{\rho'_1} \cap \Pi_{\rho'_2} = \emptyset$, if $\rho'_1 \neq \rho'_2$. Such a $\Pi_{\rho'}$ is called an L**-packet**.
- $\mathcal{A}_F(G) = \bigcup_{\rho' \in \mathcal{G}_F(G)} \Pi_{\rho'}$.
- $\forall \pi \in \Pi_{\rho'}$ and map

  $$\mathbf{r} : {}^LG^0 \to \mathrm{GL}(V),$$

  (V a $\mathbb{C}$-vector space). We have,

  $$L(s, \pi, \mathbf{r}) = L(s, \mathbf{r} \circ \rho'),$$

  in the cases where the local factor $L(s, \pi, \mathbf{r})$ is defined.

It is worth noting that this correspondence is known in some cases. Most notably, Harris and Taylor proved it for $\mathrm{GL}_n(F)$ [10]. Also note that for $G = \mathrm{Sp}_2(F)$, we are interested in (ρ, N) where ρ has the form

$$\rho : \mathcal{W}'_F \to {}^LG^0 = \mathrm{SO}_5(\mathbb{C})$$

and $\mathbf{r} = \mathbf{r}_{st}$, where $\mathbf{r}_{st}$ maps $\mathrm{SO}_5(\mathbb{C})$ into $\mathrm{GL}_5(\mathbb{C})$ as an inclusion.

2.3.3 Lusztig's criteria for choosing ρ'

While this correspondence is not known in general, Lusztig gives some criteria for how one might choose a pair $\rho' = (\rho, N)$ that parameterize constituents of the unramified principal series of various p-adic groups [25]. For such a π, his criteria can be described as follows:

For a split p-adic group G with Borel subgroup B and π a constituent of the unramified principal series of G, there exists a set of unramified quasi-characters $\{\chi_i\}$ such that π is a constituent of $\mathrm{Ind}_B^G(\chi_1, \chi_2, \ldots, \chi_n)$. Notice that the induced representation has a spherical constituent. Thus $\rho(\mathrm{Fr}_q)$ should be the Satake parameter of this spherical representation and $\rho|_{\mathcal{I}_F} = 1$. This is clearly a semisimple representation of $\mathcal{W}'_F$.

Since the Satake parameter is a equivalence class of semisimple matrices, we desire to fix a particular representative for our calculations. Since the χ_i are unramified, $\chi_i = |\cdot|^{s_i}$ for some $s_i \in \mathbb{C}$. Furthermore, because the quasicharacter $|\cdot|^s$ is $(2\pi i)/\log q$ periodic as a function of s, we can restrict our s_i to the strip $0 \leq \mathrm{Im}(s) < (2\pi i)/\log q$. So define an ordering on this strip as follows:

$$z_1 > z_2 \text{ if } \begin{cases} \mathrm{Re}(z_1) > \mathrm{Re}(z_2) \text{ or} \\ \mathrm{Re}(z_1) = \mathrm{Re}(z_2) \text{ and } \mathrm{Im}(z_1) > \mathrm{Im}(z_2) \end{cases} .$$

One may thing of this as a kind of dictionary ordering. So by convention, we will our representative of the Satake parameter with $s_i \geq s_j$ for $i < j$. For example, in the case of $\mathrm{Sp}_2(F)$, our unramified principal series representations are induced from two characters $\chi_1 = |\cdot|^{s_1}$ and $\chi_2 = |\cdot|^{s_2}$. Our convention dictates that $s_1 \geq s_2$ and the representative of the Satake parameter we use in our computations will be

$$\mathrm{diag}(|\varpi|^{s_1}, |\varpi|^{s_2}, 1, |\varpi|^{-s_1}, |\varpi|^{-s_2}).$$

While this convention is certainly not needed for one to obtain L-factors that match those in this thesis, it does allow the reader to more easily follow our intermediate steps.

According to Lusztig [25], choosing N reduces to the case that π is tempered. If π is not tempered, then it will be the Langland's quotient of some representation induced from the twist of a tempered representation of a proper parabolic subgroup's Levi factor. Once we have a method of selection for tempered representations, we can choose the appropriate nilpotent operator on the Levi factor and then take the corresponding N in $\mathrm{Lie}({}^L G^0)$. When π is tempered, we look at the set of all N satisfying

$$\rho(\mathrm{Fr}_q) N \rho(\mathrm{Fr}_q)^{-1} = |\varpi| N.$$

This set will have unique open orbit under the action of $\mathcal{W}'_F$. Choosing any element from that orbit will function as our N. Using these criteria, we have found pairs (ρ, N) corresponding to all representations of interest. The results are summarized in Table B.5 in Appendix B. This table will also include the various local L-factors

$$L(s, \mathbf{r}_{st} \circ \rho') = \det(1 - (r_{sr} \circ \rho)(\mathrm{Fr}_q)|_{V_N^{\mathcal{I}_F}} q^{-s})^{-1}.$$

In our case, the action of $\mathcal{I}_F$ is trivial, so $V_N = V_N^{\mathcal{I}_F}$.

2.3.4 An example

Now that we have described the computations on the Galois side of correspondence, let us compute an example relevant to this thesis. In particular, we will apply our calculations and Lusztig's criteria for the representation

$$\mathrm{Ind}_{P_\emptyset}^G (\nu^2 \otimes \nu).$$

This representation has four irreducible subquotients: St_G, $L(\nu^2, St_{\mathrm{Sp}_1})$, $L(\nu^{\frac{3}{2}} St_{\mathrm{GL}_2})$ and $\mathbb{1}_G$. We know that St_G is tempered (in fact, square-integrable) and the Langlands quotients are induced from twists of Steinberg representations on the Levi factors of the maximal parabolic subgroups. By our criteria, $\rho(\mathrm{Fr}_q)$ should correspond the the Satake parameter of the spherical constituent. In particular,

$$\rho(\mathcal{I}_F) = 1_{\mathrm{SO}_5} \quad \text{and} \quad \rho(\mathrm{Fr}_q) = \mathrm{diag}(|\varpi|^2, |\varpi|^1, 1, |\varpi|^{-2}, |\varpi|^{-1}) \in {}^L\mathrm{Sp}_2 = \mathrm{SO}_5(\mathbb{C}).$$

Now the set of nilpotent matrices in $\mathfrak{so}_5(\mathbb{C}) = \mathrm{Lie}(\mathrm{SO}_5(\mathbb{C}))$ that satisfy the desired property when conjugated by $\rho(\mathrm{Fr}_q)$ are given by

$$\mathbb{N}(x,y) = \{ \begin{pmatrix} 0 & x & 0 & 0 & 0 \\ 0 & 0 & y & 0 & 0 \\ 0 & 0 & 0 & 0 & -y \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -x & 0 \end{pmatrix} \mid x, y \in \mathbb{C}\}.$$

The open orbit of the space under the conjugation action of ${}^LG^0$ is clearly the subset of $\mathbb{N}(x,y)$ with $x, y \neq 0$. So we take the representative with $x = y = 1$ (call this matrix N_3); thus (ρ, N_3) should be the data corresponding to the Steinberg representation St_G. Furthermore, $\mathbb{1}_G$ is spherical, so it corresponds to the data $(\rho, 0)$. So finally, we come to the Langlands quotients. The representation $L(\nu^{\frac{3}{2}} St_{\mathrm{GL}_2})$ is the unique irreducible quotient of $\mathrm{Ind}_{P_\alpha}^G(\nu^{\frac{3}{2}} St_{\mathrm{GL}_2})$. It is induced from the twist of a tempered representation of M_α, the Levi factor for P_α. As a representation of

$\mathrm{GL}_2(F)$, St_{GL_2} corresponds to the Weil-Deligne representation with

$$\rho(\mathcal{I}_F) = 1 \quad \rho(\mathrm{Fr}_q) = \begin{pmatrix} |\varpi|^{\frac{1}{2}} & \\ & |\varpi|^{-\frac{1}{2}} \end{pmatrix} \quad N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

where ${}^L\mathrm{GL}_2^0 = \mathrm{GL}_2(\mathbb{C})$. Moreover, SO_5 has a standard parabolic subgroup with Levi factor isomorphic to GL_2, so we choose the subset of $\mathbb{N}(x, y)$ that exponentiates into that Levi factor. Our N will be a representative of that subset. In particular, we choose $x = 1$ and $y = 0$ (call this element N_1) and this finally specifies the pair (ρ, N_1). The other Langlands quotient is done similarly, when we note that it is a quotient of a representation induced from the parabolic $F^\times \times \mathrm{Sp}_1(F)$ and the Langlands dual to this group is $\mathbb{C}^\times \times \mathrm{SO}_3(\mathbb{C})$ which is the Levi factor into which $\mathbb{N}(0, y)$ exponentiates.

Chapter 3

Basic Theory Surrounding $\widetilde{\mathrm{Sp}_n}(F)$

We will also demonstrate that the method for defining L-factors offered in this thesis actually generalizes to covering groups of $\mathrm{Sp}_n(F)$ called metaplectic covers. We will now discuss one construction of the metaplectic group. However, it is worth mentioning that our method is not the only way to realize these topological covering group. For instance, [18] refers to a somewhat different construction that relies on the **Stone-von Neumann Theorem** regarding Hiesenberg groups and the uniqueness of their representations having a given central character.

3.1 The Weil Representation and the Metaplectic Group

In this section section, we will describe the metaplectic cover $\widetilde{\mathrm{Sp}_n}(F)$ of the symplectic group $\mathrm{Sp}_n(F)$. We are interested in this group for two reasons. First, it is a p-adic (topological) group in its own right. In fact, we will discuss its parabolic subgroups, the genuine principal series representations of this group and define the local L-factor for most of these representations. Second, the metaplectic group is the natural group on which to define the **Weil representation**.

The Weil representation is a very important object in the study of representation theory for several deep reasons. For instance, it is used to define theta liftings between representations forming a dual reductive pair. In fact, we will use some of these theta lifts in order to relate certain genuine principal series representations of $\widetilde{\mathrm{Sp}_2}(F)$ to constituents of the principal series for $\mathrm{SO}_5(F)$. Additionally, our doubling integral will require an auxiliary computation that proceeds directly from the Weil representation.

3.1.1 The metaplectic cover of $\mathrm{Sp}_n(F)$

Now we will construct the metaplectic cover of $\mathrm{Sp}_n(F)$. Let $G = \mathrm{Sp}(V)$, where $V = F^{2n}$ has a basis $e_1, \ldots, e_n, f_1, \ldots, f_n$ and symplectic form given by

$$< e_i, e_j >=< f_i, f_j >= 0 \quad \text{and} \quad < e_i, f_j >= \delta_{ij}$$

for all i, j. Then V has a complete polarization $V = X + Y$ where $X = \mathrm{span}_{\mathbb{C}}\{e_i\}_{i=1}^{n}$ and $Y = \mathrm{span}_{\mathbb{C}}\{f_i\}_{i=1}^{n}$ are maximal isotropic subspaces. We then define $P = P_Y$ to be the stabilizer of Y in G, where G acts on V via right multiplication. Notice that P_Y is precisely our upper triangular Siegel parabolic. Further, let $K = \mathrm{Sp}_n(\mathcal{O})$ and define

$$w_j = \begin{pmatrix} I_{n-j} & & & \\ & & & I_j \\ & & I_{n-j} & \\ & -I_j & & \end{pmatrix}$$

for $0 \leq j \leq n$. We notice that these represent distinct elements of the Weyl group W_G of G. Furthermore, $w_0 = I_G$ and w_n represents the long Weyl group element in G.

The metaplectic extension $\widetilde{G}$ of G satisfies the following exact sequence

$$1 \to \mathbb{C}^1 \to \widetilde{G} \to G \to 1.$$

There exists an obvious section $G \to \tilde{G}$ so that $\tilde{G}$ is realized as the set

$$\tilde{G} \simeq G \times \mathbb{C}^1$$

with multiplication given by

$$[g_1, z_1]_L \cdot [g_2, z_2]_L = [g_1 g_2, c_L(g_1, g_2) z_1 z_2]_L.$$

In this notation, the $c_L(\cdot, \cdot)$ is the Leray cocycle, which we describe below. For each $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$, define an operator on the Schwartz space $S(X) = S(F^n)$ by

$$r(g)\varphi(x) = \int_{F^n/\ker(c)} \psi(\frac{1}{2}(xa, xb) + (xb, yc) + \frac{1}{2}(yc, yd))\varphi(xa + yc) d_g(y),$$

where $x, y \in F^n$ are row vectors and $(x, y) = x^t y$. Furthermore, the measure is normalized to ensure that $r(g)$ is a unitary operator. While these operators do not define a representation of G, they do define a projective representation of G on $S(F^n)$ where

$$r(g_1)r(g_2) = c_L(g_1, g_2) r(g_1, g_2).$$

Notice that this defines a representation of the covering group $\tilde{G}$ on $S(F^n)$ by

$$[g.z]_L \cdot \varphi(x) = z r(g)\varphi(x).$$

A theorem of Rao [31] also shows that

$$c_L(g_1, g_2) = \gamma(\psi \circ q(g_1, g_2)),$$

where

$$q(g_1, g_2) = \mathrm{Leray}(Yg_1, Y, Yg_2^{-1}).$$

Here $\mathrm{Leray}(Y_1, Y_2, Y_3)$ denotes the Leray invariant attached to the triple of isotropic subspaces (Y_1, Y_2, Y_3) and $\gamma(\psi \circ q)$ is the Weil index of the character of second degree $\psi \circ q$ (see [31]).

The Leray cocycle is trivial on $G \times P$ and $P \times G$. This fact will be sufficient for most of the computations in this thesis. However, the cocycle is not trivial on $K \times K$, so it must be modified for use in the global setting. Kudla, Rapoport and Yang define such a modification in [22]. It is very elementary to show

$$\begin{aligned} r(m(a))\varphi(x) &= |\det(a)|^{\frac{1}{2}}\varphi(xa), \\ r(n_2(b))\varphi(x) &= \psi(\frac{1}{2}(x, xb))\varphi(x) \text{ and} \\ r(w_n)\varphi(x) &= \int_{F^n} \psi((x, y))\varphi(y)dy. \end{aligned}$$

We also note that the various $r(w_j)$ are the partial Fourier transforms for the last j coordinates of X.

It is also worth mentioning that the typical definition of covering groups for G rely on a different cocycle $c_R(\cdot, \cdot)$ called the **Rao cocycle**. For this thesis, let $\widetilde{\mathrm{Sp}_n}^*(F)$ denote the metaplectic cover

$$1 \to \mathbb{C}^1 \to \widetilde{\mathrm{Sp}_n}^*(F) \to \mathrm{Sp}_n(F) \to 1$$

defined using the Rao coordinates. It is worth noting that because the Rao cocycle is valued in $\{\pm 1\}$, that we have a character

$$\widetilde{\mathrm{Sp}_n}^*(F) \to \mathbb{C}^1 \qquad [g, z]_R \mapsto z^2$$

with kernel $\widetilde{\mathrm{Sp}_n}^{(2)}(F)$, the unique double cover of $\mathrm{Sp}_n(F)$. Furthermore, one can verify that the double cover is the commutator subgroup of $\widetilde{\mathrm{Sp}_n}^{*}(F)$. Even though the double cover has fewer splitting than the circle cover, the double cover is the group that appears most often in the literature. In particular, we will mention some conjectures regarding a correspondence between representations of $\widetilde{\mathrm{Sp}_n}^{(2)}(F)$ and representations of odd special orthogonal groups. Consequently, we will use one section to discuss this conjecture.

3.1.2 Some splittings of the metaplectic cover

We would also like to consider splittings for subgroups of G into $\widetilde{G}$. We will see that properties of these splittings are important the parabolic subgroups and intertwining operators on the metaplectic group. Furthermore, in this section, it is important that $p \neq 2$. First, if we let $P = P_Y \subset \mathrm{Sp}_n(F)$ be the Siegel parabolic subgroup, the the map $P \hookrightarrow \widetilde{\mathrm{Sp}_n}(F)$ given by

$$p \mapsto [p, 1]_L$$

is a splitting of P. This a simple consequence of the cocycle being trivial on $P \times P$. However, there is an additional splitting that will be useful. If we consider the projective representation $(r, S(F^n))$ of $\mathrm{Sp}_n(F)$ above and consider $\varphi^0 \in S(F^n)$ to be the characteristic function of $\mathcal{O}^n \subset F^n$, then we define a function λ on K by

$$r(k)\varphi^0 = \lambda(k)^{-1}\varphi^0.$$

By this definition, we notice that

$$c_L(k_1, k_2) = \lambda(k_1 k_2)\lambda(k_1)^{-1}\lambda(k_2)^{-1}.$$

Thus we get the splitting of $K \hookrightarrow \widetilde{G}$,

$$k \mapsto [k, \lambda(k)]_L.$$

Finally, we would like to explicitly compute the value of $\lambda(k)$ for a particular choice of k. This computation becomes important when we consider the intertwining operators on the metaplectic group. For $G = \mathrm{Sp}_2(F)$ and $P_\alpha = M_\alpha N_\alpha$ the Siegel parabolic subgroup, let

$$n(a) \in N_\alpha \simeq \mathrm{Sym}_2(F)$$

that corresponds to the matrix

$$\begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} \in \mathrm{Sym}_2(F).$$

Further let $\bar{n}(a) = w_n n(a) w_n^{-1}$, where w_n is the long Weyl group element. Notice that $n(a) \in M_\beta$ the Levi factor of the Long Root parabolic P_β. Let us consider $\lambda(\bar{n}(a))$. First, we realize that

$$c_L(\bar{n}(a), w_n) = \lambda(\bar{n}(a)w_n)\lambda(w_n)^{-1}\lambda(\bar{n}(a))^{-1}.$$

However, $\lambda(w_n) = 1$ can be easily verified by the definition. Also

$$\lambda(\bar{n}(a)w_n) = \lambda(w_n n(a)) = c_L(w_n, n(a))\lambda(w_n)\lambda(n(a)) = 1.$$

So we see that $\lambda(\bar{n}(a)) = c_L(\bar{n}(a), w_n)^{-1}$. Moreover, we know that

$$c_L(\bar{n}(a), w_n) = \gamma(\psi \circ q(\bar{n}(a), w_n))^{-1}$$

where $q(\bar{n}(a), w_n)$ is the Leray invariant of the triple $(Y\bar{n}(a), Y, Yw_n^{-1})$ of isotropic subspaces. This we compute using [31] and find that

$$q(\bar{n}(a), w_n)(x) = -\frac{1}{2}ax^2.$$

Moreover, Rao's appendix gives us the means to compute the Weil index of $\psi \circ q(\bar{n}(a), w_n)(x)$. In particular,

$$\gamma(\psi \circ q(\bar{n}(a), w_n)(x)) = \gamma(\psi_{-\frac{1}{2}a} \circ x^2) = \gamma(-\frac{1}{2}a, \psi \circ x^2)\gamma(\psi \circ x^2) \tag{3.1}$$

Finally, Proposition A.11 of [31] tells that

$$\gamma(\psi \circ x^2) = 1$$

for our choice of additive character and for $a = u\varpi^{\mathrm{ord}(a)}$,

$$\gamma(-\frac{1}{2}a, \psi \circ x^2) = \begin{cases} 1 & \text{if } \mathrm{ord}(a) \text{ is even} \\ \left(\frac{u}{\varpi}\right)_F \cdot \theta & \text{if } \mathrm{ord}(a) \text{ is odd} \end{cases}$$

where $\left(\frac{\cdot}{\varpi}\right)_F$ is the Legendre symbol for F and θ is a fourth root of unity depending only on ψ. So we see equation 3.1 becomes

$$\gamma(\psi \circ q(\bar{n}(a), w_n)(x)) = \begin{cases} 1 & \text{if } \mathrm{ord}(a) \text{ is even} \\ \left(\frac{u}{\varpi}\right)_F \cdot \theta & \text{if } \mathrm{ord}(a) \text{ is odd} \end{cases}. \tag{3.2}$$

3.1.3 Parabolic subgroups of $\widetilde{\mathrm{Sp}_n}(F)$

We now explore one of the more important aspects the splittings we have just discussed. In particular, by using the Leray coordinates for $\widetilde{\mathrm{Sp}_n}(F)$ and obtaining the splitting of the Siegel parabolic, there will be a very natural structure relating the parabolic subgroups of G and $\widetilde{G}$. In particular, let P be the Siegel parabolic of $\mathrm{Sp}_n(F)$. Then for any $P' \subseteq P$, we have the splitting map

$$P' \to \widetilde{\mathrm{Sp}_2}(F) \qquad p \mapsto [p, 1]_L$$

discussed previously. Now let $Q = MN$ be an arbitrary parabolic of $\mathrm{Sp}_n(F)$. Then

$$M \simeq \mathrm{GL}_{n_1}(F) \times \mathrm{GL}_{n_2}(F) \times \cdots \times \mathrm{GL}_{n_r}(F) \times \mathrm{Sp}_m(F)$$

with $m + \sum_{i=1}^{r} n_i = n$. Notice that the subgroup

$$A \simeq \mathrm{GL}_{n_1}(F) \times \mathrm{GL}_{n_2}(F) \times \cdots \times \mathrm{GL}_{n_r}(F) \times \{1_{\mathrm{Sp}_m}\}$$

is contained in the Siegel parabolic subgroup. So again we get a splitting

$$A \to \widetilde{\mathrm{Sp}_n}(F) \qquad a \mapsto [a, 1]_L.$$

If we let $\tau_n : \widetilde{\mathrm{Sp}_n}(F) \to \mathrm{Sp}_n(F)$ be the natural projection map, we see that the inverse image of M in $\widetilde{\mathrm{Sp}_n}(F)$ is given by

$$\tau_n^{-1}(M) = \widetilde{M} \simeq \mathrm{GL}_{n_1}(F) \times \mathrm{GL}_{n_2}(F) \times \cdots \times \mathrm{GL}_{n_r}(F) \times \widetilde{\mathrm{Sp}_m}(F)$$

as groups. Furthermore if $m = 0$, we will define

$$\widetilde{\mathrm{Sp}_0}(F) = \mathbb{C}^1,$$

in order to keep our notation consistent. So

$$\widetilde{Q} \simeq \widetilde{M} N$$

where N is identified with its image under the splitting $n \mapsto [n, 1]_L$. Moreover, one can easily see that for parabolic subgroups

$$Q' \subset Q \subset \mathrm{Sp}_n(F),$$

then

$$\widetilde{Q'} \subset \widetilde{Q} \subset \widetilde{\mathrm{Sp}_n}(F)$$

in the obvious way. So for our purposes, a parabolic subgroup $\widetilde{Q}$ of $\widetilde{\mathrm{Sp}_n}(F)$ will be a subgroup of the form described above having a Levi decomposition

$$\widetilde{Q} \simeq \widetilde{M}N.$$

3.1.4 Genuine principal series of metaplectic group

Now that we have discussed the parabolic subgroups for the metaplectic group, let us look at parabolic induction in this context. Let

$$\widetilde{P} = \widetilde{M}N$$

be a parabolic subgroup of $\widetilde{G}$ with

$$\widetilde{M} \simeq \mathrm{GL}_{n_1}(F) \times \mathrm{GL}_{n_2}(F) \times \cdots \times \mathrm{GL}_{n_r}(F) \times \widetilde{\mathrm{Sp}_m}(F).$$

Let (π_i, V_i) be a representation of $\mathrm{GL}_{n_i}(F)$ and let (σ, V) be a genuine representation of $\widetilde{\mathrm{Sp}_m}(F)$. Then we can define (normalized) induction in the usual way

$$\mathrm{Ind}_{\widetilde{P}}^{\widetilde{\mathrm{Sp}_n}(F)}(\pi_1 \otimes \pi_2 \otimes \cdots \otimes \pi_r \otimes \sigma)$$

with $\delta_{\widetilde{P}} = \delta_P$. If $m \geq 1$, one can routinely verify that this gives a genuine representation of $\widetilde{\mathrm{Sp}_n}(F)$. If $m = 0$, then we define the representation

$$\sigma_0 : \widetilde{\mathrm{Sp}_0}(F) \to \mathbb{C}^1 \qquad \sigma_0(z) = z.$$

So for $\widetilde{P} = \widetilde{M}N$ with

$$\widetilde{M} \simeq \mathrm{GL}_{n_1}(F) \times \mathrm{GL}_{n_2}(F) \times \cdots \times \mathrm{GL}_{n_r}(F) \times \widetilde{\mathrm{Sp}_0}(F)$$

and $\Pi = \bigotimes_{i=1}^r \pi_i$,

$$\mathrm{Ind}_{\widetilde{P}}^{\widetilde{\mathrm{Sp}_n}(F)}(\Pi \otimes \sigma_0)$$

consists of locally constant functions satisfying

$$f([p,z]_L g') = z\delta_P(p)^{\frac{1}{2}}\Pi(p)f(g').$$

For ease of notation, in the case that $m=0$, we will denote the representation

$$(\bigotimes_{i=1}^{r}\pi_r)' := (\bigotimes_{i=1}^{r}\pi_i)\otimes\sigma_0.$$

Let B be the Borel subgroup of $\mathrm{Sp}_1(F)$, then we have the following exact sequence of representations

$$1\to\tau(\varsigma\nu^{\frac{1}{2}})\to\mathrm{Ind}_{\widetilde{B}}^{\widetilde{\mathrm{Sp}_1}(F)}(\varsigma\nu^{\frac{1}{2}}\otimes\sigma_0) = \mathrm{Ind}_{\widetilde{B}}^{\widetilde{\mathrm{Sp}_1}(F)}((\varsigma\nu^{\frac{1}{2}})')\to\pi(\varsigma\nu^{\frac{1}{2}})\to 1$$

(see [22]).

It is also worth noting that using the Leray cocycle, we have the same transitivity of induction that was discussed for the linear group in the previous chapter. In particular, if $\widetilde{P'} = \widetilde{M'}N'$ and $\widetilde{P} = \widetilde{M}N$ are parabolic subgroups with $\widetilde{P'}\subset\widetilde{P}$, then $\widetilde{P'}\cap\widetilde{M}$ is a parabolic subgroup of $\widetilde{M}$. Consequently, we can extend all our results regarding Jacquet modules as well as Tadić's reducibility (resp. irreducibility) criteria to the metaplectic group in the obvious way. In fact, the table of reducibility points for the genuine principal series of $\widetilde{\mathrm{Sp}_2}(F)$ (Table B.9) is derived from these very results.

3.2 The Weil representation

Now that we have defined the metaplectic cover of G, we would also like to define the Weil representation of $\widetilde{G}$ associated to a quadratic space. Let (V,Q) be a quadratic vector space over F. We define the Weil representation $(\omega_V, S(V^n))$ of $\widetilde{G}$ associated to (V, Q) as follows

$$\omega_V([g,z]_L)\varphi(x) = \chi_V(x(g))(z\gamma(\eta)^{j(g)})^{\bullet}\gamma(\eta \circ V)^{-j(g)} r_V(g)\varphi(x) \tag{3.3}$$

where

$$\bullet = \begin{cases} 0 \text{ if } \dim_F(V) \text{ is even} \\ 1 \text{ if } \dim_F(V) \text{ is odd.} \end{cases}$$

Before we define the multitude of factors arising from this definition. Note that if $\dim_F(V)$ is even, then ω_V is trivial on $\{[1,z]_L \mid z \in \mathbb{C}^1\}$, so the representation factors through the group G. This follows from the fact that $(z\gamma(\eta)^{j(g)})$ is the only factor in (3.3) containing z. Moreover, if $\dim_F(V)$ is odd, then the Weil representation is genuine. We now define the various terms above. Let $m = \dim_F(V)$, then

$$\chi_V(t) = (t, (-1)^{\frac{m(m-1)}{2}}\det(V))_F$$

where $\det(V)$ is the determinant of the matrix of the bilinear form on V and $(\cdot,\cdot)_F$ is the Hilbert symbol of F. In order to define $x(g)$, we note that

$$G = \bigcup_{0 \le j \le n} P_Y w_j P_Y$$

is a disjoint union. So for $g \in G$, $g = p_1 w_j p_2$ where w_j is uniquely determined. Thus

$$x(g) = x(p_1 w_j p_2) := \det(p_1 p_2|_Y) \bmod (F^\times)^2.$$

If addition, if

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then $j(g) = \text{rank}(c)$. One can show that for $g \in P_Y w_i P_Y$, $j(g) = i$. So we see that $\{P_Y w_j P_Y\}_{j=0}^n$ partitions G according the the rank of the lower left block.

Next, we define $\eta = \psi_{\frac{1}{2}}$ (i.e., $\eta(t) = \psi(\frac{1}{2}t)$). As before, γ denotes the Weil index. In particular, $\gamma(\eta)$ is simply the Weil index of the character of second degree $\eta \circ Q$. Furthermore, Kudla defines

$$\gamma(\eta \circ V) := \gamma(\det(V), \eta)\gamma(\eta)^m \epsilon(V)$$

in [18]. Here $\epsilon(V)$ is the Hasse invariant of the quadratic space V and $\gamma(\det(V), \eta)$ is the relative Weil index and is computed by Rao in [31]. Ultimately, all the Weil index computations and definitions can be found in [31].

Finally, if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ let

$$r_V(g)\varphi(x) = \int\limits_{V^n/\text{Ker}(c)} \psi(\text{tr}\left(\frac{1}{2}(xa, xb) + (xb, yc) + \frac{1}{2}(yc, yd)\right))\varphi(xa + yc)d_g(y)$$

where $x, y \in V^n$, $(x, y) = [(x_i, y_j)_V]_{i,j}$ and $(x_i, y_j)_V$ is inner product defined using Q. Moreover, $d_g(y)$ is normalized so that $r_V(g)$ is unitary.

3.3 Liftings Between Representations of $\widetilde{\text{Sp}_2}(F)$ and $\text{SO}_5(F)$

While we have the Local Langlands Correspondence to give us conjectural L-factors $L(s, \mathbf{r}_{st} \circ \rho')$ for representations of $\text{Sp}_2(F)$, this conjecture does not apply to the metaplectic group $\widetilde{\text{Sp}_2}(F)$. However, we shall see that there may still be a method for us to compute conjectural L-factors for constituents of genuine principal series representations of $\widetilde{\text{Sp}_2}(F)$. Ultimately, they should correspond to L-factors of the unramified principal series of $\mathrm{SO}_5(F)$. We can then compute conjectural L-factors for these using Lusztig's criteria. The connection between these groups will come from the local theta correspondence between $\widetilde{\mathrm{Sp}_2}(F)$ and $\mathrm{SO}_5(F)$.

3.3.1 Conjectural L-values $\widetilde{\mathrm{Sp}_2}(F)$ and $\mathrm{SO}_5(F)$

We now give only a brief sketch of the local theta correspondence and cite some sources for this material. Two subgroups $G, G' \subset \mathrm{Sp}(V)$, are called a **dual reductive pair** if

1. $C_{\mathrm{Sp}(V)}(G) = G'$ and $C_{\mathrm{Sp}(V)}(G) = G'$ where $C_{\mathrm{Sp}(V)}(G)$ denotes the centralizer of the subgroup G in $\mathrm{Sp}(V)$.

2. The actions of G and G' on V are completely reducible.

The theory of local theta lifts and Howe duality predict that given a dual reductive pair (G, G'), there exists a bijection of certain subsets of admissible representations of G and G'. In [15], Kudla proves results along these lines for the dual pair $(\mathrm{O}(V), \mathrm{Sp}(W))$ with V and W arbitrary. The paper futher proves the compatibility of the theta correspondence with parabolic induction. The metaplectic group can also form half of a dual reductive pair (along with certain orthogonal and special orthogonal groups). Some general results regarding theta lifts on dual pairs of this form appear in the work of Mœglin, Vignéras and Waldspurger [26]. Moreover, Waldspurger studies the case of $(\widetilde{\mathrm{SL}_2}^{(2)}, \mathrm{SO}_3)$ in great detail [40] and [41].

In this thesis, we can compute the various reducibility points for genuine principal series representations $\mathrm{Ind}_{\widetilde{P}_\emptyset}^{\widetilde{G}}((\chi_1 \otimes \chi_2)')$. In particular, we use the Tadić's criteria for both irreducibility and reducibility to derive the results in Table B.10. For instance, consider the representation

$$\mathrm{Ind}_{\widetilde{P}_\emptyset}^{\widetilde{G}}((\varsigma\nu^{\frac{3}{2}} \otimes \varsigma\nu^{\frac{1}{2}})')$$

where $\varsigma^2 = 1$ is unramified. We notice that this representation satisfies the following two exact sequences

$$1 \to \mathrm{Ind}_{\widetilde{P}_\alpha}^{\widetilde{G}}((\varsigma\nu St_{\mathrm{GL}_2})') \to \mathrm{Ind}_{\widetilde{P}_\emptyset}^{\widetilde{G}}((\varsigma\nu^{\frac{3}{2}} \otimes \varsigma\nu^{\frac{1}{2}})') \to \mathrm{Ind}_{\widetilde{P}_\alpha}^{\widetilde{G}}((\varsigma\nu \mathbb{1}_{\mathrm{GL}_2})') \to 1$$

and

$$1 \to \mathrm{Ind}_{\widetilde{P}_\beta}^{\widetilde{G}}(\varsigma\nu^{\frac{3}{2}} \otimes \tau(\varsigma\nu^{\frac{1}{2}})) \to \mathrm{Ind}_{\widetilde{P}_\emptyset}^{\widetilde{G}}((\varsigma\nu^{\frac{3}{2}} \otimes \varsigma\nu^{\frac{1}{2}})') \to \mathrm{Ind}_{\widetilde{P}_\beta}^{\widetilde{G}}(\varsigma\nu^{\frac{3}{2}} \otimes \pi(\varsigma\nu^{\frac{1}{2}})) \to 1.$$

Each representation induced from the Siegel parabolic shares a constituent with a representation induced from the long root parabolic. Using Tadić's irreducibility criteria, we see that the shared constituents are irreducible. A similar argument works for

$$\mathrm{Ind}_{\widetilde{P}_\emptyset}^{\widetilde{G}}((\varsigma\nu^{\frac{1}{2}} \otimes \varsigma\nu^{-\frac{1}{2}})'),$$

except that we need to consider some results regarding unitarizability and complete reducibility. In particular,

$$\mathrm{Ind}_{\widetilde{P}_\emptyset}^{\widetilde{G}}((\varsigma\nu^{\frac{1}{2}} \otimes \varsigma\nu^{-\frac{1}{2}})')$$

has unitarizable subquotients

$$\mathrm{Ind}_{\widetilde{P}_\alpha}^{\widetilde{G}}((\varsigma St_{\mathrm{GL}_2})')$$

and

$$\mathrm{Ind}_{\widetilde{P}_\alpha}^{\widetilde{G}}((\varsigma \mathbb{1}_{\mathrm{GL}_2})').$$

These representations are completely reducible. Moreover, Frobenius reciprocity shows us that for either of these representations (denoted as $\tilde{\pi}$)

$$\dim_{\mathbb{C}}\left[\mathrm{Hom}_{\widetilde{G}}(\tilde{\pi},\tilde{\pi})\right] \leq 2,$$

so both may have at most two inequivalent irreducible constituents. At this point, our argument is identical to the previous example. The irreducible constituents are shared by one of the unitarizable representations induced from the Siegel parabolic and a representation induced from the long root parabolic found in the exact sequence

$$1 \to \mathrm{Ind}_{\widetilde{P}_{\beta}}^{\widetilde{G}}(\varsigma\nu^{\frac{1}{2}} \otimes \tau(\varsigma\nu^{-\frac{1}{2}})) \to \mathrm{Ind}_{\widetilde{P}_{\emptyset}}^{\widetilde{G}}((\varsigma\nu^{\frac{1}{2}} \otimes \varsigma\nu^{-\frac{1}{2}})') \to \mathrm{Ind}_{\widetilde{P}_{\beta}}^{\widetilde{G}}(\varsigma\nu^{\frac{1}{2}} \otimes \pi(\varsigma\nu^{-\frac{1}{2}})) \to 1.$$

Note that $\tau(\varsigma\nu^{-\frac{1}{2}}) \simeq \pi(\varsigma\nu^{\frac{1}{2}})$ and $\pi(\varsigma\nu^{-\frac{1}{2}}) \simeq \tau(\varsigma\nu^{\frac{1}{2}})$.

Ultimately, Tables B.10 and B.11 outline a bijection between constituents of

$$\mathrm{Ind}_{\widetilde{P}_{\emptyset}}^{\widetilde{G}}((\chi_1 \otimes \chi_2)')$$

and constituents of

$$\mathrm{Ind}_{P'_{\emptyset}}^{\mathrm{SO}_5(F)}(\chi_1 \otimes \chi_2)$$

where $P'_{\emptyset}$ is a fixed Borel subgroup of $\mathrm{SO}_5(F)$. In particular, the representation on a given line of Table B.10 corresponds to the representation on the exact same line in Table B.11. Note that our data regarding the reducibility of the principal series of $\mathrm{SO}_5(F)$ comes from the work of Jantzen [12]. As we will eventually prove, there is an equality of L-factors for representations that correspond according to this bijection. In particular, the doubling L-factor of a constituent $\tilde{\pi}$ for $\widetilde{\mathrm{Sp}_2}(F)$ matches to the

Weil-Deligne L-factor for the data ρ' in the image of the corresponding constituent π for $\mathrm{SO}_5(F)$. Again, we are using Lusztig's criteria to specify the map

$$\pi \mapsto \rho'.$$

In the future, one might ascertain whether or not our bijection is consistent with the local theta correspondence on the dual pair $(\widetilde{\mathrm{Sp}_2}(F), \mathrm{SO}_5(F))$. Finally, we would like to relate representations of $\widetilde{G}$ to representations of the double cover $\widetilde{G}^{(2)}$, since it is the double cover that appears most frequently in the literature.

3.3.2 Relating representations of $\widetilde{G}$ and $\widetilde{G}^{(2)}$.

The relevant theta correspondence results are often formulated for the dual pair $(\widetilde{\mathrm{Sp}_n}^{(2)}(F), \mathrm{O}_{2n+1}(F))$. However, the double cover of the symplectic group is naturally defined using the Rao cocycle. Consequently, the representation theory is defined in a slightly different way. One can consult [1] for a fuller explanation. Ultimately, we will take principal series representations of $\widetilde{\mathrm{Sp}_n}(F)$ relate them to representations of $\widetilde{\mathrm{Sp}_n}^{*}(F)$ and then restrict those to the double cover. This relates a representation on the Leray cover to a representation for which the conjectural liftings with $\mathrm{SO}_5(F)$ makes sense.

So now let us relate certain induced representations of $\widetilde{\mathrm{Sp}_n}^{*}(F)$ to those of $\widetilde{\mathrm{Sp}_n}(F)$. Let us specialize to the case that we are inducing from a parabolic subgroup contained in the Siegel parabolic. Let P be a parabolic subgroup of $\mathrm{Sp}_n(F)$ that is contained in the Siegel parabolic. Further we let $\widetilde{P}$ (resp. $\widetilde{P}^{*}$) be the inverse image of P in $\widetilde{\mathrm{Sp}_n}(F)$ (resp. $\widetilde{\mathrm{Sp}_n}^*(F)$). Adams employs a character

$$\chi^* : \widetilde{P}^* \to \mathbb{C}^1 \qquad \chi^*([g,z]_R) := \gamma(\det(g),\eta)z$$

where γ is the Weil index discussed in the previous sections. So for a representation (π, V) of P, we get a representation of $\widetilde{P}^*$ on V given by

$$(\chi^*\pi)([p,z]_R)v := \chi^*([p,z]_R)\pi(p)v = z\gamma(\det(p),\eta)\pi(p)v.$$

We would like to show that this is identical to the representation (π', V) on $\widetilde{P}$ given by

$$\pi'([p,z]_L)v := z\pi(p)v.$$

To see this, we notice that the Leray and Rao coordinates are related as follows

$$[g,z]_R = [g, z\beta(g)]_L$$

where

$$\beta(g) := \gamma(x(g),\eta)^{-1}\gamma(\eta)^{-j(g)}$$

which are described in detail in the Section 3.2. We also note that for $p \in P$

$$x(p) = \det(p) \qquad \text{and} \qquad j(p) = 0.$$

Thus,

$$\beta(p) = \gamma(\det(p),\eta)^{-1}.$$

So finally we see that

$$\chi^*([p, z\beta(p)^{-1}]_R) = z\beta(p)^{-1}\gamma(\det(p),\eta) = z.$$

Therefore, the representations $\chi^*\pi$ and π' are the identical representations viewed under different coordinates. As a result, we see that

$$\mathrm{Ind}_{\widetilde{P}}^{\widetilde{\mathrm{Sp}_n}(F)}(\pi')$$

and

$$\mathrm{Ind}_{\widetilde{P}^*}^{\widetilde{\mathrm{Sp}_n}^*(F)}(\chi^*\pi)$$

are also the same representation viewed in different coordinates. Because of this and using the results of Kudla [15], we feel justified in relating constituents of

$$\mathrm{Ind}_{\widetilde{P}_\emptyset}^{\tilde{G}}((\chi_1 \otimes \chi_2)')$$

with those of

$$\mathrm{Ind}_{P'_\emptyset}^{\mathrm{SO}_5(F)}(\chi_1 \otimes \chi_2).$$

The next chapter will discuss a method for computing L-factors for representations of certain p-adic (topological) groups. In particular, we will use a method that does not depend on the existence of Whittaker models (i.e., the representation being generic) so the method will apply to all of the representations discussed in this and the previous chapter.

Chapter 4

Building the Doubling Integral

In this chapter, we are going to build all the necessary machinery to produce local L-factors for constituents of the unramified principal series of $\mathrm{Sp}_2(F)$ and $\widetilde{\mathrm{Sp}_2}(F)$. Much of this chapter applies to more general groups; however, we will concentrate mostly on our rank two cases. We begin by describing the main method for defining analytic L-factors for the standard representation r_{st}, the doubling integral.

4.1 Computing $L(s, \pi, \mathbf{r}_{st})$ for π a representation of $\mathrm{Sp}_2(F)$

While there is a very general method for defining L-factors attached to Weil-Deligne representations of $\mathcal{W}'_F$, the same cannot be said regarding admissible representations of reductive p-adic groups. While there are several methods for defining such analytic L-factors, they all require some extra conditions whether they be on the group or on the representations themselves. For instance, Godement and Jacquet produces a method for determining the analytic L-factors for representations of GL_n. On the other hand, the Langlands-Shahidi method that work for more general groups but requires the representation to be generic (i.e., have a non-zero Whittaker functional).

In this section, our main tool for defining L-factors for representations of $\mathrm{Sp}_2(F)$ will be the doubling method of Piatetski-Shapiro and Rallis [8]. Moreover, we shall see that this same method works to define L-factors for representations of $\widetilde{\mathrm{Sp}_2}(F)$. To briefly summarize, the doubling integral gives a way of defining the L-factor of a representation as the normalized generator of some fractional ideal produced via these integrals. Moreover, the L-factor attached to spherical representations of more general p-adic groups are defined via their Satake parameter. In the spherical case, the L-factor provided by the Satake parameter can be realized by evaluating the doubling integral at certain "good test vectors".

4.1.1 The doubling method of Piatetski-Shapiro and Rallis

We will now explain the doubling method of Piatetski-Shapiro and Rallis [8]. This Rankin-Selberg type integral is constructed globally but unwinds as a product of local integrals for factorizable global data. Our main goal is the study of these local integrals for constituents of the unramified principal series of $G = \mathrm{Sp}_2(F)$ and $\tilde{G}$.

In order to motivate and explain the doubling integral, we begin by explaining this method as it pertains to the symplectic groups Sp_n; one should also note that this method can be generalized to classical groups defined as preserving an inner product. For this section, k will be a number field and k_v will be its completion with respect to the place v. Also, $\mathbb{A}$ will be the ring of adeles over k. Finally, for any group G defined over k and for any k-algebra R, we let G_R denote the R points of G.

Let $G = \mathrm{Sp}(V)$ where V is an $2n$-dimensional symplectic vector space over k

with skew-symmetric form $<\cdot,\cdot>_V$. We now define a new vector space $W = V \oplus V$ and endow it with the following symplectic inner form

$$<<(v_1, v_2),(v_1', v_2')>>_W := <v_1, v_1'>_V - <v_2, v_2'>_V .$$

Thus we find an obvious map $i_0 : G \times G \hookrightarrow H = \mathrm{Sp}(W)$ by considering the action of $G \times G$ on W given by

$$(v_1, v_2) \cdot (g_1, g_2) := (v_1 g_1, v_2 g_2).$$

Moreover, since we defined the inner form on W as the difference of the inner forms from V, we notice that the space $V^d = \{(v, v) \in W \mid v \in V\}$ is isotropic. In fact, V^d is a maximal isotropic subspace of W. So let $P_H \subset H$ be the subgroup that fixes the V^d, then $P_H\backslash H$ becomes the variety of maximal isotropic subspaces of W.

We now consider the $G \times G$ orbits of $X = P_H\backslash H$. Piatetski-Shapiro and Rallis define such an orbit X' to be **negligible** if $\exists\, x' \in X'$ such that the its stabilizer R' in $G \times G$ contains the unipotent radical N' of a proper parabolic subgroup $P' \subset G \times G$ with N' normal in $G \times G$. If we let x_0 be the identity coset P_H and X_0 its orbit, we see that its stabilizer is $P_H \cap (G \times G) = \{(g, g) \in G \times G \mid g \in G\} =: G^d$. Finally, Piatetski-Shapiro and Rallis prove that the action above satisfies the following conditions:

1. The stabilizer of x_0 is G^d.

2. All orbits $X' \neq X_0$ are negligible.

Piatetski-Shapiro and Rallis call X_0 the **main orbit** for reasons that will become readily apparent.

Let $\delta_P^* : P_{H,k} \to k^\times$ be the modulus character of P_H and let $\chi : I_k/k^\times \to \mathbb{C}^\times$ be any quasicharacter such that $\chi \circ \delta_P^*$ is trivial on $G_{\mathbb{A}}^d$. Next let

$$\Phi_s \in \mathrm{Ind}_{P_{H,\mathbb{A}}}^{H_{\mathbb{A}}}(|\det|^s \chi \circ \delta_P^*) =: I(s, \chi),$$

where the induction is taken to be normalized. To this Φ_s, we can associate the Eisenstein series

$$E_\Phi(h; s) = \sum_{\gamma \in P_{H,k} \backslash H_k} \Phi_s(\gamma h)$$

wherever this sum is absolutely convergent (for $\mathrm{Re}(s) >> 0$). Furthermore, let π be an irreducible cuspidal automorphic representation of G and $\pi^\vee$ its contragredient. For $f \in \pi$ and $f' \in \pi^\vee$ we may finally define

$$Z(s, \chi, \Phi, f, f') = \int\limits_{(G \times G)_k \backslash (G \times G)_{\mathbb{A}}} E_\Phi(i_0(g_1, g_2); s) f(g_1) f'(g_2) dg_1 dg_2.$$

Because π and $\pi^\vee$ are cuspidal, this integral converges; further, it inherits the analytic properties of E_Φ. Perhaps the most important property of this integral is the following identity

$$\begin{aligned} \int\limits_{(G \times G)_k \backslash (G \times G)_{\mathbb{A}}} & E_\Phi(i_0(g_1, g_2); s) f(g_1) f'(g_2) dg_1 dg_2 = \\ & = \int\limits_{G_{\mathbb{A}}} \Phi_s(i_0(g, 1)) < \pi(g) f, f' >_{\mathrm{Pet}} dg \end{aligned}$$

where $< \cdot, \cdot >_{\mathrm{Pet}}$ is the Peterson inner product. The proof of this can be found in [8], but it boils down to a interchanging the summation in the Eisenstein series with the integral. In doing this, we obtain a finite sum of integrals indexed by the orbits. The domain of integration of each term becomes $R_k^\gamma \backslash (G \times G)_{\mathbb{A}}$, where R^γ is the stabilizer of $P\gamma \in P \backslash H$. Because f and f' were chosen to be cuspidal, the integrals corresponding to the negligible orbits vanish and only the integral over the main orbit persists. Some more elementary manipulations yields the equation above. Finally, because $\chi = \otimes \chi_v$ and $< \cdot, \cdot >_{\text{Pet}} = \otimes_v < \cdot, \cdot >_v$ factor as a product of local terms, if we choose $\Phi_s = \otimes_v \Phi_{v,s}$, $f = \otimes_v f_v$ and $f' = \otimes_v f'_v$ to be factorizable, then

$$\begin{aligned} Z(s, \chi, \Phi, f, f') &= \bigotimes_v Z(s, \chi_v, \Phi_v, f_v, f'_v) \\ &= \bigotimes_v \int_{G_{k_v}} \Phi_{v,s}(i_0(g_v, 1)) < \pi_v(g_v) f_v, f'_v >_v dg_v. \end{aligned}$$

We now consider the local case of the doubling integral for a nonarchimedean place v. In the original work by Piatetski-Shapiro and Rallis, they suggest computing their doubling integral over a family of **good sections** $\{\Phi_{v,s}\}$. The corresponding family $\{Z(s, \chi_v, \Phi_v, f_v, f'_v)\}$ generates a fractional ideal in the polynomial ring $\mathbb{C}[q^{-s}]$. The normalized generator of this ideal defines the L-factor associated to π_v under the standard map $\mathbf{r}_{st}$ on the dual group. Let us consider the following definitions with $K_H = H_{k_v} \cap \mathrm{GL}_{4n}(\mathcal{O})$, (resp. $K_G = G_{k_v} \cap \mathrm{GL}_{2n}(\mathcal{O})$) a maximal compact subgroup of H_{k_v} (resp. G_{k_v}).

Definition 4.1.1. *A section $\Phi_{v,s}$ is called* **standard** *if its restriction to K_H is independent of s.*

Definition 4.1.2. *The family of* **good sections** *consists of*

1. *$\mathbb{C}[q^s, q^{-s}] \cdot$ standard sections.*

2. *$\mathbb{I}^*_{w_n}(s, \chi)\mathbb{C}[q^s, q^{-s}] \cdot$ standard sections, where $\mathbb{I}^*_{w_n}(s, \chi)$ is a normalized standard intertwining operator defined using the long Weyl group element w_n (see [20]*

for the definition of $\mathbb{I}^*_{w_n}(s,\chi)$*).*

3. *Sections of the form*

$$d_{v,H}(s,\chi)\cdot\Phi^0_{v,s}*\mu_s$$

where $\mu_s\in\mathcal{H}(H_{k_v},K_H)[q^s,q^{-s}]$*. In this notation,* $\mathcal{H}(H_{k_v},K_H)$ *is the Hecke algebra with respect to* K_H*,* $\Phi^0_{v,s}$ *is the* K_H*-invariant function with* $\Phi^0_{v,s}(1_H)=1$*. Furthermore,* $d_{v,H}(s,\chi)$ *is a normalizing factor that we will address below.*

In the various works [27] and [28], Piatetski-Shapiro and Rallis prove the following proposition.

Proposition 4.1.1. *Let* π_v *be an irreducible admissible representation of* G_{k_v} *and* $\pi_v^\vee$ *its contragredient. Let* $f_v\in\pi_v$ *and* $f'_v\in\pi_v^\vee$*.*

1. *For any good section* $\Phi_{v,s}$*,* $Z(s,\chi_v,\Phi_v,f_v,f'_v)$ *is a rational function in* q^{-s}*.*

2. *There exists* $f_v\in\pi_v$*,* $f'_v\in\pi_v^\vee$ *and there is a good section* $\Phi_{v,s}$ *with*

$$Z(s,\chi_v,\Phi_v,f_v,f'_v)=1.$$

3. *The set*

$$\mathcal{Z}(s,\pi_v,\chi_v)=\{Z(s,\chi_v,\Phi_v,f_v,f'_v)\mid f_v\in\pi_v,\ f'_v\in\pi_v^\vee,\ \Phi_{v,s}\in \textit{good sections}\}$$

is a fractional ideal for $\mathbb{C}[q^s,q^{-s}]$*, with generator* $P(q^{-s})^{-1}$*, for a unique polynomial* $P(x)\in\mathbb{C}[x]$ *with* $P(0)=1$*.*

Notice that the proposition indicates that one might be able to find "good" test vectors f and f' as well as a "good" section that computes $L(s,\pi_v,\mathbf{r}_{st})$ exactly.

In our variant of the doubling method, we seek to use "good test vectors" to realize the Weil-Deligne L-factor $L(s, \mathbf{r}_{st} \circ \rho')$ for π_v in the ideal $\mathcal{Z}(s, \pi_v, \chi_v)$. Because we restrict ourselves to the case that π_v is a constituent of the unramified principal series, we will choose $f_v \in \pi_v$ to be fixed by one of our standard parahoric subgroups. As a consequence, we will require our section $\{\Phi_{v,s}\}$ to have a related invariance property. However, we will see that the most natural choices of good section actually require a further normalization before they compute $L(s, \mathbf{r}_{st} \circ \rho')$ exactly. Let us see an example of this that will also explain the $d_{v,H}(s)$ from Definition 4.1.2.

Suppose that π_v and $\pi_v^\vee$ are spherical. In this case, choose $f_v \in \pi_v$ (resp. $f_v' \in \pi_v^\vee$) to be the normalized spherical function with $f_v(1_G) = 1$ (resp. $f_v'(1_G) = 1$). Also, choose the section $\Phi_{v,s}^0$ spherical with $\Phi_{s,v}^0(1_H) = 1$. With these rather natural choices, one might presume that $Z(s, \chi_v, \Phi_v, f_v, f_v')$ would exactly match the L-factor defined using the Satake parameter. However, this is not the case, instead

$$Z(s, \chi_v, \Phi_v, f_v, f_v') = \frac{L(s, \chi_v, \pi_v, \mathbf{r}_{st})}{d_{H,v}(s)}$$

where $L(s, \chi_v, \pi_v, \mathbf{r}_{st})$ matches the L-factor provided by the Satake parameter and

$$d_{H,v}(s) = L(s + \frac{1}{2}(n+1), \chi_v) \prod_{i=0}^{\frac{n}{2}-1} L(2s + 2i + 1, \chi_v^2).$$

Similar formulas exist for the different classical groups. So for a cuspidal representation $\pi = \otimes_v \pi_v$ of $G_{\mathbb{A}}$ we define

$$d_H^S(s) = \prod_{v \notin S} d_{H,v}(s)$$

where S is the finite set of places such that if v is nonarchimedean and π_v is unramified then $v \notin S$. Consequently, we call $d_H(s)$ the **normalizing factor** of our

Eisenstein series.

For arbitrary places v, with π_v possibly ramified, we will construct the $\Phi_{v,s}$ having a certain desired invariance property. Using such sections, our doubling integral will yield a rational function

$$\frac{L(s, \chi_v, \pi_v, \mathbf{r}_{st})}{d_{\Phi,v}(s)}$$

where $d_{\Phi,v}(s)^{-1} \in \mathbb{C}[q^{-s}]$ and $\deg(L(s, \chi_v, \pi_v, \mathbf{r}_{st})^{-1}) = \deg(d_{\Phi,v}(s)^{-1})$ as polynomials in $\mathbb{C}[q^{-s}]$. Notice that this agrees with the spherical case mentioned above. While this equality of degrees may not hold for general representations π, it certainly seems to apply toward constituents of the unramified principal series.

Finally, while we will not use this fact in this thesis, it is worth mentioning that this doubling construction yields a functional equation. The following proposition is due to Piatetski-Shapiro and Rallis as well as Lapid and Rallis (see [8], [27], [28] and [24]).

Proposition 4.1.2. *Let π be an irreducible admissible representation of $G_{\mathbb{A}}$. Then*

$$\dim(\mathrm{Hom}_{G_{\mathbb{A}} \times G_{\mathbb{A}}}[\mathrm{Ind}_{P_{H,\mathbb{A}}}^{H_A}(|\det|^s \chi \circ \delta_P), \pi \otimes \pi^\vee])$$

is at most one for all but finitely many s. Thus there exists a factor $\Gamma_{G_{\mathbb{A}}}(s, \pi, \chi)$ such that

$$Z(-s, \chi^{-1}, \mathbb{I}_{w_0}(s, \chi)\Phi, f, f') = \Gamma_{G_{\mathbb{A}}}(s, \pi, \chi) Z(s, \chi, \Phi, f, f')$$

for all $f \in \pi$, $f' \in \pi^\vee$ and Φ_s a good section.

4.1.2 The variant of Kudla, Rapoport and Yang

Now the doubling method that we will employ is similar to the one described above. Unlike the Piatetski and Rallis version of the doubling integral, this method will involve computing an integral for specific "good" test vectors in order to explicitly compute the our local L-factors. We also make some other alterations that will allow our doubling integral to be more easily computed.

One alteration to the method involves the embedding of $G \times G$ into H. This altered embedding will be more natural for applying the doubling method to the metaplectic covers of G and H. After we discuss the alterations made to the original embeddings, we will explain their extensions to $\tilde{G} \times \tilde{G}$ in $\tilde{H}$.

We now return to our original notation, so F is a p-adic local field of odd residue characteristic. In particular, we are only considering local theory, so we drop the v subscript for the remainder of the thesis. Also, we will need to define the analogous embeddings and maps more explicitly for our applications. In our case, we have the following embedding of $G \times G$ into H

$$i_0(g_1, g_2) := \begin{pmatrix} a_1 & & b_1 & \\ & a_2 & & b_2 \\ c_1 & & d_1 & \\ & c_2 & & d_2 \end{pmatrix},$$

where $g_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$ blockwise. For a particular choice of basis, this agrees with the embedding described above. Also notice that for the projective representation

$(r, S(X))$ in Section 2.2.1 , we have

$$r(\underline{i}_0(g_1, g_2)) = r(g_1) \otimes r(g_2)$$

where $S(F^n) \otimes S(F^n) \hookrightarrow S(F^{2n})$ in the canonical way. Futhermore, if we let $V = V_1 + V_2$ according to the embedding $\underline{i}_0$, then $Y \cap V = Y \cap V_1 + Y \cap V_2$ and we see that the Leray cocycle is compatible with $\underline{i}_0$. So consequently, we see that $\underline{i}_0$ lifts to a map

$$i_0 : \tilde{G} \times \tilde{G} \to \tilde{H} \qquad\qquad i_0([g_1, z_1]_L, [g_2, z_2]_L) = [\underline{i}_0(g_1, g_2), z_1 z_2]_L.$$

Next we modify this embedding via

$$i(g_1, g_2) := i_0(g_1, g_2^\vee).$$

where

$$g^\vee := \begin{pmatrix} I_n & \\ & -I_n \end{pmatrix} g \begin{pmatrix} I_n & \\ & -I_n \end{pmatrix}.$$

With this twisted embedding, we will also use a different representative for our main orbit. In our case, we represent the main orbit with the coset $P_H \delta$ such that

$$\delta := \begin{pmatrix} & & 1 & \\ & 1 & & \\ -1 & 1 & & \\ & & 1 & 1 \end{pmatrix} = \begin{pmatrix} & & 1 & \\ & 1 & & \\ -1 & & & \\ & & & 1 \end{pmatrix} m \begin{pmatrix} 1 & -1 \\ & 1 \end{pmatrix}.$$

This δ also conjugates G^d into our standard Siegel parabolic. In particular, $\delta i(g, g) =$

$p(g)\delta$, where

$$p(g) = \begin{pmatrix} d & c & -c & \\ b & a & & -b \\ & & a & -b \\ & & -c & d \end{pmatrix}.$$

Notice that this relations shows us that $P_H\delta$ is stabilized by

$$G^d = \{i(g,g) \mid g \in G\},$$

which justifies our claiming that it is in our main orbit. Moreover, the altered embedding also extends to the metaplectic case in the obvious way. In particular, we would like to show that this map has a lift to $\tilde{G}$.

Lemma 4.1.1. *The map* $^\vee : G \to G$ *has a lift to* $^\vee : \tilde{G} \to \tilde{G}$*, given in Leray coordinates by*

$$[g,z]_L^\vee := [g^\vee, z^{-1}]_L.$$

Proof. The proof reduces to showing that $c_L(g_1^\vee, g_2^\vee) = c_L(g_1, g_1)^{-1}$. But notice that

$$\begin{aligned} r(g^\vee)\overline{\varphi(x)} &= \int_{F^n/\ker(-c)} \psi\left(-\frac{1}{2}(xa,xb) + (xb,yc) - \frac{1}{2}(yc,yd)\right)\overline{\varphi(xa-yc)}d_g(y) \\ &= \int_{F^n/\ker(c)} \psi\left(-\frac{1}{2}(xa,xb) - (xb,yc) - \frac{1}{2}(yc,yd)\right)\overline{\varphi(xa+yc)}d_g(y) \\ &= \overline{r(g)\varphi(x)} \end{aligned}$$

where we simply perform the transformation $y \mapsto -y$. Our result follows when we realize that since $c_L(g_1, g_2)$ is unimodular, so conjugation is the same as inversion.

$\square$

Consequently, we see that $\underline{i} : G \times G \to H$, lifts to

$$i : \tilde{G} \times \tilde{G} \to \tilde{H} \qquad i([g_1, z_1]_L, [g_2, z_2]) := i_0([g_1, z_1]_L, [g_2, z_2]_L^\vee) = [\underline{i}(g_1, g_2), z_1 z_2^{-1}]_L.$$

Also, if we consider the λ defined in Section 3.1.2, we have the following facts.

$$\lambda(k^\vee) = \lambda(k)^{-1}$$

and

$$\lambda_H(\underline{i}_0(k_1, k_2)) = \lambda_G(k_1)\lambda_G(k_2)$$

where we use the subscript to distinguish the various λ. Putting all this together, we realize that i is consistent with the splittings

$$K_G \mapsto \widetilde{\mathrm{Sp}_2}(F) \qquad \text{and} \qquad K_H \mapsto \widetilde{\mathrm{Sp}_4}(F).$$

Our modified doubling integral also uses some slightly different data. Instead of integrating our section $\{\Phi_s\}$ against two functions $f \in \pi$ and $f' \in \pi^\vee$ to obtain a function of s, we integrate against a single function $f \in \pi$. The doubling integral will thus define another function $Z(s, \Phi, f) \in \pi$. For our local integrals, the ultimate strategy will be to choose Φ_s is such a way that $Z(s, \Phi, f)$ will have invariance properties of similar to those of the original f.

Now let σ be an admissible representation (resp. genuine admissible representation) of $G = \mathrm{Sp}_2(F)$ (resp. $\tilde{G} = \widetilde{\mathrm{Sp}_2}(F)$). Further, let χ be a character of the Siegel parabolic P_H and let $I(s, \chi)$ denoted the representation induced from $\chi|\det|^s$ (resp. $(\chi|\det|^s)'$) on P_H (resp. $\widetilde{P_H}$) to $H = \mathrm{Sp}_4(F)$ (resp. $\tilde{H} = \widetilde{\mathrm{Sp}_4}(F)$). Then for $f \in \sigma$ and $\Phi_s \in I(s, \chi)$, we define the local doubling integral

$$Z(s, \Phi, f)(g_0') = \int_{\mathrm{Sp}_2(F)} \Phi_s(\delta' i(g_0', g'))\sigma(g') f dg \tag{4.1}$$

where δ', g' are any elements projecting to $\delta \in \mathrm{Sp}_4(F)$ and $g \in \mathrm{Sp}_2(F)$ respectively. Notice that the integrand is independent of the choice of g' over g. For sufficiently large $\mathrm{Re}(s)$, the integral converges absolutely and

$$Z(s, \Phi, f) \in \sigma.$$

The main result in [22] that motivates this variant of the doubling method is the following (see Lemma 4.1 in [22]):

Lemma 4.1.2. *(i) For any choice of $\delta' \in \tilde{H}$ with image δ and $g' \in \tilde{G}$,*

$$\delta' i(g', g') = p(g')\delta'$$

where $p(g') \in \widetilde{P_H}$ satisfying $\chi(p(g')) = 1$. In particular, for g_0', g_1' and $g_2' \in \tilde{G}$,

$$\Phi_s(\delta' i(g_0' g_1', g_0' g_2')) = \Phi_s(\delta' i(g_1', g_2')).$$

(ii) For $g' \in \tilde{G}$,

$$Z(s, R(i(1, g'))\Phi, \sigma(g')f) = Z(s, \Phi, f)$$

and

$$\sigma(g')Z(s, \Phi, f) = Z(s, R(i(g', 1))\Phi, f).$$

Here R denotes right multiplication.

In particular, we have the following important corollary from [22].

Corollary 4.1.1. *Suppose there exists a subgroup $A \subset G'$ such that*

$$R(i(a, 1))\Phi_s = \Phi_s, \ \forall a \in A.$$

Then

$$Z(s, \Phi, f) \in \sigma^A.$$

In particular, if $\sigma^A = 0$*, then* $Z(s, \Phi, f) = 0$ *for all* $f \in \sigma$*.*

Perhaps equally important, if there exist a subgroup $A \subset G'$ such that

$$\dim_{\mathbb{C}}(\sigma^A) = 1$$

and

$$R(i(a,1))\Phi_s = \Phi_s, \ \forall a \in A,$$

then

$$Z(s, \Phi, f) = \Delta(s, \Phi, \sigma) f$$

for all $f \in \sigma^A$. Ultimately, our goal will be to make good choices for Φ_s and f so that $\Delta(s, \Phi, \sigma) = L(s, \mathbf{r}_{st} \circ \rho')$ for the Weil-Deligne representation $\mathbf{r}_{st} \circ \rho'$ where

$$\sigma \mapsto \rho' = (\rho, N)$$

via Lusztig's criteria.

Finally, this version of the doubling integral will employ an interpolation method to construct Φ_s. In particular, we will take the Weil representation of $\mathrm{Sp}_4(F)$ on a space of Schwartz functions and define a map from this space to $I(s, \chi_V)$. Ultimately, our choice of Φ_s will actually amount to a choice of Schwartz function.

4.1.3 The interpolation method

We will now describe an interpolation method used to construct the sections Φ_s. It is worth noting that this method is valid for our calculations on both the linear and metaplectic groups. For $G = \mathrm{Sp}_n(F)$, consider the following map

$$\lambda_V : S(V^n) \to \mathrm{Ind}_{\widetilde{P_H}}^{\tilde{H}}(\chi_V|\cdot|^{s_0}) \qquad\qquad \varphi \mapsto \lambda_V(\varphi)([g,z]_L) := \omega_V([g,z]_L)\varphi(0)$$

where $s_0 = \frac{\dim_F(V)-(n+1)}{2}$. This interpolation trick also respects the various splittings that defined above. For instance, we have the following lemma (see [22]).

Lemma 4.1.3.

$$\omega_V([g^\vee, z^{-1}]_L)\bar{\varphi}(x) = \overline{\omega_V([g,z]_L)\varphi}(x)$$

Proof. For our choice of additive character, $\chi_V(x(g))(\gamma(\eta)^j(g))^{\bullet}\gamma(\eta\circ V)^{-j(g)} \in \{\pm 1\}$, so we need only verify that

$$r_V(g^\vee)\bar{\varphi}(x) = \overline{r_V(g)\varphi}(x).$$

However, this follows from nearly the same argument as the proof of Lemma 4.1.1.

□

Consequently, we have that for $\varphi_1 \otimes \bar{\varphi}_2(x) \in S(V^n) \otimes S(V^n) \hookrightarrow S(V^{2n})$,

$$\omega_V(i([g_1,z_1]_L,[g_2,z_2]_L))[\varphi_1 \otimes \bar{\varphi}_2](x) = (\omega_V([g_1,z_1]_L)\varphi_1) \otimes (\overline{\omega_V([g_2,z_1]_L)\varphi_2})(x).$$

The Weil representation also has the following property for $V = V_1 \oplus V_2$ (an orthogonal sum) with $m_i := \dim_F(V_i)$.

Lemma 4.1.4. *The Weil representation ω_V of $\tilde{G}$ on $S(V^n) \simeq S(V_1^n) \otimes S(V_2^n)$ is given by*

$$\omega_V(g') = \omega_{V_1}(g') \otimes \omega_{V_2}(g') \cdot \begin{cases} \zeta(g')^{-1} & m_1 m_2 \text{ is odd} \\ 1 & \text{otherwise} \end{cases}$$

where $\zeta : \tilde{G} \to \mathbb{C}^1$ is a character defined in [22].

Proof. See [22]. □

Ultimately, we will create a family of functions $\{\Phi_{s_0+r}\}_{r=0}^{\infty}$ using our interpolation method while extending our vectorspace V by adjoining hyperplanes (F^2, Q_0) where $Q_0(x, y) = 2xy$. Since a product of hyperplanes is even dimensional, the previous lemma says that Weil representation factors as a tensor product according to Weil representations on V and the hyperplanes. Let $V_{r,r}$ denote the orthogonal sum of r hyperplanes and let

$$V_r = V \oplus V_{r,r}.$$

For any $\varphi = \otimes_{i=1}^{n} \varphi_i \in \bigotimes_{i=1}^{n} S(V) \hookrightarrow S(V^n)$, we define $\varphi^{(r)}$ as

$$\varphi^{(r)} = \bigotimes_{i=1}^{n} (\varphi_i \otimes \varphi_0^r)$$

where $\varphi_0^r \in S(V_{r,r})$ is the characteristic function of $\mathcal{O}^{2r} \in V_{r,r}$. We now define the function Φ_{s_0+r} as

$$\Phi_{s_0+r} = \lambda(\varphi_1^{(r)} \otimes \overline{\varphi_2^{(r)}}) \in I(s_0 + r, \chi_V)$$

where $\varphi_i \in S(V^n)$.

For our purposes, the φ_i will be characteristic functions of various **good lattices** L having the property

$$\varpi(L^{\#}) \subset L \subset L^{\#}$$

where $L^{\#}$ is the dual lattice to L via the bilinear from on the quadratic space (V, Q). It is worth noting that the choice of the various lattices matters much less than their relatively position to each other. Consider the following example.

Example 4.1.1. *Let* $(V, Q) = (M_2(F), \kappa\det)$ *with* $\kappa \in \mathcal{O}^\times$ *and let*

$$\begin{aligned} L_1 &= M_2(\mathcal{O}) \\ L_2 &= \mathrm{diag}(\varpi^{-1}, 1)M_2(\mathcal{O})\mathrm{diag}(\varpi, 1). \end{aligned}$$

Consider the functions,

$$\lambda(\varphi_{L_i \oplus L_i}).$$

One can easily verify that both functions are spherical vectors in the appropriate induced representation. So we see that

$$\lambda(\varphi_{L_1 \oplus L_1}) = c\lambda(\varphi_{L_2 \oplus L_2})$$

and a routine calculation shows that $c = 1$*.*

It is worth noting that L_2 is the image of L_1 under an element of $\mathrm{O}(V)$. As such, there is a natural action of $\mathrm{O}(V)$ on $S(V^n)$ and that

$$\lambda : S(V^n) \to I(s_0, \chi_V)$$

is $\mathrm{O}(V)$ invariant. See [18] for a more detailed discussion of this action and how it relates to the dual pair $(\mathrm{O}(V), \mathrm{Sp}(W))$.

More generally, we desire that $\lambda(\varphi_{L^{(r)} \oplus (L')^{(r)}})$ be $I_\emptyset$-fixed. This is achieved if

$$\varpi L' \subseteq L \subseteq L'. \tag{4.2}$$

Notice that this requirement involves only to the relative position of the lattices. Again, this comes from the $\mathrm{O}(V)$ invariance of λ. Once we know that $\lambda(\varphi)$ is Iwahori-fixed, then the function becomes determined by its values on representatives of the

Weyl group $W_G = W_{\mathrm{Sp}_2}$. For our chosen generators w_α and w_β, we see that

$$\lambda(\varphi_{L\oplus L'})(w_\alpha) = [\omega_V(w_\alpha)\varphi_{L\oplus L'}](0) = \varphi_{L'\oplus L}(0) = 1$$

and

$$\lambda(\varphi_{L\oplus L'})(w_\beta) = [\omega_V(w_\beta)\varphi_{L\oplus L'}](0) = \varphi_L(0)\widehat{\varphi_{L'}}(0) = q^{-m_2}$$

where m_2 is characterized by our choosing the Haar measure on V that is self-dual with respect to $\psi \circ (\cdot,\cdot)$.

4.2 Explicit Values for Iwahori Fixed Vectors

Previously, we mentioned needing some general methods for computing explicit values of Iwahori-fixed vectors; now let us describe these methods in further detail. First, the Iwasawa decomposition for split p-adic groups says that for any split p-adic group G,

$$G = BK$$

where $B = MN$ is a Borel subgroup with Levi factor M and unipotent radical N and K an appropriate choice for a maximal compact subgroup of G. In practice, K will be the $\mathcal{O}$-points of the group defining G. We can further decompose K as

$$K = \coprod_{w\in W_G} N(\mathcal{O})wJ$$

where $N(\mathcal{O}) = N \cap K$, J is an Iwahori subgroup having an Iwahori factorization with respect to (B, M). Thus,

$$G = \coprod_{w\in W_G} BwJ$$

so $f \in \mathrm{Ind}_B^G(\chi_1, \chi_2, \ldots, \chi_n)^J$ is determined by its values on representatives of the finite set of Weyl group elements representing double cosets

$$B \backslash G / J.$$

Thus in the next section, we will use some of our general representation theory in order to compute the various values of Iwahori-fixed vectors on Weyl group elements. In particular, if f is an Iwahori-fixed vector in a representation π, we will use the vanishing of certain standard intertwining operators to find various relations amongst the elements of $\{f(w)\}_{w \in W_G}$.

Recall that for regular $\chi_1 \otimes \chi_2$,

$$\dim_{\mathbb{C}} \mathrm{Hom}_G(\mathrm{Ind}_{P_\emptyset}^G(\chi_1 \otimes \chi_2), \mathrm{Ind}_{P_\emptyset}^G((\chi_1 \otimes \chi_2) \circ w)) = 1.$$

We also notice that the intertwining operator

$$\mathbb{I}_w : \mathrm{Ind}_{P_\emptyset}^G(\chi_1 \otimes \chi_2) \to \mathrm{Ind}_{P_\emptyset}^G(\chi_1 \otimes \chi_2 \circ w) \qquad (\mathbb{I}_w f)(g) := \int_{N^w} f(wng)dn,$$

where $N^w = N_\emptyset \cap w^{-1}\overline{N_\emptyset}w$ gives a basis for this space as long as the integral converges. However, even when the integral does not converge, we can still make some sense out of this construction.

Suppose $\mathrm{Ind}_{P_\emptyset}^G(\chi_1 \otimes \chi_2)$ is such a representation where $\mathbb{I}_w$ does not converge. Let us consider the family of representations $\{\mathrm{Ind}_{P_\emptyset}^G(\chi_1\nu^s \otimes \chi_2\nu^s)\}_{s \in \mathbb{C}}$ (recall $\nu(x) = |x|$). It is known that the family of intertwining maps $\mathbb{I}_w(s)$ varies analytically in the domain where the $\mathbb{I}_w(s)$ converge. Thus these intertwining maps can be meromorphically extended beyond the domain of convergence. Thus we see that the intertwining maps of interest are either given as an integral or the meromorphic continuation of an integral.

4.2.1 Constituents of induced representation with regular data

Let us start with some general results. For this section, let $G = \mathbb{G}(F)$ be the p-adic points of a split group. Let B be a fixed Borel subgroup of G with $B = MN$, $M \simeq (F^\times)^n$ its Levi factor and N its unipotent radical. Further, let $K = \mathbb{G}(\mathcal{O})$ a maximal compact subgroup. Finally, we let $J \subset K$ be an Iwahori subgroup in K having an Iwahori factorization with respect to (B, M) and let W_G be the Weyl group of G.

Now suppose that (π, V) is an irreducible constituent of the unramified principal series of G, then there exists and unramified characters $\{\chi_i\}_{i=1}^n$ such that $\pi \hookrightarrow \mathrm{Ind}_B^G(\chi_1, \chi_2, \dots, \chi_n)$. Moreover, let r_P^G represent the Jacquet functor with respect to the parabolic subgroup P with Levi factor M_P. Recall from Section 2.1.1 that both the induction and Jacquet functors are normalized to respect unitarizability. Frobenius reciprocity tells us that for π an admissible representation of G and σ an admissible representation of M_P,

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_P^G(\sigma)) \simeq \mathrm{Hom}_{M_P}(r_P^G(\pi), \sigma).$$

If we specialize to the case that π is irreducible and $P = B$, then we find

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_B^G((\chi_1, \chi_2, \dots, \chi_n) \circ w)) \simeq \mathrm{Hom}_M(r_B^G(\pi), (\chi_1, \chi_2, \dots, \chi_n) \circ w).$$

Recalling the definition of the intertwining operators that we previously introduced, we can now give a methodology for computing the explicit values of Iwahori-fixed vectors of a given irreducible constituent π of the unramified principal series for G.

1. Use Frobenius Reciprocity to find inducing data $(\chi_1, \chi_2, \ldots, \chi_n)$ with $\pi \hookrightarrow \mathrm{Ind}_B^G(\chi_1, \chi_2, \ldots, \chi_n)$. Notice that this inducing data need not be unique. Moreover, the explicit values of the Iwahori fixed vectors may depend on this choice of inducing data.

2. Find an element w of the Weyl group W_G such that

$$(\chi_1, \chi_2, \ldots, \chi_n) \circ w \not\leq r_B^G(\pi) \in \mathfrak{R}(M).$$

   Then Frobenius Reciprocity establishes that π is not a submodule of

$$\mathrm{Ind}_B^G((\chi_1, \chi_2, \ldots, \chi_n) \circ w).$$

   Note that if no such w exists, then this intertwining method cannot be employed.

3. Compute the standard intertwining operator

$$\mathbb{I}_w : \mathrm{Ind}_B^G(\chi_1, \chi_2, \ldots, \chi_n) \to \mathrm{Ind}_B^G((\chi_1, \chi_2, \ldots, \chi_n) \circ w)$$

   for an arbitrary Iwahori fixed vector

$$f \in \mathrm{Ind}_B^G(\chi_1, \chi_2, \ldots, \chi_n)^J.$$

4. Use fact that $\mathbb{I}_w(f) = 0$ for $f \in \pi$ to find relations amongst the $\{f(w)\}_{w \in W_G}$.

Since $|W_G| = 8$ for $G = \mathrm{Sp}_2(F)$, we would need $8 - \dim_{\mathbb{C}}(\pi^{I_\emptyset})$ relations to explicitly determine the set of Iwahori fixed vectors in a given π. For our purposes, the standard intertwining operators are sufficient for determining Iwahori fixed vectors when the inducing data is regular; however more tools will be needed to deal with irregular characters. However, we first compute an example.

4.2.2 An example involving $\mathrm{Sp}_2(F)$

Let $G = \mathrm{Sp}_2(F)$. We will need the various parabolic subgroups described earlier. Let us consider the representation $\mathrm{Ind}_{P_\emptyset}^G(\nu^2 \otimes \nu)$. As we see in our table of constituents, this representation has an irreducible subquotient St_{Sp_2} with $r_{P_\emptyset}^G(St_2) = \nu^2 \otimes \nu$. Frobenius Reciprocity then tells us two crucial facts. First, St_{Sp_2} is an irreducible submodule of $\mathrm{Ind}_{P_\emptyset}^G(\nu^2 \otimes \nu)$. Second, is is not an irreducible submodule of any $\mathrm{Ind}_{P_\emptyset}^G((\nu^2 \otimes \nu) \circ w)$ for any non-trivial w. Thus, this representation is in the kernel of any nontrivial standard intertwining operator. So we consider the following operators $\mathbb{I}_{w_\alpha}$ and $\mathbb{I}_{w_\beta}$:

$$\mathbb{I}_{w_j} f(g) = \int_{N^{w_j}} f(w_j n g) dn.$$

It is clear that if f is fixed by the Iwahori subgroup that $\mathbb{I}_{w_j} f$ will be as well. Furthermore, because $\mathbb{I}_{w_j} f \in \mathrm{Ind}_{P_\emptyset}^G((\nu^2 \otimes \nu) \circ w_j)$, we see that $\mathbb{I}_{w_j} f \equiv 0$ for $f \in St_{\mathrm{Sp}_2}$. So let us derive some relations on the $f(w)$ for our Iwahori fixed vector f. First, let $w_j = w_\alpha$, then for w with $\ell(w) < \ell(w_\alpha w)$ we have

$$\begin{aligned} 0 = \mathbb{I}_{w_\alpha} f(w) &= \int_{N^{w_\alpha}} f(w_\alpha n w) dn = \int_F f(w_\alpha m \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} w) dx \\ &= \int_{\mathcal{O}} f(w_\alpha w) dx + \int_{F \setminus \mathcal{O}} f(m \begin{pmatrix} x^{-1} & \\ & x \end{pmatrix} w) dx. \end{aligned}$$

Notice that for w with $\ell(w) < \ell(w_\alpha w)$, we have $w^{-1}N^{w_\alpha}w \subset N$, so the first integral follows. The second integral comes from the equality

$$w_\alpha m \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} = m \begin{pmatrix} 1 & x^{-1} \\ & 1 \end{pmatrix} m \begin{pmatrix} x^{-1} & \\ & x \end{pmatrix} m \begin{pmatrix} 1 & \\ -x^{-1} & 1 \end{pmatrix}$$

along with the fact that f is an Iwahori fixed vector. Furthermore, if we normalize dx to so that $\mathcal{O}$ has measure 1, then we find

$$\begin{aligned} 0 &= f(w_\alpha w) + \sum_{j=1}^{\infty} \int_{\mathcal{P}^{-j}\setminus\mathcal{P}^{-j+1}} |x|^{-1}|x|^{-1} f(w)dx \\ &= f(w_\alpha w) + \sum_{j=1}^{\infty} q^{-2j}q^j(1-q^{-1})f(w) = f(w_\alpha w) + q^{-1}f(w). \end{aligned}$$

Similarly, one does nearly the identical calculation for w with $\ell(w) > \ell(w_\alpha w)$ except that one replaces the $\mathcal{O}$ with $\mathcal{P}$ and the summation then begins with $j = 0$. This adjustment is necessary since for w satisfying $\ell(w) > \ell(w_\alpha w)$, $w^{-1}N^{w_\alpha}w \subset \bar{N}$. However, after this slight alteration, we see that

$$0 = q^{-1}f(w_\alpha w) + f(w).$$

We can also perform a similar calculation with the Weyl group element w_β. In this case, we find that,

$$\begin{aligned} 0 = \mathbb{I}_{w_\beta} f(w) &= \int_{N^{w_\beta}} f(w_\beta n w)dn = \int_F f(w_\beta n_2 \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix} w)dx \\ &= \int_{\mathcal{O}} f(w_\beta w)dx + \int_{F\setminus\mathcal{O}} f(m \begin{pmatrix} 1 & 0 \\ 0 & x^{-1} \end{pmatrix} w)dx. \end{aligned}$$

Analogously to the previous case, we notice that if $\ell(w) < \ell(w_\beta w)$, then $w^{-1}N^{w_\beta}w \subset$

N. We also have the decomposition

$$w_\beta n_2 \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix} = n_2 \begin{pmatrix} 0 & 0 \\ 0 & x^{-1} \end{pmatrix} m \begin{pmatrix} 1 & 0 \\ 0 & x^{-1} \end{pmatrix} \overline{n_2} \begin{pmatrix} 0 & 0 \\ 0 & -x^{-1} \end{pmatrix}.$$

Combining these with f being Iwahori invariant yields,

$$\begin{aligned} 0 &= f(w_\beta w) + \sum_{j=1}^{\infty} \int_{\mathcal{P}^{-j} \setminus \mathcal{P}^{-j+1}} |x|^{-1}|x|^{-1} f(w)dx \\ &= f(w_\beta w) + \sum_{j=1}^{\infty} q^{-2j} q^j (1-q^{-1}) f(w) = f(w_\beta w) + q^{-1} f(w). \end{aligned}$$

Furthermore, one can verify as above that for $\ell(w) > \ell(w_\beta w)$ we get

$$0 = q^{-1} f(w_\beta w) + f(w).$$

Now we consider all of the relations that we have just formulated. We notice that $f(w) = -q^{-1} f(w')$ for $w, w' \in W_G$ with $\ell(w) = \ell(w') - 1$. Consequently, we have the following proposition.

Proposition 4.2.1. *Let St_G be the Steinberg representation of $\mathrm{Sp}_2(F)$ and let $f \in St^{I_\emptyset}$ such that $f(1_G) = 1$. Then we see that*

$$f(w) = (-q^{-1})^{\ell(w)}$$

for our chosen Weyl group representatives $w \in W_G$.

4.2.3 $\mathbb{I}_w$ for $\ell(w) > 1$ and analytically continuing $\mathbb{I}_w(s)$

There a couple of points worth noting about this technique. First, for representations π with $r^G_{P_\emptyset}(\pi)$ having multiple submodules, it might be necessary to compute $\mathbb{I}_w f$ for $\ell(w) > 1$. In this case, take a reduced expression for w in terms of w_α and w_β and compose the various $\mathbb{I}_{w_\alpha}$ and $\mathbb{I}_{w_\beta}$ according to this decomposition. Thus one can use a sequence of calculations that are nearly identical to the ones above. Second, for some representations, it may be necessary to take an analytic continuation of the standard intertwining operator. For instance, consider the representation $\mathrm{Ind}_{P_\emptyset}^G(\nu^{-2} \otimes \nu^{-1})$. This representation contains the trivial representation of $\mathrm{Sp}_2(F)$ as a submodule. In particular, $\mathbb{1}_G \subset \mathrm{Ind}(\nu^{-2} \otimes \nu^{-1})$ is spherical with $r_{P_\emptyset}^G(\mathbb{1}_G) = \nu^{-2} \otimes \nu^{-1}$. When we try to follow the computations above we find

$$
\begin{aligned}
0 = \mathbb{I}_{w_\beta} f(w) &= \int_{N^\beta} f(w_\beta n w) dn = \int_F f(w_\beta n_2 \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix} w) dx \\
&= \int_{\mathcal{O}} f(w_\beta w) dx + \int_{F \setminus \mathcal{O}} f(m \begin{pmatrix} 1 & \\ & x^{-1} \end{pmatrix} w) dx \\
&= f(w_\beta w) + \sum_{j=1}^{\infty} \int_{\mathcal{P}^{-j} \setminus \mathcal{P}^{-j+1}} |x||x|^{-1} f(w) dx.
\end{aligned}
$$

Clearly the integral above does not converge, so instead we need to use the analytic continuation of this intertwining operator. To compute this, let us consider the family of intertwining operators on the representation $\mathrm{Ind}_{P_\emptyset}^G(\nu^{2s} \otimes \nu^s)$ as above.

Thus we have,

$$
\begin{aligned}
\mathbb{I}_{w_\beta}(s)f(w) &= \int_{N^\beta} f(w_\beta n w)dn = \int_F f(w_\beta n_2 \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix} w)dx \\
&= \int_{\mathcal{O}} f(w_\beta w)dx + \int_{F\setminus\mathcal{O}} f(m \begin{pmatrix} 1 & \\ & x^{-1} \end{pmatrix} w)dx \\
&= f(w_\beta w) + \sum_{j=1}^{\infty} \int_{\mathcal{P}^{-j}\setminus\mathcal{P}^{-j+1}} |x|^{-s}|x|^{-1} f(w)dx \\
&= f(w_\beta w) + \sum_{j=1}^{\infty} q^{-sj}(1-q^{-1})f(w) \\
&= f(w_\beta w) + \frac{q^{-s}}{1-q^{-s}}(1-q^{-1})f(w).
\end{aligned}
$$

Now if we consider $\mathbb{I}_{w_\beta}(s)f(w)$ to be a function of the complex variable s, then the rational function that we have just derived is the analytic continuation of the integral expression for $\mathbb{I}_{w_\beta}(s)f(w)$. This rational function has a well defined value as s approaches -1. In fact, as $s \to -1$ we see that

$$0 = f(w_\beta w) - f(w).$$

Notice that at $s = -1$, we have the analytic continuation of the standard intertwining operator

$$\mathrm{Ind}_{P_\emptyset}^G(\nu^{-2} \otimes \nu^{-1}) \to \mathrm{Ind}_{P_\emptyset}^G(\nu^{-2} \otimes \nu),$$

which vanishes for $\mathbb{1}_{\mathrm{Sp}_2}$. So we get exactly the relation that we would expect from the spherical vector. A similar technique also works for $\mathbb{I}_{w_\alpha}f(w)$. Finally, we see that using the various compositions and analytic continuations, we can compute almost any of the standard intertwining operators evaluated on our set of Weyl group representatives.

4.2.4 Intertwining operators on $\widetilde{\mathrm{Sp}_2}(F)$

There is also one alteration worth mentioning when we attempt to employ these techniques for the metaplectic group $\widetilde{\mathrm{Sp}_2}(F)$. First, we realize that the definition of the intertwining operators are completely analogous to those on the linear group. In fact,

$$\mathbb{I}_w f([g,z]_L) := \int_{N^w} f([w,1]_L[n,1]_L[g,z]_L)dn.$$

Notice that $n \to [n,1]_L$ is a splitting of $N \hookrightarrow \widetilde{\mathrm{Sp}_2}(F)$. As we saw in a previous section, we also have a splitting $K \to \widetilde{\mathrm{Sp}_2}(F)$ given by $[k,\lambda(k)]_L$. If we choose f to be fixed by $I_\emptyset \to \widetilde{\mathrm{Sp}_2}(F)$, then we can attempt to emulate the procedure described above. In the case of $\mathbb{I}_{w_\alpha}$, the above procedure proceeds without modification. This occurs because of the decomposition

$$\begin{aligned}&[w_\alpha,1]_L[m\begin{pmatrix}1 & x\\ & 1\end{pmatrix},1]_L\\ &=[m\begin{pmatrix}1 & -x^{-1}\\ & 1\end{pmatrix},1]_L[m\begin{pmatrix}x^{-1} & \\ & x\end{pmatrix},1]_L[m\begin{pmatrix}1 & \\ -x^{-1} & 1\end{pmatrix},1]_L.\end{aligned}$$

In this case, all the elements involved belong to $\tilde{P}_\alpha$ on which the Leray cocycle is trivial. However, the same is not true when we consider the intertwining map $\mathbb{I}_{w_\beta}$. In that case, we have the following matrix decomposition

$$\begin{aligned}&[w_\beta,1]_L[n_2\begin{pmatrix}0 & 0\\ 0 & x\end{pmatrix},1]_L\\ &=[n_2\begin{pmatrix}0 & 0\\ 0 & x^{-1}\end{pmatrix},1]_L[m\begin{pmatrix}1 & \\ & x^{-1}\end{pmatrix},\lambda(\bar{n}(x^{-1}))]_L[\overline{n_2}\begin{pmatrix}0 & 0\\ 0 & -x^{-1}\end{pmatrix},\lambda(\bar{n}(x^{-1}))]_L\end{aligned}$$

where

$$\bar{n}(a) = \overline{n_2}\begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}.$$

Because f belongs to a genuine principal series representation, we see that

$$f([w_\beta, 1]_L[n_2\begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix}, 1]_L) = \lambda(\bar{n}(x^{-1}))f([m\begin{pmatrix} 1 & \\ & x^{-1} \end{pmatrix}, 1]_L).$$

It is important to notice that because our inducing data is unramified

$$f([m\begin{pmatrix} 1 & \\ & x^{-1} \end{pmatrix}, 1]_L)$$

is constant on the set $\mathcal{P}^j \setminus \mathcal{P}^{j+1}$ for any j. So let us consider the integral

$$\int_{\mathcal{P}^{-j}\setminus\mathcal{P}^{-j+1}} \lambda(\bar{n}(x^{-1}))dx$$

for $j \geq 1$. Using Equation 3.2, we see that

$$\int_{\mathcal{P}^{-j}\setminus\mathcal{P}^{-j+1}} \lambda(\bar{n}(x^{-1}))dx = \begin{cases} \int_{\mathcal{P}^{-j}\setminus\mathcal{P}^{-j+1}} 1dx & \text{if } j \text{ is even} \\ \theta \int_{\mathcal{P}^{-j}\setminus\mathcal{P}^{-j+1}} \left(\frac{x^{-1}}{\varpi}\right)_F dx & \text{if } j \text{ is odd} \end{cases}.$$

In the case that j is odd, we can transform $x \mapsto \varpi^{-j}x$ and get

$$q^{-j}\theta \int_{\mathcal{O}^\times} \left(\frac{x^{-1}}{\varpi}\right)_F dx,$$

which is the integral of a ramified character over the group of units of $\mathcal{O}$; therefore, it vanishes. What we have shown is that when computing $\mathbb{I}_{w_\beta}(f)$, we will still get a sum of integrals over $\mathcal{P}^{-j} \setminus \mathcal{P}^{-j+1}$, as in the linear group case, except that the integral vanishes for all odd j.

4.2.5 Techniques for irregular inducing data

The techniques we employ above are sufficient to determine the explicit values of Iwahori fixed vectors when π is a constituent of a representation induced from regular data. However, this may no longer be the case when our representation is induced from irregular data. Fortunately for both $\mathrm{Sp}_2(F)$ and its metaplectic cover, most reducible principal series representations induced from irregular data will have certain unitarizable constituents. We can exploit this extra structure in order to compute the explicit values of the Iwahori fixed vectors.

First, let us suppose that our inducing data is not regular without any other conditions on the constituents. For $\mathrm{Sp}_2(F)$ and $\widetilde{\mathrm{Sp}_2}(F)$ such representations will have constituents that are induced from representations on the maximal parabolic subgroups. In these cases, we can use the exactness of parabolic induction as well as knowledge of Iwahori invariance in the inducing data to derive the necessary results.

We see above that for a larger parabolic subgroup P, Frobenius Reciprocity still allows us to determine representations into which π embeds. Let $P = M_P N_P$ and $Q = M_Q N_Q$ be parabolic subgroups with $Q \subset P$ and $Q \cap M_P = M_Q$. Further, consider the following exact sequence of M_P representations

$$0 \to \sigma_1 \to \pi \to \sigma_2 \to 0.$$

The exactness of parabolic induction tells us that

$$0 \to \mathrm{Ind}_P^G(\sigma_1) \to \mathrm{Ind}_P^G(\pi) \to \mathrm{Ind}_P^G(\sigma_2) \to 0$$

is also exact. Furthermore, if $\pi = \mathrm{Ind}_{Q \cap M_P}^{M_P}(\tau)$ for (τ, V) a representation of M_Q, then there is a natural isomorphism between $\mathrm{Ind}_P^G(\mathrm{Ind}_{Q\cap M_P}^{M_P}(\tau))$ and $\mathrm{Ind}_Q^G(\tau)$. Moreover, we can use this isomorphism to understand the Iwahori invariants of $\mathrm{Ind}_P^G(\sigma_1)$. Let us compute an example for $G = \mathrm{Sp}_2(F)$. For this, we consider the representations

$$\chi \otimes \mathrm{Ind}_B^{\mathrm{Sp}_1(F)}(\xi) = (\chi \otimes T_\xi^2) \oplus (\chi \otimes T_\xi^1)$$

of $F^\times \times \mathrm{Sp}_1(F)$. Now P_β is a parabolic of G with Levi factor isomorphic to $F^\times \times \mathrm{Sp}_1(F)$, so by the exactness of induction, we get a direct sum of representations on G

$$\mathrm{Ind}_{P_\beta}^G(\chi \otimes \mathrm{Ind}_B^{\mathrm{Sp}_1(F)}(\xi)) = \mathrm{Ind}_{P_\beta}^G(\chi \otimes T_\xi^2) \oplus \mathrm{Ind}_{P_\beta}^G(\chi \otimes T_\xi^1).$$

It is worth noting that for $P \in \{P_\emptyset, P_\alpha\}$,

$$r_P^G(\mathrm{Ind}_{P_\beta}^G(\chi \otimes T_\xi^1)) = r_P^G(\mathrm{Ind}_{P_\beta}^G(\chi \otimes T_\xi^2))$$

so using our techniques with intertwining operators would not seem very fruitful in this case. However, there is a natural isomorphism

$$\Lambda : \mathrm{Ind}_{P_\beta}^G(\chi \otimes \mathrm{Ind}_B^{\mathrm{Sp}_1(F)}(\xi)) \to \mathrm{Ind}_{P_\emptyset}^G(\chi \otimes \xi) \qquad f \to (\Lambda f),$$

where $(\Lambda f)(g) = [f(g)](1_{F^\times \times \mathrm{Sp}_1(F)})$. Now let us consider the double coset space $P_\beta \backslash G / I_\emptyset$. It is clear that

$$G = \bigcup_{\substack{w \in W_G \\ \ell(w_\beta w) > \ell(w)}} P_\beta w I_\emptyset$$

is a disjoint union. Let us define the set $W_\beta = \{w \in W_G \mid \ell(w_\beta w) > \ell(w)\}$. Since $|W_\beta| = 4$, we see that

$$\dim_{\mathbb{C}}[\mathrm{Ind}_{P_\beta}^G(\chi \otimes T_\xi^j)^{I_\emptyset}] = 4 \times \dim_{\mathbb{C}}[(\chi \otimes T_\chi^j)^{\mathcal{O}^\times \times I}]$$

where I is the Iwahori subgroup of $\mathrm{Sp}_1(F)$ corresponding to the upper triangular Borel. In fact, let ϕ_j be the Iwahori-fixed vector of $\chi \otimes T_\xi^j$ with $\phi_j(1_{\mathrm{Sp}_1}) = 1$. We define the functions $f_w^j \in \mathrm{Ind}_{P_\beta}^G(\chi \otimes T_\xi^j)$ with $\mathrm{supp}(f_w^j) \subset P_\beta w I_\emptyset$ and $f_w^j(w) = \phi_j$. Then one can show that $(\Lambda f_w^j) \in \mathrm{Ind}_{P_\beta}^G(\chi \otimes T_\xi^j)^{I_\emptyset}$. Moreover, this construction yields all the Iwahori-fixed vectors in $\mathrm{Ind}_{P_\beta}^G(\chi \otimes T_\xi^j)$. This is clear because it produces four 1-dimensional spaces of Iwahori fixed vectors having pairwise disjoint support.

Now let us examine the case where our representation in induced from unitarizable data. For example, consider $\pi = \mathrm{Ind}_{P_\beta}^G(\mathbb{1}_{F^\times} \otimes St_{\mathrm{Sp}_1}) \subset \mathrm{Ind}_{P_\emptyset}^G(\mathbb{1}_{F^\times} \otimes \nu)$. As before, induction is normalized so π is unitarizable. Consequently, π is completely reducible. In fact,

$$\pi = \sigma(\nu^{\frac{1}{2}} \mathbb{1}_{\mathrm{GL}_2}) \oplus \sigma(\nu^{\frac{1}{2}} St_{\mathrm{GL}_2})$$

where

$$\begin{aligned} r_{P_\emptyset}^G(\sigma(\nu^{\frac{1}{2}} \mathbb{1}_{\mathrm{GL}_2})) &= \mathbb{1}_{F^\times} \otimes \nu \\ r_{P_\emptyset}^G(\sigma(\nu^{\frac{1}{2}} St_{\mathrm{GL}_2})) &= \mathbb{1}_{F^\times} \otimes \nu + 2 \cdot \nu \otimes \mathbb{1}_{F^\times}. \end{aligned}$$

Notice that we can use the intertwining operator $\mathbb{I}_{w_\beta}$ as above to find that both constituents have Iwahori-fixed vectors satisfying

$$f(w_\beta w) = \begin{cases} -q^{-1} f(w) \text{ if } \ell(w_\beta w) > \ell(w) \\ -q f(w) \ \text{ if } \ell(w_\beta w) < \ell(w) \end{cases}.$$

Moreover, the intertwining operator $\mathbb{I}_{w_\alpha}$ can be applied to $\sigma(\nu^{\frac{1}{2}} \mathbb{1}_{\mathrm{GL}_2})$ to show that it has an Iwahori fixed vector satisfying

$$f(w_\alpha w) = f(w).$$

Consequently, we see that the representation $\sigma(\nu^{\frac{1}{2}} \mathbb{1}_{\mathrm{GL}_2})$ has a one dimensional space of Iwahori-fixed vectors spanned by f_0 with

$$\begin{aligned} f_0(1_G) &= f_0(w_\alpha) = 1 \\ f_0(w_\beta) = f_0(w_{\alpha\beta}) &= f_0(w_{\beta\alpha}) = f_0(w_{\alpha\beta\alpha}) = -q^{-1} \\ f_0(w_{\beta\alpha\beta}) &= f_0(w_{\alpha\beta\alpha\beta}) = q^{-2}. \end{aligned}$$

Alternatively, for $w \in W_G$ such that $\ell(w_\beta w) > \ell(w)$ define $f_w \in \mathrm{Ind}_{P_\beta}^G(\mathbb{1}_{F^\times} \otimes St_{\mathrm{Sp}_1})^{I_\emptyset}$ with

$$\mathrm{supp}(f_w) \subset P_\beta w I_\emptyset$$

and

$$f_w(w) = \phi$$

where ϕ is the unique Iwahori-fixed vector in $St_{\mathrm{Sp}_1} \subset \mathrm{Ind}_B^{\mathrm{Sp}_1(F)}(\nu)$ with $\phi(1_{\mathrm{Sp}_1}) = 1$. Then we see that

$$f_0 = f_{w_0} + f_{w_\alpha} - q^{-1} f_{w_{\alpha\beta}} - q^{-1} f_{w_{\alpha\beta\alpha}}$$

Though note that $\mathbb{I}_{w_\alpha}$ need not vanish on the representation $\sigma(\nu^{\frac{1}{2}} St_{\mathrm{GL}_2})$. So we would like to use the inner product on the representation $\mathrm{Ind}_{P_\beta}^G(\mathbb{1}_{F^\times} \otimes St_{\mathrm{Sp}_1})$ in order to find the explicit values of Iwahori fixed vectors in $\sigma(\nu^{\frac{1}{2}} St_{\mathrm{GL}_2})$.

In this case, we notice that $\dim_{\mathbb{C}}(\mathrm{Ind}_{P_\beta}^G(\mathbb{1}_{F^\times} \otimes St_{\mathrm{Sp}_1})^{I_\emptyset}) = 4$ and

$$\mathrm{Ind}_{P_\beta}^G(\mathbb{1}_{F^\times} \otimes St_{\mathrm{Sp}_1})^{I_\emptyset} = \sigma(\nu^{\frac{1}{2}} \mathbb{1}_{\mathrm{GL}_2})^{I_\emptyset} \oplus \sigma(\nu^{\frac{1}{2}} St_{\mathrm{GL}_2})^{I_\emptyset}.$$

Moreover this is an orthogonal direct sum with respect to a non-degenerate inner product on the induced representation. To define this inner-product, notice that the square-integrability of St_{Sp_1} affords us an non-degenerate positive-definite inner product $< \cdot, \cdot >_{\mathrm{Sp}_1}$ with

$$< \phi, \phi >_{\mathrm{Sp}_1} = 1.$$

Then the inner product on $\mathrm{Ind}_{P_\beta}^G(\mathbb{1}_{F^\times} \otimes St_{\mathrm{Sp}_1})$ is defined by

$$<< f_1, f_2 >> = \int_K < f_1(k), f_2(k) >_{\mathrm{Sp}_1} dk.$$

Thus for any $f \in \sigma(\nu^{\frac{1}{2}} St_{\mathrm{GL}_2})^{I_\emptyset}$ and f_0 as above, we have

$$0 = << f, f_0 >> = \int_K < f(k), f_0(k) >_{\mathrm{Sp}_1} dk.$$

From previous arguments, we see that f must be of the form

$$f = \lambda_0 f_{w_0} + \lambda_\alpha f_{w_\alpha} + \lambda_{\alpha\beta} f_{w_{\alpha\beta}} + \lambda_{\alpha\beta\alpha} f_{w_{\alpha\beta\alpha}}$$

with $\lambda_i \in \mathbb{C}$. When we compute $<< f, f_0 >>$, we find

$$0 = \lambda_0 + q\lambda_\alpha - q\lambda_{\alpha\beta} - q^2 \lambda_{\alpha\beta\alpha}.$$

This relation proves sufficient to determine the Iwahori invariants of $\sigma(\nu^{\frac{1}{2}} St_{\mathrm{GL}_2})$. Moreover, using the the two methods outlined in this section, we are able to compute the explicit values of our Iwahori invariants for constituents of representations induced from irregular data.

4.2.6 Tables of Parahoric Invariants

The following pages contain several tables summarizing our computations of the explicit values for Iwahori-fixed vectors. Table 4.1 lists the various irreducible constituents π of the unramified principal series for $\mathrm{Sp}_2(F)$ as well as which induced representation $\mathrm{Ind}_{P_\emptyset}^G(\chi_1 \otimes \chi_2)$ we embedded π into in order to compute the values of the parahoric invariant vectors. It is worth noting that while the dimension of the parahoric invariant vectors is independent of the embedding, the actual values of the vectors depend on which induced representation we use to realize π as a submodule. As such, Table 4.1 outlines these realizations for representations of $\mathrm{Sp}_2(F)$. Table 4.2 contains the same information for the group $\widetilde{\mathrm{Sp}_2}(F)$. Finally, Table 4.3 gives the explicit values of all the Iwahori-fixed vectors stemming from Tables 4.1 and 4.2. In this table, each row contains the data for a different Iwahori-fixed vector. Notice that the last column lists which constituents from Tables B.1 and B.9 contain the vector whose data is described in a given row.

Table 4.1: **Realizing our constituents as submodules-**$\mathrm{Sp}_2(F)$ **case**

		Constituent	**Representation**
I		$\mathrm{Ind}_{P_\emptyset}^{G}(\chi_1 \otimes \chi_2)$ (irreducible)	$\mathrm{Ind}_{P_\emptyset}^{G}(\chi_1 \otimes \chi_2)$
II	a	$\mathrm{Ind}_{P_\alpha}^{G}(\chi St_{\mathrm{GL}_2})$	$\mathrm{Ind}_{P_\emptyset}^{G}(\chi\nu^{\frac{1}{2}} \otimes \chi^{-1}\nu^{\frac{1}{2}})$
	b	$\mathrm{Ind}_{P_\alpha}^{G}(\chi \mathbb{1}_{\mathrm{GL}_2})$	$\mathrm{Ind}_{P_\emptyset}^{G}(\chi\nu^{\frac{1}{2}} \otimes \chi\nu^{-\frac{1}{2}})$
III	a	$\mathrm{Ind}_{P_\beta}^{G}(\chi \otimes St_{\mathrm{Sp}_1})$	$\mathrm{Ind}_{P_\emptyset}^{G}(\nu \otimes \chi)$
	b	$\mathrm{Ind}_{P_\beta}^{G}(\chi \otimes \mathbb{1}_{\mathrm{Sp}_1})$	$\mathrm{Ind}_{P_\emptyset}^{G}(\chi \otimes \nu)$
IV	a	$\mathrm{Ind}_{P_\beta}^{G}(\chi \otimes T_\xi^2)$	$\mathrm{Ind}_{P_\emptyset}^{G}(\xi \otimes \chi)$
	b	$\mathrm{Ind}_{P_\beta}^{G}(\chi \otimes T_\xi^1)$	$\mathrm{Ind}_{P_\emptyset}^{G}(\chi \otimes \xi)$
V	a	St_{Sp_2}	$\mathrm{Ind}_{P_\emptyset}^{G}(\nu^2 \otimes \nu)$
	b	$L(\nu^2, St_{\mathrm{Sp}_1})$	$\mathrm{Ind}_{P_\emptyset}^{G}(\nu \otimes \nu^2)$
	c	$L(\nu^{\frac{3}{2}} St_{\mathrm{GL}_2}, 1)$	$\mathrm{Ind}_{P_\emptyset}^{G}(\nu^2 \otimes \nu^{-1})$
	d	$\mathbb{1}_{\mathrm{Sp}_2}$	$\mathrm{Ind}_{P_\emptyset}^{G}(\nu^{-2} \otimes \nu^{-1})$
VI	a	$\sigma(\nu^{\frac{1}{2}} St_{\mathrm{GL}_2})$	$\mathrm{Ind}_{P_\emptyset}^{G}(\mathbb{1}_{F^\times} \otimes \nu)$
	b	$L(\nu^{\frac{1}{2}} St_{\mathrm{GL}_2}, 1)$	$\mathrm{Ind}_{P_\emptyset}^{G}(\mathbb{1}_{F^\times} \otimes \nu^{-1})$
	c	$\sigma(\nu^{\frac{1}{2}} \mathbb{1}_{\mathrm{GL}_2})$	$\mathrm{Ind}_{P_\emptyset}^{G}(\mathbb{1}_{F^\times} \otimes \nu)$
	d	$L(\nu, \mathbb{1}_{F^\times}, 1)$	$\mathrm{Ind}_{P_\emptyset}^{G}(\mathbb{1}_{F^\times} \otimes \nu^{-1})$
VII	a	$\sigma(\nu \otimes T_\xi^2)$	$\mathrm{Ind}_{P_\emptyset}^{G}(\nu \otimes \xi)$
	b	$\sigma(\nu \otimes T_\xi^1)$	$\mathrm{Ind}_{P_\emptyset}^{G}(\nu \otimes \xi)$
	c	$L(\nu, T_\xi^2)$	$\mathrm{Ind}_{P_\emptyset}^{G}(\nu^{-1} \otimes \xi)$
	d	$L(\nu, T_\xi^1)$	$\mathrm{Ind}_{P_\emptyset}^{G}(\nu^{-1} \otimes \xi)$
Table continued on nextpage.			

Table 4.1: **Realizing our constituents as submodules-**$\mathrm{Sp}_2(F)$ **case**

		Constituent	**Representation**
VIII	a	$\delta([\xi,\nu\xi],1)_{T^2_\xi}$	$\mathrm{Ind}_{P_\emptyset}^G(\xi\nu\otimes\xi)$
	b	$\delta([\xi,\nu\xi],1)_{T^1_\xi}$	$\mathrm{Ind}_{P_\emptyset}^G(\xi\nu\otimes\xi)$
	c	$L(\nu^{\frac{1}{2}}\xi St_{\mathrm{GL}_2},1)$	$\mathrm{Ind}_{P_\emptyset}^G(\xi\otimes\xi\nu)$
	d	$L(\nu\xi,T^2_\xi)$	$\mathrm{Ind}_{P_\emptyset}^G(\xi\nu^{-1}\otimes\xi)$
	e	$L(\nu\xi,T^1_\xi)$	$\mathrm{Ind}_{P_\emptyset}^G(\xi\nu^{-1}\otimes\xi)$

Table 4.2: **Realizing representations as submodules-$\widetilde{\mathrm{Sp}_2}(F)$ case**

		Constituent	**Representation**
IX		$\mathrm{Ind}_{\tilde{P}_\emptyset}^{\tilde{G}}((\chi_1 \otimes \chi_2)')$	$\mathrm{Ind}_{\tilde{P}_\emptyset}^{\tilde{G}}((\chi_1 \otimes \chi_2)')$
X	a	$\mathrm{Ind}_{\tilde{P}_\alpha}^{\tilde{G}}((\chi St_{\mathrm{GL}_2})')$	$\mathrm{Ind}_{\tilde{P}_\emptyset}^{\tilde{G}}((\chi\nu^{\frac{1}{2}} \otimes \chi^{-1}\nu^{\frac{1}{2}})')$
	b	$\mathrm{Ind}_{\tilde{P}_\alpha}^{\tilde{G}}((\chi \mathbb{1}_{\mathrm{GL}_2})')$	$\mathrm{Ind}_{\tilde{P}_\emptyset}^{\tilde{G}}((\chi\nu^{\frac{1}{2}} \otimes \chi\nu^{-\frac{1}{2}})')$
XI	a	$\mathrm{Ind}_{\tilde{P}_\beta}^{\tilde{G}}(\chi \otimes \tau(\varsigma\nu^{\frac{1}{2}}))$	$\mathrm{Ind}_{\tilde{P}_\emptyset}^{\tilde{G}}((\varsigma\nu^{\frac{1}{2}} \otimes \chi)')$
	b	$\mathrm{Ind}_{\tilde{P}_\beta}^{\tilde{G}}(\chi \otimes \pi(\varsigma\nu^{\frac{1}{2}}))$	$\mathrm{Ind}_{\tilde{P}_\emptyset}^{\tilde{G}}((\chi \otimes \varsigma\nu^{\frac{1}{2}})')$
XII	a	$\Pi((\varsigma St_{\mathrm{GL}_2})', \varsigma\nu^{\frac{1}{2}} \otimes \pi(\varsigma\nu^{\frac{1}{2}}))$	$\mathrm{Ind}_{\tilde{P}_\emptyset}^{\tilde{G}}((\varsigma\nu^{\frac{1}{2}} \otimes \varsigma\nu^{-\frac{1}{2}})')$
	b	$\Pi((\varsigma St_{\mathrm{GL}_2})', \varsigma\nu^{\frac{1}{2}} \otimes \tau(\varsigma\nu^{\frac{1}{2}}))$	$\mathrm{Ind}_{\tilde{P}_\emptyset}^{\tilde{G}}((\varsigma\nu^{\frac{1}{2}} \otimes \varsigma\nu^{-\frac{1}{2}})')$
	c	$\Pi((\varsigma \mathbb{1}_{\mathrm{GL}_2})', \varsigma\nu^{\frac{1}{2}} \otimes \tau(\varsigma\nu^{\frac{1}{2}}))$	$\mathrm{Ind}_{\tilde{P}_\emptyset}^{\tilde{G}}((\varsigma\nu^{-\frac{1}{2}} \otimes \varsigma\nu^{\frac{1}{2}})')$
	d	$\Pi((\varsigma \mathbb{1}_{\mathrm{GL}_2})', \varsigma\nu^{\frac{1}{2}} \otimes \pi(\varsigma\nu^{\frac{1}{2}}))$	$\mathrm{Ind}_{\tilde{P}_\emptyset}^{\tilde{G}}((\varsigma\nu^{-\frac{1}{2}} \otimes \varsigma\nu^{\frac{1}{2}})')$
XIII	a	$\Pi((\varsigma\nu St_{\mathrm{GL}_2})', \varsigma\nu^{\frac{3}{2}} \otimes \tau(\varsigma\nu^{\frac{1}{2}}))$	$\mathrm{Ind}_{\tilde{P}_\emptyset}^{\tilde{G}}((\varsigma\nu^{\frac{3}{2}} \otimes \varsigma\nu^{\frac{1}{2}})')$
	b	$\Pi((\varsigma\nu St_{\mathrm{GL}_2})', \varsigma\nu^{\frac{3}{2}} \otimes \pi(\varsigma\nu^{\frac{1}{2}}))$	$\mathrm{Ind}_{\tilde{P}_\emptyset}^{\tilde{G}}((\varsigma\nu^{\frac{3}{2}} \otimes \varsigma\nu^{-\frac{1}{2}})')$
	c	$\Pi((\varsigma\nu \mathbb{1}_{\mathrm{GL}_2})', \varsigma\nu^{\frac{3}{2}} \otimes \tau(\varsigma\nu^{\frac{1}{2}}))$	$\mathrm{Ind}_{\tilde{P}_\emptyset}^{\tilde{G}}((\varsigma\nu^{\frac{1}{2}} \otimes \varsigma\nu^{\frac{3}{2}})')$
	d	$\Pi((\varsigma\nu \mathbb{1}_{\mathrm{GL}_2})', \varsigma\nu^{\frac{3}{2}} \otimes \pi(\varsigma\nu^{\frac{1}{2}}))$	$\mathrm{Ind}_{\tilde{P}_\emptyset}^{\tilde{G}}((\varsigma\nu^{-\frac{3}{2}} \otimes \varsigma\nu^{-\frac{1}{2}})')$

Table 4.3: **Explicit values of Iwahori-fixed vectors**

f	$f(1_G)$	$f(w_\alpha)$	$f(w_\beta)$	$f(w_{\alpha\beta})$	$f(w_{\beta\alpha})$	$f(w_{\alpha\beta\alpha})$	$f(w_{\beta\alpha\beta})$	$f(w_n)$	**Families**
f_0	1	1	1	1	1	1	1	1	I, IIb, IIIb, IVb, Vd, VId, VIId, VIIIe, IX, Xb, XIb,XIId,XIIId
f_1	1	1	$\frac{(q-1)q^{t-1}}{q^t-q^{1/2}}$	$-\frac{q-1}{q^{t+3/2}-q^2}$	$\frac{(q-1)q^{t-1}}{q^t-q^{1/2}}$	$-\frac{q-1}{q^{t+3/2}-q^2}$	$-q^{-2}$	$-q^{-2}$	IIa
f_2	1	$\frac{q^t-q^{t-1}}{q^t-q}$	1	$\frac{q^t-q^{t-1}}{q^t-q}$	$-\frac{q-1}{q^{t+1}-q^2}$	$-q^{-2}$	$-\frac{q-1}{q^{t+1}-q^2}$	$-q^{-2}$	IIIa
f_3	1	$\frac{q^t-q^{t-1}}{q^t+1}$	1	$\frac{q^t-q^{t-1}}{q^t+1}$	$\frac{q-1}{q^{t+2}+q^2}$	$-q^{-2}$	$\frac{q-1}{q^{t+2}+q^2}$	$-q^{-2}$	IVa
f_4	1	$-q^{-1}$	$-q^{-1}$	q^{-2}	q^{-2}	$-q^{-3}$	$-q^{-3}$	q^{-4}	Va, VIIIa, XIIIa
f_5	1	1	1	1	$-q^{-2}$	$-q^{-2}$	$-q^{-2}$	$-q^{-2}$	Vb, VIIIc, XIIIc
f_6	1	1	1	$-q^{-2}$	1	$-q^{-2}$	$-q^{-2}$	$-q^{-2}$	Vc, VIIb, XIIIb
f_7	1	1	$-q^{-1}$	$-q^{-1}$	$-q^{-1}$	$-q^{-1}$	q^{-2}	q^{-2}	VIc, VIIa, VIIId XIIb, XIIc
f_8	1	$-q^{-1}$	1	$-q^{-1}$	$-q^{-1}$	q^{-2}	$-q^{-1}$	q^{-2}	VIb, VIIIb, XIIa

Table continued on next page.

Table 4.3: **Explicit values of Iwahori-fixed vectors**

f	$f(1_G)$	$f(w_\alpha)$	$f(w_\beta)$	$f(w_{\alpha\beta})$	$f(w_{\beta\alpha})$	$f(w_{\alpha\beta\alpha})$	$f(w_{\beta\alpha\beta})$	$f(w_n)$	**Families**
f_9	1	1	$-q^{-1}$	q^{-1}	$-q^{-1}$	q^{-1}	$-q^{-2}$	$-q^{-2}$	VIa, VIIc
f_{10}	1	1	$\frac{1-q^{-1}}{1-q^{-(2t-1)}}$	$\frac{q^{-(2t+1)}-q^{-2t}}{1-q^{-(2t-1)}}$	$\frac{1-q^{-1}}{1-q^{-(2t-1)}}$	$\frac{q^{-(2t+1)}-q^{-2t}}{1-q^{-(2t-1)}}$	$-q^{-2}$	$-q^{-2}$	Xa
f_{11}	1	$\frac{1-q^{-1}}{1-\varsigma(\varpi)q^{\frac{1}{2}-t}}$	1	$\frac{1-q^{-1}}{1-\varsigma(\varpi)q^{\frac{1}{2}-t}}$	$\frac{\varsigma(\varpi)(q^{-\frac{3}{2}-t}-q^{-\frac{1}{2}-t})}{1-\varsigma(\varpi)q^{\frac{1}{2}-t}}$	$-q^{-2}$	$\frac{\varsigma(\varpi)(q^{-\frac{3}{2}-t}-q^{-\frac{1}{2}-t})}{1-\varsigma(\varpi)q^{\frac{1}{2}-t}}$	$-q^{-2}$	XIa

4.3 Local Densities of Quadratic Forms

As we have seen in the first section, the interpolation method used to construct Φ_{s_0+r} requires choosing a quadratic space (V, Q). Eventually, we reduce our doubling calculation to a combination of Whittaker functions that are closely related to the local densities of the quadratic space (V, Q). In fact, by some results of Yang [42], there exists completely general formulas for local densities of quadratic forms in various low rank cases. In particular, the cases needed for the doubling integral for $\mathrm{Sp}_2(F)$ and $\widetilde{\mathrm{Sp}_2}(F)$ are completely computable via [42].

4.3.1 The work of Yang

As we compute $Z(s, \Phi, f)(g_0')$, we will find that our doubling method eventually requires us to compute integrals of the form

$$W_0(r, L, L') = \int_{\mathrm{Sym}_2(F)} \int_{V_r^2} \varphi_{L\oplus L'}^{(r)}(x)\psi(-\mathrm{tr}(b[Q(x)]))dx\, db$$

where L and $L' \subset V$ are good lattices. We will further reduce such calculations to a weighted average of integrals where $L = L'$. Such an integral is closely related to one of the form

$$\alpha_Q(T, S) = W(T, S) := \int_{\mathrm{Sym}_n(F)} \int_{\bigoplus_{i=1}^{n} L} \psi(\mathrm{tr}(b[Q(x)])\psi(-\mathrm{tr}(Tb))dx\, db.$$

In this case, S and T are symmetric matrices of degree m and n respectively. Moreover, (V, Q) is the quadratic vector space with $V = F^m$ and $Q(x) = {}^t xSx$. The matrix T has no analog in our integral.

While there are some general formulas to compute $\alpha_Q(T, S)$ for any choice of

T and S given by Hironaka and Sato [11], these are so complicated as to be computationally unusable. Some of the most general computationally useful formulas are provided by Yang [42], where he gives general formulas in the case that S is arbitrary and $T \in F$ or $T \in \mathrm{Sym}_2(F)$. The formula for $n = 2$ are vastly more complicated that when $n = 1$, thus it would seem that the generalizing Yang's results to higher dimensional T would also provide unusable results in sufficiently high dimension.

Yang also formulates his result in a particularly useful way for our application. Let V, Q, S be as before. Then as before, we can adjoin r hyperplanes to form the quadratic space V_r; in this case, we will denote the new matrix of the quadratic form as S_r. Yang then provides a formula for the integral

$$\alpha(q^{-r}, T, S) = W(T, S_r) = \int_{\mathrm{Sym}_n(F)} \int_{\bigoplus_{i=1}^{n} L^{(r)}} \psi(\mathrm{tr}(b[Q(x)])\psi(-\mathrm{tr}(Tb))dx\, db$$

when $n \in \{1, 2\}$ and where $\alpha(q^{-r}, T, S)$ is polynomial in q^{-r}. In the case that $n = 2$, Yang's expression is a linear combination of 12 different polynomial terms of various complexity. However, we will show that for $r >> 0$ and $T \to 0$, nine of these terms vanish and the others become convergent geometric series in q^{-r} (realized as rational functions). In fact, for $r >> 0$ they simplify in such a way that

$$W_0(r, L, L) = \lim_{T \to 0} \alpha(X, T, S)$$

is easily computed by hand. We will now derive the necessary formulas using Yang's results and notation in [42]. Let

$$S = \mathrm{diag}(\epsilon_i \varpi^{l_i})_{i=1}^{m}$$

with $l_i \le l_j$ when $i < j$ be the matrix of the quadratic form on V. Further, let

$$T = \begin{pmatrix} \alpha_1 \varpi^{a_1} & \\ & \alpha_2 \varpi^{a_2} \end{pmatrix}$$

where the $\alpha_i \in \mathcal{O}^\times$. The following essential quantities are defined by Yang [42]. For $k \ge 0$ an integer, we have

$$\begin{aligned} L(k,1) &= \{1 \le i \le m \,:\, l_i - k < 0 \text{ is odd}\} \\ l(k,1) &= \#L(k,1) \\ d(k) &= k + \frac{1}{2}\sum_{l_i < k}(l_i - k) \\ v(k) &= \left(\frac{-1}{\varpi}\right)_F^{[\frac{l(k,1)}{2}]} \prod_{i \in L(k,1)} \left(\frac{\epsilon_i}{\varpi}\right)_F \\ \delta^{\pm}(k) &= \frac{1 \pm (-1)^{l(k,1)}}{2}. \end{aligned}$$

Yang then proves the following equation

$$\alpha(q^{-r}, T, S) = R_0(X) + R_1(X) + R_2(X)$$

with

$$R_0(X) = 1, \quad R_1(X) = \sum_{i=1}^{4} I_{1,i}, \quad R_2(X) = (1 - q^{-1}) \sum_{i=1}^{8} I_{2,i} + q^{-1} I_{2,6}$$

and $X = q^{-r}$. The individual $I_{j,k}$ are defined as polynomials in X using the quantities defined above. While many of these terms have complicated formulas, one can see that in most cases $I_{j,k} \to 0$ with $T \to 0$. For an example, consider one such term

$$\begin{aligned} I_{2,2} &= \left(\left(\frac{\alpha_1}{\varpi}\right)_F \frac{1}{\sqrt{q}} \delta^-(a_1+1) - \frac{1}{q^2}\delta^+(a_1+1)\right) \\ &\qquad \times \sum_{0<k\le a_1} v(a_1+1)v(k)\delta^+(k) q^{a_1+1+d(a_1+1)+d(k)} X^{a_1+1+k}. \end{aligned}$$

Notice that the leading coefficient is at most 1 and the summation is at most

$$\sum_{k=1}^{a_1} q^{a_1+1+d(a_1+1)+d(k)} X^{a_1+1+k}.$$

. If we let $\mu = \frac{1}{2} \sum_{i=1}^{m} l_i$, then for a sufficiently large j, we have

$$d(j) - \left(1 - \frac{m}{2}\right) j = \mu.$$

Let us choose a T sufficiently close to 0 so that $a_1 > l_i$ for all i. Then we have

$$\begin{aligned} \sum_{k=1}^{a_1} q^{a_1+1+d(a_1+1)+d(k)} X^{a_1+1+k} &= q^{a_1+1+d(a_1+1)} X^{a_1+1} \sum_{k=1}^{a_1} q^{d(k)} X^k \\ &= q^{\mu + \left(2 - \frac{m}{2}\right)(a_1+1)} X^{a_1+1} \sum_{k=1}^{a_1} q^{d(k)} X^k \end{aligned}$$

We now want to consider what happens as $T \to 0$ or alternatively when $a_i \to \infty$ for both a_1 and a_2. First notice that

$$\sum_{k=1}^{a_1} q^{d(k)} X^k$$

differs from

$$q^{\mu} \sum_{k=1}^{a_1} (q^{1-\frac{m}{2}} X)^k$$

by only a finite number of terms. Moreover, since $X = q^{-r}$, we see that

$$q^{\mu} \sum_{k=1}^{a_1} (q^{1-\frac{m}{2}} X)^k$$

becomes a convergent geometric series as $a_1 \to \infty$ as long as $1 - m/2 - r < 0$ which holds in our cases. Furthermore, we see that

$$q^{\mu + \left(2 - \frac{m}{2}\right)(a_1+1)} X^{a_1+1} = (q^{\mu})(q^{2-\frac{m}{2}} X)^{a_1+1}$$

which vansishes as $a_i \to \infty$ as long as $2 - m/2 - r < 0$. Consequently, $\lim_{T \to \infty} I_{2,2} = 0$ as long as $2 - m/2 - r < 0$. Note that this slightly more stringent restriction on r

and m is still satisfied by our Whittaker functions. In fact, all the vanishing terms in $\alpha(X,T,S)$ are treated similarly. Either they can be written as a polynomial in X times a convergent rational function or else the term is simply a monomial in X, which obviously vanishes as $T \to 0$. Ultimately, we have the following three non-vanishing terms as $T \to 0$:

$$\begin{aligned} I_{1,1} &= (1-q^{-2}) \sum_{0<k\le a_1} v(k)\delta^+(k)q^{k+d(k)}X^k, \\ I_{2,1} &= (1-q^{-2}) \sum_{0<k_2<k_1\le a_1} v(k_1)v(k_2)\delta^+(k_1)\delta^+(k_2)q^{k_1+d(k_1)+d(k_2)}X^{k_1+k_2}, \end{aligned}$$

and

$$I_{2,8} = \sum_{0<k\le a_1} (\delta^+(k)+q^{-1}\delta^-(k))q^{k+2d(k)}X^{2k}.$$

As before, $T \to 0$ implies that $a_1 \to \infty$ and our three terms above are dominated by convergent geometric series. Let us now compute a concrete example that we will use for the doubling integral with spherical representations. In both the linear and metaplectic cases, we will be interested in computing

$$W_0(r,L_1,L_1) = \int_{\mathrm{Sym}_2(F)} \int_{L_1^{(r)}\oplus L_1^{(r)}} \psi(-\mathrm{tr}(b[Q(x)]))dx\,db$$

for

$$L_1 = \mathcal{O}^4 \quad (\text{resp. } L_1 = \mathcal{O}^5)$$

where

$$S_l = \mathrm{diag}(2\kappa,-2\kappa,2\kappa,-2\kappa) \quad (\text{resp. } S_m = \mathrm{diag}(2\kappa,2\kappa,-2\kappa,2\kappa,-2\kappa))$$

is the matrix for the quadratic form in the linear (resp. metaplectic) case and $\kappa \in \mathcal{O}^\times$. Given these matrices, we can compute all of the auxiliary quantities above (see Table 4.4).

So we substitute these quantities into our formulas and produce the rational functions in Table 4.5. By summing and simplifying all these terms, we find that

$$W_0(r, L_1, L_1) = \frac{(1 - q^{-(2r+2)})(1 - q^{-(r+2)})}{(1 - q^{-(2r+1)})(1 - q^{-r})} = \frac{\zeta(2r+1)\zeta(r)}{\zeta(2r+2)\zeta(r+2)}$$

in the linear case and

$$W_0(r, L_1, L_1) = \frac{1 - q^{-(2r+4)}}{1 - q^{-(2r+1)}} = \frac{\zeta(2r+1)}{\zeta(2r+4)}$$

in the metaplectic case.

Similar calculations yield all the local densities needed for our doubling integrals. However, to use Yang's formula, one must write the matrix of our quadratic form as if the desired lattice $L \simeq \mathcal{O}^{\dim(V)}$. For instance, if $V = M_2(F)$, $Q(X) = \kappa\det(X)$ and $L_0 = L_1 \cap L_2$ from Example 4.1.1, then we see that $L_0 \simeq \mathcal{O}^4$ with respect to the matrix $S = \mathrm{diag}(2\kappa, -2\kappa, 2\varpi\kappa, -2\varpi\kappa)$. To compute $W_0(r, L_0, L_0)$, we apply Yang's formula to this matrix. We summarize all the necessary local density calculations in the Tables 4.4 and 4.5. Table 4.6 summarizes all of the quadratic spaces that will be relevant to the thesis. Notice that in this table $B^-(F)$ represents the division quaternion algebra with

$$i^2 = \varepsilon \qquad \text{and} \qquad j^2 = \varpi$$

where $\varepsilon \in \mathcal{O}^\times$ such that $\left(\frac{\varepsilon}{\varpi}\right)_F = -1$. Table 4.7 computes

$$W_0(r, L, L)$$

for all the relevant lattices.

4.3.2 Some lemmas for computing $W_0(r, L, L')$.

Now that we have computed the various values for $W_0(r, L, L)$, we need to consider how to compute the analogous result for $W_0(r, L, L')$ where the L and L' are the various lattices mentioned in the previous section and found in Table 4.6. In particular, we have

$$W_0(r, L, L') = \int_{\mathrm{Sym}_2(F)} \int_{L^{(r)} \oplus (L')^{(r)}} \psi(\mathrm{tr}(b[Q(x)]))dx\, db.$$

We will see that for lattices L, L' and $L \cap L' = L''$, $W_0(r, L, L')$ is a weighted average of $W_0(r, L^*, L^*)$ for $L^* \in \{L, L', L''\}$. In particular, we have the following proposition.

Proposition 4.3.1. *Let L_1, L_2 be the self-dual lattices from Table 4.6 and $L_0 = L_1 \cap L_2$. Then*

$$W_0(r, L_1, L_2) = W_0(r, L_0, L_1) + W_0(r, L_0, L_2) - W_0(r, L_0, L_0).$$

Proof. Let

$$L^{(r)} = \sum_{a \in \mathcal{O}/\mathcal{P}} [a] + L_0^{(r)} \quad \text{and} \quad L_2^{(r)} = \sum_{c \in \mathcal{O}/\mathcal{P}} [c] + L_0^{(r)},$$

where

$$[a] = \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}^{(r)}$$

and

$$[c] = \begin{pmatrix} 0 & c\varpi^{-1} \\ 0 & 0 \end{pmatrix}^{(r)}.$$

Note that for $M \in L_i$, $M^{(r)}$ denotes expanding M by a zero vector in $V_{r,r}$ so that $M^{(r)}$ lies in $L_i^{(r)}$. Then we find,

$$\begin{aligned} W_0(r, L_1, L_2) &= \int_{\mathrm{Sym}_2(F)} \int_{L_1^{(r)} \oplus L_2^{(r)}} \psi(\mathrm{tr}(b[Q(x)]))dx\, db \\ &= \sum_{a,b \in \mathcal{O}/\mathcal{P}} \int_{\mathrm{Sym}_2(F)} \int_{[a]+L_0^{(r)} \oplus [c]+L_0^{(r)}} \psi(\mathrm{tr}(b[Q(x)]))dx\, db. \end{aligned}$$

Next, we would like to show that for every $a, c \notin \mathcal{P}$, then

$$W_{a,c} := \int_{\mathrm{Sym}_2(F)} \int_{[a]+L_0^{(r)} \oplus [c]+L_0^{(r)}} \psi(\mathrm{tr}(b[Q(x)]))dx\, db$$

vanishes. Notice that $[Q(x)]$ is a 2×2 matrix whose i, j^{th} entry is given by $(x_i, x_j)_V$ where $(\cdot, \cdot)_V$ denotes the inner product on our quadratic space V_r. In our case, $x_1 = [a] + x_1'$ and $x_2 = [c] + x_2'$ where $x_i' \in L_0^{(r)}$. Thus,

$$(x_i, x_j)_V = \begin{cases} ([a],[a])_V + 2([a], x_1')_V + (x_1', x_1')_V & \text{if } i = j = 1 \\ ([a],[c])_V + ([a], x_2')_V + ([c], x_1')_V + (x_1', x_2')_V & \text{if } i \neq j \\ ([c],[c])_V + 2([c], x_2')_V + (x_2', x_2')_V & \text{if } i = j = 2 \end{cases}$$

Notice that for the terms above, only $([a],[c])_V \in F \setminus \mathcal{O}$. If we set $d = ([a],[c])_V$, we have that

$$[Q(x)] = \begin{pmatrix} & d \\ d & \end{pmatrix} + X$$

where $X \in \mathrm{Sym}_2(\mathcal{O})$. Let

$$w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

then $[Q(x)] = dw + X$. So we substitute this into our integral to get

$$W_{a,c} = \int_{\mathrm{Sym}_2(F)} \int_{[a]+L_0^{(r)} \oplus [c]+L_0^{(r)}} \psi(\mathrm{tr}(bdw))\psi(\mathrm{tr}(bX))dx\, db$$

So finally, we perform the translation $b \to b+tw$ where $t \in \mathcal{O}^\times$ such that $\psi(2dt) \neq 1$. Such a t exists because ψ is our standard additive character that is trivial on $\mathcal{O}$ but not on $\mathcal{P}^{-1}$. So we have

$$W_{a,c} = \psi(2td) \int_{\mathrm{Sym}_2(F)} \int_{[a]+L_0^{(r)} \oplus [c]+L_0^{(r)}} \psi(\mathrm{tr}(bdw))\psi(\mathrm{tr}(bX))\psi(\mathrm{tr}(twX))dx\, db.$$

Notice, though, that $twX \in \mathrm{Sym}_2(\mathcal{O})$ so $\psi(\mathrm{tr}(twX)) = 1$. Therefore we see that

$$W_{a,c} = \psi(2td)W_{a,c}$$

if $a, c \notin \mathcal{P}$. So

$$W_{a,c} = \int_{\mathrm{Sym}_2(F)} \int_{[a]+L_0^{(r)} \oplus [c]+L_0^{(r)}} \psi(\mathrm{tr}(b[Q(x)]))dx\, db = 0.$$

Since these terms vanish we find that

$$\begin{aligned} W_0(r, L_1, L_2) &= \sum_{a \in \mathcal{O}/\mathcal{P}} \int_{\mathrm{Sym}_2(F)} \int_{[a]+L_0^{(r)} \oplus L_0^{(r)}} \psi(\mathrm{tr}(b[Q(x)]))dx\, db \\ &\quad + \sum_{[c] \in \mathcal{O}/\mathcal{P}} \int_{\mathrm{Sym}_2(F)} \int_{L_0^{(r)} \oplus [c]+L_0^{(r)}} \psi(\mathrm{tr}(b[Q(x)]))dx\, db \\ &\quad - \int_{\mathrm{Sym}_2(F)} \int_{L_0^{(r)} \oplus L_0^{(r)}} \psi(\mathrm{tr}(b[Q(x)]))dx\, db. \end{aligned}$$

□

So we are reduced to the case that $L \subset L'$ or vice versa. We will show that the quantity $W_0(r, L, L')$ is a weighted average of the terms $W_0(r, L, L)$ and $W_0(r, L', L')$. In particular, we have the following lemma.

Lemma 4.3.1. *Let $L \subset L' \subset V$ be lattices lattices from Table 4.6 such that one of the following holds:*

1. $V = V_s$, $L = L_0$ *and* $L' \in \{L_1, L_2, L_0^\#\}$

2. $V = V_s$, $L \in \{L_1, L_2\}$ *and* $L' = L_0^\#$.

3. $V = V_{ra}$, $L = L_{ra}$ *and* $L' = L_{ra}^\#$.

Then we have,

$$W_0(r, L, L') = \frac{1}{q+1}\left(qW_0(r, L, L) + W_0(r, L', L')\right). \tag{4.3}$$

Proof. To prove this statement, we appeal to some results of Yang's [42]. In particular, Proposition 5.1 in [42] states that for f a locally-constant bounded function on $\mathrm{Sym}_2(F)$ such that $f \in L^1(\mathrm{Sym}_2(F))$. Then

$$\begin{aligned}\int_{\mathrm{Sym}_2(F)} f(b)db \;&=\; \sum_{\alpha\in\mathbb{Z}} q^{-3\alpha}\Bigg(\frac{1}{2(1-(\frac{-1}{\varpi})_F q^{-1})}\int_{\mathrm{GL}_2(\mathcal{O})} f(q^\alpha\, b^t b)db \\ &+\; \frac{1}{2(1+(\frac{-1}{\varpi})_F q^{-1})}\int_{\mathrm{GL}_2(\mathcal{O})} f(q^\alpha b\mathrm{diag}(1,u)\,{}^t b)db\Bigg) \\ &+\; \frac{1}{4}\sum_{\substack{\alpha_1<\alpha_2 \\ u_i\in\{1,u\}}} q^{-2\alpha_1-\alpha_2}\int_{\mathrm{GL}_2(\mathcal{O})} f(b\mathrm{diag}(u_1 q^{\alpha_1}, u_2 q^{\alpha_2})\,{}^t b)db\end{aligned}$$

where $u \in \mathcal{O}^\times$ with $\left(\frac{u}{\varpi}\right)_F = -1$. Ultimately, we will apply this Theorem in a manner similar to that in Yang [42]. For our purposes,

$$f(b) = \int_{L^{(r)}\oplus(L')^{(r)}} \psi(-\mathrm{tr}(b[Q(x)]))dx,$$

whereas Yang uses a similar function twisted by a $\psi(\mathrm{Tb})$ with T a symmetric matrix. So to derive our formula, we want to compute the following integral

$$\Lambda(L, L'; t_1, t_2) = \int_{\mathrm{GL}_2(\mathcal{O})}\int_{L^{(r)}\oplus(L')^{(r)}} \psi(-\mathrm{tr}(b\mathrm{diag}(t_1,t_2)\,{}^t b[Q(x)]))dx\,db.$$

We will carefully prove (4.3) in the case $[L' : L] = q$. For now, let $L' = \sum_{a\in\mathcal{O}/\mathcal{P}} ax + L$.

Next, we will decompose $K = \mathrm{GL}_2(\mathcal{O})$ into left cosets of the standard Iwahori subgroup J (the inverse image of the upper triangular matrices in $\mathrm{GL}_2(\mathcal{O}/\mathcal{P})$). In particular,

$$K = J \cup \bigcup_{c \in \mathcal{O}/\mathcal{P}} \mathbf{n}(c) w J$$

where $w = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$ and $\mathbf{n}(c) = \begin{pmatrix} 1 & c \\ & 1 \end{pmatrix}$.

Next we notice that

$$(L^{(r)} \oplus (L')^{(r)}) \cdot b = L^{(r)} \oplus (L')^{(r)}$$

for $b \in J$. So we can then rewrite the integral according to our decomposition. This yields,

$$\begin{aligned} \Lambda(L, L'; t_1, t_2) &= \int_J \int_{L^{(r)} \oplus (L')^{(r)}} \psi(-\mathrm{tr}(b\mathrm{diag}(t_1, t_2)\, {}^t b[Q(x)]))dx\, db \\ &+ \sum_{c \in \mathcal{O}/\mathcal{P}} \int_{\mathbf{n}(c)wJ} \int_{L^{(r)} \oplus (L')^{(r)}} \psi(-\mathrm{tr}(b\mathrm{diag}(t_1, t_2)\, {}^t b[Q(x)]))dx\, db. \end{aligned}$$

Now, we have already shown that the inner most integral is invariant under transformation by J so we can integrate it out of our expression which yields,

$$\begin{aligned} \Lambda(L, L'; t_1, t_2) &= \mathrm{vol}(J) \left(\int_{L^{(r)} \oplus (L')^{(r)}} \psi(-\mathrm{tr}(\mathrm{diag}(t_1, t_2)[Q(x)]))dx \right. \\ &+ \left. \sum_{a, c \in \mathcal{O}/\mathcal{P}} \int_{L^{(r)} \oplus ax + L^{(r)}} \psi(-\mathrm{tr}(w\mathbf{n}(c)\mathrm{diag}(t_1, t_2)\mathbf{n}_-(c)w[Q(x)]))dx \right). \end{aligned}$$

Next we note that $x \to x \cdot w$ interchanges the lattices and the transformation $x \to x\mathbf{n}(-c)$ translates the lattices. Thus our expression becomes,

$$\begin{aligned} \Lambda(L, L'; t_1, t_2) &= \mathrm{vol}(J) \int_{L^{(r)} \oplus (L')^{(r)}} \psi(-\mathrm{tr}(\mathrm{diag}(t_1, t_2)[Q(x)]))dx \\ &+ \mathrm{vol}(J) \sum_{a, c \in \mathcal{O}} \int_{ax + L^{(r)} \oplus acx + L^{(r)}} \psi(-\mathrm{tr}(\mathrm{diag}(t_1, t_2)[Q(x)]))dx. \end{aligned} \tag{4.4}$$

By a simple reordering how we sum the integrals in (4.4), we realize that

$$\begin{aligned}\Lambda(L,L';t_1,t_2) &= \mathrm{vol}(J)\left(q\int_{L^{(r)}\oplus L^{(r)}}\psi(-\mathrm{tr}(\mathrm{diag}(t_1,t_2)[Q(x)]))dx\right.\\ &\quad\left.+\int_{L'^{(r)}\oplus L'^{(r)}}\psi(-\mathrm{tr}(\mathrm{diag}(t_1,t_2)[Q(x)]))dx\right) \qquad (4.5)\end{aligned}$$

When $[L':L]>q$, we find that the argument above doesn't quite follow as simply. For the general case, we need to add several integrals, having the same integrand as in (4.5), but with domains of integration of the form $y_1+L^{(r)}\oplus y_2+L^{(r)}$ where $y_1-cy_2\notin L^{(r)}$ for any $c\in\mathcal{O}/\mathcal{P}$. However, these terms are essentially the $W_{a,c}$ in Proposition 4.3.1 which vanish.

So we have reduced our problem of an integral over $L^{(r)}\oplus(L')^{(r)}$ to a problem involving a weighted average of $(L^*)^{(r)}\oplus(L^*)^{(r)}$. Further we notice that for any $b\in\mathrm{GL}_2(\mathcal{O})$, we have $(L^*)^{(r)}\oplus(L^*)^{(r)}\cdot b=(L^*)^{(r)}\oplus(L^*)^{(r)}$. Therefore,

$$\begin{aligned}&\int_{(L^*)^{(r)}\oplus(L^*)^{(r)}}\psi(-\mathrm{tr}(\mathrm{diag}(t_1,t_2)[Q(x\cdot b)]))dx\\ &\qquad=\int_{(L^*)^{(r)}\oplus(L^*)^{(r)}}\psi(-\mathrm{tr}(\mathrm{diag}(t_1,t_2)[Q(x)]))dx\end{aligned}$$

for all $b\in\mathrm{GL}_2(\mathcal{O})$. Finally, we see that

$$\begin{aligned}&\int_{(L^*)^{(r)}\oplus(L^*)^{(r)}}\psi(-\mathrm{tr}(\mathrm{diag}(t_1,t_2)[Q(x)]))dx\\ &=\mathrm{vol}(\mathrm{GL}_2(\mathcal{O}))^{-1}\int_{\mathrm{GL}_2(\mathcal{O})}\int_{(L^*)^{(r)}\oplus(L^*)^{(r)}}\psi(-\mathrm{tr}(b\mathrm{diag}(t_1,t_2)^t b[Q(x)]))dx\,db\end{aligned}$$

and

$$\Lambda(L,L';t_1,t_2)=\frac{\mathrm{vol}(J)}{\mathrm{vol}(\mathrm{GL}_2(\mathcal{O}))}\left(q\Lambda(L,L;t_1,t_2)+\Lambda(L',L';t_1,t_2)\right).$$

Finally, we realize that

$$\mathrm{vol}(J)=(q+1)^{-1}\mathrm{vol}(\mathrm{GL}_2(\mathcal{O})),$$

so we ultimately find

$$W_0(r, L, L') = \frac{1}{q+1}(qW_0(r, L, L) + W_0(r, L', L')).$$

$\square$

Finally, it is worth noting that the expression for $W_0(r, L, L')$ is invariant under the transformations $x \to xw$ and $b \to wbw$. This sequence of transformations exchange the lattices so we have that $W_0(r, L, L') = W_0(r, L', L)$. So by combining Proposition 4.3.1 and the Lemma 4.3.1, we get the following corollary.

Corollary 4.3.1. *Let L, L' and $L'' = L \cap L'$ be lattices from Table 4.6. Then we have*

$$W_0(r, L, L') = \frac{1}{q+1}((q-1)W_0(r, L'', L'') + W_0(r, L, L) + W_0(r, L', L')).$$

Proof. Simply substitute the formula from Lemma 4.3.1 into the formula from Proposition 4.3.1. $\square$

Table 4.4: **Intermediate computations for the Whittaker functions in spherical case.**

	S_l	S_m
$L(k,1)$	$= \begin{cases} \emptyset & \text{if } k \text{ even} \\ \{1,2,3,4\} & \text{if } k \text{ odd} \end{cases}$	$= \begin{cases} \emptyset & \text{if } k \text{ even} \\ \{1,2,3,4,5\} & \text{if } k \text{ odd} \end{cases}$
$l(k,1)$	$= \begin{cases} 0 \text{ if } k \text{ even} \\ 4 \text{ if } k \text{ odd} \end{cases}$	$= \begin{cases} 0 \text{ if } k \text{ even} \\ 5 \text{ if } k \text{ odd} \end{cases}$
$d(k)$	$= -k$	$= -\frac{3}{2}k$
$v(k)$	$= 1$	$= \begin{cases} 1 & \text{if } k \text{ even} \\ \left(\frac{2\kappa}{\varpi}\right) & \text{if } k \text{ odd} \end{cases}$
$\delta^+(k)$	$= 1$	$= \begin{cases} 1 \text{ if } k \text{ even} \\ 0 \text{ if } k \text{ odd} \end{cases}$
$\delta^-(k)$	$= 0$	$= \begin{cases} 0 \text{ if } k \text{ even} \\ 1 \text{ if } k \text{ odd} \end{cases}$

Table 4.5: **Rational functions used to compute $W_0(r, L, L)$, spherical case**

	S_l	S_m
$I_{1,1}$	$= \frac{(1-q^{-2})X}{1-X}$	$= \frac{(1-q^{-2})q^{-1}X^2}{1-q^{-1}X^2}$
$I_{2,1}$	$= \frac{(1-q^{-2})q^{-1}X^3}{(1-X)(1-q^{-1}X^2)}$	$= \frac{(1-q^{-2})q^{-5}X^6}{(1-q^{-1}X^2)(1-q^{-4}X^4)}$
$I_{2,8}$	$= \frac{q^{-1}X^2}{1-q^{-1}X^2}$	$= \frac{q^{-3}X^2+q^{-4}X^4}{1-q^{-4}X^4}$

Table 4.6: **Relevant quadratic spaces**

	$\mathrm{Sp}_2(F)$	$\widetilde{\mathrm{Sp}_2}(F)$
(V^s, Q)	$V^{s_l} = (M_2(F), \kappa\det)$	$V^{s_m} = (F^5, Q(x_0, \dots, x_4))$ $Q(\mathbf{x}) = \kappa(x_0^2 - x_1x_2 - x_3x_4)$
L_1	$M_2(\mathcal{O})$	$\mathcal{O}^5$
L_2	$\begin{pmatrix} 1 & \\ & \varpi \end{pmatrix} M_2(\mathcal{O}) \begin{pmatrix} 1 & \\ & \varpi^{-1} \end{pmatrix}$	$\mathcal{O} \oplus \varpi^{-1}\mathcal{O} \oplus \varpi\mathcal{O} \oplus \mathcal{O}^2$
L_0	$L_1 \cap L_2$	$L_1 \cap L_2$
(V^{ra}, Q)	$V^{ra_l} = (B^-(F), \kappa N)$	$V^{ra_m} = (F^5, Q(x_0, \dots, x_4))$ $Q(\mathbf{x}) = \kappa(\varepsilon x_0^2 + \varpi x_1^2 - \varepsilon\varpi x_2^3 - x_3x_4)$
L_{ra}	$B^-(\mathcal{O})$	$\mathcal{O}^5$

Table 4.7: **Local densities required for the doubling integrals.**

	$\mathrm{Sp}_2(F)$	$\widetilde{\mathrm{Sp}_2}(F)$
$W_0(r, L_1, L_1)$	$\frac{(1-q^{-(2r+2)})(1-q^{-(r+2)})}{(1-q^{-(2r+1)})(1-q^{-r})}$	$\frac{1-q^{-(2r+4)}}{1-q^{-(2r+1)}}$
$W_0(r, L_0, L_0)$	$\frac{1}{q^2}\frac{(1-q^{-(r+1)})(1+q^{-(r-1)}-q^{-r}+q^{-(2r-1)}-2q^{-(2r+1)})}{(1-q^{-(2r+1)})(1-q^{-r})}$	$\frac{1}{q^2}\frac{1+q^{-2r}-2q^{-(2r+2)}}{1-q^{-(2r+1)}}$
$W_0(r, L_{ra}, L_{ra})$	$\frac{1}{q^2}\frac{(1+q^{-(r+1)})(1-q^{-(r-1)})}{1-q^{-(2r+1)}}$	$\frac{1}{q^2}\frac{1-q^{-2r}}{1-q^{-(2r+1)}}$

Chapter 5

The Main Calculation

Now that we have set up the necessary machinery, we are ready to compute L-factors. Note that because the formulas in this section essentially encompass both the linear and metaplectic group cases simultaneously, G (resp. H) will be used to denote either $\mathrm{Sp}_2(F)$ or $\widetilde{\mathrm{Sp}_2}(F)$ (resp. $\mathrm{Sp}_4(F)$ or $\widetilde{\mathrm{Sp}_4}(F)$). We will also suppress the tilde notation from the various parabolic subgroups.

Let $f \in \mathrm{Ind}_{P_\emptyset}^G((\chi_1 \otimes \chi_2)')$. Then for any $\{\Phi_s\}_{\mathrm{Re}(s)>>0}$ with

$$\Phi_s \in \mathrm{Ind}_{P_H}^H((\chi_V|\cdot|^s)') = I(s, \chi_V),$$

we have the doubling integral [22]

$$Z(s, \Phi, f)(g_0') = \int_{\mathrm{Sp}_2(F)} \Phi_s(\delta' i(g_0', g'))f(g')\,dg$$

Using the Iwasawa decomposition $G = P_\emptyset K$ with $N_\emptyset = N_2 N_1$ (see Notation page), we get the following decomposition of the doubling integral,

$$\begin{aligned} Z(s, \Phi, f)(g_0') \quad = \quad & \int_K \int_{(F^\times)^2} \int_F \int_{\mathrm{Sym}_2(F)} \Phi_s(\delta' i(g_0', \mathbf{n_2}(b)\mathbf{m}\begin{pmatrix} a_1 & c \\ & a_2 \end{pmatrix} k')) \\ & \times f(\mathbf{n_2}(b)\mathbf{m}\begin{pmatrix} a_1 & c \\ & a_2 \end{pmatrix} k')db\,dc\,d^\times a\,dk. \end{aligned} \tag{5.1}$$

where $g' \in G$ is any element projecting onto $g \in \mathrm{Sp}_2(F)$.

5.1 Constructing Φ_{s_0+r}

Next we employ our interpolation method to construct our Φ_{s_0+r}. Let V be one of the quadratic spaces listed in Table 4.6. Thus our interpolation method will produce a vector in $\mathrm{Ind}_{P_H}^H(\chi_V|\cdot|^{s_0+r})$ where

$$s_0 = \begin{cases} -\frac{1}{2} \text{ if } G = \mathrm{Sp}_2(F) \\ \;\;0 \;\text{ if } G = \widetilde{\mathrm{Sp}_2}(F) \end{cases}.$$

So for $\varphi_i \in S(V^2)$, we have

$$\Phi_{s_0+r} = \lambda_V(\varphi_1^{(r)} \otimes \overline{\varphi_2^{(r)}}).$$

Lemma 5.1.1. *Suppose that* $\Phi_{s_0+r} = \lambda_V(\varphi_1^{(r)} \otimes \overline{\varphi_2^{(r)}})$*, then*

$$\Phi_{s_0+r}(\delta' i(g_0', g')) = \gamma(V_r) \int_{V_r^2} \omega_{V_r}(g_0')\varphi_1^{(r)}(x) \cdot \overline{\omega_{V_r}(g')\varphi_2^{(r)}(-x)}dx$$

where δ and γ are defined in section 4.1.2.

Proof. Let $\alpha = \begin{pmatrix} I & -I \\ & I \end{pmatrix}$. We have already seen that

$$\begin{aligned}\Phi_{s_0+r}(\delta' i(g_0', g')) &= \omega_{V_r}\left([w_1, 1]_L[\mathbf{m}(\alpha), 1]_L i(g_0', g')\right)(\varphi_1^{(r)} \otimes \overline{\varphi_2^{(r)}})(0) \\ &= \omega_{V_r}([w_1, 1]_L[\mathbf{m}(\alpha), 1]_L)(\omega_{V_r}(g_0')\varphi_1^{(r)} \otimes \overline{\omega_{V_r}(g')\varphi_2^{(r)}})(0).\end{aligned}$$

Letting $\phi_1 = \omega_{V_r}(g_0')\varphi_1^{(r)}$ and $\phi_2 = \omega_{V_r}(g')\varphi_2^{(r)}$, we get

$$\begin{aligned}\omega_{V_r}([w_1, 1]_L[\mathbf{m}(\alpha), 1]_L)(\phi_1 \otimes \overline{\phi_2})(0) &= \gamma(V_r) \int_{V_r} \omega_{V_r}(\mathbf{m}(\alpha))(\phi_1 \otimes \overline{\phi_2})(x, 0)dx \\ &= \gamma(V_r) \int_{V_r} \phi_1(x)\overline{\phi_2(-x)}dx\end{aligned}$$

Substituting the φ_i back into the equation gives us our result. □

It is also worth noting that

$$\chi_{V_r}(x) := (x, (-1)^{\frac{(m+2r)(m+2r-1)}{2}} \det(V_r))$$

where $m = \dim_F(V_r)$. However, a routine calcultion shows that

$$(-1)^{\frac{(m+2r)(m+2r-1)}{2}} \det(V_r) = (-1)^{\frac{m(m-1)}{2}} \det(V)$$

so $\chi_{V_r} = \chi_V$. Lastly, by another elementary calculation, we see that

$$\chi_V(x) = \begin{cases} 1 & \text{if } G = \mathrm{Sp}_2(F) \\ (x, 2\kappa)_F & \text{if } G = \widetilde{\mathrm{Sp}_2}(F) \end{cases} .$$

We now substitute this into our integral which yields

$$\begin{aligned} Z(s_0 + r, \Phi, f)(g_0') &= \int_K \int_{(F^\times)^2} \int_F \int_{\mathrm{Sym}_2(F)} \int_{V_r^2} \omega_{V_r}(g_0')\varphi_1^{(r)}(x) \\ &\quad \overline{\times \omega_{V_r}(\mathbf{n_2}(b)\mathbf{m}\begin{pmatrix} a_1 & c \\ & a_2 \end{pmatrix} k')\varphi_2^{(r)}(x) dx} \\ &\quad \times f(\mathbf{n_2}(b)\mathbf{m}\begin{pmatrix} a_1 & c \\ & a_2 \end{pmatrix} k') db\, dc\, d^\times a\, dk \\ &= \int_K \int_{(F^\times)^2} \int_F \int_{\mathrm{Sym}_2(F)} \int_{V_r^2} \omega_{V_r}(g_0')\varphi_1^{(r)}(x) \\ &\quad \overline{\times \omega_{V_r}(\mathbf{n_2}(b)\mathbf{m}\begin{pmatrix} a_1 & c \\ & a_2 \end{pmatrix} k')\varphi_2^{(r)}(x)\, dx} \qquad (5.2) \\ &\quad \times |a_1|^{-4}|a_2|^{-3} f(\mathbf{m}(\mathrm{diag}(a_1, a_2))\mathbf{n_1}(c)\mathbf{n_2}(b)k') db\, dc\, d^\times a\, dk \end{aligned}$$

where the factor $|a_1|^{-4}|a_2|^{-3}$ comes from the transformations

$$\mathbf{n_2}(b) \mapsto \mathbf{m}\begin{pmatrix} a_1 & c \\ & a_2 \end{pmatrix} \mathbf{n_2}(b)\mathbf{m}\begin{pmatrix} a_1 & c \\ & a_2 \end{pmatrix}^{-1}$$

and $c \mapsto ca_1$. Next, we use the fact that $f \in \mathrm{Ind}_{P_\emptyset}^G((\chi_1 \otimes \chi_2)')$ and compute the Weil representation to arrive at

$$
\begin{aligned}
&Z(s_0 + r, \Phi, f)(g_0') \\
&= \int_{(F^\times)^2} \chi_V(a_1 a_2)\chi_1(a_1)\chi_2(a_2)|a_1 a_2|^{-2}|a_1 a_2|^{\frac{5+2s_0+2r}{2}} \\
&\times \int_F \int_{\mathrm{Sym}_2(F)} \int_{V_r^2} \omega_{V_r}(g_0)\varphi_1^{(r)}(x) \\
&\times \overline{\int_K f(k)[\omega_{V_r}(k)\varphi_2^{(r)}](-x \begin{pmatrix} a_1 & c \\ & a_2 \end{pmatrix})dk} \\
&\times \psi(-\mathrm{tr}(b[Q(x)]))dx\, db\, dc\, d^\times a \\
&= \int_{(F^\times)^2} \chi_V \chi_1(a_1)\chi_V \chi_2(a_2)|a_1 a_2|^{r+\frac{1}{2}+s_0} \int_F \int_{\mathrm{Sym}_2(F)} \int_{V_r^2} \omega_{V_r}(g_0)\varphi_1^{(r)}(x) \\
&\times I(f, \varphi_2^{(r)})(-x \begin{pmatrix} a_1 & c \\ & a_2 \end{pmatrix})\psi(-\mathrm{tr}(b[Q(x)]))dx\, db\, dc\, d^\times a
\end{aligned}
\tag{5.3}
$$

Note that, in practice $g_0 = 1_{\mathrm{Sp}_2}$.

In this final expression we have

$$
I(f, \varphi^{(r)})(x) := \int_K f(k)\overline{\omega_{V_r}(k)\varphi^{(r)}(x)}dk.
$$

This calculates the action of f averaged over K on $\varphi^{(r)}(x)$ and turns out to be one of the critical computations in our method. It is also worth noting that $I(f, \varphi^{(r)})(x) \in S(V_r^2)$; moreover, we will eventually show that $I(f, \varphi^{(r)})(x) = I(f, \varphi)^{(r)}(x)$. So we see our method of augmenting the dimension of V is very natural with respect to our interpolation method.

5.1.1 Choosing the characteristic functions

At this point, we must make a choice for our functions φ_i. As has been stated previously, we will be interested in characteristic functions of lattices $L \oplus L' \subset V^2$ where $L^* \subset V$ are good latices (as defined in Section 4.1.3). In particular, we will return to the quadratic spaces (and notation) from Table 4.6. The next step in our calculation is to compute $I(f,\varphi)(x)$, when we take φ as above and f to be an Iwahori fixed vector. So let us define

$$\varphi_{L\oplus L'}(x) := \mathrm{char}_{L\oplus L'}(x)$$

with L and L' good lattices. We see that $\varphi_{L\oplus L'}(x)$ is $I_\emptyset$-invariant (resp. $\overline{I_\emptyset}$-invariant) if $L \subset L'$ (resp. $L' \subset L$).

Let us now consider the calculations $I(f,\varphi)(x)$ for $\varphi(x) \in S(V^2)$. After that, we will show that

$$I(f,\varphi^{(r)})(x) = I(f,\varphi)^{(r)}(x).$$

In order to compute our doubling integrals, we will primarily be interested in computing $I(f,\varphi)$ for the following characteristic functions: $\varphi_{L_i\oplus L_i}$, $\varphi_{L_0\oplus L_i}$ and $\varphi_{L_{ra}\oplus L_{ra}}$ with $i \in \{1,2\}$. A routine computation shows that the characteristic function for each product of lattices is invariant under $I_\emptyset$ via the Weil representation. We will see that this allows us to write our integral as a finite sum. In particular, we have that

$$\mathrm{Sp}_2(\mathcal{O}) = \bigcup_{w\in W_{\mathrm{Sp}_2}} N_\emptyset(\mathcal{O})wI_\emptyset$$

where W_{Sp_2} is the Weyl group of Sp_2 and $N_\emptyset$ is the unipotent radical of $P_\emptyset = N_\emptyset M_\emptyset$.

However, we can refine this decomposition even more. Let us define some subgroups of $N_\emptyset$ that depend on the various elements of $W_G = W_{\mathrm{Sp}_2}$.

Let W_G now denote a fixed set of Weyl group representatives that we shall explicitly describe below. Then for each $w \in W_G$, let $\mathrm{Inn}(w) : G \to G$ such that $\mathrm{Inn}(w)(g) := wgw^{-1}$. Also let $\bar{N}_\emptyset$ be the opposite unipotent subgroup of $N_\emptyset$ (alternatively, $\bar{N}_\emptyset = \mathrm{Inn}(w_{\alpha\beta\alpha\beta})N_\emptyset$ for the long Weyl group element $w_{\alpha\beta\alpha\beta} \in W_G$). Thus for any $w \in W_G$, we define $N^w := \{n \in N_\emptyset | \mathrm{Inn}(w)(n) \in \bar{N}_\emptyset\}$. This gives us our first refinement of

$$\mathrm{Sp}_2(\mathcal{O}) = \bigcup_{w \in W_G} N^w(\mathcal{O}) w I_\emptyset.$$

We also notice that for $n \in N^w(\varpi)$, $\mathrm{Inn}(w)(n) \in I_\emptyset$. Therefore,

$$\mathrm{Sp}_2(\mathcal{O}) = \bigcup_{w \in W_G} (N^w(\mathcal{O})/N^w(\varpi)) w I_\emptyset,$$

a finite union of cosets of I_ϕ.

So if we choose a Haar measure on $\mathrm{Sp}_2(F)$, in particular so $K = \mathrm{Sp}_2(\mathcal{O})$ has measure one, our integral becomes

$$I(f, \varphi)(x) = \mathrm{vol}(I_\emptyset) \sum_{w \in W_G} f(w) \sum_{n \in N^w(\mathcal{O})/N^w(\varpi)} \overline{(\omega_V(nw)\varphi)(x)}.$$

Consequently, we need only compute the action of certain Weyl group elements on the factorizable function $\varphi_{L \oplus L'}(x) = \varphi_L(x_1) \otimes \varphi_{L'}(x_2)$ via the Weil representation. We now consider this calculation for a factorizable $\varphi = \varphi_1 \otimes \varphi_2 \in S(V^2)$. To do this, let

$$w_\alpha = \begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{pmatrix} \qquad w_\beta = \begin{pmatrix} 1 & & & \\ & & & 1 \\ & & 1 & \\ & -1 & & \end{pmatrix}.$$

Via some very simple calculations we see that

$$\begin{aligned}
r_V(w_\alpha)\varphi(x) &= |\det(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})|^{\frac{m}{2}}\varphi(x\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}) = \varphi_2(x_1)\otimes\varphi_1(x_2) \\
r_V(w_\beta)\varphi(x) &= \int_V \psi(\mathrm{tr}[\frac{1}{2}(x\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, x\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}) \\
&+ (x\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, y\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix})+\frac{1}{2}(y\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, y\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix})]) \\
&\times \varphi(x\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + y\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix})dy_2 \\
&= \int_V \psi((x_2,-y_2))\varphi_1(x_1)\otimes\varphi_2(-y_2)dy_2 \\
&= \varphi_1(x_1)\otimes\left(\int_V \psi((x_2,-y_2))\varphi_2(-y_2)dy_2\right) \\
&= \varphi_1(x_1)\otimes\widehat{\varphi_2}(x_2).
\end{aligned}$$

So ultimately we see that

$$\omega_V([w_\alpha,1]_L)\varphi(x) = \chi_V(x(w_\alpha))\gamma(\eta\circ V)^{-j(w_\alpha)}\varphi_2\otimes\varphi_1(x) \tag{5.4}$$

$$\omega_V([w_\beta,1]_L)\varphi(x) = \chi_V(x(w_\beta))\gamma(\eta\circ V)^{-j(w_\beta)}\varphi_1\otimes\hat{\varphi}_2(x) \tag{5.5}$$

using our expression for $\omega_V([g,z]_L)\varphi(x)$. Also, from Section 3.2, we have

$$x(w_\alpha) = x(w_\beta) = 1,$$

$$j(w_\alpha) = 0, \qquad\qquad j(w_\beta) = 1$$

and

$$\gamma(\eta \circ V) = \gamma(\det(V), \eta)\gamma(\eta)^4\epsilon(V) = \begin{cases} 1 \text{ if } V \in \{V^{s_l}, V^{s_m}\} \\ -1 \text{ if } V \in \{V^{ra_l}, V^{ra_m}\} \end{cases}.$$

We would also like to compute the action of $\omega_V(\mathbf{n_2}(b))$ on a function $\varphi(x)$ for $n_2(b) \in N_2(\mathcal{O})$. Notice that because N_2 is an abelian subgroup of G, we can compute this action individually for each variable b_{11}, $b_{12} = b_{21}$ and b_{22}. Also notice that for $x(\mathbf{n_2}(b)) = 1$ and $j(\mathbf{n_2}(b)) = 0$, thus $\omega_V(\mathbf{n_2}(b))\varphi(x) = r_V(\mathbf{n_2}(b))\varphi(x)$. So we compute each individually (with $x = (x_1, x_2)$). For b_{11},

$$\begin{aligned} r_V(\mathbf{n_2}\begin{pmatrix} b_{11} & 0 \\ 0 & 0 \end{pmatrix})\varphi(x) &= \psi\left(\frac{1}{2}\mathrm{tr}(x, x\begin{pmatrix} b_{11} & 0 \\ 0 & 0 \end{pmatrix})\right)\varphi(x) \\ &= \psi(\frac{b_{11}}{2}Q(x_1))\varphi(x_1, x_2). \end{aligned}$$

Thus

$$\sum_{b_{11} \in \mathcal{O}/\mathcal{P}} \psi(\frac{b_{11}}{2}Q(x_1))\varphi(x_1, x_2) = q\mathrm{ch}_{\mathcal{O}}(Q(x_1))\varphi(x). \tag{5.6}$$

Notice that $r_V(\mathbf{n_2}\begin{pmatrix} 0 & 0 \\ 0 & b_{22} \end{pmatrix})\varphi(x)$ is handled almost identically yielding the following results

$$\begin{aligned} r_V(\mathbf{n_2}\begin{pmatrix} 0 & 0 \\ 0 & b_{22} \end{pmatrix})\varphi(x) &= \psi(\frac{b_{22}}{2}Q(x_2))\varphi(x_1, x_2) \\ \sum_{b_{22} \in \mathcal{O}/\mathcal{P}} \psi(\frac{b_{22}}{2}Q(x_2))\varphi(x_1, x_2) &= q\mathrm{ch}_{\mathcal{O}}(Q(x_2))\varphi(x). \end{aligned} \tag{5.7}$$

Finally, in the case where $b_{12} = b_{21}$, we find

$$\begin{aligned} r_V(\mathbf{n_2}\begin{pmatrix} 0 & b_{12} \\ b_{12} & 0 \end{pmatrix})\varphi(x) &= \psi\left(\frac{1}{2}\mathrm{tr}(x, x\begin{pmatrix} 0 & b_{12} \\ b_{12} & 0 \end{pmatrix})\right)\varphi(x) \\ &= \psi(b_{12}(x_1, x_2))\varphi(x_1, x_2). \end{aligned}$$

Thus,

$$\sum_{b_{12}\in\mathcal{O}/\mathcal{P}} \psi(b_{12}(x_1, x_2))\varphi(x_1, x_2) = q\mathrm{ch}_{\mathcal{O}}((x_1, x_2))\varphi(x). \tag{5.8}$$

Thus if we apply equations $(5.6)-(5.8)$ for a function $\varphi_{L\oplus L'}(x)$ for L and L' lattices in V, we get the following proposition.

Proposition 5.1.1. *Let L and L' be lattices in V and let $\varphi(x) = \varphi_{L\oplus L'}(x)$, then we have the following:*

$$\begin{aligned} \sum_{b_{11}\in\mathcal{O}/\mathcal{P}} \psi(\frac{b_{11}}{2}Q(x_1))\varphi_{L\oplus L'}(x) &= q\varphi_{(L\cap L^{\#})\oplus L'}(x) \\ \sum_{b_{22}\in\mathcal{O}/\mathcal{P}} \psi(\frac{b_{22}}{2}Q(x_2))\varphi_{L\oplus L'}(x) &= q\varphi_{L\oplus(L'\cap(L')^{\#})}(x) \\ \sum_{b_{12}\in\mathcal{O}/\mathcal{P}} \psi(b_{12}(x_1, x_2))\varphi_{L\oplus L'}(x) &= q\varphi_{L\oplus(L'\cap L^{\#})}(x) + q\varphi_{(L\cap(L')^{\#})\oplus L'}(x) \\ &\quad -q\varphi_{(L\cap(L')^{\#})\oplus(L'\cap L^{\#})}(x). \end{aligned}$$

Also note that $\omega_V(\mathbf{n_1}(a))\varphi(x) = \varphi_1(x_1)\otimes\varphi_2(x_2 + ax_1)$. Although the formula for

$$\sum_{a\in\mathcal{O}/\mathcal{P}} \omega_V(\mathbf{n_1}(a))\varphi(x)$$

doesn't generally simplify in any particularly nice way, there are two cases that do have simple expressions. They occur when $\varphi = \varphi_{L\oplus L'}$ and L and L' are good lattices with $L \subseteq L'$ or $L' \subset L$ and $[L : L'] = q$. In these cases,

$$\sum_{a \in \mathcal{O}/\mathcal{P}} \varphi_L(x_1) \otimes \varphi_{L'}(x_2 + ax_1) = \begin{cases} q\varphi_{L \oplus L'}(x) & \text{if } L \subseteq L' \\ \begin{pmatrix} q\varphi_{L \oplus L}(x) + \varphi_{L' \oplus L'}(x) \\ -\varphi_{L' \oplus L}(x) \end{pmatrix} & \text{if } [L : L'] = q. \end{cases} \tag{5.9}$$

It is worth noting that in order to extend our doubling method to more exotic representations, we would need to find another reasonable simplification when $L' \subset L$ and $[L : L'] = q^n$ with $n > 1$. Those sums involve much more complicated terms.

Using our formulas for the Weil representation on the necessary factorizable functions, we include a table at the end of the chapter compiling the various $I(f, \varphi)$ needed in this thesis (see Table 5.1).

There is another important property of $I(f, \varphi)(x)$ that we use over the course of our calculation. Recall that we defined

$$V_r = V + V_{r,r}$$

where $V_{r,r}$ is the orthogonal direct product of r copies of the space (F^2, Q) with $Q(x, y) = 2xy$ (i.e., $V_{r,r}$ is the orthogonal direct sum of r hyperplanes). Then, for $\varphi^{(r)} = \varphi \otimes \varphi_r^0$, we would like to show the following proposition.

Proposition 5.1.2.

$$I(f, \varphi^{(r)})(x) = I(f, \varphi)^{(r)}(x).$$

Proof. Notice that $\omega_{V_r}(g) = \omega_V(g) \otimes \omega_{V_{r,r}}(g)$ and the simple calculation

$$\begin{aligned} I(f, \varphi^{(r)})(x, x_r) &= \int_K f(k)\overline{\omega_{V_r}(k)\varphi^{(r)}(x, x_r)}dk \\ &= \int_K f(k)\overline{\omega_V(k)\varphi(x) \otimes \omega_{V_{r,r}}(k)\varphi_r^0(x_r)}dk \\ &= \int_K f(k)\overline{\omega_V(k)\varphi(x)} \otimes \varphi_r^0(x_r)dk = I(f, \varphi)(x)^{(r)}. \end{aligned}$$

Notice that φ_r^0 is a K-invariant characteristic function. $\square$

Finally, we notice that by (5.4), (5.5), (5.9) along with Propositions 5.1.1, we have the following lemma.

Lemma 5.1.2. *Let L and L' be lattices in V with $L \subset L'$ or $[L : L'] = q$ and $\varphi_{L\oplus L'}$ as above, then*

$$I(f, \varphi_{L\oplus L'})(x)$$

is a linear combination of functions $\varphi_{L_i\oplus L_j}(x)$ with L_i and L_j lattices in V.

5.1.2 Reduction of Whittaker functions to local densities

By Lemma 5.1.2, it now suffices for us to compute the integral of the form

$$\int_{(F^\times)^2} \chi_V\chi_1(a_1)\chi_V\chi_2(a_2)|a_1a_2|^{r+\frac{1}{2}+s_0} \int_F \int_{\mathrm{Sym}_2(F)} \int_{V_r^2} \varphi^{(r)}_{L^1\oplus L^2}(x)$$
$$\times\varphi^{(r)}_{L^3\oplus L^4}(-x\mathbf{m}\begin{pmatrix} a_1 & c \\ & a_2 \end{pmatrix})\psi(-\mathrm{tr}(b[Q(x)]))dx\, db\, dc\, d^\times a \qquad (5.10)$$

where L^i are all good lattices. If fact we can define a function on $\mathrm{GL}_2(F)$

$$\Lambda(L^1, L^2, L^3, L^4; g) := \int_{\mathrm{Sym}_2(F)} \int_{V_r^2} \varphi_{L^1\oplus L^2}(x)\varphi_{L^3\oplus L^4}(-xg)\psi(-\mathrm{tr}(b[Q(x)]))dx\, db.$$

Substituting this expression into (5.10) yields

$$\int_{(F^\times)^2} \chi_1(a_1)\chi_2(a_2)|a_1a_2|^{r+\frac{1}{2}+s_0} \int_F \Lambda(L^1, L^2, L^3, L^4; \begin{pmatrix} a_1 & c \\ & a_2 \end{pmatrix})dc\, d^\times a. \qquad (5.11)$$

There are two important properties of $\Lambda(L^1, L^2, L^3, L^4; g)$ worth noting. First, if

$$g = \begin{pmatrix} a_1 & \\ & a_2 \end{pmatrix},$$

then $\Lambda(L^1,L^2,L^3,L^4;g)$ is easily computable. Second, $\Lambda(L^1,L^2,L^3,L^4;g)$ is both right and left invariant with respect to compact open subgroups that depend on the relationships between the lattices. We seek to make both statements explicit and, as a consequence, will establish the connection between our doubling integral and the local densities defined by Yang in [42].

Proposition 5.1.3. *Let $\{L^i\}_{i=1}^4$ be good lattices. Then we have*

$$\Lambda(L^1,L^2,L^3,L^4;\begin{pmatrix} a_1 & \\ & a_2 \end{pmatrix}) = \begin{cases} W_0(r,L^1,L^2) \\ \qquad \text{if } L^1 \oplus L^2 \subset a_1^{-1}L^3 \oplus a_2^{-1}L^4 \\ |a_1|^{-(2r+1)} W_0(r,L^3,L^2) \\ \qquad \text{if } a_1^{-1}L^3 \oplus L^2 \subset L^1 \oplus a_2^{-1}L^4 \\ |a_2|^{-(2r+1)} W_0(r,L^1,L^3) \\ \qquad \text{if } L^1 \oplus a_2^{-1}L^4 \subset a_1^{-1}L^3 \oplus L^2 \\ |a_1a_2|^{-(2r+1)} W_0(r,L^3,L^4) \\ \qquad \text{if } a_1^{-1}L^3 \oplus a_2^{-1}L^4 \subset L^1 \oplus L^2 \end{cases}$$

where

$$W_0(r,L,L') := \int_{\mathrm{Sym}_2(F)} \int_{L\oplus L'} \psi(-\mathrm{tr}(b[Q(x)]))dx\,db.$$

Proof. The proof is nearly transparent. We notice that

$$\begin{aligned} \varphi_{L^1\oplus L^2}(x)\varphi_{L_3\oplus L_4}(x\begin{pmatrix} a_1 & \\ & a_2 \end{pmatrix}) &= (\varphi_{L^1}(x_1)\otimes\varphi_{L^2}(x_2))\,(\varphi_{L^3}(x_1a_1)\otimes\varphi_{L^4}(x_2a_2)) \\ &= [\varphi_{L^1}(x_1)\varphi_{L^3}(x_1a_1)]\otimes[\varphi_{L^2}(x_2)\varphi_{L^4}(x_2a_2)] \\ &= [\varphi_{L^1}(x_1)\varphi_{a_1^{-1}L^3}(x_1)]\otimes[\varphi_{L^2}(x_2)\varphi_{a_2^{-1}L^4}(x_2)] \end{aligned}$$

So we see that the conditions on the lattices simply determine which characteristic function dominates in each pair. Moreover, for any pair where either $a_1^{-1}L^3$ or $a_2^{-1}L^4$ dominates, then we make the appropriate transfromations $x_i \mapsto x_i a_i^{-1}$ and

$$b \mapsto \begin{pmatrix} a_1^* & \\ & a_2^* \end{pmatrix} b \begin{pmatrix} a_1^* & \\ & a_2^* \end{pmatrix}$$

where $a_i^* = a_i$ when we transform the x_i as above and $a_i^* = 1$ otherwise. $\square$

So to compute the integral

$$\int_F \Lambda(L^1, L^2, L^3, L^4; \begin{pmatrix} a_1 & c \\ & a_2 \end{pmatrix})dc$$

we will use the invariance properties of $\Lambda(L^1, L^2, L^3, L^4; g)$ to reduce our calculation to the previous one. In particular, if $L^1 \subset L^2$, then we see that

$$\begin{aligned} &\int_{\mathrm{Sym}_2(F)} \int_{V_r^2} \varphi_{L^1 \oplus L^2}(x) \varphi_{L^3 \oplus L^4}(-xkg) \psi(-\mathrm{tr}(b[Q(x)]))\, dx\, db \\ = &\int_{\mathrm{Sym}_2(F)} \int_{V_r^2} \varphi_{L^1 \oplus L^2}(xk^{-1}) \varphi_{L^3 \oplus L^4}(-xg) \psi(-\mathrm{tr}(b[Q(xk^{-1})]))\, dx\, db \\ = &\int_{\mathrm{Sym}_2(F)} \int_{V_r^2} \varphi_{L^1 \oplus L^2}(x) \varphi_{L^3 \oplus L^4}(-xg) \psi(-\mathrm{tr}(b[Q(x)]))\, dx\, db \end{aligned}$$

for all $k \in J$ where J is the standard Iwahori subgroup of $\mathrm{GL}_2(F)$. Notice we perform the transforms $x \mapsto xk^{-1}$ and $b \mapsto k^{-1}b^t k$, which both preserve the measure. Thus $\Lambda(L^1, L^2, L^3, L^4; g)$ is left J-invariant. Similarly, one can show that for $L^2 \subset L^1$, then $\Lambda(L^1, L^2, L^3, L^4; g)$ is left $\bar{J}$-invariant, where $\bar{J}$ is the opposite Iwahori (i.e., $\bar{J} = {}^t J$). Identical conditions on L^3 and L^4 gives $\Lambda(L^1, L^2, L^3, L^4; g)$ right-invariance under J or $\bar{J}$.

Finally, we consider the double coset space $J_1 \backslash \mathrm{GL}_2(F)/J_2$ with $J_i \in \{J, \bar{J}\}$. We can now show that any upper triangular matrix in $\mathrm{GL}_2(F)$ belongs to a double coset represented by a diagonal matrix or a matrix having only entries on the anti-diagonal. In particular, we have the following proposition.

Proposition 5.1.4.

$$\begin{pmatrix} a_1 & c \\ & a_2 \end{pmatrix} \in \begin{cases} J \begin{pmatrix} a_1 & \\ & a_2 \end{pmatrix} J & \text{if } \exists i, \operatorname{ord}(c) \geq \operatorname{ord}(a_i) \\ J \begin{pmatrix} & c \\ -c^{-1}a_1a_2 & \end{pmatrix} J & \text{if } \forall i, \operatorname{ord}(c) < \operatorname{ord}(a_i) \end{cases}$$

Proof. If $\operatorname{ord}(a_1) \leq \operatorname{ord}(c)$, then we have

$$\begin{pmatrix} a_1 & c \\ & a_2 \end{pmatrix} = \begin{pmatrix} a_1 & \\ & a_2 \end{pmatrix} \begin{pmatrix} 1 & ca_1^{-1} \\ & 1 \end{pmatrix}.$$

Similarly, if $\operatorname{ord}(a_2) \leq \operatorname{ord}(c)$, then we find,

$$\begin{pmatrix} a_1 & c \\ & a_2 \end{pmatrix} = \begin{pmatrix} 1 & ca_2^{-1} \\ & 1 \end{pmatrix} \begin{pmatrix} a_1 & \\ & a_2 \end{pmatrix}.$$

Finally, if $\operatorname{ord}(a_i) > \operatorname{ord}(c)$ for both $i = 1$ or 2, then we find,

$$\begin{pmatrix} a_1 & c \\ & a_2 \end{pmatrix} = \begin{pmatrix} 1 & \\ c^{-1}a_2 & 1 \end{pmatrix} \begin{pmatrix} & c \\ -c^{-1}a_1a_2 & \end{pmatrix} \begin{pmatrix} 1 & \\ c^{-1}a_1 & 1 \end{pmatrix}.$$

□

We get nearly identical results if we have left or right $\bar{J}$ invariance. In fact, only the conditions on $\operatorname{ord}(a_i)$ and $\operatorname{ord}(c)$ shift by ± 1.

These previous two propositions finally allow us to compute the integral

$$\int_F \Lambda(L^1, L^2, L^3, L^4; \begin{pmatrix} a_1 & c \\ & a_2 \end{pmatrix})dc =$$

$$\int_F \int_{\mathrm{Sym}_2(F)} \int_{V_r^2} \varphi^{(r)}_{L^1 \oplus L^2}(x)\varphi^{(r)}_{L^3 \oplus L^4}(-x \begin{pmatrix} a_1 & c \\ & a_2 \end{pmatrix})\psi(-\mathrm{tr}(b[Q(x)]))dx\, db\, dc.$$

For the sake of completeness, we will compute one example; the rest follow with only minor adjustments.

Let $V \in \{V^{s_l}, V^{s_m}\}$ and let $L^1 = L^2 = L^3 = L_0^{(r)}$ and $L^4 = L_1^{(r)}$ (see Table 4.6). In this case, we break our integral into four pieces according to whether $\mathrm{ord}(a_i) \geq 0$ or $\mathrm{ord}(a_i) < 0$. First, let $\mathrm{ord}(a_i) \geq 0$ for both $i = 1$ and $i = 2$, then let $\mathcal{P}^\beta = a_1\mathcal{O} \cup a_2\mathcal{O}$. Using the results of Propositions 5.1.3 and 5.1.4, we have

$$\int\limits_F \Lambda(r, L_0, L_0, L_0, L_1; \begin{pmatrix} a_1 & c \\ & a_2 \end{pmatrix})dc$$

$$= \int\limits_F \int\limits_{\mathrm{Sym}_2(F)} \int\limits_{V_r^2} \varphi^{(r)}_{L_0 \oplus L_0}(x)\varphi^{(r)}_{L_0 \oplus L_1}(-x \begin{pmatrix} a_1 & c \\ & a_2 \end{pmatrix})\psi(-\mathrm{tr}(b[Q(x)]))dx\, db\, dc$$

$$= \int\limits_{\mathcal{P}^\beta} \int\limits_{\mathrm{Sym}_2(F)} \int\limits_{V_r^2} \varphi^{(r)}_{L_0 \oplus L_0}(x)\varphi^{(r)}_{L_0 \oplus L_1}(-x \begin{pmatrix} a_1 & \\ & a_2 \end{pmatrix})\psi(-\mathrm{tr}(b[Q(x)]))dx\, db\, dc$$

$$+ \int\limits_{F\setminus\mathcal{P}^\beta} \int\limits_{\mathrm{Sym}_2(F)} \int\limits_{V_r^2} \varphi^{(r)}_{L_0 \oplus L_0}(x)\varphi^{(r)}_{L_0 \oplus L_1}(-x \begin{pmatrix} & c \\ -c^{-1}a_1a_2 & \end{pmatrix}) \times\psi(-\mathrm{tr}(b[Q(x)]))dx\, db\, dc.$$

In the second integral we make the substitution $x \mapsto x \cdot w^{-1}$. Then we follow this with the substitution $b \mapsto wbw^{-1}$ and notice that $\varphi_{L_0^{(r)} \oplus L_0^{(r)}}$ is invariant via right translation by w, we get

$$\begin{aligned}
&\int\limits_{F} \Lambda(r, L_0, L_0, L_0, L_1; \begin{pmatrix} a_1 & c \\ & a_2 \end{pmatrix})dc \\
&= \int\limits_{\mathcal{P}^\beta} \int\limits_{\mathrm{Sym}_2(F)} \int\limits_{V_r^2} \varphi^{(r)}_{L_0\oplus L_0}(x)\varphi^{(r)}_{L_0\oplus L_1}(-x \begin{pmatrix} a_1 & \\ & a_2 \end{pmatrix})\psi(-\mathrm{tr}(b[Q(x)]))dx\,db\,dc \\
&+ \int\limits_{F\setminus\mathcal{P}^\beta} \int\limits_{\mathrm{Sym}_2(F)} \int\limits_{V_r^2} \varphi^{(r)}_{L_0\oplus L_0}(x)\varphi^{(r)}_{L_0\oplus L_1}(-x \begin{pmatrix} c^{-1}a_1a_2 & \\ & c \end{pmatrix}) \\
&\qquad \times\psi(-\mathrm{tr}(b[Q(x)]))dx\,db\,dc \\
&= \int\limits_{\mathcal{P}^\beta} W_0(r, L_0, L_0)dc + \int\limits_{\mathcal{O}\setminus\mathcal{P}^\beta} W_0(r, L_0, L_0)dc + \int\limits_{F\setminus\mathcal{O}} |c|^{-(2r+2+2s_0)}W_0(r, L_0, L_1)dc
\end{aligned}$$

where the $W_0(r, L_i, L_j)$ is a Whittaker function that we evaluate using Yang's work [42]. The extra factor of $|c|^{-(2r+2+2s_0)}$ comes from the substitutions $x_2 \mapsto c^{-1}x_2$ and $b \mapsto \mathrm{diag}(1, c)b\mathrm{diag}(1, c)$. So ultimately we find,

$$\int\limits_{F} \Lambda(r,L_0,L_0,L_0,L_1; \begin{pmatrix} a_1 & c \\ & a_2 \end{pmatrix})dc = W_0(r,L_0,L_0)+\frac{(1-q^{-1})q^{-(2r+1+2s_0)}}{1-q^{-(2r+1+2s_0)}}W_0(r,L_0,L_1)$$

for $a_i \in \mathcal{O}$ for both i. The other three cases are evaluated in a similar fashion using the similar substitutions as above. We will compute those cases with less detail than before.

Next let $\mathrm{ord}(a_1) < 0$ and $\mathrm{ord}(a_2) \geq 0$. Furthermore, let $\mathcal{P}^{-\alpha_1} = a_1\mathcal{O}$. In this case,

$$
\begin{aligned}
&\int\limits_{F} \Lambda(r, L_0, L_0, L_0, L_1; \begin{pmatrix} a_1 & c \\ & a_2 \end{pmatrix})dc \\
=& \int\limits_{\mathcal{P}^{-\alpha_1}} \int\limits_{\mathrm{Sym}_2(F)} \int\limits_{V_r^2} \varphi^{(r)}_{L_0\oplus L_0}(x)\varphi^{(r)}_{L_0\oplus L_1}(-x \begin{pmatrix} a_1 & \\ & a_2 \end{pmatrix})\psi(-\mathrm{tr}(b[Q(x)]))dx\, db\, dc \\
+& \int\limits_{F\backslash\mathcal{P}^{-\alpha_1}} \int\limits_{\mathrm{Sym}_2(F)} \int\limits_{V_r^2} \varphi^{(r)}_{L_0\oplus L_0}(x)\varphi^{(r)}_{L_0\oplus L_1}(-x \begin{pmatrix} & c \\ -c^{-1}a_1a_2 & \end{pmatrix}) \\
&\qquad \times\psi(-\mathrm{tr}(b[Q(x)]))dx\, db\, dc \\
=& q^{-(2r+2+2s_0)\alpha_1} \int\limits_{\mathcal{P}^{-\alpha_1}} W_0(r, L_0, L_0)dc + \int\limits_{F\backslash\mathcal{P}^{-\alpha_1}} |c|^{-(2r+2+2s_0)} W_0(r, L_0, L_1)dc \\
=& q^{-(2r+1+2s_0)\alpha_1} W_0(r, L_0, L_0) + q^{-(2r+1+2s_0)\alpha_1} \frac{(1-q^{-1})q^{-(2r+1+2s_0)}}{1-q^{-(2r+1+2s_0)}} W_0(r, L_0, L_1) \\
=& q^{-(2r+1+2s_0)\alpha_1} \left(W_0(r, L_0, L_0) + (1-q^{-1}) \frac{q^{-(2r+1+2s_0)}}{1-q^{-(2r+1+2s_0)}} W_0(r, L_0, L_1) \right).
\end{aligned}
$$

Now let $\mathrm{ord}(a_1) \geq 0$ and $\mathrm{ord}(a_2) < 0$. As above, let $\mathcal{P}^{-\alpha_2} = a_2\mathcal{O}$. We will see this case is quite similar to the previous case. In particular,

$$\int_F \Lambda(r, L_0, L_0, L_0, L_1; \begin{pmatrix} a_1 & c \\ & a_2 \end{pmatrix})dc$$

$$= \int_{\mathcal{P}^{-\alpha_2}} \int_{\mathrm{Sym}_2(F)} \int_{V_r^2} \varphi^{(r)}_{L_0\oplus L_0}(x)\varphi^{(r)}_{L_0\oplus L_1}(-x \begin{pmatrix} a_1 & \\ & a_2 \end{pmatrix})\psi(-\mathrm{tr}(b[Q(x)]))dx\,db\,dc$$

$$+ \int_{F\setminus\mathcal{P}^{-\alpha_2}} \int_{\mathrm{Sym}_2(F)} \int_{V_r^2} \varphi^{(r)}_{L_0\oplus L_0}(x)\varphi^{(r)}_{L_0\oplus L_1}(-x \begin{pmatrix} & c \\ -c^{-1}a_1a_2 & \end{pmatrix})$$

$$\times\psi(-\mathrm{tr}(b[Q(x)]))dx\,db\,dc$$

$$= q^{-(2r+2+2s_0)\alpha_2}\int_{\mathcal{P}^{-\alpha_2}} W_0(r, L_0, L_1)dc + \int_{F\setminus\mathcal{P}^{-\alpha_2}} |c|^{-(2r+2+2s_0)}W_0(r, L_0, L_1)dc$$

$$= q^{-(2r+1+2s_0)\alpha_2}W_0(r, L_0, L_1) + q^{-(2r+1+2s_0)\alpha_2}\frac{(1-q^{-1})q^{-(2r+1+2s_0)}}{1-q^{-(2r+1+2s_0)}}W_0(r, L_0, L_1)$$

$$= q^{-(2r+1+2s_0)\alpha_2}\frac{1-q^{-(2r+2+2s_0)}}{1-q^{-(2r+1+2s_0)}}W_0(r, L_0, L_1).$$

Finally, let $\mathrm{ord}(a_i) < 0$ for both i. As in the first case, we let $\mathcal{P}^{\beta} = a_1\mathcal{O} \cup a_2\mathcal{O}$ and $\mathcal{P}^{-\alpha_i} = a_i\mathcal{O}$. So we evaluate our integral,

$$\begin{aligned}
&\int\limits_F \Lambda(r, L_0, L_0, L_0, L_1; \begin{pmatrix} a_1 & c \\ & a_2 \end{pmatrix})dc \\
=\ & \int\limits_{\mathcal{P}^\beta} \int\limits_{\mathrm{Sym}_2(F)} \int\limits_{V_r^2} \varphi^{(r)}_{L_0\oplus L_0}(x)\varphi^{(r)}_{L_0\oplus L_1}(-x \begin{pmatrix} a_1 & \\ & a_2 \end{pmatrix})\psi(-\mathrm{tr}(b[Q(x)]))dx\, db\, dc \\
+\ & \int\limits_{F\backslash\mathcal{P}^\beta} \int\limits_{\mathrm{Sym}_2(F)} \int\limits_{V_r^2} \varphi^{(r)}_{L_0\oplus L_0}(x)\varphi^{(r)}_{L_0\oplus L_1}(-x \begin{pmatrix} & c \\ -c^{-1}a_1a_2 & \end{pmatrix}) \\
& \quad \times\psi(-\mathrm{tr}(b[Q(x)]))dx\, db\, dc \\
=\ & q^{-(2r+2+2s_0)(\alpha_1+\alpha_2)} \int\limits_{\mathcal{P}^\beta} W_0(r, L_0, L_1)dc \\
+\ & \int\limits_{\mathcal{P}^{-\alpha_1-\alpha_2}\backslash\mathcal{P}^\beta} |c|^{-(2r+2+2s_0)}|c^{-1}a_1a_2|^{-(2r+1)}W_0(r, L_0, L_1)dc \\
+\ & \int\limits_{F\backslash\mathcal{P}^{-\alpha_1-\alpha_2}} |c|^{-(2r+2+2s_0)}W_0(r, L_0, L_1)dc \\
=\ & q^{-(2r+2+2s_0)(\alpha_1+\alpha_2)} \int\limits_{\mathcal{P}^{-\alpha_1-\alpha_2}} W_0(r, L_0, L_1)dc + \int\limits_{F\backslash\mathcal{P}^{-\alpha_1-\alpha_2}} |c|^{-(2r+2+2s_0)}W_0(r, L_0, L_1)dc \\
=\ & q^{-(2r+1+2s_0)(\alpha_1+\alpha_2)} \frac{1-q^{-(2r+2+2s_0)}}{1-q^{-(2r+1+2s_0)}} W_0(r, L_0, L_1).
\end{aligned}$$

We include Table 5.2 at the end of the chapter to summarize all of the computations of

$$\int_F \Lambda(r, L, L', L'', L'''; \begin{pmatrix} a_1 & c \\ & a_2 \end{pmatrix})\, dc$$

needed in this thesis.

5.1.3 Examples of doubling integrals for spherical representations

Until now we have evaluated all of the individual steps in the calculation without considering the overall picture. We now use our previous results to compute a few doubling integrals to achieve some local L-factors. The author originally computed the spherical case by hand and the others followed later using Mathematica; so let us start with the spherical case. We will follow the calculation for a representation of the linear group $\mathrm{Sp}_2(F)$ and then mention which slight modifications are needed in the metaplectic case.

Let $\pi = \mathrm{Ind}_{P_\emptyset}^G(\chi_1 \otimes \chi_2)$ be irreducible. Because π is spherical, we choose our Iwahori-fixed vector to be the spherical vector f normalized so $f(1_G) = 1$. Next we choose our $\{\Phi_{s_0+r}\}$. In order to match the invariance properties of f, we want Φ_{s_0+r} to be invariant under $i(K \times \{1_G\})$. Consequently, we choose Φ_{s_0+r} as

$$\Phi_{s_0+r}(g) = \lambda(\varphi^{(r)}_{L_1\oplus L_1} \otimes \overline{\varphi^{(r)}_{L_1\oplus L_1}})(g).$$

Recall that $L_1 = M_{2\times 2}(\mathcal{O}) \subset V^{s_l}$, where V^{s_l} has dimension 4. As one might expect, since $\varphi^{(r)}_{L_1\oplus L_1}(x)$ is K-invariant under the Weil representation, we find that

$$I(f, \varphi^{(r)}_{L_1\oplus L_1})(x) = \varphi^{(r)}_{L_1\oplus L_1}(x).$$

Furthermore, for $\Lambda(g) := \Lambda(r, L_1, L_1, L_1, L_1; g)$, following the example from Section 5.1.2, one can easily verify that

$$\int_F \Lambda(\begin{pmatrix} a_1 & c \\ & a_2 \end{pmatrix})dc = |a_1|^{-2r\iota(a_1)}|a_2|^{-2r\iota(a_2)}\frac{1-q^{-(2r+1)}}{1-q^{-2r}}W_0(r, L_1, L_1)$$

where

$$\iota(a) = \begin{cases} 0 \text{ if } a \in \mathcal{O} \\ \\ 1 \text{ if } a \in F \setminus \mathcal{O} \end{cases} .$$

Substituting these into (5.11) , we see that

$$\begin{aligned} & Z(r - \frac{1}{2}, \Phi, f)(1_G) \\ &= \frac{1 - q^{-(2r+1)}}{1 - q^{-2r}} W_0(r, L_1, L_1) \int_{(F^\times)^2} \chi_1(a_1)\chi_2(a_2)|a_1|^{r-2r\iota(a_1)}|a_2|^{r-2r\iota(a_2)}\, d^\times a. \end{aligned}$$

This integral breaks up according to whether the a_i are in $\mathcal{O}$ or not. So we get four integrals,

$$\begin{aligned} & Z(r - \frac{1}{2}, \Phi, f)(1_G) \\ &= \frac{1 - q^{-(2r+1)}}{1 - q^{-2r}} W_0(r, L_1, L_1) \int_{\mathcal{O}\setminus\{0\}} \int_{\mathcal{O}\setminus\{0\}} \chi_1(a_1)\chi_2(a_2)|a_1|^{r}|a_2|^{r}\, d^\times a_1\, d^\times a_2 \\ &+ \frac{1 - q^{-(2r+1)}}{1 - q^{-2r}} W_0(r, L_1, L_1) \int_{\mathcal{O}\setminus\{0\}} \int_{F\setminus\mathcal{O}} \chi_1(a_1)\chi_2(a_2)|a_1|^{r}|a_2|^{-r}\, d^\times a_1\, d^\times a_2 \\ &+ \frac{1 - q^{-(2r+1)}}{1 - q^{-2r}} W_0(r, L_1, L_1) \int_{F\setminus\mathcal{O}} \int_{\mathcal{O}\setminus\{0\}} \chi_1(a_1)\chi_2(a_2)|a_1|^{-r}|a_2|^{r}\, d^\times a_1\, d^\times a_2 \\ &+ \frac{1 - q^{-(2r+1)}}{1 - q^{-2r}} W_0(r, L_1, L_1) \int_{F\setminus\mathcal{O}} \int_{F\setminus\mathcal{O}} \chi_1(a_1)\chi_2(a_2)|a_1|^{-r}|a_2|^{-r}\, d^\times a_1\, d^\times a_2. \end{aligned}$$

So finally, with both Haar measures $d^\times a_i$ normalized so that $\mathcal{O}^\times$ has measure 1 and both χ_j being unramified, we see that

$$\begin{aligned} Z(r - \frac{1}{2}, \Phi, f)(1_G) &= \frac{1}{1 - \chi_1(\varpi)q^{-r}} \frac{1}{1 - \chi_2(\varpi)q^{-r}} \frac{1 - q^{-(2r+1)}}{1 - q^{-2r}} W_0(r, L_1, L_1) \\ &+ \frac{\chi_1^{-1}(\varpi)q^{-r}}{1 - \chi_1^{-1}(\varpi)q^{-r}} \frac{1}{1 - \chi_2(\varpi)q^{-r}} \frac{1 - q^{-(2r+1)}}{1 - q^{-2r}} W_0(r, L_1, L_1) \\ &+ \frac{1}{1 - \chi_1(\varpi)q^{-r}} \frac{\chi_2^{-1}(\varpi)q^{-r}}{1 - \chi_2^{-1}(\varpi)q^{-r}} \frac{1 - q^{-(2r+1)}}{1 - q^{-2r}} W_0(r, L_1, L_1) \\ &+ \frac{\chi_1^{-1}(\varpi)q^{-r}}{1 - \chi_1^{-1}(\varpi)q^{-r}} \frac{\chi_2^{-1}(\varpi)q^{-r}}{1 - \chi_2^{-1}(\varpi)q^{-r}} \frac{1 - q^{-(2r+1)}}{1 - q^{-2r}} W_0(r, L_1, L_1). \end{aligned}$$

This simplifies to

$$Z(r-\frac{1}{2},\Phi,f)(1_G)=\frac{L(r,\chi_1)L(r,\chi_2)L(r,\chi_1^{-1})L(r,\chi_2^{-1})}{\zeta(2r+1)\zeta(2r)}W_0(r,L_1,L_1)$$

in an elementary way. Finally, using our previous calculations for the local density term, we make the substitution

$$W_0(r,L_1,L_1)=\frac{\zeta(2r+1)\zeta(r)}{\zeta(2r+2)\zeta(r+2)}$$

and we finally arrive at the following proposition.

Proposition 5.1.5. *Let π be a spherical constituent of $\mathrm{Ind}_{P_\emptyset}^G(\chi_1\otimes\chi_2)$. For the choices of "good test vectors" outlined above, we have*

$$Z(r-\frac{1}{2},\Phi,f)(1_G)=\frac{L(r,\chi_1)L(r,\chi_2)L(r,\mathbb{1}_{F^\times})L(r,\chi_1^{-1})L(r,\chi_2^{-1})}{\zeta(2r+2)\zeta(2r)\zeta(r+2)}. \tag{5.12}$$

There are two points worth noticing about this particular result. First,

$$L(r,\chi_1)L(r,\chi_2)L(r,\mathbb{1}_{F^\times})L(r,\chi_1^{-1})L(r,\chi_2^{-1})$$

is exactly the factor defined by the Satake parameter for a spherical representation of $\mathrm{Sp}_2(F)$. Second, the extra term $\zeta(2r+2)\zeta(2r)\zeta(r+2)$ is predicted by Piatetski-Shapiro and Rallis coming from the normalizing factor for the Eisenstein series in their original work on the doubling integral [8].

From Section 4.1, we saw that the normalizing factor of the spherical Eisenstein series had the following terms at the unramified places,

$$d_{H,v}(s)=L(s+\frac{2n+1}{2},\chi)\prod_{i=0}^{n-1}L(2s+2i+1,\chi^2).$$

In our case $n = 2$, and $\chi = \chi_V = 1$. Finally substituting $s = r - \frac{1}{2}$, we see that

$$d_{H,v}(r - \frac{1}{2}) = L(r+2, \mathbb{1}_{F^\times}) \prod_{i=0}^{1} L(2r - 1 + 2i + 1, \mathbb{1}_{F^\times}) = \zeta(r+2)\zeta(2r)\zeta(2r+2),$$

which conforms to our doubling result.

One of the truly outstanding aspects of this doubling calculation is the simplicity in with which one can modify the integral to work for genuine principal series representations of $\widetilde{\mathrm{Sp}_2}(F)$. In this case, let $\tilde{\pi} = \mathrm{Ind}_{\tilde{P}_\emptyset}^{\tilde{G}}((\chi_1 \otimes \chi_2)')$ be a irreducible genuine principal series representation of $\tilde{G}$. So for $f \in \tilde{\pi}$,

$$f([\begin{pmatrix} a_1 & c \\ & a_2 \end{pmatrix}, z]_L \tilde{g}) = z\chi_1(a_1)\chi_2(a_2)|a_1|^2|a_2| f(\tilde{g}).$$

The only change in setting up the integral in this case involves the quadratic space V. In this case, $V = V^{s_m} \simeq F^5$ and

$$Q_{s_m}(x_o, x_1, x_2, x_3, x_4) = \kappa(x_0^2 + x_1x_4 - x_2x_3).$$

So (V^{s_m}, Q_{s_m}) is an analog to (V^{s_l}, Q_{s_l}) that we used above. Furthermore, let $L_1 = \mathcal{O}^5$. This change in the quadratic vector space has three effects on our doubling calculation. First, because V_r is now one dimension larger than previously, we have a shift $r \mapsto r + \frac{1}{2}$. Second, we now have

$$W_0(r, L_1, L_1) = \frac{\zeta(2r+1)}{\zeta(2r+4)}.$$

Third, note that in the metaplectic case

$$\chi_V(x) = (x, 2\kappa)_F$$

with $\kappa \in \mathcal{O}^\times$, so χ_V is unramified but possibly nontrivial. Besides those adjustments, the calculation is identical with the spherical case for $\mathrm{Sp}_2(F)$. Thus ultimately we find have the following proposition.

Proposition 5.1.6. *Let $\tilde{\pi}$ be a spherical constituent of $\mathrm{Ind}_{\widetilde{P}_\emptyset}^{\widetilde{G}}((\chi_1 \otimes \chi_2)')$. For the choices of "good" test vectors outlined above, we have*

$$Z(r, \Phi, f)([1_G, 1]_L) = \frac{L(r+\frac{1}{2}, \chi_V\chi_1)L(r+\frac{1}{2}, \chi_V\chi_2)L(r+\frac{1}{2}, \chi_V\chi_1^{-1})L(r+\frac{1}{2}, \chi_V\chi_2^{-1})}{\zeta(2r+4)\zeta(2r+2)}. \tag{5.13}$$

It is worth remarking that due to a cancellation, the numerator and denominator are one degree less than the $\mathrm{Sp}_2(F)$ case. In fact, the results resembles the L-factor defined by the Satake parameter for a spherical representation of $\mathrm{SO}_5(F)$.

5.1.4 An example with a ramified representation

Next, let us consider an example of a non-spherical representation of $\mathrm{Sp}_2(F)$ to illustrate how the calculation changes when a representation has a unique vector fixed by a smaller compact open subgroup. Consider the representation

$$\pi = L(\nu^{\frac{3}{2}} St_{\mathrm{GL}_2}, 1) \subset \mathrm{Ind}_{P_\emptyset}^{G}(\nu^2 \otimes \nu^{-1}).$$

We see that $\dim_{\mathbb{C}}(\pi^{I_\alpha}) = 1$ and, for the given embedding of π into $\mathrm{Ind}_{P_\emptyset}^{G}(\nu^2 \otimes \nu^{-1})$, π^{I_α} is spanned by the vector f with

$$\begin{aligned} f(1_G) = f(w_\alpha) &= f(w_\beta) = f(w_{\beta\alpha}) = 1 \\ f(w_{\alpha\beta}) = f(w_{\alpha\beta\alpha}) &= f(w_{\beta\alpha\beta}) = f(w_{\alpha\beta\alpha\beta}) = -q^{-2}. \end{aligned}$$

Because we want to produce a doubling integral $Z(s,\Phi,f)$ that is also I_α-invariant, we have two appropriate choices of characteristic functions to use in our interpolation method. They are

$$\varphi_{ra}(x) = \varphi_{L_{ra}\oplus L_{ra}}(x) \qquad \text{and} \qquad \varphi_0(x) = \varphi_{L_0\oplus L_0}(x).$$

Using (5.4) – (5.9), one can show

Lemma 5.1.3. *Let φ_0 and φ_{ra} be as above. Then,*

$$\begin{aligned} I(f,\varphi_{ra})(x) &= 0 \\ I(f,\varphi_0)(x) &= R_1(q)(\varphi_{L_0\oplus L_1}(x)+\varphi_{L_0\oplus L_2}(x)) - R_2(q)(\varphi_{L_1\oplus L_1}(x)+\varphi_{L_2\oplus L_2}(x)) \end{aligned}$$

where $R_i(q)$ non-zero rational functions in $\mathbb{C}(q)$.

Thus we see that only one choice of lattice functions afford us the desired invariance properties while not vanishing. So we let

$$\Phi_{s_0+r} = \lambda(\varphi^{(r)}_{L_0\oplus L_0} \otimes \overline{\vartheta\varphi^{(r)}_{L_0\oplus L_0}})$$

with

$$\vartheta = \frac{q^3}{2\mathrm{vol}(I_\emptyset)(q^2-1)}.$$

Now that we have selected $\{\Phi_{s_0+r}\}$, the computation proceeds in a similar way as the spherical case. As before, we will divide the doubling integral $Z(s_0+r,\Phi,f)$ in (5.11) into four parts depending on whether $a_i \in \mathcal{O}$ or $a_i \in F\setminus\mathcal{O}$. There is one slight difference. For π, we need to compute the integrals

$$\int_F \Lambda(r,L_0,L_0,L_0,L_i;\begin{pmatrix} a_1 & c \\ & a_2 \end{pmatrix})dc$$

and

$$\int_F \Lambda(r, L_0, L_0, L_i, L_i; \begin{pmatrix} a_1 & c \\ & a_2 \end{pmatrix})dc$$

for $i \in \{1,2\}$. Fortunately, one can verify quite routinely that

$$\int_F \Lambda(r, L_0, L_0, L_0, L_1; \begin{pmatrix} a_1 & c \\ & a_2 \end{pmatrix})dc = \int_F \Lambda(r, L_0, L_0, L_0, L_2; \begin{pmatrix} a_1 & c \\ & a_2 \end{pmatrix})dc$$

and

$$\int_F \Lambda(r, L_0, L_0, L_1, L_1; \begin{pmatrix} a_1 & c \\ & a_2 \end{pmatrix})dc = \int_F \Lambda(r, L_0, L_0, L_2, L_2; \begin{pmatrix} a_1 & c \\ & a_2 \end{pmatrix})dc,$$

so only two integrals need be evaluated. One of these we already computed in the example at the end of Section 5.1.2 and the other can be computed in a similar manner. So now let us compute $Z(s, \Phi, f)(1_G)$ as we before. If $\mathrm{ord}(a_i) \geq 0$ for $i \in \{1,2\}$, we have

$$\begin{aligned} Z_{+,+}(s_0 + r, \Phi, f)(1_G) &= \int_{\mathcal{O}\setminus\{0\}} \int_{\mathcal{O}\setminus\{0\}} |a_1|^2 |a_2|^{-1} |a_1 a_2|^r \\ &\quad \times \int_F \frac{q(q^2+1)}{(q-1)} \Lambda(r, L_0, L_0, L_0, L_1; \begin{pmatrix} a_1 & c \\ & a_2 \end{pmatrix}) \\ &\quad - \frac{q(q+1)}{(q-1)} \Lambda(r, L_0, L_0, L_1, L_1; \begin{pmatrix} a_1 & c \\ & a_2 \end{pmatrix})dc\, d^\times a_1\, d^\times a_2. \end{aligned}$$

By evaluating the innermost integrals we see this expression becomes

$$\begin{aligned}
&\int_{\mathcal{O}\setminus\{0\}}\int_{\mathcal{O}\setminus\{0\}} |a_1|^2|a_2|^{-1}|a_1a_2|^r \\
&\times\frac{q(q^2+1)}{(q-1)}\left(W_0(r,L_0,L_0)+(1-q^{-1})\frac{q^{-2r}}{1-q^{-2r}}W_0(r,L_0,L_1)\right) \\
&-\frac{q(q+1)}{(q-1)}\left(W_0(r,L_0,L_0)+(1-q^{-1})\frac{q^{-2r}}{1-q^{-2r}}W_0(r,L_0,L_1)\right)d^\times a_1\,d^\times a_2 \\
=\;& \frac{q^2\left(W_0(r,L_0,L_0)+(1-q^{-1})\frac{q^{-2r}}{1-q^{-2r}}W_0(r,L_0,L_1)\right)}{(1-|\varpi|^2q^{-r})(1-|\varpi|^{-1}q^{-r})}. \qquad (5.14)
\end{aligned}$$

If $\mathrm{ord}(a_1)<0$ and $\mathrm{ord}(a_2)\geq 0$, we have

$$\begin{aligned}
Z_{-,+}(s_0+r,\Phi,f)(1_G) \;=\;& \int_{F\setminus\mathcal{O}}\int_{\mathcal{O}\setminus\{0\}} |a_1|^2|a_2|^{-1}|a_1|^{-r}|a_2|^r \\
&\times\int_F \frac{q(q^2+1)}{(q-1)}\Lambda(r,L_0,L_0,L_0,L_1;\begin{pmatrix} a_1 & c \\ & a_2\end{pmatrix}) \\
&-\frac{q(q+1)}{(q-1)}\Lambda(r,L_0,L_0,L_1,L_1;\begin{pmatrix} a_1 & c \\ & a_2\end{pmatrix})dc\,d^\times a_1\,d^\times a_2.
\end{aligned}$$

Evaluating this we see find,

$$\begin{aligned}
&\int_{F\setminus\mathcal{O}}\int_{\mathcal{O}\setminus\{0\}} |a_1|^2|a_2|^{-1}|a_1|^{-r}|a_2|^r \\
&\times\frac{q(q^2+1)}{(q-1)}\left(W_0(r,L_0,L_0)+(1-q^{-1})\frac{q^{-2r}}{1-q^{-2r}}W_0(r,L_0,L_1)\right) \\
&-\frac{q(q+1)}{(q-1)}\frac{1-q^{-(2r+1)}}{1-q^{-2r}}W_0(r,L_0,L_1)\,d^\times a_1\,d^\times a_2 \\
=\;& \Bigg[\frac{q(q^2+1)}{(q-1)}\left(W_0(r,L_0,L_0)+(1-q^{-1})\frac{q^{-2r}}{1-q^{-2r}}W_0(r,L_0,L_1)\right) \\
&-\frac{q(q+1)}{(q-1)}\frac{1-q^{-(2r+1)}}{1-q^{-2r}}W_0(r,L_0,L_1)\Bigg]\frac{|\varpi|^{-2}q^{-r}}{(1-|\varpi|^{-2}q^{-r})(1-|\varpi|^{-1}q^{-r})} \qquad (5.15)
\end{aligned}$$

Next, we have $\operatorname{ord}(a_1) \geq 0$ and $\operatorname{ord}(a_2) < 0$, which yields

$$\begin{aligned} Z_{+,-}(s_0+r,\Phi,f)(1_G) &= \int_{\mathcal{O}\backslash\{0\}}\int_{F\backslash\mathcal{O}} |a_1|^2|a_2|^{-1}|a_1|^r|a_2|^{-r} \\ &\quad \times \int_F \frac{q(q^2+1)}{(q-1)}\Lambda(r,L_0,L_0,L_0,L_1;\begin{pmatrix} a_1 & c \\ & a_2 \end{pmatrix}) \\ &\quad -\frac{q(q+1)}{(q-1)}\Lambda(r,L_0,L_0,L_1,L_1;\begin{pmatrix} a_1 & c \\ & a_2 \end{pmatrix})dc\,d^\times a_1\,d^\times a_2. \end{aligned}$$

This evaluates to

$$\begin{aligned} &\int_{\mathcal{O}\backslash\{0\}}\int_{F\backslash\mathcal{O}} |a_1|^2|a_2|^{-1}|a_1|^r|a_2|^{-r} \\ &\times\frac{q(q^2+1)}{(q-1)}\left(\frac{1-q^{-(2r+1)}}{1-q^{-2r}}W_0(r,L_0,L_1)\right) \\ &-\frac{q(q+1)}{(q-1)}\frac{1-q^{-(2r+1)}}{1-q^{-2r}}W_0(r,L_0,L_1)\,d^\times a_1\,d^\times a_2 \\ = \;& \frac{q^2\left(\frac{1-q^{-(2r+1)}}{1-q^{-2r}}W_0(r,L_0,L_1)\right)|\varpi|q^{-r}}{(1-|\varpi|^2q^{-r})(1-|\varpi|q^{-r})}. \end{aligned} \tag{5.16}$$

Finally, we have the case that $\operatorname{ord}(a_i) < 0$ for $i \in \{1,2\}$. Here we find

$$\begin{aligned} Z_{-,-}(s_0+r,\Phi,f)(1_G) &= \int_{F\backslash\mathcal{O}}\int_{F\backslash\mathcal{O}} |a_1|^2|a_2|^{-1}|a_1|^{-r}|a_2|^{-r} \\ &\quad \times \int_F \frac{q(q^2+1)}{(q-1)}\Lambda(r,L_0,L_0,L_0,L_1;\begin{pmatrix} a_1 & c \\ & a_2 \end{pmatrix}) \\ &\quad -\frac{q(q+1)}{(q-1)}\Lambda(r,L_0,L_0,L_1,L_1;\begin{pmatrix} a_1 & c \\ & a_2 \end{pmatrix})dc\,d^\times a_1\,d^\times a_2. \end{aligned}$$

Evaluating this we see find,

$$
\begin{aligned}
& \int_{F\backslash\mathcal{O}} \int_{F\backslash\mathcal{O}} |a_1|^2|a_2|^{-1}|a_1|^{-r}|a_2|^{-r} \\
& \times \frac{q(q^2+1)}{(q-1)} \frac{1-q^{-(2r+1)}}{1-q^{-2r}} W_0(r, L_0, L_1) \\
& - \frac{q(q+1)}{(q-1)} \left(q^{-1} W_0(r, L_1, L_1) + \frac{1-q^{-1}}{1-q^{-2r}} W_0(r, L_0, L_1) \right) d^\times a_1 \, d^\times a_2 \\
= \; & \left[\frac{q(q^2+1)}{(q-1)} \frac{1-q^{-(2r+1)}}{1-q^{-2r}} W_0(r, L_0, L_1) - \frac{q(q+1)}{(q-1)} (q^{-1} W_0(r, L_1, L_1) \right. \\
& \left. + \frac{1-q^{-1}}{1-q^{-2r}} W_0(r, L_0, L_1)) \right] \frac{(|\varpi|^{-2}q^{-r})(|\varpi|q^{-r})}{(1-|\varpi|^{-2}q^{-r})(1-|\varpi|q^{-r})}. \qquad (5.17)
\end{aligned}
$$

While these terms are fairly complicated, we can use Mathematica to sum them and simplify the result. Consequently, we find that

$$
Z(r - \frac{1}{2}, \Phi, f)(1_G) = \frac{(1+q^{-r})(1-q^{-(r+1)})}{(1-q^{-(r-1)})(1-q^{-(r+2)})}.
$$

If we multiply the numerator and denominator by $(1-q^{-r})$, we have the following theorem.

Theorem 5.1.1. *Let $\pi = L(\nu^{\frac{3}{2}} St_{\mathrm{GL}_2}, 1)$, then for the "good" test vector f and "good" section $\{\Phi_{s_0+r}\}_{r=1}^{\infty}$ described above we get*

$$
\begin{aligned}
Z(r - \frac{1}{2}, \Phi, f)(1_G) &= \frac{(1-q^{-2r})(1-q^{-(r+1)})}{(1-q^{-(r-1)})(1-q^{-r})(1-q^{-(r+2)})} \\
&= \frac{(r, \nu^{-1}) L(r, \mathbb{1}_{F^\times}) L(r, \nu^2)}{\zeta(2r)\zeta(r+1)}.
\end{aligned}
$$

Notice that as polynomials in $\mathbb{C}[q^{-r}]$,

$$
\deg(L(r, \pi, \mathbf{r}_{st})^{-1}) = \deg(\zeta(2r)^{-1}\zeta(r+1)^{-1})
$$

as was mentioned in a previous section.

Note that this example is by no means the most complicated case of the doubling integral. For several representations with a 1-dimensional space invariant under the long-root parahoric, we actually need to break $Z(s_0+r, \Phi, f)$ into six cases depending on whether $\mathrm{ord}(a_i) < 0$, $= 0$ or > 0. Except in the spherical case, which can be computed by hand, computation of the L-factors was aided by Mathematica. In particular, the various terms akin to $Z_{\pm,\pm}(s_0 + r, \Phi, f)(1_G)$ were computed by hand and then summed and simplified using Mathematica.

5.2 An unresolved case

We would now like to say a few words about a the case of the doubling integral which is still unresolved. If one refers to our table containing the various dimensions of parahoric invariance, we see that there are two representations of $\mathrm{Sp}_2(F)$ with a one-dimensional space of $I_\emptyset$-fixed vectors which are not fixed under any larger parahoric. They are the square-integrable representations of $\mathrm{Sp}_2(F)$:

- St_{Sp_2}, the Steinberg representation for $\mathrm{Sp}_2(F)$, and
- $\delta([\xi, \nu\xi], 1)_{T_\xi^2}$, a constituent of $\mathrm{Ind}_{P_\emptyset}^G(\nu\xi \otimes \xi)$.

In both cases, we can show that the space of Iwahori-fixed vectors is spanned by f with

$$f(w) = (-\frac{1}{q})^{\ell(w)}.$$

The identical situation exists for the representations

$$\Pi((\varsigma\nu St)', \varsigma\nu^{\frac{3}{2}} \otimes \sigma(\varsigma\nu^{\frac{1}{2}}))$$

for $\widetilde{\mathrm{Sp}_2}(F)$. The difficulty in resolving this case comes with computing the doubling integral for our choice of $\{\Phi_{s_0+r}\}$.

As with the other cases, we would like to apply our interpolation trick to a product of good lattices. Recall that we have,

$$\Phi_{s_0+r} = \lambda(\varphi^{(r)}_{L^1 \oplus L^2} \otimes \overline{\varphi^{(r)}_{L^3 \oplus L^4}})$$

where L^i are all good lattices. Moreover, we would like our Φ is be fixed under $i(I_\emptyset \times \{1\})$, but not $i(I \times \{1\})$ for any larger parahoric $I \supset I_\emptyset$. In order to prevent invariance of the latter type, L^1 and L^2 must satisfy the following conditions:

1. $\mathrm{vol}(L^1) \neq \mathrm{vol}(L^2)$ and

2. $(L^i)^\# \neq (L^j)$, for $i, j \in \{1, 2\}$.

One consequence of these requirements is that one must use the quadratic space $(M_2(F), \kappa\det)$ to find a sufficient supply of lattices. Notice that the quadratic space $(B^-(F), \kappa N(x))$ simply does not have enough good lattices to satisfy the condition above.

Now let us consider the various good lattices in $M_2(F)$. In particular, we have the following self dual lattices.

$$\begin{aligned} L_1 &= M_2(\mathcal{O}) = \left\{ \begin{pmatrix} x & y \\ z & w \end{pmatrix} \mid x, y, z, w \in \mathcal{O} \right\} \\ L_2 &= \left\{ \begin{pmatrix} x & \varpi^{-1} y \\ \varpi z & w \end{pmatrix} \mid x, y, z, w \in \mathcal{O} \right\} \\ L_3 &= \left\{ \begin{pmatrix} \varpi^{-1} x & y \\ z & \varpi w \end{pmatrix} \mid x, y, z, w \in \mathcal{O} \right\} \\ L_4 &= \left\{ \begin{pmatrix} \varpi^{-1} x & \varpi^{-1} y \\ \varpi z & \varpi w \end{pmatrix} \mid x, y, z, w \in \mathcal{O} \right\} \end{aligned}$$

If we normalize the additive Haar measure on $M_2(F)$ so that the volume of the self dual lattices are one, then we also have the following good lattices with volume q^{-1}:

$$L_{ij} = L_i \cap L_j$$

where $i < j$ and $i + j \neq 5$. Notice that L_{12} is the same as the previously defined L_0. Finally, we have a minimal good lattice given by

$$L_{00} = L_1 \cap L_4 = L_2 \cap L_3 = \left\{ \begin{pmatrix} x & y \\ \varpi z & \varpi w \end{pmatrix} \mid x, y, z, w \in \mathcal{O} \right\}.$$

In order for Φ_{s_0+r} to have the desired invariance properties, we must choose $L^1 = L_{00}$ and $L^2 = L_{ij}$ for any lattice with $\mathrm{vol}(L_{ij}) = q^{-1}$.

Thus we need to compute $I(f, \varphi_{L_{00} \otimes L_{12}})(x)$. While computing this, we need to evaluate summations such as

$$f(w_{\alpha\beta}) \sum_{a \in \mathcal{O}/\mathcal{P}} q \omega_V(\mathbf{n}_1(a)) \mathrm{char}_{\mathcal{O}}(Q(x_1)) \widehat{\varphi_{L_{12}}}(x_1) \varphi_{L_{00}}(x_2)$$

and

$$f(w_{\beta\alpha})q^2\mathrm{char}_{\mathcal{O}}((x_1,x_2)_V)\varphi_{L_{12}}(x_1)\mathrm{char}_{\mathcal{O}}(Q(x_2))\widehat{\varphi_{L_{00}}}(x_2).$$

While these are somewhat manageable, they do produce some problematic terms. For instance, we get several functions similar to

$$\varphi_{L_{12}\oplus L_{13}}(x).$$

Because of such terms, computing the doubling integral would involve us evaluate the integral

$$\int_F\int_{\mathrm{Sym}_2(F)}\int_{V_r^2}\varphi_{L\oplus L'}(x)\varphi_{L_{12}\oplus L_{13}}(-x\begin{pmatrix} a_1 & c \\ & a_2 \end{pmatrix})\psi(\mathrm{tr}(-b[Q(x)]))\,dx\,db\,dc.$$

It is here that we finally find where our method stalls. Let us define a function on $\mathrm{GL}_2(F)$ by

$$\eta^*(g) := \int_F\int_{\mathrm{Sym}_2(F)}\int_{V_r^2}\varphi_{L\oplus L'}(x)\varphi_{L_{12}\oplus L_{13}}(-xg))\psi(\mathrm{tr}(-b[Q(x)]))\,dx\,db\,dc. \quad (5.18)$$

Notice that our method computes

$$\eta^*\begin{pmatrix} a_1 & c \\ & a_2 \end{pmatrix}$$

as long as $\eta^* \in \mathcal{H}(J_1\backslash\mathrm{GL}_2(F)/J_2)$ where $J_i \in \{J, \bar{J}\}$ and $J \subset \mathrm{GL}_2(F)$ is the Iwahori-subgroup. Unfortunately, η^* does not have the proper right-invariance. Notice that

$$\mathrm{vol}(L_{12}) = \mathrm{vol}(L_{13})$$

and

$$L_{12}\cap L_{13} \notin \{L_{12}, L_{13}\}.$$

Therefore, $\varphi_{L_{12}\oplus L_{13}}(x)$ is not right-invariant for either J or $\bar{J}$.

Ultimately, there is no reason to believe that our doubling method fails for these representations. In fact, we have not even established that our choices for $\{\Phi_{s_0+r}\}$ are the wrong ones. What can be said is that the current methods and results used to compute the doubling integral are not sufficiently general so as to let us compute $Z(s_0 + r, \Phi, f)$ for in this case.

Because, the conditions on f are very rigid in this case, there seems to be just two ways to resolve this difficulty. First, one might generalize the intermediate computations so that they apply to η^*. However, even if one were to generalize the necessary results, this method seems to require too much brute force. In particular, evaluating $I(f, \varphi)(x)$ in this case produces a linear combination with several dozen terms. Another avenue that one might attempt is finding a different set of candidates for Φ_{s_0+r}. Ideally, it would be somekind of refinement of the present interpolation method that would either reduce to or simplify our interpolation method for the cases where we have already established the "good test vectors".

Table 5.1: **Computing $I(f,\varphi)(x)$ for various φ.**

φ	$\mathrm{Vol}(I_\emptyset)^{-1}I(f,\varphi)$
$\varphi_{L_i\oplus L_i}(x)$, $i\in\{1,2\}$	$[f(1)+qf(w_\alpha)+qf(w_\beta)+q^2f(w_{\alpha\beta})+q^2f(w_{\beta\alpha})+q^3f(w_{\alpha\beta\alpha})$ $+q^3f(w_{\beta\alpha\beta})+q^4f(w_{\alpha\beta\alpha\beta})]\varphi_{L_i\oplus L_i}(x)$
$\varphi_{L_0\oplus L_0}$	$[f(1)+qf(w_\alpha)-f(w_\beta)+qf(w_{\alpha\beta})-qf(w_{\beta\alpha})+q^2f(w_{\alpha\beta\alpha})$ $-qf(w_{\beta\alpha\beta})-q^2f(w_{\alpha\beta\alpha\beta})]\varphi_{L_0\oplus L_0}$ $+[f(w_\beta)-f(w_{\alpha\beta})+qf(w_{\beta\alpha})-qf(w_{\alpha\beta\alpha})](\varphi_{L_0\oplus L_1}+\varphi_{L_0\oplus L_2})$ $+[f(w_{\alpha\beta})+qf(w_{\alpha\beta\alpha})+qf(w_{\beta\alpha\beta})+q^2f(w_{\alpha\beta\alpha\beta})](\varphi_{L_1\oplus L_1}+\varphi_{L_2\oplus L_2})$
$\varphi_{L_0\oplus L_1}$	$[qf(w_\alpha)+q^2f(w_{\alpha\beta})-qf(w_{\beta\alpha})-q^2f(w_{\beta\alpha\beta})]\varphi_{L_0\oplus L_0}$ $+[f(1)-f(w_\alpha)+qf(w_\beta)-qf(w_{\alpha\beta})]\varphi_{L_0\oplus L_1}$ $+[qf(w_{\beta\alpha})-qf(w_{\alpha\beta\alpha})+q^2f(w_{\beta\alpha\beta})-q^2f(w_{\alpha\beta\alpha\beta})]\varphi_{L_0\oplus L_2}$ $+[f(w_\alpha)+qf(w_{\alpha\beta})+qf(w_{\beta\alpha})+q^2f(w_{\alpha\beta\alpha})$ $+q^2f(w_{\beta\alpha\beta})+q^3f(w_{\alpha\beta\alpha\beta})]\varphi_{L_1\oplus L_1}$ $+[qf(w_{\alpha\beta\alpha})+q^2f(w_{\alpha\beta\alpha\beta})]\varphi_{L_2\oplus L_2}$
$\varphi_{L_{ra}\oplus L_{ra}}$	$[f(1)+qf(w_\alpha)-f(w_\beta)-qf(w_{\alpha\beta})-qf(w_{\beta\alpha})$ $-q^2f(w_{\alpha\beta\alpha})+qf(w_{\beta\alpha\beta})+q^2f(w_{\alpha\beta\alpha\beta})]\varphi_{L_{ra}\oplus L_{ra}}$

Table 5.2: **Computing $\int_F \Lambda \, dc$ for necessary lattices.**

$\Lambda(r, L, L', L'', L'''; \cdot)$	$\mathrm{ord}(a_1) \leq 0$ $\mathrm{ord}(a_2) < 0$	$\mathrm{ord}(a_1) \leq 0$ $\mathrm{ord}(a_2) = 0$	$\mathrm{ord}(a_1) \leq 0$ $\mathrm{ord}(a_2) > 0$	$\mathrm{ord}(a_1) > 0$ $\mathrm{ord}(a_2) < 0$	$\mathrm{ord}(a_1) > 0$ $\mathrm{ord}(a_2) = 0$	$\mathrm{ord}(a_1) > 0$ $\mathrm{ord}(a_2) > 0$
$\Lambda(r, L_*, L_*, L_*, L_*; \cdot)$	$C_1 W_0(*, *)$	$C_1 W_0(*, *)$	$C_1 W_0(*, *)$	$C_1 W_0(*, *)$	$C_1 W_0(*, *)$	$C_1 W_0(*, *)$
$\Lambda(r, L_0, L_0, L_0, L_i; \cdot)$	$W_0(0,0)$ $+C_2 W_0(0, i)$	$W_0(0,0)$ $+C_2 W_0(0, i)$	$C_1 W_0(0, i)$	$W_0(0,0)$ $+C_2 W_0(0, i)$	$W_0(0,0)$ $+C_2 W_0(0, i)$	$C_1 W_0(0, i)$
$\Lambda(r, L_0, L_0, L_i, L_i; \cdot)$	$W_0(0,0)$ $+C_2 W_0(0, i)$	$W_0(0,0)$ $+C_2 W_0(0, i)$	$C_1 W_0(0, i)$	$C_1 W_0(0, i)$	$C_1 W_0(0, i)$	$C_1 W_0(0, i)$
$\Lambda(r, L_0, L_i, L_0, L_0; \cdot)$	$C_1 W_0(0, i)$	$W_0(0,0)$ $+C_2 W_0(0, i)$	$W_0(0,0)$ $+C_2 W_0(0, i)$	$C_1 W_0(0, i)$	$W_0(0,0)$ $+C_2 W_0(0, i)$	$W_0(0,0)$ $+C_2 W_0(0, i)$
$\Lambda(r, L_0, L_i, L_0, L_i; \cdot)$	$W_0(0, i)$ $+C_2 W_0(i, i)$	$W_0(0, i)$ $+C_2 W_0(i, i)$	$W_0(0, i)$ $+C_2 W_0(i, i)$	$W_0(0, i)$ $+C_2 W_0(i, i)$	$W_0(0, i)$ $+C_2 W_0(i, i)$	$W_0(0, i)$ $+C_2 W_0(i, i)$

Table continued on next page.

Table 5.2: **Computing $\int_F \Lambda\, dc$ for necessary lattices.**

$\Lambda(r, L, L', L'', L'''; \cdot)$	$\mathrm{ord}(a_1) \leq 0$ $\mathrm{ord}(a_2) < 0$	$\mathrm{ord}(a_1) \leq 0$ $\mathrm{ord}(a_2) = 0$	$\mathrm{ord}(a_1) \leq 0$ $\mathrm{ord}(a_2) > 0$	$\mathrm{ord}(a_1) > 0$ $\mathrm{ord}(a_2) < 0$	$\mathrm{ord}(a_1) > 0$ $\mathrm{ord}(a_2) = 0$	$\mathrm{ord}(a_1) > 0$ $\mathrm{ord}(a_2) > 0$
$\Lambda(r, L_0, L_i, L_0, L_j; \cdot)$	$W_0(0,i)$ $+C_2W_0(i,j)$	$W_0(0,0)$ $+C_2W_0(i,j)$	$W_0(0,j)$ $+C_2W_0(i,j)$	$W_0(0,i)$ $+C_2W_0(i,j)$	$W_0(0,0)$ $+C_2W_0(i,j)$	$W_0(0,j)$ $+C_2W_0(i,j)$
$\Lambda(r, L_0, L_i, L_i, L_i; \cdot)$	$W_0(0,i)$ $+C_2W_0(i,i)$	$W_0(0,i)$ $+C_2W_0(i,i)$	$W_0(0,i)$ $+C_2W_0(i,i)$	$C_1W_0(i,i)$	$C_1W_0(i,i)$	$C_1W_0(i,i)$
$\Lambda(r, L_0, L_i, L_j, L_j; \cdot)$	$W_0(0,i)$ $+C_2W_0(i,j)$	$W_0(0,0)$ $+C_2W_0(i,j)$	$W_0(0,i)$ $+C_2W_0(i,j)$	$C_1W_0(i,j)$	$W_0(0,j)$ $+C_2W_0(i,j)$	$q^{-1}W_0(j,j)$ $+(1-q^{-1})W(0,j)$ $+C_2W_0(i,j)$
$* \in \{0,1,2,ra\}$ $\quad i,j \in \{1,2\}$ with $i \neq j$ $\quad W_0(s,t) = W_0(r, L_s, L_t)$ $\quad C_1 = \frac{1-q^{-(2r+1)}}{1-q^{-2r}}$ $\quad C_2 = \frac{q^{-2r}-q^{-(2r+1)}}{1-q^{-2r}}$						

Appendix A

Computing Weil Indices

Because we are working the Weil representation, there is a question of computing various Weil indices for characters of second degree. However, since these computations are less critical than some of the others in the thesis, we included Weil index calculations as well as some related computations in this appendix rather than the main section of the text. To perform these calculations, we largely follow the works of Rao [31] and Kudla [18] with some supporting definitions from Serre [34]. For this section, ψ is a fixed continuous character of $(F, +)$ for a local field F. Note that for the purposes of this section, finite fields will have the discrete topology in order that they be locally compact abelian groups. For any $a \in F$, define

$$\psi_a(x) := \psi(ax).$$

Let X be a vector space over F with non-degenerate symmetric bilinear form $(\cdot, \cdot)_X$. Further, let $X^* = \mathrm{Hom}_F(X, F)$ be the dual vector space and $[x, x^*] = x^*(x)$ be the canonical pairing. Since $(\cdot, \cdot)_X$ is non-degenerate, we have an isomorphism $\varrho : X \to X^*$ such that

$$(x, y)_X = [x, \varrho(y)].$$

Let dx and dx^* denote the Haar measures on X and X^* that are dual with respect to Fourier transform defined using $\psi \circ [\cdot, \cdot]$. In particular, for $f \in S(X)$,

$$\mathcal{F}f(x^*) := \int_X \psi([x, x^*])f(x)dx.$$

We also have a similar transform $\mathcal{F}^*$ on $S(X^*)$. Moreover, if η is a tempered distribution on $S(X)$, we define the Fourier transform $\mathcal{F}\eta$ as

$$< f, \mathcal{F}\eta >=< \mathcal{F}f, \eta > .$$

Finally, for $Q(x) = \frac{1}{2}(x, x)$, we get a character of second degree $\phi_Q(x) = \psi(Q(x))$. Notice that it is called a character of second degree because it satisfies

$$\phi_Q(x + y)\phi_Q(x)^{-1}\phi_Q(y)^{-1} = \psi((x, y)_X)$$

and $\psi((x, y)_X)$ is a bicharacter of X. Because ϕ_Q is valued $\{|z| = 1\}$, we can define a tempered distribution $\phi_Q dx$ on $S(X)$ via

$$< f, \phi_Q dx >= \int_X f(x)\phi_Q(x)dx.$$

Analogously, we have a quadratic form on X^* given by

$$Q^*(x^*) = \psi(\frac{1}{2}[\varrho^{-1}(x^*), x^*])$$

with which we can define a character of second degree ϕ_{Q^*} and distribution $\phi_{Q^*}^{-1} dx^*$ on $S(X^*)$. Finally, we see that $\phi_Q dx$ and $\phi_{Q^*}^{-1} dx^*$ are related by Fourier transform in the following way:

$$\mathcal{F}(\phi_Q dx) = \gamma(\psi)|\varrho|^{-\frac{1}{2}}\phi_{Q^*}^{-1} dx^*.$$

Here $\gamma(\psi)$ is called the Weil index of ψ and $|\varrho|$ is a constant such that

$$\mathcal{F}^*\mathcal{F}f(x) = |\varrho|f(-x).$$

Rao uses the following notation

$$\begin{aligned} \gamma(\psi) &= \text{Weil index of: } x \to \psi(x^2), \\ \gamma(a, \psi) &= \frac{\gamma(\psi_a)}{\gamma(\psi)} \end{aligned}$$

The main theorem regarding $\gamma(a, \psi)$ is the following (see [31]).

Theorem A.0.1. *$\gamma(ac^2, \psi) = \gamma(a, \psi)$ and the function $a \to \gamma(a, \psi)$ is a character of second degree on $F^\times/(F^\times)^2$ with*

$$\gamma(ab, \psi)\gamma(a, \psi)^{-1}\gamma(b, \psi)^{-1} = (a, b)_F$$

where $(a, b)_F$ is the Hilbert symbol of F.

Recall that for $a, b \in F^\times$

$$(a, b)_F = \begin{cases} 1 & \text{if } \exists (z, x, y) \in F^3 \, (z, x, y) \neq (0, 0, 0), \text{ with } z^2 - ax^2 - by^2 = 0 \\ -1 & \text{otherwise} \end{cases}.$$

Further, recall that for a quadratic space (V, Q) and orthogonal basis $\{e_1, e_2, \ldots, e_n\}$ such that $a_i = (e_i, e_i)_V$. Then the **Hasse invariant** of (V, Q) is given by

$$\epsilon(Q) := \prod_{i<j}(a_i, a_j)_F.$$

Given the theorem, the next corollary is a routine computation by the definitions.

Corollary A.0.1. *We have the following identities:*

1. $\gamma(a, \psi_b) = (a, b)_F \gamma(a, \psi)$.
2. $\gamma(-1, \psi) = \gamma(\psi)^{-2}$.
3. $\gamma(a, \psi)^2 = (-1, a)_F = (a, a)_F$.
4. $\gamma(a, \psi)^4 = 1$ *and* $\gamma(\psi)^8 = 1$.

So let us summarize Rao's formulas for $\gamma(a, \psi)$ for the various local fields F.

Proposition A.0.1. *1. For $F = \mathbb{C}$,*

$$\gamma(\psi) = (a, b)_F = \epsilon(Q) = 1$$

for all choices of character ψ, quadratic form Q and all $a, b \in \mathbb{C}^\times$.

2. For $F = \mathbb{R}$

$$(a, b)_F = \begin{cases} -1 & \text{if } a, b < 0 \\ 1 & \text{otherwise} \end{cases}$$

For $\psi(t) = \exp(2\pi t\sqrt{-1})$, then

$$\gamma(\psi_a) = \psi(\frac{\operatorname{sign}(a)}{8}).$$

Finally, if Q has quadratic form of signature (a, b), then

$$\epsilon(Q) = (-1)^{\frac{b(b-1)}{2}}.$$

The next result will deal with the case of finite fields.

Proposition A.0.2. *Let F be a finite field with* char $\neq 2$. *Then*

1. *$\gamma(a, \psi) = \left(\frac{a}{F}\right)$, where $\left(\frac{\cdot}{F}\right)$ is the Legendre symbol for F.*

2. *$a \mapsto \gamma(a, \psi)$ is a homomorphism and $(a, b)_F = 1$ for all $a, b \in F^\times$.*

3. *$\epsilon(Q) = 1$ for any quadratic space (V, Q).*

4. *If $\mathbb{F}_p$ is the prime subfield of F and $[F : \mathbb{F}_p] = n$ and $\psi' = \psi \circ \operatorname{tr}$, then*

$$\gamma_F(\psi') = \gamma_{\mathbb{F}_p}(\psi)^n.$$

5. *If ψ is the character $t \mapsto \exp(\frac{2\pi t\sqrt{-1}}{p})$ on $\mathbb{F}_p$, then*

$$\gamma(\psi) = \begin{cases} 1 \text{ if } p \equiv 1 \mod 4 \\ \sqrt{-1} \text{ if } p \equiv 3 \mod 4 \end{cases}.$$

Finally, we come to the case that F is a p-adic field with residue characteristic different from 2. For an additive character ψ, let $\mathrm{ord}(\psi)$ be the largest integer m such that ψ is trivial on $\varpi^{-m}\mathcal{O}$. Futhermore, let ι represent a parity function defined for both characters and elements of $F^\times$ in the following way:

$$\begin{aligned} \iota(\psi) &:= \begin{cases} 1 \text{ if } \mathrm{ord}(\psi) \text{ is odd} \\ 0 \text{ if } \mathrm{ord}(\psi) \text{ is even} \end{cases} \\ \iota(a) &:= \begin{cases} 1 \text{ if } \mathrm{ord}(a) \text{ is odd} \\ 0 \text{ if } \mathrm{ord}(a) \text{ is even} \end{cases} \end{aligned}.$$

So with this, we have the following result.

Proposition A.0.3. *Let F be a p-adic local field with residue characteristic $\neq 2$ and let $\mathbb{F}_q$ be its residue field. For a fixed character ψ, let $m = \mathrm{ord}(\psi)$ Then we can define a character $\tilde{\psi}$ on $\mathcal{O}/\mathcal{P}$ by*

$$\tilde{\psi}(x + \mathcal{P}) := \psi(\varpi^{-(m+1)}x).$$

This is a non-trivial character on $\mathbb{F}_q$ and

$$\gamma_F(\psi) = \gamma_{\mathbb{F}_q}(\tilde{\psi})^{\iota(\psi)}.$$

Furthermore,

$$\gamma_F(a, \psi) = \left[\left(\frac{[u]}{\mathbb{F}_q}\right)\gamma_{\mathbb{F}_q}(\tilde{\psi})\right]^{\iota(a)}$$

where $a = \varpi^{\mathrm{ord}(a)}u$ and $[u]$ is image of u under the isomorphism $\mathcal{O}/\mathcal{P} \simeq \mathbb{F}_q$.

We notice that computing the Weil indices in the p-adic case, reduces to the computation in the finite field case.

Finally, because it is a factor in many of our calculations, we will include a formula for $(\cdot,\cdot)_F$ with F a p-adic field. The formula comes from[34].

Proposition A.0.4. *Let F be a p-adic local field with $p \neq 2$. For $a, b \in F$, let $a = \varpi^{\alpha} u$, $b = \varpi^{\beta} v$ with $u, v \in \mathcal{O}^{\times}$. Then we have*

$$(a,b)_F = \left(\frac{-1}{\mathbb{F}_q}\right)^{\alpha\beta} \left(\frac{[u]}{\mathbb{F}_q}\right)^{\beta} \left(\frac{[v]}{\mathbb{F}_q}\right)^{\alpha}.$$

It is worth noting that $(\cdot,\cdot)_F$ is trivial on $\mathcal{O}^{\times} \times \mathcal{O}^{\times}$, which is a case that frequently appears in our applications.

Appendix B

Tables of Results

In this Appendix, we compile much of data relevant to this thesis. Tables B.1 and B.2 contain all of the irreducible constituents π of the unramified principal series of $\mathrm{Sp}_2(F)$ along with the inducing data for π is a constituent of the induced representation. Much of this data along with the conditions for π to be tempered or L^2 was distilled from the work of Sally and Tadić [33]. The dimension of the parahoric invariants in Table B.1 and the Jacquet modules in Tables B.3 and B.4 were computed by the author. The Weil-Deligne data to which each constituent (in Table B.1) maps are contained in Tables B.5 and B.6. Table B.7 outlines all of the choices of "good test vectors" used to compute $Z(s_0 + r, \Phi, f)$. In this table, the column for f refers to the Iwahori fixed vectors from Table 4.3 and the quadratic spaces V refer the spaces listed in Table 4.5. Finally, Table B.8 contains the computed values for

$$Z(s, \phi, f) = C(q) \frac{L(s, \pi, \mathbf{r}_{st})}{d_\Phi(s)}$$

where $C(q) \in \mathbb{C}(q^{\frac{1}{2}})$.

Tables B.9 and B.10 contain data regarding the constituents of the principal series for both $\widetilde{\mathrm{Sp}_2}(F)$ and $\mathrm{SO}_5(F)$. It is arranged so that representations that correspond via our bijection appear in the identical rows in their respective tables. The reducibility points of $\mathrm{SO}_5(F)$ were described in Jantzen [12] and [13]. The reducibility points for $\widetilde{\mathrm{Sp}_2}(F)$ were derived by the author using Tadić's criteria. All dimensions of parahoric invariants as well as all Jacquet modules (in Tables B.11-B.14) were computed by the author. Finally, Tables B.15-B.16 (resp. Tables B.17-B.18) are the analogs to Tables B.5-B.6 for $\mathrm{SO}_5(F)$ (resp. to Tables B.7-B.8 for $\widetilde{\mathrm{Sp}_2}(F)$).

For the tables pertaining to $\mathrm{Sp}_2(F)$ we fix the following notation:

$P_\emptyset$: Borel Subgroup	$P_\emptyset = M_\emptyset N_\emptyset$ with $M_\emptyset \cong \mathrm{GL}_1(F)^2$
$I_\emptyset$: Iwahori Subgroup	
P_α: Siegel Parabolic Subgroup	$P_\alpha = M_\alpha N_\alpha$ with $M_\alpha \cong \mathrm{GL}_2(F)$
I_α: Siegel Parahoric Subgroup	
P_β: Long Root Parabolic Subgroup	$P_\beta = M_\beta N_\beta$ with $M_\beta \cong \mathrm{GL}_1(F) \times \mathrm{Sp}_1(F)$
I_β: Long Root Parahoric Subgroup	
$K = \mathrm{Sp}_2(\mathcal{O})$.	
$\mathrm{Ind}_B^{\mathrm{Sp}_1(F)}(\xi) = T_\xi^1 \oplus T_\xi^2$	

Table B.1: **Constituents of Unramified Principal Series for $\mathrm{Sp}_2(F)$ and the Dimension of Parahoric Invariants**

		Representation	**Constituents**	$I_\emptyset$	I_α	I_β	K
I		$\mathrm{Ind}_{P_\emptyset}^G(\chi_1 \otimes \chi_2)$ (irreducible)		8	4	4	1
II	a	$\mathrm{Ind}_{P_\emptyset}^G(\chi\nu^{\frac{1}{2}} \otimes \chi\nu^{-\frac{1}{2}})$	$\mathrm{Ind}_{P_\alpha}^G(\chi St_{\mathrm{GL}_2})$	4	1	2	0
	b	$\chi \notin \{\nu^{\pm\frac{3}{2}}, \nu^{\pm\frac{1}{2}}\varsigma\}$ with $\varsigma^2 = 1$	$\mathrm{Ind}_{P_\alpha}^G(\chi \mathbb{1}_{\mathrm{GL}_2})$	4	3	2	1
III	a	$\mathrm{Ind}_{P_\emptyset}^G(\chi \otimes \nu)$	$\mathrm{Ind}_{P_\beta}^G(\chi \otimes St_{\mathrm{Sp}_1})$	4	2	1	0
	b	$\chi \notin \{\nu^{\pm 2}, \varsigma\}$ with $\varsigma^2 = 1$	$\mathrm{Ind}_{P_\beta}^G(\chi \otimes \mathbb{1}_{\mathrm{Sp}_1})$	4	2	3	1
IV	a	$\mathrm{Ind}_{P_\emptyset}^G(\chi \otimes \xi),\ \xi = \lvert\cdot\rvert^{\frac{\pi i}{\log q}}$	$\mathrm{Ind}_{P_\beta}^G(\chi \otimes T_\xi^2)$	4	2	1	0
	b	$\chi \notin \{\varsigma\nu^{\pm 1}\}$ with $\varsigma^2 = 1$	$\mathrm{Ind}_{P_\beta}^G(\chi \otimes T_\xi^1)$	4	2	3	1
V	a	$\mathrm{Ind}_{P_\emptyset}^G(\nu^2 \otimes \nu)$	St_{Sp_2}	1	0	0	0
	b		$L(\nu^2, St_{\mathrm{Sp}_1})$	3	2	1	0
	c		$L(\nu^{\frac{3}{2}} St_{\mathrm{GL}_2}, 1)$	3	1	2	0
	d		$\mathbb{1}_{\mathrm{Sp}_2}$	1	1	1	1
VI	a	$\mathrm{Ind}_{P_\emptyset}^G(\nu \otimes \mathbb{1}_{F^\times})$	$\sigma(\nu^{\frac{1}{2}} St_{\mathrm{GL}_2})$	3	1	1	0
	b		$L(\nu^{\frac{1}{2}} St_{\mathrm{GL}_2}, 1)$	1	0	1	0
	c		$\sigma(\nu^{\frac{1}{2}} \mathbb{1}_{\mathrm{GL}_2})$	1	1	0	0
	d		$L(\nu, \mathbb{1}_{F^\times}, 1)$	3	2	2	1
VII	a	$\mathrm{Ind}_{P_\emptyset}^G(\nu \otimes \xi)$	$\sigma(\nu \otimes T_\xi^2)$	2	1	0	0
	b	$\xi = \lvert\cdot\rvert^{\frac{\pi i}{\log q}}$	$\sigma(\nu \otimes T_\xi^1)$	2	1	1	0
	c		$L(\nu, T_\xi^2)$	2	1	1	0
	d		$L(\nu, T_\xi^1)$	2	1	2	1
Table continued on next page.							

Table B.1: **Constituents of Unramified Principal Series for $\mathrm{Sp}_2(F)$ and the Dimension of Parahoric Invariants**

		Representation	**Constituents**	$I_\emptyset$	I_α	I_β	K
VIII	a	$\mathrm{Ind}_{P_\emptyset}^G(\nu\xi \otimes \xi)$ $\xi = \lvert\cdot\rvert^{\frac{\pi i}{\log q}}$	$\delta([\xi, \nu\xi], 1)_{T_\xi^2}$	1	0	0	0
	b		$\delta([\xi, \nu\xi], 1)_{T_\xi^1}$	1	0	1	0
	c		$L(\nu^{\frac{1}{2}}\xi St_{\mathrm{GL}_2}, 1)$*	2	1	1	0
	d		$L(\nu\xi, T_\xi^2)$	1	1	0	0
	e		$L(\nu\xi, T_\xi^1)$	1	1	1	1
* denotes a representation having multiplicity 2 in $\mathrm{Ind}_{P_\emptyset}^G(\nu\xi \otimes \xi)$							

Table B.2: **Tempered and L^2 Representations for** $\mathrm{Sp}_2(F)$

		Representation	**Constituents**	**Tempered**	L^2
I		$\mathrm{Ind}_{P_\emptyset}^G(\chi_1 \otimes \chi_2)$ (irreducible)		χ_i unitary	
II	a	$\mathrm{Ind}_{P_\emptyset}^G(\chi\nu^{\frac{1}{2}} \otimes \chi\nu^{-\frac{1}{2}})$	$\mathrm{Ind}_{P_\alpha}^G(\chi St_{\mathrm{GL}_2})$	χ unitary	
	b	$\chi \notin \{\nu^{\pm\frac{3}{2}}, \nu^{\pm\frac{1}{2}}\varsigma\}$ with $\varsigma^2 = 1$	$\mathrm{Ind}_{P_\alpha}^G(\chi \mathbb{1}_{\mathrm{GL}_2})$		
III	a	$\mathrm{Ind}_{P_\emptyset}^G(\chi \otimes \nu)$	$\mathrm{Ind}_{P_\beta}^G(\chi \otimes St_{\mathrm{Sp}_1})$	χ unitary	
	b	$\chi \notin \{\nu^{\pm 2}, \varsigma\}$ with $\varsigma^2 = 1$	$\mathrm{Ind}_{P_\beta}^G(\chi \otimes \mathbb{1}_{\mathrm{Sp}_1})$		
IV	a	$\mathrm{Ind}_{P_\emptyset}^G(\chi \otimes \xi),\ \xi = \lvert\cdot\rvert^{\frac{\pi i}{\log q}}$	$\mathrm{Ind}_{P_\beta}^G(\chi \otimes T_\xi^2)$	χ unitary	
	b	$\chi \notin \{\varsigma\nu^{\pm 1}\}$ with $\varsigma^2 = 1$	$\mathrm{Ind}_{P_\beta}^G(\chi \otimes T_\xi^1)$	χ unitary	
V	a	$\mathrm{Ind}_{P_\emptyset}^G(\nu^2 \otimes \nu)$	St_{Sp_2}	•	•
	b		$L(\nu^2, St_{\mathrm{Sp}_1})$		
	c		$L(\nu^{\frac{3}{2}} St_{\mathrm{GL}_2}, 1)$		
	d		$\mathbb{1}_{\mathrm{Sp}_2}$		
VI	a	$\mathrm{Ind}_{P_\emptyset}^G(\nu \otimes \mathbb{1}_{F^\times})$	$\sigma(\nu^{\frac{1}{2}} St_{\mathrm{GL}_2})$	•	
	b		$L(\nu^{\frac{1}{2}} St_{\mathrm{GL}_2}, 1)$		
	c		$\sigma(\nu^{\frac{1}{2}} \mathbb{1}_{\mathrm{GL}_2})$	•	
	d		$L(\nu, \mathbb{1}_{F^\times}, 1)$		
VII	a	$\mathrm{Ind}_{P_\emptyset}^G(\nu \otimes \xi)$, $\xi = \lvert\cdot\rvert^{\frac{\pi i}{\log q}}$	$\sigma(\nu \otimes T_\xi^2)$	•	
	b		$\sigma(\nu \otimes T_\xi^1)$	•	
	c		$L(\nu, T_\xi^2)$		
	d		$L(\nu, T_\xi^1)$		

Table continued on next page.

Table B.2: **Tempered and L^2 Representations for** $\mathrm{Sp}_2(F)$

		Representation	**Constituents**	**Tempered**	L^2
VIII	a	$\mathrm{Ind}_{P_\emptyset}^G(\nu\xi \otimes \xi)$ $\xi = \lvert\cdot\rvert^{\frac{\pi i}{\log q}}$	$\delta([\xi, \nu\xi], 1)_{T_\xi^2}$	•	•
	b		$\delta([\xi, \nu\xi], 1)_{T_\xi^1}$	•	•
	c		$L(\nu^{\frac{1}{2}}\xi St_{\mathrm{GL}_2}, 1)$		
	d		$L(\nu\xi, T_\xi^2)$		
	e		$L(\nu\xi, T_\xi^1)$		

Table B.3: **Jacquet Modules-$\mathrm{Sp}_2(F)$-Siegel Parabolic**

		Representation	$r^G_{P_\alpha}(\pi) \in \mathfrak{R}(M_\alpha)$	#
I		$\mathrm{Ind}^G_{P_\emptyset}(\chi_1 \otimes \chi_2)$ (irreducible)	$\mathrm{Ind}^{\mathrm{GL}_2}_B(\chi_1 \otimes \chi_2) + \mathrm{Ind}^{\mathrm{GL}_2}_B(\chi_1 \otimes \chi_2^{-1})$ $+\mathrm{Ind}^{\mathrm{GL}_2}_B(\chi_1^{-1} \otimes \chi_2) + \mathrm{Ind}^{\mathrm{GL}_2}_B(\chi_1^{-1} \otimes \chi_2^{-1})$	4
II	a	$\mathrm{Ind}^G_{P_\alpha}(\chi St_{\mathrm{GL}_2})$	$\chi St_{\mathrm{GL}_2} + \chi^{-1} St_{\mathrm{GL}_2}$ $+\mathrm{Ind}^{\mathrm{GL}_2}_B(\chi\nu^{\frac{1}{2}} \otimes \chi^{-1}\nu^{\frac{1}{2}})$	3
	b	$\mathrm{Ind}^G_{P_\alpha}(\chi \mathbb{1}_{\mathrm{GL}_2})$	$\chi \mathbb{1}_{\mathrm{GL}_2} + \chi^{-1} \mathbb{1}_{\mathrm{GL}_2}$ $+\mathrm{Ind}^{\mathrm{GL}_2}_B(\chi\nu^{-\frac{1}{2}} \otimes \chi^{-1}\nu^{-\frac{1}{2}})$	3
III	a	$\mathrm{Ind}^G_{P_\beta}(\chi \otimes St_{\mathrm{Sp}_1})$	$\mathrm{Ind}^{\mathrm{GL}_2}_B(\chi \otimes \nu) + \mathrm{Ind}^{\mathrm{GL}_2}_B(\chi^{-1} \otimes \nu)$	2
	b	$\mathrm{Ind}^G_{P_\beta}(\chi \otimes \mathbb{1}_{\mathrm{Sp}_1})$	$\mathrm{Ind}^{\mathrm{GL}_2}_B(\chi \otimes \nu^{-1}) + \mathrm{Ind}^{\mathrm{GL}_2}_B(\chi^{-1} \otimes \nu^{-1})$	2
IV	a	$\mathrm{Ind}^G_{P_\beta}(\chi \otimes T^2_\xi)$	$\mathrm{Ind}^{\mathrm{GL}_2}_B(\chi \otimes \xi) + \mathrm{Ind}^{\mathrm{GL}_2}_B(\chi^{-1} \otimes \xi)$	2
	b	$\mathrm{Ind}^G_{P_\beta}(\chi \otimes T^1_\xi)$	$\mathrm{Ind}^{\mathrm{GL}_2}_B(\chi \otimes \xi) + \mathrm{Ind}^{\mathrm{GL}_2}_B(\chi^{-1} \otimes \xi)$	2
V	a	St_{Sp_2}	$\nu^{\frac{3}{2}} St_{\mathrm{GL}_2}$	1
	b	$L(\nu^2, St_{\mathrm{Sp}_1})$	$\nu^{\frac{3}{2}} \mathbb{1}_{\mathrm{GL}_2} + \mathrm{Ind}^{\mathrm{GL}_2}_B(\nu \otimes \nu^{-2})$	2
	c	$L(\nu^{\frac{3}{2}} St_{\mathrm{GL}_2}, 1)$	$\nu^{-\frac{3}{2}} St_{\mathrm{GL}_2} + \mathrm{Ind}^{\mathrm{GL}_2}_B(\nu^2 \otimes \nu^{-1})$	2
	d	$\mathbb{1}_{\mathrm{Sp}_2}$	$\nu^{-\frac{3}{2}} \mathbb{1}_{\mathrm{GL}_2}$	1
VI	a	$\sigma(\nu^{\frac{1}{2}} St_{\mathrm{GL}_2})$	$2 \cdot \nu^{\frac{1}{2}} St_{\mathrm{GL}_2} + \nu^{\frac{1}{2}} \mathbb{1}_{\mathrm{GL}_2}$	3
	b	$L(\nu^{\frac{1}{2}} St_{\mathrm{GL}_2}, 1)$	$\nu^{-\frac{1}{2}} St_{\mathrm{GL}_2}$	1
	c	$\sigma(\nu^{\frac{1}{2}} \mathbb{1}_{\mathrm{GL}_2})$	$\nu^{\frac{1}{2}} \mathbb{1}_{\mathrm{GL}_2}$	1
	d	$L(\nu, \mathbb{1}_{F^\times}, 1)$	$2 \cdot \nu^{-\frac{1}{2}} \mathbb{1}_{\mathrm{GL}_2} + \nu^{-\frac{1}{2}} St_{\mathrm{GL}_2}$	3
VII	a	$\sigma(\nu \otimes T^2_\xi)$	$\mathrm{Ind}^{\mathrm{GL}_2}_B(\nu \otimes \xi)$	1
	b	$\sigma(\nu \otimes T^1_\xi)$	$\mathrm{Ind}^{\mathrm{GL}_2}_B(\nu \otimes \xi)$	1
	c	$L(\nu, T^2_\xi)$	$\mathrm{Ind}^{\mathrm{GL}_2}_B(\nu^{-1} \otimes \xi)$	1
	d	$L(\nu, T^1_\xi)$	$\mathrm{Ind}^{\mathrm{GL}_2}_B(\nu^{-1} \otimes \xi)$	1

Table continued on next page.

Table B.3: **Jacquet Modules-$\mathrm{Sp}_2(F)$-Siegel Parabolic**

		Representation	$r^G_{P_\alpha}(\pi) \in \mathfrak{R}(M_\alpha)$	#
VIII	a	$\delta([\xi,\nu\xi],1)_{T^2_\xi}$	$\nu^{\frac{1}{2}}\xi St_{\mathrm{GL}_2}$	1
	b	$\delta([\xi,\nu\xi],1)_{T^1_\xi}$	$\nu^{\frac{1}{2}}\xi St_{\mathrm{GL}_2}$	1
	c	$L(\nu^{\frac{1}{2}}\xi St_{\mathrm{GL}_2},1)$	$\nu^{\frac{1}{2}}\xi \mathbb{1}_{\mathrm{GL}_2} + \nu^{-\frac{1}{2}}\xi St_{\mathrm{GL}_2}$	2
	d	$L(\nu\xi, T^2_\xi)$	$\nu^{-\frac{1}{2}}\xi \mathbb{1}_{\mathrm{GL}_2}$	1
	e	$L(\nu\xi, T^1_\xi)$	$\nu^{-\frac{1}{2}}\xi \mathbb{1}_{\mathrm{GL}_2}$	1

Table B.4: **Jacquet Modules-$\mathrm{Sp}_2(F)$-Long Root Parabolic**

		Representation	$r^G_{P_\beta}(\pi) \in \mathfrak{R}(M_\beta)$	#
I		$\mathrm{Ind}^G_{P_\emptyset}(\chi_1 \otimes \chi_2)$ (irreducible)	$\chi_1 \otimes \mathrm{Ind}^{\mathrm{Sp}_1}_B(\chi_2) + \chi_1^{-1} \otimes \mathrm{Ind}^{\mathrm{Sp}_1}_B(\chi_2)$ $\chi_2 \otimes \mathrm{Ind}^{\mathrm{Sp}_1}_B(\chi_1) + \chi_2^{-1} \otimes \mathrm{Ind}^{\mathrm{Sp}_1}_B(\chi_1)$	4
II	a	$\mathrm{Ind}^G_{P_\alpha}(\chi St_{\mathrm{GL}_2})$	$\chi\nu^{\frac{1}{2}} \otimes \mathrm{Ind}^{\mathrm{Sp}_1}_B(\chi\nu^{-\frac{1}{2}})$ $+\chi^{-1}\nu^{\frac{1}{2}} \otimes \mathrm{Ind}^{\mathrm{Sp}_1}_B(\chi\nu^{\frac{1}{2}})$	2
	b	$\mathrm{Ind}^G_{P_\alpha}(\chi \mathbb{1}_{\mathrm{GL}_2})$	$\chi\nu^{-\frac{1}{2}} \otimes \mathrm{Ind}^{\mathrm{Sp}_1}_B(\chi\nu^{\frac{1}{2}})$ $+\chi^{-1}\nu^{-\frac{1}{2}} \otimes \mathrm{Ind}^{\mathrm{Sp}_1}_B(\chi\nu^{-\frac{1}{2}})$	2
III	a	$\mathrm{Ind}^G_{P_\beta}(\chi \otimes St_{\mathrm{Sp}_1})$	$\chi \otimes St_{\mathrm{Sp}_1} + \chi^{-1} \otimes St_{\mathrm{Sp}_1}$ $+\nu \otimes \mathrm{Ind}^{\mathrm{Sp}_1}_B(\chi)$	3
	b	$\mathrm{Ind}^G_{P_\beta}(\chi \otimes \mathbb{1}_{\mathrm{Sp}_1})$	$\chi \otimes \mathbb{1}_{\mathrm{Sp}_1} + \chi^{-1} \otimes \mathbb{1}_{\mathrm{Sp}_1}$ $+\nu^{-1} \otimes \mathrm{Ind}^{\mathrm{Sp}_1}_B(\chi)$	3
IV	a	$\mathrm{Ind}^G_{P_\beta}(\chi \otimes T^2_\xi)$	$\chi \otimes T^2_\xi + \chi^{-1} \otimes T^2_\xi$ $+\xi \otimes \mathrm{Ind}^{\mathrm{Sp}_1}_B(\chi)$	3
	b	$\mathrm{Ind}^G_{P_\beta}(\chi \otimes T^1_\xi)$	$\chi \otimes T^1_\xi + \chi^{-1} \otimes T^1_\xi$ $+\xi \otimes \mathrm{Ind}^{\mathrm{Sp}_1}_B(\chi)$	3
V	a	St_{Sp_2}	$\nu^2 \otimes St_{\mathrm{Sp}_1}$	1
	b	$L(\nu^2, St_{\mathrm{Sp}_1})$	$\nu^{-2} \otimes St_{\mathrm{Sp}_1} + \nu \otimes \mathrm{Ind}^{\mathrm{Sp}_1}_B(\nu^2)$	2
	c	$L(\nu^{\frac{3}{2}} St_{\mathrm{GL}_2}, 1)$	$\nu^2 \otimes \mathbb{1}_{\mathrm{Sp}_1} + \nu^{-1} \otimes \mathrm{Ind}^{\mathrm{Sp}_1}_B(\nu^2)$	2
	d	$\mathbb{1}_{\mathrm{Sp}_2}$	$\nu^{-2} \otimes \mathbb{1}_{\mathrm{Sp}_1}$	1
Table continued on next page.				

Table B.4: **Jacquet Modules-$\mathrm{Sp}_2(F)$-Long Root Parabolic**

		Representation	$r^G_{P_\beta}(\pi) \in \mathfrak{R}(M_\beta)$	#
VI	a	$\sigma(\nu^{\frac{1}{2}} St_{\mathrm{GL}_2})$	$\mathbb{1}_{F^\times} \otimes St_{\mathrm{Sp}_1} + \nu \otimes \mathrm{Ind}^G_B(\mathbb{1}_{F^\times})$	2
	b	$L(\nu^{\frac{1}{2}} St_{\mathrm{GL}_2}, 1)$	$\mathbb{1}_{F^\times} \otimes \mathbb{1}_{\mathrm{Sp}_1}$	1
	c	$\sigma(\nu^{\frac{1}{2}} \mathbb{1}_{\mathrm{GL}_2})$	$\mathbb{1}_{F^\times} \otimes St_{\mathrm{Sp}_1}$	1
	d	$L(\nu, \mathbb{1}_{F^\times}, 1)$	$\mathbb{1}_{F^\times} \otimes \mathbb{1}_{\mathrm{Sp}_1} + \nu^{-1} \otimes \mathrm{Ind}^{\mathrm{Sp}_1}_B(\mathbb{1}_{F^\times})$	2
VII	a	$\sigma(\nu \otimes T^2_\xi)$	$\nu \otimes T^2_\xi + \xi \otimes St_{\mathrm{Sp}_1}$	2
	b	$\sigma(\nu \otimes T^1_\xi)$	$\nu \otimes T^1_\xi + \xi \otimes St_{\mathrm{Sp}_1}$	2
	c	$L(\nu, T^2_\xi)$	$\nu^{-1} \otimes T^2_\xi + \xi \otimes \mathbb{1}_{\mathrm{Sp}_1}$	2
	d	$L(\nu, T^1_\xi)$	$\nu^{-1} \otimes T^1_\xi + \xi \otimes \mathbb{1}_{\mathrm{Sp}_1}$	2
VIII	a	$\delta([\xi, \nu\xi], 1)_{T^2_\xi}$	$\nu\xi \otimes T^2_\xi$	1
	b	$\delta([\xi, \nu\xi], 1)_{T^1_\xi}$	$\nu\xi \otimes T^1_\xi$	1
	c	$L(\nu^{\frac{1}{2}} \xi St_{\mathrm{GL}_2}, 1)$	$\xi \otimes \mathrm{Ind}^{\mathrm{Sp}_1}_B(\nu\xi)$	1
	d	$L(\nu\xi, T^2_\xi)$	$\nu^{-1}\xi \otimes T^2_\xi$	1
	e	$L(\nu\xi, T^1_\xi)$	$\nu^{-1}\xi \otimes T^1_\xi$	1

Table B.5: **Weil-Deligne data and conjectural L-values– $\mathrm{Sp}_2(F)$ case**

		$\rho(\mathrm{Fr}_q)$	N	**L-factor**
I		$\mathrm{diag}(\chi_1(\varpi), \chi_2(\varpi), 1, \chi_1^{-1}(\varpi), \chi_2^{-1}(\varpi))$	0	$L(s,\chi_1)L(s,\chi_2)L(s,\mathbb{1})L(s,\chi_1^{-1})L(s,\chi_2^{-1})$
II	a	$\mathrm{diag}(\chi\nu^{\frac{1}{2}}(\varpi), \chi\nu^{-\frac{1}{2}}(\varpi), 1, \chi^{-1}\nu^{-\frac{1}{2}}(\varpi), \chi^{-1}\nu^{\frac{1}{2}}(\varpi))$	N_1	$L(s,\chi\nu^{\frac{1}{2}})L(s,\mathbb{1})L(s,\chi^{-1}\nu^{\frac{1}{2}})$
	b		0	$L(s,\chi\nu^{\frac{1}{2}})L(s,\chi\nu^{-\frac{1}{2}})L(s,\mathbb{1})L(s,\chi^{-1}\nu^{-\frac{1}{2}})L(s,\chi^{-1}\nu^{\frac{1}{2}})$
III	a	$\mathrm{diag}(\chi(\varpi), \nu(\varpi), 1, \chi^{-1}(\varpi), \nu^{-1}(\varpi))$	N_2	$L(s,\chi)L(s,\nu)L(s,\chi^{-1})$
	b		0	$L(s,\chi)L(s,\nu)L(s,\mathbb{1})L(s,\chi^{-1})L(s,\nu^{-1})$
IV	a	$\mathrm{diag}(\chi(\varpi), \xi(\varpi), 1, \chi^{-1}(\varpi), \xi(\varpi))$	0	$L(s,\chi)L(s,\xi)L(s,\mathbb{1})L(s,\chi^{-1})L(s,\xi)$
	b		0	$L(s,\chi)L(s,\xi)L(s,\mathbb{1})L(s,\chi^{-1})L(s,\xi)$
V	a	$\mathrm{diag}(\nu^2(\varpi), \nu(\varpi), 1, \nu^{-2}(\varpi), \nu^{-1}(\varpi))$	N_3	$L(s,\nu^2)$
	b		N_2	$L(s,\nu^2)L(s,\nu)L(s,\nu^{-2})$
	c		N_1	$L(s,\nu^2)L(s,\mathbb{1})L(s,\nu^{-1})$
	d		0	$L(s,\nu^2)L(s,\nu)L(s,\mathbb{1})L(s,\nu^{-2})L(s,\nu^{-1})$

Table continued on next page.

Table B.5: **Weil-Deligne data and conjectural L-values–$\mathrm{Sp}_2(F)$ case**

		$\rho(\mathrm{Fr}_q)$	N	**L-factor**
VI	a	$\mathrm{diag}(\nu(\varpi), 1, 1, \nu^{-1}(\varpi), 1)$	N_4	$L(s,\nu)L(s,\mathbb{1})^2$
	b		N_1	$L(s,\nu)L(s,\mathbb{1})^2$
	c		N_4	$L(s,\nu)L(s,\mathbb{1})^2$
	d		0	$L(s,\nu)L(s,\mathbb{1})^3L(s,\nu^{-1})$
VII	a	$\mathrm{diag}(\nu(\varpi), \xi(\varpi), 1, \nu^{-1}(\varpi), \xi(\varpi))$	N_5	$L(s,\nu)L(s,\xi)^2$
	b		N_5	$L(s,\nu)L(s,\xi)^2$
	c		0	$L(s,\nu)L(s,\xi)^2L(s,\mathbb{1})L(s,\nu^{-1})$
	d		0	$L(s,\nu)L(s,\xi)^2L(s,\mathbb{1})L(s,\nu^{-1})$
VIII	a	$\mathrm{diag}(\nu\xi(\varpi), \xi(\varphi), 1, \nu^{-1}\xi(\varpi), \xi(\varpi))$	N_6	$L(s,\nu\xi)L(s,\xi)L(s,\mathbb{1})$
	b		N_6	$L(s,\nu\xi)L(s,\xi)L(s,\mathbb{1})$
	c		N_1	$L(s,\nu\xi)L(s,\xi)L(s,\mathbb{1})$
	d		0	$L(s,\nu\xi)L(s,\xi)^2L(s,\mathbb{1})L(s,\nu^{-1}\xi)$
	d		0	$L(s,\nu\xi)L(s,\xi)^2L(s,\mathbb{1})L(s,\nu^{-1}\xi)$

Table B.6: **Table of relevant nilpotent operators for $\mathfrak{so}_5(\mathbb{C})$**

$$N_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix} \quad N_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad N_3 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix} \quad N_4 = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

$$N_5 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad N_6 = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

Table B.7: **Good Test Vectors for Computing** $L(s,\pi,\mathbf{r})$

		f^*	V^{**}	$\varphi=\varphi_1\otimes\varphi_2$	
I		f_0	V^{s_l}	$\varphi_{L_1\otimes L_1}\otimes C_0\varphi_{L_1\otimes L_1}$	$C_0=1$
II	a	f_1	V^{s_l}	$\varphi_{L_0\oplus L_0}\otimes C_1\varphi_{L_0\otimes L_0}$	$C_1=\left((q^{t+\frac{3}{2}}-q^2)\mathrm{vol}(I_\emptyset)^{-1}\right)/(q+1)$
	b	f_0	V^{s_l}	$\varphi_{L_1\otimes L_1}\otimes C_0\varphi_{L_1\otimes L_1}$	$C_0=1$
III	a	f_2	V^{s_l}	$\varphi_{L_0\oplus L_1}\otimes C_2\varphi_{L_0\oplus L_1}$	$C_2=\left((q^{t+1}-q^2)\mathrm{vol}(I_\emptyset)^{-1}\right)/(q+1)$
	b	f_0	V^{s_l}	$\varphi_{L_1\otimes L_1}\otimes C_0\varphi_{L_1\otimes L_1}$	$C_0=1$
IV	a	f_3	V^{s_l}	$\varphi_{L_0\oplus L_1}\otimes \mathrm{vol}(I_\emptyset)^{-1}\left(\varphi_{L_0\oplus L_0}-C_3(\varphi_{L_0\oplus L_1}+\varphi_{L_0\oplus L_2})\right)$	$C_3=q/(q+1)$
	b	f_0	V^{s_l}	$\varphi_{L_1\otimes L_1}\otimes C_0\varphi_{L_1\otimes L_1}$	$C_0=1$
V	a	f_4	-	-	-
	b	f_5	V^{s_l}	$\varphi_{L_0\oplus L_1}\otimes C_5\varphi_{L_0\oplus L_1}$	$C_5=\left(q\mathrm{vol}(I_\emptyset)^{-1}\right)/(q+1)$
	c	f_6	V^{s_l}	$\varphi_{L_0\oplus L_0}\otimes C_6\varphi_{L_0\oplus L_0}$	$C_6=\left(q^2\mathrm{vol}(I_\emptyset)^{-1}\right)/(2q+2)$
	d	f_0	V^{s_l}	$\varphi_{L_1\otimes L_1}\otimes C_0\varphi_{L_1\otimes L_1}$	$C_0=1$

Table continued on next page

Table B.7: **Good Test Vectors for Computing** $L(s,\pi,\mathbf{r})$

		f^*	V^{**}	$\varphi=\varphi_1\otimes\varphi_2$	
VI	a	f_9	V^{s_l}	$\varphi_{L_0\oplus L_0}\otimes C_9\varphi_{L_0\oplus L_0}$	$C_9=\left(q\mathrm{vol}(I_\emptyset)^{-1}\right)/(2q+2)$
	b	f_8	V^{s_l}	$\varphi_{L_0\oplus L_1}\otimes C_8\varphi_{L_0\oplus L_1}$	$C_8=\left(q\mathrm{vol}(I_\emptyset)^{-1}\right)/(q+1)$
	c	f_7	V^{ra_l}	$\varphi_{L_{ra}\otimes L_{ra}}\otimes C_7\varphi_{L_{ra}\otimes L_{ra}}$	$C_7=\left(q\mathrm{vol}(I_\emptyset)^{-1}\right)/\left(2(q+1)^2\right)$
	d	f_0	V^{s_l}	$\varphi_{L_1\otimes L_1}\otimes C_0\varphi_{L_1\otimes L_1}$	$C_0=1$
VII	a	f_7	V^{ra_l}	$\varphi_{L_{ra}\otimes L_{ra}}\otimes C_7\varphi_{L_{ra}\otimes L_{ra}}$	$C_7=\left(q\mathrm{vol}(I_\emptyset)^{-1}\right)/\left(2(q+1)^2\right)$
	b	f_6	V^{s_l}	$\varphi_{L_0\oplus L_0}\otimes C_6\varphi_{L_0\oplus L_0}$	$C_6=\left(q^2\mathrm{vol}(I_\emptyset)^{-1}\right)/(2q+2)$
	c	f_9	V^{s_l}	$\varphi_{L_0\oplus L_0}\otimes C_9\varphi_{L_0\oplus L_0}$	$C_9=\left(q\mathrm{vol}(I_\emptyset)^{-1}\right)/(2q+2)$
	d	f_0	V^{s_l}	$\varphi_{L_1\otimes L_1}\otimes C_0\varphi_{L_1\otimes L_1}$	$C_0=1$
VIII	a	f_4	-	-	-
	b	f_8	V^{s_l}	$\varphi_{L_0\oplus L_1}\otimes C_8\varphi_{L_0\oplus L_1}$	$C_8=\left(q\mathrm{vol}(I_\emptyset)^{-1}\right)/(q+1)$
	c	f_5	V^{s_l}	$\varphi_{L_0\oplus L_1}\otimes C_5\varphi_{L_0\oplus L_1}$	$C_5=\left(q\mathrm{vol}(I_\emptyset)^{-1}\right)/(q+1)$
	d	f_7	V^{ra_l}	$\varphi_{L_{ra}\otimes L_{ra}}\otimes C_7\varphi_{L_{ra}\otimes L_{ra}}$	$C_7=\left(q\mathrm{vol}(I_\emptyset)^{-1}\right)/\left(2(q+1)^2\right)$
	e	f_0	V^{s_l}	$\varphi_{L_1\otimes L_1}\otimes C_0\varphi_{L_1\otimes L_1}$	$C_0=1$

$*$=See Table 4.3; $**$=See Table 4.5

Table B.8: **Computed L-factors for** $\mathrm{Sp}_2(F)$

		$C(q)$	$L(s,\pi,\mathbf{r}_{st})$	$d_\Phi(s)$
I		1	$L(s,\chi_1)L(s,\chi_2)L(s,\mathbb{1})L(s,\chi_1^{-1})L(s,\chi_2^{-1})$	$\zeta(2s)\zeta(2s+2)\zeta(s+2)$
II	a	$\left(2(q-1)(q^t-q^{\frac{1}{2}})\right)/q^{\frac{3}{2}}$	$L(s,\chi\nu^{\frac{1}{2}})L(s,\mathbb{1})L(s,\chi^{-1}\nu^{\frac{1}{2}})$	$\zeta(2s)\zeta(s+1)$
	b	1	$L(s,\chi\nu^{\frac{1}{2}})L(s,\chi\nu^{-\frac{1}{2}})L(s,\mathbb{1})L(s,\chi^{-1}\nu^{-\frac{1}{2}})L(s,\chi^{-1}\nu^{\frac{1}{2}})$	$\zeta(2s)\zeta(2s+2)\zeta(s+2)$
III	a	$((q-1)(q^t-q))/q$	$L(s,\chi)L(s,\nu)L(s,\chi^{-1})$	$\zeta(2s)\zeta(s+1)$
	b	1	$L(s,\chi)L(s,\nu)L(s,\mathbb{1})L(s,\chi^{-1})L(s,\nu^{-1})$	$\zeta(2s)\zeta(2s+2)\zeta(s+2)$
IV	a	$(q-1)/q$	$L(s,\chi)L(s,\xi)L(s,\mathbb{1})L(s,\chi^{-1})L(s,\xi)$	$\zeta(2s)\zeta(s+1)L(r-1,\xi)L(r,\xi)$
	b	1	$L(s,\chi)L(s,\xi)L(s,\mathbb{1})L(s,\chi^{-1})L(s,\xi)$	$\zeta(2s)\zeta(2s+2)\zeta(s+2)$
V	a	-	-	-
	b	$(q-1)/q$	$L(s,\nu^2)L(s,\nu)L(s,\nu^{-2})$	$\zeta(2s)\zeta(s+1)$
	c	$(q-1)/q$	$L(s,\nu^2)L(s,\mathbb{1})L(s,\nu^{-1})$	$\zeta(2s)\zeta(s+1)$
	d	1	$L(s,\nu^2)L(s,\nu)L(s,\mathbb{1})L(s,\nu^{-2})L(s,\nu^{-1})$	$\zeta(2s)\zeta(2s+2)\zeta(s+2)$

Table continued on next page

Table B.8: **Computed L-factors for $\mathrm{Sp}_2(F)$**

		$C(q)$	$L(s,\pi,\mathbf{r}_{st})$	$d_\Phi(s)$
VI	a	$(q-1)/q^2$	$L(s,\nu)L(s,\mathbb{1})^2$	$\zeta(2s)\zeta(s+1)$
	b	$(q-1)/q$	$L(s,\nu)L(s,\mathbb{1})^2$	$\zeta(2s)\zeta(s+1)$
	c	$1/q^2$	$L(s,\nu)L(s,\mathbb{1})^2$	$\zeta(2s)\zeta(s+1)$
	d	1	$L(s,\nu)L(s,\mathbb{1})^3L(s,\nu^{-1})$	$\zeta(2s)\zeta(2s+2)\zeta(s+2)$
VII	a	$1/q^2$	$L(s,\nu)L(s,\xi)^2$	$\zeta(2s)L(s+1,\xi)$
	b	$(q-1)/q$	$L(s,\nu)L(s,\xi)^2$	$\zeta(2s)\zeta(s+1)$
	c	$(q-1)/q^2$	$L(s,\nu)L(s,\xi)^2L(s,\mathbb{1})L(s,\nu^{-1})$	$\zeta(2s)\zeta(s+1)L(s,\xi)L(s-1,\xi)$
	d	1	$L(s,\nu)L(s,\xi)^2L(s,\mathbb{1})L(s,\nu^{-1})$	$\zeta(2s)\zeta(2s+2)\zeta(s+2)$
VIII	a	-	-	-
	b	$(q-1)/q$	$L(s,\nu\xi)L(s,\xi)L(s,\mathbb{1})$	$\zeta(2s)\zeta(s+1)$
	c	$(q-1)/q$	$L(s,\nu\xi)L(s,\xi)L(s,\mathbb{1})$	$\zeta(2s)\zeta(s+1)$
	d	$1/q^2$	$L(s,\nu\xi)L(s,\xi)^2L(s,\mathbb{1})L(s,\nu^{-1}\xi)$	$\zeta(2s)\zeta(s)\zeta(s-1)L(s+1,\xi)$
	e	1	$L(s,\nu\xi)L(s,\xi)^2L(s,\mathbb{1})L(s,\nu^{-1}\xi)$	$\zeta(2s)\zeta(2s+2)\zeta(s+2)$

For tables for $\widetilde{\mathrm{Sp}_2}(F)$, we have the following notation:

$$\begin{array}{lll}
\tilde{P}_\emptyset = \tilde{M}_\emptyset N_\emptyset & \tilde{M}_\emptyset \simeq (F^\times)^2 \times \mathbb{C}^1 \subset \tilde{G} & I^*_\emptyset = \{[k, \lambda(k)]_L \mid k \in I_\emptyset\} \\
\tilde{P}_\alpha = \tilde{M}_\alpha N_\alpha & \tilde{M}_\alpha \simeq \mathrm{GL}_2(F) \times \mathbb{C}^1 \subset \tilde{G} & I^*_\alpha = \{[k, \lambda(k)]_L \mid k \in I_\alpha\} \\
\tilde{P}_\beta = \tilde{M}_\beta N_\beta & \tilde{M}_\beta \simeq F^\times \times \widetilde{\mathrm{Sp}_1}(F) \subset \tilde{G} & I^*_\beta = \{[k, \lambda(k)]_L \mid k \in I_\beta\} \\
 & & K^* = \{[k, \lambda(k)]_L \mid k \in K\}
\end{array}$$

For results pertaining to $\mathrm{SO}_5(F)$, we have the following notation:

$P'_\emptyset$: Borel Subgroup $\quad P'_\emptyset = M'_\emptyset N'_\emptyset$ with $M'_\emptyset \cong \mathrm{GL}_1(F)^2$

$I'_\emptyset$: Iwahori Subgroup

P'_α: Siegel Parabolic Subgroup $\quad P'_\alpha = M'_\alpha N'_\alpha$ with $M'_\alpha \cong \mathrm{GL}_2(F)$

I'_α: Siegel Parahoric Subgroup

P'_β: Short Root Parabolic Subgroup $\quad P'_\beta = M'_\beta N'_\beta$ with $M'_\beta \cong \mathrm{GL}_1(F) \times \mathrm{S0}_3(F)$

I'_β: Short Root Parahoric Subgroup

$K' = \mathrm{SO}_5(\mathcal{O})$.

Table B.9: **Constituents of Unramified Principal Series for $\widetilde{\mathrm{Sp}_2}(F)$ and the Dimension of Parahoric Invariants**

		Representation	**Constituents**	$I_\emptyset^*$	I_α^*	I_β^*	K^*
IX		$\mathrm{Ind}_{\tilde{P}_\emptyset}^{\tilde{G}}((\chi_1 \otimes \chi_2)')$ (irreducible)		8	4	4	1
X	a	$\mathrm{Ind}_{\tilde{P}_\emptyset}^{\tilde{G}}((\chi\nu^{\frac{1}{2}} \otimes \chi\nu^{-\frac{1}{2}})')$	$\mathrm{Ind}_{\tilde{P}_\alpha}^{\tilde{G}}((\chi St_{\mathrm{GL}_2})')$	4	1	2	0
	b	$\chi \notin \{\varsigma, \varsigma\nu^{\pm 1}\}$	$\mathrm{Ind}_{\tilde{P}_\alpha}^{\tilde{G}}((\chi \mathbb{1}_{GL_2})')$	4	3	2	1
XI	a	$\mathrm{Ind}_{\tilde{P}_\emptyset}^{\tilde{G}}((\chi \otimes \varsigma\nu^{\frac{1}{2}})')$	$\mathrm{Ind}_{\tilde{P}_\beta}^{\tilde{G}}(\chi \otimes \tau(\varsigma\nu^{\frac{1}{2}}))$	4	2	1	0
	b	$\chi \notin \{\varsigma\nu^{\pm\frac{1}{2}}, \varsigma\nu^{\pm\frac{3}{2}}\}$	$\mathrm{Ind}_{\tilde{P}_\beta}^{\tilde{G}}(\chi \otimes \pi(\varsigma\nu^{\frac{1}{2}}))$	4	2	3	1
XII	a	$\mathrm{Ind}_{\tilde{P}_\emptyset}^{\tilde{G}}((\varsigma\nu^{\frac{1}{2}} \otimes \varsigma\nu^{-\frac{1}{2}})')$	$\Pi((\varsigma St_{\mathrm{GL}_2})', \varsigma\nu^{\frac{1}{2}} \otimes \pi(\varsigma\nu^{\frac{1}{2}}))$	1	0	1	0
	b		$\Pi((\varsigma St_{\mathrm{GL}_2})', \varsigma\nu^{\frac{1}{2}} \otimes \tau(\varsigma\nu^{\frac{1}{2}}))$	3	1	1	0
	c		$\Pi((\varsigma \mathbb{1}_{\mathrm{GL}_2})', \varsigma\nu^{\frac{1}{2}} \otimes \tau(\varsigma\nu^{\frac{1}{2}}))$	1	1	0	0
	d		$\Pi((\varsigma \mathbb{1}_{\mathrm{GL}_2})', \varsigma\nu^{\frac{1}{2}} \otimes \pi(\varsigma\nu^{\frac{1}{2}}))$	3	2	2	1
XIII	a	$\mathrm{Ind}_{\tilde{P}_\emptyset}^{\tilde{G}}((\varsigma\nu^{\frac{3}{2}} \otimes \varsigma\nu^{\frac{1}{2}})')$	$\Pi((\varsigma\nu St_{\mathrm{GL}_2})', \varsigma\nu^{\frac{3}{2}} \otimes \tau(\varsigma\nu^{\frac{1}{2}}))$	1	0	0	0
	b		$\Pi((\varsigma\nu St_{\mathrm{GL}_2})', \varsigma\nu^{\frac{3}{2}} \otimes \pi(\varsigma\nu^{\frac{1}{2}}))$	3	1	2	0
	c		$\Pi((\varsigma\nu \mathbb{1}_{\mathrm{GL}_2})', \varsigma\nu^{\frac{3}{2}} \otimes \tau(\varsigma\nu^{\frac{1}{2}}))$	3	2	1	0
	d		$\Pi((\varsigma\nu \mathbb{1}_{\mathrm{GL}_2})', \varsigma\nu^{\frac{3}{2}} \otimes \pi(\varsigma\nu^{\frac{1}{2}}))$	1	1	1	1
Note: $\varsigma^2 = 1$ and ς is unramified.							

Table B.10: **Constituents of Unramified Principal Series for $\mathrm{SO}_5(F)$ and the Dimension of Parahoric Invariants**

		Representation	**Constituents**	$I'_\emptyset$	I'_α	I'_β	K'
IX		$\mathrm{Ind}_{P'_\emptyset}^{G'}(\chi_1 \otimes \chi_2)$ (irreducible)		8	4	4	1
X	a	$\mathrm{Ind}_{P'_\emptyset}^{G'}(\chi\nu^{\frac{1}{2}} \otimes \chi\nu^{-\frac{1}{2}})$	$\mathrm{Ind}_{P'_\alpha}^{G'}(\chi St_{\mathrm{GL}_2})$	4	1	2	0
	b	$\chi \notin \{\varsigma, \varsigma\nu^{\pm 1}\}$ $\varsigma^2 = 1$	$\mathrm{Ind}_{P'_\alpha}^{G'}(\chi \mathbb{1}_{GL_2})$	4	3	2	1
XI	a	$\mathrm{Ind}_{P'_\emptyset}^{G'}(\chi \otimes \varsigma\nu^{\frac{1}{2}})$	$\mathrm{Ind}_{P'_\beta}^{G'}(\chi \otimes \varsigma St_{\mathrm{SO}_3})$	4	2	1	0
	b	$\chi \notin \{\varsigma\nu^{\pm\frac{1}{2}}, \varsigma\nu^{\pm\frac{3}{2}}\}$ $\varsigma^2 = 1$	$\mathrm{Ind}_{P'_\beta}^{G'}(\chi \otimes \varsigma\mathbb{1}_{\mathrm{SO}_3})$	4	2	3	1
XII	a		$\sigma(\varsigma\nu^{\frac{1}{2}} \otimes \varsigma\mathbb{1}_{\mathrm{SO}_3})$	1	0	1	0
	b	$\mathrm{Ind}_{P'_\emptyset}^{G'}(\varsigma\nu^{\frac{1}{2}} \otimes \varsigma\nu^{-\frac{1}{2}})$	$\sigma(\varsigma\nu^{\frac{1}{2}} \otimes \varsigma St_{\mathrm{SO}_3})$	3	1	1	0
	c	$\varsigma^2 = 1$	$L(\varsigma\nu^{\frac{1}{2}}, \varsigma St_{\mathrm{SO}_3})$	1	1	0	0
	d		$L(\varsigma\nu^{\frac{1}{2}}, \varsigma\nu^{\frac{1}{2}})$	3	2	2	1
XIII	a		$\varsigma St_{\mathrm{SO}_5}$	1	0	0	0
	b	$\mathrm{Ind}_{P'_\emptyset}^{G'}(\varsigma\nu^{\frac{3}{2}} \otimes \varsigma\nu^{\frac{1}{2}})$	$L(\varsigma\nu St_{\mathrm{GL}_2}, 1)$	3	1	2	0
	c	$\varsigma^2 = 1$	$L(\varsigma\nu^{\frac{3}{2}}, \varsigma St_{\mathrm{S0}_3})$	3	2	1	0
	d		$L(\varsigma\nu^{\frac{3}{2}}, \varsigma\nu^{\frac{1}{2}})$	1	1	1	1

Note: $\varsigma^2 = 1$ and ς is unramified.

Table B.11: **Jacquet Modules-$\widetilde{\mathrm{Sp}_2}(F)$-Siegel Parabolic**

		Representation	$r^{\tilde{G}}_{\tilde{P}_\alpha}(\pi) \in \mathfrak{R}(\tilde{M}_\alpha)$	#
IX		$\mathrm{Ind}^{\tilde{G}}_{\tilde{P}_\emptyset}((\chi_1 \otimes \chi_2)')$ (irreducible)	$\mathrm{Ind}^{\mathrm{GL}_2}_B(\chi_1 \otimes \chi_2)' + \mathrm{Ind}^{\mathrm{GL}_2}_B(\chi_1 \otimes \chi_2^{-1})'$ $+\mathrm{Ind}^{\mathrm{GL}_2}_B(\chi_1^{-1} \otimes \chi_2)' + \mathrm{Ind}^{\mathrm{GL}_2}_B(\chi_1^{-1} \otimes \chi_2^{-1})'$	4
X	a	$\mathrm{Ind}^{\tilde{G}}_{\tilde{P}_\alpha}((\chi St_{\mathrm{GL}_2})')$	$(\chi St_{\mathrm{GL}_2})' + (\chi^{-1} St_{\mathrm{GL}_2})'$ $+\mathrm{Ind}^{\mathrm{GL}_2}_B(\chi\nu^{\frac{1}{2}} \otimes \chi^{-1}\nu^{\frac{1}{2}})'$	3
	b	$\mathrm{Ind}^{\tilde{G}}_{\tilde{P}_\alpha}((\chi \mathbb{1}_{\mathrm{GL}_2})')$	$(\chi \mathbb{1}_{\mathrm{GL}_2})' + (\chi^{-1} \mathbb{1}_{\mathrm{GL}_2})'$ $+\mathrm{Ind}^{\mathrm{GL}_2}_B(\chi\nu^{-\frac{1}{2}} \otimes \chi^{-1}\nu^{-\frac{1}{2}})'$	3
XI	a	$\mathrm{Ind}^{\tilde{G}}_{\tilde{P}_\beta}(\chi \otimes \tau(\varsigma\nu^{\frac{1}{2}}))$	$\mathrm{Ind}^{\mathrm{GL}_2}_B(\chi \otimes \varsigma\nu^{\frac{1}{2}})' + \mathrm{Ind}^{\mathrm{GL}_2}_B(\chi^{-1} \otimes \varsigma\nu^{\frac{1}{2}})'$	2
	b	$\mathrm{Ind}^{\tilde{G}}_{\tilde{P}_\beta}(\chi \otimes \pi(\varsigma\nu^{\frac{1}{2}}))$	$\mathrm{Ind}^{\mathrm{GL}_2}_B(\chi \otimes \varsigma\nu^{-\frac{1}{2}})' + \mathrm{Ind}^{\mathrm{GL}_2}_B(\chi^{-1} \otimes \varsigma\nu^{-\frac{1}{2}})'$	2
XII	a	$\Pi((\varsigma St_{\mathrm{GL}_2})', \varsigma\nu^{\frac{1}{2}} \otimes \pi(\varsigma\nu^{\frac{1}{2}}))$	$(\varsigma St_{\mathrm{GL}_2})'$	1
	b	$\Pi((\varsigma St_{\mathrm{GL}_2})', \varsigma\nu^{\frac{1}{2}} \otimes \tau(\varsigma\nu^{\frac{1}{2}}))$	$(\varsigma St_{\mathrm{GL}_2})' + \mathrm{Ind}^{\mathrm{GL}_2}_B(\varsigma\nu^{\frac{1}{2}} \otimes \varsigma\nu^{\frac{1}{2}})'$	2
	c	$\Pi((\varsigma \mathbb{1}_{\mathrm{GL}_2})', \varsigma\nu^{\frac{1}{2}} \otimes \tau(\varsigma\nu^{\frac{1}{2}}))$	$(\varsigma \mathbb{1}_{\mathrm{GL}_2})'$	1
	d	$\Pi((\varsigma \mathbb{1}_{\mathrm{GL}_2})', \varsigma\nu^{\frac{1}{2}} \otimes \pi(\varsigma\nu^{\frac{1}{2}}))$	$(\varsigma \mathbb{1}_{\mathrm{GL}_2})' + \mathrm{Ind}^{\mathrm{GL}_2}_B(\varsigma\nu^{-\frac{1}{2}} \otimes \varsigma\nu^{-\frac{1}{2}})$	2
XIII	a	$\Pi((\varsigma\nu St_{\mathrm{GL}_2})', \varsigma\nu^{\frac{3}{2}} \otimes \tau(\varsigma\nu^{\frac{1}{2}}))$	$(\varsigma\nu St_{\mathrm{GL}_2})'$	1
	b	$\Pi((\varsigma\nu St_{\mathrm{GL}_2})', \varsigma\nu^{\frac{3}{2}} \otimes \pi(\varsigma\nu^{\frac{1}{2}}))$	$(\varsigma\nu^{-1} St_{\mathrm{GL}_2})' + \mathrm{Ind}^{\mathrm{GL}_2}_B(\varsigma\nu^{\frac{3}{2}} \otimes \varsigma\nu^{-\frac{1}{2}})'$	2
	c	$\Pi((\varsigma\nu \mathbb{1}_{\mathrm{GL}_2})', \varsigma\nu^{\frac{3}{2}} \otimes \tau(\varsigma\nu^{\frac{1}{2}}))$	$(\varsigma\nu \mathbb{1}_{\mathrm{GL}_2})' + \mathrm{Ind}^{\mathrm{GL}_2}_B(\varsigma\nu^{\frac{1}{2}} \otimes \varsigma\nu^{-\frac{3}{2}})'$	2
	d	$\Pi((\varsigma\nu \mathbb{1}_{\mathrm{GL}_2})', \varsigma\nu^{\frac{3}{2}} \otimes \pi(\varsigma\nu^{\frac{1}{2}}))$	$(\varsigma\nu^{-\frac{1}{2}} \mathbb{1}_{\mathrm{GL}_2})'$	1

Table B.12: **Jacquet Modules-$\widetilde{\mathrm{Sp}_2}(F)$-Long Root Parabolic**

		Representation	$r^{\tilde{G}}_{\tilde{P}_\beta}(\pi) \in \mathfrak{R}(\tilde{M}_\beta)$	#
IX		$\mathrm{Ind}^{\tilde{G}}_{\tilde{P}_\emptyset}((\chi_1 \otimes \chi_2)')$ (irreducible)	$\chi_1 \otimes \mathrm{Ind}^{\widetilde{\mathrm{Sp}_1}}_B(\chi_2') + \chi_1^{-1} \otimes \mathrm{Ind}^{\widetilde{\mathrm{Sp}_1}}_B(\chi_2')$ $+\chi_2 \otimes \mathrm{Ind}^{\widetilde{\mathrm{Sp}_1}}_B(\chi_1') + \chi_2^{-1} \otimes \mathrm{Ind}^{\widetilde{\mathrm{Sp}_1}}_B(\chi_1')$	4
X	a	$\mathrm{Ind}^{\tilde{G}}_{\tilde{P}_\alpha}((\chi St_{\mathrm{GL}_2})')$	$\chi\nu^{\frac{1}{2}} \otimes \mathrm{Ind}^{\widetilde{\mathrm{Sp}_1}}_B((\chi\nu^{-\frac{1}{2}})')$ $+\chi^{-1}\nu^{\frac{1}{2}} \otimes \mathrm{Ind}^{\widetilde{\mathrm{Sp}_1}}_B((\chi\nu^{\frac{1}{2}})')$	2
	b	$\mathrm{Ind}^{\tilde{G}}_{\tilde{P}_\alpha}((\chi \mathbb{1}_{\mathrm{GL}_2})')$	$\chi\nu^{-\frac{1}{2}} \otimes \mathrm{Ind}^{\widetilde{\mathrm{Sp}_1}}_B((\chi\nu^{\frac{1}{2}})')$ $+\chi^{-1}\nu^{-\frac{1}{2}} \otimes \mathrm{Ind}^{\widetilde{\mathrm{Sp}_1}}_B((\chi\nu^{-\frac{1}{2}})')$	2
XI	a	$\mathrm{Ind}^{\tilde{G}}_{\tilde{P}_\beta}(\chi \otimes \tau(\varsigma\nu^{\frac{1}{2}}))$	$\chi \otimes \tau(\varsigma\nu^{\frac{1}{2}}) + \chi^{-1} \otimes \tau(\varsigma\nu^{\frac{1}{2}})$ $+\varsigma\nu^{\frac{1}{2}} \otimes \mathrm{Ind}^{\widetilde{\mathrm{Sp}_1}}_B(\chi')$	3
	b	$\mathrm{Ind}^{\tilde{G}}_{\tilde{P}_\beta}(\chi \otimes \pi(\varsigma\nu^{\frac{1}{2}}))$	$\chi \otimes \pi(\varsigma\nu^{\frac{1}{2}}) + \chi^{-1} \otimes \pi(\varsigma\nu^{\frac{1}{2}})$ $+\varsigma\nu^{-\frac{1}{2}} \otimes \mathrm{Ind}^{\widetilde{\mathrm{Sp}_1}}_B(\chi')$	3
XII	a	$\Pi((\varsigma St_{\mathrm{GL}_2})', \varsigma\nu^{\frac{1}{2}} \otimes \pi(\varsigma\nu^{\frac{1}{2}}))$	$\varsigma\nu^{\frac{1}{2}} \otimes \pi(\varsigma\nu^{\frac{1}{2}})$	1
	b	$\Pi((\varsigma St_{\mathrm{GL}_2})', \varsigma\nu^{\frac{1}{2}} \otimes \tau(\varsigma\nu^{\frac{1}{2}}))$	$\varsigma\nu^{\frac{1}{2}} \otimes \pi(\varsigma\nu^{\frac{1}{2}}) + 2 \cdot \varsigma\nu^{\frac{1}{2}} \otimes \tau(\varsigma\nu^{\frac{1}{2}})$	3
	c	$\Pi((\varsigma \mathbb{1}_{\mathrm{GL}_2})', \varsigma\nu^{\frac{1}{2}} \otimes \tau(\varsigma\nu^{\frac{1}{2}}))$	$\varsigma\nu^{-\frac{1}{2}} \otimes \tau(\varsigma\nu^{\frac{1}{2}})$	1
	d	$\Pi((\varsigma \mathbb{1}_{\mathrm{GL}_2})', \varsigma\nu^{\frac{1}{2}} \otimes \pi(\varsigma\nu^{\frac{1}{2}}))$	$\varsigma\nu^{-\frac{1}{2}} \otimes \tau(\varsigma\nu^{\frac{1}{2}}) + 2 \cdot \varsigma\nu^{-\frac{1}{2}} \otimes \pi(\varsigma\nu^{\frac{1}{2}})$	3
XIII	a	$\Pi((\varsigma\nu St_{\mathrm{GL}_2})', \varsigma\nu^{\frac{3}{2}} \otimes \tau(\varsigma\nu^{\frac{1}{2}}))$	$\varsigma\nu^{\frac{3}{2}} \otimes \tau(\varsigma\nu^{\frac{1}{2}})$	1
	b	$\Pi((\varsigma\nu St_{\mathrm{GL}_2})', \varsigma\nu^{\frac{3}{2}} \otimes \pi(\varsigma\nu^{\frac{1}{2}}))$	$\varsigma\nu^{\frac{3}{2}} \otimes \pi(\varsigma\nu^{\frac{1}{2}}) + \varsigma\nu^{\frac{1}{2}} \otimes \mathrm{Ind}^{\widetilde{\mathrm{Sp}_1}}_B((\varsigma\nu^{\frac{3}{2}})')$	2
	c	$\Pi((\varsigma\nu \mathbb{1}_{\mathrm{GL}_2})', \varsigma\nu^{\frac{3}{2}} \otimes \tau(\varsigma\nu^{\frac{1}{2}}))$	$\varsigma\nu^{-\frac{3}{2}} \otimes \tau(\varsigma\nu^{\frac{1}{2}}) + \varsigma\nu^{\frac{1}{2}} \otimes \mathrm{Ind}^{\widetilde{\mathrm{Sp}_1}}_B((\varsigma\nu^{\frac{3}{2}})')$	2
	d	$\Pi((\varsigma\nu \mathbb{1}_{\mathrm{GL}_2})', \varsigma\nu^{\frac{3}{2}} \otimes \pi(\varsigma\nu^{\frac{1}{2}}))$	$\varsigma\nu^{-\frac{3}{2}} \otimes \pi(\varsigma\nu^{\frac{1}{2}})$	1

Table B.13: **Jacquet Modules-$\mathrm{SO}_5(F)$-Siegel Parabolic**

		Representation	$r^{G'}_{P'_\alpha}(\pi) \in \mathfrak{R}(M'_\alpha)$	#
IX		$\mathrm{Ind}^{G'}_{P'_\emptyset}(\chi_1 \otimes \chi_2)$ (irreducible)	$\mathrm{Ind}^{\mathrm{GL}_2}_B(\chi_1 \otimes \chi_2) + \mathrm{Ind}^{\mathrm{GL}_2}_B(\chi_1 \otimes \chi_2^{-1})$ $+\mathrm{Ind}^{\mathrm{GL}_2}_B(\chi_1^{-1} \otimes \chi_2) + \mathrm{Ind}^{\mathrm{GL}_2}_B(\chi_1^{-1} \otimes \chi_2^{-1})$	4
X	a	$\mathrm{Ind}^{G'}_{P'_\alpha}(\chi St_{\mathrm{GL}_2})$	$\chi St_{\mathrm{GL}_2} + \chi^{-1} St_{\mathrm{GL}_2}$ $+\mathrm{Ind}^{\mathrm{GL}_2}_B(\chi\nu^{\frac{1}{2}} \otimes \chi^{-1}\nu^{\frac{1}{2}})$	3
	b	$\mathrm{Ind}^{G'}_{P'_\alpha}(\chi \mathbb{1}_{\mathrm{GL}_2})$	$\chi \mathbb{1}_{\mathrm{GL}_2} + \chi^{-1} \mathbb{1}_{\mathrm{GL}_2}$ $+\mathrm{Ind}^{\mathrm{GL}_2}_B(\chi\nu^{-\frac{1}{2}} \otimes \chi^{-1}\nu^{-\frac{1}{2}})$	3
XI	a	$\mathrm{Ind}^{G'}_{P'_\beta}(\chi \otimes \varsigma St_{\mathrm{S0}_3})$	$\mathrm{Ind}^{\mathrm{GL}_2}_B(\chi \otimes \varsigma\nu^{\frac{1}{2}}) + \mathrm{Ind}^{\mathrm{GL}_2}_B(\chi^{-1} \otimes \varsigma\nu^{\frac{1}{2}})$	2
	b	$\mathrm{Ind}^{G'}_{P'_\beta}(\chi \otimes \varsigma \mathbb{1}_{\mathrm{S0}_3})$	$\mathrm{Ind}^{\mathrm{GL}_2}_B(\chi \otimes \varsigma\nu^{-\frac{1}{2}}) + \mathrm{Ind}^{\mathrm{GL}_2}_B(\chi^{-1} \otimes \varsigma\nu^{-\frac{1}{2}})$	2
XII	a	$\sigma(\varsigma\nu^{\frac{1}{2}} \otimes \varsigma \mathbb{1}_{\mathrm{SO}_3})$	$\varsigma St_{\mathrm{GL}_2}$	1
	b	$\sigma(\varsigma\nu^{\frac{1}{2}} \otimes \varsigma St_{\mathrm{SO}_3})$	$\varsigma St_{\mathrm{GL}_2} + \mathrm{Ind}^{\mathrm{GL}_2}_B(\varsigma\nu^{\frac{1}{2}} \otimes \varsigma\nu^{\frac{1}{2}})$	2
	c	$L(\varsigma\nu^{\frac{1}{2}}, \varsigma St_{\mathrm{SO}_3})$	$\varsigma \mathbb{1}_{\mathrm{GL}_2}$	1
	d	$L(\varsigma\nu^{\frac{1}{2}}, \varsigma\nu^{\frac{1}{2}})$	$\varsigma \mathbb{1}_{\mathrm{GL}_2} + \mathrm{Ind}^{\mathrm{GL}_2}_B(\varsigma\nu^{-\frac{1}{2}} \otimes \varsigma\nu^{-\frac{1}{2}})$	2
XIII	a	$\varsigma St_{\mathrm{SO}_5}$	$\varsigma\nu St_{\mathrm{GL}_2}$	1
	b	$L(\varsigma\nu St_{\mathrm{GL}_2}, 1)$	$\varsigma\nu^{-1} St_{\mathrm{GL}_2} + \mathrm{Ind}^{\mathrm{GL}_2}_B(\varsigma\nu^{\frac{3}{2}} \otimes \varsigma\nu^{-\frac{1}{2}})$	2
	c	$L(\varsigma\nu^{\frac{3}{2}}, \varsigma St_{\mathrm{SO}_3})$	$\varsigma\nu \mathbb{1}_{\mathrm{GL}_2} + \mathrm{Ind}^{\mathrm{GL}_2}_B(\varsigma\nu^{\frac{1}{2}} \otimes \varsigma\nu^{-\frac{3}{2}})$	2
	d	$L(\varsigma\nu^{\frac{3}{2}}, \varsigma\nu^{\frac{1}{2}})$	$\varsigma\nu^{-\frac{1}{2}} \mathbb{1}_{\mathrm{GL}_2}$	1

Table B.14: **Jacquet Modules-$\mathrm{SO}_5(F)$-Short Root Parabolic**

		Representation	$r^{G'}_{P'_\beta}(\pi) \in \mathfrak{R}(M'_\beta)$	#
IX		$\mathrm{Ind}^{G'}_{P'_\emptyset}(\chi_1 \otimes \chi_2)$ (irreducible)	$\chi_1 \otimes \mathrm{Ind}^{\mathrm{SO}_3}_B(\chi_2) + \chi_1^{-1} \otimes \mathrm{Ind}^{\mathrm{SO}_3}_B(\chi_2)$ $+\chi_2 \otimes \mathrm{Ind}^{\mathrm{SO}_3}_B(\chi_1) + \chi_2^{-1} \otimes \mathrm{Ind}^{\mathrm{SO}_3}_B(\chi_1)$	4
X	a	$\mathrm{Ind}^{G'}_{P'_\alpha}(\chi St_{\mathrm{GL}_2})$	$\chi\nu^{\frac{1}{2}} \otimes \mathrm{Ind}^{\mathrm{SO}_3}_B(\chi\nu^{-\frac{1}{2}})$ $+\chi^{-1}\nu^{\frac{1}{2}} \otimes \mathrm{Ind}^{\mathrm{SO}_3}_B(\chi\nu^{\frac{1}{2}})$	2
	b	$\mathrm{Ind}^{G'}_{P'_\alpha}(\chi \mathbb{1}_{\mathrm{GL}_2})$	$\chi\nu^{-\frac{1}{2}} \otimes \mathrm{Ind}^{\mathrm{SO}_3}_B(\chi\nu^{\frac{1}{2}})$ $+\chi^{-1}\nu^{-\frac{1}{2}} \otimes \mathrm{Ind}^{\mathrm{SO}_3}_B(\chi\nu^{-\frac{1}{2}})$	2
XI	a	$\mathrm{Ind}^{G'}_{P'_\beta}(\chi \otimes \varsigma St_{\mathrm{SO}_3})$	$\chi \otimes \varsigma St_{\mathrm{SO}_3} + \chi^{-1} \otimes \varsigma St_{\mathrm{SO}_3}$ $+\varsigma\nu^{\frac{1}{2}} \otimes \mathrm{Ind}^{\mathrm{SO}_3}_B(\chi)$	3
	b	$\mathrm{Ind}^{G'}_{P'_\beta}(\chi \otimes \varsigma \mathbb{1}_{\mathrm{SO}_3})$	$\chi \otimes \varsigma \mathbb{1}_{\mathrm{SO}_3} + \chi^{-1} \otimes \varsigma \mathbb{1}_{\mathrm{SO}_3}$ $+\varsigma\nu^{-\frac{1}{2}} \otimes \mathrm{Ind}^{\mathrm{SO}_3}_B(\chi)$	3
XII	a	$\sigma(\varsigma\nu^{\frac{1}{2}} \otimes \varsigma \mathbb{1}_{\mathrm{SO}_3})$	$\varsigma\nu^{\frac{1}{2}} \otimes \varsigma \mathbb{1}_{\mathrm{SO}_3}$	1
	b	$\sigma(\varsigma\nu^{\frac{1}{2}} \otimes \varsigma St_{\mathrm{SO}_3})$	$\varsigma\nu^{\frac{1}{2}} \otimes \varsigma \mathbb{1}_{\mathrm{SO}_3} + 2 \cdot \varsigma\nu^{\frac{1}{2}} \otimes \varsigma St_{\mathrm{SO}_3}$	3
	c	$L(\varsigma\nu^{\frac{1}{2}}, \varsigma St_{\mathrm{SO}_3})$	$\varsigma\nu^{-\frac{1}{2}} \otimes \varsigma St_{\mathrm{SO}_3}$	1
	d	$L(\varsigma\nu^{\frac{1}{2}}, \varsigma\nu^{\frac{1}{2}})$	$\varsigma\nu^{-\frac{1}{2}} \otimes \varsigma St_{\mathrm{SO}_3} + 2 \cdot \varsigma\nu^{-\frac{1}{2}} \otimes \varsigma \mathbb{1}_{\mathrm{SO}_3}$	3
XIII	a	$\varsigma St_{\mathrm{SO}_5}$	$\varsigma\nu^{\frac{3}{2}} \otimes \varsigma St_{\mathrm{SO}_3}$	1
	b	$L(\varsigma\nu St_{\mathrm{GL}_2}, 1)$	$\varsigma\nu^{\frac{3}{2}} \otimes \varsigma \mathbb{1}_{\mathrm{SO}_3} + \varsigma\nu^{\frac{1}{2}} \otimes \mathrm{Ind}^{\mathrm{SO}_3}_B(\nu^{\varsigma\frac{3}{2}})$	2
	c	$L(\varsigma\nu^{\frac{3}{2}}, \varsigma St_{\mathrm{SO}_3})$	$\varsigma\nu^{-\frac{3}{2}} \otimes \varsigma St_{\mathrm{SO}_3} + \varsigma\nu^{\frac{1}{2}} \otimes \mathrm{Ind}^{\mathrm{SO}_3}_B(\varsigma\nu^{\frac{3}{2}})$	2
	d	$L(\varsigma\nu^{\frac{3}{2}}, \varsigma\nu^{\frac{1}{2}})$	$\varsigma\nu^{-\frac{3}{2}} \otimes \varsigma \mathbb{1}_{\mathrm{SO}_3}$	1

Table B.15: **Weil-Deligne data and conjectural L-values–$\mathrm{SO}_5(F)$ case**

		$\rho(\mathrm{Fr}_q)$	N	**L-factor**
IX		$\mathrm{diag}(\chi_1(\varpi), \chi_2(\varpi), \chi_1^{-1}(\varpi), \chi_2^{-1}(\varpi))$	0	$L(s,\chi_1)L(s,\chi_2)L(s,\chi_1^{-1})L(s,\chi_2^{-1})$
X	a	$\mathrm{diag}(\chi\nu^{\frac{1}{2}}(\varpi), \chi\nu^{-\frac{1}{2}}(\varpi), \chi^{-1}\nu^{-\frac{1}{2}}(\varpi), \chi^{-1}\nu^{\frac{1}{2}}(\varpi))$	N_1'	$L(s,\chi\nu^{\frac{1}{2}})L(s,\chi^{-1}\nu^{\frac{1}{2}})$
	b		0	$L(s,\chi\nu^{\frac{1}{2}})L(s,\chi\nu^{-\frac{1}{2}})L(s,\chi^{-1}\nu^{\frac{1}{2}})L(s,\chi^{-1}\nu^{-\frac{1}{2}})$
XI	a	$\mathrm{diag}(\chi(\varpi), \varsigma\nu^{\frac{1}{2}}(\varpi), \chi^{-1}(\varpi), \varsigma\nu^{-\frac{1}{2}}(\varpi))$	N_2'	$L(s,\chi)L(s,\varsigma\nu^{\frac{1}{2}})L(s,\chi^{-1})$
	b		0	$L(s,\chi)L(s,\varsigma\nu^{\frac{1}{2}})L(s,\chi^{-1})L(s,\varsigma\nu^{-\frac{1}{2}})$
XII	a	$\mathrm{diag}(\varsigma\nu^{\frac{1}{2}}(\varpi), \varsigma\nu^{-\frac{1}{2}}(\varpi), \varsigma\nu^{-\frac{1}{2}}(\varpi), \varsigma\nu^{\frac{1}{2}}(\varpi))$	N_3'	$L(s,\varsigma\nu^{\frac{1}{2}})L(s,\varsigma\nu^{-\frac{1}{2}})$
	b		N_3'	$L(s,\varsigma\nu^{\frac{1}{2}})L(s,\varsigma\nu^{-\frac{1}{2}})$
	c		N_4'	$L(s,\varsigma\nu^{\frac{1}{2}})^2L(s,\varsigma\nu^{-\frac{1}{2}})$
	d		0	$L(s,\varsigma\nu^{\frac{1}{2}})L(s,\varsigma\nu^{-\frac{1}{2}})L(s,\varsigma\nu^{-\frac{1}{2}})L(s,\varsigma\nu^{\frac{1}{2}})$
XIII	a	$\mathrm{diag}(\varsigma\nu^{\frac{3}{2}}(\varpi), \varsigma\nu^{\frac{1}{2}}(\varpi), \varsigma\nu^{-\frac{3}{2}}(\varpi), \varsigma\nu^{-\frac{1}{2}}(\varpi))$	N_5'	$L(s,\varsigma\nu^{\frac{3}{2}})$
	b		N_6'	$L(s,\varsigma\nu^{\frac{3}{2}})L(s,\varsigma\nu^{-\frac{1}{2}})$
	c		N_2'	$L(s,\varsigma\nu^{\frac{3}{2}})L(s,\varsigma\nu^{\frac{1}{2}})L(s,\varsigma\nu^{-\frac{3}{2}})$
	d		0	$L(s,\varsigma\nu^{\frac{3}{2}})L(s,\varsigma\nu^{\frac{1}{2}})L(s,\varsigma\nu^{-\frac{3}{2}})L(s,\varsigma\nu^{-\frac{1}{2}})$

Table B.16: **Table of relevant nilpotent operators for $\mathfrak{sp}_2(\mathbb{C})$**

$$N_1' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad N_2' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad N_3' = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$N_4' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad N_5' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad N_6' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

Table B.17: **Good Test Vectors for Computing** $L(s, \tilde{\pi}, \mathbf{r})$

		f^*	V^{**}	$\varphi = \varphi_1 \otimes \varphi_2$	
IX		f_0	V^{s_m}	$\varphi_{L_1 \otimes L_1} \otimes C_0 \varphi_{L_1 \otimes L_1}$	$C_0 = 1$
X	a	f_{10}	V^{s_m}	$\varphi_{L_0 \oplus L_0} \otimes C_{10} \varphi_{L_0 \otimes L_0}$	$C_{10} = \left(q\mathrm{vol}(I_\emptyset)^{-1}\right)/(q+1)$
	b	f_0	V^{s_m}	$\varphi_{L_1 \otimes L_1} \otimes C_0 \varphi_{L_1 \otimes L_1}$	$C_0 = 1$
XI	a	f_{11}	V^{s_m}	$\varphi_{L_0 \oplus L_1} \otimes C_{11} \varphi_{L_0 \oplus L_1}$	$C_{11} = \left(q\mathrm{vol}(I_\emptyset)^{-1}\right)/(q+1)$
	b	f_0	V^{s_m}	$\varphi_{L_1 \otimes L_1} \otimes C_0 \varphi_{L_1 \otimes L_1}$	$C_0 = 1$
XII	a	f_8	V^{s_m}	$\varphi_{L_0 \oplus L_1} \otimes C_8 \varphi_{L_0 \oplus L_1}$	$C_8 = \left(q\mathrm{vol}(I_\emptyset)^{-1}\right)/(q+1)$
	b	f_7	V^{ra_m}	$\varphi_{L_{ra} \otimes L_{ra}} \otimes C_7 \varphi_{L_{ra} \otimes L_{ra}}$	$C_7 = \left(q\mathrm{vol}(I_\emptyset)^{-1}\right) / \left(2(q+1)^2\right)$
	c	f_7	V^{ra_m}	$\varphi_{L_{ra} \otimes L_{ra}} \otimes C_7 \varphi_{L_{ra} \otimes L_{ra}}$	$C_7 = \left(q\mathrm{vol}(I_\emptyset)^{-1}\right) / \left(2(q+1)^2\right)$
	d	f_0	V^{s_m}	$\varphi_{L_1 \otimes L_1} \otimes C_0 \varphi_{L_1 \otimes L_1}$	$C_0 = 1$

Table continued on next page.

Table B.17: **Good Test Vectors for Computing** $L(s,\tilde{\pi},\mathbf{r})$

		f^*	V^{**}	$\varphi=\varphi_1\otimes\varphi_2$	
XII	a	f_4	-	-	-
	b	f_6	V^{s_m}	$\varphi_{L_0\oplus L_0}\otimes C_6\varphi_{L_0\oplus L_0}$	$C_6=\left(q^2\mathrm{vol}(I_\emptyset)^{-1}\right)/(2q+2)$
	c	f_5	V^{s_m}	$\varphi_{L_0\oplus L_1}\otimes C_5\varphi_{L_0\oplus L_1}$	$C_5=\left(q\mathrm{vol}(I_\emptyset)^{-1}\right)/(q+1)$
	d	f_0	V^{s_m}	$\varphi_{L_1\otimes L_1}\otimes C_0\varphi_{L_1\otimes L_1}$	$C_0=1$
$*$=See Table 4.3; $**$=See Table 4.5					

Table B.18: **Computed L-factors for $\widetilde{\mathrm{Sp}_2}(F)$**

		$C(q)$	$L(s,\tilde{\pi},\mathbf{r}_{st})$	$d_\Phi(s)$
IX		1	$L(s+\frac{1}{2},\chi_V\chi_1)L(s+\frac{1}{2},\chi_V\chi_2)L(s+\frac{1}{2},\chi_V\chi_1^{-1})L(s+\frac{1}{2},\chi_V\chi_2^{-1})$	$\zeta(2s+2)\zeta(2s+4)$
X	a	$(2(q-1))/q^2$	$L(s+\frac{1}{2},\chi_V\nu^{\frac{1}{2}}\chi)L(s+\frac{1}{2},\chi_V\nu^{\frac{1}{2}}\chi^{-1})$	$\zeta(2s+2)$
	b	1	$L(s+\frac{1}{2},\chi_V\chi\nu^{\frac{1}{2}})L(s+\frac{1}{2},\chi_V\chi\nu^{-\frac{1}{2}})L(s+\frac{1}{2},\chi_V\chi^{-1}\nu^{\frac{1}{2}})L(s+\frac{1}{2},\chi_V\chi^{-1}\nu^{-\frac{1}{2}})$	$\zeta(2s+2)\zeta(2s+4)$
XI	a	$(q-1)/q$	$L(s+\frac{1}{2},\chi_V\chi\nu^{\frac{1}{2}})L(s+\frac{1}{2},\chi_V\chi^{-1}\nu^{\frac{1}{2}})L(s+\frac{1}{2},\varsigma\nu^{\frac{1}{2}})$	$\zeta(2s+2)L(s,\varsigma)$
	b	1	$L(s+\frac{1}{2},\chi_V\chi)L(s+\frac{1}{2},\chi_V\varsigma\nu^{\frac{1}{2}})L(s+\frac{1}{2},\chi_V\chi^{-1})L(s+\frac{1}{2},\chi_V\varsigma\nu^{-\frac{1}{2}})$	$\zeta(2s+2)\zeta(2s+4)$
XII	a	$(q-1)/q$	$L(s+\frac{1}{2},\chi_V\varsigma)^2$	$\zeta(2s+2)$
	b	$(q-1)/q$	$L(s+\frac{1}{2},\chi_V\varsigma\nu^{\frac{1}{2}})^2$	$\zeta(2s+2)$
	c	$1/q^2$	$L(s+\frac{1}{2},\chi_V\varsigma\nu^{\frac{1}{2}})^2L(s+\frac{1}{2},\chi_V\varsigma\nu^{-\frac{1}{2}})$	$\zeta(2s+2)L(s+2,\varsigma\xi)$
	d	1	$L(s+\frac{1}{2},\chi_V\varsigma\nu^{\frac{1}{2}})^2L(s+\frac{1}{2},\chi_V\varsigma\nu^{-\frac{1}{2}})^2$	$\zeta(2s+2)\zeta(2s+4)$
XIII	a	–	–	–
	b	$(q-1)/q$	$L(s+\frac{1}{2},\chi_V\varsigma\nu^{\frac{3}{2}})L(s+\frac{1}{2},\chi_V\varsigma\nu^{-\frac{1}{2}})$	$\zeta(2s+2)$
	c	$(q-1)/q$	$L(s+\frac{1}{2},\chi_V\varsigma\nu^{\frac{3}{2}})L(s+\frac{1}{2},\chi_V\varsigma\nu^{\frac{1}{2}})L(s+\frac{1}{2},\chi_V\varsigma\nu^{-\frac{3}{2}})$	$\zeta(2s+2)\zeta(s)$
	d	1	$L(s+\frac{1}{2},\chi_V\varsigma\nu^{\frac{3}{2}})L(s+\frac{1}{2},\chi_V\varsigma\nu^{\frac{1}{2}})L(s+\frac{1}{2},\chi_V\varsigma\nu^{-\frac{3}{2}})L(s+\frac{1}{2},\chi_V\varsigma\nu^{-\frac{1}{2}})$	$\zeta(2s+2)\zeta(2s+4)$

Bibliography

[1] Jeffrey Adams. **Lifting of characters on orthogonal and metaplectic groups.** Duke Math. J. 92 (1998), no. 1, 129-178.

[2] I.N. Bernstein, A.V. Zelevinsky. **Representations of the group** $\mathrm{GL}(n, F)$**, where** F **is a local non-archimedean field**. Uspekhi Mat. Nauk 31 (1976), 5-70.

[3] Armand Borel. **Automorphic** L**-functions**. *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2, pp. 27-61*, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, RI, 1979.

[4] Daniel Bump. *Automorphic Forms and Representations*. Cambridge Studies in Advanced Mathematics 55, Cambridge University Press, 1998.

[5] William Casselman. **The unramified principal series of** $\mathfrak{p}$**-padic groups. I. The spherical function**. Compositio Mathematica 40 (1980), no. 3, 387-406.

[6] William Casselman. **Introduction to the theory of admissible representations of** p**-adic reductive groups**, preprint.

[7] Stephen Gelbart. *Automorphic Forms on Adele Groups*. Annals of Mathematics Studies 83, Princeton University Press, 1975.

[8] Stephen Gelbart, Ilya Piatetski-Shapiro and Stephen Rallis. **Explicit Constructions of Automorphic L-Functions**. Lecture Notes in Mathematics no. 1254. Springer-Verlag, Berlin Heidelberg, 1987.

[9] Robert Gustafson. **The degenerate principal series for** $\mathrm{Sp}(2n)$**.** *Memiors of the American Mathematical Society* **33** (1981), no. 248, vi+81 pp.

[10] Michael Harris and Richard Taylor. *The Geometry and Cohomology of Some Simple Shimura Varieties*. The Annals of Mathematical Studies 151, Princeton University Press, 2001.

[11] Y. Hironaka and F. Sato. **Local Densities of representations of quadratic forms over** p**-adic integers**, J. Number Theory **83** (2002), 106-136.

[12] Chris Jantzen. **Degenerate principal series for orthogonal groups.** Journal für die Reine und Angewandte Mathematik 441 (1993), 61-98.

[13] Chris Jantzen. **Some remarks on degenerate principal series**. Pacific Journal of Mathematics, 186 (1998) no. 1, 67-87.

[14] Yoshiyuki Kitaoka. **A note on local densities of quadratic forms**. Nagoya Mathematical Journal, 92 (1983), 145–152.

[15] Stephen S. Kudla. **On the local theta-correspondence**. Inventiones Mathematicae., 83 (1986), no. 2, 229-255.

[16] Stephen S. Kudla. **Splitting metaplectic covers of dual reductive pairs**. Israel Journal of Math., 87 (1994), no. 1-3, 361-401.

[17] Stephen S. Kudla. **The local Langlands correspondence: the non-Archimedean case**. *Motives (Seattle, WA, 1991),* Proceedings in Symposia of Pure Math., 55, Part 2, AMS, Providence, RI, 1994, 365-391.

[18] Stephen S. Kudla. **On the Theta Correspondence**. Lectures at European School of Group Theory, Beilngries 1996.

[19] Stephen S. Kudla. **Central Derivatives of Eisenstein Series and Height Pairings**. Annals of Mathematics 146 (1997), no. 3, 545-646.

[20] Stephen S. Kudla and Stephen Rallis. **On the Weil-Siegel Formula**. Journal für die Reine und Angewandte Mathematik 391 (1988), 65-84.

[21] Stephen S. Kudla and Stephen Rallis. **Degenerate Principal Series and Invariant Distributions**. Israeli Journal of Mathematics 69 (1990), no. 1, 25-45.

[22] Stephen S. Kudla, Michael Rapoport and Tonghai Yang. **Modular Forms and Special Cycles on Elliptic Curves**. Annals of Mathematics Studies, 161. Princeton University Press, 2006.

[23] Stephen S. Kudla, Michael Rapoport and Tonghai Yang. **Derivatives of Eisenstein series and Faltings heights**, Compos. Math., 140 (2004), 887-951.

[24] Erez M. Lapid and Stephen Rallis. **On the local factors of representations of classical groups.** *Automorphic representations, L-functions and applications: progress and prospects*, 309-359. Ohio State University Mathematical Research Institute Publications, de Gruyter, Berlin, 2005.

[25] George Luszting. **Some examples of square integrable representations of semisimple p-adic groups.** Transactions of the American Mathematical Society 277 (1983), no. 2, 623-653.

[26] Colette Mœglin, Marie-France Vignéras and Jean-Loup Waldspurger. **Correspondance de Howe sur un corps p-adique**. Lecture Notes in Mathematics no. 1291. Springer-Verlag, Berlin Heidelberg, 1988.

[27] Ilya Piatetski-Shapiro and Stephen Rallis. **ϵ-factors of representations of classical groups**, Proc. Symp. Nat. Acad. Sci. 83 ((186), 4589-4593.

[28] Ilya Piatetski-Shapiro and Stephen Rallis. **Rankin triple L-functions**. Compositio Math., 64 (1987), 333-399.

[29] Dipendra Prasad. **On the local Howe duality correspondence.** IMRN, No. 11 (1993), 279-287.

[30] Dipendra Prasad. **Weil Representation, Howe Duality and the Theta Correspondence**. *Theta functions:from the classical to the modern*, 105-127, CRM Proc. Lecture Notes, 1, Amer. Math. Soc., Providence, RI, 1993.

[31] R Ranga Rao. **On Some Explicit Formulas in the Theory of Weil Representation**. Pacific Journal of Math 157 (1993) no. 2, 335-371.

[32] Brooks Roberts, Ralf Schmidt. **Tables for Representations of** $\mathrm{GSp}(4)$. Avaiable at http://www.math.ou.edu/ rschmidt/.

[33] Paul J. Sally, Jr. and Marko Tadić. **Induced Representations and Classifications for** $\mathrm{GSp}(2,F)$ **and** $\mathrm{Sp}(2,F)$**.** Mémoires de le Société Mathématique de France. Nouvelle Série, 52 (1993), 75-133.

[34] Jean-Pierre Serre. *A Course in Arithmetic*. Graduate Texts in Mathematics 7, Springer-Verlag, New York, 1973.

[35] G. Shimura. *Arithmetic Theory of Automorphic Functions*. Publications of the Mathematical Society of Japan, 11. Iwanami Shoten, Publishers and Princeton University Press, 1971.

[36] Marko Tadić. **Parabolically Induced Representations and Unitarizability for** $\mathrm{Sp}(2,F)$. Preprint (1998).

[37] Marko Tadić. **Representations of p-adic Symplectic Groups**. Compositio Math., 90 (1994), 123-181.

[38] Marko Tadić. **Representations of Classical p-adic Groups**. *Representations of Lie groups and quantum groups*, Pitman Research Notes in Mathematics Series 311, Longman Sci. Tech., Harlow, 1994.

[39] John Tate. **Number theoretic background**. *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2, pp. 3-26*, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, RI, 1979.

[40] J.-L. Waldspurger. **Représentation métaplectic et conjectures de Howe**. *Séminaire Bourbaki, Vol. 1986/87.* Astérisque No. 142-153 (1987), 3, 85-99 (1988)

[41] J.-L. Waldspurger. **Démonstration d'une conjecture de dualité de Howe dans le cas p-adique,** $p \neq 2$. *Festschrift in honor of I.I.Piatetski-Shapiro on the occasion of his sixtieth birthday, Part I (Ramat Aviv, 1989)*, 267-324, Israel Math. Conf. Proc., 2, Weizmann, Jerusalem, 1990.

[42] Tonghai Yang. **An Explicit Formula For Local Densities of Quadratic Forms**. J. Number Theory 72 (1998), No. 2, 309-356.