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Dynamics and Control of Constrained Nonlinear Systems with Application to Robotics

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DYNAMICS AND CONTROL OF CONSTRAINED NONLINEAR SYSTEMS WITH APPLICATION TO ROBOTICS*

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Abstract

A class of nonlinear constrained dynamic systems is studied. We first characterize the constrained submanifold and the constrained dynamics without using the vector relative degree. Applying the nonlinear feedback and exact linearization techniques to constrained systems, we discuss several control problems for the constrained dynamics such as asymptotically stabilization, asymptotically tracking reference outputs. Our results for the control of constrained nonlinear systems extend previous results, which is based on linear approximation and linear feedback.

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1. Introduction

There has been considerable research on constrained dynamic systems (CDS), in which the interaction between the environment and the system is essential. In recent years, nonlinear CDS have become an important area in the study of CDS. Zero-output constrained dynamics or clamped dynamics are introduced for control systems [1][2]. Nonlinear descriptor systems defined by singular implicit differential equations in which constraint relations are not explicit are studied [3]. Stabilizability, controllability and other control problems are discussed for constrained mechanical systems (CMS) such as constrained Hamiltonian systems [4], holonomic systems [5], and classical nonholonomic systems [6][7]. Applications are found in the control system design of a robot manipulator during execution of tasks characterized by physical contact between the manipulator and the environment [8][9]. More recently, controller design based on linear approximation is addressed for a class of nonlinear CDS in which the relative degree vector is well defined and input variables are separated as constraint inputs and control inputs [10].

In this paper we will investigate a class of nonlinear CDS defined by differential algebraic equations. The goal of this paper is to extend previous results on dynamics and control problems for nonlinear CDS. First, we will characterize the constrained dynamics for nonlinear CDS without using the relative degree vector. Second, we will develop special techniques to solve control problems for constrained systems such as stabilization, tracking, etc., based on nonlinear feedback and input-output exact linearization. Finally, we return to applications in robotics. We will develop a new control scheme for the tracking problem of a robot manipulator with end-effector moving along a constraint surface — a problem for which there is no natural decomposition into constraint inputs and control inputs, and for which control by approximate linearization has significant limitations.

2. Dynamics of Constrained Nonlinear Systems

Consider a class of nonlinear CDS defined by the following differential algebraic equations:

$$\dot{x} = f(x) + \sum_{j=1}^{m} g_j(x)u_j, \; x \in U \subset R^n \qquad \textit{system equation}, \tag{2.1a}$$

$$z_i = k_i(x) = 0, \; i = 1, ..., l, \qquad \textit{constraint relations}, \tag{2.1b}$$

$$y_i = h_i(x), \;\; i = 1, ..., p, \qquad \textit{output functions}, \tag{2.1c}$$

where $f(x)$ and $g_i(x)$ are smooth vector fields, $k_i(x)$ and $h_i(x)$ smooth functions. For convenience we set $G(x) = [g_1(x), ..., g_m(x)]$, $k(x) = [k_1(x), ..., k_l(x)]^\top$, $h(x) = [h_1(x), ..., h_p(x)]^\top$, and $u = [u_1, ..., u_m]^\top$. Suppose that for all $x \in U$, $g_1(x), ..., g_m(x)$ are linearly independent vector fields, $\{dk_1(x), ..., dk_l(x)\}$ and $\{dh_1(x), ..., dh_p(x)\}$ are each linearly independent sets of covector fields, and $m = l + p$. In this model there is no need to separate input variables into constraint inputs and control inputs.

We are interested in characterizing the dynamics of (2.1a) which satisfy the constraint relations (2.1b). It is natural to consider the following problem:

Find proper sets of initial state x^0 and inputs u such that the corresponding dynamics $x(x^0, u, t)$ of (1.1a) satisfy the constraints $k_i(x(x^0, u, t)) = 0$, $1 \le i \le l$, for all time.

This problem is the same as the problem of zeroing the output if $k_i(x)$ are regarded as output functions . Thus we can use the same approach as that in the theory of zero dynamics or clamped dynamics [1][2]. Differentiate the constraint relations (1.1b) and suppose that input variables explicitly appear after r_i times differentiation of the i^{th} constraint relation, $1 \le i \le l$:

$$\begin{aligned} &dz_i/dt = L_f k_i(x) = 0, \\ &\vdots \\ &d^{(r_i-1)}z_i/dt^{(r_i-1)} = L_f^{r_i-1}k_i(x) = 0, \\ &d^{r_i}z_i/dt^{r_i} = L_f^{r_i}k_i(x) + L_{g_1}L_f^{r_i-1}k_i(x)u_1 + \cdots + L_{g_m}L_f^{r_i-1}k_i(x)u_m = 0. \end{aligned} \tag{2.2}$$

Thus, necessary conditions for the dynamics $x(x^0, u, t)$ of (2.1a) to satisfy the constraint (2.1b) are

$$k_i(x) = L_f k_i(x) = \cdots = L_f^{r_i-1}k_i(x) = 0,\ i = 1, ..., l, \tag{2.3}$$

and

$$\begin{aligned} &b(x) + A(x)u = 0 \\ &where\ b(x) = \left[L_f^{r_1}k_1(x), ..., L_f^{r_l}k_l(x)\right]^\top, \\ &\qquad A(x) = [a_{ij}(x)]_{l\times m} = \left[L_{g_j}L_f^{r_i-1}k_i(x)\right]_{l\times m}. \end{aligned} \tag{2.4}$$

Therefore we say that CDS (2.1) has the *constraint relative degree vector* $[r_1, \ldots, r_l]$ on U if the functions $L_{g_j}L_f^{\rho}k_i(x) = 0,\ 1 \le j \le m,\ 0 \le \rho \le r_i - 2,$ for all $x \in U$ and the matrix $A(x) = [a_{ij}(x)]_{l\times m} = \left[L_{g_j}L_f^{r_i-1}k_i(x)\right]_{l\times m}$ has full row rank for all $x \in U$. The submanifold

$$C^* = \left\{x \in U | L_f^{\rho}k_i(x) = 0,\ 1 \le i \le l,\ 0 \le \rho \le r_i - 1\right\} \tag{2.5}$$

is called *constraint submanifold.* If nonlinear CDS (2.1) has the relative degree vector $[r_1, \ldots, r_l]$, it is not difficult to see that sufficient conditions for dynamics $x(x^0, u, t)$ to satisfy the constraint relations (2.1b) are also that the initial state satisfies (2.3) and the inputs satisfy (2.4) [1].

Note that A(x) has full row rank for all $x \in U$. The solutions of (2.4) can be written as

$$u = -A^{\dagger}(x)b(x) + \Big(I - A^{\dagger}(x)A(x)\Big)\tilde{u}, \qquad \tilde{u} \in R^m \tag{2.6}$$

where $A^{\dagger}(x) = A^{\top}(x)\big(A(x)A^{\top}(x)\big)^{-1}$ is the pseudo-inverse of A(x). In (2.6) u is expressed as a feedback law and the closed loop equation is

$$\dot{x} = \Big(f(x) - G(x)A^{\dagger}(x)b(x)\Big) + G(x)\Big(I - A^{\dagger}(x)A(x)\Big)\tilde{u} = \tilde{f}(x) + \tilde{G}(x)\tilde{u} \tag{2.7}$$

Since the vector field $\tilde{f}(x) + \tilde{G}(x)\tilde{u}$ is tangent to C^* for all $\tilde{u} \in R^m$, it follows that for any $x^0 \in C^*$, the dynamics generated by the vector field $\tilde{f}(x) + \tilde{G}(x)\tilde{u}$ stay in C^* for all time, which is called the *constrained dynamics.* In other words, constraint relations (1.1b) are imposed by the feedback (2.6).

Remark: Since $A(x)$ has full row rank, we can rearrange inputs $u^{\top} = [w^{\top}, v^{\top}], w \in R^l, v \in R^{m-l}$ such that $A(x) = [A_1(x), A_2(x)]$, where $A_1(x)$ has linearly independent column vectors. In order to impose the constraint, the inputs can be chosen as $w = -A_1^{-1}(x)(b(x) + A_2(x)v)$. But the linear independently column vectors in $A(x)$ might change as x varies in U. Therefore the feedback (2.6) can impose the constraint in a larger region. Using the terminology of [10], the partition into control inputs and constraint inputs can change as x varies. By avoiding such a partition, the control law (2.6) imposes the constraints without being sensitive to such changes.

Now we can develop the state space description for the constrained dynamics of (2.1) under the assumption that the relative degree vector $[r_1, \ldots, r_l]$ exists. Letting $\zeta = [\zeta_1^1, ..., \zeta_{r_l}^l]^{\top}$ where $\zeta_1^i = k_i(x), \zeta_2^i = L_f k_i(x), ..., \zeta_{r_i}^i = L_f^{r_i - 1} k_i(x)$, $1 \leq i \leq l$, there exists a smooth vector function $\eta = [\phi_1(x), ..., \phi_{n-r}(x)]^{\top} = \phi(x)$, $r = r_1 + \ldots + r_l$, such that $(\zeta, \eta) = \Phi(x)$ is a local coordinate transformation near x^0. Then in (ζ,η) coordinates the constraint submanifold C^* is $\{(\zeta, \eta) \in R^n | \zeta = 0\}$ and (2.7) becomes

$$\begin{aligned} \dot{\zeta} &= N\zeta, \qquad\qquad N \; a \; nilpotent \; block \; matrix \\ \dot{\eta} &= L_{\tilde{f}}\phi(x)|_{x=\Phi^{-1}(\zeta,\eta)} + L_{\tilde{G}}\phi(x)|_{x=\Phi^{-1}(\zeta,\eta)}\tilde{u} \end{aligned} \tag{2.8}$$

Note that the matrix $(I - A^{\dagger}(x)A(x))$ has constant rank $(m-l)$ and $g_1(x), ..., g_m(x)$ are linearly independent. By a state-dependent change of basis in the input space, we may assume $\tilde{G}(x)\tilde{u} = \tilde{G}_1(x)v$, where $v \in R^{m-l}$ for x near x^0. Therefore the reduced state space equation for the constrained dynamics is

$$
\begin{aligned}
\dot{\eta} &= f^*(\eta) + \sum_{j=1}^{m-l} g_j^*(\eta)v_j, \quad \eta \in R^{n-r}, \ \eta \ near\ the\ origin \\
y_i &= h_i^*(\eta), \qquad i = 1, ..., p
\end{aligned}
\tag{2.9}
$$

where $f^*(\eta) = L_{\tilde{f}}\phi(x)|_{x=\Phi^{-1}(0,\eta)}$, $[g_1^*(\eta), ..., g_{m-l}^*(\eta)] = L_{\tilde{G}_1}\phi(x)|_{x=\Phi^{-1}(0,\eta)}$, $h_i^*(\eta) = h_i(x)|_{x=\Phi^{-1}(0,\eta)}$. It can be shown that the reduced state space equation (2.9) is unique up to a transformation group of diffeomorphisms on C^* and state feedback.

Now we discuss a more general problem: *How can the constrained dynamics of CDS (2.1) be determined without the using the relative degree vector*? We want to extend the preceding results to a class of CDS in which the relative degree vectors do not exist. For convenience we introduce some definitions from nonlinear control theory [1][2]. A smooth connected submanifold M of U is said to be *controlled invariant* in U if there exists a smooth feedback $u : M \to R^m$ such that the vector field of the closed loop system $\tilde{f}(x) = f(x) + G(x)u$ is tangent to M for all $x \in M$. A submanifold C of U is said to be a *constraint nulling submanifold* if it is controlled invariant and the constraint functions $k_i(x)$ are zero on C. The maximal constraint nulling submanifold is called the *constraint submanifold*, denoted as C^*.

According to the theory of zero dynamics or clamped dynamics, we can modify the Hirschorn structure algorithm to calculate the explicit expression for C^* and the feedback $u(x)$ which makes $\tilde{f}(x)$ tangent to C^* [2].

Constraint Submanifold Algorithm:

$Step\ 0: \quad C_0 = \{x \in U | k(x) = 0\}$

$Step\ i: \quad C_i = \{x \in C_{i-1} | f(x) \in span\{g_1(x), ..., g_m(x)\} + T_x C_{i-1}\}$

If every C_i is a smooth submanifold with constant dimension this algorithm will terminate because $\{C_i;\ i = 1, 2, ...\}$ is a sequence of nested submanifolds and $\dim C_0 = n - l$.

Let $k^0(x) = k(x)$ and $s_0 = l$. Then C_1 is the set of x in C_0 such that there exists u which makes $f(x) + G(x)u \in T_x C_0$ or equivalently

$$\langle dk^0(x), f(x) + G(x)u \rangle = L_f k^0(x) + L_G k^0(x)u = 0 \tag{2.10}$$

has solution for u. Suppose that the matrix $L_G k^0(x)$ has constant rank r_0 on C_0. Then there is a smooth matrix $D_0(x) \in R^{(s_0 - r_0)\times s_0}$ such that $D_0(x) L_G k^0(x) = 0$, $\forall x \in C_0$. Letting $\varphi^0(x) = D_0(x) L_f k^0(x)$ gives $C_1 = \{x \in U | k^0(x) = 0, \varphi^o(x) = 0\}$. Suppose $\left[dk^0(x)^\top, d\varphi^0(x)^\top\right]$ has constant rank $(s_0 + s_1)$ and the first s_1 rows of φ^0 have linearly independent differentials. There exists a constant matrix S_0 such that $S_0\varphi^0$ selects the first s_1 rows of φ^0. Letting $k^1(x)^\top = \left[k^0(x)^\top, S_0\varphi^0(x)^\top\right]$ gives $C_1 = \{x \in U | k^1(x) = 0\}$ and $s_1 \le s_0 - r_0$.

Given $C_i = \{x \in U | k^i(x) = 0\} = \{x \in U | k^{i-1}(x) = 0, S_{i-1}\varphi^{i-1}(x) = 0\}$, C_{i+1} is the subset of C_i such that $\langle dk^i(x), f(x) + G(x)u \rangle = L_f k^i(x) + L_G k^i(x)u = 0$ is solvable for u. Assume that $L_G k^i(x)$ has constant rank r_i on C_i. Then there is a smooth matrix $D_i(x)$ such that

$$D_i(x) L_G k^i(x) = \begin{bmatrix} D_{i-1}(x) & 0 \\ P_{i-1}(x) & Q_{i-1}(x) \end{bmatrix} \begin{bmatrix} L_G k^{i-1}(x) \\ L_G S_{i-1}\varphi^{i-1}(x) \end{bmatrix} = 0, \quad \forall x \in C_i. \tag{2.11}$$

Therefore we can obtain $C_{i+1} = \{x \in U | k^i(x) = 0, \varphi^i(x) = 0\}$ where $\varphi^i = P_{i-1} L_f k^{i-1} + Q_{i-1} L_f S_{i-1}\varphi^{i-1}$. If $\left[dk^i(x)^\top, d\varphi^i(x)^\top\right]$ has constant rank $(s_1 + \cdots + s_i + s_{i+1})$ and the first s_{i+1} rows of φ^i have linearly independent differentials, we can find a constant matrix S_i such that $S_i\varphi^i$ selects the first s_{i+1} rows of φ^i. Letting $k^{i+1}(x)^\top = \left[k^i(x)^\top, S_i\varphi^i(x)^\top\right]$, then C_{i+1} can be written as $C_{i+1} = \{x \in U | k^{i+1}(x) = 0\}$.

If the algorithm terminates at step i^*, $C_{i^*} = \{x \in U | k^{i^*}(x) = 0\}$ is the constraint submanifold and the input variables must satisfy the equation

$$L_f k^{i^*}(x) + L_G k^{i^*}(x)u = 0. \tag{2.12}$$

Moreover the smooth solutions $u(x)$ of the equation (2.12) keep the vector fields $\tilde{f}(x) = f(x) + G(x)u(x)$ tangent to C^*.

Suppose that in equation (2.12) $L_G k^{i^*}(x)$ has constant rank and $L_f k^{i^*}(x) \in \mathrm{Im}\big(L_G k^{i^*}(x)\big)$. So the solutions of (2.12) can be written as $u(x) = \tilde{u}(x) + R(x)v$, where $\tilde{u}(x)$ is the minimal norm solution which is a smooth vector function on U, and $R(x)v$, $v \in R^{m-l}$, is a vector in the null space of $L_G k^{i^*}(x)$. The column vectors of $R(x)$ are a basis of the null space which are smooth at least locally for each $x \in C^*$. Thus $u(x)$ can be expressed as a feedback law and the closed loop equation is

$$\dot{x} = \tilde{f}(x) + \tilde{G}(x)v, \ \tilde{f}(x) = f(x) + G(x)\tilde{u}(x), \ \tilde{G}(x) = G(x)R(x) \tag{2.13}$$

If the initial state x^0 is on the constraint submanifold C^*, the dynamics of the closed loop system (2.13) stay on C^* for all time. We call it the *constrained dynamics*. It is not difficult to derive the reduced state space equation for the constrained dynamics. For the details of the above geometric approach we refer to [1][2].

3. Control of Constrained Nonlinear Systems

We have shown that the constrained dynamics of CDS (2.1) can be characterized by the reduced state space equation (2.9) which is affine nonlinear. There have been various control techniques to deal with this kind of system. However if we apply these techniques directly to (2.9) we need to perform the transformation $\bar{x} = \Phi(x)$ to obtain the local coordinate expression for $f^*, g_1^*, ..., g_{m-l}^*$ and C^*, which is extremely difficult for practical problems because in general partial differential equations must be solved to compute new coordinates. In this section we will show how to develop control schemes for CDS based on the original system equation and constraint relations, —i.e. on (f, G, k, h) instead of (f^*, G^*, h^*)

A. Input-output Linearization and Nonlinear Zeros

A new technique, input-output exact linearization, has been developed for the analysis and the design of nonlinear control systems such as asymptotic stabilization, tracking, noninteracting control, etc. [1]. This technique uses nonlinear feedback to achieve the exact cancellation of the effects of nonlinearities on input-output behavior. In the same way that the transmission zeros are crucial in linear systems, the notion of "nonlinear zeros" is fundamental for this approach. "Nonlinear zeros", or zero dynamics, are internal dynamics of a system when its inputs and initial state are chosen to make output functions identically zero. We are interested in analyzing "nonlinear zeros" of the constrained dynamics of CDS (2.1) in terms of (f, G, k, h) instead of (f^*, G^*, h^*). We also want to find the feedback to linearize the input-output behavior of the constrained dynamics while avoiding the computation of $\bar{x} = \Phi(x)$. Throughout this section we assume that CDS (2.1) satisfies:

(A1) Both the constraint relative degree vector $[r_1^c, ..., r_l^c]$ *with respect to* (f, G, k) *and the output relative degree vector* $[r_1^o, ..., r_p^o]$ *with respect to* (f, G, h) *exist.*

(A2) $[A^c(x)^\top \quad A^o(x)^\top] \in R^{m \times m}$ *is nonsingular where* A^c *is the decoupling matrix for* (f, G, k) *and* A^o *is the decoupling matrix for* (f, G, h).

There are a large class of physical systems which satisfy (A_1) and (A_2). However these assumptions can be relaxed. Under assumptions (A_1) and (A_2) there is a coordinate change $\bar{x} = (\zeta, \xi, \eta) = \Phi(x)$ to transform (1.1a) into

$$
\begin{aligned}
&\dot{\zeta}_1^1 = \zeta_2^1 \\
&\dot{\zeta}_2^1 = \zeta_3^1 \\
&\vdots \\
&\dot{\zeta}_{r_1^c}^1 = L_f^{r_1^c} k_1(x) + \left(L_{g_1} L_f^{r_1^c - 1} k_1(x)\right) u_1 + \cdots + \left(L_{g_m} L_f^{r_1^c - 1} k_1(x)\right) u_m \\
&\vdots \\
&\dot{\zeta}_{r_l^c}^l = L_f^{r_l^c} k_l(x) + \left(L_{g_1} L_f^{r_l^c - 1} k_l(x)\right) u_1 + \cdots + \left(L_{g_m} L_f^{r_l^c - 1} k_l(x)\right) u_m \\
&\dot{\xi}_1^1 = \xi_2^1 \\
&\dot{\xi}_2^1 = \xi_3^1 \\
&\vdots \\
&\dot{\xi}_{r_1^o}^1 = L_f^{r_1^o} h_1(x) + \left(L_{g_1} L_f^{r_1^o - 1} h_1(x)\right) u_1 + \cdots + \left(L_{g_m} L_f^{r_1^o - 1} h_1(x)\right) u_m \\
&\vdots \\
&\dot{\xi}_{r_p^o}^p = L_f^{r_p^o} h_p(x) + \left(L_{g_1} L_f^{r_p^o - 1} h_p(x)\right) u_1 + \cdots + \left(L_{g_m} L_f^{r_p^o - 1} h_p(x)\right) u_m \\
&\dot{\eta}_i = L_f \phi_i(x) + (L_{g_1} \phi_i(x)) u_1 + \cdots + (L_{g_m} \phi_i(x)) u_m, \ i = 1, ..., n - r^c - r^o
\end{aligned} \tag{3.1}
$$

where $r^c = r_1^c + ... + r_l^c$ and $r^o = r_1^o + ... + r_p^o$. Constraint relations (1.1b) become $\zeta_1^1 = 0, ..., \zeta_1^l = 0$ and output functions become $y_1 = \xi_1^1, ..., y_p = \xi_1^p$. (3.1) is called the *Normal Form* . It is easy to see that the input-output behavior of the constrained dynamics can be exact linearized by

$$
u = \alpha(x) = \begin{bmatrix} A^c(x) \\ A^o(x) \end{bmatrix}^{-1} \left(\begin{bmatrix} 0 \\ v \end{bmatrix} - \begin{bmatrix} b^c(x) \\ b^o(x) \end{bmatrix} \right). \tag{3.2}
$$

with A^c, A^o, b^c, b^o defined by (3.4). The "nonlinear zeros" of the constrained dynamics can be described by

$$
\dot{x} = f(x) + G(x)\tilde{u}^*, \quad x \in C^* \cap Z^*, \tag{3.3}
$$

where $C^* = \left\{x \in U | L_f^\rho k_i(x) = 0, 1 \le i \le l, 0 \le \rho \le r_i^c - 1\right\}$, $Z^* = \{x \in U | L_f^\rho h_i(x) = 0, 1 \le i \le l, 0 \le \rho \le r_i^o - 1\}$, and input $\tilde{u}^*$ satisfies equations

$$b^c(x) + A^c(x)u = 0, \qquad A^c = \left[L_{g_j} L_f^{r_i^c - 1} k_i\right]_{l\times m}, b^c = \left[L_f^{r_1^c} k_1, ..., L_f^{r_l^c} k_l\right]^\top, \tag{3.4a}$$

$$b^o(x) + A^o(x)u = 0, \qquad A^o = \left[L_{g_j} L_f^{r_i^o - 1} h_i\right]_{p\times m}, b^o = \left[L_f^{r_1^o} h_1, ..., L_f^{r_p^o} h_p\right]^\top. \tag{3.4b}$$

Since the row vectors of A^c and A^o are linearly independent, to solve equation (3.4) we can either solve $\tilde{u}$ from (3.4a) first or solve u^* from (3.4b) first. Thus we have the following conclusion:

Suppose a nonlinear CDS satisfies assumptions (A1) and (A2). The operations of calculating "nonlinear zeros" and imposing constraints commute. I.e., the zero dynamics of the constrained dynamics is the same as the constrained dynamics of the zero dynamics.

B. Feedback Stabilization

We now discuss the stabilization problem of CDS (2.1). We introduce stabilizability concepts for the constrained dynamics first: x^e is said to be an *equilibrium point (solution) of the constrained dynamics* of CDS (2.1) if $\eta^e = \phi(x^e)$ is an equilibrium point of the reduced state space equation (2.9), which is equivalent to saying that $\tilde{f}(x^e) + \tilde{G}(x^e)\tilde{u} = 0$ for some $\tilde{u} \in R^m$. An equilibrium point x^e of the constrained dynamics is said to be *locally asymptotically stable* provided that there exists a neighborhood V of x^e on C^* such that $\forall x^0 \in V$, the solution $\tilde{x}(x^0, t, 0)$ of equation (2.7) remains in C^* and approaches x^e as $t \to \infty$. The constrained dynamics of CDS (2.1) is said to be *locally asymptotically stabilizable* to an equilibrium point $x^e \in C^*$ if there exists a smooth feedback $\alpha : R^n \to R^m$ with $\alpha(x^e) = 0$ such that $\forall x^0 \in V \subset C^*$ for some open neighborhood V of x^e on C^*, the solution $x(x^0, t)$of the closed-loop system $\dot{x} = f(x) + G(x)\alpha(x)$ remains in C^* for all time and approaches x^e as $t \to \infty$. It is equivalent to saying that the reduced state space equation (2.9) can be stabilized to $\eta^e = \phi(x^e)$.

One of the widely used techniques to solve the stabilization problem is to design feedback based on the linear approximation. To apply this technique, we need to find the linear approximation of the reduced state space equation (2.9). It is well known that *the linear approximation of zero dynamics coincides with the zero dynamics of the linear approximation of the original system* [1]. By the similarity between the constrained dynamics and the zero dynamics, the linear approximation of constrained dynamics can be obtained without computing the nonlinear transformation $\eta = \phi(x)$. For the calculation details we refer to [11].

However in many situations using linear feedback cannot achieve the stabilization task because the linear approximation has uncontrollable modes associated with eigenvalues on the imaginary axis. This is called the critical case. Another approach to solve the stabilization problem is the input-output exact linearization, which can deal with the critical case. According to the Normal Form (3.1) we can choose v in the linearization feedback (3.2) as

$$v = [\gamma_1(x), ..., \gamma_p(x)]^\top, \; \gamma_i(x) = \sum_{j=1}^{r_i^o} -\gamma_{j-1}^i L_f^{j-1} k_i(x), \; \gamma_j^i \in R, \tag{3.5}$$

where γ_j^i are chosen such that polynomials $\Gamma_i(s) = s^{r_i^o} + \gamma_{r_i^o-1}^i s^{r_i^o-1} + \cdots + \gamma_0^i$, $1 \le i \le p$, are Hurwitz. Then in (ζ, ξ, η) coordinates the closed-loop equation can be decomposed into three parts: (1) $\dot{\zeta} = N\zeta$ with N a nilpotent block matrix, (2) $\dot{\xi} = \Gamma\xi$ with Γ a companion block matrix, (3) $\dot{\eta} = L_{\tilde{f}(x)}\phi(x)|_{x=\Phi^{-1}(\zeta,\xi,\eta)}$ with $\tilde{f}(x) = f(x) + G(x)\alpha(x)$. Note that $h(x^e) = 0$ and $\phi(x)$ in the coordinate change can be chosen such that $\phi(x^e) = 0$. Thus $\Phi(x^e) = (0,0,0)$ and $\Phi(x^0) = (0, \xi^0, \eta^0)$ for all $x^0 \in C^*$. It is easy to see that the dynamics of the first part is identically zero for any $x^0 \in C^*$, which means that the closed-loop dynamics remain in C^* for all time. The dynamics of the second part approaches $\xi = 0$ as $t \to \infty$. The dynamics of the third part also approaches $\eta = 0$ provided that $\dot{\eta} = f^*(\eta) = L_{\tilde{f}(x)}\phi(x)|_{x=\Phi^{-1}(0,0,\eta)}$ is locally asymptotically stable at $\eta = 0$. This condition is equivalent to the asymptotic stability of the "nonlinear zeros". Therefore we have the following conclusion:

Suppose x^e is an equilibrium point of the constrained dynamics of CDS (2.1) with $h(x^e) = 0$. If the constrained system satisfies assumptions (A1) and (A2) and the "nonlinear zeros" defined by (3.3) are asymptotically stable at x^e, then the constrained dynamics can be locally asymptotically stabilized by a nonlinear feedback in the form of (3.2) with v satisfying (3.5).

C. Output Tracking

Now we consider the output tracking problem. Given a nonlinear CDS (2.1) and a reference output $\hat{y}(t) = [\hat{y}_1(t), ..., \hat{y}_p(t)]^\top$, find a smooth feedback $u = u(x,t)$ such that (a) the constraint relations (1.1b) are imposed, (b) the real outputs of the system converge asymptotically to the reference output, i.e., $\|y(t) - \hat{y}(t)\| = \left\|h\big(x(x^0, u(x,t), t)\big) - \hat{y}(t)\right\| \to 0$.

In order to achieve the output tracking task we can use the feedback in the form of (3.2) with

$$v = [\gamma_1(x,t), ..., \gamma_p(x,t)]^{\mathrm{T}}, \ \gamma_i(x,t) = \hat{y}_i^{(r_i^o)}(t) - \sum_{j=1}^{r_i^o} \gamma_{j-1}^i \left(L_f^{j-1} k_i(x) - \hat{y}_i^{(j-1)}(t) \right). \quad (3.6)$$

Then we have the similar decompostion for the closed loop dynamics in (ζ, ξ, η) coordinates as that in Section B. The first and the third part have the same form. The second part becomes

$$\begin{aligned}
\dot{\xi}_1^1 &= \xi_2^1, \\
\dot{\xi}_2^1 &= \xi_3^1, \\
&\vdots \\
\dot{\xi}_{r_1^o}^1 &= \hat{y}_1^{(r_1^o)}(t) - \sum_{j=1}^{r_1^o} \gamma_{j-1}^1 \left(\xi_j^1 - \hat{y}_1^{(j-1)}(t) \right), \\
&\vdots \\
\dot{\xi}_{r_p^o}^p &= \hat{y}_p^{(r_p^o)}(t) - \sum_{j=1}^{r_p^o} \gamma_{j-1}^p \left(\xi_j^p - \hat{y}_p^{(j-1)}(t) \right).
\end{aligned} \quad (3.7)$$

Let the output error vector be $e(t) = y(t) - \hat{y}(t)$. According to (3.7), the components of the error vector satisfy the following linear differential equations

$$e_i^{(r_i^o)}(t) + \gamma_{r_i^o-1}^i e_i^{(r_i^o-1)}(t) + \cdots + \gamma_0^i e_i(t) = 0, \ \ i = 1, ..., p. \quad (3.8)$$

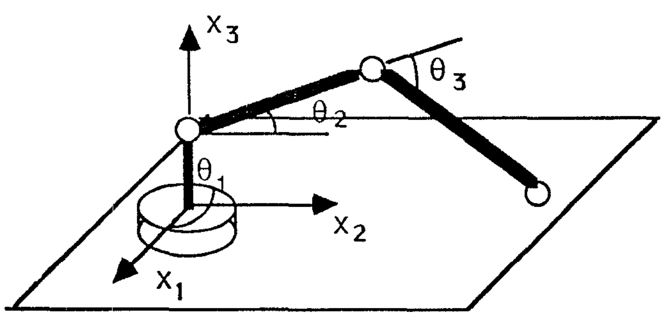
If γ_j^i are chosen such that polynomials $\Gamma_i(s) = s^{r_i^o} + \gamma_{r_i^o-1}^i s^{r_i^o-1} + \cdots + \gamma_0^i$, $1 \le i \le p$, are Hurwitz, then the component of the error vector $e_i(t) = y_i(t) - \hat{y}_i(t) \to 0$, $1 \le i \le p$. Therefore by using this feedback the asymptotic tracking problem can be solved. Obviously we are also concerned about the internal behavior of the constrained dynamics when using feedback (3.6) to solve tracking problem. We need to check the third part of the decomposed system: $\dot{\eta} = f^*\left(\hat{\xi}(t), \eta\right)$ for the stability at $\eta = 0$

Remark. This result can be extended to the problem of tracking the output of a reference linear system.

4. Application to Robotic Contour Following Problem

Now we apply our theoretical framework for CDS to a typical control problem in robotics — the robotic contour following problem. The robotic contour following problem is to control the motion of the robot manipulator while the end-effector of the manipulator maintains contact with a rigid unilateral surface. There are two essential aspects of the robotic contour following problem: (1) the constraint between the end-effector and the surface is actively imposed by the robot manipulator system instead of some external forces; (2) the contact force applied by the end-effector to the constraint surface is under control. In addition to the satisfaction of these requirements, we also want to control the manipulator to perform more complicated tasks such as tracking a moving object on the constraint surface, regulating the end-effector to a desired position, etc.

We consider a three revolute joint manipulator with end-effector moving along the horizontal coordinate plane. Suppose that there are two output functions which are the coordinate functions of the plane. (For more general constraint surfaces it is also reasonable to assume that output functions are the coordinate functions on the constraint surface since these functions can be observed easily.) Then this contour following problem can be characterized by the Lagrangian formulation [9]:



$$
\begin{aligned}
\dot{\theta} &= \omega \\
\dot{\omega} &= M^{-1}(F + \lambda J_3) + M^{-1}\tau \\
x_3 &= h_3(\theta) = 0 \qquad\qquad (4.1) \\
x_1 &= h_1(\theta) \\
x_2 &= h_2(\theta)
\end{aligned}
$$

where $\theta = [\,\theta_1 \quad \theta_2 \quad \theta_3\,]^{\mathsf{T}}$ is the joint angle vector; τ is the joint torque vector; M is the inertia matrix; F consists of the Coriolis term, centrifugal term, gravitational term, and payload term; $[\,x_1 \quad x_2 \quad x_3\,]^{\mathsf{T}} = [\,h_1(\theta) \quad h_2(\theta) \quad h_3(\theta)\,]^{\mathsf{T}}$ is the transformation from joint coordinates to Cartesian coordinates; $J^{\mathsf{T}}(\theta) = [\,J_1(\theta) \quad J_2(\theta) \quad J_3(\theta)\,] \in R^{3\times 3}$ is the transpose of the manipulator Jacobian; λ is the Lagrangian multiplier; $\lambda J_3(\theta)$ is the vector of joint torques associated with the contact force vector $[\,0 \quad 0 \quad \lambda\,]^{\mathsf{T}}$.

For all regular kinematic configurations the constraint relative degree is 2, the output relative degree vector is $[\,2 \quad 2\,]$, and the constraint submanifold is $C^* = \{(\theta,\omega) | h_3(\theta) = 0, J_3^{\mathsf{T}}(\theta)\omega = 0\}$. Moreover the decoupling matrices A^c and A^o have linearly independent row vectors because these vectors are the row vectors of the nonsingular matrix $J(\theta)M^{-1}(\theta)$. Using the coordinate change $\xi_1 = h_1(\theta), \xi_2 = J_1^{\top}(\theta)\omega, \xi_3 = h_2(\theta), \xi_4 = J_2^{\top}(\theta)\omega, \zeta_1 = h_3(\theta), \zeta_2 = J_3^{\top}(\theta)\omega$, (4.1) can be transformed into the Normal Form

$$
\begin{aligned}
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= \omega^{\top}\left[\partial^2 h_1/\partial\theta^2\right]\omega + J_1^{\top}M^{-1}(F+\lambda J_3) + J_1^{\top}M^{-1}\tau \\
\dot{\xi}_3 &= \xi_4 \\
\dot{\xi}_4 &= \omega^{\top}\left[\partial^2 h_2/\partial\theta^2\right]\omega + J_2^{\top}M^{-1}(F+\lambda J_3) + J_2^{\top}M^{-1}\tau \\
\dot{\zeta}_1 &= \zeta_2 \\
\dot{\zeta}_2 &= \omega^{\top}\left[\partial^2 h_3/\partial\theta^2\right]\omega + J_3^{\top}M^{-1}(F+\lambda J_3) + J_3^{\top}M^{-1}\tau.
\end{aligned}
\tag{4.2}
$$

In order to maintain the constraint and contact force, the joint torque vector must satisfy

$$
\begin{aligned}
&JM^{-1}\tau + b + JM^{-1}(F+\lambda J_3) = \begin{bmatrix} v_1 & v_2 & 0 \end{bmatrix}^{\top} = v \\
&b = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}^{\top},\ b_i(\theta,\omega) = \omega^{\top}\left[\partial^2 h_i/\partial\theta^2\right]\omega,\ i = 1,2,3.
\end{aligned}
\tag{4.3}
$$

For any $v = \begin{bmatrix} v_1 & v_2 & 0 \end{bmatrix}^{\top}$ with continuous function components, (4.3) has a unique solution

$$
\tau = -(F(\theta,\omega) + \lambda J_3(\theta)) + M(\theta)J^{-1}(\theta)(v - b(\theta,\omega)). \tag{4.4}
$$

Using this feedback we can achieve the following goals: (a) The constraint relation is imposed by joint actuator; (b) the desired contact force is maintained; (c) the input-output behavior of the constrained dynamics is linearized and decoupled because $\ddot{x}_1 = v_1$, $\ddot{x}_2 = v_2$. Moreover we can achieve more complicated tasks by proper choice of v_1 and v_2.

Regulation: One of the important tasks in robotics is to regulate the end-effector of the robot manipulator to a desired position in the constraint surface while keeping proper contact force. Suppose that we want to regulate the end-effector of the manipulator system (4.1) to $(x_1^0, x_2^0) = (h_1(\theta^0), h_2(\theta^0))$. Assume that θ^0 is an equilibrium configuration of (4.1), i.e., $F(\theta^0, 0) + \lambda J_3^{\top}(\theta^0) = 0$. Choose

$$
v_i = -\gamma_1^i J_i^{\top}(\theta)\omega - \gamma_0^i\big(h_i(\theta) - x_i^0\big),\ i = 1,2, \tag{4.5}
$$

with $\gamma_j^i \in R$ such that $\Gamma^i(s) = s^2 + \gamma_1^i s + \gamma_0^i$, $i = 1,2$, are Hurwitz polynomials. It is easy to see that the feedback (4.4) and (4.5) impose the constraint, maintain the contact force, stabilize the constrained dynamics, and drive the end-effector to the desired position from any initial position on the constraint surface, i.e. $x_1(t) \to x_1^0$, $x_2(t) \to x_2^0$ as $t \to \infty$.

Tracking: Now we consider the problem of tracking a moving object on the constraint surface while keeping proper contact force. Assume that the position of the moving object can be observed as $[\, x_1^R(t) \quad x_2^R(t) \quad 0\,]^\top$. This problem is the output tracking problem discussed in Section 3. Choosing

$$v_i = \ddot{x}_i^R(t) - \gamma_1^i \Big(J_i^\top(\theta)\omega - \dot{x}_i^R(t) \Big) - \gamma_0^i \Big(h_i(\theta) - x_i^R(t) \Big), \quad i = 1, 2, \tag{4.6}$$

with $\gamma_j^i \in R$ such that $\Gamma^i(s) = s^2 + \gamma_1^i s + \gamma_0^i$, $i = 1, 2$, are Hurwitz polynomials, then feedback (4.4) and (4.6) solve this tracking problem.

Remark: Our control scheme is different from the previous schemes in the literature [12][9][8]. It has following advantages: (1) *The feedback scheme (4.4) is valid for all regular configurations* of the robot manipulator, which is not the case with the linear approximation approach. (2) *There is no separation of input variables.* It is impossible to separate joint torques as control inputs and constraint inputs even for the task of imposing the constraint and maintaining the contact force. (3) *It is easy to implement.* The calculation of (4.4), (4.5) and (4.6) only requires evaluation of the manipulator Jacobian, the inertia matrix, and the coordinate transformation, all of which can be computed in real–time.

5. Conclusion

Constrained nonlinear systems can model a large class of physical phenomena which cannot be handled by regular state space systems. Our theoretical work concerns two aspects of constrained nonlinear systems: (1) Characterization of the dynamic behavior under the constraints: We determine the constraint submanifold and the constrained dynamics without requiring partition into constraint and control input variables and without requiring the existence of vector relative degree. (2) Control of the constrained dynamics: We use nonlinear feedback and exact linearization techniques for the nonlinear CDS so that improved results can be obtained compared to controller design via linear approximation and linear feedback. The application to the robotic contour following problem illustrates the advantages of our extended model and new control scheme for nonlinear CDS.

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