ABSTRACT

Title of dissertation: A Gauge-Theoretic Approach to the Chern Form of the Canonical Bundle on the Moduli Space of Stable Parabolic Bundles

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In this thesis we apply a gauge-theoretic approach to construct the moduli space of stable parabolic bundles on a closed Riemann surface using weighted Sobolev spaces. We study the metric properties of the moduli space, and in particular, we compute the L^2 curvature of its canonical bundle. By identifying the canonical bundle with the index bundle of a suitable family of Dolbeault operators, we define a spectral Quillen metric on the canonical bundle via a relative analytic torsion construction first introduced by Müller. We compute the curvature of the canonical bundle with respect to this Quillen metric and find that it consists of the standard Atiyah-Singer term along with a cuspidal contribution coming from the parabolic structure and depending upon the parabolic weights. This gives a new proof of a result of Zograf-Takhtajan.

A GAUGE-THEORETIC APPROACH TO THE CHERN FORM OF THE CANONICAL BUNDLE ON THE MODULI SPACE OF STABLE PARABOLIC BUNDLES

by

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Dedication

谨以此文献给我的父亲母亲，感谢你们无私的爱和支持; 也感谢我太太和女儿，让我的生活充满阳光。

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Table of Contents

Chapter 1: Introduction

1.1 Background

Moduli spaces arise naturally as one is trying to classify a class of mathematical objects subject to an equivalence relation. Often, the set of equivalence classes carries topological and geometric structures. The objective of the theory of moduli spaces is the study of these structures from both local and global aspects.

Let us consider the problem of classifying holomorphic vector bundles over a compact Riemann surface X. This problem was solved for $\mathbb{P}^1(\mathbb{C})$ by Grothendieck [59], and in the case of elliptic curves by Atiyah [4]. From now on, we will restrict our attention to Riemann surfaces of genus greater or equal to 2.

Since C^∞ complex vector bundles are determined by the rank and degree (equals the first Chern number of E), we will consider the space $\mathscr{C}$ of holomorphic structures on a fixed smooth vector bundle E with rank n and degree d. The complex gauge group $\mathscr{G}^{\mathbb{C}}$ of complex automorphisms of E acts on $\mathscr{C}$ by pullback. The orbit space $\mathscr{C}/\mathscr{G}^{\mathbb{C}}$ clearly parametrizes isomorphism classes of holomorphic bundles over X. However, in general it is not Hausdorff due to the jump phenomenon, see [91]. Moreover, the proper definition of such a moduli space and how to construct it remain in question.

To answer these questions, Mumford [91] introduced the notion of stable and semi-stable holomorphic bundles.

Definition 1. Let the slope of E be given by

$$\mu(E) = \frac{\deg E}{\operatorname{rank} E}.$$

We say that E is a stable (resp. semi-stable) holomorphic bundle if for every proper holomorphic subbundle F of E, the following inequality

$$\mu(F) < \mu(E) \quad (\text{resp.}\, \mu(F) \leq \mu(E)),$$

holds.

Let $\mathscr{C}^s$ (resp. $\mathscr{C}^{ss}$) denote the subspace of $\mathscr{C}$ consists of stable (resp. semi-stable) holomorphic structures on E. Narasimhan and Seshadri [92] constructed the moduli space $\mathscr{C}^s/\mathscr{G}^{\mathbb{C}}$ of stable holomorphic bundles algebraically, and proved their famous theorem that $\mathscr{C}^s/\mathscr{G}^{\mathbb{C}}$ can be identified with the moduli space of irreducible projective unitary representations of $\pi_1(X)$. The geometric invariant theory of Mumford [91] provides a projective compactification of $\mathscr{C}^s/\mathscr{G}^{\mathbb{C}}$, which is given by $\mathscr{C}^{ss}/\!\!/\mathscr{G}^{\mathbb{C}}$. Donaldson [37] gave another proof of the Narasimhan-Seshadri theorem from the gauge-theoretic point of view of Atiyah and Bott [5], and showed that every stable holomorphic bundle has an essentially unique irreducible Hermitian-Einstein (or Hermitian Yang-Mills) connection, i.e., the Chern connection d_A associated to a Hermitian holomorphic stable bundle (E, h) satisfies the Hermitian Yang–Mills equation:

$$F(A) = -i\mu\,\omega_g\,\mathrm{Id}_E, \tag{1.1}$$

where ω_g is the Kähler form on X normalized to have volume 2π. Let $\mathscr{A}^*_{HE}$ denote the space of all irreducible unitary connections satisfying (1.1), and let $\mathscr{G}$ denote the gauge group of unitary transformations of (E, h). Donaldson's result states that the following map

$$\begin{aligned} \Phi_h : \mathscr{A}^*_{HE}/\mathscr{G} &\to \mathscr{C}^s/\mathscr{G}^{\mathbb{C}}, \\ [d_A] &\mapsto [\bar{\partial}_A := d_A^{0,1}]. \end{aligned} \tag{1.2}$$

is a bijection.

To endow these moduli spaces with symplectic and Kähler structures, the traditional approach is to apply the process called Kähler reduction, see [5]. More precisely, let (E, h) be a fixed smooth complex Hermitian vector bundle over a compact Riemann surface X. The set $\mathscr{A}$ of all unitary connections on (E, h) is an affine space modeled on $\Omega^1(\mathfrak{u}(E))$, which is equipped with a symplectic structure defined by

$$\Omega(\mu, \nu) = \int_X \mathrm{tr}(\mu \wedge \nu), \tag{1.3}$$

for any $d_A \in \mathscr{A}$ and $\mu, \nu \in T_A\mathscr{A}$. This form, referred to as the Atiyah-Bott-Goldman-Narasimhan form, is closed since it is constant. The set $\mathscr{C}$ of holomorphic structures on E is an affine space modeled on $\Omega^{0,1}(\mathrm{End}\,E)$, and it has a complex structure, given by multiplication by i. The symplectic structure of $\mathscr{A}$ and the complex structure induced from $\mathscr{C}$ by the Chern connection construction, see Lemma 18, defines a Kähler structure on $\mathscr{A}$. Moreover, the gauge group $\mathscr{G}$ preserves this Kähler structure, and there is a moment map for this action [6] given by

$$\mu : \mathscr{A} \to \Omega^2(\mathfrak{u}(E)),$$

$$d_A \mapsto F(A).$$

Here we have used the fact that $\mathscr{G}$ is a Banach Lie group with Lie algebra $\Omega^0(u(E))$ and dual Lie algebra $\Omega^2(u(E))$. Take the central element $-i\mu\,\omega_g\,\mathrm{Id}_E$ as in the right hand side of (1.1) and consider $\mu^{-1}(-i\mu\,\omega_g\,\mathrm{Id}_E)$, which equals to $\mathscr{A}_{HE}$. By the work of Mumford, Kempf–Ness, Guillemin and Sternberg and others, (see the appendix to [91], written by Kirwan), the quotient $\mu^{-1}(-i\mu\,\omega_g\,\mathrm{Id}_E)/\mathscr{G}$ is a Kähler manifold. In particular, as an open subspace, $\mathscr{A}^*_{HE}/\mathscr{G}$ is also Kähler. Moreover, the correspondence (1.2) proved by Donaldson can be understood formally as an infinite dimensional analogue of the isomorphism between the symplectic and the algebraic quotients in finite dimensions as studied in the geometric invariant theory. More precisely, (1.2) is the restriction of

$$\mu^{-1}(\lambda)/\mathscr{G} = \mathscr{C}^{ss} /\!\!/ \mathscr{G}^{\mathbb{C}}, \tag{1.4}$$

to the stable loci $\mathscr{C}^s \subset \mathscr{C}^{ss}$.

With the above discussion of moduli spaces of vector bundles, we are ready to introduce the main objects of interest in this thesis. In the proof of [92], Narasimhan and Seshadri were led to consider bundles on ramified covers of a Riemann surface, where the fundamental group lifts to the bundles. In terms of the bundle on the initial Riemann surface, one has extra structure at the branch points, a reduction of the local automorphisms to a parabolic subgroup. This leads to the notion of a parabolic bundle over a compact surface $\overline{X}$ with a finite set S of marked points. More precisely, we define a parabolic bundle as a holomorphic bundle E over $\overline{X}$ together with descending flags

$$E_{p_k} = E_{1,k} \supset E_{2,k} \supset \cdots \supset E_{s_k,k} \supset 0, \tag{1.5}$$

in the fiber of E_{p_k} and associated parabolic weights $0 \le \alpha_{1,k} \le \alpha_{2,k} \le \cdots \le \alpha_{n,k} < 1$ for each $p_k \in S$. Set $m_{i,k} = \dim E_{i,k} - \dim E_{i+1,k}$, we will refer the following partitions

$$n = m_{1,k} + \cdots + m_{s_k,k}, \;\; k = 1, \cdots, m,$$

as the *multiplicity type* of E. Let P_k denote the stabilizer of (1.5), which is a parabolic subgroup in $\mathrm{GL}(E_{p_k})$. Their quotient $\mathrm{GL}(E_{p_k})/P_k \cong \mathscr{F}_k$ is the flag manifold parametrizing decreasing flags on E_{p_k} of the given multiplicity type.

Let us consider the classification of all parabolic bundles over $\overline{X}$ with fixed multiplicity type and parabolic weights at each marked point $p_k \in S$. We say that two parabolic bundles E and E' are isomorphic if there is a bundle isomorphism $E \to E'$ mapping the corresponding descending flags into each other for all $p_k \in S$. Since $\mathrm{GL}(E_{p_k})$ acts on $\mathscr{F}_k$ transitively, the above classification problem can be rephrased as to study the orbit space of the space $\mathscr{C}$ of all holomorphic structures on E with a fixed parabolic structure modulo the group $\mathscr{G}^{\mathbb{C}}$ of complex automorphisms that preserves the descending flags at each marked points. Therefore, we are in an analogous position as the classification of holomorphic bundles over a compact Riemann surface.

Let us remark that parabolic weights are also important in the understanding of moduli space of parabolic bundles. First, Mehta and Seshadri [85] used these weights to introduce suitable notions of parabolic degree, and stable and semi-stable parabolic bundles, and constructed the moduli space of stable parabolic bundles of parabolic degree d with a fixed parabolic structure of rational weights at the marked points. Moreover, they generalized Narasimhan-Seshadri theorem and showed that the moduli space of stable parabolic bundles of parabolic degree 0 can be identified with the moduli space of irreducible unitary representations of $\pi_1(X)$ with local monodromies in a fixed conjugacy class about each puncture. Note that conjugacy classes in $U(n)$ are labeled by eigenvalues and multiplicities. By exponentiating the eigenvalues in the unit interval, the conjugacy class of the local monodromy of a representation is thus also encoded by parabolic weights and choices of a flag, and this is the precise correspondence in the Mehta-Seshadri theorem. Later Biquard [12], following the argument of Donaldson [37], improved the result of Mehta and Seshadri [85] to allow real parabolic weights.

In the following, we will denote by $\mathscr{M}_{\mathrm{P}}^{s}$ the moduli space of parabolic bundles on $\overline{X}$ with fixed parabolic structure of parabolic degree 0, and by $\mathscr{M}_{HE}^{*}$ the moduli space of irreducible flat unitary connections on X with fixed holonomy conjugacy classes. So far, most research concerning $\mathscr{M}_{\mathrm{P}}^s$ has been to understand its topology and application in algebraic geometry, see [15], [55], [111], and [105], while leaving the differential geometric properties of $\mathscr{M}_{\mathrm{P}}^s$ less documented. One objective of this thesis is to partially fill this gap in current literature. That is, in the first part this thesis, we will apply gauge-theoretic methods as those used in [5] and [73], rather than the process of Kähler reduction, to understand the complex and Kähler properties of these moduli spaces.

1.2 Main Results

In order to study these moduli spaces in a gauge theoretic setup, the first question is how to encode the parabolic structure of a parabolic bundle E on $\overline{X}$. Our approach here is to equip E with the adapted Hermitian metric, which is singular at the marked points with specific behavior, see Definition 14. It is therefore more convenient to work with the punctured surface X instead. In this work, we will equip X with a complete Riemannian metric of cusp type near the punctures, see (2.3), and use the theory of weighted Sobolev spaces as discussed in [77] to define the weighted complex gauge group $\mathscr{G}_\delta^{\mathbb{C}}$ (resp. the weighted gauge group $\mathscr{G}_\delta$ of unitary transformations) acting on the space $\mathscr{C}_\delta^s$ of stable Dolbeault operators $\mathscr{C}_\delta^s$ of parabolic degree 0 (resp. the space $\mathscr{A}_{F,\delta}^*$ of irreducible flat unitary connections), both adapted to a fixed parabolic structure on E in suitable sense.

By a key result of Biquard, see Theorem 45, concerning the Fredholmness of the Dolbeault Laplacians, we can apply an implicit function theorem argument to show the existence of local slices around any point in these moduli spaces, and these slices are the key to our analysis of the local structures of these moduli spaces. Take the action of $\mathscr{G}_\delta$ on $\mathscr{A}^*_{F,\delta}$ as an example. We will show that, locally around any $d_A \in \mathscr{A}^*_{F,\delta}$, there is a (non-linear) Hermitian-Einstein slice $U_{A,\epsilon} \subset \mathscr{A}^*_{F,\delta}$ such that it is invariant under the stabilizer $\mathrm{Stab}(A)$ of d_A, and the following natural map

$$U_{A,\epsilon} \times_{\mathrm{Stab}(A)} \mathscr{G}_\delta \to \mathscr{A}^*_{F,\delta}, \tag{1.6}$$

is a local diffeomorphism. Moreover, we will define a Riemannian metric on $U_{A,\epsilon}$ that is invariant under different choices of local slice representation, hence the metric patches to a global metric on $\mathscr{M}^*_{HE}$. Similarly, we will construct, around any $\bar{\partial}_A \in \mathscr{C}^s_\delta$, a local (linear) Dolbeault slice $V_{A,\epsilon}$, see Definition 62, which provide a compatible system of local holomorphic coordinate charts on $\mathscr{M}^s_{\mathrm{P}}$. Then we will show that there exists a local diffeomorphism between $V_{A,\epsilon}$ and the image of $U_{A,\epsilon}$ in $\mathscr{M}^s_{\mathrm{P}}$ under (3.6), which will be denoted by $U^{0,1}_{A,\epsilon}$, hence enable us to define a Hermitian metric on $\mathscr{M}^s_{\mathrm{P}}$. In chapter 4, we prove that the Hermitian metric just defined are in fact Kähler and compute the curvature of the canonical bundle of $\mathscr{M}^s_{\mathrm{P}}$ with the induced metric.

Theorem 2 (see Theorem 71 below)**.** The curvature of the canonical line bundle $\lambda = \det T^*\mathscr{M}^s_{\mathrm{P}}$ with respect to the induced L^2-Hermitian metric is given by

$$\Theta(\mu,\bar{\nu}) = -\operatorname{Tr}\Big(\operatorname{ad} f_{\mu,\bar{\nu}} \circ P_A - \operatorname{ad}\mu \circ \Delta_A^{-1} \circ * \operatorname{ad}\nu * \circ P_A\Big),$$

for any $\mu,\nu \in H^{0,1}_{A,\delta} \cong T_{\bar{\partial}_A} V_{A,\epsilon}$ on $V_{A,\epsilon}$. Here Tr denotes the operator trace on

$L^2(\Omega^{0,1}(\mathrm{End}E))$ and $\mathrm{ad}\,\mu$ denotes the adjoint action $[\mu,\cdot]$.

Let us remark that the choice of the canonical line bundle in Theorem 2, besides the fact that λ is geometrically interesting, is due to the following identification of λ with the determinant of cohomology. That is, given any $\bar\partial_\gamma = \bar\partial_A + \gamma$ in the Hermitian-Einstein slice $U^{0,1}_{A,\epsilon}$, by Proposition 133, the following L^2 complex

$$0 \to L^2(\mathrm{End}\,E) \xrightarrow{\bar\partial_\gamma} L^2(\Lambda^{0,1}\otimes \mathrm{End}\,E) \to 0,$$

is Fredholm. We may define the following determinant line bundle on $U^{0,1}_{A,\epsilon}$,

$$\begin{aligned}\det(\mathrm{ind}\,\bar\partial_\gamma) &= \Lambda^{\max}\ker\bar\partial_\gamma \otimes (\Lambda^{\max}\mathrm{coker}\,\bar\partial_\gamma)^{-1},\\ &= (\Lambda^{\max}\mathrm{coker}\,\bar\partial_\gamma)^{-1}.\end{aligned}$$

where we have used the stability condition of the associated parabolic bundle $(E,\bar\partial_\gamma)$. As the dimension of $\mathrm{coker}\,\bar\partial_\gamma$ remains constant on $U^{0,1}_{A,\epsilon}$, the induced L^2 metric is a well-defined Hermitian metric. Moreover, by the local diffeomorphism between $U^{01}_{A,\epsilon}$ and $V_{A,\epsilon}$ and Definition 3.19, we have the following identification

$$\lambda \coloneqq \det(T^*V_{A,\epsilon}) \cong \det(\mathrm{ind}\,\bar\partial_\gamma). \tag{1.7}$$

When $X = \mathbb{H}/\Gamma$ is a hyperbolic surface, Takhtajan and Zograf [113] defined a Quillen metric on λ in terms of the Selberg zeta function, see [119]. More precisely, for any $d_{A+a}\in U_{A,\epsilon}$ with $a^{0,1}=\gamma$, let ρ_γ denote the associated holonomy representation of d_{A+a}. They define

$$Z(s,\Gamma;\mathrm{Ad}\,\rho_\gamma)=\prod_{\{P\}}\prod_{n}\det\left(I-\mathrm{Ad}\,\rho_\gamma(P)N(P)^{-n-s}\right), \tag{1.8}$$

where P runs over the set of all primitive conjugacy classes of hyperbolic elements of Γ, and $N(P)>1$ is the norm of the element $P\in\Gamma$. As $Z(s,\Gamma;\mathrm{Ad}\,\rho_\gamma)$ has a simple pole at $s=1$, and they define the regularized determinant of Δ_γ as

$$\det{}_{TZ}\,\Delta_\gamma=\frac{\partial}{\partial s}\bigg|_{s=1}Z(s,\Gamma;\mathrm{Ad}\,\rho_\gamma),$$

and the Quillen metric on λ by

$$\|\cdot\|_{TZ}:=\|\cdot\|_{L^2}(\det{}_{TZ}\,\Delta_\gamma)^{-1/2}. \tag{1.9}$$

They compute the curvature form of this Quillen metric, which consists of the usual Atiyah-Singer term and a cuspidal defect. Such a definition of metric is justified by the relation between the Selberg zeta function and the functional determinant of the Laplacian of the following form,

$$\det(\bar\partial_\gamma^*\bar\partial_\gamma+s(1+s))=\phi(s)Z(s,\Gamma;\mathrm{Ad}\,\rho),$$

where $\phi(s)$ is some universal meromorphic function depending only on g, n and the parabolic weights. In the case of compact Riemann surface, such a relation was first discovered by D'Hoker and Phong [34], see also Sarnak [104] and Voros [121].

Later, Efrat generalized this to the case of torsion-free hyperbolic surface of finite volume. The method he use to construct the spectral zeta function is to use not only discrete eigenvalues but also poles of the scattering determinant in the Selberg trace formula.

This is another motivation of this thesis, that is, we wish to construct the Quillen metric using the heat kernel techniques along the idea of Quillen [98], see also [100], [101].

Let us recall the zeta function regularization of the determinant of an elliptic operator on compact manifolds, which was first introduced by Ray and Singer [100]. Let E be a Hermitian vector bundle over a closed n-dimensional smooth manifold M. Let $A : C^\infty(M, E) \to C^\infty(M, E)$ be a non-negative elliptic self-adjoint differential operator of order m, whose action extends uniquely to $L^2(M, E)$. Then the spectral zeta function of A is defined by

$$\zeta_A(s) = \sum_{\lambda_j > 0} \lambda_j^{-s},$$

where λ_j runs over the nonzero eigenvalues of A, counted with multiplicity. By Seeley [109], the above series converges absolutely in the half-plane $\mathrm{Re} s > n/m$ and admits a meromorphic continuation to the entire complex plane, regular at $s = 0$. As it is well-known that the heat operator e^{-tA} is of trace class for $t > 0$ on compact manifolds, we can apply the Mellin transform

$$\lambda^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-\lambda t} dt$$

to express the zeta function of A as

$$\zeta_A(s) = \frac{1}{\Gamma(s)} \int_0^\infty (\mathrm{Tr}(e^{-tA}) - \dim\ker A)\, t^{s-1}\, dt,$$

for $\mathrm{Re} s > n/m$, which admits a meromorphic continuation by the following

1. As $t \to 0^+$, there exists an asymptotic expansion

$$\mathrm{Tr}(e^{-tA}) \sim t^{n/m} \sum_{j\geq 0} a_j t^j,$$

2. As $t \to \infty$, we have

$$\mathrm{Tr}(e^{-tA}) = \dim\ker A + O(e^{-ct}).$$

One then defines the regularized determinant of A as

$$\det A = \exp\Big(- \frac{d}{d\,s}\Big|_{s=0} \zeta_A(s)\Big).$$

Since X is noncompact, the elliptic operator Δ_γ has continuous spectrum and $e^{-t\Delta_\gamma}$ is not of trace class, hence the zeta function regularization discussed above is not applicable. In this thesis, we apply the the method of Müller [89] and define a relative determinant

$$\det \Delta_\gamma := \det(\Delta_\gamma, \Delta_A),$$

with respect to the "reference" Dolbeault Laplacian Δ_A, as long as the relative heat trace $\mathrm{Tr}(e^{-t\Delta_\gamma} - e^{-t\Delta_A})$ is finite for any $t > 0$ and has similar asymptotic expansions as discussed above in the compact case. With the relative determinant, we may define a Quillen metric on λ by

$$\|\cdot\|_Q := \|\cdot\|_{L^2}(\det \Delta_\gamma)^{-1/2}. \tag{1.10}$$

Moreover, using the method of [98], we prove that

Theorem 3 (see Theorem 119 below)**.** The first Chern form Ω of the canonical bundle λ equipped with the Quillen metric 1.10 is given by

$$\Omega(\mu,\bar\nu) = -\frac{i}{4\pi^2}\int_X \operatorname{ad}\mu\wedge\operatorname{ad}*\nu + \frac{i}{2\pi}\sum_{k=1}^{m}\sum_{i\neq j=1}^{s_k}\operatorname{sgn}(\alpha_{i,k}-\alpha_{j,k})(1-2|\alpha_{i,k}-\alpha_{j,k}|)m_{i,k}\Theta_{j,k}(\mu,\bar\nu), \tag{1.11}$$

for any $\mu,\nu\in H^{0,1}_{A,\delta}\cong T_{\bar\partial_A}V_{A,\epsilon}$, and $\Omega_{i,k}$ is the curvature form of the line bundles $\lambda_{i,k}$, see Definition 4.12.

Lastly, we comment that this thesis is inspired by the work of Wolpert [125], in which a similar question about cuspidal contribution to the index bundle over the Teichmüller space of punctured Riemann surfaces with cusp ends was considered.

1.3 Outline

In Chapter 1, we introduce the necessary background of moduli space of holomorphic bundles over a compact Riemann surface and the notion of a parabolic bundle. Then we describe our motivation and the main results of this thesis.

In Chapter 2, we will introduce basics of parabolic bundles, and the notion of an adapted Hermitian metric on a parabolic bundle on a Riemann surface with cusp ends. Applying the theory of weighted Sobolev spaces as in [78] and [77], we define various spaces of sections for $\operatorname{End} E$ and $\mathfrak{u}(E)$. Using a result of Biquard [13], we will show the Fredholmness property of the Dolbeault operators acting on the aforementioned weighted spaces. This provides us with the key analytic tool for applying the implicit function theorem type argument used in the next chapter.

In Chapter 3, we present gauge theoretical constructions of the moduli space of irreducible flat unitary connections, denoted by $\mathscr{M}^*_{\mathrm{HE}}$, and of the moduli space of stable parabolic bundles, denoted by $\mathscr{M}^s_{\mathrm{P}}$. More importantly, we construct various local slices and discuss the complex and Hermitian metric of the moduli space of stable parabolic bundles over these slices.

In Chapter 4, we prove that the L^2 Hermitian metric on the moduli space is Kähler, moreover, the Dolbeault slices, see definition 62, provide an atlas of normal coordinate charts on $\mathscr{M}^s_P$. Then we proceed to compute the L^2-curvature of the canonical bundle of $\mathscr{M}^s_P$ over any Dolbeault slice.

In Chapter 5, we apply the relative zeta function regularization, first proposed by Müller [89], to define a relative regularized determinant for Dolbeault Laplacians associated with the Dolbeault operators in a given Dolbeault slice. This enable us to define a Quillen metric, see Definition 1.10, on the canonical line bundle of the moduli space of stable parabolic bundles.

In Chapter 6, we use the method as in Quillen [98] to compute the curvature of the canonical line bundle of the moduli space of parabolic bundles with respect to the Quillen metric.

Chapter 2: Preliminaries

In this chapter, we will introduce basics of parabolic bundles, and the notion of an adapted Hermitian metric on a parabolic bundle $E \to X$, where X is a Riemann surface with cusp ends. Later, we apply the theory of weighted Sobolev spaces as in [78] and [77] to define various spaces of sections for $\operatorname{End} E$ and $\mathfrak{u}(E)$. Using results of Lockhart and McOwen [78] and Biquard [13], we establish the Fredholmness property of the Dolbeault operators acting on the aforementioned weighted spaces. This provides us with the key analytic tool for the implicit function theorem type argument used in the next chapter.

2.1 Parabolic Bundles

Let $\overline{X}$ be a closed Riemann surface of genus h. Let $S = \{p_1, \cdots, p_m\}$ be a finite set of marked points in $\overline{X}$. Let E be a smooth complex vector bundle over $\overline{X}$ of rank n.

Definition 4. A *parabolic structure* on E consists of, at each $p_k \in S$, a decreasing flag

$$E_{p_k} = E_{1,k} \supset E_{2,k} \supset \cdots \supset E_{s_k,k} \supset 0,$$

with weights

$$0 \leq \alpha_{1,k} \leq \alpha_{2,k} \leq \cdots \leq \alpha_{n,k} < 1.$$

We set $m_{i,k} := \dim(E_{i,k}/E_{i+1,k})$ the multiplicity of the weight $\alpha_{i,k}$ and s_k the number of distinct weights at p_k.

Definition 5. We define the *parabolic degree* of E by

$$\text{par-deg}\, E = \deg E + \sum_{k=1}^{m} \sum_{j=1}^{n} \alpha_{j,k},$$

and the *parabolic slope* of E by

$$\mu(E) = \frac{\text{par-deg}\, E}{\operatorname{rank} E}.$$

Given any subbundle $F \subset E$ with the quotient $Q = E/F$, they inherit canonical induced parabolic structures as follows: the flag structures at each p_k are given by

$$F_{1,k} = E_{1,k} \cap F_{p_k} \supset F_{2,k} = E_{2,k} \cap F_{p_k} \supset \cdots \supset F_{n,k} = E_{n,k} \cap F_{p_k} \supset 0,$$

and

$$Q_{1,k} = E_{1,k}/F_{p_k} \supset Q_{2,k} = E_{2,k}/F_{p_k} \supset \cdots \supset Q_{n,k} = E_{n,k}/F_{p_k} \supset 0.$$

As to the induced parabolic weights, if different intersections (resp. quotients) would coincide, it is the smallest (resp. largest) possible weights of the corresponding flag containing the sub-flag (resp. quotient flag) that we choose.

By the Newlander-Nirenberg Theorem [94], we define a holomorphic structure on E by

Definition 6. A *Dolbeault operator* $\bar{\partial}_E$ is a $\mathbb{C}$-linear map

$$\bar{\partial}_E : \Omega^0(E) \to \Omega^{0,1}(E),$$

satisfying the Leibniz rule

$$\bar{\partial}_E(fs) = \bar{\partial} f \otimes s + f \bar{\partial}_E s,$$

for all $f \in C^\infty(\overline{X})$ and $s \in \Omega^0(\overline{X}, E)$.

Let $\mathscr{C}$ denote the space of all Dolbeault operators (or holomorphic structures) on E. Since the difference of any two Dolbeault operators $\bar{\partial}_E$ and $\bar{\partial}'_E$ satisfies

$$(\bar{\partial}_E - \bar{\partial}'_E)(fs) = f(\bar{\partial}_E - \bar{\partial}'_E)s,$$

it is an $\operatorname{End} E$-valued $(0,1)$-form. Conversely, given a Dolbeault operator $\bar{\partial}_E$ and $\gamma \in \Omega^{0,1}(\overline{X}, \operatorname{End} E)$, $\bar{\partial}'_E = \bar{\partial}_E + \gamma$ is another Dolbeault operator. Hence $\mathscr{C}$ is an infinite-dimensional affine space modeled on $\Omega^{0,1}(\overline{X}, \operatorname{End} E)$.

Definition 7. A *parabolic bundle* E over $\overline{X}$ is a choice of a parabolic structure and a Dolbeault operator $\bar{\partial}_E \in \mathscr{C}$.

In the rest of this thesis, we will fix a parabolic structure on E and consider different holomorphic structures on E.

Definition 8. A parabolic bundle E is *stable* (resp. *semi-stable*) if, for every proper holomorphic sub-bundle F with the induced parabolic structure,

$$\mu(F) < \mu(E) \;\; (\text{resp.}\, \mu(F) \leq \mu(E)).$$

We denote by $\mathscr{C}^s$ and $\mathscr{C}^{ss}$ the spaces of stable and semi-stable bundles.

Let $\mathrm{GL}(E)$ denote the group of (smooth) complex gauge transformations of the vector bundle E on $\overline{X}$. Set

$$\mathscr{G}^{\mathbb{C}} = \{g \in \mathrm{GL}(E) \,|\, g(E_{i,k}) \subset E_{i,k}\},$$

the group of complex gauge transformations of E which respect the fixed parabolic structure on E. We say that two parabolic bundles $(E, \bar\partial)$ and $(E, \bar\partial')$ are isomorphic if there exists a $g \in \mathscr{G}^{\mathbb{C}}$ such that

$$\bar\partial'_E = g(\bar\partial_E) = g \circ \bar\partial_E \circ g^{-1} = \bar\partial_E - \bar\partial_E g g^{-1}.$$

Due to the jumping phenomenon [91], the naive quotient $\mathscr{C}/\mathscr{G}^{\mathbb{C}}$ is in general not even Hausdorff. One way to circumvent this trouble is via the notion of S-equivalence as introduced by Seshadri. More explicitly, Let E be a semi-stable parabolic bundle, its Jordan-Hölder filtration

$$0 \subset E_1 \subset \cdots \subset E_d = E$$

satisfies that the successive quotient E_{i+1}/E_i is stable with the induced parabolic structure. Though such a sequence of filtration is not canonical in general, the associated graded object

$$\mathrm{Gr}(E) = \bigoplus_{i=1}^{d-1} E_{i+1}/E_i$$

is canonical and we can therefore identify two semi-stable parabolic bundles to be S-equivalent if their associated graded objects are isomorphic in the category of parabolic bundles.

For a fixed parabolic structure on E, we can then construct the moduli space of stable (resp. semi-stable) parabolic bundles, denoted by $\mathscr{M}_{\mathrm{P}}^{s}$ (resp. $\mathscr{M}_{\mathrm{P}}^{ss}$) as the set of isomorphism classes of stable (resp. S-equivalent semi-stable) parabolic bundles.

Theorem 9 (Mehta-Seshadri [85])**.** $\mathscr{M}_{\mathrm{P}}^{ss}$ is a nonsingular projective variety of dimension

$$\dim_{\mathbb{C}} \mathscr{M}_{\mathrm{P}}^{ss} = n^2(h-1) + 1 + \sum_{p_k \in S} d_k \tag{2.1}$$

where

$$d_k := \dim_{\mathbb{C}} \mathscr{F}_k = \frac{1}{2}(n^2 - \sum_{i=1}^{s_k} m_{i,k}^2), \tag{2.2}$$

and $\mathscr{F}_k \cong U(n)/U(m_{1,k}) \times \cdots \times U(m_{s_k,k})$ the flag variety of the prescribed type.

Remark 10. When the choice of the parabolic weights is generic, see [15], the parabolic bundle E with the given weights is stable if and only if it is semi-stable.

2.2 Local Geometry

2.2.1 Cusp Ends

Let $X = \overline{X} \backslash S$ be the punctured Riemann surface. We equip X with a *conformal* Riemannian metric g such that it is a *Riemann surface with cusp ends*, i.e., there is a decomposition of the form

$$X = M \cup Z_1 \cup \cdots \cup Z_m, \tag{2.3}$$

where M is a compact surface with m copies of circles as its boundary and each Z_k is isometric to

$$\mathbb{S}^1 (= \mathbb{R}/2\pi\mathbb{Z}) \times [a_k, \infty), \quad ds^2_{\text{hyp}} = \frac{dx^2 + dy^2}{y^2}, \tag{2.4}$$

with $a_k > 0$ referred as the level of the cusp end Z_k and $z = x + iy$ the conformal coordinate on Z_k.

If $2 - 2h - m < 0$, by the uniformization theorem, X admits a complete hyperbolic metric with constant curvature -1, and, in particular, provides an example of a Riemann surface with cusp ends.

Remark 11. We want to point out at the beginning that the choice of the complete metric of cusp type as discussed above is not necessary for the gauge-theoretic construction of the moduli space of stable parabolic bundles, and any metric that is admissible along the ends (see [77] for more details) would suffice. However, such a choice simplifies our discussion of the long time behavior of the heat kernels as the bottom of the continuous spectrum of the related Laplacians is strictly positive and hence no scattering resonance at 0. Moreover, the explicit expression of the cusp metric along each end simplifies our discussion of the related weighted cohomology groups and the identification with their L^2 counterpart. Lastly, we can obtain the main results of this thesis in "almost" the same setup as in [113].

Lemma 12. (see [86, Lemma 1.3]) X is a complete Riemannian manifold. Moreover,

1. $\mathrm{Vol}(X) < \infty$;
2. For any z, z' on each Z_k,

$$|\log(\frac{y}{y'})| \leq d(z, z') \leq 2 \operatorname{arccosh} \sqrt{u(z, z')}. \tag{2.5}$$

where $d(z, z')$ denotes the Riemannian distance on X and

$$u(z, z') = \frac{(y - y')^2 + d(x, x')^2}{4yy'}$$

is an important invariant of the pair of points $z, z' \in Z_k$

2.2.2 Adapted Hermitian Metrics

Identify each cusp end Z_k with the punctured disk $\mathbb{D}^*(e^{-a_k})$ via the biholomorphic map $w = e^{-z}$.

Definition 13. A local trivialization $\{f_{i,k}\}_{i=1}^n$ along each cusp end Z_k is said to be adapted to the fixed parabolic structure of E if, under the identification with the

the trivial bundle $\mathbb{D}^*(e^{-a_k}) \times \mathbb{C}^n$, it generates the flag structure, that is

$$E_{i,k} = \operatorname{span}\{f_{n-\dim E_{i,k}+1,k}, \cdots, f_{n,k}\}$$

for $i = 1, \cdots, s_k$.

Definition 14. A Hermitian metric h on E is called adapted to the parabolic structure if there exists a frame $\{f_{i,k}\}_{i=1}^n$ adapted to the parabolic structure of E such that

$$\{e_{i,k}(z) := e^{\alpha_{i,k} y} f_{i,k}(z)\}_{i=1}^n,$$

is an orthonormal frame with respect to h near the cusp end Z_k.

Remark 15. In terms of the adapted frame $\{f_{i,k}\}_{i=1}^n$, a local complex gauge transformation $g \in \mathscr{G}^{\mathbb{C}}$ satisfies that

$$g_{ij}(z) = O(e^{-y}), \text{ if } \alpha_{i,k} < \alpha_{j,k}.$$

With respect to the temporal frame $e_{i,k} = e^{\alpha_{i,k} y} f_{i,k}$, then g is given by

$$e^{-(\alpha_{i,k} - \alpha_{j,k}) y} g_{ij}(z),$$

which suggests that sections of E whose norm under h are $e^{-(\alpha_{i,k} - \alpha_{j,k}) y}$ if $\alpha_{i,k} \geq \alpha_{j,k}$ and $e^{-(1+\alpha_{i,k} - \alpha_{j,k}) y}$ if $\alpha_{i,k} < \alpha_{j,k}$ should lie in the weighted Sobolev space completion of gauge groups to be defined later.

Example 16 (Local Model of the Non-abelian Correspondence)**.** Suppose $\{f_{i,k}\}_{i=1}^{n}$ is a local holomorphic basis adapted to the parabolic structure of E equipped with the metric

$$h = \begin{bmatrix} e^{-2\alpha_{1,k}y} & & \\ & \ddots & \\ & & e^{-2\alpha_{n,k}y} \end{bmatrix}$$

then $\{e_{i,k} = e^{\alpha_{i,k}y} f_{i,k}^{(*)}\}_{i=1}^{n}$ is an orthonormal frame, with respect to which, the Dolbeault operator is given by

$$\bar{\partial}_{\alpha_k} = \bar{\partial} + \frac{i}{2}\alpha_k \, d\bar{z}. \tag{2.6}$$

Here α_k denotes the diagonal matrix of parabolic weights along Z_k. The Chern connection of $\bar{\partial}_{\alpha_k}$ is given by

$$d_{\alpha_k} = d + i\alpha_k \, dx. \tag{2.7}$$

which is flat with its holonomy representation of the following unique (up to permutation) representative

$$\exp(-2\pi i \alpha_k) = \exp \begin{bmatrix} -2\pi i \alpha_{1,k} & & \\ & \ddots & \\ & & -2\pi i \alpha_{n,k} \end{bmatrix} \tag{2.8}$$

Recall that

Definition 17. A $\mathbb{C}$-linear map $d_A : \Omega^0(E) \to \Omega^1(E)$ on E over X is called a unitary connection if it satisfies

1. $d_A(fs) = df \otimes s + f d_A s$;

2. $dh(s, s') = h(d_A s, s') + h(s, d_A s')$.

for any $f \in C^\infty(X)$ and $s, s' \in \Omega^0(E)$.

There exists a canonical extension of any Dolbeault operator on (E, h).

Lemma 18. (see [122, Theorem 2.1]), Let E be a Hermitian holomorphic vector bundle, there exists a unique unitary connection d_A such that it is compatible with the holomorphic structure,

$$\bar{\partial}_A := d_A^{0,1} = \bar{\partial}_E.$$

This canonical extension is usually referred as the *Chern connection* of (E, h).

Since $\pi_1(Z_k) \cong \mathbb{Z}$, extending the observation made in Example 16, we have the following

Lemma 19. (see [32, Lemma 2.7]) For any smooth flat unitary connection d_A on $E \to X$, if its holonomy along each of Z_k is conjugate to (2.8). Then there exists a global *temporal gauge* such that d_A is of the form in (2.7) along each end Z_k.

Remark 20. The temporal gauge, which should be identified with the orthonormal frames in Definition 14, is equivalent to a local translation invariant trivialization of E together with an translation equivalent Hermitian metric. To say that it is adapted to the parabolic structure, it is a matter of the identification of the temporal gauge with that $\mathbb{D} \times \mathbb{C}^n$.

Let $\mathscr{A}$ denote the space of unitary connections on $(E, h) \to X$, which is modeled on the infinite-dimensional space $\Omega^1(X, \mathfrak{u}(E))$. Let $\mathscr{A}_{F,\alpha} \subset \mathscr{A}$ denote the space of all flat unitary connections d_A on E with its holonomy representation along each cusp end given by (2.8). Let $\mathscr{G} \subset \mathscr{G}^{\mathbb{C}}$ denote the unitary gauge transformations of (E, h) on X. We can thus form the moduli space of flat unitary connections as

$$\mathscr{M}_{\mathrm{HE}} = \mathscr{A}_{F,\alpha}/\mathscr{G}.$$

In terms of the holonomy representation, $\rho : \pi_1(X) \to U(n)\}/U(n)$ is admissible if its restriction to the generator of $\pi_1(Z_k)$ is the conjugate of (2.8). Then the holonomy representation gives us the following bijective map

$$\mathscr{M}_{\mathrm{HE}} \to \{\text{admissible representation } \rho : \pi_1(X) \to U(n)\}/U(n),$$

which is known as the Riemann-Hilbert correspondence.

Let us discuss how to construct extensions of $(E, \bar{\partial}_E)$ as holomorphic bundles over $\overline{X}$ from an admissible flat unitary connection d_A over X. Set $\bar{\partial}_A := d_A^{0,1}$. By a result attributed to Hans Grauert and Helmut Röhrl (1956), $(E, \bar{\partial}_A)$ is trivial as a holomorphic bundle on X. There exist, a priori, many different extensions of E to $\overline{X}$. In the presence of the flat unitary connection d_A, we can extend E in an essentially unique way.

Because the extension problem is local, we will restrict our study to each cusp end Z_k with respect to the fixed temporal frames $\{e_{1,k}, \cdots, e_{n,k}\}$. Let $f_{i,k}(z)$ be a new frame defined by $e^{(-\alpha_{i,k} y)} e_{i,k}(z)$, for $i = 1, \cdots, n$. By simple computation using (2.6), we have

$$\bar{\partial}_A f_{i,k}(z) = 0,$$

which implies that these define a local holomorphic trivialization of E over Z_k. Therefore we can extend E to $\overline{X}$ by identifying $\{f_{j,k}, j = 1, \cdots, n\}$ with the standard basis of $\mathbb{D}(e^{-a_k}) \times \mathbb{C}^n$. We will denote the resulting holomorphic bundle over $\overline{X}$ by E when there is no risk of confusion, and such an extension is called the the Deligne's extension or Mehta-Seshadri extension.

In terms of the punctured disk model $\mathbb{D}^*(e^{-a_k})$,the norm of the holomorphic section $f_{j,k}$ equals $|w|^{\alpha_{j,k}}$. In fact, let $E_{\alpha_{j,k}}$ denote the span of the germs of any local holomorphic sections around p_k with norm less or equal to $|w|^{\alpha_{j,k}}$, these defines a parabolic structure on the extension of E to $\overline{X}$, see [111] for more detail.

2.3 Weighted Sobolev Spaces on Surfaces with Cusp Ends

In this part, we first discuss briefly some properties of the relevant weighted Sobolev spaces and our treatment largely follows [77], [78], [13]. Later, we introduce various Banach groups and their action on the related Banach manifolds over X.

2.3.1 Function Spaces

Let (E, h) be the bundle with fixed adapted Hermitian metric on the surface with cusp ends X. We will denote by $\Lambda^k X$ (resp. $\Lambda^{p,q} X$) the smooth bundles of real k-forms (resp. complex (p,q)-forms) on X.

Let P denote the associated $U(n)$ principal bundle of E. We will denote by

$\mathfrak{u}(E)$ and End$E$ the adjoint bundles $P \times_{\mathrm{ad}} \mathfrak{u}(n)$ and $P \times_{\mathrm{ad}} \mathrm{End}(n)$, where Ad and ad denote the adjoint representation of $U(n)$ and $\mathfrak{u}(n)$.

Let ∇ be any unitary connection on E which is flat with respect to the fixed orthonormal frames $\{e_{i,k}\}_{i=1}^{n}$ adapted to the Hermitian metric h along the cusp ends Z_k. Let $\{\tau_k : X \to [0,\infty), k = 1, \cdots, m\}$ be an m-tuple of smooth positive functions defined as

$$\tau_k(x,y) = \begin{cases} 1, & \text{for } z \in M, \\ y, & \text{for } z = (x,y) \in Z_k, \end{cases}$$

Set $\rho = -\ln(\tau)$, in particular, $g = e^{\rho}(dx^2 + dy^2)$ along each cusp end.

Let $\delta \in \mathbb{R}^m$ be m-tuples of real numbers. We denote by $\delta\tau$ their scalar product. For $\delta_1, \delta_2 \in \mathbb{R}^m$, we say that $\delta_1 \leq$ (resp. $<$) δ_2 if $\delta_{1,k} \leq$ (resp. $<$) $\delta_{2,k}$ for $k = 1, \cdots, m$.

Definition 21. For $1 < p < \infty$, $\delta \in \mathbb{R}^m$, $s \in \mathbb{R}$, $k \in \mathbb{N}$. The weighted Sobolev space $L^{k,p}_{\delta,s}(\Lambda^{0,*}X \otimes \mathrm{End}\, E, g)$ is defined as the space of all sections $u \in L^{k,p}_{\mathrm{loc}}(\Lambda^{0,*}X \otimes \mathrm{End}\, E)$ such that

$$\|u\|_{L^{k,p}_{\delta,s}} = \Big(\sum_{i=0}^{k} \int_X \|e^{\delta\tau + (i+s)\rho} \nabla^i u\|_g^p \, dv_g \Big)^{\frac{1}{p}},$$

is finite.

Remark 22. The Sobolev spaces $L^{k,p}_{\delta,s}(\Lambda^* X \otimes \mathfrak{u}(E))$ can be defined in similar way.

Remark 23. Even though in [77], the authors defined weighted Sobolev spaces only for the bundle of exterior powers of the tangent and cotangent bundle of the underlying manifold, since the bundle E in our case are equipped with a fixed translation invariant trivialization (the temporal gauge) and a translation invariant Hermitian metric (the adapted Hermitian metric), therefore their definitions and results in [77] can be generalized to our case.

Remark 24. The above definition involves the choice of trivialization of E along the cusp ends, the function τ, and the connection ∇. Different choices of trivialization define inequivalent norms.

Remark 25. The convention in the choice of δ and a is that, in terms of the variable $r = e^{-\tau}$, we have $r^a \in L^2_\delta$ if $a > \delta$ and for any $s \in \mathbb{R}$. As the flags in our definition of parabolic structure are upper-semicontinuous, i.e., the germ of any local holomorphic section σ whose norm $\|\sigma\| \lesssim e^{-\alpha_{i,k}y}$ belongs to $E_{i,k}$, therefore we will always require the choice of s to be non-positive.

It is easy to verify that

Lemma 26. The weighted Sobolev space $L^{k,p}_{\delta,s}(\Lambda^{0,*}X \otimes \operatorname{End} E, g)$ is a Banach space.

To better understand the factor $e^{s\rho}$ in our definition of weighted Sobolev spaces, we need the following. Let g_0 be another Riemannian metric on X that is cylindrical along each end, that is

$$g_0 = dx^2 + dy^2, \text{ and } g = e^\rho g_0,$$

along each Z_k. We define the following weighted Sobolev spaces on (X, g_o).

Definition 27. The weighted Sobolev spaces $L^{k,p}_\delta(\Lambda^{0,*}X \otimes \operatorname{End} E, g_0)$ is defined as the space of all sections $u \in L^{k,p}_{\mathrm{loc}}(\operatorname{End} E)$ such that

$$\|u\|_{L^{k,p}_{\delta}} = (\sum_{i=0}^{k} \int_X \|e^{\delta\tau} \nabla^i u\|^p_{g_0} \, dv_{g_0})^{1/p}$$

is finite.

We can then define the following maps between the related weighted spaces:

Lemma 28. [77, Proposition and Definition 4.4] For any $k \in \mathbb{Z}$, the following map

$$\begin{aligned} K \colon L^{k,p}_{\delta,s}(\Lambda^{0,r} X \otimes \operatorname{End} E, g) &\to L^{k,p}_{\delta}(\Lambda^{0,r} X \otimes \operatorname{End} E, g_0) \\ u &\mapsto e^{(s+r+\frac{n}{p})\rho} u. \end{aligned} \tag{2.9}$$

is an isomorphism between Banach spaces.

Proof (from [77, Proposition and Definition 4.4]). When $k \geq 0$, we have

$$\|e^{(s+r+\frac{n}{p})\rho} u\|_{L^{k,p}_{\delta}} \leq \Big(\sum_{i=0}^{k} \sum_{j=0}^{i} c(i,j) \int_X \|e^{\delta\tau + (s+r+\frac{n}{p})\rho} \nabla^{i-j} \rho \nabla^j u\|^p_{g_0} \, dv_{g_0} \Big)^{\frac{1}{p}},$$

As $\lim_{y\to\infty} \|\nabla^i \rho\|_{g_0} = \lim_{y\to\infty} \frac{1}{y^{i+1}} = 0$, we therefore get

$$\begin{aligned} &\leq c\Big(\sum_{i=0}^{k} \int_X \|e^{\delta\tau + (s+r+\frac{n}{p})\rho} \nabla^i u\|^p_{g_0} \, dv_{g_0} \Big)^{\frac{1}{p}}, \\ &\leq c\Big(\sum_{i=0}^{k} \int_X \|e^{\delta\tau + (s+i)\rho} \nabla^i u\|^p_{g} \, dv_{g} \Big)^{\frac{1}{p}}, \\ &= c\|u\|_{L^{k,p}_{\delta,s}}. \end{aligned}$$

Define K^{-1} by

$$K^{-1}\sigma = e^{-(s+r+\frac{n}{p})\rho}\sigma,$$

we can show in similar way that it is continuous and this completes the proof for $k \geq 0$. The case of $k < 0$ is similar. $\square$

Remark 29. In fact, in [77], the authors defined the weighted Sobolev spaces for a more general class of admissible metrics on X, see [77, Definition 3.6] for more details. It is easy to see that the metric g, which is of cusp type along the ends of X, is admissible.

Though on an open manifold, from [44, pp. 14-15], the closure of smooth functions with compact support, the closure of smooth function in the Sobolev space, and the Sobolev space are all different in general. For the weighted Sobolev spaces defined above, we have

Lemma 30. $C_c^\infty(\Lambda^{0,*}X \otimes \mathrm{End}\, E)$ is dense in $L^{k,p}_{\delta,s}(\Lambda^{0,*}X \otimes \mathrm{End}\, E, g)$ and $L^{k,p}_{\delta}(\Lambda^{0,*}X \otimes \mathrm{End}\, E, g_0)$.

Proof. In the case of $L^{k,p}_{\delta}(\Lambda^{0,*}X \otimes \mathrm{End}\, E, g_0)$, we can cover X with the union of an atlas of finite open covers of the compact submanifold M and the ends Z_k, then as the geometry of the bundles involved are translation invariant and hence the corresponding statement on density of compactly supported smooth functions on $\mathbb{R}^n$ applies. Then use the above map K and we can therefore prove the statement for $L^{k,p}_{\delta,s}(\Lambda^{0,*}X \otimes \mathrm{End}\, E, g)$. $\square$

With the isomorphism K, the multiplication theorem and Sobolev embedding theorem on the space $L^{k,p}_{\delta}$, which is accounted in detail in Appendix A, have the following form on the space $L^{k,p}_{\delta,s}$.

Lemma 31 (Multiplication Theorem)**.** Let E, E' be two bundles over X equipped with adapted Hermtian metrics. The tensor product on smooth sections induces an continuous map

$$L^{k_1,p_1}_{\delta_1,s_1}(E) \times L^{k_2,p_2}_{\delta_2,s_2}(E') \to L^{k,p}_{\delta,s}(E \otimes E')$$

provided $k \leq \min(k_1, k_2)$, $\delta_1 + \delta_2 > \delta$, and $k - 2/p < k_1 - 2/p_1 + k_2 - 2/p_2$.

Lemma 32 (Weighted Sobolev Embedding)**.** Let E be a bundle equipped with an adapted Hermtian metric. Let $k, l \in \mathbb{N}$, $1 < p, q < \infty$, $a \in \mathbb{R}$, $\delta, \delta' \in \mathbb{R}^m$,

$$L^{k,p}_{\delta,s}(E) \to L^{l,q}_{\delta',s+n(1/p-1/q)}(E)$$

is continuous when

1. $k \geq l$, and $k - 2/p \geq l - 2/q$,

2. Either $1 < p \leq q$ and $\delta \geq \delta'$ or $1 < q < p$ and $\delta > \delta'$;

Moreover, we have

1. If $k > l \geq 0$, $k - 2/p > l - 2/q$ and $\delta > \delta'$, then the above map is compact;

2. If $k - 2/p > 0$, and $\delta > 0$, then $L^{k,p}_{\delta,s}(E) \hookrightarrow e^{-s\rho} C^0_{\delta}(E)$ is continuous, where the weighted C^0_{δ}-norm is given by

$$\|u\|_{C^0_{\delta}} = \sup_{z \in X} \{ e^{\delta\tau} |u(z)| \}.$$

Proof of Lemma 31 and Lemma 32. These follow directly from Lemma 28, and the corresponding results of $L^{k,p}_{\delta}$ as stated in Lemma 125 and Lemma 126. □

As we are interested in operators that are close to the local model (2.7) and (2.6), we need the following: with respect to the fixed orthonormal frames along the cusp end Z_k and considering the following diagonal matrix of the parabolic weights

$$\alpha_k = \begin{bmatrix} \alpha_{1,k} & & \\ & \ddots & \\ & & \alpha_{n,k} \end{bmatrix},$$

we get the decomposition

$$\operatorname{End} E = \operatorname{End} E^D \oplus \operatorname{End} E^H$$

where $\operatorname{End} E^D = \ker(\operatorname{ad} \alpha_k)$ and $\operatorname{End} E^H = \ker(\operatorname{ad} \alpha_k)^{\perp}$.

With respect to such a decomposition, the action of the model covariant derivative in (2.7) is given by

$$d_{\alpha_k} u = \begin{cases} du^D, \\ du^H + i\,[\alpha_k, u^H]\,dx. \end{cases}$$

Also, if the weight $\delta \in \mathbb{R}^m > 0$, then any $u \in L^{k,p}_{\delta,s}(\operatorname{End} E)$ vanishes at infinity of X. In order to define complex gauge transformations that preserve the flag structure, we need u^D, the block diagonal of u, to have limiting values. For that purpose, we define

Definition 33. For $1 < p < 1$, $\hat{L}^{1,p}_{\delta,s}(\mathrm{End}E)$ is the space of $u \in L^{1,p}_{\mathrm{loc}}(\mathrm{End}E)$ satisfying $u^H \in L^{1,p}_{\delta,s}(\mathrm{End}E)$ and u^D such that

$$(\|e^{-\tau}u\|^p_{L^p_{\delta,s}} + \|\nabla u\|^p_{L^p_{\delta,s+1}})^{1/p}$$

is finite. Similarly, we define $\hat{L}^{2,2}_{\delta,s}(\mathrm{End}E)$ as the space of $u \in L^{2,2}_{\mathrm{loc}}(\mathrm{End}E)$ satisfying $u^H \in L^{2,2}_{\delta,s}(\mathrm{End}E)$ and u^D such that

$$(\|e^{-\tau}u\|^2_{L^2_{\delta,s}} + \|\nabla u\|^2_{L^{1,2}_{\delta,s+1}})^{1/2}$$

is finite.

Lemma 34. (cf. [13, p.10-11])

1. (Radial Poincare Inequality.) For any $p > 1$, if $\delta < 0$ and a local section u of End E on the end Z_k vanishes on the boundary of Z_k, or if $\delta > 0$ and u vanishes near the infinity of Z_k, then

$$\|\frac{du}{dy}\|_{L^p_{\delta,s+1}} \leq c\|u\|_{L^p_{\delta,s}}.$$

   In particular, if $\delta < 0$, then

$$L^{1,p}_{\delta-1,s} = \hat{L}^{1,p}_{\delta-1,s}.$$

2. (Existence of Limiting Value.) For $p > 2$ and $\delta > 0$ such that $1 - 2/p < \delta$, then

any $u \in \hat{L}^{1,p}_{\delta,s} \subset C^0(\mathbb{D}(e^{-a_k}))$ and

$$u(z) - u(\infty) \in L^{1,p}_{\delta,s}.$$

Proof. This follows directly from Lemma 28 and Lemma 128. □

Corollary 34.1. The space $\hat{L}^{2,2}_{\delta,s}(\operatorname{End} E)$ is a Banach space. Moreover, the subspace $L^{2,2}_{\delta,s}(\operatorname{End} E)$ is of finite codimension $\sum_{i,k} m^2_{i,k}$.

Proof. By definition and Lemma 32, any section $u \in \hat{L}^{2,2}_{\delta,s}(E)$ lies in $L^p_{\delta-1,s}$ for any $p > 2$ and $\nabla u \in L^p_\delta$, hence by Lemma 34, u has a limiting values $u_k(\infty)$ along each cusp Z_k. Let $\chi_k : Z \to [0,1]$ be smooth cutoff functions such that it equals 1 along the cusp end Z_k for $y \geq 2$ and vanishes outside Z_k for $y \leq 1$. By Lemma 34,

$$u - \sum_k \chi_k u_k(\infty)$$

belongs to $L^{2,2}_{\delta,s}$. Let P denote this fixed projection from $\hat{L}^{2,2}_{\delta,s}$ to $L^{2,2}_\delta$. We see that, $\hat{L}^{2,2}_\delta$ contains $L^{2,2}_\delta$ as a finite codimension subspace, which implies that $\hat{L}^{2,2}_\delta$ is a Banach space and the codimension is determined by the size of the block diagonal as prescribed by the parabolic structure. □

Let us also define

Definition 35. The space of reduced sections of $\operatorname{End} E$ is given by

$$S^{2,2}_{\delta,s}(\operatorname{End} E) := \{u \in \hat{L}^{2,2}_{\delta,s}(\operatorname{End} E), \int_X \operatorname{tr}(u)\, d\, v_g = 0\},$$

which is the L^2-orthogonal complement of the space of constant sections $\mathbb{C} \otimes \mathrm{Id}_E$ in $\hat{L}^{2,2}_{\delta}(\mathrm{End}\, E)$.

Remark 36. Note that $S^{2,2}_{\delta,s}(\mathrm{End}\, E)$ is an ideal as [u,v] = 0 for all $u, v \in S^{2,2}_{\delta,s}(\mathrm{End}\, E)$, and can therefore be identified with the Lie algebra of $\tilde{\mathscr{G}}^{\mathbb{C}}_{\delta}$.

We conclude this part of discussion by the following a priori estimate

Lemma 37. (cf. [77, Theorem 3.7])

For all $1 < p < \infty$, $k \in \mathbb{Z}$, and $\delta \in \mathbb{R}^m$, given any Dolbeault Laplacian Δ_A on $\mathrm{End}\, E$,

1. Δ_A is continuous from $L^{k,p}_{\delta,s}(\mathrm{End}\, E)$ to $L^{k-2,p}_{\delta,s+2}(\mathrm{End}\, E)$,
2. For all $u \in L^{k,p}_{\delta,s}(\mathrm{End}\, E)$, there exists a constant $c > 0$ independent of u such that

$$\|u\|_{L^{k,p}_{\delta,s}} \le c(\|\Delta_A u\|_{L^{k-2,p}_{\delta,s+2}} + \|u\|_{L^{k-2,p}_{\delta,s}}).$$

Remark 38. In the case of noncompact manifold, Fredholmness does not follow directly from the a priori inequality.

2.3.2 Module Structure

Let d_0 be any unitary connection on E that agrees with (2.7) with respect to the fixed adapted orthonormal frames of E along each cusp end Z_k. Let $\bar{\partial}_0$ denote the $(0,1)$-part of d_0.

Definition 39. We define the space of Dolbeault operators adapted to the parabolic structure of E as

$$\mathscr{C}_\delta = \{\bar{\partial}_0 + \gamma \,|\, \gamma \in L^{1,2}_{\delta,s+1}(\Lambda^{0,1}X \otimes \mathrm{End}(E)\},$$

and the space of unitary connections adapted to the parabolic structure of E as

$$\mathscr{A}_\delta = \{d_0 + a \,|\, a \in L^{1,2}_{\delta,s+1}(\Lambda^{0,1}X \otimes \mathfrak{u}(E)\}.$$

Also, we define the complex gauge group as

$$\mathscr{G}^{\mathbb{C}}_\delta = \{g \in \hat{L}^{2,2}_{\delta,s}(\mathrm{End}\, E) \,|\, \det g \neq 0\},$$

and the unitary gauge group as

$$\mathscr{G}_\delta = \{g \in \hat{L}^{2,2}_{\delta,s}(U(E)) \,|\, g^* g = \mathrm{Id}\}.$$

Note that these definitions depend on the choice of adapted orthonormal frames along each cusp end, for them to be independent of such a choice, we define

Definition 40. Any choice of weight $\delta \in \mathbb{R}^m$ is called admissible with respect to the fixed parabolic structure of E if

$$0 < \delta < \inf_{k=1,\cdots,m,\alpha_i > \alpha_j} \big(\alpha_{i,k} - \alpha_{j,k}, 1 - \alpha_{i,k} + \alpha_{j,k}\big).$$

From Remark 15, it is clear that

Lemma 41. For admissible choice of weights $\delta \in \mathbb{R}^m$, the complex gauge transformation (resp. unitary gauge transformation) between any two the adapted bases of E (resp. To (E, h)) belongs to $\mathscr{G}_\delta^{\mathbb{C}}$ (resp. $\mathscr{G}_\delta$).

The following lemma shows that smooth operators with the right type of singularity are included in the weighted Sobolev spaces of operators just defined.

Lemma 42. (cf. [12, Proposition 2.7]) Given any choice of admissible $\delta \in \mathbb{R}^m$, we have the following inclusion

$$\mathscr{C} \subset \mathscr{C}_\delta, \text{ and } \mathscr{G}^{\mathbb{C}} \subset \mathscr{G}_\delta^{\mathbb{C}}.$$

Similarly, we also have

$$\mathscr{A} \subset \mathscr{A}_\delta, \text{ and } \mathscr{G} \subset \mathscr{G}_\delta.$$

Proof. Consider the holomorphic bundle $E \to \overline{X}$. Let $\{f_{i,k}\}_{i=1}^n$ be a local smooth frame adapted to the corresponding flag structure around the marked points. Any smooth Dolbeault operator $\bar{\partial}_E$ can be written locally with respect to such a frame as

$$\bar{\partial}_E = \bar{\partial} + \gamma_k \, d\bar{z},$$

where $\gamma_{ij,k}$ are smooth functions on the closed disk $\mathbb{D}(0, e^{-a_k})$. Now in terms of the orthonormal frames $e_{i,k} = |z|^{\alpha_{i,k}} f_{i,k}$ of the adapted Hermitian metric h, we obtain component-wise

$$\bar{\partial}_E = \bar{\partial} - \frac{\alpha_k}{2} \frac{d\bar{z}}{\bar{z}} + |z|^{\alpha_{i,k} - \alpha_{j,k}} \gamma_{ij,k} \, d\bar{z},$$

which belongs to $L^{1,2}_\delta$ provided δ is admissible. The other claims can be proved similarly. $\square$

The following lemma is key to define the action of $\mathscr{G}^{\mathbb{C}}_\delta$ (resp. $\mathscr{G}_\delta$) on $\mathscr{C}_\delta$ (resp. $\mathscr{A}_\delta$).

Lemma 43. (cf. [13, Lemma 3.5]) Let $\delta \in \mathbb{R}^m$ satisfy $0 < \delta < 1$ and $i = 0, 1$, the space $L^{1,2}_{\delta,a+i}(\Lambda^{0,i}X \otimes \operatorname{End} E)$ is a Banach module over the Banach algebra $\hat{L}^{2,2}_{\delta,s}(\operatorname{End} E)$. Similar statements hold when $\operatorname{End} E$ is replaced by $\mathfrak{u}(E)$.

Proof. To show that $\hat{L}^{2,2}_{\delta,s}(\operatorname{End} E)$ is a Banach algebra, we need to show that for any $u, v \in \hat{L}^{2,2}_{\delta}(\operatorname{End} E)$, uv also belongs to $\hat{L}^{2,2}_{\delta,s}(\operatorname{End} E)$.

For $u^D \in \hat{L}^{2,2}_{\delta,s}$, from its definition, we have $u^D \in L^{1,2}_{\delta-1,s}$ and $\nabla u^D \in L^{1,2}_{\delta,s+1}$, by Lemma 32, $u^D \in L^p_{\delta-1,s}$ for some $p > 2$ and $\nabla u^D \in L^p_{\delta,s+1}$, in particular, $u^D \in \hat{L}^{1,p}_{\delta,s}$, which by Lemma 34 implies that $u^D \in e^{-\rho s} C^0_\delta$. In a similar way, $u^H \in e^{-\rho s} C^0_\delta$. Therefore, the pointwise multiplication and inversion are well-defined. Hence, it remains to check the regularity. Let's look at $\nabla^2(uv)$ as an example. As $\nabla^2(uv) = \nabla^2 uv + u\nabla^2 v + \nabla u \nabla v$, the first two terms (both their "D" and "H" parts) belongs to $L^2_{\delta,s}$ due to the fact $u, v \in e^{-\rho s} C^0_\delta(X)$ that we just discussed. For the third term, as $\nabla u^D, \nabla v^D \in L^p_{\delta,s+1}$ for any $p > 2$ as we just discussed, by Lemma 32, we conclude that their product lies in $L^2_{\delta,s}$. The rest part of the lemma can be proven in a similar manner.

$\square$

Lemma 44. (cf. [13, Lemma 2.1]) $\mathscr{G}_\delta$ is a Banach Lie group which acts smoothly on $\mathscr{A}_\delta$ by

$$g(d_A) = g \circ d_A \circ g^{-1}$$

and its Lie algebra is given by

$$\mathrm{Lie}(\mathscr{G}_\delta) = \hat{L}^{2,2}_{\delta,s}(\mathfrak{u}(E)).$$

Its action on is smooth. In analogue, $\mathscr{G}^{\mathbb{C}}_\delta$ is a Banach Lie group that acts smoothly on $\mathscr{C}_\delta$, and its Lie algebra is given by

$$\mathrm{Lie}(\mathscr{G}^{\mathbb{C}}_\delta) = \hat{L}^{2,2}_{\delta,s}(\mathrm{End}E).$$

Proof. We will prove the above statement for $\mathscr{G}$, while other statements can be proved easily in a similar way. Define the following smooth function

$$\begin{aligned}\mathrm{F} : \hat{L}^{2,2}_{\delta,s}(\mathrm{End}E) &\to \mathrm{S} := \{a \in \hat{L}^{2,2}_{\delta,s}(\mathrm{End}E),\ a = a^*\}\\ u &\mapsto uu^* - \mathrm{id}\end{aligned}$$

and we will prove that $0 \in \mathrm{S}$ is a regular value of F and hence F is an submersion at every point of $\mathscr{G}_\delta$. For that purpose, we compute the differential of F at $a \in \mathscr{G}_\delta$ with respect to the direction $h \circ a \in \hat{L}^{2,2}_{\delta,s}(\mathrm{End}E)$:

$$\begin{aligned}\mathrm{D(F)}_a(h \circ a) &= \frac{d}{dt}|_{t=0} F(a + t\, h \circ a)\\ &= a \circ a^* \circ h^* + h \circ a \circ a^*\\ &= h + h^*\end{aligned}$$

Therefore, D(F) is surjective at every point of $\mathscr{G}_\delta$ to S, by Implicit function theorem,

$\mathscr{G}_\delta$ is a smooth Banach manifold. Furthermore, the tangent space to $\mathscr{G}_\delta$ is given by the kernel of D(F) at $a = \mathrm{id}$, which is given exactly by $\hat{L}^{2,2}_{\delta,s}(\mathfrak{u}(E))$.

To see that $\mathscr{G}_\delta$ is a Lie group, we need to show that pointwise multiplication $\mathscr{G}_\delta \times \mathscr{G}_\delta \to \mathscr{G}_\delta$ is well-defined and continuous, and see Lemma 43 for the proof. Similarly, the inverse map $u \mapsto u^{-1}$ is also smooth since $u^{-1} = u^*$ in $\mathscr{G}_\delta$.

As to the complex gauge group $\mathscr{G}^{\mathbb{C}}$, it is an open subset of the Banach space $\hat{L}^{2,2}_{\delta,s}(\operatorname{End} E)$ and the claim follows easily. □

2.3.3 Fredholmness

In the following, given any unitary connection $d_A \in \mathscr{A}_\delta$ and Dolbeault operator $\bar{\partial}_A \in \mathscr{C}_\delta$ on E, we investigate the Fredholmness of the related Laplacian operators. We will only discuss the case with the Dolbeault operator as the argument for the case of a unitary connection is similar. In fact, we show that

Theorem 45. (cf. [13, Lemma 5.1]) For admissible choice of weight $\delta \in \mathbb{R}^m$ and any $\bar{\partial}_A = \bar{\partial}_0 + A \in \mathscr{C}_\delta$, the Dolbeault Laplacian

$$\Delta_A := \bar{\partial}_A^* \bar{\partial}_A : \hat{L}^{2,2}_{\delta,s}(\operatorname{End} E) \to L^2_{\delta,s+2}(\operatorname{End} E) \tag{2.10}$$

is Fredholm, of index 0.

Proof. First note that, by Lemma 32 and Lemma 31, the perturbation A is a compact operator, therefore we are reduced to prove Theorem 45 for $\Delta_0 = \bar{\partial}_0^* \bar{\partial}_0$.

To see the Fredholmness of Δ_0, as along each cusp end Z_k, with respect to the fixed orthonormal frames $\{e_i\}_{i=1}^n$ of E, the action of Δ_0 on $\operatorname{End} E$ is of the form

$$\Delta_0 u = \sum_{i,j} \Big\{ - y^2 \frac{d^2}{dy^2} - y^2 \Big(\frac{d}{dx} + i(\alpha_i - \alpha_j) \Big)^2 \Big\} u_{ij} e_i \otimes e_j^*,$$

i.e., it is the Beltrami Laplacian of underlying hyperbolic metric twisted by the parabolic weights. By [77, Theorem 5.2], it is Fredholm provided δ is away from its indicial indices, which in our case, is the lattice generated by the eigenvalues of the operator $i\frac{d}{dx} - (\alpha_i - \alpha_j)$, hence by the definition of admissible weights, Δ_0 is Fredholm.

As to the vanishing of the index of Δ_0, this follows from Lemma 5.1 of [13] and Lemma 28. □

Remark 46. By Weitzenböck formula and the same argument as above, for any unitary connection $d_A \in \mathscr{A}_\delta$,

$$\Delta_A := d_A^* d_A : \hat{L}^{2,2}_{\delta,s}(\mathfrak{u}(E)) \to L^2_{\delta,s+2}(\mathfrak{u}(E)), \tag{2.11}$$

is Fredholm of index 0.

We conclude this part by recording the following result on small perturbation of an elliptic operator.

Lemma 47 ([10], Proposition 1.11.)**.** Suppose that $P : X \to Y$ is a semi-Fredholm map between Banach spaces X, Y. Then there are constants $C, \epsilon > 0$ depending only on P such that if $P' : X \to Y$ is any semi-Fredholm map satisfying $\|P - P'\|_{\mathrm{op}} < \epsilon$, then

$$\dim \ker P' \leq \dim \ker P.$$

Chapter 3: Gauge Theory of the Moduli Space of Stable Parabolic Bundles

In this chapter, we present gauge theoretical constructions of the moduli space of irreducible flat unitary connections, denoted by $\mathscr{M}^*_{\mathrm{HE}}$, and of the moduli space of stable parabolic bundles, denoted by $\mathscr{M}^s_{\mathrm{P}}$. The program of using gauge theory to investigate moduli problems was initiated by Atiyah and Bott in [6]. The construction of moduli space of stable parabolic bundles over a Riemann surface with cylindrical ends has been carried out by Daskalopoulos and Wentworth in [32] and by Poritz in [96]. The construction of moduli space of stable parabolic Higgs bundles has been carried out by Konno in [74]. In the second part of this chapter, we study the Riemannian structure on $\mathscr{M}^*_{\mathrm{HE}}$ and the holomorphic structure on $\mathscr{M}^s_{\mathrm{P}}$. Our treatment here is inspired by that of Itoh in [67] and [68], where the moduli space of Yang-Mills connections over a closed Kähler surface was studied.

3.1 Construction of Moduli Spaces

3.1.1 Local Model of Dolbeault Operators

Before we dive into the construction of moduli space, let us digress and understand that any Dolbeault operator $\bar{\partial}_\gamma = \bar{\partial}_0 + \gamma \in \mathscr{C}_\delta$ in fact defines the same parabolic structure. In fact, we will show that there exists a local complex gauge transformation of E in a neighborhood of each cusp end Z_k such that $\bar{\partial}_\gamma$ can be put into the standard form $\bar{\partial}_0$ as in (2.6), and then Deligne's extension applies to get the desired extension.

Since the problem is local in nature, we will work with a small punctured disk $\mathbb{D}^*(\epsilon)$ model of the cusp end Z_k together with an adapted trivialization of E such that γ is represented as a matrix of $(0,1)$-forms, our goal is to find over $\mathbb{D}^*(\epsilon)$, by shrinking it if needed, a complex gauge transformation

$$g : \mathbb{D}^*(\epsilon) \to \mathrm{GL}(n, \mathbb{C}),$$

with $g\gamma g^{-1} = \bar{\partial}_0 g g^{-1}$ on $\mathbb{D}^*(\epsilon)$. In fact, we will prove that,

Lemma 48. Given any choice of admissible weight $\delta \in \mathbb{R}^m$, there exists $\eta > 0$ such that if $\|\gamma\|_{L^{1,2}_{\delta,0}} < \eta$, there exist local continuous complex gauge transformation g_k such that $g_k^H \in C^0_{\delta',-1}$ for some $\delta' < \delta$ with

$$g(\bar{\partial}_0 + \gamma) = \bar{\partial}_0$$

on each cusp end Z_k.

To prove this, we need the following technical lemma concerning the Cauchy kernel of $\frac{\partial}{\partial \bar{z}}$ on $\mathbb{D}^*(\epsilon)$:

Lemma 49. For any $0 < \alpha < 1$, for any weight $0 < \delta < \alpha$ and $p > 2$, and any $f d\bar{z} \in L^p_{\delta,1}(\Lambda^{0,1}\mathbb{D}^*(\epsilon))$, where $\mathbb{D}^*(\epsilon)$ is equipped with the Poincaré metric, convolution with the Cauchy kernel

$$u(z) = \frac{i}{2\pi} \int \frac{f(w)}{z - w} dz \wedge d\bar{z} \tag{3.1}$$

provides an inverse to the following problem

$$(\bar{z}\frac{\partial}{\partial \bar{z}} - \frac{\alpha}{2})\frac{d\bar{z}}{\bar{z}}\, u = f\bar{z},$$

satisfying

$$\|u\|_{C^0_{\delta'}} \leq C\|f\bar{z}\|_{L^p_{\delta,1}},$$

for some $0 < \delta' < \delta$

Proof. By density of $C^\infty_c \subset L^p_\delta$, we will prove the statement for any compactly supported $(0,1)$-form $f\, d\bar{z}$.

Note that

$$\frac{\partial}{\partial \bar{z}} - \frac{\alpha}{2}\frac{d\bar{z}}{\bar{z}} = |z|^\alpha \circ \frac{\partial}{\partial \bar{z}} \circ |z|^{-\alpha},$$

therefore we are reduced to solve

$$\frac{\partial}{\partial \bar{z}}(\frac{u}{|z|^\alpha}) = \frac{f}{|z|^\alpha},$$

to which we can apply (3.1). By Hölder inequality, we get

$$
\begin{aligned}
|u(z)| &\leq |z|^{\alpha} \int_{\mathbb{D}(\epsilon)} \frac{|f|}{|w|^{-1+\delta+\frac{2}{p}}} \frac{|w|^{((1+\delta-\alpha)-2\frac{p-1}{p})} \ln |w|^{\frac{2}{p}}}{|z-w|} |dw|^2 \\
&\leq c|z|^{\alpha} \|f\bar{z}\|_{L^p_\delta} \Big\{ \int_{\mathbb{D}(\epsilon)} \Big(\frac{|w|^{((1+\delta'-\alpha)\frac{p}{p-1}-2)}}{|z-w|^{\frac{p}{p-1}}} \Big) |dw|^2 \Big\}^{\frac{p-1}{p}} \\
&\leq c' \|f\bar{z}\|_{L^p_\delta} |z|^{\alpha} \frac{1}{|z|^{\alpha-\delta'}} \\
&\leq c' |z|^{\delta'} \|f\bar{z}\|_{L^p_\delta}.
\end{aligned}
$$

Hence the claim follows. $\square$

Proof of Lemma 48. (cf. [13, Lemma]) Define the following Banach spaces

$$
\begin{aligned}
U_{\delta'} &= \{u \in C^0(\operatorname{End} E) \,|\, u^H \in C^0_{\delta',-1}(\operatorname{End} E)\}; \\
A_\delta &= \{\gamma \in L^p_{\delta,0}(\operatorname{End} E)\}.
\end{aligned}
$$

Observe that since $\gamma \in L^{1,2}_{\delta,0}$, by Lemma 32, it belongs to $L^p_{\delta",0}$ for some $\delta" < \delta$ and p large. We then pick a $\delta' < \delta"$.

Write $g = 1 + u$, then we need to solve

$$
\bar{\partial}_0 u = u\gamma + \gamma \tag{3.2}
$$

near each cusp end. As $\bar{\partial}_0$ is along each cusp end is of the form studied in Lemma 49, there is a bounded right inverse between $U\delta'$ and A_δ. This enables us to use the contraction mapping theorem to find a solution of (3.2). That is, we look at

$$u \mapsto G(u\gamma + \gamma).$$

As

$$\|G(u\gamma + \gamma) - G(v\gamma + \gamma)\|_{U_{\delta'}} \le c\|u - v\|_{U_{\delta'}}\|\gamma\|_{A_\delta},$$

we conclude the proof by using the freedom of scaling

$$\gamma(z) \mapsto r\gamma(rz),$$

and take r sufficiently small. □

3.1.2 Mehta-Seshadri Theorem

Lemma 50. (cf. [74, Proposition 1.3, Theorem 1.5]) For any admissible choice of weight $\delta \in \mathbb{R}^m$, the following map

$$\mathscr{C}/\mathscr{G}^{\mathbb{C}} \to \mathscr{C}_\delta/\mathscr{G}_\delta^{\mathbb{C}}, \tag{3.3}$$

and

$$\mathscr{A}/\mathscr{G} \to \mathscr{A}_\delta/\mathscr{G}_\delta, \tag{3.4}$$

are well-defined and bijective.

Proof. By Lemma 44 and Lemma 42, the maps defined above is well-defined. The injectivity follows from Lemma 41. To see that these maps are surjective, it follows from Theorem 45 and the argument of Atiyah and Bott in Lemma 14.8 in [5]. See also Proposition 2.8 of [12]. □

Definition 51. A unitary connection d_A is called irreducible if its covariant derivative

$$d_A : L^{2,2}_{\delta,s}(\mathfrak{u}(E)) \to L^{1,2}_{\delta,s+1}(\Omega^1(\mathfrak{u}(E))),$$

$$\psi \mapsto d_A\psi,$$

has trivial kernel. A unitary connection is called reducible if it is not irreducible.

Set

$$\mathrm{Stab}_A := \{g \in \mathscr{G}_\delta,\, d_A\, g = 0\},$$

to be the stabilizer of d_A. It is easy to check that Stab_A is a group. Let $Z\,(= \mathbb{S}^1)$ denote the center of $U(n)$, where we identify element of Z with constant sections with value in Z, then $Z \subset \mathrm{Stab}_A$ for all $d_A \in \mathscr{A}_\delta$. On the other direction, given a $g \in \mathrm{Stab}_A$, in a local trivialization, $g^{-1} \circ d_A \circ g - d_A = g^{-1}\, d_A\, g = 0$. This implies that $g = c \cdot \mathrm{id}_E$, $c \in \mathbb{S}^1$, therefore, d_A is irreducible precisely when $\mathrm{Stab}_A = Z$. Then the first covariant derivative d_A in (3.11) can be interpreted as the infinitesimal action of the gauge group $\mathscr{G}_\delta$ on $\mathscr{A}_\delta$, and $H^0_{A,\delta}$ measures the reducibility of d_A. Let $\mathscr{A}^*_\delta$ denote the space of irreducible unitary connections. As the action of $\tilde{\mathscr{G}}_\delta := \mathscr{G}_\delta/Z$ is free, we call the quotient space

$$\mathscr{B}^* = \mathscr{A}_\delta^* / \tilde{\mathscr{G}}_\delta$$

as the moduli space of irreducible unitary connections.

Let $\mathscr{A}_{F,\delta}^*$ denote the space of flat irreducible unitary connections. As $F_{g(A)} = F_A$, the action of $\tilde{\mathscr{G}}_\delta$ preserves $\mathscr{A}_{F,\delta}^*$. The quotient space

$$\mathscr{M}_{\mathrm{HE}}^* = \mathscr{A}_{F,\delta}^* / \tilde{\mathscr{G}}_\delta$$

is called the moduli space of irreducible flat Hermitian connections. Infinitesimally speaking, $H_{A,\delta}^1$ represents the deformation of d_A in $\mathscr{M}_{\mathrm{HE}}^*$. We will denote by $[d_A]$ the image of d_A in the quotient space.

Similarly, we will define that

Definition 52. A Dolbeault operator $\bar{\partial}_A \in \mathscr{C}_\delta$ is *stable* if there is a smooth representative in its $\mathscr{G}_\delta^{\mathbb{C}}$-orbit, which exists by Lemma 42, is stable.

Let $\mathscr{C}_\delta^s$ denote the space of stable Dolbeault operators on E. We will denote by

$$\mathscr{M}_{\mathrm{P}}^s = \mathscr{C}_\delta^s / \mathscr{G}_\delta^{\mathbb{C}}$$

as the moduli space of stable parabolic bundles.

To see the relation between $\mathscr{M}_{\mathrm{HE}}^*$ and $\mathscr{M}_{\mathrm{P}}^s$, we consider the following map obtained with respect to the fixed adapted Hermitian metric h on E,

$$\Phi_h : \mathscr{A}_\delta \longrightarrow \mathscr{C}_\delta,$$
$$d_A \mapsto \bar{\partial}_A := d_A^{0,1}.$$

with its inverse given by the Chern connection construction, see Lemma 18, with respect to the fixed adapted metric h.

By the same arguments as in Theorem 2.5 in [122] and Proposition 2.13 in [12], we can show that

Lemma 53. Any Dolbeault operator whose Chern connection is flat and irreducible is itself stable.

Hence the above map Φ_h has a well-defined restriction to

$$\Phi_h : \mathscr{A}^*_{F,\delta} \longrightarrow \mathscr{C}^s_\delta. \tag{3.5}$$

On the other hand, Mehta and Seshadri [85] proved the following in the case of rational parabolic weights,

Theorem 54 (Mehta-Seshadri [85])**.** Let E be a holomorphic bundle over $\overline{X}$ with fixed parabolic structure, which is indecomposable and par-deg$(E) = 0$, then E is stable if and only if there exists an irreducible unitary representation $\rho : \pi_1(x) \to U(n)$ such that $E \cong E^\rho$. Here, E^ρ denotes the Deligne's extension to $\overline{X}$.

Biquard [12] improved the above theorem to the case of real weights, and showed that within each $\mathscr{G}^{\mathbb{C}}_\delta$-orbit of any stable $\bar{\partial}_A \in \mathscr{C}_\delta$, there exists a $\bar{\partial}'_A$, unique up to $\mathscr{G}_\delta$, such that the latter's Chern connection with respect to h is irreducible and flat. Hence the map Φ_h descends to the following bijection

$$\Phi_h : \mathscr{M}_{\mathrm{P}}^s \to \mathscr{M}_{HE}^*. \tag{3.6}$$

Remark 55. Such a bijection as proved in [12] is between sets of equivalence classes. Later on, we will improve this result and show that they are in fact a diffeomorphism so that various differential geometric structures can be identified.

3.2 Geometry of Moduli Spaces: the Local Theory

3.2.1 Riemannian Metric on $\mathscr{M}_{\mathrm{HE}}^*$

In this part, we construct the Riemannian metric explicitly on a local (non-linear) Hermitian-Einstein slice of $\mathscr{M}_{\mathrm{HE}}^*$.

By Lemma 47, irreducibility is an open condition, hence the tangent space of $\mathscr{A}_\delta^*$ at any given point is given by $L_{\delta,s+1}^{1,2}(\Lambda^1 X \otimes \mathfrak{u}(E))$, which is a subspace of $L^2(\Lambda^1 X \otimes \mathfrak{u}(E))$. Therefore, we may define the L^2-metric by

$$< \phi, \psi > := \int_X \mathrm{tr}(\phi \wedge *\psi),$$

for any $\phi, \psi \in L_{\delta,s+1}^{1,2}(\Lambda^1 X \otimes \mathfrak{u}(E))$. Here tr denote the fiberwise matrix trace. Since this definition is $\mathscr{G}_\delta$-invariant, it induces a natural metric on the quotient space $\mathscr{B}_\delta^* = \mathscr{A}_\delta^* / \tilde{\mathscr{G}}_\delta$. To understand the induced metric better, we first construct, for any

$[d_A] \in \mathscr{B}_\delta^*$, a local (infinite dimensional) slice in $\mathscr{A}_\delta^*$ for the $\mathscr{G}_\delta$-action.

The proof of this claim is a typical implicit function theorem argument which will be used extensively, hence we record it in the following for reference.

Theorem 56 (Implicit Function Theorem)**.** Let X, Y, Z be Banach spaces, $\mathscr{U} \subset X$, $\mathscr{V} \subset Y$ open sets and

$$F : \mathscr{U} \times \mathscr{V} \to Z$$

a smooth map. Let $(x_0, y_0) \in U \times V$ and $z_0 := F(x_0, y_0)$. Suppose that its Frechet derivative of the second variable

$$D_{y_0} F(x_0, \cdot) \in \mathscr{L}(Y, Z),$$

is invertible. Then there exist open neighborhoods $U \subset \mathscr{U}$ of x_0 in X, $V \subset \mathscr{V}$ of y_0 in Y, and a smooth map $G : U \to V$ such that the set S of solution (x, y) of the equation $F(x, y) = z_0$ which lie inside $U \times V$ can be identified with the graph of G, i.e.,

$$\{(x, y) \in U \times V; F(x, y) = z_0\} = \{(x, G(x)) \in U \times V; x \in U\}.$$

Proposition 57. For any admissible choice of weight $\delta \in \mathbb{R}^m$ and any $d_A \in \mathscr{A}_\delta^*$, define the slice neighborhood of d_A by $S_A = \{a \in L^{1,2}_{\delta, s+1}(\mathfrak{u}(E)),\ d_A^* a = 0\}$. Then the following map

$$\pi_A : S_A \longrightarrow \mathscr{B}_\delta^*,$$

$$a \mapsto [d_A + a],$$

is a local homeomorphism near a neighborhood of $0 \in S_A$ and that of $[d_A]$ in $\mathscr{B}^*$ with the induced topology. Here, d_A^* is the formal adjoint of d_A with respect to the L^2-inner product.

Proof. Firstly, we show that the map π is a locally surjective map near $a = 0$. That is to show that for any $d_A + b \in \mathscr{A}_\delta^*$ with $\|b\|_{L_\delta^{1,2}}$ sufficiently small, there exists a unitary gauge transformation $g \in \tilde{\mathscr{G}}_\delta$ such that

$$g(d_A + b) - d_A \in S_A.$$

Consider the following smooth map

$$\begin{aligned} F_A : \tilde{\mathscr{G}}_\delta \times S_A &\to \mathscr{A}_\delta^*, \\ (g, a) &\mapsto g^{-1} \circ (d_A + a) \circ g. \end{aligned}$$

Its differential at $(\mathrm{Id}_E, 0)$ is given by

$$\begin{aligned} dF_A(0,0) : S_{\delta,s}^{2,2}(\mathfrak{u}(E)) \times \ker d_A^* &\to \mathscr{A}_\delta(\mathfrak{u}(E)), \\ (u, a) &\mapsto d_A u + a \end{aligned}$$

Due to fact that $L_{\delta,s}^{2,2} \subset C_\delta^0$ for $s \leq 0$ and that $L_{\mathrm{loc}}^{1,2}$ has a continuous restriction to any hypersurface on X, we can apply integration by parts and considering Remark 46, we get that $\operatorname{im} d_A$ and $\ker d_A^*$ are orthogonal with respect to the L^2 inner product. Hence dF_A is injective. Let G_A denote the Green's operator of the Bochner Laplacian

$\Delta_A = d_A^* d_A$, which exists again by Remark 46. For any $d_A + b \in \mathscr{A}_\delta$, we have

$$d_A^* \left(b - d_A \circ G_A \circ d_A^* b\right) = 0$$

and $d F_A$ is surjective. Therefore, by implicit function theorem, there exists a sufficient small $\epsilon > 0$ such that for any $a \in L^{1,2}_{\delta,s+1}(\mathfrak{u}(E))$ with $\|a\|_{L^{1,2}_{\delta,s+1}} < \epsilon$, there exist unique $b \in S_A$ and $u \in S^{2,2}_{\delta,s}$ both close to 0, such that

$$e^{-u} \circ (d_A + a) \circ e^u = d_A + b.$$

Here we use the fact that $\exp : \mathfrak{u}(n) \to U(n)$ is a local homeomorphism. Therefore, π is locally surjective on $S_{A,\epsilon}$.

Now we prove that π is locally injective. That is to say, there exists $\epsilon > 0$ sufficiently small, for given any $a_1, a_2 \in S_A$ with $\|a_i\|_{L^{1,2}_{\delta,s+1}} < \epsilon$ and a unitary gauge transformation $g \in \mathscr{G}_\delta$ such that $g(d_A + a_1) = d_A + a_2$, then $g = c \cdot \mathrm{Id}_E$ for some $c \in \mathbb{S}^1$. Without loss of generality, we may assume that $g = \mathrm{Id}_E + g_0$, where $g_0 \in S^{2,2}_{\delta,s}(\mathfrak{u}(E))$. By Remark 46, there exists a constant $c > 0$ such that

$$\|g_0\|_{L^{2,2}_{\delta,s}} \le c \|d_A\, g_0\|_{L^{1,2}_{\delta,s+1}}.$$

Therefore, using the fact that $d_A\, g = d_A\, g_0 = g a_1 - a_2 g$, where $a_i = d_{A_i} - d_A$ and Lemma 32, we have

$$\|g_0\|_{L^{2,2}_{\delta,s}} \le c' (\|\mathrm{Id}_E\|_{L^{2,2}_{\delta,s}} + \|g_0\|_{L^{2,2}_{\delta,s}})(\|a_1\|_{L^{1,2}_{\delta,s}} + \|a_2\|_{L^{1,2}_{\delta,s+1}}).$$

which implies that by choosing a_1, a_2 sufficiently small, g_0 is close to 0 and we can invoke the proof of the surjectivity to conclude that π is locally injective. $\square$

Let us now discuss the Riemannian metric on the slice $S_{A,\epsilon}$, the latter being interpreted as a local chart on $\mathscr{B}^*_\delta$. For each $a \in S_{A,\epsilon}$, by Remark 46, we have the following orthogonal decomposition with respect to $d_A + a$, that is

$$L^{1,2}_{\delta,s+1}(\Omega^1(\mathfrak{u}(E))) = \operatorname{im} d_{A+a} \oplus \ker d^*_{A+a},$$

Thus any $b \in T_a S_{A,\epsilon} \cong \ker d^*_A$ splits into its vertical part and horizontal part

$$b = b^v + b^h, \tag{3.7}$$

with $b^v \in \operatorname{im} d_{A+a} \cap \ker d^*_A$ and $b^h \in \ker d^*_{A+a} \cap \ker d^*_A$. Therefore, the L^2-inner metric at $a \in S_{A,\epsilon}$ is given by

$$\langle b_1, b_2 \rangle_{A+a} = \langle b^h_1, b^h_2 \rangle, \tag{3.8}$$

where the "h" refers to the decomposition (3.7) with respect to $d_A + a$.

The following concerns the dependence of the above L^2-metric on different choices of slice representations, that is

Lemma 58. The L^2-metric as defined in (3.8) is independent of the choice of slice representations and defines a Riemannian metric on $\mathscr{B}^*$.

Proof. Suppose that $S_{A_1,\epsilon}$ and $S_{A_2,\epsilon}$ are two ϵ-slices of $\mathscr{B}^*$ such that $S_{A_1,\epsilon} \cap \pi^{-1}_{A_1} \circ \pi_{A_2}(S_{A_2,\epsilon}) \neq \varnothing$, with $d_{A_1} + a_1 = d_{A_2} + a_2$, then for a small neighborhood of a_1, there exists a smooth gauge transformation $g_a \in \mathscr{G}_\delta$ for each a close to a_1 in $S_{A_1,\epsilon} \cap \pi_{A_1}^{-1} \circ \pi_{A_2}(S_{A_2,\epsilon})$ such that $g_a(A_1 + a) = A_2 + a' \in S_{A_2,\epsilon}$, i.e., g_a is the "change of coordinate" from the neighborhood of a_1 to a neighborhood of a_2. We may define

$$F : S_{A_1,\epsilon} \cap \pi_{A_1}^{-1} \circ \pi_{A_2}(S_{A_2,\epsilon}) \to S_{A_2,\epsilon} \cap \pi_{A_2}^{-1} \circ \pi_{A_1}(S_{A_1,\epsilon}),$$
$$a \mapsto g_a(d_{A_1} + a) - d_{A_2}.$$

Its differential dF_a can be computed by

$$\begin{aligned} dF_a(b) &= \frac{d}{dt} F(d_{A_1} + a + tb)\Big|_{t=0} \\ &= \frac{d}{dt} g_{(a+tb)}(d_{A_1} + a + tb)\Big|_{t=0}. \end{aligned}$$

with $g_{(a+tb)} = g_a \cdot \exp \psi_t$ and $\dot{\psi} = \frac{d}{dt}\big|_{t=0} \psi_t$, we thus have

$$\begin{aligned} &= \frac{d}{dt}\Big|_{t=0} g_{(a+tb)}(d_{A_1} + a) + \frac{d}{dt}\Big|_{t=0} g_{(a+tb)}(tb) \\ &= d_{A_2+a_2} \dot{\psi} + g_a(b). \end{aligned}$$

Thus the horizontal part of $dF_a(b)$ is given by $g_a(b^h)$, which implies the desired statement. $\square$

Because $F_{g(A)} = g \circ F_A \circ g^{-1}$, $\mathscr{M}^*_{\mathrm{HE}}$ is a finite dimensional submanifold of the Banach manifold $\mathscr{B}^*$, and the Riemannian metric (3.8) of $\mathscr{B}^*$ induces one on $\mathscr{M}^*_{\mathrm{HE}}$. As the local expression of the metric depends on the local charts chosen on $\mathscr{M}^*_{\mathrm{HE}}$.

Definition 59. For any $[d_A] \in \mathscr{M}^*_{\mathrm{HE}}$, we will call the local (nonlinear) slice around

$[d_A]$ given by

$$U_{A,\epsilon} = \mathscr{M}^*_{\mathrm{HE}} \cap S_{A,\epsilon} = \{d_A + a,\ \|a\|_{L^{1,2}_{\delta,s+1}} < \epsilon,\ d_A^* a = 0,\ d_A a + a \wedge a = 0\} \tag{3.9}$$

with the induced topology from $\mathscr{B}^*$ as the *Hermitian-Einstein* slice.

As the tangent space at $[d_A + a] \in U_{A,\epsilon}$ is given by

$$T_a U_{A,\epsilon} = \{b \in L^{1,2}_{\delta,s+1}(\Lambda^1 X \otimes \mathfrak{u}(E));\ d_A^* b = 0, d_{A+a} b = 0\},$$

the restriction of the L^2-metric (3.8) on $U_{A,\epsilon}$ has the following expression

$$\langle\, b_1, b_2 \,\rangle_{A+a} = \langle\, P_a(b_1), P_a(b_2) \,\rangle, \tag{3.10}$$

where $b_1, b_2 \in T_a U_{A,\epsilon}$ and P_a denotes the projection onto $H^1_{A+a,\delta}$ with respect to $d_A + a$.

Remark 60. A more traditional choice of slice near $[d_A]$ is to use the first cohomology group of the de Rham complex

$$0 \to \hat{L}^{2,2}_{\delta,s}(\Omega^0(\mathfrak{u}(E))) \xrightarrow{d_A} L^{1,2}_{\delta,s+1}(\Omega^1(\mathfrak{u}(E))) \xrightarrow{d_A} L^2_{\delta,s+2}(\Omega^2(\mathfrak{u}(E))) \to 0. \tag{3.11}$$

That is, there exists a sufficiently small $\epsilon > 0$ such that

$$\tilde{U}_{A,\epsilon} = \{d_A + a,\ \|a\|_{L^{1,2}_{\delta,s+1}} < \epsilon,\ d_A^* a = 0,\ d_A a = 0\} \subset H^1_{A,\delta},$$

is a linear local slice near $[d_A] \in \mathscr{M}^*_{\mathrm{HE}}$. To understand the Riemannian metric on this slice involves the Kuranishi map, which is not suitable for variation computation in the Dolbeault coordinate to be defined later, hence we will refer the reader to [68, pp.16-17] for more detailed discussion.

We conclude this part of discussion by the following regularity result on the connections in the Hermitian-Einstein slice.

Lemma 61. Suppose $d_A \in \mathscr{A}_\delta^*$ is any smooth irreducible flat connection on E, then any connection $d_A + a \in U_{A,\epsilon}$ lies in $C^\infty \cap C^0_{\delta'}$ for any $0 < \delta' < \delta$.

Proof. This follows from the a priori estimate in Lemma 37, the Sobolev multiplication theorem 31, and a bootstrap argument as in [38, Proposition (2.3.4)]. □

3.2.2 Complex Structure of $\mathscr{M}_{\mathrm{P}}^s$

In this part, we will show that $\mathscr{M}_{\mathrm{P}}^s$ is a (smooth) complex manifold. That is, we construct near any $[\bar{\partial}_A] \in \mathscr{M}_{\mathrm{P}}^s$ a local complex chart and show that the transformation between any two such local coordinate charts is holomorphic.

To begin with, given any $\bar{\partial}_A \in \mathscr{C}_\delta^s$, we consider the following two term Dolbeault complex

$$0 \to \hat{L}^{2,2}_{\delta,s}(\Omega^0(\mathrm{End}E)) \xrightarrow{\bar{\partial}_A} L^{1,2}_{\delta,s+1}(\Omega^{0,1}(\mathrm{End}E))) \to 0. \tag{3.12}$$

By a similar argument as in Lemma 57 and the fact that stability is an open condition which follows from Lemma 47.

Definition 62. There exists, near any $\bar{\partial}_A \in \mathscr{C}_\delta^s$, the following *Dolbeault slice* of the form

$$V_{A,\epsilon} = \{\bar\partial_A + a,\ \|a\|_{L^{1,2}_{\delta,s+1}} < \epsilon,\ \bar\partial_A^* a = 0\} \subset H^{0,1}_{A,\delta}, \tag{3.13}$$

for the $\mathscr{G}^{\mathbb{C}}_\delta$ action on $\mathscr{C}^s_\delta$.

Here $H^{0,1}_{A,\delta}$ denotes the first cohomology group of the above complex, with a canonical complex structure given by multiplication by i.

Suppose $\bar\partial_{A_1}, \bar\partial_{A_2} \in \mathscr{C}^s_\delta$ with their respective local Dolbeault slices satisfy that $V_{A_1,\epsilon_1} \cap V_{A_2,\epsilon_2} \neq \varnothing$, i.e., there exist $a, b \in \mathscr{C}^s_\delta$ such that $\bar\partial_{A_1} + a = \bar\partial_{A_2} + b$. Moreover, by an implicit function theorem type argument as in Lemma 67, for any a' in a sufficiently small neighborhood of a, there exists a unique $g_{a'} \in \tilde{\mathscr{G}}^{\mathbb{C}}_\delta$ close to identity such that $g_{a'}(\bar\partial_{A_1} + a') \in V_{A_2,\epsilon}$. Moreover, the dependence of $g_{a'}$ on a' is smooth.

In the following, we show that, $g_{a'}$ depends on a' holomorphically, that is

Proposition 63. The Dolbeault slices defined as above endow $\mathscr{M}^s_{\mathrm{P}}$ with a holomorphically compatible system of complex local charts.

We will need the following technical lemma below.

Lemma 64. [61, Proposition 3.35] Let X be any element in $M_n(\mathbb{C})$, we define

$$\mathrm{ad}_X : M_n(\mathbb{C}) \to M_n(\mathbb{C}),$$

by $\mathrm{ad}_X(Y) = [X, Y]$. Then for any $Y \in M_n(\mathbb{C})$, we have

$$\exp(X) Y \exp(-X) = \sum_{n\geq 0} \mathrm{ad}^n_X(Y) \frac{t^n}{n!} = Y + [X, Y] + \frac{1}{2}[X, [X, Y]] + \cdots$$

Remark 65. Note that in the above formula, every term depends on Y linearly. Furthermore, the above expression converges with infinite radius in $\|X\|$.

Proof of Proposition 63. Let $\bar{\partial}_{A_1} + a(t)$ be a holomorphic family in V_{A_1,ϵ_1} such that $a(0) = a$. Let g_t be a family of complex gauge transformations such that

$$g_t(\bar{\partial}_{A_1} + a(t)) = \bar{\partial}_{A_2} + b(t);\; g_0(\bar{\partial}_{A_1} + a_0) = \bar{\partial}_{A_2} + b_0.$$

We first claim that when $\|b_t\|_{L^{1,2}_{\delta,s+1}}$ is small, such a family is unique and close to the g_0 in $\hat{L}^{2,2}_{\delta,s}$ norm. Set $h_t = g_t \circ (g_0)^{-1}$, $h_0 = \mathrm{Id}$, and there exists a family $u_t \in S^{2,2}_{\delta,s}(\mathrm{End}E)$ such that $h_t = \exp(u_t)$ with $u_0 = 0$. Moreover, we have

$$\begin{aligned} g_t(\bar{\partial}_{A_1} + a(t)) &= \bar{\partial}_{A_2} + b + \bar{\partial}_{A_2+b}(h_t)h_t^{-1} + h_t g_0\, a(t)\, g_0^{-1} h_t^{-1}, \\ &= \bar{\partial}_{A_2} + b + \bar{\partial}_{A_2+b}(u_t) + g_0\, a(t)\, g_0^{-1} + R(u_t, a_t). \end{aligned}$$

In the second equality, we have used Lemma 64, and $R(u_t, a_t)$ denotes the terms of order greater or equal to two. Since the it lies in V_{A_2,ϵ_2}, by definition,

$$\bar{\partial}^*_{A_2}\bar{\partial}_{A_2+b_0}u_t + \bar{\partial}^*_{A_2}(g_0\, a(t)\, g_0^{-1}) + \bar{\partial}^*_{A_2}R(u_t, a_t) = 0$$

By Theorem 45, we can apply $\Delta^{-1}_{A_2}$ on $S^{2,2}_{\delta,s}(\mathrm{End}E)$ and we get,

$$F(u_t, a_t) := u_t + \Delta^{-1}_{A_2}\bar{\partial}^*_{A_2}([b_0, u_t]) + \Delta^{-1}_{A_2}\bar{\partial}^*_{A_2}(g_0\, a(t)\, g_0^{-1}) + \Delta^{-1}_{A_2}\bar{\partial}^*_{A_2}R(u_t, a_t) = 0$$

Note that $F(0,0) = 0$ and $D_{u_t=0}F(\cdot, a_t)$ is invertible provided $\|b_0\|_{L^{1,2}_{\delta,s+1}}$ is sufficiently small by the following estimate

$$\|\Delta_{A_2}^{-1}\bar{\partial}_{A_2}^{*}([b_0,u_t])\|_{S_{\delta,s}^{2,2}} \lesssim \|b_0\|_{L_{\delta,s+1}^{1,2}}\|u_t\|_{S_{\delta,s}^{2,2}},$$

which in turn results from Theorem 45 and Lemma 31. By the implicit function theorem, u_t is unique and depends on t smoothly.

Now, apply $\frac{\partial}{\partial \bar{t}}$ to F and by the fact that a_t is holomorphic, we get that

$$\dot{u}_t + \Delta_{A_2}^{-1}\bar{\partial}_{A_2}^{*}([b_0,\dot{u}_t]) + \Delta_{A_2}^{-1}\bar{\partial}_{A_2}^{*}R(\dot{u}_t,a_t) = 0,$$

where $\cdot$ denotes $\frac{\partial}{\partial \bar{t}}$. Again, the last two terms can be made sufficiently small provided $\|a_0\|_{L_{\delta,s+1}^{1,2}}$ and $\|b_0\|_{L_{\delta,s+1}^{1,2}}$ are sufficiently small. Therefore, we have that

$$\|\dot{u}_t\|_{S_{\delta,s}^{2,2}} \leq c(\|a_0\|_{L_{\delta,s+1}^{1,2}} + \|b_0\|_{L_{\delta,s}^{1,2}})\|\dot{u}_t\|_{S_{\delta,s}^{2,2}} < \|\dot{u}_t\|_{S_{\delta,s}^{2,2}},$$

which implies that $\dot{u}_t = 0$ and our claim follows. $\square$

To sum up, we have showed that $\mathscr{M}_{\mathrm{P}}^{s}$ is a complex manifold, and the restriction of the canonical line bundle on $V_{A,\epsilon}$ is given by

$$\lambda\big|_{V_{A,\epsilon}} = \det T\mathscr{M}_{\mathrm{P}}^{s}\big|_{V_{A,\epsilon}} = \det T^{*}V_{A,\epsilon}. \tag{3.14}$$

3.2.3 Hermitian Metric on $\mathscr{M}_{\mathrm{P}}^s$

Given any irreducible flat unitary connection $d_A \in \mathscr{A}_\delta^*$, we denote its $(0,1)$ part by $\bar{\partial}_A$, which defines a stable parabolic structure on E by Lemma 53. We have seen that there exist two different local slices: the Hermitian-Einstein slice $U_{A,\epsilon} \subset \mathscr{M}_{\mathrm{HE}}^*$ and the Dolbeault slice $V_{A,\epsilon} \subset \mathscr{M}_{\mathrm{P}}^s$.

Definition 66. We will refer the image of $U_{A,\epsilon}$ in $\mathscr{M}_{\mathrm{P}}^s$ under the map Φ_h, denoted by $U_{A,\epsilon}^{0,1}$, as the Hermitian-Einstein slice in $\mathscr{M}_{\mathrm{P}}^s$.

In particular, $D\Phi_h$ induces a $\mathbb{R}$-linear isomorphism between the tangent bundles of these two different Hermitian-Einstein slices. For $d_A + a \in U_{A,\epsilon}$ with $\bar{\partial}_A + a^{0,1}$ its $(0,1)$-part in $U_{A,\epsilon}^{0,1}$, we define the following Hermitian metric,

$$\langle\, b_1^{0,1}, b_2^{0,1} \,\rangle_{a^{0,1}} = \langle\, P_a(b_1), P_a(b_2) \,\rangle, \tag{3.15}$$

for any $b_1^{0,1}, b_2^{0,1} \in T_a U_{A,\epsilon}^{0,1}$, which is canonically the $(0,1)$-part of $T_a U_{A,\epsilon}$, and P_a is the orthogonal projection with respect to $\bar{\partial}_A + a^{0,1}$. In the following, we will construct a local diffeomorphism from $V_{A,\epsilon}$ to $U_{A,\epsilon}^{0,1}$ to pull-back the above Hermitian metric to the Dolbeault slice $V_{A,\epsilon}$.

Theorem 67. Let $d_A \in \mathscr{A}_\delta^*$ be a fixed irreducible flat unitary connection. For $\epsilon > 0$ sufficiently small, and for any $\bar{\partial}_A + \mu \in V_{A,\epsilon}$, there exists a unique $g_\mu \in \tilde{\mathscr{G}}_\delta^{\mathbb{C}}$ close to the identity which depends on μ smoothly satisfying

$$g_\mu^{-1}(\bar{\partial}_A + \mu) = g_\mu \circ (\bar{\partial}_A + \mu) \circ g_\mu^{-1} \in U_{A,\epsilon}^{0,1}.$$

Proof. In order to find $g_\mu \in \tilde{\mathscr{G}}_\delta^{\mathbb{C}}$ such that $g_\mu^{-1}(\bar{\partial}_A + \mu)$ lies in $U_{A,\epsilon}^{0,1}$, by Kähler identity, we only need

$$\bar{\partial}_A^*(-\bar{\partial}_A g_\mu g_\mu^{-1} + g_\mu \mu g_\mu^{-1}) = 0. \tag{3.16}$$

Set $g_\mu = \exp(u)$ with $u \in S_{\delta,s}^{2,2}(\mathrm{End}E)$, by Lemma 64, (3.16) simplifies to be

$$\bar{\partial}_A^* \bar{\partial}_A u - \bar{\partial}_A^*[u,\mu] + \bar{\partial}_A^* R(u,\mu) = 0. \tag{3.17}$$

Applying Δ_A^{-1} on $S_{\delta,s}^{2,2}(\mathrm{End}E)$ to (3.17), we get

$$u - \Delta_A^{-1}\bar{\partial}_A^*[u,\mu] + \Delta_A^{-1}\bar{\partial}_A^* R(u,\mu) = 0.$$

Applying implicit function theorem to the linearization of (3.17), we can argue as in Lemma 57 and prove the existence and uniqueness of u, and the smooth dependence of u on μ. □

To sum up, for any smooth $\bar{\partial}_A \in \mathscr{C}_\delta^s$, if its Chern connection d_A satisfies $F(A) = 0$, then we have constructed near it two different local slices, $U_{A,\epsilon}^{0,1}$ and $V_{A,\epsilon}$, for the action of $\mathscr{G}_\delta^{\mathbb{C}}$ on $\mathscr{C}_\delta^s$. That is, they both provide a local coordinate patch of $\mathscr{M}_P^s$ near $[\bar{\partial}_A]$. Moreover, we have constructed a unique smooth map

$$\Psi_A : V_{A,\epsilon} \to \tilde{\mathscr{G}}_\delta^{\mathbb{C}}, \tag{3.18}$$

$$\bar{\partial}_A + \mu \mapsto g_\mu,$$

satisfying

$$g_\mu(\bar{\partial}_A + \mu) = g_\mu \circ (\bar{\partial}_A + \mu) \circ g_\mu^{-1} \in U^{0,1}_{A,\epsilon}.$$

In particular, $g_0 = \mathrm{Id}$. Via this identification, we pull back the Hermitian metric (3.15) of $U^{0,1}_{A,\epsilon}$ to $V_{A,\epsilon}$.

Definition 68. The Hermitian metric on $T_{\bar{\partial}_A+\mu} V_{A,\epsilon} \cong H^{0,1}_{A,\delta}(\mathrm{End} E)$ is given for any $X, Y \in H^{0,1}_{A,\delta}(\mathrm{End} E)$ by

$$\langle X, Y \rangle_{A+\mu} := \langle P_\mu(g_\mu X g_\mu^{-1}), P_\mu(g_\mu Y g_\mu^{-1}) \rangle, \tag{3.19}$$

where

$$P_\mu : L^{1,2}_{\delta,s+1}(\Omega^{0,1}(\mathrm{End} E)) \to H^{0,1}_{A+a,\delta}(\mathrm{End} E),$$

is the orthogonal projection with respect to decomposition associated with the Dolbeault operator $g_\mu \circ (\bar{\partial}_A + \mu) \circ g_\mu^{-1} \in U^{0,1}_{A,\delta}$.

By the same argument as in Lemma 58, we can show that the above Hermitian metric is well-defined on $\mathscr{M}^s_{\mathrm{P}}$ under change of coordinate, hence patch together to endow $\mathscr{M}^s_{\mathrm{P}}$ with a global Hermtian metric.

Chapter 4: Kähler Metric on the Moduli Space of Stable Parabolic Bundles

In this chapter, we prove that the Hermitian metric (3.19) is Kähler, moreover, the Dolbeault slices provide an atlas of normal coordinate charts on $\mathscr{M}_P^s$. Then we proceed to compute the L^2-curvature of the canonical bundle of $\mathscr{M}_P^s$ over a Dolbeault slice $V_{A,\epsilon}$.

4.1 Kählerian Property

Fix a basis $\{\mu_1, \cdots, \mu_d\} \subset H^{0,1}_{A,\delta}(\mathrm{End}E)$, where d denotes the complex dimension of $\mathscr{M}_{\mathrm{P}}^s$. For any point $\bar{\partial}_A + t\mu \in V_{A,\epsilon}$, the Hermitian metric (3.19), with respect to the fixed basis is then given by

$$h_{i\bar{j}}(t) := \left\langle P_t(g_{t\mu} \circ \mu_i \circ g_{t\mu}^{-1}), P_t(g_{t\mu} \circ \mu_j \circ g_{t\mu}^{-1}) \right\rangle, \tag{4.1}$$

where $g_{t\mu}$ is the unique complex gauge transformation close to identity satisfying

$$\bar{\partial}_A + \gamma(t) := g_{t\mu} \circ (\bar{\partial}_A + t\mu) \circ g_{t\mu}^{-1} \in U^{0,1}_{A,\epsilon},$$

which exists and depends on t smoothly as a result of Theorem 67. In the following, we will show that $h_{i\bar{j}}$ osculates to second order at $\bar{\partial}_A \in V_{A,\epsilon}$, i.e., the following equality

$$h_{i\bar{j}}(0) = \delta_{i,j}, \; \frac{\partial}{\partial \mu_k} h_{i\bar{j}}(0) = \frac{\partial}{\partial \bar{\mu}_k} h_{i\bar{j}}(0) = 0,$$

is valid for any $i, j, k = 1, \cdots, d$. This is one of the equivalent definition for a Hermitian metric to be Kähler. As the choice of $\bar{\partial}_A$ is arbitrary, this implies that

Proposition 69. The Hermitian metric defined in (3.19) is Kähler on $\mathscr{M}^s_{\mathrm{P}}$. Moreover, the Dolbeault slice $V_{A,\epsilon}$ is a local normal coordinate.

Lemma 70. $\left.\frac{\partial}{\partial t}(g^*_{t\mu_k} g_{t\mu_k})\right|_{t=0} = \left.\frac{\partial}{\partial \bar{t}}(g^*_{t\mu_k} g_{t\mu_k})\right|_{t=0} = 0.$

Proof. Set

$$\bar{\partial}_A + \gamma = g_\mu(\bar{\partial}_A + \mu) g_\mu^{-1}, \tag{4.2}$$

where $\bar{\partial}_A + \gamma$ satisfies that

$$-\bar{\partial}_A(\gamma^*) + \partial_A(\gamma) - [\gamma, \gamma^*] = 0, \tag{4.3}$$

with $\gamma(0) = 0$ and $g(0) = \mathrm{Id}$. Differentiate (4.2) with respect to t and set $t = 0$, we have

$$\bar{\partial}_A \dot{g}_{\mu_k} = \mu_k - \dot{\gamma}_k^{0,1}. \tag{4.4}$$

where $\dot{x}$ means $\frac{dx}{dt}$. Differentiate (4.2) with respect to $\bar{t}$ at $t = 0$, and taking its adjoint with respect to h, we get

$$\partial_A \dot{g}^*_{\mu_k} = \dot{\gamma}_k^{1,0}. \tag{4.5}$$

Differentiating (4.3) with respect to t and take $t = 0$, we get

$$\bar{\partial}_A \dot{\gamma}_k^{1,0} + \partial_A \dot{\gamma}_k^{0,1} = 0.$$

Now, apply $\bar{\partial}^*_A$ and ∂^*_A to (4.4) and (4.5), respectively and use Kähler identities, we can sum these together and get

$$\bar{\partial}^*_A \bar{\partial}_A (\dot{g}_{\mu_k} + \dot{g}^*_{\mu_k}) = 0.$$

As $\dot{g}_{\mu_k} \in S^{2,2}_{\delta,s}(\mathrm{End}E)$, by Theorem 45, we have

$$\frac{\partial}{\partial t}(g^*_{t\mu_k} g_{t\mu_k})\bigg|_{t=0} = \dot{g}_{\mu_k} + \dot{g}^*_{\mu_k} = 0.$$

The case of $\frac{\partial}{\partial t}\big|_{t=0}(g^*_{t\mu_k} g_{t\mu_k}) = 0$ is similar and this completes the proof. $\square$

Proof of Proposition 69. Note that

$$\begin{aligned}
\frac{\partial}{\partial t} h_{i\bar{j}}(t)\bigg|_{t=0} &= \frac{\partial}{\partial t}\langle \operatorname{Ad} g^*_{t\mu} \circ (1 - \bar{\partial}_\gamma \Delta_\gamma^{-1} \bar{\partial}^*_\gamma) \circ \operatorname{Ad} g_{t\mu}(\mu_i), \mu_j \rangle\bigg|_{t=0}, \\
&= \frac{\partial}{\partial t}\langle \operatorname{Ad} g^*_{t\mu} g_{t\mu}(\mu_i), \mu_j \rangle\bigg|_{t=0} - \frac{\partial}{\partial t}\langle \operatorname{Ad} g^*_{t\mu} \circ \bar{\partial}_\gamma \Delta_\gamma^{-1} \bar{\partial}^*_\gamma \circ \operatorname{Ad} g_{t\mu}(\mu_i), \mu_j \rangle\bigg|_{t=0}, \\
&= 0.
\end{aligned}$$

where in the last equality, we have used Lemma 70 to the first term, while for the second term, note that $g_0 = \mathrm{Id}$ and $\mathrm{Im}\,\bar{\partial}_A$ is orthogonal to μ_j with respect to the L^2 inner product. □

4.2 L^2-Curvature of the Canonical Bundle

Recall that the canonical line bundle

$$\lambda := \det(T^* V_{A,\epsilon}) \cong \det T^* \mathscr{M}^s_{\mathrm{P}}\big|_{V_{A,\epsilon}}, \tag{4.6}$$

is equipped with a Hermitian metric induced from (3.19) on $V_{A,\epsilon}$. Fix an orthonormal basis $\{\mu_i,\, i = 1, \cdots, d\} \subset H^{0,1}_{A,\delta}(\mathrm{End}E)$ to simplify our computation. Let M denote the $d \times d$ matrix valued function on $V_{A,\epsilon}$ with its (i,j)-th entry given by $h_{i\bar{j}}(t)$, see (4.1). The curvature form of λ can be written as

$$\Theta_{L^2}(\lambda) = \partial\bar{\partial} \log \det M. \tag{4.7}$$

In the following, we evaluate (4.7) over the Dolbeault slice $V_{A,\epsilon}$. More explicitly,

Theorem 71. The curvature of the canonical line bundle $\lambda = \det T^* \mathscr{M}^s_{\mathrm{P}}$ with respect to the induced L^2-Hermitian metric is given by

$$\Theta_{L^2}(\mu, \bar{\nu}) = -\operatorname{Tr}\Big(\operatorname{ad} f_{\mu,\bar{\nu}} \circ P_A - \operatorname{ad}\mu \circ \Delta_A^{-1} \circ * \operatorname{ad}\nu * \circ P_A\Big),$$

for any $\mu, \nu \in H^{0,1}_{A,\delta} \cong T_{\bar{\partial}_A} V_{A,\epsilon}$ on $V_{A,\epsilon}$. Here Tr denotes the operator trace on

$L^2(\Omega^{0,1}(\mathrm{End}E))$ and $\mathrm{ad}\,\mu$ denotes the adjoint action $[\mu,\cdot]$.

For the proof of Theorem 71, we will need the following technical lemmas.

Lemma 72. For any $\mu,\nu \in H_A^{0,1}(\mathrm{End}E)$, we have

$$\frac{\partial^2}{\partial t \partial \bar{s}}(g_{t\mu+s\nu}^* \circ g_{t\mu+s\nu})\Big|_{t=s=0} = -\Delta_A^{-1}(*[*\mu,\nu]). \tag{4.8}$$

As the proof of Lemma 72 is lengthy and computational, we postpone its proof in Appendix C. In the following, we will denote $\Delta_A^{-1}(*[*\mu,\nu])$ by $f_{\mu,\bar{\nu}}$, and since it belongs to $L_{\delta,s}^{2,2}(\mathrm{End}\,E)$, by Lemma 34, its limiting value along each cusp end Z_k exists and we will denote it by $F_{\mu\bar{\nu}}^k$ which lies in $\mathrm{End}\,E_{p_k}$. Moreover, we have the following symmetry property,

$$f_{\mu\bar{\nu}}^* = f_{\nu\bar{\mu}}. \tag{4.9}$$

Lemma 73. The second variation of $h_{i\bar{j}}$ is given by

$$\frac{\partial^2}{\partial \bar{s} \partial t} h_{i\bar{j}}\Big|_{t=s=0} = -\langle\, [\, f_{\mu,\bar{\nu}},\, \mu_i\,] - [\,\mu, \Delta_A^{-1} *\, [\,\nu,\, *\mu_i\,]],\, \mu_j\,\rangle.$$

Proof.

$$\begin{aligned}
\frac{\partial^2}{\partial \bar{s} \partial t} h_{i\bar{j}}\Big|_{t=s=0} &= \frac{\partial^2}{\partial \bar{s} \partial t}\langle\, \mathrm{Ad}(g^*g)(\mu_i),\, \mu_j\,\rangle\Big|_{t=s=0} \\
&\quad - \frac{\partial^2}{\partial \bar{s} \partial t}\langle\, \mathrm{Ad}(g^*g)(\bar{\partial}_A + t\mu + s\nu)\,\mathrm{Ad}(g^{-1})\Delta_\gamma^{-1}\,\mathrm{Ad}(g) \\
&\quad \mathrm{Ad}(g^*g)^{-1}(\bar{\partial}_A + t\mu + s\nu)^*\,\mathrm{Ad}(g^*g)(\mu_i),\, \mu_j\,\rangle\Big|_{t=s=0}, \\
&= -\langle\, [f_{\mu,\bar{\nu}}, \mu_i],\, \mu_j\,\rangle - \langle\, [\mu, \Delta_A^{-1} * [\nu, *\mu_i]],\, \mu_j\,\rangle.
\end{aligned}$$

In the last equality, we have used Lemma 70 and Lemma 72. $\square$

Proof of Theorem 71. For $\mu, \nu \in H^{0,1}_A(\mathrm{End} E)$,

$$\begin{aligned}\Theta_{L^2}(\mu, \bar{\nu}) &= \partial_t \partial_{\bar{s}} \log \det M\Big|_{t=s=0}, \\ &= \partial_t \operatorname{tr} M^{-1} \partial_{\bar{s}} M\Big|_{t=s=0}, \\ &= \left(\operatorname{tr} M^{-1} \partial_t \partial_{\bar{s}} M - \operatorname{tr} M^{-1} \partial_t M M^{-1} \partial_{\bar{s}} M\right)\Big|_{t=s=0}.\end{aligned}$$

where tr denote the ordinary matrix trace. Since $\{\mu_i\}$ chosen to be orthonormal at $\bar{\partial}_A$, the result follows from Proposition 69, Lemma 72, and Lemma 73, and the following fact that for a finite dimensional subspace V in a Hilbert space E, with any orthonormal basis $e_1, , e_d$, and linear operator F, we have

$$\sum_{i,j=1}^{d} \langle\, F e_i,\, e_j \,\rangle = \operatorname{Tr}(F \circ P),$$

where P is the orthogonal projection onto V. $\square$

4.3 Identification With the Index Bundle

For any $\bar{\partial}_\gamma = \bar{\partial}_A + \gamma$ in the Hermitian-Einstein slice $U^{0,1}_{A,\epsilon}$, by Proposition 133, the following L^2 complex

$$0 \to L^2(\mathrm{End}\, E) \xrightarrow{\bar{\partial}_\gamma} L^2(\Lambda^{0,1} \otimes \mathrm{End}\, E) \to 0,$$

is Fredholm. We may define the determinant line bundle on $U^{0,1}_{A,\epsilon}$ as follows,

$$\det(\operatorname{ind}\bar{\partial}_\gamma) = (\Lambda^{\max}\ker\bar{\partial}_\gamma) \otimes (\Lambda^{\max}\operatorname{coker}\bar{\partial}_\gamma)^{-1},$$
$$= (\Lambda^{\max}\operatorname{coker}\bar{\partial}_\gamma)^{-1}.$$

where we use the stability condition of the associated parabolic bundle $(E, \bar{\partial}_\gamma)$. As the dimension of $\operatorname{coker}\bar{\partial}_\gamma$ remains constant on $U^{0,1}_{A,\epsilon}$, the above is a well-defined line bundle and it is equipped with a L^2-Hermitian metric.

Moreover, by the local diffeomorphism between $U^{01}_{A,\epsilon}$ and $V_{A,\epsilon}$ as is shown in Theorem (67) and Definition 3.19, we get the first part of the following identification as an isometry between Hermitian line bundles,

$$TV_{A,\epsilon} \cong TU^{0,1}_{A,\epsilon} = L^2\operatorname{coker}\bar{\partial}_\gamma. \tag{4.10}$$

where as the second part of the above identification is discussed below.

One direction of inclusion is easy. Given any $\sigma \in H^{0,1}_{A,\delta} = T_{[\bar{\partial}_A]}\mathscr{M}^s_{\mathrm{P}}$, by Lemma 37 and Lemma 32, it belongs to $C^0_\delta \cap C^\infty(\Lambda^{0,1}X \otimes \operatorname{End} E)$. In particular, $TU^{0,1}_{A,\epsilon} \subset L^2\operatorname{coker}\bar{\partial}_\gamma$. Moreover, as we have the following Hodge decomposition, where

$$H^1_{A,\delta} \otimes \mathbb{C} = H^{1,0}_{A,\delta} \oplus H^{0,1}_{A,\delta}, \;\; H^{0,1}_{A,\delta} \cong \overline{H^{1,0}_{A,\delta}}.$$

With respect to the fixed holomorphic adapted frame $\{f_{i,k}\}$ along each cusp end Z_k, the (i,j)-th component $\sigma_{ij,k}$ of σ is in particular a anti-holomorphic function which satisfies

$$\|\sigma_{ij,k} f_{i,k} \otimes f_{j,k}^* d\bar{z}\|_{C^0} \lesssim e^{-\delta_k y}.$$

This implies that in terms of the punctured disk model, $\sigma_{ij,k}$ extends to a anti-holomorphic function on $\mathbb{D}(\epsilon)$ with

$$\sigma_{ij,k}(0) = \begin{cases} 0, & \alpha_i \geq \alpha_j; \\ \text{otherwise}, & \alpha_i < \alpha_j. \end{cases}$$

As $d\bar{z}$ transforms to $\frac{d\bar{w}}{\bar{w}}$, we see that there exists a conjugate linear isomorphism between the tangent space $H^{0,1}_{A,\delta}$ and the space of $(1,0)$- meromorphic forms with values in $\operatorname{End} E$ such that it has at most a simple pole at the origin with its residues respecting the flag structure strictly, i.e.,

$$\operatorname{res}(\sigma)_{p_k}(F_{i,k}) \subset F_{j,k},$$

if and only if $\alpha_i < \alpha_j$, or equivalently $\operatorname{res}(\sigma)_{p_k}$ is strictly lower triangular. Considering similar description of $L^2 \operatorname{coker} \bar{\partial}_\gamma$ in Appendix D. of [45], in particular Proposition D.4 (b), we have $H^{0,1}_{A,\delta} = L^2 \operatorname{coker} \bar{\partial}_\gamma$. Therefore, we have seen that

Lemma 74. With respect to the Riemannian metric g of cusp type on X and the adapted Hermitian metric h on E, the first L^2-cohomology group of the following complex

$$0 \to L^2(X, \operatorname{End} E) \xrightarrow{\bar{\partial}_A} L^2(X, \Lambda^{0,1} X \otimes \operatorname{End} E) \to 0,$$

satisfies that $H^{0,1}_{A,\delta} \cong L^2 H^{0,1}_A$.

As for any $\bar{\partial}_\gamma \in U^{0,1}_{A,\delta}$, there exists a temporal gauge by Lemma 19, the above argument applies without any change, and we have shown that

$$TU^{0,1}_{A,\epsilon} = L^2 \operatorname{coker} \bar{\partial}_\gamma,$$

as a Hermitian bundle over $U^{0,1}_{A,\epsilon}$. Therefore, we get the following identification of the associated determinant lines,

$$\lambda := \det(T^* V_{A,\epsilon}) \cong \det(\operatorname{ind} \bar{\partial}_\gamma) \tag{4.11}$$

Later on, we will use this identification to pull-back the Quillen metric defined on the latter to the canonical line λ and compute its curvature form.

Remark 75. It is easy to see from the above that local existence of universal family of parabolic bundles is always valid on $\mathscr{M}^s_P$. On the other hand, Boden and Yokogawa in [15] and [17] showed that for a generic choice of parabolic weights, semi-stability implies stability, in particular, the moduli space $\mathscr{M}^s_P$ of stable parabolic bundles is compact; moreover, for a generic choice of parabolic weights, the moduli space $\mathscr{M}^s_P$ of stable parabolic bundles of vanishing parabolic degree admits a universal parabolic vector bundle.

Let us finish this part with the discussion of the following Hermitian holomorphic line bundles defined on $V_{A,\epsilon}$. Namely, for the fixed temporal frame $\{e_{i,k}\}_{i=1}^n$

adapted to the fixed descending flag structure of E

$$E_{p_k} = E_{1,k} \supset E_{2,k} \supset \cdots \supset E_{s_k,k} \supset 0,$$

i.e., the span of $\{e_{n-m_{1,k}-\cdots-m_{i-1,k},k}, \cdots, e_{n,k}\}$ equals $E_{i,k}$. Therefore, we can form the following trivial holomorphic line bundle over $V_{A,\epsilon}$:

$$\lambda_{i,k} = \det(E_{i,k}/E_{i+1,k}), \tag{4.12}$$

with its basis given by $u_{i,k} := e_{n-m_{1,k}-\cdots-m_{i-1,k},k} \wedge \cdots \wedge e_{n-m_{1,k}-\cdots-m_{i,k}+1,k}$. As to its metric, at the point $\bar{\partial}_A + \mu \in V_{A,\epsilon}$, we set

$$\|u_{i,k}\|^2_{i,k} = \det\left(g^*_\mu g_\mu\right)_{i,k}, \tag{4.13}$$

where g_μ is the complex gauge transformation defined in Lemma 67. Since $g^*_\mu g_\mu$ varies smoothly depending on μ, it has a limiting value in $\mathrm{Herm}^+(m_{1,k}) \times \cdots \times \mathrm{Herm}^+(m_{s_k,k})$ along the cusp end Z_k, whose existence is guaranteed by Lemma 34, and the above definition is independent of the chosen temporal frame, therefore the definition (4.13) makes sense. $\mathrm{Herm}^+(n)$ denotes the space of positive definite Hermitian matrices on a complex vector space of dimension n. The sub-index indicates that we are taking the i-block along the k-th cusp ends. By Lemma 72, we have the following

Lemma 76. The curvature form $\Theta_{i,k}$ of the Hermitian holomorphic line bundle

$(\lambda_{i,k}, \|\cdot\|_{i,k})$ on the Dolbeault slice $V_{A,\epsilon}$ is given by

$$\Theta_{i,k}(\mu, \bar{\nu}) = -\operatorname{tr} F^{i,k}_{\mu,\bar{\nu}}, \tag{4.14}$$

where $F^{i,k}_{\mu,\bar{\nu}}$ denotes the i-th diagonal block of $F^{k}_{\mu,\bar{\nu}} := \lim_{\operatorname{Im} z\to\infty} \Delta_{\gamma}^{-1}(*[*\mu, \nu])$ along the k-th cusp end.

Remark 77. By the uniqueness of the transformation map g_μ on the Dolbeault slice $V_{A,\epsilon}$, if $V_{A_1,\epsilon} \cap V_{A_2,\epsilon} \neq \varnothing$, then there exists unique unitary gauge transformations connecting these coordinate charts such that the above defined metric is preserved and hence define a global Hermitian metric on these line bundles $\lambda_{i,k}$ on $\mathscr{M}^s_P$.

Chapter 5: The Quillen Metric

In this chapter, we apply the relative zeta function regularization, which was first proposed by Müller [89], to define a relative regularized determinant $\det(\Delta_\gamma, \Delta_A)$ for the pair of Dolbeault Laplacians Δ_γ associated with any $\bar{\partial}_\gamma \in U^{0,1}_{A,\epsilon}$ and the reference operator Δ_A. Such construction have been studied extensively, see e.g. [86], [48], [24], [70], and [19].

5.1 Formal Definition of the Relative Determinant

Suppose a pair of non-negative self-adjoint operators A and A_0 on a separable Hilbert space $\mathscr{H}$ satisfies that

1. $e^{-tA} - e^{-tA_0}$ is of trace class for any $t > 0$.

2. As $t \to 0$, there exists an asymptotic expansion of the form

$$\mathrm{Tr}(e^{-tA} - e^{-tA_0}) \sim \sum_{j=0}^{\infty} \sum_{k=0}^{k(j)} a_{j,k} t^{\alpha_j} (\log t)^k,$$

where $-\infty < \alpha_0 < \alpha_1 < \cdots$ and $\alpha_j \to \infty$. In particular, we assume that $\alpha_{j,k} = 0$ if $\alpha_j = 0$ and $k \geq 1$.

3. As $t \to \infty$, there exist $h \in \mathbb{C}$ and $c > 0$ such that

$$\operatorname{Tr}(e^{-tA} - e^{-tA_0}) \sim h + O(e^{-ct}).$$

We can define

Definition 78. The relative zeta function of the pair of operators A and A_0 is given by

$$\zeta(A, A_0, s) = \frac{1}{\Gamma(s)} \int_0^\infty \operatorname{Tr}(e^{-tA} - e^{-tA_0})\, t^{s-1}\, dt,$$

which is a meromorphic function on $\mathbb{C}$. In particular, $\zeta(A, A_0, s)$ is holomorphic at $s = 0$. The relative determinant of A and A_0 is then defined as

$$\det(A, A_0) = \exp\Big(- \frac{d}{ds}\zeta(A, A_0, s)\Big|_{s=0}\Big).$$

Note that the existence of a spectral gap implies condition (3), that is

Lemma 79. [89, Lemma 2.2.] Suppose that $P_{\text{ess}}(A_0) \subset [c, \infty)$ where $c > 0$. Then $\ker A$ and $\ker A_0$ are both finite dimensional and there exists $c_1 > 0$ such that

$$\operatorname{Tr}(e^{-tA} - e^{-tA_0}) = \dim \ker A - \dim \ker A_0 + O(e^{-c_1 t}),$$

as $t \to \infty$.

5.2 Heat Kernels and Estimates

By a theorem of Chernoff [27], the Dolbeault Laplacian Δ_γ associated with any $\bar{\partial}_\gamma = \bar{\partial}_A + \gamma \in U^{0,1}_{A,\epsilon}$ is essentially selfadjoint on $L^2(\operatorname{End} E)$, see Theorem 133. By spectral theorem, we can construct the heat semigroup $e^{-t\Delta_\gamma}$. Its heat kernel $K_{\Delta_\gamma}(t;z,z')$ is a (smooth) family of sections of $\operatorname{End} E \boxtimes \operatorname{End} E^*$ depending on $t > 0$ and satisfies the following properties

1. It is C^1 in the time variable t and C^2 in the space variables z, z';

2. Denote by $\Delta_{\gamma,z}$ the Laplacian Δ_γ acting on the variable z, then

$$\frac{d}{dt} K_{\Delta_\gamma}(t;z,z') + \Delta_{\gamma,z} K_{\Delta_\gamma}(t;z,z') = 0;$$

3. For any compactly supported smooth section s of $\operatorname{End} E$, we have

$$\lim_{t \to 0} \int_X K_{\Delta_\gamma}(t;z,z') s(z') dv_g(z') = s(z).$$

   Since X is noncompact, in order to obtain uniqueness, we will also require that $K_{\Delta_\gamma}(t;z,z')$ is "good" (cf. [39, Page 488, P.4.]);

4. For any $T > 0$ and $0 < t \leq T$, one has

$$\left|\frac{\partial^i}{\partial t^i} \nabla_z^j \nabla_{z'}^k K_{\Delta_\gamma}(t;z,z')\right| < C t^{-1-i-j-k} (i(z) i(z'))^{\frac{1}{2}} e^{\frac{-d(z,z')^2}{8t}},$$

   where $i, j, k \in \mathbb{N}$, C depends only on T, and $i(z)$ is the smoothing of the variable

y along cusp ends, which is defined as

$$i(z) = \begin{cases} 1, & \text{for } z \in M_1 \\ y, & \text{for } z \in \cup_{k=1}^m Z_k. \end{cases}$$

Lemma 80. A smooth kernel $K_{\Delta_\gamma}(z, z', t)$ which satisfies (l)- (4) is uniquely determined by these properties.

Proof. First, we show that for any $f \in C_c^\infty(X, \operatorname{End} E)$,

$$u(z,t) := \int_X K_{\Delta_\gamma}(z, z', t) f(z')\, dv_g(z') \in L^2(\operatorname{End} E).$$

From condition (4) and a substitution $y' = vy$, we have

$$\begin{aligned} \|u(z,t)\|_{L^2}^2 = \int_X |u(z,t)|^2 dv_g(z) &\lesssim t^{-2} \int_1^\infty \left(\int_1^\infty e^{-\frac{\log^2 v}{8t}} |f(vy)| \frac{dv}{v^{\frac{3}{2}}} \right)^2 \frac{dy}{y^3}, \\ &\lesssim t^{-2} \Big(\int_1^\infty \int_1^\infty e^{-\frac{\log^2 v}{4t}} \frac{dv}{v} \frac{dy}{y^3} \Big) \|f\|_{L^2}^2 \\ &\lesssim t^{-\frac{3}{2}} \|f\|_{L^2}^2. \end{aligned}$$

Now, suppose $K_1(t; z, z')$ and $K_2(t; z, z')$ are two kernels satisfying the above conditions (1)-(4), and set $v(z,t) := \int_X (K_1(t; z, z') - K_2(t; z, z')) f(z')\, dv_g(z')$ for any $f \in C_c^\infty(X, \operatorname{End} E)$. We have

$$\frac{1}{2} \frac{d}{dt} \|v(z,t)\|^2 = \langle \frac{d}{dt} v(z,t)\, , v(z,t)\, \rangle = -\|\bar{\partial}_\gamma v\| \le 0.$$

Because of condition (2), the initial value of $v(z,t)$ is 0 and the uniqueness follows.

□

In the remaining part of this section, we present a construction of the heat kernel $K_{\Delta_\gamma}(t;z,z')$ for any $\bar\partial_A+\gamma\in U^{0,1}_{A,\epsilon}$.

5.2.1 Heat Kernels on $\mathbb{H}$

In this part, using the work of Fay [47] and Phong and D'Hoker [34], [35], we exhibit explicit formulae for the heat kernels of the standard Dolbeault Laplacians on the upper half plane $\mathbb{H}$ satisfying certain Gaussian upper bounds.

Let

$$\bar\partial=\partial_{\bar z}d\,\bar z:\Omega^0(\mathbb{H})\to\Omega^{0,1}(\mathbb{H}),$$

and

$$\bar\partial^*=-2y^2\partial_z d\,z:\Omega^{0,1}(\mathbb{H})\to\Omega^{1,1}(\mathbb{H})\cong\Omega^0(\mathbb{H}),$$

denote the standard Cauchy-Riemann operator and its adjoint on $\mathbb{H}$, where we have used the Hodge duality in the last isomorphism, one obtains the following non-negative self-adjoint Laplacians

$$\Delta_{\mathbb{H},0}=2\bar\partial^*\bar\partial=-y^2(\frac{d^2}{d\,x^2}+\frac{d^2}{d\,y^2}),$$

and

$$\Delta_{\mathbb{H},1}=2\bar\partial\bar\partial^*=-y^2(\frac{d^2}{d\,x^2}+\frac{d^2}{d\,y^2})-2iy(\frac{d}{d\,x}-i\frac{d}{d\,y}).$$

For $n = 0, 1$, we define the following self-adjoint second order operators

$$D_n := -y^2\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2}\right) + 2iny\frac{d}{dx},$$

acting on $\Omega^0(\mathbb{H})$ and the anti-linear isometries

$$I_n : f(z)d\bar{z}^n \mapsto y^n \bar{f}(z).$$

By simple computation, we have

$$D_n y^n \bar{f}(z) = y^n \Delta_{\mathbb{H},n} \bar{f}(z), \ n = 0, 1. \tag{5.1}$$

We will denote by $K_{\Delta_{\mathbb{H},n}}(t; z, z')$ and $K_{D_n}(t; z, z')$ the related heat kernels of $\Delta_{\mathbb{H},n}$ and D_n.

Theorem 81 (see e.g. [84], [47], [34] [35])**.** Let $d = d(z, z')$ denote the hyperbolic distance function,

$$K_{D_n}(t; z, z') = \frac{\sqrt{2}e^{-\frac{t}{4}}}{(4\pi t)^{\frac{3}{2}}} \int_d^\infty \frac{ue^{-\frac{u^2}{4t}}}{\sqrt{\cosh(u) - \cosh(d)}} \mathrm{T}_{2n}\left(\frac{\cosh(\frac{u}{2})}{\cosh(\frac{d}{2})}\right) du,$$

where $\mathrm{T}_{2n}(t)$ is the $2n$-th Chebyshev polynomial.

Remark 82. [33, page 29.] In the case of $n = 0$, this formula was proved by McKean in [84], and was later generalized by Fay in [47] to arbitrary $n \in \mathbb{Z}$. In [34], Phong and D'Hoker noted that the complicated factor appearing in the integrand of Fay's formula can be simplified in terms of the Chebyshev polynomial, but there was a mistake as the contribution of the discrete spactrum was computed twice, which was pointed out by Fay to Phong and D'Hoker in [35] on page 1004.

We record the following technical lemma.

Lemma 83. For any $a > 0$, and $b, d, l \in \mathbb{R}$, we have

$$\int_d^l e^{-ax^2+bx}\,dx = \frac{e^{b^2/4a}}{\sqrt{a}} \int_{\sqrt{a}(d+b/2a)}^{\sqrt{a}(l+b/2a)} e^{-v^2+\frac{b}{\sqrt{a}}v}\,dv \leq \sqrt{\pi}\frac{e^{b^2/4a}}{\sqrt{a}}.$$

Proof. This follows from a simple change of variable computation. □

Proposition 84. For $n = 0, 1$ and each $T > 0$, there exists constant $C > 0$ depending only on T such that for $0 < t \leq T$, we have

$$|(\frac{d}{dt})^i \nabla_z^j \nabla_{z'}^k K_{D_n}(t; z, z')| < Ct^{-1-i-j-k} e^{\frac{-d(z,z')^2}{8t}},$$

where $i, j, k \in \mathbb{N}$.

Proof. The case of $n = 0$ and $i = j = k = 0$ is proved in [23], Lemma 7.4.26. We can adapt the proof to the case $n = 1$ and $i = j = k = 0$ since

1. $\mathrm{T}_{2n}(t) \approx t^{2n}$ for any $t \geq 1$;
2. $\frac{\cosh(\frac{u}{2})}{\cosh(\frac{d}{2})} \approx e^{u-d}$;

The result then follows from Lemma 83 and induction on i, j, k. □

As to the heat kernel $K_{\Delta_{\mathbb{H},n}}(t; z, z')$, we have

Lemma 85. For $t > 0$, and $z, z' \in \mathbb{H}$,

$$K_{\Delta_{\mathbb{H},n}}(t; z, z') = \frac{\operatorname{Im}(z)^n}{\operatorname{Im}(z')^n} K_{D_n}(t; z, z').$$

Proof. By 5.1 and the fact that $K_{D_n}(t; z, z')$ is real valued, we have

$$\begin{aligned} \frac{d}{dt} \frac{\operatorname{Im}(z)^n}{\operatorname{Im}(z')^n} K_{D_n}(t; z, z') &= \frac{\operatorname{Im}(z)^n}{\operatorname{Im}(z')^n} D_{n,z} K_{D_n}(t; z, z') \\ &= (\Delta_{\mathbb{H},n})_z \frac{\operatorname{Im}(z)^n}{\operatorname{Im}(z')^n} K_{D_n}(t; z, z'). \end{aligned}$$

Hence it solves the heat equation for $\Delta_{\mathbb{H},n}$. It is easy to see that $K_{\Delta_{\mathbb{H},n}}(t; z, z')$ satisfies the initial condition. By Proposition 84 and Corollary 85.1, these kernels are "good" and this completes the proof. □

Corollary 85.1. For $n = 0, 1$, for each $T > 0$, there exists constant $C > 0$ such that for $0 < t \leq T$, we have

$$|(\frac{d}{dt})^i \nabla_z^j \nabla_{z'}^k K_{\Delta_{\mathbb{H},n}}(t; z, z')| < C t^{-1-i-j-k} e^{\frac{-d(z,z')^2}{8t}},$$

where $i, j, k \in \mathbb{N}$.

Proof. This follows from Lemma 84 and the observation that $\frac{y}{y'} \leq e^{d(z,z')}$, which results from Lemma 12. □

Remark 86. Donnelly [39] presented a construction of the above heat kernels on $\mathbb{H}$ using the standard parametric construction as in [11], which is justified by the fact that $\mathbb{H}$ is of bounded geometry and the finite speed of propagation technique as used in [25].

5.2.2 Heat Kernels on the Cusp

By Lemma 19, there exists a temporal gauge for each $\bar{\partial}_\gamma = \bar{\partial}_A + \gamma \in U^{0,1}_{A,\epsilon}$ and the dependence is smooth. In the following, we will construct the heat kernel associated with the model Dolbeault operator (2.6), denoted as $K_{\Delta_{Z,n}}(t; z, z')$, on the complete cusp Z.

Let Γ denote $\pi_1(Z) \cong \mathbb{Z}$, with its generator denoted by $P : z \mapsto z + 2\pi$. Let $\rho : \pi_1(Z) \to U(n)$ be the unitary representation such that $\rho(P)$ is given by (2.8). Given the covering map $\pi : \mathbb{H} \to Z$, the endomorphism bundle $\operatorname{End} E$ on Z can be identified with

$$\operatorname{End} E \cong \mathbb{H} \times_{\operatorname{Ad}\rho} \operatorname{End}(\mathbb{C}^n), \quad (z \cdot P, s) \sim (z, \operatorname{Ad}\rho(P)s),$$

hence we can define

$$K_{\Delta_{Z,n}}(t; \bar{z}, \bar{z}') = \sum_{j \in \mathbb{Z}} K_{\Delta_{\mathbb{H},n}}(t; z, P^j z') \operatorname{Ad}\rho(P^j). \tag{5.2}$$

for any $z, z' \in \mathbb{H}$ with their images denoted by $\bar{z}, \bar{z}' \in Z$, It is clear that these kernels satisfy the heat equation. Hence it remains to show that they are "good" as defined in 5.2 (4).

Lemma 87. (c.f. [87, Lemma 5.3]) For any $T, a > 0$, there exists constant $C > 0$

such that for $0 < t \le T$, and $z, z' \in \mathbb{H}$ that projects to $\bar{z}, \bar{z}' \in \mathbb{S}^1 \times [a, \infty)$, we have

$$\sum_{j \in \mathbb{Z}} e^{-\frac{\mathrm{d}(z, P^j z')^2}{4t}} \le C(yy')^{\frac{1}{2}} e^{-\frac{\mathrm{d}(z,z')^2}{8t}}.$$

Proof. First, we estimate

$$\mathrm{N}(d) := \mathrm{N}(z, z', d) = |\{j \in \mathbb{Z} \,|\, d(z, P^j z') < d\}|,$$

for $d > 0$. That is, $\mathrm{N}(d)$ counts the number of geodesic paths on $\mathbb{H}_{y \ge a}$ connecting z and $P^j z'$ with length at most d. Given any point $z = x + iy$, let $B(z, d)$ denote the ball of radius d centered at z, and let $B(z)$ denote the hyperbolic rectangle $[x - \frac{1}{2}, x + \frac{1}{2}] \times [y/2, 2y]$. By our definition of $B(z')$, we have $B(z') \cap P^j B(z') = \varnothing$ if $j \ne 0$. Assume that d is larger than the diameter of $B(z')$, then if $B(z, d)$ contains p Γ-translations of z', $B(z, 2d)$ would contain at least p non-overlapping Γ-translations of $B(z')$. Using the following facts that

1. The diameter of $B(z')$ is uniformly bounded on $\mathbb{H}_{y \ge a}$;

2. The volume of $B(z, d)$ is uniformly bounded by $c_1 e^{c_2 d}$ for some constants $c_1, c_2 > 0$;

3. The volume of $B(z')$ is uniformly bounded by $c_3 y'$ for some constant $c_3 > 0$.

Combining these altogether, there exists constants $c_4, c_5 > 0$ such that

$$\mathrm{N}(d) \le \mathrm{vol}(B(z, d)) / \mathrm{vol}(B(z')) \le c_4 y' e^{c_5 d}. \tag{5.3}$$

Now, choose $d > \text{diam}(B(z'))$. using estimate (5.3), we have

$$\sum_{j\in\mathbb{Z}} e^{-\frac{\mathrm{d}(z,P^j z')^2}{4t}} = \sum_{n=0}^{\infty} \sum_{nd\leq d(z,P^j z'\leq(n+1)d)} e^{-\frac{\mathrm{d}(z,P^j z')^2}{4t}} \tag{5.4}$$

$$\leq c_6 y' \sum_{n=0}^{\infty} e^{c_7(n+1)d - \frac{n^2 d^2}{8t} - \frac{d^2(z,z')}{8t}} \tag{5.5}$$

Now, using the fact that

$$\sum_{n=0}^{\infty} e^{c_8 nd - \frac{n^2 d^2}{8t}} \lesssim \int_0^{\infty} e^{cxd - \frac{x^2 d^2}{8t}} \lesssim \frac{1}{d} e^{c_9 t}.$$

This implies that

$$\sum_{j\in\mathbb{Z}} e^{-\frac{\mathrm{d}(z,P^j z')^2}{4t}} \leq c_{10} e^{c_7 d} y' e^{-\frac{d^2(z,z')}{8t}},$$

for $0 < t \leq T$ and $z, z' \in \mathbb{H}_a$. Finally, for $y' \geq a$, the diameter of $B(z')$ is uniformly bounded, and use symmetry between z and z', we get our desired estimate. □

By (5.2), the fact that $\mathrm{Ad}\,\rho$ is unitary, and Lemma 87, we obtain

$$|K_{\Delta_{Z,n}}(t;\bar{z},\bar{z}')| \leq Ct^{-\frac{n}{2}} \sum_{j\in\mathbb{Z}} e^{-\frac{\mathrm{d}(z,P^j z')^2}{4t}}$$

$$\leq C' t^{-\frac{n}{2}} (yy')^{\frac{1}{2}} e^{-\frac{d^2(z,z')}{8t}}$$

The estimates of higher derivatives of $K_{\Delta_{Z,n}}(t;\bar{z},\bar{z}')$ follows from the estimate in [11, p. 86]. To summarize, we have proved

Proposition 88. The heat kernels $K_{\Delta_{Z,n}}(t;\bar{z},\bar{z}')$ of the Dolbeault Laplacians $\Delta_{Z,n}$, $n = 0, 1$, is given by (5.2). In particular, for any $T > 0$ and $a >$, there exist constants

$C, c > 0$ such that for $0 < t \le T$ and $\mathrm{Im}(z), \mathrm{Im}(z') \ge a$, we have

$$|\frac{\partial^i}{\partial t^i} \nabla_z^j \nabla_{z'}^k K_{\Delta_{Z,n}}(t; z, z')| \le C\, t^{-1-i-j-k} (yy')^{\frac{1}{2}} \exp(-\frac{c\, d^2(z, z')}{t}). \tag{5.6}$$

for $i, j, k \in \mathbb{N}$.

Remark 89. With respect to the fixed temporal frame along each cusp end, a good local parametrix of $K_{\Delta_\gamma, n}(t; z, z')$ on X is provided by

$$K^{Z_k}_{\Delta_{\gamma,n}}(t; z, z') \coloneqq U_\gamma K_{\Delta_{Z,n}}(t; z, z') U_\gamma^{-1},$$

along the cusp end where U_γ is a unitary endomorphism which depends on γ and the choice of base point x for $\pi_1(X, x)$ smoothly. We will see that since any two $\bar{\partial}_{\gamma_1}$ and $\bar{\partial}_{\gamma_2}$ in $U^{0,1}_{A,\epsilon}$ are unitary gauge equivalent, in particular, their heat trace along each cusp Z_k agrees up to $O(t^\infty)$, which follows from a finite speed of propagation argument which essentially says that the long range contribution from the compact part of the surface to the cusp ends is negligible when t is small.

5.2.3 Heat Kernels on Riemann Surfaces with Cusp Ends

In this part, we will restrict our attention to the heat kernel of $\Delta_\gamma = \Delta_{\gamma,0} = \bar{\partial}_\gamma^* \bar{\partial}_\gamma$, as the case of $\Delta_{\gamma,1}$ is similar.

First, we construct a good interior parametrix $Q^N_{\Delta^+_\gamma}(t; z, z')$ for the heat kernel $K_{\Delta_\gamma}(t; z, z')$ associated with any $\bar{\partial}_\gamma = \bar{\partial}_A + \gamma \in U^{0,1}_{A,\epsilon}$.

For any $\ell > 0$, we define the truncated surface of X at level ℓ by

$$M_\ell = X \setminus \bigcup_{k=1}^{m} Z_{k,y>\ell}, \tag{5.7}$$

with its injectivity radius $\rho(\ell) > 0$. Let $f(a,b)$ be an increasing smooth function on $\mathbb{R}$ such that $f(x) = 0$ for $x \le a$, and $f(x) = 1$ for $x \ge b$. Set $\psi = f(\frac{1}{2}, 1)$.

For any $N \in \mathbb{N}$, apply the procedure in Theorem 2.26 [11] and we can construct the following parametrix on $M_{\ell+1}$,

$$Q^{N,\ell+1}_{\Delta_\gamma}(t; z, z') = \psi(\frac{d(z,z')}{\rho})\frac{1}{4\pi t}\exp(\frac{-d(z,z')^2}{4t})\sum_{i=0}^{N} t^i F_i(z,z'), \tag{5.8}$$

where F_i are smooth sections of $\operatorname{End} E \boxtimes \operatorname{End} E^*$ supported in a neighborhood of the diagonal of $M_{\ell+1} \times M_{\ell+1}$, in particular, $F_0(z,z') = F_{\nabla_\gamma}(z,z')$ is the parallel transport from z' to z with respect to the unitary connection ∇_γ. Moreover,

Proposition 90. [11, Cf. Theorem 2.26]

1. For any $T > 0$ and $k \in \mathbb{N}$, the kernel $Q^{N,\ell+1}_{\Delta_\gamma}(t; z, z')$ define a uniformly bounded family of operators on $C^k(X_{\ell+1}, \operatorname{End} E)$ such that

$$\|Q^{N,\ell+1}_{\Delta_\gamma}(s) - s\|_{C^k} \to 0,\ s \in C^k(X_{\ell+1}, \operatorname{End} E),$$

over any compact subset of $X_{\ell+1}$.

2. The "defect" $r^{N,\ell+1}(t;z,z') := (\frac{\partial}{\partial t} + \Delta_{\gamma,z})Q^{N,\ell+1}_{\Delta_\gamma}(t;z,z')$ satisfies

$$|\frac{\partial^i}{\partial t^i}\nabla^j_z\nabla^k_{z'}(t;z,z')r^{N,\ell+1}(t;z,z')| \le Ct^{N-1-i-k-j},$$

on any compact subset of $X_{\ell+1}$ supported away from the boundary.

Remark 91. In Proposition 90, we are a little sloppy in not imposing any boundary condition at $\partial X_{\ell+1}$. By the principle of not feeling the boundary by Kac [71] and the fact that we will be gluing using some cut off function supported away from the boundary of $X_{\ell+1}$, this won't cause a problem.

We now construct a parametrix of the heat kernel $K_{\Delta_\gamma}(t;z,z')$ on X with good estimates. Our treatment here follows that of [18] and [11].

Let $\Phi_1, \Phi_2, \Psi_1, \Psi_2$ be the following gluing functions on Z satisfying

1. $\Phi_1 = f(\ell, \ell+1/4)$; $\Psi_1 = f(\ell+3/8, \ell+5/8)$;
2. $\Phi_2 = 1 - f(\ell+3/4, \ell+1)$; $\Psi_2 = 1 - \Psi_1$.
3. $\mathrm{dist}(\mathrm{supp}(\nabla\Phi_i), \mathrm{supp}\,\Psi_i) \ge 1/8$.

We can define

$$K^N_{\Delta_\gamma}(t;z,z') = \Phi_1(z)Q^{N,\ell+1}_{\Delta_\gamma}(t;z,z')\Psi_1(z') + \sum_{k=1}^{m}\Phi_2(z)K^{Z_k}_{\Delta_\gamma}(t;z,z')\Psi^{(k)}_2(z'). \quad (5.9)$$

with its "defect" denoted by

$$R^N_{\Delta_\gamma}(t;z,z') = (\frac{\partial}{\partial t} + \Delta_{\gamma,z})K^N_{\Delta_\gamma}(t;z,z').$$

By its construction, $R^N_{\Delta_\gamma}(t;z,z')$ vanishes whenever $z,z' \in \mathring{Z}_{k,y\geq \ell+1}$ and $d(z,z') > \rho(\ell)$, hence it satisfies similar Gaussian type upper bound as given in Proposition 84. Because of the following elementary inequality

$$\frac{d(x,z)^2}{4t} \leq \frac{d(x,y)^2}{4s} + \frac{d(y,z)^2}{4(t-s)},\ 0 < s < t,$$

the Volterra series construction as used in Theorem 2.19 [11] is still valid. Therefore we get,

Theorem 92 (cf. [11], Theorem 2.23)**.** Set $N > k+1$.

1. The following Volterra series

$$K^N_{\Delta_\gamma}(t;z,z') + \sum_{i=1}^{\infty}(-1)^i (K^N_{\Delta_\gamma} * R^{N,i}_{\Delta_\gamma})(t;z,z'), \tag{5.10}$$

   converges to $K_{\Delta_\gamma}(t;z,z')$ in the C^k-norm on X. Here

$$R^{N,1}_{\Delta_\gamma} := R^N_{\Delta_\gamma},\ R^{N,i}_{\Delta_\gamma} = R^N_{\Delta_\gamma} * R^{N,i-1}_{\Delta_\gamma},$$

   and $\alpha * \beta(t;z,z')$ denotes the convolution

$$\alpha * \beta(t;z,z') = \int_0^t \int_X \alpha(s;z,w)\beta(t-s;w,z')\, dw ds.$$

2. For any $T > 0$, there exists a constant $C, c > 0$ such that

$$|K_{\Delta_\gamma}(t;z,z')| \leq Ct^{-1}(i(z)i(z'))^{\frac{1}{2}} e^{-C'\frac{d^2(z,z')}{t}}, \tag{5.11}$$

where $i(z)$ is the smoothing of the variable y along cusp ends, which is defined as

$$i(z) = \begin{cases} 1, & \text{for } z \in M_1 \\ y, & \text{for } z \in \cup_{k=1}^{m} Z_k. \end{cases}$$

3. The kernel $K^N_{\Delta_\gamma}$ approximates K_{Δ_γ} in the sense that

$$|\frac{\partial^l}{\partial t^l}(K^N_{\Delta_\gamma} - K_{\Delta_\gamma})(t; z, z')| \sim O(t^{N-l-k}), \tag{5.12}$$

   for small time $t > 0$.

4. Moreover, for $0 < t \leq T$, there exists a constant $C > 0$ such that

$$|\sum_{i=1}^{\infty}(-1)^i(K^N_{\Delta_\gamma} * R^{N,i}_{\Delta_\gamma})(t; z, z')| \leq Ct^{N-1}e^{-\frac{d(z,z')^2}{4t}}. \tag{5.13}$$

5.3 Relative Heat Trace

Let R, S, T be bounded operators defined on a separable Hilbert space H, we recall the following basic facts about Hilbert–Schmidt operators and trace-class operators.

1. Let R be a trace-class operator and S a bounded operator, then $T := RS$ is trace class with

$$\|T\|_1 \leq \|R\|_1 \|S\|.$$

2. Let R, S be two Hilbert-Schmidt operators. Then $T := RS$ is of trace-class

with

$$\|T\|_1 \leq \|R\|_2 \|S\|_2,$$

where $\|\cdot\|_1$ and $\|\cdot\|_2$ denote the trace norm and the Hilbert-Schmidt norm.

3. For an integral operator R with integral kernel $r(z, z')$, its Hilbert-Schmidt norm is then given by

$$\|R\|_2 = \int_X \int_X |r(z, z')|^2 \, dv_g(z) \, dv_g(z').$$

5.3.1 Trace Class Property

Without loss of generality, we assume till the end of this chapter that, with respect to the fixed temporal frame along each cusp end Z_k, the Dolbeault operator $\bar{\partial}_A$ is given by the model Dolbeault operator as in (2.6). The main result of this section is the following,

Theorem 93. The relative heat operator

$$e^{-t\Delta_\gamma} - e^{-t\Delta_A}$$

is trace class for any $t > 0$.

To prove this statement, we apply the technique as used by Müller and Salomonsen [90] and the Duhamel principle,

$$e^{-t\Delta_\gamma} - e^{-t\Delta_A} = \int_0^t e^{-s\Delta_\gamma} (\Delta_A - \Delta_\gamma) e^{-(t-s)\Delta_A} \, ds. \tag{5.14}$$

Taking the respective trace norm and Hilbert-Schmidt norm, we get

$$\begin{aligned}\|e^{-t\Delta_\gamma} - e^{-t\Delta_A}\|_1 &\le \int_0^{t/2} \|e^{-s\Delta_\gamma}\| \|(\Delta_\gamma - \Delta_A)e^{-(t-s)\Delta_A}\|_1 \, ds \\ &+ \int_{t/2}^t \|e^{-s\Delta_\gamma}(\Delta_\gamma - \Delta_A)\|_1 \|e^{-(t-s)\Delta_A}\| \, ds \\ &\le \int_0^{t/2} \|(\Delta_\gamma - \Delta_A)e^{-(t-s)\Delta_A}\|_1 \, ds \\ &+ \int_{t/2}^t \|e^{-s\Delta_\gamma}(\Delta_\gamma - \Delta_A)\|_1 \, ds. \end{aligned} \tag{5.15}$$

where the decomposition avoids the singularity $t = 0$. Therefore, we have reduced to question to show the uniform trace-norm estimate of the above two integrands, which we deal in the following lemma.

Lemma 94. (cf. [90, Lemma 6.3]) The following operators

$$(\Delta_\gamma - \Delta_A)e^{-t\Delta_A} \ \text{ and } \ e^{-t\Delta_\gamma}(\Delta_\gamma - \Delta_A)$$

are trace class and their trace norms are uniformly bounded for t in any closed interval of $(0, \infty)$.

We record the following technical lemma that will be used repeatedly below.

Lemma 95. For any $\delta > 0$, $0 < t < T$, $k, l \in \mathbb{Z}$, $c > 0$, we have

$$\int_1^\infty \int_1^\infty e^{-\delta y'} y^k y'^l e^{-\frac{c}{t}\log(y/y')^2} \, dy \, dy' \lesssim \sqrt{t} e^{(1+k)^2 \frac{t}{c}}.$$

Similar result holds when $\delta = 0$, and any $k, l \le 0$ with $k + l < -2$.

Proof. Let $v = \log(y/y')$, then $y = y'e^v$ and such change of variable reduces the integral into

$$\begin{aligned}\int_1^\infty \int_1^\infty e^{-\delta y'} y^k {y'}^l e^{-\frac{c}{t}\log(y/y')^2}\, dy dy' &= \int_1^\infty \int_{-\log y'}^\infty e^{-\delta y'} {y'}^{l+k+1} e^{-\frac{c}{t}v^2+(1+k)v}\, dv dy' \\ &\le \int_1^\infty e^{-\delta y'} {y'}^{l+k+1} dy' \int_{-\infty}^\infty e^{-\frac{c}{t}v^2+(1+k)v}\, dv \\ &\lesssim \sqrt{t} e^{(1+k)^2 \frac{t}{c}}.\end{aligned}$$

where the last inequality follows from Lemma 83. The case of $\delta = 0$ and $k + l < -2$ follows by similar argument. □

Proof. Let M_ϕ and M_ϕ^{-1} denote the multiplication by $\phi(z) := i(z)^{-1/2}$ and $\phi(z)^{-1} = i(z)^{1/2}$, respectively. As the proof of the two cases are analogous, we work the second case here.

By the semi-group property of $e^{-t\Delta_\gamma}$ and the trick of Deift-Simon as in [86] and [69], we have

$$e^{-t\Delta_\gamma}(\Delta_\gamma - \Delta_A) = e^{-t/2\Delta_\gamma} M_\phi \circ M_\phi^{-1} e^{-t/2\Delta_\gamma}(\Delta_\gamma - \Delta_A),$$

and we will thus show that these two factors both have uniform Hilbert-Schmidt norm for t in a compact set of $(0, \infty)$.

Let us start with $e^{-t/2\Delta_\gamma} M_\phi$. Its Hilbert-Schmidt norm is given by

$$\int_X \int_X i(z')^{-1} |K_\gamma(t; z, z')|^2\, dv_g(z) dv_g(z').$$

Given the decomposition of X in (2.3), the above integral can be decomposed into four parts:

1. The short range contribution of each cusp end Z_k is estimated by

$$1/t^2 \int_1^\infty \int_1^\infty y e^{-c/t \log(y/y')^2} \frac{dy}{y^2} \frac{dy'}{y'^2} \lesssim 1/t^{1/2},$$

   where we have used $|\log(y/y')| \leq d(z, z')$ in Lemma 12, the estimate in part 2 of Theorem 92, and Lemma 95. Note that we need the M_ϕ decay in this part.

2. The short range contribution of M is bounded by

$$1/t^2 \int_M \int_M e^{-c\frac{d^2(z,z')}{t}} dv_g(z) dv_g(z') \lesssim 1/t^2,$$

   as a result of the estimate in Proposition 90 and part 3 of Theorem 92

3. The long range contribution from the interaction between any cusp end Z_k and M is estimated by

$$1/t^2 \int_M \int_{\mathbb{S}^1} \int_0^\infty \frac{1}{y'^3} e^{-c\frac{d^2(z,z')}{t}} dy' dx' dz \lesssim 1/t^2 \int_0^\infty \frac{1}{y'^3} e^{-c/t \log(y')^2} dy' \lesssim 1/t^{3/2},$$

   where we have used part 2 of Theorem 92, the fact that $|\log(y')| \lesssim d(\partial M, z') \leq d(z, z')$, and Lemma 95.

4. The long range contribution from the interaction between any two cusp ends Z_k and Z_j can be estimated analogously to the above case.

This concludes that the Hilbert-Schmidt norm of $e^{-t/2\Delta_\gamma} M_\phi$ is uniformly bounded away from $t = 0$.

Remark 96. In [21, page 70 -71], Bunke estimated the above long range contribution using a combination of the semi-group domination principle as in Hess, Schrader, Uhlenbrock in [63] and the heat kernel upper bound from Cheng, Li, and Yau [26], and it is at this point the geometry of the manifold enters the argument, while the short range estimates are localized. For us, due to the explicit heat kernel upper bound in Theorem 92, our estimates are reduced to local computations.

To show that $M_\phi^{-1} e^{-t/2\Delta_\gamma}(\Delta_\gamma - \Delta_A)$ has uniform bounded Hilbert-Schmidt norm for t in any compact subset of $(0, \infty)$, first note that since Δ_γ and Δ_A are self-adjoint, the kernel of $e^{-t\Delta_\gamma}(\Delta_\gamma - \Delta_A)$ is given by $(\Delta_\gamma - \Delta_A)_{z'} K_\gamma(t; z, z')$; Then use Kähler identity, we get

$$\Delta_\gamma - \Delta_A = i\Lambda_g(\gamma^*(z')\bar{\partial}_{A,z'} + \gamma(z')\partial_{A,z'} - \partial_A(\gamma)(z') + \gamma^* \wedge \gamma(z')),$$

which lies in $L^2_{\delta,s+2}$. We can apply the similar argument as above together with Lemma 95 and Hölder inequality to show that $\|M_\phi^{-1} e^{-t/2\Delta_\gamma}(\Delta_\gamma - \Delta_A)\|_2$ is uniformly bounded when t lies in a compact part of $(0, \infty)$. This concludes the second part. $\square$

To sum up, we have finished the proof of Theorem 93. With similar argument, we can also show the following

Proposition 97. Given any section $\beta \in L^{1,2}_{\delta,s+1}(\mathrm{End}\, E)$, the following operator

$$M_\beta(z)\nabla^i_z \nabla^j_{z'} e^{-t\Delta_\gamma}(z,z')$$

is trace class for $t > 0$. Here M_β denotes multiplication by β.

5.3.2 Asymptotics of the Relative Heat Trace

In order to define the relative regularized determinant for the pair of Dolbeault Laplacians Δ_γ and Δ_A, we still need to show that

Proposition 98. There exists an asymptotic expansion of the relative heat trace of the following form,

$$\mathrm{Tr}(e^{-t\Delta_\gamma} - e^{-t\Delta_A}) = a_{-1}\frac{1}{t} + a_0 + a_1 t + O(t^2), \tag{5.16}$$

as $t \to 0$.

Proof. By Theorem 92, to find the asymptotic behavior of $\mathrm{Tr}(e^{-t\Delta_\gamma} - e^{-t\Delta_A})$, we could use the parametrices $K^N_{\Delta_\gamma}(t;z,z')$ and $K^N_{\Delta_A}(t;z,z')$ instead to find the right asymptotic expansion.

On the compact part M_ℓ, the short time asymptotic expansion follow from Theorem 2.30 [11], whereas on each cusp ends, by Remark 89, the relative trace of the above parametrices is therefore of $O(t^\infty)$, hence the result follows. $\square$

Lemma 99. There exists a constant $c > 0$,

$$\mathrm{Tr}(e^{-t\Delta_\gamma} - e^{-t\Delta_A}) \sim e^{-ct}, \tag{5.17}$$

as $t \to \infty$.

Proof. This follows from Lemma 79. □

Combining Lemma 99, Theorem 93, and Proposition 98, based on our discussion of relative determinant in Section 5.1, we may define

Definition 100. For any $\bar{\partial}_\gamma \in U^{0,1}_{A,\epsilon}$, we define the relative zeta function of the pair Δ_γ and Δ_A as

$$\zeta(\Delta_\gamma, \Delta_A, s) = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{Tr}(e^{-t\Delta_\gamma} - e^{-t\Delta_A})\, t^{s-1}\, dt. \tag{5.18}$$

Furthermore, we define the zeta regularized determinant of Δ_γ with respect to Δ_A as

$$\det \Delta_\gamma := \det(\Delta_\gamma, \Delta_A) = \exp\Big(- \frac{d}{ds}\Big|_{s=0} \zeta(\Delta_\gamma, \Delta_A, s)\Big). \tag{5.19}$$

Remark 101. It is clear that we can define the relative determinant $\det(\Delta_\gamma, \Delta_A)$ as a function on $\mathscr{A}^*_\delta$, with Dolbeault Laplacian changed to Bochner Laplacian of the unitary connections. Moreover, such a relative determinant is $\mathscr{G}_\delta$-invariant and descends to a function on $\mathscr{M}^*_{\mathrm{HE}}$. Also, we could have chosen the reference Dolbeault Laplacian associated with any point in $\mathscr{C}^s_\delta$.

Recall that we have shown that the operator $M_\beta e^{-t\Delta_\gamma}$ with $\beta \in L^{1,2}_{\delta,s+1}(\Lambda^{0,1} \otimes$

$\operatorname{End} E)$ is trace class for any $t > 0$. We conclude this part of discussion by investigating the related long time and short time asymptotic behavior of its trace. For the rest of this part, we will assume that $\beta \in L^{1,2}_{\delta,s+1}(\Lambda^{0,1} \otimes \operatorname{End} E) \cap C^\infty(\Lambda^{0,1} \otimes \operatorname{End} E)$ as for our Dolbeault operator $\bar{\partial}_A + \gamma \in U^{0,1}_{A,\epsilon}$, γ always satisfies this assumption. Let us consider the long time behavior first.

Proposition 102. There exists a constant $c > 0$ such that

$$\operatorname{Tr}\left(M_\beta(e^{-t\Delta_\gamma} - P_{\operatorname{Ker}\Delta_\gamma})\right) = O(e^{-ct}),$$

as $t \to \infty$.

Proof. By Theorem 133, the essential spectral of Δ_γ is bounded from below by a constant $C > 0$. Therefore, we have

$$\begin{aligned}\|\left(M_\beta(e^{-t\Delta_\gamma} - P_{\operatorname{Ker}\Delta_\gamma})\right)\|_1 &\leq \|M_\beta e^{-t/2\Delta_\gamma}\|_1 \|e^{-t/2\Delta_\gamma}(1 - P_{\operatorname{Ker}\Delta_\gamma})\| \\ &\lesssim O(e^{-ct}),\end{aligned}$$

where in the last inequality, we have used spectral theorem to infer that $e^{-t/2\Delta_\gamma}(1 - P_{\operatorname{Ker}\Delta_\gamma})$ is a bounded operator with norm bounded by e^{-ct}, where c is the infinimum of nonzero spectrum of Δ_γ. □

On the other hand, as to the short time asymptotic behavior, we have

Proposition 103. The trace of $M_\beta(e^{-t\Delta_\gamma} - P_{\operatorname{Ker}\Delta_\gamma})$ has the following asymptotics

$$\operatorname{Tr}\big(M_\beta(e^{-t\Delta_\gamma} - P_{\operatorname{Ker}\Delta_\gamma})\big) \sim \int_X \operatorname{tr}\big(\beta(z)(\frac{\mathrm{Id}}{4\pi t} + \frac{\Omega_E}{4\pi} + \frac{R_g}{24} - \frac{\mathrm{Id}}{\operatorname{Vol}(X)})\big)\, dv_g(z) + O(t),$$

as $t \to 0$. Here Ω_E denotes the Hermitian-Einstein tensor of E.

Proof. By Theoerm 92, we know that the asymptotic behavior can be determined by the parametrix we constructed, hence the contribution to short time asymptotics splits into two parts.

For the relative compact part $M_\ell \subset M_{\ell+1}$, by Proposition 5.8, Theorem 2.30 [11], and [117], we have the following asymptotic behavior

$$\begin{aligned}&\int_{M_\ell} \operatorname{tr}\big(\beta(z)(Q^{N,\ell}_{\Delta_\gamma}(t;z,z') - \frac{\mathrm{Id}}{\operatorname{Vol}(X)})\big)\, dv_g(z)\\ &\sim \int_{M_\ell} \operatorname{tr}\big(\beta(z)(\frac{\mathrm{Id}}{4\pi t} + \frac{\Omega_E}{4\pi} + \frac{R_g}{24\pi} - \frac{\mathrm{Id}}{\operatorname{Vol}(X)})\big)\, dv_g(z) + O(t),\end{aligned}$$

uniformly on M_ℓ.

For each cusp end Z_k, by the discussion in 5.5.2, we know that

$$\begin{aligned}K^{Z_k}_{\Delta_\gamma}(t;z,z') &= U_\gamma \sum_{j\in\mathbb{Z}} K_{\Delta_{\mathbb{H}}}(t;\tilde z, P^j \tilde z')\, \mathrm{Ad}\,\rho(P^j) U_\gamma^{-1}\\ &= U_\gamma K_{\Delta_{\mathbb{H}}}(t;\tilde z, P^j\tilde z') U_\gamma^{-1} + U_\gamma \sum_{j\neq 0} K_{\Delta_{\mathbb{H}}}(\tilde z, \tilde z' + 2j\pi) U_\gamma^{-1},\end{aligned}$$

where the second term of this expression is of the order $O(t^k)$ for any $k \in \mathbb{N}$. This follows from the fact

$$|K_{\Delta_{\mathbb{H}}}(\tilde z, \tilde z')| \lesssim \frac{1}{t} e^{-c\frac{d(\tilde z, \tilde z')}{t}},$$

and the explicit distance formula on the upper half plane that

$$d((\tilde{x}+m,\tilde{y}),(\tilde{x},\tilde{y})) = \cosh^{-1}(1+\frac{2m^2}{\tilde{y}^2}) \geq \log(1+\frac{2m^2}{\tilde{y}^2}),$$

where the second inequality follows from the fact that $\cosh^{-1}(s) = \log(s+\sqrt{s^2-1})$ when $s > 1$. Then we can apply the argument as in Lemma 87 to get the desired estimate. For the first term in the above expression, by the local expansion of $K_{\Delta_{\mathbb{H}}}(\tilde{z},\tilde{z}')$ as discussed in Theorem 3.3 in Donnelly [39] and the fact that these local expressions are univeral polynomials involving covariant derivatives of the curvature of (X,g) and $(\operatorname{End} E, h)$, see [20] for more details, which vanishes by our assumption of the geometry along the cusp ends. Therefore, we have

$$\begin{aligned}&\int_X \operatorname{tr}\big(M_\beta e^{-t\Delta_\gamma} - \frac{\mathrm{Id}}{\mathrm{Vol}(X)}\big)(t;z,z)\,dv_g(z)\\ \sim &\int_X \operatorname{tr}\big(\beta(z)(\frac{\mathrm{Id}}{4\pi t}+\frac{\Omega_E}{4\pi}+\frac{R_g}{24\pi}-\frac{\mathrm{Id}}{\mathrm{Vol}(X)})\,\mathrm{Id}\big)\,dv_g(z)+O(t),\end{aligned}$$

as $t \to 0$. This completes the proof of the statement. □

Remark 104. Similar asymptotic expansion holds for $\Delta_\gamma^- := \bar{\partial}_\gamma\bar{\partial}_\gamma^*$, that is, as $t \to 0$,

$$\begin{aligned}\operatorname{Tr}\big(M_\beta(e^{-t\Delta_\gamma^-} - P_{\operatorname{Ker}\Delta_\gamma^-})\big) \sim \int_X \operatorname{tr}\big\{\beta(z)\wedge\big((\frac{\mathrm{Id}}{4\pi t}+\frac{\Omega_{E\otimes K^*}}{4\pi}+\frac{R_g}{24\pi})\\ -\sum_i \omega_{\gamma,i}^*(z)\otimes\omega_{\gamma,i}(z)\big)\big\}\,dv_g(z)+O(t),\end{aligned}$$

where $\beta \in L^{1,2}_{\delta,s+1}(\Lambda^{1,0}\otimes\operatorname{End} E)$ and $\{\omega_{\gamma,i}(z)\}$ is any orthonormal basis of $\operatorname{Ker}\Delta_\gamma^-$.

Chapter 6: The Curvature of the Quillen Metric

In this chapter, we apply the method as used by Quillen [98] and study the heat regularization of the Cauchy kernel of $\bar{\partial}_\gamma \in U^{0,1}_{A,\epsilon}$, both in the interior of X and on its cusp ends, to compute the curvature of the canonical line bundle

$$\lambda = \det(T^* V_{A,\epsilon}) \cong \det(\operatorname{ind} \bar{\partial}_\gamma), \tag{6.1}$$

with respect to the Quillen metric

$$\| \cdot \|_Q^2 = \| \cdot \|_{L^2}^2 (\det \Delta_\gamma)^{-1}, \tag{6.2}$$

where $\det \Delta_\gamma$ is the relative determinant, see Definition 100.

We remark that similar computation of the first variation of the Quillen metric interpreted as the anomaly associated with change of certain complex structure has been considered in [101] and [103]. Our treatment of the heat regularization of the Cauchy kernels in the interior of X follows that of [98], [97], and [106].

6.1 First Variation of the Quillen Metric

Let $\bar{\partial}_A + \epsilon_1\mu + \epsilon_2\nu$ be a holomorphic family of Dolbeault operators in $V_{A,\epsilon}$ with $\mu, \nu \in H^{0,1}_{A,\delta}$. By Lemma 67, there exists a unique smooth family of complex gauge transformation $g(\epsilon_1, \epsilon_2)$ close to the identity and a family of Dolbeault operators $\bar{\partial}_\gamma := \bar{\partial}_A + \gamma(\epsilon_1, \epsilon_2)$ in the Hermtian-Einstein slice $U^{0,1}_{A,\epsilon}$ satisfying $\bar{\partial}_\gamma = g \circ (\bar{\partial}_A + \epsilon_1\mu + \epsilon_2\nu) \circ g^{-1}$. With respect to the their action on $\operatorname{End} E$, we have

$$\bar{\partial}_\gamma = \operatorname{Ad} g \circ (\bar{\partial}_A + \epsilon_1\mu + \epsilon_2\nu) \circ \operatorname{Ad} g^{-1}. \tag{6.3}$$

where $\operatorname{Ad} g(\omega) := g \circ \omega \circ g^{-1}$ for any section $\omega \in \Omega^{0,*}(\operatorname{End} E)$. Furthermore, the formal adjoint of $\bar{\partial}_\gamma$ satisfies

$$\bar{\partial}_\gamma^* = \operatorname{Ad} g^{*-1} \circ (\bar{\partial}_A + \epsilon_1\mu + \epsilon_2\nu)^* \circ \operatorname{Ad} g^*, \tag{6.4}$$

In the rest of this chapter, we will denote by $\Delta_\gamma^+ = \bar{\partial}_\gamma^*\bar{\partial}_\gamma$ and $\Delta_\gamma^- = \bar{\partial}_\gamma\bar{\partial}_\gamma^*$. Similarly, we will denote by $P_\gamma^\pm$ the projection to $\ker \Delta_\gamma^\pm$, respectively.

Recall that the relative zeta function associated with the family of Dolbeault Laplacians Δ_γ^+ is given by

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \operatorname{Tr}(e^{-t\Delta_\gamma^+} - e^{-t\Delta_A^+})\, t^{s-1}\, dt. \tag{6.5}$$

In order to compute the curvature of the Quillen metric, we need to compute the following,

$$- \delta_{\bar{\epsilon_2}}\delta_\epsilon \log\det\Delta_\gamma = \delta_{\bar{\epsilon_2}}\delta_\epsilon\zeta'(0). \tag{6.6}$$

Let us start by considering the first variation of $\zeta(s)$ as follows,

$$\begin{aligned}
\delta_{\epsilon_1}\zeta(s) &= \frac{-1}{\Gamma(s)}\int_0^\infty \mathrm{Tr}\{\delta_{\epsilon_1}(\bar\partial_\gamma^*\bar\partial_\gamma)\, e^{-t\bar\partial_\gamma^*\bar\partial_\gamma}\}\, t^s\, dt, \\
&= \frac{1}{\Gamma(s)}\int_0^\infty \mathrm{Tr}\{\mathrm{ad}(g^{*-1}\frac{\partial g^*}{\partial\epsilon_1})\bar\partial_\gamma^*\bar\partial_\gamma\, e^{-t\bar\partial_\gamma^*\bar\partial_\gamma}\}\, t^s\, dt \\
&+ \frac{-1}{\Gamma(s)}\int_0^\infty \mathrm{Tr}\{\mathrm{ad}(g^{*-1}\frac{\partial g^*}{\partial\epsilon_1})\bar\partial_\gamma\bar\partial_\gamma^*\, e^{-t\bar\partial_\gamma\bar\partial_\gamma^*}\}\, t^s\, dt \\
&+ \frac{-1}{\Gamma(s)}\int_0^\infty \mathrm{Tr}\{\mathrm{ad}(\frac{\partial g}{\partial\epsilon_1}g^{-1})\bar\partial_\gamma\bar\partial_\gamma^*\, e^{-t\bar\partial_\gamma\bar\partial_\gamma^*}\}\, t^s\, dt \\
&+ \frac{1}{\Gamma(s)}\int_0^\infty \mathrm{Tr}\{\mathrm{ad}(\frac{\partial g}{\partial\epsilon_1}g^{-1})\bar\partial_\gamma^*\bar\partial_\gamma\, e^{-t\bar\partial_\gamma^*\bar\partial_\gamma}\}\, t^s\, dt \\
&+ \frac{-1}{\Gamma(s)}\int_0^\infty \mathrm{Tr}\{\bar\partial_\gamma^*\,\mathrm{Ad}\, g\,\mathrm{ad}\,\mu\,\mathrm{Ad}\, g^{-1}\, e^{-t\bar\partial_\gamma^*\bar\partial_\gamma}\}\, t^s\, dt,
\end{aligned} \tag{6.7}$$

where we have used the following identities in the second equality,

$$e^{-t\bar\partial_\gamma^*\bar\partial_\gamma}\bar\partial_\gamma^* = \bar\partial_\gamma^*\, e^{-t\bar\partial_\gamma\bar\partial_\gamma^*},$$

and

$$\begin{aligned}
\mathrm{Tr}\{\bar\partial_\gamma^*\,\mathrm{ad}(\frac{\partial g}{\partial\epsilon_1}g^{-1})\bar\partial_\gamma\, e^{-t\bar\partial_\gamma^*\bar\partial_\gamma}\} &= \mathrm{Tr}\{\mathrm{ad}(\frac{\partial g}{\partial\epsilon_1}g^{-1})\,\bar\partial_\gamma e^{-t\bar\partial_\gamma^*\bar\partial_\gamma}\,\bar\partial_\gamma^*\} \\
&= \mathrm{Tr}\{\mathrm{ad}(\frac{\partial g}{\partial\epsilon_1}g^{-1})\bar\partial_\gamma\bar\partial_\gamma^*\, e^{-t\bar\partial_\gamma\bar\partial_\gamma^*}\},
\end{aligned}$$

which follow from the uniqueness of the heat kernels for $\Delta_\gamma^\pm$, see Lemma 80, and Proposition 97. Since on the space complementary to their kernel, the Dolbeault

Laplacians $\Delta_\gamma^\pm$ are invertible, therefore by Proposition 103 and Remark 104, we can perform an integration by parts with respect to t-variable to (6.7) and get the following,

$$\begin{aligned}\delta_{\epsilon_1}\zeta(s) = &-\frac{s}{\Gamma(s)}\int_0^\infty \operatorname{Tr}\{\operatorname{ad}(\frac{\partial g}{\partial\epsilon_1}g^{-1}+g^{*-1}\frac{\partial g^*}{\partial\epsilon_1})\,e^{-t\bar\partial_\gamma^*\bar\partial_\gamma}(1-P_\gamma^+)\}\,t^{s-1}\,dt\\ &+\frac{s}{\Gamma(s)}\int_0^\infty \operatorname{Tr}\{\operatorname{ad}(\frac{\partial g}{\partial\epsilon_1}g^{-1}+g^{*-1}\frac{\partial g^*}{\partial\epsilon_1})\,e^{-t\bar\partial_\gamma\bar\partial_\gamma^*}(1-P_\gamma^-)\}\,t^{s-1}\,dt\\ &+\frac{s}{\Gamma(s)}\int_0^\infty \operatorname{Tr}\{\operatorname{Ad}g\operatorname{ad}\mu\operatorname{Ad}g^{-1}\,e^{-t\bar\partial_\gamma^*\bar\partial_\gamma}(\bar\partial_\gamma^*\bar\partial_\gamma)^{-1}\bar\partial_\gamma^*\}\,t^{s-1}\,dt.\end{aligned}\tag{6.8}$$

Since $\Gamma(s)=1/s+c+O(s)$, $\delta_{\epsilon_1}\zeta'(0)$ equals the constant term of the short time asymptotic expansion of the right hand side of (6.8). By Proposition 103 and Remark 104, such short time asymptotic expansions exist and we get the following,

$$\begin{aligned}\delta_{\epsilon_1}\zeta'(0) = &-\int_X \operatorname{ad}(\frac{\partial g}{\partial\epsilon_1}g^{-1}+g^{*-1}\frac{\partial g^*}{\partial\epsilon_1})(z)(a_0^+(z)-a_0^-(z)-\beta^+(z,z)+\beta^-(z,z))\,dv_g(z)\\ &+\operatorname*{LIM}_{t\to0}\operatorname{Tr}\{\operatorname{Ad}g\operatorname{ad}\mu\operatorname{Ad}g^{-1}\,e^{-t\bar\partial_\gamma^*\bar\partial_\gamma}(\bar\partial_\gamma^*\bar\partial_\gamma)^{-1}\bar\partial_\gamma^*\}.\end{aligned}\tag{6.9}$$

Here $a_0^\pm$ denote the constant terms in the asymptotic expansion of the heat kernels of $\Delta_\gamma^\pm$. $\beta_\gamma^\pm(z,z')=\sum\omega_i^\pm(z)\otimes\omega_i^{\pm*}(z')$ denote the integral kernels of $P_\gamma^\pm$ with $\omega_i^\pm$ being any chosen orthonormal basis of $\ker\Delta_\gamma^\pm$. In particular, the orthonormal basis for $\ker\Delta_\gamma^+$ is given by $\omega_1^+=\frac{\mathrm{Id}}{\sqrt{\mathrm{vol}(X)}}$. By Proposition 103, the second term in (6.9) has a short time asymptotic expansion of the form $a_{-1}t^{-1}+a_0+\cdots$, and the notation $\operatorname*{LIM}_{t\to0}$ indicates that we are to pick out the constant term in the expansion.

Proposition 105. The first variation of the relative zeta function (6.5) with respect to ϵ_1 is given by

$$\delta_{\epsilon_1}\zeta'(0) = -\operatorname{Tr}\{\operatorname{ad}(\frac{\partial g}{\partial \epsilon_1}g^{-1} + g^{*-1}\frac{\partial g^*}{\partial \epsilon_1})P_\gamma^-\} + \operatorname*{LIM}_{t\to 0}\operatorname{Tr}\{\operatorname{ad}(P_\gamma^-(g\mu g^{-1}))\, e^{-t\bar{\partial}_\gamma^*\bar{\partial}_\gamma}(\bar{\partial}_\gamma^*\bar{\partial}_\gamma)^{-1}\bar{\partial}_\gamma^*\}. \tag{6.10}$$

Proof. By Proposition 103, we know that $a_0^+(z) - a_0^-(z) = -\frac{\Omega_{K^*}(z)}{24\pi}$, which is a scalar multiple of the identity map on $\operatorname{End} E$. From the following fact

$$\operatorname{tr}(\operatorname{ad} a \cdot \operatorname{Id}_{\mathrm{End}}) = 0, \tag{6.11}$$

we see that P_γ^- is the only part that contributes non-trivially in the first term of (6.9). For the second part of the expression, note that we have $\operatorname{Ad} g \operatorname{ad}\mu \operatorname{Ad} g^{-1} = \operatorname{ad}(g\mu g^{-1})$, and for any $s \in \ker^\perp(\Delta_\gamma)$, we have

$$\begin{aligned}
(1 - P_\gamma^-)\circ \operatorname{ad}(g\mu g^{-1})(s) &= \bar{\partial}_\gamma \Delta_\gamma^{-1}\bar{\partial}_\gamma^* \circ \operatorname{ad}(g\mu g^{-1})(s), \\
&= -\bar{\partial}_\gamma \Delta_\gamma^{-1}\bar{\partial}_\gamma^*(\operatorname{ad}(g\mu g^{-1}))(s) + \operatorname{ad}(g\mu g^{-1})(\bar{\partial}_\gamma \Delta_\gamma^{-1}\bar{\partial}_\gamma^* s), \\
&= P_\gamma^-(\operatorname{ad}(g\mu g^{-1}))(s), \\
&= \operatorname{ad}(P_\gamma^-(g\mu g^{-1}))(s).
\end{aligned} \tag{6.12}$$

Therefore our desired expression follows. □

6.2 The Heat Regularization

In this part, our objective is to understand the term

$$\operatorname*{LIM}_{t\to 0} \operatorname{Tr}\{\operatorname{ad}(P_\gamma^-(g\mu g^{-1}))\, e^{-t\bar\partial_\gamma^*\bar\partial_\gamma} (\bar\partial_\gamma^*\bar\partial_\gamma)^{-1}\bar\partial_\gamma^*\}. \tag{6.13}$$

Let $G_\gamma(z,z')$ denote the integral kernel of $(\Delta_\gamma^+)^{-1}\bar\partial_\gamma^*$ ($=(\bar\partial_\gamma^*\bar\partial_\gamma)^{-1}\bar\partial_\gamma^*$), which will be identified with a section of $(\operatorname{End} E)_z \otimes (\operatorname{End} E^*\otimes\Lambda^{1,0}X)_{z'}$. By the argument in Part II below, we may define

$$J_\gamma(z) = \operatorname*{LIM}_{t\to 0} \int_X K_{\Delta_\gamma}(t;z,z')G_\gamma(z',z)\, dv_g(z'), \tag{6.14}$$

and it is this term that determines the finite part of (6.13). Since $\Delta_\gamma^{-1}\bar\partial_\gamma^* = \bar\partial_\gamma^{-1}$ on $(\ker\bar\partial_\gamma^*)^\perp$, these two operators share the same parametrix and we will try to understand $G_\gamma(z,z')$ via constructing a parametrix $G_\gamma^\#(z,z')$ of $\bar\partial_\gamma^{-1}$ instead. Moreover, we will show that, by a cancellation mechanism, we have

$$J_\gamma(z) = \lim_{t\to 0} \int_X K_{\Delta_\gamma}(t;z,z')G_\gamma(z',z)\, dv_g(z').$$

and we will use this expression to study the second variation of $\delta_{\epsilon_1}\zeta'(0)$.

6.2.1 A Parametrix of the Dolbeault Operator

Given any $\bar\partial_\gamma \in U_{A,\epsilon}^{0,1}$, $\ell > 0$, and any point $z' \in M_\ell$, we pick a neighborhood $\mathscr{U}_{z'}$ with coordinate z and an orthonormal frame of E on $\mathscr{U}_{z'}$. With respect to these choices,

$$\bar{\partial}_\gamma = (\frac{\partial}{\partial \bar{z}} + \alpha_\gamma) d\bar{z}.$$

By the Cauchy integral formula, the locally integrable function $\frac{1}{z}$ satisfies the distributional equation

$$\frac{\partial}{\partial \bar{z}}(\frac{1}{\pi(z-z')}) = \delta(z-z'),$$

with respect to the volume form $dx \wedge dy$. As the integral kernel of $\Delta_\gamma^{-1} \circ \bar{\partial}_\gamma^*$, $G_\gamma(z,z')$ satisfies

$$(\bar{\partial}_\gamma)_z((z-z')G_\gamma(z,z')) = 0,$$

by Poincaré lemma, $G_\gamma(z,z')$ is of the following form on $\mathscr{U}_{z'}$,

$$G_\gamma(z,z') = \frac{i}{2\pi} \frac{dz'}{(z-z')} F_\gamma(z,z'),$$

where $F_\gamma(z,z')$ is a smooth section of $(\operatorname{End} E)_z \otimes (\operatorname{End} E^*)_{z'}$ close to the identity near the diagonal of $\mathscr{U}_{z'} \times \mathscr{U}_{z'}$. Taking the Taylor expansion around z', we get

$$F_\gamma(z,z') = 1 + A(z')(z-z') + B(z')(\overline{z-z'}) + O((z-z')^2),$$

and

$$\alpha_\gamma(z) = \alpha_\gamma(z') + \partial_z \alpha_\gamma(z')(z-z') + \partial_{\bar{z}} \alpha_\gamma(z')\overline{(z-z')} + O((z-z')^2),$$

because

$$\bar{\partial}_\gamma G_\gamma(z,z') = \delta(z-z'),$$

we find that $B(z') = -\alpha_\gamma(z')$ and in particular,

$$F_\gamma(z,z') = \mathrm{Id} + \beta_\gamma(z')(z-z') - \alpha_\gamma(z')(\overline{z-z'}) + O((z-z')^2),$$

for some smooth section $\beta_\gamma(z') \in \operatorname{End}\operatorname{End} E$ depending holomorphically on α_γ. Moreover, any parametrix of $\Delta_\gamma^{-1} \circ \bar{\partial}_\gamma^*$ is of the following form on $\mathscr{U}_{z'}$:

$$G_\gamma(z,z') = \frac{i}{2\pi}(\underbrace{\frac{1}{z-z'} - \alpha_\gamma(z')\frac{\overline{(z-z')}}{z-z'}}_{\text{singular part}} + \underbrace{\text{const} + \text{terms vanishing at } z = z'}_{\text{regular part}})dz'.$$

Remark 106. The expression of $\beta_\gamma(z')$ depends on the choice of the local coordinate z. For if we take $w = z + cz^2 + \cdots$ to be another coordinate near z', since

$$\frac{dw}{w} = \frac{dz}{z} + c\,dz + O(z),$$

this would add a constant $-c$ to $\beta_\gamma(z')$.

Let ∇_γ denote the Chern connection associated with $\bar{\partial}_\gamma$ on E. The parallel transport $F_{\nabla_\gamma}(z,z') \in \operatorname{Hom}(\operatorname{End} E_{z'}, \operatorname{End} E_z)$ along geodesic ray from z' to z has the following expansion:

$$F_{\nabla_\gamma}(z,z') = \mathrm{Id} + \alpha_\gamma^*(z')(z-z') - \alpha_\gamma(z')(\overline{z-z'}) + O((z-z')^2),$$

and this implies that

$$G^{\#}_{\gamma,z'}(z,z') := \frac{i}{2\pi}\psi(\frac{d(z,z')}{\rho})(\partial_{z'}\log d^2(z,z'))F_{\nabla_\gamma}(z,z') \text{ on } \mathscr{U}_{z'}$$

is a local parametrix of $G_\gamma(z,z')$ near z'. Here, ψ is the cut off function defined in (5.8)

In Theorem 141 in Appendix D, we showed that there exists a local parametrix of G_γ, denoted by $G_{\gamma,k}$, on each cusp end Z_k. Actually, we know that along each cusp end Z_k, the following difference

$$\lim_{z'\to z}|G_{\gamma,k}(z,z') - \frac{i}{2\pi(z-z')}dz| \sim O(1),$$

is smooth and uniformly bounded along each cusp end Z_k. This follows from the fact that the Cauchy kernel on the upper half plane is explicitly given by

$$Q(z,z') = \frac{i}{2\pi}(\frac{1}{z-z'} - \frac{1}{\bar{z}-z'}),$$

with respect to the volume form $dz \wedge d\bar{z}$ and the method of periodization to construct the exact kernel of $\bar{\partial}_\gamma^{-1}$ on Z_k (see Proposition 111 below and the related discussion). Hence we will take $G_{\gamma,k}$ to be the $\frac{i}{2\pi(z-z')}$ in the following.

As M_ℓ is compact, there exists a finite covering $\{\mathscr{U}_{z'_j}\}$ of M_ℓ. Let $\{\psi_{z'_j}(z)\}$ and $\{\psi_k(z)\}$ denote a partition of unity subordinate to the covering $\{\mathscr{U}_{z'_j}\}_j$ together with

$\{Z_{k,y\geq\ell-1}\}_k$. With an abuse of notation, we define the following global parametrix

$$G_\gamma^{\#}(z,z') = \sum G^{\#}_{\gamma,z'_j}(z,z')\psi_{z'_j}(z') + \sum G_{\gamma,k}\psi_k(z'),$$

of $G_\gamma(z,z')$ on X. It is not hard to see that the following lemma holds and hence we will omit its proof.

Lemma 107. There exists a global parametrix $G_\gamma^{\#}(z,z')$ of $\Delta_\gamma^{-1}\bar{\partial}_\gamma^*$ on X, which satisfies that

1. On each $\mathscr{U}_{z'_j}$,

$$G_\gamma^{\#}(z,z') = \frac{i}{2\pi}\psi(\frac{d(z,z')}{\rho})(\partial_{z'}\log d^2(z,z'))F_{\nabla_\gamma}(z,z');$$

2. Along each cusp end Z_k, with respect to the fixed cusp coordinate and temporal gauge, $G_\gamma^{\#}(z,z') = \frac{i}{2\pi}\frac{dz'}{z-z'}$.

3. $\lim_{z'\to z}|G_\gamma(z,z') - G_\gamma^{\#}(z,z')|$ is smooth and uniformly bounded on X

Remark 108. The construction of $G_\gamma^{\#}(z,z')$ clearly depends on the level $\ell > 0$. In the following, we will show that

$$J_\gamma(z) = \lim_{\ell\to\infty}\lim_{z'\to z}(G_\gamma(z,z') - G_\gamma^{\#}(z,z')),$$

exists and is uniformly bounded on X. Moreover, we will find an explicit expression for its limiting value as z goes to infinity along each cusp end.

6.2.2 An Invariant Section J_γ

By Theorem 92, for sufficiently large $N \in \mathbb{N}$, we know that $|K^N_{\Delta_\gamma}(t;z,z') - K_{\Delta_\gamma}(t;z,z')|$ is uniformly bounded and tends to zero near the diagonal uniformly as $t \to 0$. Due to the fact that $G_\gamma(z,z')$ is locally integrable and its behavior along each cusp end, see Lemma 107, it follows that we have

$$\lim_{t\to 0}\int_X |(K_{\Delta_\gamma}(t;z,z') - K^N_{\Delta_\gamma}(t;z,z'))G_\gamma(z',z)|\, dv_g(z') = 0.$$

Therefore,

$$J_\gamma(z) = \operatorname*{LIM}_{t\to 0}\int_X K^N_{\Delta_\gamma}(t;z,z')G_\gamma(z',z)\, dv_g(z'). \tag{6.15}$$

which reduces our computation of J_γ to the interior of X and along the cusp ends. In the following, we analyze the behavior of $J_\gamma(z)$ when either $z \in M_\ell$ for ℓ in any bounded interval of $[1,\infty)$ or $z \in Z_k$ for $k = 1,,m$. The first part was done by Quillen [98] and we provide the details here for the reader's convenience. In the second part, the key is to prove uniform convergence of (6.14) and find explicit expression of J_γ along the cusp ends. These form the key results of this part.

Part I. Given any $\ell > 0$, for any point $z \in M_\ell$, we study the existence and some properties of the limit $J_\gamma(z)$ defined in (6.14).

For any point $z \in M_\ell$, fix a local coordinate z and an orthonormal frame of E as discussed in the construction of $G^\#_\gamma(z,z')$. By definition, the section $J_\gamma(z)$ is the constant term of the following expression,

$$J_\gamma(z) = \underset{t\to 0}{\mathrm{LIM}} \int_{\mathbb{C}} \frac{1}{4\pi t} \exp(-\frac{d^2(z,w)}{4t})\Big(F_{\nabla_\gamma}(z,w) + \cdots + t^N F_N(z,w)\Big)(\frac{dz}{w-z})$$

$$\Big(\mathrm{Id} + \beta_\gamma(z)(w-z) - \alpha_\gamma(z)\overline{(w-z)} + O(|w-z|^2)\Big) g(w) dw \wedge d\bar{w},$$

$$= \underset{t\to 0}{\mathrm{LIM}} \int_{\mathbb{C}} \frac{1}{4\pi t} \exp(-\frac{d^2(z,w)}{4t})(\frac{dz}{w-z})(\mathrm{Id} + \beta_\gamma(z)(w-z) - \alpha^*_\gamma(z)(w-z)$$

$$+ O(|w-z|^2)) g(w) dw \wedge d\bar{w}.$$

Here, $g(z) = |\frac{\partial}{\partial z}|^2$. The expansions of g and $d(z,w)$ is given by

$$g(w) = g(z) + \partial_w g(z)(w-z) + \partial_{\bar{w}g(z)}(\overline{w-z}) + O(|w-z|^2),$$

$$d^2(z,w) = |z-w|^2(g(z) + \frac{1}{2}\partial_w g(z)(w-z) + \frac{1}{2}\partial_{\bar{w}g(z)}(\overline{w-z}) + O(|w-z|^2).$$

To simplify the above computation, we set $z = 0$ and use the substitution $u = \frac{w}{2\sqrt{t}}$, we get

$$J_\gamma(0) = \lim_{t\to 0} \int_{\mathbb{C}} \exp(-g(0)|u|^2)\Big(1 - 2\sqrt{t}|u|^2(\frac{\partial_w g(0)u}{2} + \frac{\partial_{\bar{w}} g(0)\bar{u}}{2}) + O(t)\Big)(\frac{1}{2\sqrt{t}u})$$

$$\Big(1 + 2\sqrt{t}\beta_\gamma(0)u - 2\sqrt{t}\alpha^*_\gamma(0)\bar{u} + O(t)\Big)\Big(g(0) + 2\sqrt{t}(\partial_w g(0)u + \partial_{\bar{w}} g(0)\bar{u}) + O(t)\Big) du \wedge d\bar{u}\frac{idz}{2\pi},$$

$$= \int_{\mathbb{C}} \exp(-g(0)|u|^2)\Big(-|u|^2\frac{\partial_w g(0)}{2} + \beta_\gamma(0) - \alpha^*_\gamma(0)\Big) g(0) du \wedge d\bar{u}\frac{idz}{2\pi},$$

where the singular coefficients involving t vanish because for any $k \in \mathbb{N}$, the following integrals vanish by symmetry,

$$\int_{\mathbb{C}} \frac{e^{-|u|^2}}{u}|u|^k \frac{dx\,dy}{\pi} = 0 \quad \text{and} \quad \int_{\mathbb{C}} e^{-|u|^2}|u|^k \frac{\bar{u}}{u}\frac{dx\,dy}{\pi} = 0. \tag{6.16}$$

Together with the following identities,

$$\int_{\mathbb{C}} e^{-g(0)|u|^2} du \wedge d\bar{u} = \frac{1}{g(0)} \quad \text{and} \quad \int_{\mathbb{C}} e^{-g(0)|u|^2} |u|^2 du \wedge d\bar{u} = \frac{1}{g(0)^2}, \tag{6.17}$$

we conclude that for any $z \in \mathring{M}_\ell$,

$$J_\gamma(z) = (\beta_\gamma(z) - \alpha_\gamma^*(z) + \frac{1}{2}\partial_z \log g(z)) \frac{idz}{2\pi}. \tag{6.18}$$

Proposition 109. Given any $\ell > 0$ and any $z \in M_\ell$, the constant term in (6.14) exists and is a smooth section of $\operatorname{End} E \otimes \operatorname{End} E^* \otimes \Lambda^{1,0} X$ uniformly bounded on M_ℓ. It is locally given by

$$J_\gamma(z) = (\beta_\gamma(z) - \alpha_\gamma^* + \frac{\partial_z g(z)}{2g(z)}) \frac{idz}{2\pi} = (\beta_\gamma(z) - \alpha_\gamma^* + \frac{1}{2}\partial_z \log g(z)) \frac{idz}{2\pi},$$

with such an expression invariant under change of coordinates.

Proof. The exact local expression has been found as above, it remains to prove the rest of the statements. To show the invariance under change of coordinate, note that under any new coordinate $w = z + cz^2 + \cdots$, we have

$$\tilde{g} = |\frac{\partial}{\partial w}|^2 = |\frac{\partial z}{\partial w}|^2 g(z).$$

Hence

$$\partial \log \tilde{g}(z) - \partial \log g(z) = \frac{d^2 w}{dz^2} / \frac{dw}{dz} = 2c.$$

By Remark 106, we finish the proof of invariance of coordinate representation. $\square$

Corollary 109.1. For any given $\ell > 0$ and any $z \in M_\ell$, we have

$$\lim_{t\to 0}\int_X K^{\#}_{\Delta_\gamma}(t;z,z')G^{\#}_\gamma(z',z)\,dv_g(z') = 0. \tag{6.19}$$

In particular, $J_\gamma(z) = \lim_{z'\to z}(G_\gamma(z',z) - G^{\#}_\gamma(z,z'))$ on M_ℓ.

Proof. This follows essentially from the following identities

$$\begin{aligned} F_{\nabla_\gamma}(z,z') &= \mathrm{Id} + \alpha^*_\gamma(z')(z-z') - \alpha_\gamma(z')(\overline{z-z'}) + O((z-z')^2),\\ F_{\nabla_\gamma}(z',z) &= \mathrm{Id} - \alpha^*_\gamma(z')(z-z') + \alpha_\gamma(z')(\overline{z-z'}) + O((z-z')^2), \end{aligned} \tag{6.20}$$

and therefore the product that appears in the regularization (6.19) contains only the singular term $\frac{1}{z-z'}$ and terms of order greater or equal to 2, hence the statement follows from the above computation. $\square$

Part II. Let us first consider the contribution of the singular part of $G_{\gamma,k}$ along the cusp end Z_k. For any $z \in Z_k$ with $\operatorname{Im} z \geq \ell + 1$, we have

$$\operatorname*{LIM}_{t\to 0}\int_X K^{\#}_{\Delta_\gamma}(t;z,w)G^{\#}_\gamma(w,z)dv_g(w) = \operatorname*{LIM}_{t\to 0}\left(\int_{Z_k} K^{Z_k}_{\Delta_\gamma}(t;z,w)(\frac{1}{w-z})\frac{dw}{w^2}\right)\frac{i}{2\pi}dz,$$

and by Lemma 88 and Remark 89, it becomes

$$U_\gamma\cdot\operatorname*{LIM}_{t\to 0}\left(\sum_{j\in\mathbb{Z}}\int_{Z_k} K_{\Delta_{\mathbb{H}}}(t;z,P^jw)\,\mathrm{Ad}\,\rho(P^j)(\frac{1}{w-z})\frac{dw}{w^2}\right)\cdot U_\gamma^{-1}\frac{i}{2\pi}dz. \tag{6.21}$$

We claim that the above expression vanishes as t goes to 0. Moreover, the following expression

$$\int_{Z_k} \operatorname{Ad} g \operatorname{ad} \mu \operatorname{Ad} g^{-1} K_{\Delta_\gamma}^{Z_k}(t; z, w)(\frac{1}{w-z})\frac{dw}{w^2}\frac{i}{2\pi}dz, \tag{6.22}$$

goes to 0 uniformly along the cusp Z_k. To prove these claims, we split (6.21) into two parts:

$$\underset{t\to 0}{\mathrm{LIM}}\left(\int_{Z_k} K_{\Delta_{\mathbb{H}}}(t; z, w)\frac{1}{w-z}\frac{dw}{w^2}\right)\frac{i}{2\pi}dz, \tag{6.23}$$

and

$$\underset{t\to 0}{\mathrm{LIM}}\left(\sum_{j\neq 0}\int_{Z_k} K_{\Delta_{\mathbb{H}}}(t; z, P^j w)\frac{1}{w-z}\frac{dw}{w^2}\right)\cdot U_\gamma \cdot \operatorname{Ad}\rho(P^j)U_\gamma^{-1}\frac{i}{2\pi}dz. \tag{6.24}$$

(6.23) vanishes by exactly the same argument we have used in the computation of J_γ and the fact that $\mathbb{H}$ is of bounded geometry. To estimate (6.24), we use the following estimate

$$|K_{\Delta_{\mathbb{H}}}(t; z, P^j w)| \lesssim \frac{1}{t}e^{-c\frac{d^2(z, w+2j\pi)}{4t}}.$$

and the fact that the distance between (x, y) and $(x+2j\pi, y)$ is given by

$$d((x+2j\pi, y), (x, y)) = \cosh^{-1}(1+\frac{8j^2\pi^2}{y^2}) \geq \log(1+\frac{8j^2\pi^2}{y^2}),$$

where the second inequality follows from the fact that $\cosh^{-1}(s) = \log(s+\sqrt{s^2-1})$

when $s > 1$. Set $a^2 = \frac{y^2}{8\pi^2}$ and use the fact that $\log(1+s) \geq s/2$ when $0 \leq s \leq 1$, we get

$$\begin{aligned}
|\int_{Z_k} K_{\Delta_{\mathbb{H}}}(t; z, P^j w)\frac{1}{w-z}\frac{dw}{w^2}| &\leq \frac{1}{t}e^{-c\frac{\log^2(1+\frac{8j^2\pi^2}{y^2})}{4t}} \int_{Z_k} |\frac{1}{w-z}|\frac{dw}{w^2} \\
&\lesssim \frac{1}{t}e^{-c\frac{\log^2(1+\frac{j^2}{a^2})}{4t}} \\
&\leq \frac{1}{t}e^{-c\frac{1}{8a^4 t}} e^{-c\frac{\log^2(1+\frac{j^2}{a^2})}{8t}}.
\end{aligned}$$

Set $v = \log(x/a)$ and by integral estimate,

$$\begin{aligned}
\sum_{j=1}^{\infty} e^{-c\frac{\log^2(1+\frac{j^2}{a^2})}{4t}} &\leq \int_1^{\infty} e^{-c\frac{\log^2(1+\frac{x^2}{a^2})}{4t}} dx \\
&\leq \int_1^{a} e^{-c\frac{\log^2(1+\frac{x^2}{a^2})}{4t}} dx + \int_a^{\infty} e^{-c\frac{\log^2(1+\frac{x^2}{a^2})}{4t}} dx \\
&\leq (a-1) + \int_1^{\infty} e^{-4c\frac{\log^2(\frac{x}{a})}{4t}} dx \\
&\leq (a-1) + \int_0^{\infty} e^{-4c\frac{v^2}{4t}} e^v dv \\
&< a(1+\sqrt{t}e^{c't}) \lesssim a.
\end{aligned}$$

Therefore, we have shown that

$$\sum_{j=1}^{\infty} |\int_{Z_{k,y\geq \ell+1}} K_{\Delta_{\mathbb{H}}}(t; z, P^j w)\frac{1}{w-z}\frac{dw}{w^2}| \leq \frac{y^2}{t}e^{-\frac{c'}{y^4 t}} \to 0.$$

Though the above expression does not vanish uniformly on Z_k, since $P_\gamma^-(g\mu g^{-1}) \in C^\infty \cap L_\delta^\infty$ (this follows from Lemma 34 and Lemma 37), their product will satisfy that

$$\int_{Z_k} \mathrm{ad}(P_\gamma^-(g\mu g^{-1})) K_{\Delta_\gamma}^{Z_k}(t; z, w)\frac{i}{2\pi}\frac{1}{w-z}\frac{dw}{w^2} dz \to 0,$$

uniformly for any $z \in Z_k$ with $\operatorname{Im} z \geq \ell + 1$. This implies that

$$\begin{aligned} J_\gamma(z) &= \operatorname*{LIM}_{t\to 0} \int_{Z_k} K^{Z_k}_{\Delta_\gamma}(t; z, w)\left(G_{\gamma,k}(w, z) - \frac{i}{2\pi}\frac{1}{w - z}\right)\frac{dw}{w^2} dz. \\ &= \lim_{z'\to z} \left(G_{\gamma,k}(w, z) - \frac{i}{2\pi}\frac{1}{w - z}\right) dz, \end{aligned} \tag{6.25}$$

where in the second equality, we have used the same analysis as above to conclude that only the term $j = 0$ of $K^{Z_k}_{\Delta_\gamma}(t; z, z')$ as discussed above contributes non-trivially. With the aid of (6.25), let us find the explicit behavior of $J_\gamma(z)$ along a cusp end Z_k.

Proposition 110. Along each cusp end Z_k, we have that

$$J_\gamma(z) = U_{\gamma,k}\bigg(\operatorname{sgn}(\alpha_{i,k} - \alpha_{j,k})(\frac{1}{2} - |\alpha_{i,k} - \alpha_{j,k}|)\bigg)_{ij} U^{-1}_{\gamma,k}\frac{dz}{2\pi} + o(1), \tag{6.26}$$

where U_γ is the constant unitary gauge transformation that puts $\bar\partial_\gamma$ into the model operator (2.6). In particular, it depends smoothly on γ and $U_{A,k} = \operatorname{Id}$.

Proof. By Lemma 19, there exists a unitary gauge transformation $U_{\gamma,k}$ along each Z_k that takes $\bar\partial_\gamma$ into the model operator (2.6), that is with respect to the fix a local temporal frame $\{e_{i,k}\}_i$ and any section $f \in C^\infty_c(\operatorname{End} E)$, we have

$$\big(U_{\gamma,k} \circ \bar\partial_\gamma \circ U^{-1}_{\gamma,k}\big) f_{ij,k}(z) e_{i,k} \otimes e^*_{j,k} = (\frac{\partial}{\partial \bar z} + \frac{i}{2}(\alpha_{i,k} - \alpha_{j,k})) f_{ij,k}(z) e_{i,k} \otimes e^*_{j,k} d\bar z,$$

where $f_{ij,k}$ is the ij-th component of f with respect to the fixed temporal frame along Z_k. Therefore, the action of $\bar\partial_{\gamma,k}$ on $\operatorname{End} E$ is diagonalized by $U_{\gamma,k}$. Moreover, the adjoint representation $\operatorname{Ad}\rho$ of its local holonomy is also diagonalized simultaneously by U_γ with respect to the fixed temporal frame. It is easy to see that $U_{\gamma,k}$ depends on γ smoothly and $U_{A,k}$ is the identity by our assumption. As the Cauchy kernel of $\frac{\partial}{\partial \bar{z}}$ on the upper half plane is explicitly given by

$$G(z,z') = \frac{i}{2\pi}\left(\frac{1}{z-z'} - \frac{1}{\bar{z}-z'}\right),$$

hence by Proposition 111, we can take the nonsingular part of its periodization with respect to the vertical strip $[0,2\pi)\times[a,\infty)$ to find an expression of J_γ along the cusp end. That is, for any $(x,y)\in[0,2\pi)\times[a,\infty)$ and in terms of the ij-th component, we have

$$\begin{aligned} J_\gamma(z) &= \left(U_\gamma \sum_{n\neq 0} G(z, z+2n\pi)\operatorname{Ad}\rho^n(P)U_\gamma^{-1}\right)_{ij} - \frac{idz}{2\pi}\left(\frac{1}{\bar{z}-z}\right) \\ &= U_\gamma\left(\sum_{n\neq 0}\left(\frac{1}{2\pi n + 2iy} - \frac{1}{2\pi n}\right)e^{2n\pi i(\alpha_i-\alpha_j)}\right)_{ij} U_\gamma^{-1}\frac{idz}{2\pi} - \frac{idz}{2\pi}\left(\frac{1}{\bar{z}-z}\right) \\ &= U_\gamma\left(\frac{1}{\pi}\sum_{n>0}\frac{\sin 2\pi n(\alpha_i-\alpha_j)}{n} + \sum_{n\neq 0}\frac{e^{2\pi n i(\alpha_i-\alpha_j)}}{2\pi n + 2iy}\right)_{ij} U_\gamma^{-1}\frac{dz}{2\pi} - \frac{idz}{2\pi}\left(\frac{1}{\bar{z}-z}\right) \\ &= U_\gamma\left(\operatorname{sign}(\alpha_i-\alpha_j)\left(\frac{1}{2} - |\alpha_i-\alpha_j|\right)\right)_{ij} U_\gamma^{-1}\frac{dz}{2\pi} + o(1). \end{aligned} \tag{6.27}$$

where we have used the fact that

$$\begin{aligned} \sum_{n>0}\frac{\sin n\alpha}{n} &= \operatorname{Im}\left(\sum_{n=1}^{\infty}\frac{e^{in\alpha}}{n}\right) = \operatorname{Im}(-\log(1-e^{i\alpha})) \\ &= \left(\frac{\pi}{2} - \frac{\alpha}{2}\right) \mod \pi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right]. \end{aligned}$$

and

$$\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2+y^2}\mathrm{d}x = \frac{\pi\exp(-y)}{y}; \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{x\sin(x)}{x^2+y^2}\mathrm{d}x = \pi\exp(-y). \tag{6.28}$$

This in particular gives us the explicit limiting value of J_γ as $y \to \infty$. □

Part III. For $z \in M_{\ell+1} \cap Z_{k,y>\ell}$, by the local integrability of $G_\gamma(z,z')$ and the above analysis, we know that for any z

$$|J_\gamma(z) - \lim_{z'\to z}(G_\gamma(z,z') - G_\gamma^{\#}(z,z'))| \to 0,$$

uniformly as $\ell \to \infty$. Therefore, we have shown that

Proposition 111. The smooth section $J_\gamma \in \operatorname{End} E \otimes \operatorname{End} E^* \otimes \Lambda^{1,0}X$ defined in (6.14) is uniformly bounded. Furthermore, it is given by

$$\begin{aligned} J_\gamma(z) &= \lim_{\ell\to\infty}\lim_{t\to 0}\int_X K_{\Delta_\gamma}(t;z,z')(G_\gamma(z',z) - G_\gamma^{\#}(z,z'))\,dv_g(z'), \\ &= \lim_{\ell\to\infty}\left(G_\gamma(z,z') - G^{\#}(z,z')\right)\Big|_{z=z'}. \end{aligned} \tag{6.29}$$

Corollary 111.1. The first variation of the Quillen metric can be written as

$$\begin{aligned} \delta_{\epsilon_1}\zeta'(0) &= -\operatorname{Tr}\{\operatorname{ad}(\frac{\partial g}{\partial \epsilon_1}g^{-1} + g^{*-1}\frac{\partial g^*}{\partial \epsilon_1}) \circ P_\gamma^-\} + \int_X \operatorname{ad}(P_\gamma^-(g\mu g^{-1})) \wedge J_\gamma, \\ &= -\operatorname{Tr}\{\operatorname{ad}(\frac{\partial g}{\partial \epsilon_1}g^{-1} + g^{*-1}\frac{\partial g^*}{\partial \epsilon_1}) \circ P_\gamma^-\} \\ &\quad + \lim_{\ell\to\infty}\int_{M_\ell} \operatorname{ad}(P_\gamma^-(g\mu g^{-1})) \wedge (G_\gamma(z,z') - G^{\#}(z,z'))\big|_{z=z'}. \end{aligned} \tag{6.30}$$

We conclude this section with the following property of J_γ first noted by

Quillen [97].

Lemma 112. (cf. [97]) For any $\ell > 0$, the restriction of J_γ on M_ℓ, as a section of $\operatorname{End} E \otimes \operatorname{End} E^* \otimes \Lambda^{1,0} X$, satisfies

$$\bar{\partial}_\gamma(J_\gamma) = \frac{1}{2}\omega_g,$$

where ω_g is the curvature form associated with g. Moreover, $\bar{\partial}_\gamma(J_\gamma)(z)$ is uniformly bounded on each cusp end Z_k.

Proof. The first part of this Lemma is a local computation and we work on a small neighborhood $\mathscr{U}_z$ around z. Take a gauge transformation locally that takes $\bar{\partial}_\gamma = (\frac{\partial}{\partial \bar{z}} + \alpha_\gamma)d\bar{z}$ to the standard Cauchy-Riemann operator $\frac{\partial}{\partial \bar{z}}d\bar{z}$, this implies that

$$\alpha_\gamma = -\frac{\partial}{\partial \bar{z}} g g^{-1}. \tag{6.31}$$

Moreover, in this new frame, $F(z,z')$ is replaced by $g(z)^{-1}F(z,z')g(z')$. As the fundamental solution of $\frac{\partial}{\partial \bar{z}}d\bar{z}$, we have

$$\frac{\partial}{\partial \bar{z}} \frac{i}{2\pi} \frac{g(z)^{-1}F(z,z')g(z')}{z - z'} = \delta(z - z'). \tag{6.32}$$

In particular, (6.32) implies that $g(z)^{-1}F(z,z')g(z')$ is a holomorphic function of the z-variable, and together with (6.31), implies that

$$\bar{\partial}_\gamma \beta_\gamma = \frac{\partial}{\partial \bar{z}}\beta_\gamma + [\alpha_\gamma, \beta_\gamma] = \frac{\partial}{\partial z}\alpha_\gamma.$$

Since $\bar{\partial}_\gamma(\alpha_\gamma^*) = \bar{\partial}\alpha_\gamma^* + [\alpha_\gamma, \alpha_\gamma^*]$, we therefore find that

$$\bar{\partial}_\gamma(\beta_\gamma - \alpha_\gamma^*) = F(\nabla_\gamma) = 0.$$

The result now follows from the fact that the curvature form of (X, g) is given by $\omega_g = -\partial_{\bar{z}}\partial_z \log|\frac{\partial}{\partial z}|^2$.

As to the second part of the Lemma, this follows from Proposition 110, and the estimate (6.28), in particular. □

Remark 113. Note the analogy of Lemma 112 to the following index formula:

$$\operatorname{ind} F = \operatorname{trace}[F, G],$$

where $F : X \to Y$ are Fredholm operator between Hilbert spaces with its pseudoinverse G satisfying $1 - GF$ and $1 - FG$ are both trace class operator.

6.3 Second Variation of the Quillen Metric

With the previous preparation, we are ready to evaluate the following,

$$
\begin{aligned}
&\frac{\partial^2}{\partial\bar{\epsilon}_2\partial\epsilon_1}\log\det\Delta_\gamma\Big|_{\epsilon_1=\epsilon_2=0} = -\frac{\partial^2}{\partial\bar{\epsilon}_2\partial\epsilon_1}\zeta'(0)\Big|_{\epsilon_1=\epsilon_2=0},\\
=&\frac{\partial}{\partial\bar{\epsilon}_2}\,\mathrm{Tr}\{\mathrm{ad}(\frac{\partial\,g}{\partial\,\epsilon_1}g^{-1}+g^{*-1}\frac{\partial\,g^*}{\partial\,\epsilon_1})\circ P_\gamma^-\}\Big|_{\epsilon_2=0} - \frac{\partial}{\partial\bar{\epsilon}_2}\int_X \mathrm{ad}(P_\gamma^-(\mathrm{Ad}\,g(\mu)))\wedge J_\gamma\Big|_{\epsilon_2=0},\\
=&\frac{\partial}{\partial\bar{\epsilon}_2}\,\mathrm{Tr}\{\mathrm{ad}(\frac{\partial\,g}{\partial\,\epsilon_1}g^{-1}+g^{*-1}\frac{\partial\,g^*}{\partial\,\epsilon_1})\circ P_\gamma^-\}\Big|_{\epsilon_2=0}\\
&-\lim_{\ell\to\infty}\left(\frac{\partial}{\partial\bar{\epsilon}_2}\int_{M_\ell}\mathrm{ad}(P_\gamma^-(g\mu g^{-1}))\wedge(G_\gamma(z,z')-G_\gamma^{\#}(z,z'))\big|_{z=z'}\Big|_{\epsilon_2=0}\right),
\end{aligned}
\tag{6.33}
$$

The following technical lemmas will be needed.

Lemma 114.

$$
\frac{\partial}{\partial\bar{\epsilon}_2}\,\mathrm{Tr}\left\{\mathrm{ad}(\frac{\partial\,g}{\partial\,\epsilon_1}g^{-1}+g^{*-1}\frac{\partial\,g^*}{\partial\,\epsilon_1})P_\gamma^-\right\}\Big|_{\epsilon_2=0} = -\,\mathrm{Tr}\left(\mathrm{ad}(f_{\mu\bar{\nu}})\circ P_A^-\right). \tag{6.34}
$$

Proof. From Lemma 70 and Lemma 72, we have

$$
\frac{\partial}{\partial\bar{\epsilon}_2}(\frac{\partial\,g}{\partial\,\epsilon_1}g^{-1}+g^{*-1}\frac{\partial\,g^*}{\partial\,\epsilon_1})\Big|_{\epsilon_2=0} = \frac{\partial^2}{\partial\bar{\epsilon}_2\partial\epsilon_1}(g^*g)\Big|_{\epsilon_1=\epsilon_2=0} = -f_{\mu\bar{\nu}},
$$

and result follows. □

Lemma 115. We have that

$$
\frac{\partial}{\partial\bar{\epsilon}_2}(\Delta_\gamma^+)^{-1}\bar{\partial}_\gamma^*\Big|_{\epsilon_2=0} = -(\Delta_A^+)^{-1}\circ *\,\mathrm{ad}\,\nu * \circ P_A^-. \tag{6.35}
$$

Therefore, we have the following

$$\lim_{\ell\to\infty}\left(\int_{M_\ell}\mathrm{ad}(P_\gamma^-(g\mu g^{-1}))\wedge\frac{\partial}{\partial\bar\epsilon_2}G_\gamma(z,z')\big|_{z=z'}\Big|_{\epsilon_2=0}\right)=-\mathrm{Tr}\{\mathrm{ad}\,\mu\circ(\Delta_A^+)^{-1}\circ*\,\mathrm{ad}\,\nu*\circ P_A^-\}. \tag{6.36}$$

Proof. Because of Lemma 70, the first statement follows from the following formal computation,

$$\begin{aligned}\frac{\partial}{\partial\bar\epsilon_2}(\Delta_\gamma^+)^{-1}\bar\partial_\gamma^*\Big|_{\epsilon_2=0}&=-(\Delta_\gamma^+)^{-1}\frac{\partial}{\partial\bar\epsilon_2}(\Delta_\gamma^+)(\Delta_\gamma^+)^{-1}\bar\partial_\gamma^*\Big|_{\epsilon_2=0}+(\Delta_\gamma^+)^{-1}\frac{\partial}{\partial\bar\epsilon_2}\bar\partial_\gamma^*\Big|_{\epsilon_2=0},\\&=(\Delta_\gamma^+)^{-1}\frac{\partial}{\partial\bar\epsilon_2}(\bar\partial_\gamma^*)\Big(1-\bar\partial_\gamma(\Delta_\gamma^+)^{-1}\bar\partial_\gamma^*\Big)\Big|_{\epsilon_2=0},\\&=-(\Delta_A^+)^{-1}\circ*\,\mathrm{ad}\,\nu*\circ P_A^-.\end{aligned}$$

For the second equality, note that $G_\gamma(z,z')$ is the integral kernel of $\Delta_\gamma^{-1}\bar\partial_\gamma^*$, whose singularity along the diagonal is of the form $\frac{1}{z-z'}$, hence the integral converges uniformly as $\ell\to\infty$ as μ decays exponentially to 0. ☐

Lemma 116.

$$\lim_{\ell\to\infty}\left(\int_{M_\ell}\mathrm{ad}(P_\gamma^-(g\mu g^{-1}))\wedge\frac{\partial}{\partial\bar\epsilon_2}G_\gamma^\#(z,z')\big|_{z=z'}\Big|_{\epsilon_2=0}\right)=-\frac{1}{2\pi}\int_X\mathrm{ad}\,\mu\wedge\mathrm{ad}*\nu. \tag{6.37}$$

Proof. This follows from the Lemma 107. ☐

Lemma 117. We have the following,

$$\frac{\partial}{\partial\bar\epsilon_2}P_\gamma^-(g\mu g^{-1})\Big|_{\epsilon_2=0}=-\bar\partial_A f_{\mu\bar\nu}. \tag{6.38}$$

and

$$\operatorname{ad}(\bar{\partial}_A f_{\mu\bar{\nu}}) = \bar{\partial}_A(\operatorname{ad} f_{\mu\bar{\nu}}).$$

Proof. For the first equality, we have

$$\begin{aligned}
\frac{\partial}{\partial\bar{\epsilon}_2} P_\gamma^-(g\mu g^{-1})\Big|_{\epsilon_2=0} &= \frac{\partial}{\partial\bar{\epsilon}_2}\big(1 - \bar{\partial}_\gamma \Delta_\gamma^{-1} \bar{\partial}_\gamma^*\big)(g\mu g^{-1})\Big|_{\epsilon_2=0}, \\
&= \bar{\partial}_A \Delta_A^{-1} * \operatorname{ad}\nu * \mu + P_A^-\big([\frac{\partial}{\partial\bar{\epsilon}_2} g\big|_{\epsilon_2=0}, \mu]\big), \\
&= -\bar{\partial}_A f_{\mu\bar{\nu}} + P_A^-\big([\frac{\partial}{\partial\bar{\epsilon}_2} g\big|_{\epsilon_2=0}, \mu]\big).
\end{aligned}$$

The result follows from the claim that $\frac{\partial}{\partial\bar{\epsilon}_2} g\big|_{\epsilon_2=0} = 0$. By definition, g satisfies

$$g(\bar{\partial}_A + \epsilon_2\nu)g^{-1} = \bar{\partial}_A + \gamma \in U_{A,\epsilon}^{0,1},$$

where γ satisfies the following

$$\bar{\partial}_A^*\gamma - \partial_A^*\gamma^* = 0,$$

and

$$-\bar{\partial}_A\gamma + \partial_A\gamma - [\gamma, \gamma^*] = 0,$$

which is essentially the defining condition on tangent space to $U_{A,\epsilon}^{0,1}$, see Definition 59. Differentiate these conditions with respect to ϵ_2 and let $\epsilon_2 = 0$, and apply Kähler identities, we get

$$\Delta_A\big(\frac{\partial}{\partial\bar{\epsilon}_2} g\Big|_{\epsilon_2=0}\big) = 0.$$

Then the claim follows as $\frac{\partial}{\partial\bar{\epsilon}_2} g\Big|_{\epsilon_2=0}$ belongs to $S^{2,2}_{\delta,s}(\operatorname{End} E)$, on which Δ_A is invertible. For the second equality, note that for any $s \in L^2(\operatorname{End} E)$, by definition we have

$$\begin{aligned}
\bar{\partial}_A \circ \operatorname{ad} f_{\mu\bar{\nu}}(s) - \operatorname{ad} f_{\mu\bar{\nu}} \circ \bar{\partial}_A(s) &= \bar{\partial}_A[f_{\mu\bar{\nu}}, s] - [f_{\mu\bar{\nu}}, \bar{\partial}_A s] \\
&= [\bar{\partial}_A f_{\mu\bar{\nu}}, s] \\
&= \bar{\partial}_A(\operatorname{ad} f_{\mu\bar{\nu}})(s).
\end{aligned} \tag{6.39}$$

□

Theorem 118. Given the holomorphic family of Dolbeault operators $\bar{\partial}_A + \epsilon_1\mu + \epsilon_2\nu$, the curvature of the canonical line bundle λ with respect to the Quillen metric (1.10) is given by:

$$\begin{aligned}
\Theta(\mu, \bar{\nu}) &= -\frac{\partial^2}{\partial\bar{\epsilon}_2\partial\epsilon_1}\zeta'(0)\Big|_{\epsilon_1=\epsilon_2=0}, \\
&= -\operatorname{Tr}\left(\operatorname{ad} f_{\mu\bar{\nu}} \circ P_A^+\right) + \operatorname{Tr}\left(\operatorname{ad}\mu \circ \Delta_A^{-1} \circ * \operatorname{ad}\nu * \circ P_A^-\right) \\
&\quad + \int_X \operatorname{tr}(\bar{\partial}_A(\operatorname{ad} f_{\mu\bar{\nu}}) \wedge J_A) - \frac{1}{2\pi}\int_X \operatorname{tr}(\operatorname{ad}\mu \wedge \operatorname{ad} *\nu), \\
&= \Theta_{L^2}(\mu, \bar{\nu}) + \int_X \operatorname{tr}(\bar{\partial}_A(\operatorname{ad} f_{\mu\bar{\nu}}) \wedge J_A) - \frac{1}{2\pi}\int_X \operatorname{tr}(\operatorname{ad}\mu \wedge \operatorname{ad} *\nu).
\end{aligned} \tag{6.40}$$

Proof. Combining Lemma 114, Lemma 115, Lemma 116, and Lemma 117, we get the desired result. □

Theorem 119. The first Chern form Ω of the canonical bundle λ equipped with the Quillen metric 1.10 is given by

$$\Omega(\mu,\bar{\nu}) = -\frac{i}{4\pi^2}\int_X \mathrm{ad}\,\mu\wedge\mathrm{ad}*\nu+\frac{i}{2\pi}\sum_{k=1}^{m}\sum_{i\neq j=1}^{s_k}\mathrm{sgn}(\alpha_{i,k}-\alpha_{j,k})(1-2|\alpha_{i,k}-\alpha_{j,k}|)m_{i,k}\Theta_{j,k}(\mu,\bar{\nu}), \tag{6.41}$$

for any $\mu,\nu\in H^{0,1}_{A,\delta}\cong T_{\bar{\partial}_A}V_{A,\epsilon}$. Here $\Omega_{i,k}$ is the curvature form of the line bundles $\lambda_{i,k}$, see Definition 4.13.

Proof. By Theorem 71 and Proposition 118, we get that

$$\delta_{\bar{\epsilon}_2}\delta_{\epsilon_1}\zeta'(0)(\mu,\bar{\nu}) = \Theta(\mu,\bar{\nu}) + \int_X \mathrm{tr}(\bar{\partial}_A\,\mathrm{ad}\,f_{\mu\bar{\nu}}\wedge J_A) + \frac{-1}{2\pi}\int_X \mathrm{tr}(\mathrm{ad}\,\mu\wedge\mathrm{ad}*\nu).$$

Applying integration by parts to the second term in the above formula, together with Lemma 109, Lemma 112, we see that

$$\begin{aligned}\int_X \mathrm{tr}(\bar{\partial}_A\,\mathrm{ad}\,f_{\mu\bar{\nu}}\wedge J_A) &= \lim_{\ell\to\infty}\int_{M_\ell}\mathrm{tr}(\bar{\partial}_A\,\mathrm{ad}\,f_{\mu\bar{\nu}}\wedge J_A)\\ &= -\lim_{\ell\to\infty}\int_{M_\ell}\mathrm{tr}(\mathrm{ad}\,f_{\mu\bar{\nu}}\wedge\bar{\partial}_\gamma J_A) + \lim_{\ell\to\infty}\int_{y=\ell}\mathrm{tr}(\mathrm{ad}\,f_{\mu\bar{\nu}}\wedge J_A),\\ &= -\sum_{k=1}^{m}\sum_{i\neq j=1}^{s_k}\mathrm{sgn}(\alpha_{i,k}-\alpha_{j,k})(1-2|\alpha_{i,k}-\alpha_{j,k}|)m_{i,k}F^{j,k}_{\mu\bar{\nu}},\\ &= \sum_{k=1}^{m}\sum_{i\neq j=1}^{s_k}\mathrm{sgn}(\alpha_{i,k}-\alpha_{j,k})(1-2|\alpha_{i,k}-\alpha_{j,k}|)m_{i,k}\Theta_{j,k}(\mu,\bar{\nu}).\end{aligned}$$

In the third equality, we have used the following fact that

$$\mathrm{ad}\,M(e_{ij}) = \sum_k (M_{ki}e_{kj} - M_{jk}e_{ki}),$$

where M and e_{ij} are $n\times n$ matrix with e_{ij} the basis matrix with 1 at ij-entry and

0 otherwise. The result follows then from the limiting behavior of J_γ in (6.27). $\square$

Remark 120. Our formula is the same as those obtained by Takhtajan and Zograf [113], and Albin and Rochon [1], and it should be noted that there is a typographical error in the statement of Theorem 2 of [113]: the sum should be only for $i \neq j$, also the normalization constant of in Theorem 5.4 of [1] is wrong and should be corrected to $\frac{i}{2\pi}$ in equation (5.15).

Appendix A: Weighted Sobolev Spaces on Surfaces with Cylindrical Ends

In this part, we provide an account of the analytic preliminaries of weighted Sobolev spaces on a surface with cylindrical ends. Much of the material presented here is well-known and we refer the reader to [78], [77], [12], [114], and [95] for more details.

Let E be a Hermitian vector bundle over (X, g_0) the surface with cylindrical ends together and we fix a trivialization $E|_{Z_k} \cong Z_k \times \mathbb{C}^n$. Let $\nabla : \Omega(E) \to \Omega^1(E)$ denote a fixed covariant derivative that is trivial along each cylindrical ends.

Let $\{\tau_k : X \to [0, \infty), k = 1, \cdots, m\}$ be an m-tuple of smooth functions defined as

$$\tau_k(x,y) = \begin{cases} 0, & \text{for } z \in M \\ y, & \text{for } z = (x,y) \in Z_k \end{cases}$$

Let $\delta \in \mathbb{R}^m$ be an m-tuple of real numbers called the weights. We denote by $\delta\tau$ the scalar product. For $\delta_1, \delta_2 \in \mathbb{R}^m$, we say that $\delta_1 \leq$ (resp. $<$) δ_2 if $\delta_1^k \leq$ (resp. $<$) δ_2^k for $k = 1, \cdots, m$.

Definition 121. Let $1 < p < \infty$, $\delta \in \mathbb{R}^m$, $k \in \mathbb{N}$. The weighted Lebesgue space $L^{k,p}_\delta(E)$ is defined as the space of all sections $u \in L^{k,p}_{\text{loc}}(E)$ such that

$$\|u\|_{L^{k,p}_\delta} := \|e^{\delta\tau}u\|_{L^{k,p}} = (\sum_{i=0}^{k}\int_X \|\nabla^i(e^{\delta\tau}u)\|^p \, dv_g)^{1/p}$$

is finite.

Lemma 122 ([75] Theorem 1.3)**.** These spaces $(L^{k,p}_\delta(X), \|\cdot\|_{L^{k,p}_\delta})$ are Banach spaces.

Lemma 123. The space $C^\infty_0(E)$ is dense in $L^{k,p}_\delta(E)$.

In the following, we generalize the Sobolev embedding and multiplication theorems to Riemann surfaces with cusp ends.

Lemma 124 (Weighted Hölder Inequality)**.** Let $1 < p, q, r < \infty$ and $\delta, \delta_1, \delta_2 \in \mathbb{R}^m$ satisfy $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ and $\delta_1 + \delta_2 = \delta$. Let $u \in L^p_{\delta_1}(E)$ and $v \in L^q_{\delta_2}(E)$, then

$$\|uv\|_{L^r_\delta} \leq \|u\|_{L^p_{\delta_1}} \|v\|_{L^q_{\delta_2}}$$

Proof. This is a direct application of the usual Hölder inequality. □

Lemma 125 (Multiplication Theorem)**.** Let E, E' be two bundles with adapted metrics over X. The tensor product on smooth sections induces an continuous map

$$L^{k_1,p_1}_{\delta_1}(E) \times L^{k_2,p_2}_{\delta_2}(E') \to L^{k,p}_\delta(E \otimes E')$$

provided $k \leq \min(k_1, k_2)$, $\delta > \delta_1 + \delta_2$ and $k - 2/p < k_1 - 2/p_1 + k_2 - 2/p_2$.

Proof. This follows from the decomposition of $Z_k = \bigcup_{i=1}^{\infty} C_{j,k}$ with $C_{j,k} := \mathbb{S}^1 \times [j, j+1]$ and applying Sobolev multiplication to $e^{\delta t}u$ on each $C_{j,k}$. Hence we omit the details. □

Lemma 126 (Weighted Sobolev Embeddings)**.** Let $k, l \in \mathbb{N}$, $1 < p, q < \infty$, $\delta, \delta' \in \mathbb{R}^m$, we have

1. If $k \geq l$, $k - 2/p \geq l - 2/q$, either $p < q$ and $\delta > \delta'$ or $p \geq q$ and $\delta \geq \delta'$, we have $L^{k,p}_{\delta}(E) \hookrightarrow L^{l,q}_{\delta'}(E)$ is continuous;

2. If $k > l$, $k - 2/p > l - 2/q$ and $\delta > \delta'$, then $L^{k,p}_{\delta}(E) \hookrightarrow L^{l,q}_{\delta'}(E)$ is compact;

3. If $k - 2/p > 0$ and $\delta' < \delta$, then $L^{k,p}_{\delta}(E) \hookrightarrow C^0_{\delta'}(E)$, where the weighted $C^0_{\delta'}$-norm is defined by
$$\|u\|_{C^0_{\delta'}} = \sup_{z \in X}\{ e^{\delta' \tau} |u(z)|\}.$$
We have,
$$\|u\|_{C^0_{\delta'}(C_{j,k})} = o(1) \text{ as } j \to \infty.$$

Proof.

Part 1. Covering M by a finite charts, together with the m cusp ends, this form an atlas of X. Take a partition of unity subordinate to this covering and apply the standard Sobolev embedding to the interior of X, we are reduced to the proof to sections supported on each cusp end Z_k.

When $k \geq l$, $k - 2/p \geq l - 2/q$, there exists a constant $c > 0$ (independent of j) such that for any $u \in \Omega(Z_k, E)$, its restriction to each $C_{j,k}$ satisfies that

$$\|u\|_{L^{l,q}(C_{j,k},dA)} \leq c\|u\|_{L^{p,k}(C_{j,k},dA)}, \tag{A.1}$$

where dA stands for the Euclidean volume element.

Assume further that $q \geq p$ and $\delta > \delta'$. Multiplying the left and right sides of $A.1$ by $e^{(j+1)\delta'}$ and $e^{j\delta}$, respectively, and for N sufficiently large and $j \geq N$, there exist constants $C > 0$ and $0 < c < 1$ such that

$$\|u\|_{L^{l,q}_{\delta'}(C_{j,k},\, dv_g)} \leq C\, e^{cj(\delta'-\delta)}\, \|u\|_{L^{k,p}_{\delta}(C_{j,k},\, dv_g)}. \tag{A.2}$$

Summing (A.2) over $j \geq N$, we get that

$$\begin{aligned}
\|u\|^q_{L^{l,q}_{\delta'}(Z_{j\geq N},\, dv_g)} &= \sum_{j\geq N} \|u\|^q_{L^{l,q}_{\delta'}(C_j,\, dv_g)} \\
&\leq c' e^{cN(\delta'-\delta)} \big(\sum_{j\geq N} \|u\|^p_{L^{k,p}_{\delta}(C_j,\, dv_g)}\big)^{q/p} \\
&= c' e^{cN(\delta'-\delta)} \|u\|^q_{L^{k,p}_{\delta}(Z_{j\geq N},\, dv_g)}
\end{aligned} \tag{A.3}$$

In the second inequality, we have used the following Hölder inequality

$$\Big(\int \|e^{\delta' t} f\|^q dv_g\Big)^{\frac{1}{q}} \leq \Big(\int \|e^{\delta t} f\|^p dv_g\Big)^{\frac{1}{p}} \Big(\int e^{r(\delta'-\delta)t} dv_g\Big)^{\frac{1}{r}},$$

where $1/q = 1/p + 1/r$, and the fact that for a positive sequence a_n and $q > p$,

$$\sum_{n=1}^{\infty} a_n^{q/p} \leq \big(\sum_{n=0}^{\infty} a_n\big)^{q/p} \tag{A.4}$$

Hence we have proved the case when $k \geq l$, $k - 2/p \geq l - 2/q$ and $q \geq p$ and $\delta > \delta'$. In the case when $l = k = 0$, $\delta \geq \delta'$, and $p > q$, by Hölder inequality, we have that

$$\Big(\int_{Z_{j\geq N}} |e^{\delta'\tau}u|^q \, dv_g\Big)^{\frac{1}{q}} \leq \Big(\int_{Z_{j\geq N}} e^{\frac{pq}{p-q}(\delta'-\delta)\tau} \, dv_g\Big)^{\frac{p-q}{pq}} \Big(\int_{Z_{j\geq N}} |e^{\delta\tau}u|^p \, dv_g\Big)^{1/p}$$
$$\leq c\, e^{N(\delta'-\delta)} \Big(\int_{Z_{j\geq N}} |e^{\delta\tau}u|^p \, dv_g\Big)^{1/p} \tag{A.5}$$

The general case follows from applying Equations (A.1) and (A.5) to the covariant derivatives of u inductively.

Part 2 (see also [28], Lemma 2.1)**.** The idea is to verify the totally boundedness of the image of unit ball $B \subset L^{k,p}_{\delta}$ in $L^{l,q}_{\delta'}$, i.e. for any $\epsilon > 0$, there exists a finite covering of the image of B by balls of radius less than ϵ in $L^{l,q}_{\delta'}$.

Again, by the patching argument with the Rellich-Kondrachov lemma, we are reduced to show that for any $\epsilon > 0$, there exists N sufficiently large, such that for any $u \in B$,

$$\|u\|_{L^{l,q}_{\delta'}(Z_{j\geq N}, dV_g)} \leq \epsilon$$

The statement follows from Equations (A.3) and (A.5).

Part 3. By Sobolev embedding on $(C_{j,k}, dA)$, there exists a constant $c > 0$ such that for any $u \in C^{\infty}_c(Z_{j\geq 1})$ and any $\delta' < \delta$,

$$\|e^{\delta'\tau}u\|_{C^0(C_{j,k})} \leq c\, \|e^{\delta'\tau}u\|_{L^{k,p}(C_{j,k}, dA)}$$
$$\leq c\,(j+1)^{1/p} \|e^{\delta'\tau}u\|_{L^{k,p}(C_{j,k}, dV_g)}$$
$$\leq c' \|e^{\delta\tau}u\|_{L^{k,p}(C_{j,k}, dV_g)}.$$

Summing over $j \geq N$, and since $\|e^{\delta\tau}u\|_{L^{k,p}(Z_{j\geq N}, dV_g)}$ is finite, this implies that

$$\{\|e^{\delta'\tau}u\|_{C^0(C_{j,k})},\, j \in \mathbb{N}\}$$

form a Cauchy sequence that approaches 0.

□

Now we define the following weighted Sobolev spaces where different weights are assigned to different degree of covariant derivatives.

Definition 127. We denote by $\hat{L}^{1,p}_{\delta}(E)$ the space of all sections $u \in L^{1,p}_{\mathrm{loc}}(E)$ such that its $\hat{L}^{1,p}_{\delta}$-norm

$$\|u\|_{\hat{L}^{1,p}_{\delta}} = (\|e^{-\tau}u\|^p_{L^p_\delta} + \|\nabla_0\, u\|^p_{L^p_\delta})^{1/p}.$$

is finite.

Lemma 128 (Weighted Poincaré Inequality)**.** 1. For any $p > 1$ and a local section u of E supported on the end Z_k, if $\delta < 0$ and u vanishes on the boundary of Z_k, or if $\delta > 0$ and u vanishes near the infinity of Z_k, then

$$\|\frac{du}{dy}\|_{L^p_\delta} \leq c\|u\|_{L^p_\delta}.$$

In particular, if $\delta < 0$, then

$$L^{1,p}_{\delta-1} = \hat{L}^{1,p}_{\delta-1}.$$

2. For $p > 2$ and $\delta > 0$ such that $1 - 2/p < \delta$, then any $u \in \hat{L}^{1,p}_\delta \subset C^0(\mathbb{D}(e^{-a_k}))$ and

$$u - u(0) \in L^{1,p}_\delta.$$

Proof.

Part 1. For any smooth section u supported on Z_k, applying integration by parts, we get

$$p\delta \int_{Z_k} e^{p\delta\tau}|u|^p dy \wedge *dy = -\int_{Z_k} e^{p\delta\tau} d\,|u|^p \wedge *d\,y - \int_{y=1} e^{p\delta\tau}|u|^p * d\,y \tag{A.6}$$
$$+ \lim_{A\to\infty} \int_{y=A} e^{p\delta\tau}|u|^p * d\,y,$$

If $\delta > 0$ and u vanishes at infinity, from Kato's inequality,

$$\le p \int_{Z_k} e^{p\delta\tau}|u|^{p-1}\,|\frac{d\,u}{dy}|\,dy \wedge *dy$$

By Cauchy-Schwarz inequality, we get the desired inequality,

$$\int_{Z_k} e^{p\delta\tau}|u|^p\ dv_g \le c \int_{Z_k} e^{p\delta\tau}|\frac{d\,u}{dy}|^p\ dv_g.$$

The above argument also works for the case of $\delta < 0$ and u vanishing near the boundary of Z_k.

For the third statement in part 1, let $\chi : [1,\infty) \to [0,1]$ be a fixed smooth cut-off function defined by

$$\chi(y) = \begin{cases} 0, & \text{for } a_k \le y \le a_k + 2, \\ 1, & \text{for } y \ge a_k + 3. \end{cases}$$

Applying (A.6) to χu and we get

$$-p\delta \int_{Z_k} e^{p\delta\tau} |\chi u|^p \, dv_g \le \Big| \int_{Z_k} e^{p\delta\tau} d\, |\chi u|^p \wedge *d\, y \Big|,$$

by Kato's inequality, there exists a constant $c > 0$,

$$\int_{Z_k} e^{p\delta\tau} |\chi u|^p \ dv_g \le c \int_{Z_k} e^{p\delta\tau} (|d\chi u| + |\chi du|) |u|^{p-1} \ dv_g.$$

Since $d\,\chi$ is bounded with its support contained in $\mathbb{S}^1 \times [a_K + 2 \le y \le a_K + 3]$, applying Hölder inequality to the second term, there exists a constant $c' > 0$,

$$\int_{Z_k} e^{p\delta\tau} |u|^p \ dv_g \le c' \Big(\int_{Z_k} e^{p(\delta-1)\tau} |u|^p \ dv_g + \int_{Z_k} e^{p(\delta-1)\tau} |d\, u|^p \ dv_g \Big).$$

In the last step, we need $\delta < 0$, and this finishes the proof of part 1.

Part 2. By identifying the punctured disk $\mathbb{D}^*(e^{-a_k})$ with the end Z_k, the $L^{1,p}$-norm on $\mathbb{D}(e^{-a_k})$ can be written as

$$\|u\|^p_{L^{1,p}(\mathbb{D}(e^{-a_k}), dA)} = \int_{Z_k} |e^{-2/p\tau} u|^p + |e^{(1-2/p)\tau} d\, u|^p \, dv_g,$$

for any $u \in C_c^\infty(\mathbb{D}(e^{-a_k}))$. Therefore, for any δ satisfying $1 - 2/p < \delta < 1$, we have

$$\|u\|_{L^{1,p}(\mathbb{D}(e^{-a_k}))} \le c \|u\|_{\hat{L}^{1,p}_\delta(Z_k, dv_g)}.$$

In this case, $C_c^\infty(\mathbb{D}(e^{-a_k}), E)$ is dense in $\hat{L}^{1,p}_\delta(Z_k)$. By Taylor expansion, we have

$$|u - u(\infty)| \sim O(e^{-\tau})$$

This enable us to apply the same integration by parts argument as in (A.6) to $|u - u(\infty)|$ as the boundary term approaches to 0 by the above estimate, hence our result follows.

$\square$

Definition 129. We denote by $\hat{L}^{2,2}_\delta(E)$ the space of all sections $u \in L^{2,2}_{\mathrm{loc}}(E)$ such that its $\hat{L}^{2,2}_\delta$-norm

$$\|u\|_{\hat{L}^{2,2}_\delta} = (\|e^{-2\tau} u\|^2_{L^2_\delta} + \|\nabla_0 u\|^2_{L^{1,2}_\delta})^{1/2}.$$

is finite.

Corollary 129.1. The spaces $\hat{L}^{2,2}_\delta(E)$ are Banach spaces. Moreover, the subspace $L^{2,2}_\delta(E)$ of finite codimension.

Proof. By definition, any section $u \in \hat{L}^{2,2}_\delta(E)$ satisfies that, for any $p > 2$, $u \in L^p_{-1+\delta}$ and $\nabla_0 u \in L^p_\delta$, which in turn by Lemma 128 implies that u has limiting values $u_k(\infty)$ along each cusp Z_k. Let $\chi_k : Z \to [0,1]$ be smooth cutoff functions such that it equals 1 along the cusp end Z_k for $y \geq 2$ and vanishes outside Z_k for $y \leq 1$. Furthermore, by Lemma 128,

$$u - \sum_k \chi_k u_k(\infty)$$

belongs to $L^{2,2}_\delta$. Let P denote this fixed projection from $\hat{L}^{2,2}_\delta$ to $L^{2,2}_\delta$. We see that,

$\hat{L}^{2,2}_\delta$ contains $L^{2,2}_\delta$ as a finite codimension subspace, which implies that $\hat{L}^{2,2}_\delta$ is a Banach space.

□

Lemma 130 (Weighted Elliptic L^2 Estimate and Regularity)**.** Let D be an second order elliptic operator, and let $\delta \in \mathbb{R}^m$. There exists a constant $c(\delta) > 0$ such that for all $u \in L^2_\delta$ satisfying $Du \in L^2_\delta$,

$$\|u\|_{L^{s,2}_\delta} \leq c(\|Du\|_{L^2_\delta} + \|u\|_{L^2_\delta}).$$

Proof. Due to the interior L^2 estimate, by a patching argument, we only need to prove the above estimate along the cusp end. Let $u, v \in L^2_\delta(Z, E)$, such that $Du = v$ in the weak sense. Restricting to $Z_{j\leq y\leq j+1}$, by local L^2 estimate, there exists a $u_j \in W^{2,2}_0(Z_{j\leq y\leq j+1}, E)$ such that $Du_j = v$ on $Z_{j\leq y\leq j+1}$. In particular, $D(u - u_j) = 0$ and hence $u - u_j$ is smooth, which implies that $u|_{j\leq y\leq j+1} \in W^{2,2}(j \leq y \leq j+1, E)$. We have in this case

$$\|u\|_{L^{2,2}(Z_{j\leq y\leq j+1})} \leq c\left(\|v\|_{L^2(Z_{j-1\leq y\leq j+2})} + \|u\|_{L^2(Z_{j-1\leq y\leq j+2})}\right).$$

Multiplying both side by $e^{\delta\tau}$, and summing over $j \in \mathbb{N}$, we conclude that

$$\|u\|_{L^{2,2}_\delta} \leq c'\left(\|v\|_{L^2_\delta} + \|u\|_{L^2_\delta}\right).$$

This completes the proof of the result. □

Appendix B: L^2-Index of Dolbeault Operators

Given the complete cusp metric g on X and the induced Hermitian metric on $\operatorname{End} E$ with respect to the fixed temporal frame of E, we define the L^2-product for any $\sigma_1, \sigma_2 \in C_c^\infty(X, \operatorname{End} E)$ as

$$(\sigma_1, \sigma_2)_{L^2} := \int_X \langle \sigma_1, \sigma_2 \rangle dv_g,$$

with the L^2-norm denoted by $\| \cdot \|$.

Definition 131. For $k \in \mathbb{N}$, we define the Sobolev space $H^k(X, \operatorname{End} E)$ as the space of sections $\sigma \in L^2_{\text{loc}}(X, \operatorname{End} E)$ such that the i-th order covariant derivative $\nabla^i \sigma \in L^2(X, \Lambda^i X \otimes \operatorname{End} E)$ for all $i \leq k$. The Sobolev H^k-norm is defined as

$$\|\sigma\|^2_{H^k} := \sum_{i=0}^{k} \|\nabla^i \sigma\|^2_{L^2}.$$

Given any Dolbeault operator $\bar{\partial}_\gamma \in U^{0,1}_{A,\epsilon}$, consider the following L^2 Dolbeault complex

$$\bar{\partial}_\gamma : L^2(\operatorname{End} E) \to L^2(\Lambda^{0,1} X \otimes \operatorname{End} E),$$

with its initial domain $C_c^\infty(X, \operatorname{End} E)$. There are two possible closed extensions of $\bar{\partial}_\gamma$. The first is the minimal extension $\bar{\partial}_{\gamma,\min}$ whose domain is defined as the completion of $C_c^\infty(X, \operatorname{End} E)$ with respect to the graph norm defined below,

$$\|\sigma\|^2_{\bar{\partial}_\gamma} := \|\sigma\|^2_{L^2} + \|\bar{\partial}_\gamma \sigma\|^2_{L^2}.$$

The second is the maximal extension $\bar{\partial}_{\gamma,\max a}$ whose domain is defined via duality as the space of all $\sigma \in L^2(X, \operatorname{End} E)$ such that there exists a section $\eta \in L^2(X, E)$ such that

$$\langle \bar{\partial}_{\gamma,\max}\sigma\, , f\rangle_{L^2} = \langle \eta\, , \bar{\partial}^*_\gamma f\rangle_{L^2}, \ \ \forall f \in C_c^\infty(X, \operatorname{End} E).$$

It is obvious that $C_c^\infty(X, \operatorname{End} E) \subset \operatorname{dom}(\bar{\partial}_{\gamma,\max})$. In our case, the minimal and maximal extensions of $\bar{\partial}_\gamma$ coincide, that is $C_c^\infty(X, \operatorname{End} E)$ is dense in the domain of $\bar{\partial}_{\gamma,\max}$ with respect to the graph norm; moreover, the Dolbeault Laplacians $\Delta^\pm_\gamma$ are essentially self-adjoint, which follow from the following,

Theorem 132 ([27] [53] [54])**.** Let X be a complete Riemannian manifold with compact boundary Y (possibly empty) and E, F two Hermitian bundles. Suppose D is a first order differential opearator acting between E and F, whose symbol satisfies that the principal symbol $|\sigma(D)(x,\xi)| \le C(1+|\xi|)$ uniformly on X, then

1. $\operatorname{dom}(D_{\min}) = \operatorname{dom}(D_{\max})$
2. D^*D and DD^* are essentially self-adjoint.

In particular, any Dirac-type operators on X satisfies condition (1).

To define the L^2-index of the above Dolbeault complex, we need the following,

Proposition 133. [87, Corollary 6.26] The following L^2 Dolbeault complex

$$\bar{\partial}_\gamma : L^2(\operatorname{End} E) \to L^2(\Lambda^{0,1}X \otimes \operatorname{End} E),$$

is Fredholm.

Alternate proof of this proposition can be found in Ballmann and Brüning in [8], Stern [112], and Lott [79]. Since $\ker \bar{\partial}_\gamma \cap L^2$ and $\ker \bar{\partial}_\gamma^* \cap L^2$ are both finite dimensional, we may define

$$\operatorname{Ind}(\bar{\partial}_\gamma) = \dim(\ker \bar{\partial}_\gamma \cap L^2) - \dim(\ker \bar{\partial}_\gamma^* \cap L^2),$$

and it is our goal in this part to compute the L^2-index of the Dolbeault complex (3.12).

Our strategy here follows that of [99]. That is, as the spectral invariant $L^2 -$ $\operatorname{ind}(D)$ is stable under surgery, we decompose the surface X at any given level $\ell > 0$ along the cusp ends, that is we have the decomposition of the following form

$$X_\ell := M_\ell \bigcup_Y \cup_{k=1}^m Z_{k,y\geq\ell},$$

where Y is a disjoint union of m-circles at level ℓ. By imposing suitable boundary conditions at Y, we prove a gluing formula for L^2-index which reduces our computation to the compact part and the cusp ends where the geometry is explicit.

As to the computation of the L^2-index of the compact part, the key observation, due to Alvarez [3] and Wentworth [123], is that when considered as a real Cauchy-Riemann operator, the totally real boundary condition, see Definition 135, is elliptic for $\bar{\partial}_\gamma$; More importantly, the Dolbeault complex becomes a Fredholm complex when imposed with these boundary conditions along Y, hence making the heat kernel approach of index computation available.

The first appearance of the totally real boundary condition was in [118], and we refer the reader to [18, Section 2.1] for an interesting discussion of the Hellwig-Vekua index theorem, which can be seen as a predecessor of the result to be proven in this part. The ellipticity of the totally real boundary condition was discussed in [108] and [124]. We refer the reader to Appendix C of [83] for a readable introduction of the analytic preliminaries concerning these boundary conditions. For the general theory of elliptic boundary value problem of a first order Dirac type operator, we refer the reader to [9] for a comprehensive discussion.

B.1 Totally Real Boundary Condition

In this part, we introduce the totally real boundary condition for the Dolbeault operator $\bar{\partial}_\gamma$ on the complete surface with boundary $X_\ell \bigcup_Y \cup_{k=1}^m Z_{k,y\geq\ell}$ and discuss its ellipticity in the sense of Shapiro-Lopatinskii. We will refer the reader to the general theory of elliptic boundary value problems to [57], [124], and [9].

First, we remark that there do not exist any ($\mathbb{C}$ -linear) elliptic boundary conditions for the Cauchy-Riemann operator $\bar{\partial}_\gamma$. Suppose that the boundary Y of M_ℓ is totally real, i.e. there exist coordinates $z = x + iy$ in a neighborhood of the boundary such that the equation of Y is $y = 0$. Without loss of generality, we assume that E is a trivial line bundle on this coordinate patch, therefore we have

$$\bar{\partial}_\gamma = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}) = \frac{i}{2}(\frac{\partial}{\partial y} + \frac{1}{\mathrm{i}}\frac{\partial}{\partial x})$$

with its symbol given by

$$\sigma(\bar{\partial}_\gamma)(x,0) = -\frac{1}{2}(p - i\xi)$$

Consider solutions of the following ordinary differential equation

$$\sigma(\bar{\partial}_\gamma)(x,0;-i\frac{\partial}{\partial t},\xi)\phi(t) = 0.$$

By basic knowledge of initial value problem in ordinary differential equations, for each ξ, $\sigma(\bar{\partial}_\gamma)(x,0,\xi)$ defines an isomorphism from the space of all initial data in $\mathbb{C}$ to the corresponding solution of this equation. Let $L^+(x,\xi)$ denote the space of initial data of the solution satisfying $\lim\limits_{t\to\infty}\phi(t) = 0$. We have

$$\mathrm{L}^+ = \begin{cases} 0, & \text{for } \mathrm{Re}(\xi) < 0, \\ \mathbb{C}, & \text{for } \mathrm{Re}(\xi) > 0, \end{cases}$$

which does not define a bundle on Y. Therefore, by the Shapiro-Lopatinskii criterion [57, page 102 - 103, (1.11.76) and (1.11.77)], there does not exist any local elliptic boundary condition for $\bar{\partial}_\gamma$.

Now, in order to define an elliptic boundary problem for the Dolbeault complex, there are two possible approaches. One is to ues the Atiyah-Patodi-Singer global boundary condition [7]. Another is to use the totally real boundary condition to be defined below.

Let M_ℓ be the compact surface with boundary Y. By restriction of scalar to $\mathbb{R}$, we will consider E as a real bundle equipped with the real part of the Hermitian metric h, denoted as $(\,,)$, which is clearly an inner product. Let J denote the complex structure of E, it is in particular orthogonal with respect to $(\,,)$ and it satisfies $J^2 = -1$. ∇_γ also descends to a metric connection and commutes with J. In the real setting, $\bar{\partial}_\gamma$ can be written formally as

$$\bar{\partial}_\gamma := (\nabla_\gamma)^{(0,1)} = \frac{1}{2}(\nabla_\gamma + J \circ \nabla_\gamma \circ j) \tag{B.1}$$

where j is the complex structrue on X.

We will assume that, with respect to the fixed temporal frame $\{e_{i,k}\}_i$ near Y, $\bar{\partial}_\gamma$ coincides with the model Dolbeault operator (2.6). Let $\vec{n} = y\frac{\partial}{\partial y}$ be the unit normal on Y. We will denote by $i_{\vec{n}}$ the inner contraction by $\vec{n}^{0,1} := \frac{1}{2}(\vec{n} + J \circ j\vec{n})$.

Lemma 134 (Green's Formula)**.** For any $s \in C_c^\infty(M_\ell, \operatorname{End} E)$ and $\sigma \in C_c^\infty(M_\ell, \Lambda^{0,1} \otimes_{\mathbb{R}} \operatorname{End} E)$, we have

$$\int_{M_\ell} (\bar{\partial}_\gamma s, \sigma)\, dv_g - \int_{M_\ell} (s, \bar{\partial}_\gamma^* \sigma)\, dv_g = \int_Y (s, i_{\vec{n}}\sigma)\ dY,$$

where dY denotes the induced volume form on Y.

Given the Riemannian vector bundle E with a compatible complex structure J over M_ℓ, we define

Definition 135. A totally real frame of E on Y is a choice of an orthonormal frame $(e_1, \cdots, e_r)$ of $E|_Y$ such that $F := \mathrm{span}_{\mathbb{R}}\{e_1, \cdots, e_r\} \subset E|_Y$ satisfies $F \perp JF$.

Trivially, the fixed temporal frame $\{e_{i,k} \otimes e_{j,k}^*\}$ is totally real. In the following, we will fix them as the totally real frame of $\mathrm{End}\, E$ along Y.

Definition 136. A section $s \in C_c^\infty(M_\ell, \mathrm{End}\, E)$ is said to satisfy the *totally real boundary condition* on Y if $s(z) \in F_z$ for all $z \in Y$.

Given the totally real frame $\{e_{i,k} \otimes e_{j,k}^*\}$, this gives rise to an involution τ of $\mathrm{End}\, E|_Y$ such that

$$\tau^2 = 1; \quad \tau \circ J + J \circ \tau = 0; \;\; (\tau\cdot, \cdot) = (\cdot, \tau\cdot);$$

with the corresponding projection $\Pi := \frac{1}{2}(\mathrm{Id} + \tau)$.

In the following, we give a direct proof that, in our above setup, the totally real boundary condition is a (local) elliptic boundary condition. As ∇_γ is flat, by the Weitzenböck formula, the following is true,

Proposition 137. For any $s_1, s_2 \in C_c^\infty(M_\ell, \mathrm{End}\, E)$, we have

$$\begin{aligned}(s_1, s_2)_{H^1} &:= \int_{M_\ell} (s_1, s_2)\, dv_g + \int_{M_\ell} (\nabla_\gamma s_1, \nabla_\gamma s_2)\, dv_g, \\ &= \int_{M_\ell} (s_1, s_2)\, dv_g + 2\int_{M_\ell} (\bar{\partial}_\gamma s_1, \bar{\partial}_\gamma s_2)\, dv_g + \int_Y (s_1, A s_2)\, dY,\end{aligned}$$

where $A = y(i\frac{d}{dx} - [\alpha, \cdot])$ is the boundary operator satisfying

$$\Pi \circ A - A \circ (1 - \Pi) = [\alpha, \cdot] \circ \tau.$$

Proposition 138. For any $s \in \mathrm{dom}(\bar{\partial}_{\gamma,\max})$ satisfying $(1-\Pi)s = 0$. For any $\varphi \in C_c^\infty(M_\ell)$ and any $\delta > 0$, there exists $C > 0$ such that the following

$$\|\varphi s\|_{H^1}^2 \leq (c + 2\delta)\|\varphi \bar{\partial}_\gamma(s)\|_{L^2}^2 + C\|\varphi s\|_{L^2}^2 \tag{B.2}$$

always holds.

Proof. The case when $\mathrm{supp}(\varphi) \cap Y = \varnothing$ follows from the interior elliptic regularity of the Dolbeault operator. It remains to prove the above statement when $\mathrm{supp}(\varphi) \cap Y \neq \varnothing$, i.e. regularity up to the boundary. Without loss of generality, let us assume that $Y \subset \mathrm{supp}(\varphi) \subset U$, where U is a compact domain of M_ℓ.

From Proposition 137 and the fact that $(1-\Pi)s = 0$, we get that

$$\|\varphi s\|_{H^1} = \int_U (\varphi s, \varphi s)\, dv_g + 2\int_U (\bar{\partial}_\gamma(\varphi s), \bar{\partial}_\gamma(\varphi s))\, dv_g + \alpha \int_Y (\varphi s, \tau(\varphi s)) d\mu_\partial \tag{B.3}$$

Since the restriction to the boundary extends to a compact operator from $H^1(U, E)$ to $L^2(Y, E|_Y)$, by an inequality of Ehrling type, that is, for each $\delta > 0$ there is $C > 0$ such that

$$\|\varphi s\|_{L^2(Y,E|_Y)}^2 \leq \delta \|\varphi s\|_{H^1(U,E)} + C\|\varphi s\|_{L^2(U,E)}, \tag{B.4}$$

holds for all $s \in C^\infty(U, E)$. The desired estimate now follows. □

Remark 139. According to Corollary 7.22 in [9], we know that $(\bar{\partial}_\gamma, 1 - \Pi)$ is an elliptic boundary value problem along each cusp end Z_k.

B.2 A Gluing Formula of L^2-Index

For $\theta \in [0, \pi/4]$, for any $s^+ \in H^1_{\mathrm{loc}}(M_\ell, \mathrm{End}\, E)$ and $s^- \in H^1_{\mathrm{loc}}(\cup_{k=1}^m Z_{k,y\geq\ell}, \mathrm{End}\, E)$, we define the following continuous family of elliptic boundary value problems on X_ℓ, that is

$$\begin{cases} \bar{\partial}_\gamma s^+ = \bar{\partial}_\gamma s^- = 0, \text{on } X_\ell, \\ \cos(\theta)\Pi(s^+) = \sin(\theta)(1-\Pi)(s^-); \\ \cos(\theta)\Pi(s^-) = \sin(\theta)(1-\Pi)(s^+); \end{cases}$$

Let us denote the boundary condition as Π_θ, then note that when $\theta = 0$, we get the totally real boundary condition on s^+ and s^-, whereas when $\theta = \pi/4$, we get the transmission boundary condition, which is equivalent to solving for $s \in H^1_{X,\mathrm{loc}}$ such that $\bar{\partial}_\gamma s = 0$ on X. By Theorem 8.12 in [9], we get that for $\theta \in [0, \pi/4]$,

$$\begin{aligned} \mathrm{Ind}_{\mathbb{R}}(\bar{\partial}_\gamma, \Pi, M_\ell) + \mathrm{Ind}_{\mathbb{R}}(\bar{\partial}_\gamma, \Pi, \cup_{k=1}^m Z_{k,y\geq\ell}) &= \mathrm{Ind}_{\mathbb{R}}(\bar{\partial}_\gamma, \Pi_0) \\ &= \mathrm{Ind}_{\mathbb{R}}(\bar{\partial}_\gamma, \Pi_{\pi/4}) \\ &= 2\,\mathrm{Ind}_{\mathbb{C}}(\bar{\partial}_\gamma). \end{aligned}$$

Let us determine $\mathrm{Ind}_{\mathbb{R}}(\bar{\partial}_\gamma, \Pi, M_\ell)$ first. For this purpose, we apply C.1.10 (ii) in [83], and get that

$$\mathrm{Ind}_{\mathbb{R}}(\bar{\partial}_\gamma, \Pi, M_\ell) = n^2(2-2h),$$

because in our case of the temporal framing, which by definition trivially extends to a global complex frame, hence the Maslov index $\mu(E, F)$ vanishes.

Remark 140. Wentworth [123] derived the above formula using an interesting feature of the totally real boundary condition, that is, when equipped with the totally real boundary condition, the Dolbeault complex on M_ℓ is a Fredholm complex and we can apply the heat kernel method to compute the related index.

As for the term $\mathrm{Ind}_{\mathbb{R}}(\bar{\partial}_\gamma, \Pi, \cup_{k=1}^m Z_{k,y\geq\ell})$, this reduces to compute the index of the following operator $(\frac{\partial}{\partial\bar{z}} + \frac{i}{2}\alpha)$ over Z_k acting on function $f(z)$ subjecting to the totally real boundary condition on ∂Z_k.

First note that if any function $f(z)$ lying in the kernel of $\frac{\partial}{\partial\bar{z}} + \frac{i}{2}\alpha$, then $e^{-\alpha y} f(z)$ is a holomorphic function on Z_k. By the L^2 integrability with respect to the hyperbolic metric on Z_k, this implies that in the case when $\alpha \geq 0$, the kernel is of real dimensional one and its dimension is zero when $\alpha < 0$. On the other hand, the codimension of the cokernel is constant equals to one by the Poisson formula. Therefore, we have proved that

$$\mathrm{Ind}_{\mathbb{R}}(\bar{\partial}_\gamma, \Pi, \cup_{k=1}^m Z_{k,y\geq\ell}) = -\sum_{i,j,k} \alpha_{i,k} < \alpha_{j,k} = -2\sum_k \dim_{\mathbb{C}} \mathscr{F}_k,$$

as in (2.2). Putting all these together, we have found that the real index of the Dolbeault complex (3.12) is given by $(2-2h)n^2 - 2\sum_k \dim_{\mathbb{C}} \mathscr{F}_k$, and the dimension of the moduli space of stable parabolic bundles is then given by

$$\dim \mathscr{M}_P^s = n^2(h-1) + 1 + \sum_k \dim_{\mathbb{C}} \mathscr{F}_k.$$

Appendix C: Proof of Lemma 72

In this part, under the assumption of Lemma 72, we show that the following equality holds.

$$\bar{\partial}_A^* \bar{\partial}_A \frac{\partial^2}{\partial t \partial \bar{s}}(g^* g)\Big|_{t=s=0} = - * [*\mu, \nu].$$

Proof. Given the holomorphic family of Dolbeault operators $\bar{\partial}_A + t\mu + s\nu$, with t and s sufficiently small, by Lemma 67, there exists a unique smooth family $g(t,s) \in \tilde{\mathscr{G}}_\delta^{\mathbb{C}}$ and $\bar{\partial}_A + \gamma(t,s) \in U_{A,\epsilon}$ satisfying

$$\gamma = -\bar{\partial}_A(g)g^{-1} + g(t\mu + s\nu)g^{-1}, \tag{C.1}$$

$$\gamma^* = -(g^*)^{-1}\partial_A(g^*) + g^{*-1}(\bar{t}\mu^* + \bar{s}\nu^*)(g^*), \tag{C.2}$$

with $g(0,0) = \mathrm{Id}$ and $\gamma(0,0) = 0$. Since $\bar{\partial}_\gamma \in U_{A,\epsilon}^{0,1}$, we additionally have

$$-\bar{\partial}_A \gamma^* + \partial_A \gamma - [\gamma, \gamma^*] = 0, \tag{C.3}$$

$$\bar{\partial}_A \gamma^* + \partial_A \gamma = 0. \tag{C.4}$$

First, we show that the first order derivatives of g and g^* vanish. Take g as an example, differentiate (C.1), (C.3) and (C.4) with respect to t and let $t = s = 0$, we then get

$$\mu - \bar{\partial}_A(\frac{\partial}{\partial t}g\big|_{t=s=0}) = \frac{\partial}{\partial t}\gamma\big|_{t=s=0}, \tag{C.5}$$

$$-\bar{\partial}_A(\frac{\partial}{\partial t}\gamma^*\big|_{t=s=0}) + \partial_A(\frac{\partial}{\partial t}\gamma\big|_{t=s=0}) = 0, \tag{C.6}$$

$$\bar{\partial}_A(\frac{\partial}{\partial t}\gamma^*\big|_{t=s=0}) + \partial_A(\frac{\partial}{\partial t}\gamma\big|_{t=s=0}) = 0. \tag{C.7}$$

Apply ∂_A to (C.5) and use (C.6) and (C.7), we get

$$-\partial_A\bar{\partial}_A(\frac{\partial}{\partial t}g\big|_{t=s=0}) = \bar{\partial}_A(\frac{\partial}{\partial t}\gamma^*\big|_{t=s=0}) = \partial_A(\frac{\partial}{\partial t}\gamma\big|_{t=s=0}) = 0.$$

Since $\frac{\partial}{\partial t}g\big|_{t=s=0} \in S^{2,2}_{\delta,s}(\operatorname{End} E)$, on which Δ_A is invertible, hence $\frac{\partial}{\partial t}g\big|_{t=s=0} = 0$. The other claims about g and g^* follow similarly. From these properties of g and g^*, we get

$$\frac{\partial^2}{\partial t\partial\bar{s}}(g^*g)\Big|_{t=s=0} = \frac{\partial^2}{\partial t\partial\bar{s}}g\Big|_{t=s=0} + \frac{\partial^2}{\partial t\partial\bar{s}}g^*\Big|_{t=s=0}. \tag{C.8}$$

To compute this, we apply $\frac{\partial^2}{\partial t\partial\bar{s}}$ to (C.3) and let $t = s = 0$, and get

$$\partial_A\frac{\partial^2\gamma}{\partial t\partial\bar{s}}\Big|_{t=s=0} - \bar{\partial}_A\frac{\partial^2\gamma^*}{\partial t\partial\bar{s}}\Big|_{t=s=0} - [\frac{\partial\gamma}{\partial t}, \frac{\partial\gamma^*}{\partial\bar{s}}] - [\frac{\partial\gamma}{\partial\bar{s}}, \frac{\partial\gamma^*}{\partial t}] = 0.$$

Use the following $\partial_A\bar{\partial}_A\frac{\partial^2 g}{\partial t\partial\bar{s}}\Big|_{t=s=0} = -\partial_A\frac{\partial^2\gamma}{\partial t\partial\bar{s}}\Big|_{t=s=0}$ and $\bar{\partial}_A\partial_A\frac{\partial^2 g^*}{\partial t\partial\bar{s}}\Big|_{t=s=0} = -\bar{\partial}_A\frac{\partial^2\gamma^*}{\partial t\partial\bar{s}}\Big|_{t=s=0}$, and we apply the contraction by the Kähler form ω_g on (X,g), we get

$$\bar{\partial}_A^* \bar{\partial}_A \frac{\partial^2}{\partial t \partial \bar{s}} (g^* g)\Big|_{t=s=0} = -i\Lambda_{\omega_g} [\frac{\partial \gamma}{\partial t}, \frac{\partial \gamma^*}{\partial \bar{s}}] \tag{C.9}$$

$$= - * [*\mu, \nu]. \tag{C.10}$$

We have used the following equality in the last equality's $\frac{\partial}{\partial t}\gamma = \mu$ and $\frac{\partial}{\partial \bar{s}}\gamma = \nu^*$, the latter follows from (C.5) and its analogous result. □

Appendix D: A Parametrix Along the Cusp Ends

In this part, we use the method of [31, page 431-433] to show the existence of an right inverse $G_{\gamma,k}$ to the Dolbeault operator $\bar{\partial}_\gamma \in U^{0,1}_{A,\epsilon}$ along the cusp end $Z_k = [a,\infty) \times \mathbb{S}^1$.

Proposition 141. For any $f d\bar{z} \in C^\infty_c(\Lambda^{0,1}(\mathring{Z}_k) \otimes \operatorname{End} E)$, there exists a smooth section $u = G_{\gamma,k}(f d\bar{z}) \in L^2(Z_k, \operatorname{End} E)$ such that

$$\bar{\partial}_\gamma u = f\bar{z}.$$

Proof. Fix a temporal frame $\{e_i\}_{i=1}^n$ for the Dolbeault operator $\bar{\partial}_\gamma$ along Z_k, which exists due to Lemma 19. Hence we only need to prove the Proposition for the model Dolbeault operator (2.6), which is given componentwise as follows,

$$\bar{\partial}_0 u = \sum_{i,j} \Big(\frac{\partial}{\partial \bar{z}} u_{ij} + \frac{\mathrm{i}}{2}(\alpha_i - \alpha_j) u_{ij} \Big) d\bar{z} \otimes e_i \otimes e_j^* = \sum_{i,j} f_{ij} d\bar{z} \otimes e_i \otimes e_j^*. \tag{D.1}$$

In terms of Fourier expansion,

$$u_{ij}(x,y) = \sum_n u_{ij,n}(y) e^{\mathrm{i}nx} \text{ and } f_{ij}(x,y) = \sum_n f_{ij,n}(y) e^{\mathrm{i}nx},$$

and (D.1) becomes

$$\Big(\frac{d}{dy} + (n + \alpha_i - \alpha_j)\Big)u_{ij,n} = 2f_{ij,n}.$$

By knowledge of ordinary differential equations, we construct the "candidate" operator G_0 of $\bar{\partial}_0$ as follows:

$$u_{ij,n}(y) := G_0(fd\bar{z})_{ij} = \begin{cases} 2e^{-(n+\alpha_i-\alpha_j)y}\int_a^y e^{(n+\alpha_i-\alpha_j)t} f_{ij,n}(t)\,dt, \text{ if } n+\alpha_i-\alpha_j \geq 0; \\ 2e^{-(n+\alpha_i-\alpha_j)y}\int_y^\infty e^{(n+\alpha_i-\alpha_j)t} f_{ij,n}(t)\,dt, \text{ if } n+\alpha_i-\alpha_j < 0. \end{cases}$$

Clearly, it satisfies $\bar{\partial}_0 u = fd\bar{z}$. Now, we verify that G_0 is a bounded right inverse, i.e., there exists a constant $C > 0$ such that

$$\|G(f\bar{z})\| \leq C\|fd\bar{z}\|.$$

For the case $n + \alpha_i - \alpha_j > 0$, we have

$$\begin{aligned}\int_a^\infty |u_{ij,n}(y)|^2\frac{1}{y^2}\,dy &= 4\int_a^\infty e^{-2(n+\alpha_i-\alpha_j)y}\Big(\int_a^y e^{(n+\alpha_i-\alpha_j)t} f_{ij,n}(t)\,dt\Big)^2\frac{dy}{y^2} \\ &\leq C\int_a^\infty e^{-2(n+\alpha_i-\alpha_j)y}\int_a^y e^{2(n+\alpha_i-\alpha_j)t} f_{ij,n}^2(t)\,dt\,dy\end{aligned}$$

where we have used Hölder inequality in the second line. By change of order of integration,

$$\begin{aligned}&\leq C\int_a^\infty\int_t^\infty e^{-2(n+\alpha_i-\alpha_j)y}\,dy\,e^{2(n+\alpha_i-\alpha_j)t} f_{ij,n}^2(t)\,dt \\ &\leq \frac{C}{2(n+\alpha_i-\alpha_j)}\int_a^\infty f_{ij,n}^2(t)\,dt.\end{aligned}$$

In the case when $n+\alpha_i-\alpha_j = 0$, we set $F(y) = \int_a^y f_{ij,n}(t)\,dt$. Note that $F(a) = 0$ and

$\lim_{y\to\infty} F(y) = \text{const}$. By integration by parts, we get

$$\begin{aligned}\int_a^\infty F^2(y)\,\frac{dy}{y^2} &= \int_a^\infty \frac{d}{dy}F^2(y)\frac{1}{y}\,dy - 2\int_a F(y)F'(y)\frac{1}{y}\,dy\\ &\leq C\Big(\int_a^\infty f_{ij,n}^2(y)\,dy\Big)^{\frac{1}{2}}\Big(\int_a^\infty F^2(y)\,\frac{dy}{y^2}\Big)^{\frac{1}{2}}\\ &\leq C\int_a^\infty f_{ij,n}^2(y)\,dy.\end{aligned}$$

The case of $n + \alpha_i - \alpha_j < 0$ follows similarly, and we get our desired estimate. $\square$

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