SRC TR 87-145

Digital Controller Design for Multivariable Systems with Structural Closed-Loop Performance Specifications

by

E. Zafiriou and M. Morari

DIGITAL CONTROLLER DESIGN FOR MULTIVARIABLE SYSTEMS WITH STRUCTURAL CLOSED–LOOP PERFORMANCE SPECIFICATIONS

Evanghelos Zafiriou†

Manfred Morari*

California Institute of Technology

Chemical Engineering, 206-41

Pasadena, CA 91125

(818)356-4186

December 1986

Submitted to the International Journal of Control

† Current affiliation: Chemical and Nuclear Engineering Department and Systems Research Center, University of Maryland, College Park, MD 20742

* To whom all correspondence should be addressed

Abstract

The problem of the direct design of the closed-loop transfer function matrix is addressed for multivariable discrete systems. The limitations imposed by unstable zeros, time delays and the structure associated with these are quantified. A design procedure is formulated that provides the designer with quantitative measures for evaluating the tradeoffs between different closed-loop interaction structures and durations. The problem of intersample rippling is also considered. The procedure requires only linear algebra operations, includes the eventual construction of the feedback controller in state space and is presented in a way that allows its straightforward computer implementation.

1. Introduction

One can find in the control literature numerous different types of criteria for synthesizing or evaluating a control system. In most cases a number of performance considerations is lumped together into some objective function, which is then optimized with respect to the control system. Such approaches have been proven satisfactory in many cases. However there are situations in which one cannot simply optimize a single scalar objective function. In process control, such a case is that of setpoint tracking for multivariable systems. Quite often it is necessary to look at the closed-loop transfer function matrix relating the setpoints to the process outputs and require that certain elements of the matrix are equal to zero, so that setpoint changes in some outputs do not upset other important ones. Also, one may sometimes wish to allow such closed-loop interactions in order to improve setpoint tracking for the important outputs at the expense of upsetting less valuable ones. The same arguments carry over to certain cases of disturbance rejection. The paper treats setpoint tracking and disturbance rejection in a uniform way.

2. Achievable Input/Output Mappings

The discretized plant is described by the transfer matrix P(z), which is obtained by adding a zero order hold in front of the continuous plant and then taking the z–transform $P(z)$ is assumed to be square.

Let H_{oi}(z) denote the transfer matrix between output o and input i. We can define the following relations with respect to Fig. 1.

$$H_{ur} = C(I + PC)^{-1} \tag{2.0.1}$$

$$H_{ud} = -H_{ur} \tag{2.0.2}$$

$$H_{yr} = PC(I + PC)^{-1} = PH_{ur} \tag{2.0.3}$$

$$H_{yd} = (I + PC)^{-1} = I - H_{yr} = I - PH_{ur} \tag{2.0.4}$$

From (2.0.4) it follows that if the control system provides good setpoint tracking ($H_{yr}r \approx r$) then one has also good disturbance rejection ($H_{yd}d \approx 0$), provided that the disturbance d is of a type similar to the setpoint r. If this is not the case, then one has to design a Two- Degree-of-Freedom controller (Vidyasagar, 1985), whose design can actually be separated into designing two different controllers C, one for setpoint tracking and one for disturbance rejection and then appropriately combine them in one unified block structure (see, e.g., Morari et al., 1987). Hence, it is sufficient to cover here only the design of C (Fig. 1) for good setpoint tracking or disturbance rejection.

From (2.0.1) we can obtain

$$C = H_{ur}(I - PH_{ur})^{-1} \tag{2.0.5}$$

and so designing C is equivalent to designing H_{ur}, which is the IMC controller (Garcia & Morari, 1982) or the parameter of the Q -Parametrization (Zames, 1981). It can be shown (e.g., Callier and Desoer, 1982) that necessary and sufficient conditions for the internal stability of the system in Fig. 1 are

Condition C1:

i) H_{ur} stable

ii) PH_{ur} stable

iii) $H_{ur}P$ stable

iv) $(I - PH_{ur})P$ stable

C1.ii,iii,iv are implied by C1.i if P is stable. Hence the following assumption, which will be made throughout this paper allows to consider only C1.i:

Assumption A1: P is stable.

It should be pointed out however, that for setpoint tracking, the above assumption need not be made. In that case, the use of the Two-Degree-of-Freedom structure makes it sufficient to consider C1.i only, even when P is unstable. The problem is then reduced to the one discussed in this paper in which A1 holds (Vidyasagar, 1985; Morari et al. 1987).

The controller $C(z)$ has to be causal since future measurements of the plant output are not known. It follows from (2.0.5) that an equivalent condition is

Condition C2: H_{ur} causal

One can see from the above discussion that the control objective can be reduced to finding an H_{yr}(z) with the desired structure and properties, which can be produced through (2.0.3) by an H_{ur}(z) that satisfies C1.i and C2. However looking only at H_{yr}(z) for checking the performance of the control system may be insufficient because of the phenomenon of intersample rippling. This phenomenon is present when H_{ur}(z) has poles near (-1,0) which are cancelled by zeros of $P(z)$ in (2.0.3). Hence, in order to make it sufficient to judge performance by looking at H_{yr}(z) only, H_{ur}(z) must also satisfy the following condition.

Condition C3: H_{ur} cancels no zeros of P that are "near" (-1,0).

One can use a number of different regions on the z–plane to define "near" (-1,0) (Astrom and Wittenmark, 1984). A simple and satisfactory in practice way to do that, is to include all zeros with negative real part (Zafiriou and Morari, 1985).

3. Characterization of All Permissible $\mathbf{H_{yr}(z)}$

From (2.0.3) it follows

$$H_{ur} = P^{-1} H_{yr} \tag{3.0.1}$$

Hence the conditions of section 2 on H_{ur} can be translated into the following condition on H_{yr}:

Condition C4: H_{yr} is a stable, causal transfer matrix that makes $P^{-1} H_{yr}$ causal and cancels the poles of P^{-1} (zeros of P) that are outside the unit circle or near (-1,0).

The time delays in $P(z)$, which make P^{-1} non–causal, appear as zeros at infinity. We shall now exploit this fact to make the treatment of time delays and undesirable zeros of P uniform. The transformation $\lambda = z^{-1}$ will be used. Define

$$\hat{P}(\lambda) \stackrel{\text{def}}{=} P(\lambda^{-1}) \leftrightarrow P(z) \tag{3.0.2}$$

$$\hat{H}_{yr}(\lambda) \stackrel{\text{def}}{=} H_{yr}(\lambda^{-1}) \leftrightarrow H_{yr}(z) \tag{3.0.3}$$

Let $a_1, \ldots, a_f$ be the zeros of P(z), which according to C4 we do not wish to appear as poles of $P(z)^{-1}H_{yr}(z)$. These will appear in $\hat{P}(\lambda)^{-1}$ as poles at $b_1, \ldots, b_f$ where

$$b_i = 1/a_i, \qquad i = 1, \ldots, f \tag{3.0.4}$$

The time delays in $P(z)$ will give rise to zeros at 0 in $\hat{P}(\lambda)$ and consequently the non–causal terms in $P(z)^{-1}$ will produce poles at 0 in $\hat{P}(\lambda)^{-1}$. Hence C4 is equivalent to:

Condition C5:

i) $H_{yr}(z)$ is a stable, causal transfer matrix

ii) $\hat{P}(\lambda)^{-1}\hat{H}_{yr}(\lambda)$ has no poles at $b_0, b_1, \ldots, b_f$.

In the above the following notation was used:

$$b_0 = 0 \tag{3.0.5}$$

Some additional notation and definitions are now needed. $P(z)$ (and $\hat{P}(\lambda)$) is assumed to have dimension $r \times r$ and to be of normal rank r. In the following it will be assumed that $\hat{P}(\lambda)$ has no poles at $b_0, \ldots, b_f$. This is certainly the case for b_0 since all elements of $P(z)$ are proper, but in general P(z) may have poles at $\alpha_1, \ldots, \alpha_f$ resulting in poles at $b_1, \ldots, b_f$ in $\hat{P}(\lambda)$. The existence of poles and zeros at the same location is a clearly multivariable characteristic (Kailath, 1980). The assumption that this is not the case for $P(z)$ serves in considerably simplifying the notation and it is not restrictive since such a phenomenon is caused by exact cancellations in $det[P(z)]$ which will not happen if a slight perturbation in the terms of $P(z)$ is introduced. Let $\{n_0, n_1, \ldots, n_f\}$ be a set of integers greater or equal to zero, such that

$$\hat{P}^{(k)}(b_i) = 0, \quad ((k = 0, \ldots, n_i - 1), i = 0, \ldots, f) \tag{3.0.6}$$

$$rank[\hat{P}^{(n_i)}(b_i)] \neq 0, \qquad i = 0, \ldots, f \tag{3.0.7}$$

where $\hat{P}^{(k)}(\lambda)$ is the kth derivative of $\hat{P}(\lambda)$. Also let $m_i, i = 0, \ldots, f$, be the order of the zero b_i of $\hat{P}(\lambda)$ as this order is defined from the Smith–McMillan form of $\hat{P}(\lambda)$ (Desoer and Schulman, 1974). The computation of m_i without going through the Smith–McMillan form is briefly discussed in Section 4.2. From (3.0.6), (3.0.7) and the definition of the order of a zero, it follows that

$$m_i \geq n_i, \qquad i = 0, \ldots, f \tag{3.0.8}$$

The following theorem quantifies C5.ii:

Theorem 1.

Condition C5.ii holds if and only if both (a) and (b) hold:

a) $\hat{H}_{yr}(\lambda) = (\lambda - b_i)^{n_i} \hat{H}_i(\lambda), \qquad i = 0, \ldots, f$

where $\hat{H}_i(\lambda)$ is a rational $r \times r$ matrix in λ, with no poles at b_i.

b) for any $i = 0, \ldots, f$ such that $m_i > n_i$, the columns of

$$[\hat{H}_i^{(0)}(b_i)^T \ldots \frac{1}{(m_i - n_i - 1)!} \hat{H}_i^{(m_i - n_i - 1)}(b_i)^T]^T$$

are in the column space of

$$M_i \stackrel{\text{def}}{=} \begin{bmatrix} \frac{1}{n_i!}\hat{P}^{(n_i)}(b_i) & 0 & \cdots & 0 \\ \frac{1}{(n_i+1)!}\hat{P}^{(n_i+1)}(b_i) & \frac{1}{n_i!}\hat{P}^{(n_i)}(b_i) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{(m_i-1)!}\hat{P}^{(m_i-1)}(b_i) & \frac{1}{(m_i-2)!}\hat{P}^{(m_i-2)}(b_i) & \cdots & \frac{1}{n_i!}\hat{P}^{(n_i)}(b_i) \end{bmatrix} \tag{3.0.9}$$

where the superscript (k) indicates k^{th} derivative and T the transpose of a matrix.

Proof: See Appendix A.

The value of Theorem 1 lies in the fact that it provides a characterization of all acceptable $\hat{H}_{yr}(\lambda)$ without requiring the inversion of $\hat{P}(\lambda)$. The theorem applies to the general case. However in practice one is usually faced with a situation where the order of the zeros $a_1, \ldots, a_f$ of the model $P(z)$ is equal to 1. Hence of the zeros $b_0, b_1, \ldots, b_f$ of $\hat{P}(\lambda)$ only b_0 has an order larger than 1. The fact that b_0 is equal to zero (Eq. (3.0.5)) can then be used to obtain a simpler form of Theorem 1. The following two Corollaries describe these situations:

Corollary 1.

Let the order of the zero a_i of $P(z)$ be equal to one. Then $\hat{P}(\lambda)^{-1}\hat{H}_{yr}(\lambda)$ has no poles at b_i if and only if the columns of $\hat{H}_{yr}(b_i)$ are in the column space of $\hat{P}(b_i)(= P(a_i))$.

Proof: It follows directly from Theorem 1 for $m_\ell = 1$.

Corollary 2.

Let $P(z)$ have the impulse response coefficient description

$$P(z) = z^{-N}(A_0 + A_1 z^{-1} + A_2 z^{-2} + \ldots) \tag{3.0.10}$$

where

$$rank[A_0] \neq 0 \tag{3.0.11}$$

$$N \geq 0 \tag{3.0.12}$$

Then

$$n_0 = N \tag{3.0.13}$$

$$M_0 = \begin{bmatrix} A_0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ A_{m_0-1-N} & \ldots & A_0 \end{bmatrix} \tag{3.0.14}$$

and $\hat{P}(\lambda)^{-1}\hat{H}_{yr}(\lambda)$ has no poles at $b_0 = 0$ if and only if both (a) and (b) hold:

a) $\hat{H}_{yr}(\lambda) = \lambda^N \hat{H}_0(\lambda)$ where $\hat{H}_0(\lambda)$ is a rational matrix in λ with no poles at $b_0 = 0$.

b) if $m_0 > N$, the columns of $[\hat{H}_0^{(0)}(0)^T \ldots \frac{1}{(m_0-N-1)!}\hat{H}_0^{(m_0-N-1)}(0)^T]^T$ are in the column space of M_0.

Proof: From (3.0.2) we get $\hat{P}(\lambda) = P(\lambda^{-1}) = \lambda^N(A_0 + A_1\lambda + A_2\lambda^2 + \ldots)$. Equations (3.0.13) and (3.0.14) can now be obtained by repeated differentiation and evaluation at $\lambda = 0$. The rest follows as a restatement of Theorem 1 for this special case.

4. Construction of $H_{yr}(z)$.

4.1 The Form of $H_{yr}(z)$.

Theorem 1 and its Corollaries quantify the restrictions that are imposed on H_{yr} from the zeros and time delays. The designer can select any overall transfer function H_{yr} he considers appropriate for the particular system, provided that it satisfies those restrictions. The choice can be made between decoupled and non–decoupled response and the location of the non–zero elements of $H_{yr}(z)$ can be directly specified. A detailed procedure for doing so will be developed in this section and quantitative criteria for the evaluation of different designs will be obtained. Before proceeding, the form of the non–zero elements of $H_{yr}(z)$ should be discussed. The possibilities are of course infinite but three simple rules will be stated and the reasoning behind them briefly explained.

Rule 1. For a given set of locations for the non–diagonal elements of H_{yr} which are allowed to be non–zero, the design should be such that in each diagonal element of $\hat{H}_{yr}(\lambda)$, every term $(\lambda - b_i)^{\kappa_i}, i = 0, \ldots, f$, has the smallest possible power κ_i.

Rule 2. If in a diagonal element of $\hat{H}_{yr}(\lambda)$, a factor $(\lambda - b_i)^{\kappa_i}$ has to appear, then one should use the factor $\frac{(1-b_i^{-1})^{\kappa_i}(\lambda-b_i)^{\kappa_i}}{(1-b_i)^{\kappa_i}(\lambda-b_i^{-1})^{\kappa_i}}$ if b_i has positive real part and the term $\frac{(\lambda-b_i)^{\kappa_i}}{(1-b_i)^{\kappa_i}}$ otherwise.

Rule 3. The non–zero, non–diagonal elements of $\hat{H}_{yr}(\lambda)$ should have the form $\lambda^{\delta}(\beta_0+\beta_1\lambda+\ldots+\beta_\nu\lambda^\nu)(1-\lambda)$.

The reasoning behind Rule 1 is that one wishes the effect of the undesirable zeros and time delays on the response of an element of the output vector to the corresponding setpoint or disturbance, to be as small as possible.

The problem that Rule 2 addresses is exactly the same as the one for the SISO case. Rule 2 is a rule obtained by Zafiriou and Morari (1985). Briefly, it introduces the pole at the inverse of the zero in order to minimize the sum of the squared errors to an external step input. In the case where the zero has negative real part this action would introduce intersample rippling, which is avoided by making a deadbeat type selection. At the same time no significant overshoot or undershoot appears. It is also possible to do the design for external inputs other than steps. This would result in a different expression for the factor in Rule 2 (Zafiriou and Morari, 1986a), which however can be used without any changes in the procedure that will be developed in the following sections.

Rule 3 makes sure that the steady–state gain of the non–diagonal elements of $H_{yr}(z)$ is zero, by including the term $(1-\lambda)\leftrightarrow(1-z^{-1})$. Also the parameters $\beta_0,\ldots,\beta\nu$ have physical meaning because $z^{-\delta}(\beta_0+\beta_1 z^{-1}+\ldots+\beta_\nu^{-\nu})$ is the step response for the corresponding pair of system output and external input. Hence one wishes to have ν small and at the same time the magnitudes of $\beta_0,\ldots,\beta_\nu$ to be small. The trade–off between these two goals will be discussed in section 4.3.iii.

The above three rules are not really restrictive and they will simplify the design procedure. It should be noted that as a result of those rules, the closed–loop steady–state gain will be

$$H_{yr}(1)=I \tag{4.1.1}$$

Hence the control system will be such that no steady–state offset is produced for external inputs r (or d) with one pole at z = 1 (step–like inputs). For inputs with more poles at z = 1 (ramp–like, etc.), the no–offset property holds when the appropriate factor is used in Rule 2 (Zafiriou and Morari, 1986).

4.2. Zeros of P(z)

The first step towards the construction of H_{yr} is clearly the computation of the zeros of $P(z)$ and of their respective orders as well as the computation of the order m_0 of the zero $b_0 = 0$ of $\hat{P}(\lambda)$. For a square system $P(z)$ the zeros can be computed as the roots of $det[P(z)] = 0$, provided that there are no cancellations with any poles. Numerically better techniques for the computation of the zeros can be found in the literature (Laub and Moore, 1978) and a number of software packages for this computation exist.

The following theorem provides a method for computing the order of a zero without having to find the Smith–McMillan form.

Theorem 2. (Van Dooren et al., 1979; rephrased)

Let

$$M_{i,k} \stackrel{\text{def}}{=} \begin{bmatrix} \frac{1}{n_i!}\hat{P}^{(n_i)}(b_i) & \dots & 0 \\ \vdots & \ddots & \vdots \\ \frac{1}{k!}\hat{P}^{(k)}(b_i) & \dots & \frac{1}{n_i!}\hat{P}^{(n_i)}(b_i) \end{bmatrix} \tag{4.2.1}$$

then

$$m_i = min\{k | rank[M_{i,k}] - rank[M_{i,k-1}] = r\}, \qquad i = 0, \dots, f \tag{4.2.2}$$

It was mentioned in Section 3 that usually in practice the order of the zeros $b_1, \dots, b_f$ is one. Theorem 2 is then useful in computing the order m_0 of b_0. In this case $M_{0,k}$ can be written in terms of the impulse response matrices defined in (3.0.10):

$$M_{0,k} = \begin{bmatrix} A_0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ A_{k-N} & \dots & A_0 \end{bmatrix} \tag{4.2.3}$$

The discussion of some computational aspects is necessary at this point. Theorem 2 explicitly requires the computation of the rank of $M_{i,k}$ for all $k = n_i, \dots, m_i$. Also in order to use Theorem 1 effectively in a design procedure it is necessary to reduce $M_i (= M_{i,m_i-1})$ in (3.0.9) to a form with linearly independent columns so that a basis for its column space is available. The Singular Value Decomposition (SVD) is a very reliable method for both purposes. However, its application on the matrices $M_{i,k}$ whose dimension can grow very large might be difficult and time–consuming. Van Dooren et al. (1979) have exploited the Toeplitz matrix form of $M_{i,k}$ to develop a fast recursive algorithm that performs the rank search in a numerically stable way. In each step the rank of $M_{i,k}$ is computed for some k by obtaining the SVD of an $r \times r$ matrix. At the same time $M_{i,k}$ is reduced to a form with linearly independent columns. Hence to obtain m_i and an orthonormal basis for the column space of M_i one has to obtain the SVD of only $(m_i - n_i + 1)$ matrices of dimension $r \times r$.

4.3 Design of a Column of $\mathbf{H_{yr}(z)}$

The requirements of Theorem 1 apply to each column of H_{yr} separately and so each column can be designed independently. Let us write

$$H_{yr}(z) = [\, h_1(z) \quad \dots \quad h_r(z) \,] \tag{4.3.1}$$

where $h_j(z)$ has dimension $r \times 1, j = 1, \dots, r$. Also let

$$\hat{h}_j(\lambda) = h_j(\lambda^{-1}) \leftrightarrow h_j(z), \qquad j = 1, \dots, r \tag{4.3.2}$$

We shall now proceed with the design of $\hat{h}_j(\lambda)$ for some j. Let U_i be a matrix whose columns form an orthonormal basis for the column space of M_i given in (3.0.9). U_i can be obtained from M_i with the procedure of Van Dooren et al. (1979) briefly discussed in Section 4.2. Also let

$$\rho_i = rank[M_i] = rank[U_i] \tag{4.3.3}$$

According to Theorem 1 we must have

$$\hat{h}_j(\lambda) = (\lambda - b_i)^{n_i} \hat{h}_{j,i}(\lambda), \qquad i = 0, \ldots, f \tag{4.3.4}$$

where

$$\eta_{j,i} \stackrel{\text{def}}{=} [\hat{h}_{j,i}^{(0)}(b_i)^T \cdots \frac{1}{(m_i - n_i - 1)!} \hat{h}_{j,i}^{(m_i - n_i - 1)}(b_i)^T]^T \tag{4.3.5}$$

is a linear combination of the columns of U_i, i.e.,

$$\eta_{j,i} = U_i \chi_i^1 \tag{4.3.6}$$

where χ_i^1 is any vector of dimension ρ_i. The freedom allowed in the choice of χ_i^1 will now be gradually reduced by requiring certain properties for $\hat{h}_j(\lambda)$, according to the designer's specifications and decisions. First, the limitations imposed by the desired structure of h_j will be quantified. Then, the undesirable zeros and time delays that have to be present in the diagonal element will be determined and the design of this element will be reduced to that of a SISO system. Finally, the non-diagonal elements will be designed so that the closed-loop interactions are minimized. It should be pointed that if for some i we have $m_i = n_i$, then part (b) of Theorem 1 and therefore (4.3.6) do not apply for that i and so all equations in this section corresponding to that particular i should be ignored.

i) Structure of h_j

Let the design specification be that the $\ell_1, \ell_2, \ldots, \ell_g$ elements of $h_j(z)$ be identically equal to zeros, where

$$g \leq r - 1 \tag{4.3.7}$$

$$\ell_k \neq j, \qquad k = 1, \ldots, g \tag{4.3.8}$$

We shall use ℓ to denote the set

$$\ell \stackrel{\text{def}}{=} \{\ell_1, \ldots, \ell_g\} \tag{4.3.9}$$

Define

$$\bar{\ell} \stackrel{\text{def}}{=} \{1,2,\ldots,r\} - \{j\} - \ell \tag{4.3.10}$$

Let

$$e_k \stackrel{\text{def}}{=} [0\ldots0 \quad 1 \quad 0\ldots0]^T \tag{4.3.11}$$

where the 1 is the k^{th} element and e_k has dimension $r \times 1$.

Define

$$e^{\ell} \stackrel{\text{def}}{=} \begin{bmatrix} e_{\ell_1}^T \\ \vdots \\ e_{\ell_g}^T \end{bmatrix} \tag{4.3.12}$$

$$\Lambda_i^{\ell} \stackrel{\text{def}}{=} diag[\underbrace{e^{\ell},\ldots,e^{\ell}}_{(m_i-n_i)}], \qquad i = 1,\ldots,f \tag{4.3.13}$$

In order for the specified elements of h_j to be zero, the vector χ_i^1 must solve:

$$\Lambda_i^{\ell}\eta_{j,i} = 0$$

or

$$\Lambda_i^{\ell}U_i\chi_i^1 = 0 \tag{4.3.14}$$

Let

$$\rho_i^{\ell} = rank[\Lambda_i^{\ell}M_i] = rank[\Lambda_i^{\ell}U_i] \tag{4.3.15}$$

Then $\rho_i \geq \rho_i^{\ell}$. Hence the null space of $\Lambda_i^{\ell}U_i$ has dimension

$$\xi_i^{\ell} = \rho_i - \rho_i^{\ell} \tag{4.3.16}$$

Let V_i^{ℓ} be a matrix whose columns form an orthonormal basis for the null space of $\Lambda_i^{\ell}U_i$. Both V_i^{ℓ} and ρ_i^{ℓ} can be obtained from an SVD of $\Lambda_i^{\ell}U_i$. Then the solutions to (4.3.14) are:

$$\chi_i^1 = V_i^{\ell}\chi_i^2 \tag{4.3.17}$$

where χ_i^2 can be any vector of dimension ξ_i^{ℓ} when $\xi_i^{\ell} \neq 0$. If $\xi_i^{\ell} = 0$ then of course $V_i^{\ell} = 0$ and $\chi_i^1 = 0$.

Hence we must have

$$\eta_{j,i} = U_i V_i^{\ell} \chi_i^2 \tag{4.3.18}$$

where $\eta_{j,i}$ was defined in (4.3.5). Note that (4.3.18) includes the case $\xi_i^{\ell} = 0$, where $V_i^{\ell} = 0$ yields $\eta_{j,i} = 0$.

ii) Diagonal element of h_j

We shall now proceed with the determination of the j^{th} element of h_j. Up to this point no assumption has been made on the order of the zeros $b_0, b_1, \ldots, b_f$. However if more than one zero has order larger than 1, then the number of possible choices to be examined at this point could grow enormously. On the other hand, in practice one is usually faced with a situation where the order and degree (as defined from the Smith–McMillan form (Desoer and Schulman, 1974); also see Lemma A.1 in Appendix A) of the zeros $a_1, \ldots, a_f$ of $P(z)$ and therefore of the zeros $b_1, \ldots, b_f$ of $\hat{P}(\lambda)$, is equal to 1. The following assumption will be made here to simplify the procedure.

Assumption A2: The degree of the zeros $a_1, \ldots, a_f$ of $P(z)$ is equal to 1.

No assumption is made however about the zero $b_0 = 0$ of $\hat{P}(\lambda)$ corresponding to time delays in P(z). We shall examine the two cases separately.

a) $b_i, i = 1, \ldots, f$

It follows from A2 that for the order of the zeros m_i we have $m_i = 1$. Also since $r \geq 2$, A2 implies that $n_i = 0$. Then from (4.3.4), (4.3.5), (4.3.6) it follows that since 0 is a linear combination of the columns of U_i, the highest power of $(\lambda - b_i)$ that is sufficient to include in the elements of $\hat{h}_j(\lambda)$ is $(\lambda - b_i)^1$. However according to Rule 1 we wish to have the smallest possible power in the j^{th} element, and that is $(\lambda - b_i)^0 = 1$. In order for this to be possible, the following equation must have a solution

$$e_j^T \eta_{j,i} = 1$$

or

$$e_j^T U_i V_i^{\ell} \chi_i^2 = 1 \tag{4.3.19}$$

where e_j is defined in (4.3.11). Eq. (4.3.19) will have no solution only if the matrix $e_j^T U_i V_i^{\ell}$ is identically zero. If this happens for some i, then the factor $(\lambda - b_i)$ must be included in the j^{th} element of $\hat{h}_j(\lambda)$. Let us assume that this matrix is zero only for $i = 1, \ldots, \phi_1$. Also let the zeros $b_1, \ldots, b_{\phi_2} (\phi_2 \leq \phi_1)$ have positive real part and the zeros $b_{\phi_2+1}, \ldots, b_{\phi_1}$ have negative real part. Then according to Rule 2, the factor

$$\varsigma_0(\lambda) = \prod_{i=1}^{\phi_2} \frac{(1 - b_i^{-1})(\lambda - b_i)}{(1 - b_i)(\lambda - b_i^{-1})} \prod_{i=\phi_2+1}^{\phi_1} \frac{(\lambda - b_i)}{(1 - b_i)} \tag{4.3.20}$$

should be included in the j^{th} element.

Note that one does not always have to follow Rule 1. One may wish to include the factor $(\lambda - b_i)$ for some i in the j^{th} element even when one does not have to do it, if that will result in significantly smaller interactions (non–diagonal elements) and if the j^{th} output is not so important. The procedure for determining the magnitude of the interactions will then be exactly the same (see Section 4.3.iii) and at the end the designer can decide whether inclusion of $(\lambda - b_i)$ pays off. A simple qualitative way to figure out a priori whether it may pay off, without going through the whole design procedure, is the following. For $m_i = 1$ and $n_i = 0$ we have $rank[U_i] = r - 1$. U_i consists of the first $(r - 1)$ columns of the left singular vector matrix in an SVD of M_i. The r^{th} column u_i is orthogonal to all the columns of U_i. If the j^{th} element of u_i is large compared to the k^{th} elements where k belongs to the set $\bar{\ell}$ defined in (4.3.10), then it is likely that inclusion of $(\lambda - b_i)$ in the j^{th} element will result in significantly smaller interactions in the non–zero non–diagonal elements of h_j.

b) $\underline{b_0 (= 0)}$ (Time delays)

Define

$$\varsigma_\tau(\lambda) = \lambda^\tau \varsigma_0(\lambda) \tag{4.3.21}$$

where τ is an integer. Then according to Rule 1 we need to find the smallest τ such that $\varsigma_\tau(\lambda)$ is possible as the j^{th} element of $\hat{h}_{j,0}(\lambda)$. From (4.3.5), (4.3.6) it follows that in order for a τ to be possible, the following equation must have a solution

$$\epsilon_j \eta_{j,0} = Z_\tau$$

or

$$\epsilon_j U_0 V_0^{\ell} \chi_0^2 = Z_\tau \tag{4.3.22}$$

where

$$\epsilon_j = diag[\underbrace{e_j^T, \ldots, e_j^T}_{(m_0 - n_0)}] \tag{4.3.23}$$

and

$$Z_\tau = [\varsigma_\tau^{(0)}(0) \cdots \frac{1}{(m_0 - n_0 - 1)!} \varsigma_\tau^{(m_0 - n_0 - 1)}(0)]^T \tag{4.3.24}$$

Hence one can obtain the smallest possible τ as

$$\tau_0 = min\{\tau \in N_0 | rank[\epsilon_j U_0 V_0^{\ell} | Z_\tau] = rank[\epsilon_j U_0 V_0^{\ell}]\} \tag{4.3.25}$$

where N_0 is the set of positive integers, including zero.

Still, contrary to Rule 1, one may wish to choose a τ larger than τ_0 if that results in smaller interactions for a given set ℓ. Eq. (4.3.22) should, of course, have a solution for this τ, i.e., the rank condition in (4.3.25) should hold. In the following paragraph τ_0 is used, but any other possible τ can be used instead, without affecting the procedure for determining the magnitude of the interactions in Section 4.3.iii.

The j^{th} element of $\hat{h}_j(\lambda)$ has been completely determined at this point as

$$e_j^T \hat{h}_j(\lambda) = \lambda^N \varsigma_{\tau_0}(\lambda) \tag{4.3.26}$$

Let us now quantify the limitations that the selection of this diagonal element imposes on $\chi_i^2, i = 0, 1, \ldots, f$. The following equations have to be satisfied.

$$e_j^T U_i V_i^\ell \chi_i^2 = b_i^{n_o} \varsigma_{\tau_0}(b_i), \qquad i = 1, \ldots, f \tag{4.3.27}$$

$$\epsilon_j U_0 V_0^\ell \chi_0^2 = Z_{\tau_0} \tag{4.3.28}$$

Let $\chi_i^0, i = 0, \ldots, f$, be a particular solution for each corresponding equation, obtained with some method for solving systems of linear equations. Also let $W_i^\ell, i = 1, \ldots, f$ be a matrix whose columns form an orthonormal basis for the null space of $e_j^T U_i V_i^\ell$, and W_0^ℓ the corresponding matrix for $\epsilon_j U_0 V_0^\ell$. These matrices and their ranks w_i^ℓ, can be obtained from an SVD. Then the χ_i^2's that solve the set of equations (4.3.27), (4.3.28) are:

$$\chi_i^2 = \chi_i^0 + W_i^\ell \chi_i^3, \qquad i = 0, 1, \ldots, f \tag{4.3.29}$$

where χ_i^3 is any vector of dimension w_i^ℓ, when $w_i^\ell \neq 0$. If $w_i^\ell = 0$ then $W_i^\ell = 0$ and $\chi_i^2 = \chi_i^0$.

From (4.3.18) and (4.3.29) we obtain

$$\eta_{j,i} = U_i V_i^\ell \chi_i^0 + U_i V_i^\ell W_i^\ell \chi_i^3, \qquad i = 0, 1, \ldots, f \tag{4.3.30}$$

iii). Non-diagonal elements of h_j

The part of the procedure that was developed in Section 4.3.i makes sure that the elements of h_j corresponding to the set ℓ defined in (4.3.9), are identically equal to zero. We shall now proceed to compute the terms in the non–zero non–diagonal elements of h_j, i.e., the elements corresponding to the set $\bar{\ell}$, defined in (4.3.10). To do so the freedom allowed in the choice of χ_i^3 will be used.

Let $\bar{\ell}_1, \bar{\ell}_2, \ldots, \bar{\ell}_q$ be the elements of the set $\bar{\ell}$. According to Rule 3 the $\bar{\ell}_k^{th}$ element of $\hat{h}_j(\lambda)$ should be of the form

$$e_{\bar{\ell}_k}^T \hat{h}_j(\lambda) = B_k(\lambda) \stackrel{\text{def}}{=} \lambda^{\delta_k}(\beta_{k,0} + \beta_{k,1}\lambda + \ldots + \beta_{k,\nu}\lambda^\nu)(1-\lambda), \qquad k = 1, \ldots, q \tag{4.3.31}$$

From Corollary 2.a it follows:

$$\delta_k = n_0 = N, \qquad k = 1, \ldots, q \tag{4.3.32}$$

The values of $\beta_{k,0}, \ldots, \beta_{k,\nu}, k = 1, \ldots, q$ will be computed from (4.3.5) and (4.3.30). Note that any of the $\beta's$ can be zero, including the first ones, β_0, β_1, etc.

Define

$$e^{\bar{\ell}} \stackrel{\text{def}}{=} \begin{bmatrix} e^T_{\bar{\ell}_1} \\ \vdots \\ e^T_{\bar{\ell}_q} \end{bmatrix} \tag{4.3.33}$$

$$\epsilon^{\bar{\ell}} \stackrel{\text{def}}{=} \begin{bmatrix} \epsilon_{\bar{\ell}_1} \\ \vdots \\ \epsilon_{\bar{\ell}_q} \end{bmatrix} \tag{4.3.34}$$

where $\epsilon_{\bar{\ell}_k}$ is defined as in (4.3.23) for $\bar{\ell}_k$ instead of j. As explained in Section 4.3.ii.a, we have $n_i = 0$ for $i = 1, \ldots, f$. From (4.3.4), (4.3.5), (4.3.31), (4.3.32) it follows:

$$e^{\bar{\ell}}\eta_{j,i} = \begin{bmatrix} B_1(b_i) \\ \vdots \\ B_q(b_i) \end{bmatrix} = diag[\underbrace{\gamma_{i,\nu}, \ldots, \gamma_{i,\nu}}_{q}] \begin{bmatrix} \theta_{1,\nu} \\ \vdots \\ \theta_{q,\nu} \end{bmatrix}, \qquad i = 1, \ldots, f \tag{4.3.35}$$

where

$$\gamma_{i,\nu} \stackrel{\text{def}}{=} \left[(1-b_i)b_i^N \quad (1-b_i)b_i^{(N+1)} \quad \ldots \quad (1-b_i)b_i^{(N+\nu)} \right], \qquad i = 1, \ldots, f \tag{4.3.36}$$

$$\theta_{k,\nu} \stackrel{\text{def}}{=} [\beta_{k,0} \quad \beta_{k,1} \quad \ldots \quad \beta_{k,\nu}]^T, \qquad k = 1, \ldots, q \tag{4.3.37}$$

It also follows that

$$\epsilon^{\bar{\ell}}\eta_{j,0} = \begin{bmatrix} B_1^0 \\ \vdots \\ B_q^0 \end{bmatrix} = diag\,[\underbrace{\Gamma_\nu, \ldots, \Gamma_\nu}_{q}] \begin{bmatrix} \theta_{1,\nu} \\ \vdots \\ \theta_{q,\nu} \end{bmatrix} \tag{4.3.38}$$

where

$$B_k^0 = [\,\beta_{k,0} \quad (\beta_{k,1} - \beta_{k,0}) \quad \ldots \quad (\beta_{k,m_0-N-1} - \beta_{k,m_0-N-2})\,]^T, \qquad k = 1, \ldots, q \tag{4.3.39}$$

$$\beta_{k,\mu} = 0 \qquad for \qquad \mu > \nu, \qquad k = 1, \ldots, q \tag{4.3.40}$$

and Γ_ν is a matrix containing the first $\nu + 1$ columns of $[\Gamma 000 \ldots]$ with

$$\Gamma = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 & 0 & 0 \\ -1 & 1 & 0 & \ldots & 0 & 0 & 0 \\ 0 & -1 & 1 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & -1 & 1 \end{bmatrix}_{(m_0 - N)\times(m_o - N)} \tag{4.3.41}$$

Then the use of (4.3.30) in (4.3.35), (4.3.38) and combination of the resulting equations yields:

$$K_\nu \theta_\nu = T_1 + T_2 X \tag{4.3.42}$$

where

$$\theta_\nu \stackrel{\text{def}}{=} \begin{bmatrix} \theta_{1,\nu} \\ \vdots \\ \theta_{q,\nu} \end{bmatrix} \tag{4.3.43}$$

$$K_\nu \stackrel{\text{def}}{=} \begin{bmatrix} diag[\Gamma_\nu, \ldots, \Gamma_\nu] \\ diag[\gamma_{1,\nu}, \ldots, \gamma_{1,\nu}] \\ \vdots \\ diag[\underbrace{\gamma_{f,\nu}, \ldots, \gamma_{f,\nu}}_{q}] \end{bmatrix} \tag{4.3.44}$$

$$X \stackrel{\text{def}}{=} [(\chi_0^3)^T \qquad (\chi_1^3)^T \ldots (\chi_f^3)^T]^T \tag{4.3.45}$$

$$T_1 \stackrel{\text{def}}{=} \begin{bmatrix} \epsilon^{\bar{\ell}} U_0 V_0^\ell \chi_0^0 \\ e^{\bar{\ell}} U_1 V_1^\ell \chi_1^0 \\ \vdots \\ e^{\bar{\ell}} U_f V_f^\ell \chi_f^0 \end{bmatrix} \tag{4.3.46}$$

$$T_2 \stackrel{\text{def}}{=} diag[\epsilon^{\bar{\ell}} U_0 V_0^\ell W_0^\ell, e^{\bar{\ell}} U_1 V_1^\ell W_1^\ell, \ldots, e^{\bar{\ell}} U_f V_f^\ell W_f^\ell] \tag{4.3.47}$$

Equation (4.3.42) can also be written as

$$[K_\nu | - T_2] \begin{bmatrix} \theta_\nu \\ X \end{bmatrix} = T_1 \tag{4.3.48}$$

Hence the smallest possible ν for h_j and for the particular choice of set ℓ, can be obtained as the smallest ν for which (4.3.48) has a solution, i.e.,

$$\nu_{min} = min\{\nu \in N_0 | rank[K_\nu | - T_2] = rank[K_\nu | - T_2 | T_1]\} \tag{4.3.49}$$

However instead of trying to minimize ν, a better alternative to use a larger ν and use the extra degrees of freedom to minimize the sum of the squared errors for the step response of the $\bar{\ell}_1, \ldots, \bar{\ell}_q$ system outputs to the j^{th} external input (j^{th} element of r or d). This means minimizing

$$J_\nu \stackrel{\text{def}}{=} \sum_{k=1}^{q} (\phi_k \sum_{\mu=0}^{\nu} \beta_{k,\mu}^2) = \theta_\nu^T \Phi^T \Phi \theta_\nu \tag{4.3.50}$$

where $\phi_k, k = 1, \ldots, q$ are optional weights (positive real numbers) and

$$\Phi = diag[\phi_1^{1/2} I_{\nu+1}, \ldots, \phi_q^{1/2} I_{\nu+1}] \tag{4.3.51}$$

where $I_{\nu+1}$ is the $(\nu + 1) \times (\nu + 1)$ identity matrix.

Equation (4.3.42) can be written as

$$K_\nu \Phi^{-1}(\Phi \theta_\nu) = T_1 + T_2 X \tag{4.3.52}$$

Hence the $\Phi\theta_\nu$ that minimizes J_ν can be obtained as the minimum norm solution to (4.3.52). For ν large enough, K_ν is full rank, i.e., $rank[K_\nu] = q(f + m_0 - N)$, and for a given X, the solution is

$$\theta_\nu(X) = \Phi^{-1} F_\nu^* (F_\nu F_\nu^*)^{-1} (T_1 + T_2 X) \tag{4.3.53}$$

where

$$F_\nu \stackrel{\text{def}}{=} K_\nu \Phi^{-1} \tag{4.3.54}$$

and the superscript * indicates complex conjugate transpose. Note that although the matrices involved may be complex, the solution θ_ν will be real because any complex zeros of P(z) come in complex conjugate pairs. However the form in which the solution is given in (4.3.53) may cause numerical problems in some cases. One can avoid them by computing the pseudo–inverse $F_\nu^\dagger = F_\nu^* (F_\nu F_\nu^*)^{-1}$ from an SVD of F_ν (Stewart (1973), p. 324).

One can now compute X by minimizing $J_\nu(X)$ for the solution $\theta_\nu(X)$ of (4.3.53). From (4.3.53) we get

$$\begin{aligned} J_\nu(X) &= (T_1 + T_2X)^*(F_\nu F_\nu^*)^{-1}(T_1 + T_2X) \\ &= (T_1 + T_2X)^*(F_\nu^\dagger)^* F_\nu^\dagger(T_1 + T_2X) \end{aligned} \tag{4.3.55}$$

By setting the gradient of $J_\nu(X)$ equal to zero we get

$$T_2^*(F_\nu^\dagger)^* F_\nu^\dagger T_2 X = -T_2^*(F_\nu^\dagger)^* F_\nu^\dagger T_1 \tag{4.3.56}$$

from which a solution X which minimizes $J_\nu(X)$ can be obtained. The optimum θ_ν can then be computed from (4.3.53).

It is clear that by increasing ν, the value of the obtained minimum of J_ν will either be reduced or it will remain the same. Hence the designer has the option to choose interactions with smaller magnitude in exchange for a longer duration of the interactions. The knowledge of the value of this minimum at the limit as $\nu \to \infty$ would be quite helpful in making this decision. From (4.3.55) we see that we need to compute $F_\nu F_\nu^*$ as $\nu \to \infty$. The fact that the elements of $\gamma_{i,\nu}, i = 1, \ldots, f$, are terms in a geometric progression, allows us to do so easily when $b_1, \ldots, b_f$ are inside the unit circle. We cannot do so however if some of them are outside the unit circle, i.e., when some of the undesirable zeros of $P(z)$ are inside the unit circle. This is actually a situation, where for numerical reasons it would be strongly recommended to compute $F_\nu^\dagger$ from an SVD of F_ν as mentioned above.

5. Construction of $\mathbf{H_{ur}(z)}$ and $\mathbf{C(z)}$

After the desired $H_{yr}(z)$ has been designed, $H_{ur}(z)$ can be obtained from (3.0.1):

$$H_{ur}(z) = P(z)^{-1} H_{yr}(z) \tag{3.0.1}$$

Substitution of (3.0.1) into (2.0.5) yields:

$$C(z) = H_{ur}(z)[I - H_{yr}(z)]^{-1} \tag{5.0.1}$$

If one attempted to construct $H_{ur}(z)$ and $C(z)$ from (3.0.1), (5.0.1) by doing the computations in terms of transfer function matrices, the procedure would be extremely tedious. Instead, the computations can be made quite simply by working in the state space. One can obtain realizations of $P(z)$, $H_{yr}(z)$ to get the state space descriptions:

$$P(z) = C(zI - A)^{-1}B + D \tag{5.0.2}$$

$$H_{yr}(z) = C_0(zI - A_0)^{-1}B_0 + D_0 \tag{5.0.3}$$

$P(z)$ represents a physical system and so it can be assumed to be strictly proper, i.e., D = 0. Then from Corollary 2 it follows $D_0 = 0$. Construction of $H_{yr}(z)$, $C(z)$ involves the following steps.

Step 1. Inversion of $P(z)$.

Silverman (1969) developed a computationally simple algorithm for the inversion of a linear multivariable system, whose state space description is known. The result of the inversion will be

$$P(z)^{-1} = (C_1(zI - A_1)^{-1}B_1 + D_1)(K_0 + K_1 z + \ldots + K_{m_0} z^{m_0})z^N \tag{5.0.4}$$

where

$$A_1 = A - B\bar{D}^{-1}\bar{C} \tag{5.0.5}$$

$$B_1 = B\bar{D}^{-1} \tag{5.0.6}$$

$$C_1 = -\bar{D}^{-1}\bar{C} \tag{5.0.7}$$

$$D_1 = \bar{D}^{-1} \tag{5.0.8}$$

m_0 is the order of the zero b_0 of $\hat{P}(\lambda)$ obtained from Corollary 2 and N is defined in (3.0.10). The matrices $\bar{C}, \bar{D}, K_0, \ldots, K_{m_o}$ are determined with Silverman's (1969) procedure.

Step 2. Computation of $H_{ur}(z)$.

The following Theorem will be used.

Theorem 3

Let

$$G(z) = C(zI - A)^{-1}B + D \tag{5.0.9}$$

then

$$G(z)z^k = C(zI - A)^{-1}A^k B + \sum_{\ell=1}^{k} CA^{\ell-1}Bz^{k-\ell} + Dz^k, \quad \forall \quad k \geq 1 \tag{5.0.10}$$

Proof. See Appendix B.

We can now apply Theorem 3 to $P(z)^{-1}$, to obtain

$$\begin{aligned} P(z)^{-1} =& C_1(zI - A_1)^{-1}(\sum_{i=0}^{m_0} A_1^{i+N} B_1 K_i) \\ &+ \sum_{i=0}^{m_0} \sum_{\ell=1}^{i+N} C_1 A_1^{\ell-1} B_1 K_i z^{i+N-\ell} + \sum_{i=0}^{m_0} D_1 K_i z^{i+N} \\ =& C_2(zI - A_2)^{-1}B_2 + D_{2,0} + D_{2,1}z + \ldots + D_{2,m_0+N} z^{m_0+N} \end{aligned} \tag{5.0.11}$$

where

$$A_2 = A_1 \tag{5.0.12}$$

$$B_2 = \sum_{i=0}^{m_0} A_1^{i+N} B_1 K_i \tag{5.0.13}$$

$$C_2 = C_1 \tag{5.0.14}$$

$$D_{2,k} = \sum_{i=0}^{m_0} C_1 A^{N-1-k+i} B_1 K_i, \qquad k = 0, \ldots, N-1 \tag{5.0.15}$$

$$D_{2,k} = D_1 K_{k-N} + \sum_{i=k-N+1}^{m_0} C_1 A^{N-1-k+i} B_1 K_i, \quad k = N, \ldots, N + m_0 \tag{5.0.16}$$

Then from (3.0.1), (5.0.3), (5.0.11) we get

$$\begin{aligned} H_{ur}(z) =& (C_2(zI - A_2)^{-1} B_2 C_0 + D_{2,0} C_0)(zI - A_0)^{-1} B_0 \\ &+ (\sum_{i=1}^{m_0+N} D_{2,i} z^i) C_0 (zI - A_0)^{-1} B_0 \end{aligned} \tag{5.0.17}$$

Application of Theorem 3 on the second term of the right–hand side yields the term

$$(\sum_{i=1}^{m_0+N} D_{2,i}C_0A_0^i)(zI-A_0)^{-1}B_0+\sum_{i=1}^{m_0+N} D_{2,i}C_0A_0^{i-1}B_0+\sum_{i=1}^{m_0+N-1}\Psi_i z^i$$

where the fact $(zI-A)^{-1}A^k = A^k(zI-A)^{-1}$ was used. However, by construction, $P(z)^{-1}H_{yr}(z)$ is proper. Therefore $\Psi_i = 0$ for all $i = 1,\ldots,m_0+N-1$. Hence

$$\begin{aligned} H_{ur}(z) = (C_2(zI-A_2)^{-1}B_2C_0 + D_{2,0}C_0 \\ + \sum_{i=1}^{m_0+N} D_{2,i}C_0A_0^i)(zI-A_0)^{-1}B_0 \\ + \sum_{i=1}^{m_0+N} D_{2,i}C_0A_0^{i-1}B_0 \end{aligned} \tag{5.0.18}$$

All that is necessary now is to compute the product of two proper transfer function matrices, whose state space descriptions are known. The following Theorem takes care of that:

Theorem 4. (Doyle, 1984)

Let

$$G_1(z) = C_1(zI-A_1)^{-1}B_1 + D_1 \tag{5.0.19}$$

$$G_2(z) = C_2(zI-A_2)^{-1}B_2 + D_2 \tag{5.0.20}$$

then

$$G_1(z)G_2(z) = C(zI-A)^{-1}B + D \tag{5.0.21}$$

where

$$A = \begin{bmatrix} A_1 & B_1C_2 \\ 0 & A_2 \end{bmatrix}, B = \begin{bmatrix} B_1D_2 \\ B_2 \end{bmatrix}, C = [C_1|D_1C_2], D = [D_1D_2]$$

Application of Theorem 4 on (5.0.18) yields a state space description for H_{ur}.

Step 3. Computation of $C(z)$

All that is needed is to compute a state space description of $(I - H_{yr}(z))^{-1}$. After that, application of Theorem 4 on (5.0.1) will give a state–space description for $C(z)$. From (5.0.3) we get

$$I - H_{yr}(z) = -C_0(zI - A_0)^{-1}B_0 + I \tag{5.0.22}$$

and a state space description of $(I - H_{yr}(z))^{-1}$ can be easily computed as

$$(I - H_{yr}(z))^{-1} = C_0(zI - (A_0 + B_0C_0))^{-1}B_0 + I \tag{5.0.23}$$

The result of the described procedure is state space descriptions of $H_{ur}(z)$ and $C(z)$. One can always obtain a matrix transfer function form, but since the control law can be easily implemented with a state space description it would be advisable to avoid further computations by implementing it as such. It is important to point out however that the realizations obtained for $H_{ur}(z)$ and $C(z)$ are not minimal. It is essential to obtain minimal realizations of them before the implementation so that the undesirable zeros $a_1, \ldots, a_f$ of $P(z)$ do not appear as poles of $H_{ur}(z)$.

6. Illustrations

The first example in this section is used to illustrate the tradeoff between the time duration of the closed loop interactions and the magnitude of the sum of squared errors that they cause. This simple example is also used to demonstrate the procedure step by step. The second example examines different structures for H_{yr} and illustrates how the structure associated with a zero outside the UC can produce large or small closed-loop interactions, depending on the structure chosen for H_{yr}.

6.1 Example 1

Consider the system

$$P(z) = \begin{bmatrix} \frac{0.6}{z-0.4} & \frac{0.5}{z-0.5} \\ 1.2\frac{0.5}{z-0.5} & \frac{0.6}{z-0.4} \end{bmatrix} \tag{6.0.1}$$

Computation of the roots of $det[P(z)]$ shows that the system has one zero outside the UC, at $z = 1.547$. Garcia and Morari (1985) pointed out that an acceptable lower triangular H_{yr} is

$$H_{yr,1}(z) = \begin{bmatrix} z^{-1} & 0 \\ 3.095(1 - z^{-1})z^{-1} & \frac{-z+1.547}{1.547z-1}z^{-1} \end{bmatrix} \tag{6.0.2}$$

Clearly the interactions in output 2 for a setpoint change in output 1, are very large in magnitude (over 300% the setpoint change) although of short duration.

We shall now use the procedure of Section 4.3, to design a lower triangular $H_{yr}(z)$. For the time delays ($b_0 = 0$) we have $n_0 = m_0 = 1$. Hence part (b) of Theorem 1 (or Corollary 2) does not apply for $i = 0$ and therefore (4.3.6) does not apply for $i = 0$. Thus none of the equations or subsections of Section 4.3 that correspond to $i = 0$ should be considered. For the zero $a_1 = 1.547$ ($b_1 = a_1^{-1}$) we have $n_1 = 0, m_1 = 1$. Also

$$U_1 = \begin{bmatrix} 0.675 \\ 0.739 \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{6.0.3}$$

and $\rho_1 = 1$.

i) <u>Design of $h_1(z)$.</u>

In this case, ℓ is the empty set and $\bar{\ell} = \{2\}$. Hence $\rho_1^{\ell} = 0$ and therefore $\xi_1^{\ell} = 1, V_1^{\ell} = 1$. Thus (4.3.19) has a solution for $i = 1$ and as a result b_1 should not be included in $\varsigma_0(\lambda)$. Thus from (4.3.20) it follows that $\varsigma_0(\lambda) = 1$. The case of b_0 should not be considered as mentioned above. Hence $\varsigma_{r_0}(\lambda) = \varsigma_0(\lambda) = 1$ and from (4.3.26) it follows that the first element of the column (diagonal element) is equal to $\lambda(= z^{-1})$. Then (4.3.27) need be satisfied for $i = 1$. Since the null space of $e_1^T U_1 V_1^{\ell}$ is the empty set, it follows that $W_1^{\ell} = 0$ and

$$\chi_1^2 = \chi_1^0 = b_1/u_1 \tag{6.0.4}$$

We shall now proceed with the design of the second element of the column. In (4.3.31), (4.3.32) we have $q = 1, \bar{\ell}_1 = 2, \delta_1 = 1$. In (4.3.44),(4.3.46) the part corresponding to b_0 is omitted and so from (4.3.36), (4.3.44), (4.3.46), (6.0.4), it follows:

$$K_\nu = [\,(1-b_1)b_1 \quad (1-b_1)b_1^2 \quad \ldots \quad (1-b_1)b_1^{\nu+1}\,] \tag{6.0.5}$$

$$T_1 = b_1 u_2 / u_1 \tag{6.0.6}$$

From $W_1^\ell = 0$ and (4.3.47) it follows that $T_2 = 0$. In this case $\Phi = I$ and from (4.3.53) we get

$$\theta_\nu = K_\nu^*(K_\nu K_\nu^*)^{-1}T_1 \tag{6.0.7}$$

(4.3.37), (4.3.50), (6.0.5), (6.0.6), (6.0.7) yield

$$\beta_{i,j} = \frac{(1+b_1)b_1^j u_2}{(1-b_1^{2\nu+2})u_1}, \qquad j = 0, \ldots, \nu \tag{6.0.8}$$

$$J_\nu = \frac{(1+b_1)u_2^2}{(1-b_1)(1-b_1^{2\nu+2})u_1^2} \tag{6.0.9}$$

For $\nu = 0$ we get $\beta_{1,0} = 3.095$, i.e., the design in (6.0.2). However the error caused by the interactions in this case is $J_0 = 9.58$, which is quite large. Increasing the duration ν of the interactions reduces J_ν as (6.0.9) indicates. Since $b_1 < 1$ we can compute the limit:

$$\lim_{\nu \to \infty} J_\nu = \frac{(1+b_1)u_2^2}{(1-b_1)u_1^2} = 5.58 \tag{6.0.10}$$

The designer can of course select a relatively small ν, for which J_ν is sufficiently close to the limit given by (6.0.10). A plot of J_ν as a function of ν is given in Fig. 2. One can see that a selection of $\nu = 4$, is satisfactory. It yields $J_\nu = 5.65$. For $\nu = 4$, the second element of $h_1(z)$ becomes equal to $(1.82 + 1.18z^{-1} + 0.76z^{-2} + 0.49z^{-3} + 0.32z^{-4})(1 - z^{-1})z^{-1}$.

ii) Design of $h_2(z)$.

Since we require the first element of the column to be zero, we have $\ell = \{1\}$ and therefore $\bar{\ell}$ is the empty set. Hence $\rho_1^\ell = 1, \xi_1^\ell = 0, V_1^\ell = 0$. Then (4.3.19) does not have a solution for $i = 1$. As a result the second element (diagonal element) has to have a zero at $\lambda = b_1$. From (4.3.20) we get

$$\varsigma_0(\lambda) = \frac{(1 - b_1^{-1})(\lambda - b_1)}{(1 - b_1)(\lambda - b_1^{-1})} \tag{6.0.11}$$

Then, since $\varsigma_{r_0}(\lambda) = \varsigma_0(\lambda)$, (4.3.26) implies that the diagonal element is $\lambda\varsigma_0(\lambda)$. Substitution of z^{-1} for λ and $a_1(= 1.547)$ for b_1^{-1}, produces the expression in (6.0.2).

6.2 Example 2.

Consider the system:

$$P(z) = \begin{bmatrix} \frac{0.90}{z-0.35} & \frac{0.50}{z-0.35}z^{-1} & \frac{1.00}{z-0.35} \\ \frac{2.70}{z-0.60}z^{-1} & \frac{5.80}{z-0.60}z^{-1} & \frac{0.60}{z-0.60}z^{-1} \\ \frac{0.40}{z-0.50} & \frac{-0.45}{z-0.50} & \frac{1.00}{z-0.50}z^{-1} \end{bmatrix} \tag{6.0.12}$$

The computation of the roots of $det[P(z)]$ yields one zero outside the UC at $z = a_1 = 1.3088$. We shall limit ourselves to the design of the first column of H_{yr}. Two different structures will be examined:

$$h_1 = \begin{bmatrix} x \\ x \\ 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} x \\ 0 \\ x \end{bmatrix} \tag{6.0.13}$$

The SVD of $P(a_1)$ yields the following left singular vector matrix:

$$U = \begin{bmatrix} 0.125 & -0.700 & -0.703 \\ 0.992 & 0.0689 & 0.107 \\ -0.0267 & -0.711 & 0.703 \end{bmatrix} \tag{6.0.14}$$

The two first columns of U form U_1. The third, u, is orthogonal to U_1. Then from Corollary 1 it follows that $u^* H_{yr}(a_1) = 0$ for all acceptable H_{yr}'s. (6.0.14) suggests that if the first structure of (6.0.13) is selected, the value of the non-diagonal element at $z = a_1$ will have to be larger than the one for the second structure, because of the smaller corresponding element in u.

The consideration of the time delays (b_0) makes the situation even more favorable for the second structure. We have $n_0 = 1, m_0 = 2$ and

$$U_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{6.0.15}$$

The fact that the second row of U_0 is zero, allows in the case of the second structure satisfaction of (4.3.14) for $i = 0$ without using any of the available degrees of freedom. This results in a nonzero w_0^ℓ and T_2 and the additional freedom in choosing χ_0^3 through (4.3.56) reduces J_ν even more.

The above qualitative observations are confirmed from the quantitative results of the design procedure. The corresponding plots of J_ν vs. ν shown in Fig. 3 for both structures of (6.0.13) show a huge difference in the closed-loop interactions for the two structures.

7. Conclusions

The results in this paper quantify the effects of the undesirable zeros and time delays of a multivariable discrete system on its closed-loop performance, in a way that can be used for the direct design of the closed loop transfer function matrix. The designer is provided with quantitative criteria for comparing different designs and evaluating the tradeoffs. The entire procedure is based on linear algebra operations and its implementation on the computer is straightforward.

The design is based on the knowledge of a system model. Hence it may not be robust to model-plant mismatch. However it can be used in the first step of the controller design for the standard two-step Internal Model Control design procedure, in which robustness properties are incorporated in the second step with the design of a low pass filter. Details on the filter design can be found in the literature (Zafiriou and Morari, 1986b,c).

APPENDIX

A) Proof of Theorem 1

The following Lemma will be used in the proof:

Lemma A.1. (Van Dooren et al., 1979; Vandewalle et al., 1974). Let a rational matrix $A(\lambda)$ of normal rank r have the following Laurent expansion at α:

$$A(\lambda) = \sum_{k=-\ell}^{\infty} (\lambda - \alpha)^k A_k(\alpha) \tag{A.1}$$

Define

$$T_k(\alpha) = \begin{bmatrix} A_{-\ell}(\alpha) & A_{-\ell+1}(\alpha) & \dots & A_k(\alpha) \\ 0 & A_{-\ell}(\alpha) & \dots & A_{k-1}(\alpha) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_{-\ell}(\alpha) \end{bmatrix} \tag{A.2}$$

$$\rho_k(\alpha) \stackrel{\text{def}}{=} rank[T_k(\alpha)] - rank[T_{k-1}(\alpha)] \tag{A.3}$$

Let p and z be a pole and a zero respectively of $A(\lambda)$ of orders ω_p, ω_z and degrees δ_p, δ_z, as these are defined from the Smith–McMillan form of $A(\lambda)$ (Van Dooren et al., 1979; Desoer and Schulman, 1974).

The following hold:

i) $\omega_p = -min\{k | \rho_k \neq 0\}$

ii) $\omega_z = min\{k | \rho_k = r\}$

iii) $\delta_p = \sum_{k=-\omega_p}^{-1} \rho_k$

iv) $\delta_z = \sum_{k=0}^{\omega_z} (r - \rho_k)$

Proof of Theorem 2. $\hat{P}(\lambda)^{-1}$ has as its poles exactly the zeros of $\hat{P}(\lambda)$ with the same order and degree (Desoer and Schulman, 1974). Hence since b_i is a zero of $\hat{P}(\lambda)$ of order m_i, it is also a pole of $\hat{P}(\lambda)^{-1}$ and we can write

$$\hat{P}(\lambda)^{-1} = \sum_{k=1}^{m_i} (\lambda - b_i)^{-k} R_{i,k} + G_i(\lambda), \qquad i = 0, 1, \dots, f \tag{A.4}$$

where

$$rank[R_{i,m_i}] \neq 0, \qquad i = 0, 1, \dots, f \tag{A.5}$$

and $G_i(\lambda)$ has no poles at $b_i, i = 0, 1, \ldots, f$.

Postmultiplication of (A.4) with $\hat{H}_{yr}(\lambda)$ yields

$$\hat{P}(\lambda)^{-1}\hat{H}_{yr}(\lambda) = \sum_{k=1}^{m_i}(\lambda - b_i)^{-k}R_{i,k}\hat{H}_{yr}(\lambda) + G_i(\lambda)\hat{H}_{yr}(\lambda) \qquad (A.6)$$

Now take a partial fraction expansion for each term in the sum of the right–hand side of (A.6) to obtain:

$$\begin{aligned}\hat{P}(\lambda)^{-1}\hat{H}_{yr}(\lambda) &= \sum_{k=1}^{m_i}[\sum_{h=0}^{k-1}(\lambda - b_i)^{-k+h}R_{i,k}\frac{1}{h!}\hat{H}_{yr}^{(h)}(b_i) + R_{i,k}G_k^H(\lambda)] + G_i(\lambda)\hat{H}_{yr}(\lambda) \\ &= \sum_{k=1}^{m_i}((\lambda - b_i)^{-k}\sum_{h=k}^{m_i}R_{i,h}\frac{1}{(h-k)!}\hat{H}_{yr}^{(h-k)}(b_i)) \\ &+ \sum_{k=1}^{m_i}R_{i,k}G_k^H(\lambda) + G_i(\lambda)\hat{H}_{yr}(\lambda) \qquad (A.7)\end{aligned}$$

where $G_k^H(\lambda)$ has no poles at b_i. Also recall that $G_i(\lambda)$ has no poles at b_i either. Hence in order for Condition C5.ii to hold we must have for all $i = 0, \ldots, f$:

$$\sum_{h=k}^{m_i}R_{i,h}\frac{1}{(h-k)!}\hat{H}_{yr}^{(h-k)}(b_i) = 0, \qquad k = 1, \ldots, m_i \qquad (A.8)$$

Satisfaction of (A.8) is equivalent to requiring that the columns of $[\hat{H}_{yr}^{(0)}(b_i)^T \ldots \frac{1}{(m_i-1)!}\hat{H}_{yr}^{(m_i-1)}(b_i)^T]^T$ are in the null space of N_i, where:

$$N_i \stackrel{\text{def}}{=} \begin{bmatrix} R_{i,m_i} & 0 & \ldots & 0 \\ R_{i,m_{i-1}} & R_{i,m_i} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ R_{i,1} & R_{i,2} & \ldots & R_{i,m_i} \end{bmatrix} \qquad (A.9)$$

We shall now proceed to determine the null space of N_i.

Postmultiply both sides of (A.4) with $\hat{P}(\lambda)$ to obtain:

$$I = \sum_{k=1}^{m_i}(\lambda - b_i)^{-k}R_{i,k}\hat{P}(\lambda) + G_i(\lambda)\hat{P}(\lambda) \qquad (A.10)$$

Since I has no poles at $b_0, \ldots, b_f$, taking a partial fraction expansion leads to a condition similar to (A.8), in exactly the same manner. Hence (A.10) yields

$$\sum_{h=k}^{m_i} R_{i,h} \frac{1}{(h-k)!} \hat{P}^{(h-k)}(b_i) = 0, \qquad k = 1, \ldots, m_i \tag{A.11}$$

The equations implied by (A.11) for $k = \ell, \ldots, m_i$ can be put together in the matrix form:

$$N_i [\underbrace{0 \ldots 0}_{\ell-1} \quad \hat{P}^{(0)}(b_i)^T \quad \ldots \quad \frac{1}{(m_i-\ell)!} \hat{P}^{(m_i-\ell)}(b_i)^T]^T = 0, \qquad \ell = 1, \ldots, m_i \tag{A.12}$$

The equations obtained from (A.12) for $\ell = 1, \ldots, m_i$ can be written together as:

$$N_i L_i = 0, \qquad i = 0, \ldots, f \tag{A.13}$$

where

$$L_i \stackrel{\text{def}}{=} \begin{bmatrix} \hat{P}^{(0)}(b_i) & 0 & \ldots & 0 \\ \hat{P}^{(1)}(b_i) & \hat{P}^{(0)}(b_i) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{(m_i-1)!} \hat{P}^{(m_i-1)}(b_i) & \frac{1}{(m_i-2)!} \hat{P}^{(m_i-2)}(b_i) & \ldots & \hat{P}^{(0)}(b_i) \end{bmatrix} \tag{A.14}$$

Hence the column space of L_i is a subspace of the null space of N_i. It will now be shown that it is exactly the null space of N_i.

As explained earlier, the order ω_p of the pole b_i of $\hat{P}(\lambda)^{-1}$ is equal to the order m_i of the zero b_i of $\hat{P}(\lambda)$, i.e., equal to m_i:

$$\omega_p(b_i) = m_i \tag{A.15}$$

Lemma A.1 will now be applied on $A(\lambda) = \hat{P}(\lambda)^{-1}$, for $\alpha = b_i$. From (A.1), (A.4), (A.5) it follows that $\ell = m_i$ and $A_{-k} = R_{i,k}$ for $k = 1, \ldots, m_i$. By using (A.15) and Lemma A.1.iii we get

$$\delta_p(b_i) = \sum_{k=-m_i}^{-1} \rho_k(b_i) = rank[T_{-1}(b_i)] = rank[N_i] \tag{A.16}$$

since $T_{m_i-1}(b_i) = 0$ and $T_{-1}(b_i)$ can be obtained from N_i by simply permuting its rows and columns.

By definition the order ω_z of the zero b_i of $\hat{P}(\lambda)$ is equal to m_i:

$$\omega_z(b_i) = m_i \tag{A.17}$$

Lemma A.1 will now be applied on $A(\lambda) = \hat{P}(\lambda)$, for $\alpha = b_i$. In this case, since $\hat{P}(\lambda)$ is assumed to have no poles at b_i, we have $\ell \leq 0$ and $A_k = \frac{1}{k!}\hat{P}^{(k)}(b_i)$ for $k = 1, \ldots, m-1$. By using (A.17) and Lemma A.1.iv we obtain

$$\begin{aligned} \delta_z(b_i) &= \sum_{k=0}^{m_i}(r - \rho_k(b_i)) = \sum_{k=0}^{m_i-1}(r - \rho_k(b_i)) \\ &= m_i r - rank[T_{m_i-1}(b_i)] = m_i r - rank[L_i] \end{aligned} \tag{A.18}$$

since from Lemma A.1.ii we have $\rho_{m_i}(b_i) = r$, and we also have $T_{-1}(b_i) = 0$ and $T_{m_i-1}(b_i)$ can be obtained from L_i by permutting its rows and columns.

The degree δ_z of the zero b_i of $\hat{P}(\lambda)$ is the same as the degree δ_p of the pole b_i of $\hat{P}(\lambda)^{-1}$ and so from (A.16), (A.18) we get

$$rank[N_i] + rank[L_i] = m_i r \tag{A.19}$$

But N_i and L_i are matrices of dimension $m_i r \times m_i r$. Therefore (A.13) and (A.19) imply that the column space of L_i is exactly the null space of N_i.

Hence from (A.8) it follows that Condition C5.ii is satisfied if and only if the columns of $[\hat{H}_{yr}^{(0)}(b_i)^T \ldots \frac{1}{(m_i-1)!}\hat{H}_{yr}^{(m_i-1)}(b_i)^T]^T$ are in the column space of L_i. From (3.0.6) it follows that the first $n_i r$ rows and the last $n_i r$ columns of L_i are identically zero. Hence

$$\hat{H}_{yr}^{(k)}(b_i) = 0, \qquad k = 0, \ldots, n_i - 1$$

which implies that

$$\hat{H}_{yr}(\lambda) = (\lambda - b_i)^{n_i}\hat{H}_i(\lambda) \tag{A.20}$$

where $\hat{H}_i(\lambda)$ has no poles at b_i. (A.20) completes the proof of part (a) of Theorem 1. If $m_i = n_i$, then this is the only requirement since then $L_i = 0$. If however $m_i > n_i$ then $rank[L_i] \neq 0$ and additional requirements on $\hat{H}_i(\lambda)$ are necessary. We have for $\ell = n_i, \ldots, m_i - 1$:

$$\frac{1}{\ell!}\hat{H}_{yr}^{(\ell)}(b_i) = \frac{1}{\ell!}\binom{\ell}{n_i} n_i! \hat{H}_i^{(\ell-n_i)}(b_i) = \frac{1}{(\ell-n_i)!}\hat{H}_i^{(\ell-n_i)}(b_i)$$

and so the requirement on $\hat{H}_i(\lambda)$ is that the columns of $[\hat{H}_i^{(0)}(b_i)^T \ldots \frac{1}{(m_i-n_i-1)!}\hat{H}_i^{(m_i-n_i-1)}(b_i)^T]^T$ are in the column space of M_i, where M_i is defined in (3.0.9). QED

B) Proof of Theorem 3

The proof is by induction

$k = 1$:

$$\begin{aligned} G(z)z &= C(zI-A)^{-1}zB + Dz \\ &= C(zI-A)^{-1}(A + zI - A)B + Dz \\ &= C(zI-A)^{-1}AB + CB + Dz \end{aligned}$$

$k = n$: Let

$$G(z)z^n = C(zI-A)^{-1}A^nB + \sum_{\ell=1}^{n} CA^{\ell-1}Bz^{n-\ell} + Dz^n \qquad (B.1)$$

hold.

$k = n+1$: From (B.1) it follows that

$$G(z)z^{n+1} = C(zI-A)^{-1}A^n zB + \sum_{\ell=1}^{n} CA^{\ell-1}Bz^{n+1-\ell} + Dz^{n+1}$$

and by using the result for $k = 1$ we obtain

$$G(z)z^{n+1} = C(zI-A)^{-1}A^{n+1}B + \sum_{\ell-1}^{n+1} CA^{\ell=1}Bz^{n+1-\ell} + Dz^{n+1} \qquad QED$$

Acknowledgements

Support from the National Science Foundation and the Department of Energy is gratefully acknowledged.

References.

- ASTROM, K. J. and B. WITTENMARK, *Computer Controlled Systems*, (Englewood Cliffs, NJ: Prentice Hall), 1984.
- CALLIER, F. M. and C. A. DESOER, *Multivariable Feedback Systems*, (New York: Springer-Verlag), 1982.
- DESOER, C. A. and J. D. SCHULMAN, "Zeros and Poles of Matrix Transfer Functions and Their Dynamical Interpetation", *I.E.E.E. Trans. on Circuits ans Systems*, CAS-21, 3, 1974.
- DOYLE, J. C., *Lecture Notes*, ONR/Honeywell Workshop on Advances on Multivariable Control, 1984.
- GARCIA, C. E. and M. MORARI, "Internal Model Control. A Review and Some New Results", *Ind. and Eng. Chem., Proc. Des. and Dev.*, 21, 308, 1982.
- GARCIA, C. E. and M. MORARI, "Internal Model Control 2. Design Procedure for Multivariable Systems", *Ind. and Eng. Chem., Proc. Des. and Dev.*, 24, 472, 1985.
- KAILATH, T., *Linear Systems*, (Englewood Cliffs, NJ:Prentice Hall), 1980.
- LAUB, A. J. and B. C. MOORE,"Calculation of Transmission Zeros Using QZ Techniques", *Automatica*, 14, 557, 1978.
- MORARI, M., E. ZAFIRIOU and C. G. ECONOMOU, *An Introduction to Internal Model Control* (accepted for publication in the series *Lecture Notes in Control and Information Sciences*, Springer-Verlag), 1987.
- SILVERMAN, L. M., "Inversion of Multivariable Linear Systems", *I.E.E.E. Trans. on Autom. Control*, AC-14, 270, 1969.
- STEWART, G. C., *Introduction to Matrix Computations*, (New York: Academic Press), 1973.
- VANDEWALLE, J. and P. DEWILDE, "On the Determination of the Order

and Degree of a Zero of a Rational Matrix", *I.E.E.E. Trans. on Autom. Control*, AC-19, 608, 1974.

- VAN DOOREN, P. M., P. DEWILDE and J. VANDEWALLE, "On the Determination of the Smith-McMillan Form of a Rational Matrix from its Laurent Expansion", *I.E.E.E. Trans. on Circuits and Systems*, CAS-26, 180, 1979.
- VIDYASAGAR, M., *Control System Synthesis*, MIT Press, Cambridge, MA, 1985.
- ZAFIRIOU, E. and M. MORARI, "Digital Controllers for SISO Systems: A Review and a New Algorithm", *Int. J. of Control*, 42, 855, 1985.
- ZAFIRIOU, E. and M. MORARI, "Design of Robust Digital Controllers and Sampling Time Selection for SISO Systems", *Int. J. of Control*, 44, 711, 1986a.
- ZAFIRIOU, E. and M. MORARI, "Synthesis of the IMC Filter by Using the Structured Singular Value Approach", *Proc. of the Amer. Control Conf.*, p. 1, Seattle, WA, 1986b.
- ZAFIRIOU, E. and M. MORARI, "Internal Model Control: Robust Digital Controller Synthesis for Multivariable Open-loop Stable or Unstable Systems", *submitted to I.E.E. Proc.*, Part D, 1986c.
- ZAMES, G., "Feedback and Optimal Sensitivity: model reference transformations, Multiplicative semi-norms and approximate inverses", *I.E.E.E. Trans. Autom. Control*, AC-26, 301, 1981.

Figure Captions

Figure 1 . Feedback control structure.

Figure 2 . Example 1; J_ν for column 1.

Figure 3 . Example 2; J_ν for column 1.

(a) $h_1 = [\,x \quad x \quad 0\,]^T$

(b) $h_1 = [\,x \quad 0 \quad x\,]^T$

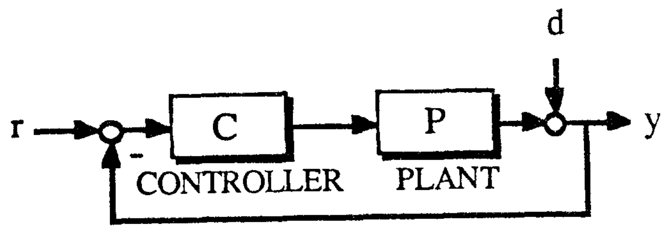


Figure 1 . Feedback control structure.

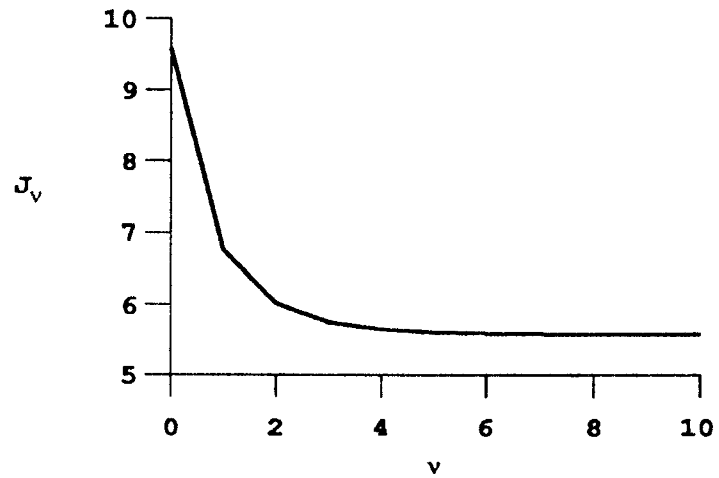
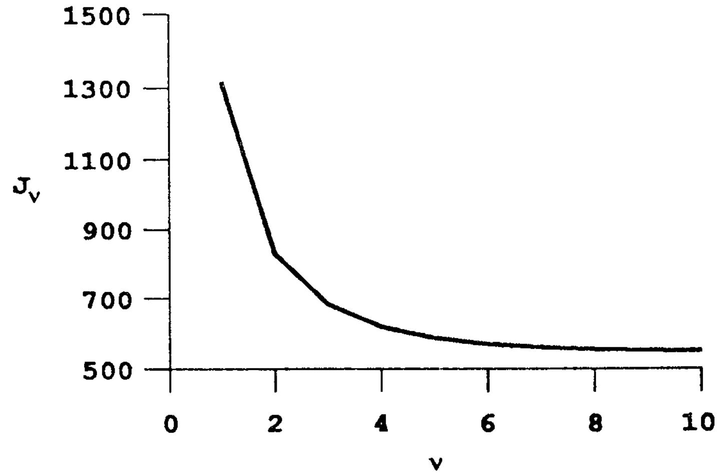
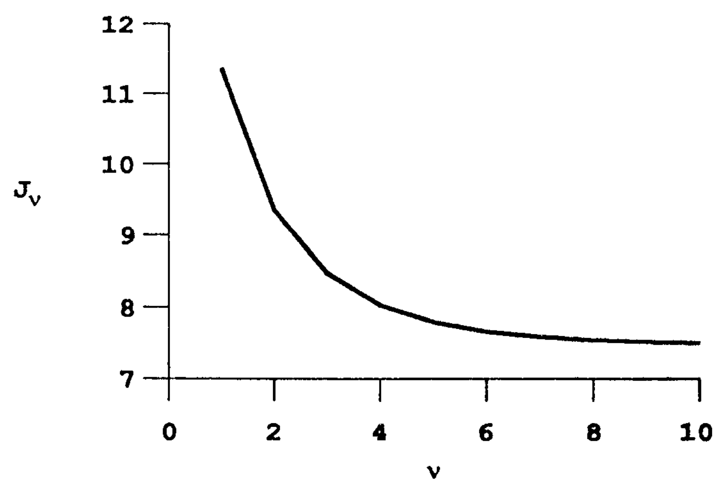


Figure 2 . Example 1; J_ν for column 1.



(a)



(b)

Figure 3 . Example 2; J_ν for column 1.

(a) $h_1 = [\,x \quad x \quad 0\,]^T$

(b) $h_1 = [\,x \quad 0 \quad x\,]^T$