

SRC TR 87-146

**Internal Model Control: Robust
Digital Controller Synthesis for
Multivariable Open-Loop Stable for
Unstable Systems**

by

E. Zafiriou and M. Morari

INTERNAL MODEL CONTROL: ROBUST DIGITAL
CONTROLLER SYNTHESIS FOR MULTIVARIABLE
OPEN-LOOP STABLE OR UNSTABLE SYSTEMS

Evangelos Zafiriou

Manfred Morari

Chemical Engineering, 206-41

California Institute of Technology

Pasadena, CA 91125

Abstract

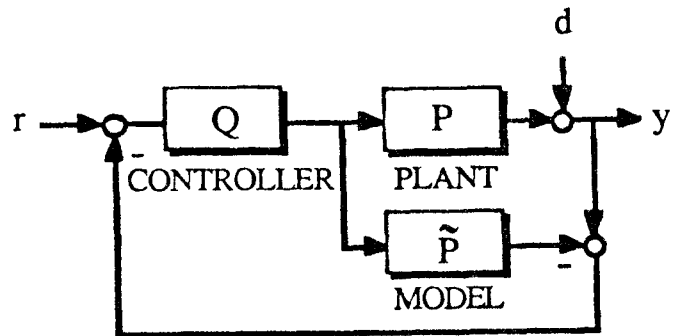
The two-step Internal Model Control procedure is used for the synthesis of multivariable discrete controllers for open-loop stable or unstable plants. The plant models used in the proposed method are transfer function matrices. In the first step the controller is designed so that the L_2 -error (sum of squared errors) is minimized for every setpoint or disturbance vector in a set and their linear combinations. A modification is then introduced to avoid the potential problem of intersample rippling. In the second step a low-pass filter is designed so that stability and good performance characteristics are maintained in the presence of model-plant mismatch. The continuous plant output is considered in order to avoid bad intersample behavior. The filter parameters are obtained as the result of a minimization of a non-conservative robustness measure, the Structured Singular Value. Special filter structures have to be used for open-loop unstable or ill-conditioned plants.

The design of a control system requires the use of a process model, either explicitly or implicitly. However, modeling error is unavoidable and it results in a mismatch between the model and the actual plant. Other reasons for such a mismatch are nonlinearities that manifest themselves as modeling error when a linear model is used for the controller design. Whatever the reasons for the modeling error, the result is that a controller designed for a particular model may perform quite differently when it is implemented on the actual process. Modern control theory addresses this problem with the design of controllers that are robust with respect to model-plant mismatch, i.e., of control systems that will perform within certain design specifications, provided that the modeling error does not exceed certain bounds.

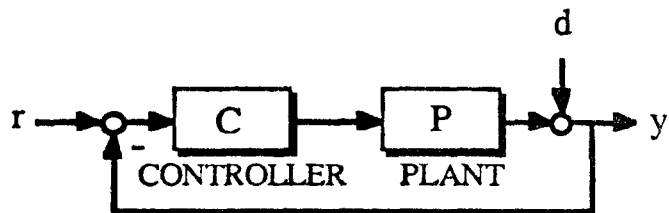
The Internal Model Control (IMC) structure, introduced by Garcia and Morari (1982), has been widely recognized as very useful in clarifying the issues related to the mismatch between the model used for controller design and the actual process. An important characteristic of the structure is that it gives rise to a two step controller synthesis procedure. In the first step the assumption is made that the model and the plant are the same, and as a result the design of an IMC controller with desired performance characteristics in this first step is significantly simplified. The second step deals with the design of a low-pass filter such that robustness with respect to model-plant mismatch is guaranteed, in the sense that the system performance will remain close to the one for the nominal case (no modeling error), even when such a mismatch exists.

1. Internal Model Control.

The IMC structure (Fig. 1a), is mathematically equivalent to the classical feedback structure (Fig. 1b). $\tilde{P}(z)$ represents the model and $P(z)$ the actual discretized plant. $P(z)$ is obtained by adding a zero order hold in front of the continuous plant and then taking the z-transform. $P(z)$ is assumed to be a square



(a)



(b)

Figure 1.

(a) Internal Model Control structure.

(b) Feedback Control structure.

$n \times n$ matrix of full normal rank, i.e., it may be singular only at a finite number of z 's. The IMC controller $Q(z)$ and the feedback $C(z)$ are related through

$$Q = C(I + \tilde{P}C)^{-1} \quad (1.0.1)$$

$$C = Q(I - \tilde{P}Q)^{-1} \quad (1.0.2)$$

where \tilde{P} is the process model. Note that throughout this paper, unless otherwise pointed, all the transfer function matrices are z transforms.

Some advantages of using the IMC structure can be seen by examining the structure for $P = \tilde{P}$ and $P \neq \tilde{P}$:

$$\underline{P = \tilde{P}.}$$

In this case the overall transfer function connecting the setpoints $r(z)$ and disturbances $d(z)$ to the errors $e = y - r$, where $y(z)$ are the process outputs is

$$e = y - r = (I - PQ)(d - r) \stackrel{\text{def}}{=} \tilde{E}(d - r) \quad (1.0.3)$$

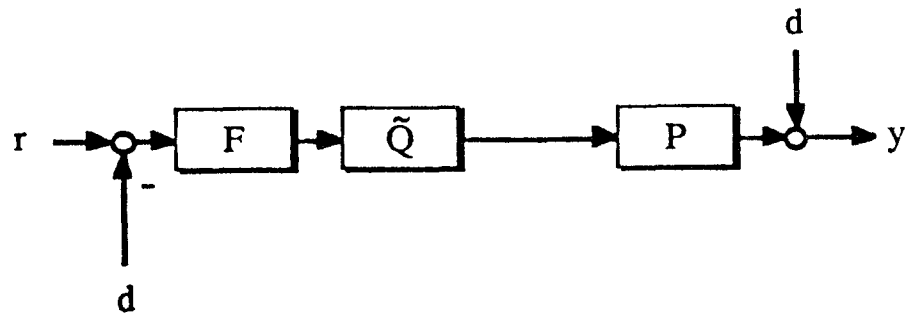
Hence the IMC structure becomes effectively open-loop (Fig. 2a) and the design of Q is simplified. Note that the IMC controller is identical to the parameter of the Q -parametrization (Zames, 1981). Also the addition of a diagonal filter F by writing

$$Q = \tilde{Q}F \quad (1.0.4)$$

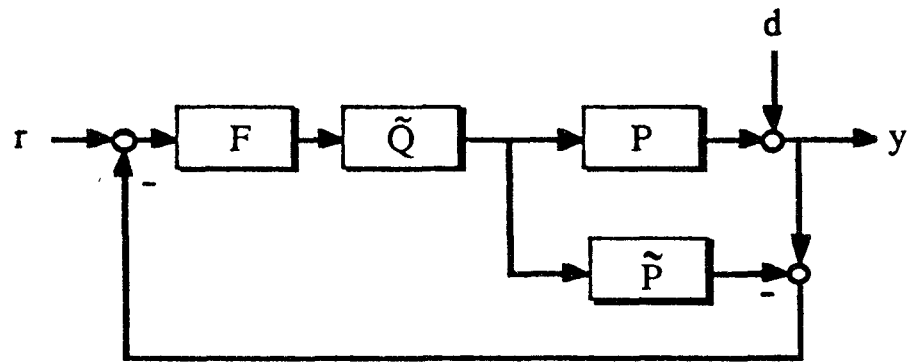
introduces parameters (the filter time constants) which can be used for adjusting on-line the speed of response for each process output.

$$\underline{P \neq \tilde{P}.}$$

The model-plant mismatch generates a feedback signal in the IMC structure which can cause performance deterioration or even instability. Since the relative modeling error is larger at higher frequencies, the addition of the low-pass filter F (Fig. 2b) adds robustness characteristics into the control system. In this case



(a)



(b)

Figure 2. IMC structure with the filter F .

(a) $P = \tilde{P}$.

(b) $P \neq \tilde{P}$.

the closed-loop transfer function is

$$e = y - r = (I - P\tilde{Q}F)(I - (P - \tilde{P})\tilde{Q}F)^{-1}(d - r) \stackrel{\text{def}}{=} E(d - r) \quad (1.0.5)$$

Hence the IMC structure gives rise rather naturally to a two-step design procedure:

Step 1: Design \tilde{Q} , assuming $P = \tilde{P}$.

Step 2: Design F so that the closed-loop characteristics that \tilde{Q} produces in Step 1, are preserved in the presence of model-plant mismatch ($P \neq \tilde{P}$).

In previous IMC work, Garcia and Morari (1985a) proposed for open-loop stable plants the direct design of the closed-loop transfer function. In Step 1 this approach was rigorously quantified in its general form by Zafriou and Morari (1986b) by using the concept of zero directions, who also took into account inter-sample rippling and modified the approach so that time delays and outside the unit circle (UC) zeros were considered in one single step. However, the extension of the above approach or of the impulse response formulation of the problem (Garcia and Morari, 1985b) to open-loop unstable systems is very awkward. The method proposed in this paper takes care of both open-loop stable and unstable plants in a general way by minimizing the Sum of Squared Errors (SSE) for a set of setpoints or disturbance vectors and their linear combinations, if possible.

With respect to the second step, this paper proposes the design of the IMC filter by minimizing an appropriate robustness condition. Potential modeling errors, described as uncertainty associated with the process model, can appear in different forms and places in a multivariable model. This fact makes the derivation of non-conservative conditions that guarantee robustness with respect to model-plant mismatch quite difficult. The Structured Singular Value (SSV), introduced by Doyle (1982), has gained a lot of popularity recently, because it takes into account the structure of the model uncertainty and it allows the non-conservative

quantification of the concept of robust performance. The objective function for the minimization problem is formulated in such a way so that the continuous plant output is considered and the problem of intersample rippling does not occur.

2. Step 1: Design of \tilde{Q} .

Throughout this section the assumption is made that $P = \tilde{P}$. Details on the definition of multivariable zeros and poles and their degrees and orders can be found in the literature (Desoer and Schulman, 1974). In general a pole of an element of $P(z)$ is also a pole of $P(z)$ and the roots of $\det[P(z)] = 0$ are the zeros of the matrix $P(z)$.

2.1. Stabilizing Q's.

It will be assumed that there are no open-loop unstable poles of the continuous plant, which after sampling do not appear in the discretized plant, i.e., no open-loop poles become unobservable because of sampling. If this is not the case, then a small change of sampling time will make those poles observable.

2.1.1 Internal Stability.

The concept of Internal Stability can be motivated by pointing out that the signals between blocks constituting a control system are subject to (possibly very small) errors. In practice it cannot be tolerated that these small errors lead to unbounded signals at some other location in the control system.

Definition 1: A linear time invariant control system is internally stable, if the transfer functions between any two points of the control system are stable, i.e., have all poles inside the UC (for discrete systems).

In a control system many different points can be selected for signal injection and observation but most of the choices are equivalent. For example, for the system in Fig. 3a, y and e differ only by a bounded signal r and their observation reveals the same information about internal stability. Also, from the point of view of internal stability the effect of d and r on u is equivalent. Simple arguments

of this type reveal that there are only two “independent” outputs, which can be chosen as y and u and two “independent” inputs which can be chosen as r and u' . Thus the classic feedback system is stable if and only if all elements in the transfer matrix in (2.1.1) have all their poles inside the UC.

$$\begin{pmatrix} y \\ u \end{pmatrix} = \begin{pmatrix} PC(I+PC)^{-1} & (I+PC)^{-1}P \\ C(I+PC)^{-1} & -C(I+PC)^{-1}P \end{pmatrix} \begin{pmatrix} r \\ u' \end{pmatrix} \quad (2.1.1)$$

Equivalently the internal stability requirements for the classical feedback structure is that all elements in the matrix IS1 in (2.1.2) are stable:

$$IS1 = \begin{pmatrix} C(I+PC)^{-1} & PC(I+PC)^{-1} & C(I+PC)^{-1}P & (I+PC)^{-1}P \end{pmatrix} \quad (2.1.2)$$

The internal stability condition clarifies the fact that mere cancellation of unstable poles by zeros is not enough to guarantee the stability of the system. It becomes evident that instability arising from unstable pole-zero cancellations is not due to inexact cancellation (as it has been argued in the past) but is solely due to the fact that the cancellation does not satisfy the internal stability requirements.

Use of (1.0.1) or (1.0.2) in (2.1.2) yields

$$IS1 = \begin{pmatrix} Q & PQ & QP & (I-PQ)P \end{pmatrix} \quad (2.1.3)$$

Note that stability of each element in (2.1.3) implies internal stability when the control system is implemented as the feedback structure in Fig. 1b, where C is obtained from the Q used in (2.1.3) through (1.0.2).

In order for the control system to be stable when implemented in the IMC structure of Fig. 1a, we have to examine the transfer functions between all possible system inputs and outputs. From the block diagram in Fig. 3b we note that there are three independent system inputs r, u_1 and u_2 and three independent outputs y, u and \tilde{y} . For no model error ($P = \tilde{P}$) the inputs and outputs are related through

the following transfer matrix.

$$\begin{pmatrix} y \\ u \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} PQ & (1 - PQ)P & P \\ Q & -QP & 0 \\ PQ & -PQP & P \end{pmatrix} \begin{pmatrix} r \\ u_1 \\ u_2 \end{pmatrix} \quad (2.1.4)$$

In order for the matrix in (2.1.4) to be stable we need that the matrix $IS2$ be stable:

$$IS2 = \begin{pmatrix} Q & PQ & QP & (I - PQ)P & PQP & P \end{pmatrix} \quad (2.1.5)$$

Hence if the process P is open-loop unstable, $IS2$ will also be unstable and the control system has to be implemented in the feedback structure of Fig. 1b. Still, the two step IMC design procedure can be used for the design of Q , as described in the following sections. C can then be obtained from (1.0.2) and the structure in Fig. 1b implemented. In this case, special attention should be paid to the construction of C , so that all the common on or outside the UC zeros of Q and $(I - PQ)$ are cancelled in (1.0.2). Minimal or balanced realization software can be used to accomplish that.

Note that when the process is open-loop stable, it follows from (2.1.3) or (2.1.5) that the only requirement for internal stability is that Q is stable.

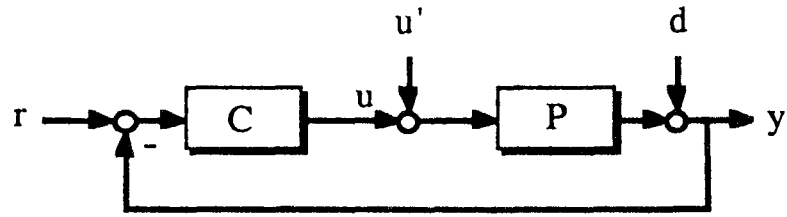
2.1.2. Parametrization of All Stabilizing Q's.

The process P can in general be open-loop unstable. The following assumption simplifies the solution of the optimization problem:

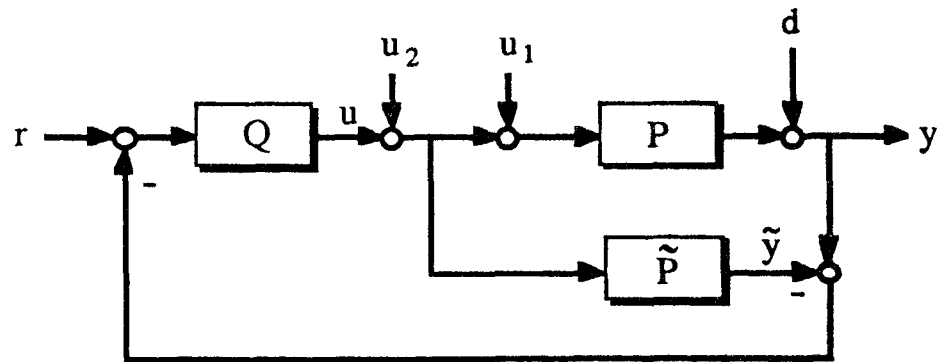
Assumption A.1. If π is a pole of the model \tilde{P} outside the UC, then:

- a. The order of π is equal to 1.
- b. \tilde{P} has no zeros at $z = \pi$.
- c. The residual matrix that corresponds to π is full rank.

Assumption A.1.a is made to simplify the notation and it is the usual case. The results in this paper can be extended to higher order poles. A.1.b is true for SISO systems but not necessarily for MIMO (Kailath, 1980). However, the



(a)



(b)

Figure 3. Internal Stability.

(a) Feedback structure.

(b) IMC structure.

assumption is not restrictive because the presence of a zero at $z = \pi$ implies an exact cancellation in $\det[\tilde{P}(z)]$, which usually does not happen after a slight perturbation in the coefficients of \tilde{P} is introduced. A.1.c is always true for SISO systems, but it can be quite restrictive for MIMO systems. Instead of A.1.c however, an additional assumption can be made on the external input for which the optimal controller is designed. This is discussed in Section 2.2.1.

Assumption A.1 is not made for poles at $z = 1$ because more than one such poles may appear in an element of \tilde{P} , introduced by capacitances that are present in the process. The following assumption true for all practical process control problems is made:

Assumption A.2. Any poles of \tilde{P} or P on the UC are at $z = 1$. Also \tilde{P} has no zeros on the UC.

Let π_1, \dots, π_k be the poles of each element of \tilde{P} outside the UC. Define the allpass

$$b_p(z) = \prod_{i=1}^k \frac{(1 - (\pi_i^*)^{-1})(z - \pi_i)}{(1 - \pi_i)(z - (\pi_i^*)^{-1})} \quad (2.1.6)$$

where the superscript (*) denotes complex conjugate (and transpose when applied to a matrix). If A.1.c does not hold, define

$$b_p(z) = 1 \quad (2.1.6')$$

The following Theorem holds, where “proper” means that the degree of the numerator in any element of $P(z)$ is less than or equal to that of its denominator.

Theorem 2.1.1.

Assume that $Q_0(z)$ is a proper transfer matrix that satisfies the internal stability requirements of Section 1.2, i.e., it produces a matrix IS1 with stable elements. Then all proper Q 's that make IS1 stable are given by

$$Q(z) = Q_0(z) + b_p(z)^2 Q_1(z) \quad (2.1.7)$$

where Q_1 is any proper and stable transfer matrix such that:

- i) If A.1.c holds, PQ_1P has no poles at $z = 1$.
- ii) If A.1.c does not hold, PQ_1P has no poles on or outside the UC.

Proof: See Appendix A.1.

Note that if $P(z)$ is stable, then Theorem 2.1.1 implies that any proper and stable $Q(z)$ is acceptable, as it was remarked in Section 2.1.1.

2.2. H_2 -Optimal $\tilde{Q}(z)$.

2.2.1. Definitions.

We define L_2 as the Hilbert space of complex valued functions $f(z)$ defined on the unit circle ($UC = \{e^{i\theta} | -\pi \leq \theta < \pi\}$) and square-integrable with respect to θ . For a vector function f , the norm on L_2 is given by:

$$\|f\|_2 = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f^*(e^{i\theta}) f(e^{i\theta}) d\theta \right)^{1/2} \quad (2.2.1)$$

L_2 can be decomposed into two subspaces, H_2 the closed subspace of functions having analytic continuations inside the UC and its orthogonal complement H_2^\perp . Note that with these definitions a constant function is in H_2 . H_2 also includes all rational z -transfer functions that are strictly unstable, i.e., which have all their poles strictly outside the UC (including poles at $z = \infty$ (improper transfer functions)). All strictly proper (numerator degree less than denominator degree), stable rational z -transfer functions, are in H_2^\perp .

If $f(z)$ is proper, stable and $\{f_k\}$ is the time domain sequence of vectors corresponding to it, i.e.,

$$\{f_k\} = Z^{-1}\{f(z)\} \quad (2.2.2)$$

then we also have

$$\|f\|_2 = \left(\sum_{k=0}^{\infty} f_k^T f_k \right)^{1/2} \quad (2.2.3)$$

where the superscript (T) denotes transpose. Hence if the error signal $e(z)$ is stable, then the square of its L_2 -norm is equal to the SSE.

For a specified external system input v ($v = d$ for $r = 0$; $v = -r$ for $d = 0$), define by using (1.0.3)

$$\phi(v) \stackrel{\text{def}}{=} \|e\|_2^2 = \|\tilde{E}v\|_2^2 = \|(I - P\tilde{Q})v\|_2^2 \quad (2.2.4)$$

Then $\phi(v)$ is the SSE for the particular input v . For every external input v that will be considered in this paper the following assumption can be made without loss of generality:

Assumption A.3.

- a. The poles of each nonzero element of v outside the UC (if any) are the first k' poles π_i of the plant outside the UC, each with degree 1.
- b. If A.1.c does not hold, then every nonzero element of v (or \hat{v}) includes all the outside the UC poles of \tilde{P} , each with degree 1.

To simplify the arguments in the paper, we shall assume that if A.3.b is satisfied, then A.1.c is not. In this way the proper choices in the definitions and the proofs will be made on the basis of A.1.c. If both A.1.c and A.3.b hold, then the results that apply to the case where A.1.c does not hold but A.3.b does, are still correct.

Define

$$b_v(z) = \prod_{i=1}^{k'} \frac{(1 - (\pi_i^*)^{-1})(z - \pi_i)}{(1 - \pi_i)(z - (\pi_i^*)^{-1})} \quad (2.2.5)$$

If A.1.c does not hold, define

$$b_v(z) = 1 \quad (2.2.5')$$

A different assumption is made for the poles of v at $z = 1$:

Assumption A.4. Let l_i be the maximum number of poles at $z = 1$ that an element of the i^{th} row of P has. Then the i^{th} element of v has at least l_i poles at $z = 1$. Also v has no other poles on the UC and its elements have no zeros on the UC.

The above assumptions are not restrictive in the case where v is an output disturbance d , because in a practical situation we want to design for an out-

put disturbance produced by a disturbance that has passed through the process. Hence, d usually includes the unstable process poles (e.g., an output disturbance d produced by a disturbance in the manipulated variables). Note that the control system will still reject other disturbances with fewer unstable poles, without producing steady-state offset. The assumption is different for poles at $z = 1$ because their number in each row of \tilde{P} can be different, since capacitances may be associated with only certain process outputs. Also the output disturbance may have more poles at $z = 1$ than the process (e.g., a persistent disturbance in the manipulated variables).

The assumptions may be restrictive in the case of setpoints though. However for setpoint tracking the use of the Two-Degree-of-Freedom structure, which will be discussed briefly in Section 2.5, allows us to disregard the existence of any unstable poles of P and therefore this assumption need not be made for setpoints.

The plant P can be factored into an allpass portion P_A and a minimum phase (MP) portion P_M :

$$P = P_A P_M \quad (2.2.6)$$

P_A is stable and such that $P_A^*(e^{i\theta})P_A(e^{i\theta}) = I$. Also P_M^{-1} is stable. P_M has the additional property that both P_M and P_M^{-1} are proper. In the case where P is scalar, this factorization can be easily accomplished by writing P_A as a scalar allpass (similarly to b_p or b_v) containing as zeros the outside the UC zeros of P , times the time delays of the plant so that P_M is semi-proper (numerator degree = denominator degree). In the general multivariable case, this "inner-outer" factorization can be accomplished through the spectral factorization of $P(z^{-1})^T P(z)$, where (T) denotes transpose. Details on these problems can be found in the literature (Motyka and Cadzow, 1967; Anderson et al., 1974; Chu, 1985; Doyle et al., 1984).

2.2.2. Minimization of $\phi(\mathbf{v})$ for One Specific \mathbf{v} .

The objective in this section is to consider only one specific input $v(z)$ and solve the problem:

$$\min_{\tilde{Q} \in \mathcal{Q}} \phi(v) \quad (P1)$$

where \mathcal{Q} denotes the set of all stabilizing Q 's described by Theorem 2.1.1.

Let $v_0(z)$ be the scalar allpass with the property $v_0(1) = 1$, which includes the common outside the UC zeros and time delays of the elements of $v(z)$. Write

$$v(z) = v_0(z)\hat{v}(z) \quad (2.2.7)$$

where $\hat{v}(z)$ is a vector. Hence \hat{v} is proper with at least one element semi-proper and there is no point z on or outside the UC where \hat{v} becomes identically zero.

The following theorem holds:

Theorem 2.2.1.

Any stabilizing \tilde{Q} that solves (P1) satisfies

$$\tilde{Q}\hat{v} = zb_p b_v^{-1} P_M^{-1} \{z^{-1} b_p^{-1} b_v P_A^{-1} \hat{v}\}_* \quad (2.2.8)$$

where the operator $\{\cdot\}_*$ denotes that after a partial fraction expansion of the operand, only the strictly proper terms are retained except for those corresponding to poles of P_A^{-1} . Furthermore, for $n \geq 2$ the number of stabilizing controllers that satisfy (2.2.8) is infinite. Guidelines for the construction of such a controller are given in the proof.

Proof: See Appendix A.2.

Note that not every \tilde{Q} satisfying (2.2.8) is necessarily a stabilizing controller.

2.2.3. Minimization of $\phi(v)$ for a Set of v 's.

Minimizing the SSE just for one vector v is not very meaningful, because of the different directions in which the disturbances enter the process or the setpoints are changed. What is desirable is to find a \tilde{Q} , that minimizes $\phi(v)$ for every single

v in a set of external inputs v of interest for the particular process. For an $n \times n$ P , let us consider the n vectors $v^i(z), i = 1, \dots, n$. Define

$$V \stackrel{\text{def}}{=} (v^1 \quad v^2 \quad \dots \quad v^n) \quad (2.2.9)$$

where v^1, \dots, v^n satisfy assumption A.3. An additional assumption on V is needed:

Assumption A.5.

- a. V has no zeros at the location of its unstable poles or on the UC and V^{-1} cancels the poles of \tilde{P} at $z = 1$ in $V^{-1}\tilde{P}$.
- b. If A.1.c holds, the residual matrices for the outside the UC poles of V are full rank; if A.1.c does not hold, then V^{-1} cancels the outside the UC poles of \tilde{P} in $V^{-1}\tilde{P}$.

Note that satisfaction of assumptions A.3.b and A.4 for each column of V does not necessarily imply satisfaction of A.5. However such a V can be easily constructed. One way is to obtain V as \tilde{P} times a matrix with no outside the UC poles and no zeros on the UC. This case corresponds to the physically meaningful situation, where the output disturbances are produced by disturbances in the manipulated variables. Another simple way is to use a diagonal V , in which case satisfaction of A.3 and A.4 by every column of V implies satisfaction of A.5 by V . This situation is discussed further in Corollary 2.2.1.

Factor V similarly to P (use $V(z)V(z^{-1})^T$ if spectral factorization theory is used):

$$V = V_M V_A \quad (2.2.10)$$

Let us now consider the problem:

$$\min_{\tilde{Q} \in \mathcal{Q}} [\phi(v^1) + \phi(v^2) + \dots + \phi(v^n)] \quad (P2)$$

Hence the \tilde{Q} that solves (P2) minimizes the sum of the squares of the L_2 -errors (SSE) that each of the inputs v^i would cause when applied to the system separately.

Theorem 2.2.2.

The controller

$$\tilde{Q} = zb_p b_v^{-1} P_M^{-1} \{z^{-1} b_p^{-1} b_v P_A^{-1} V_M\} \cdot V_M^{-1} \quad (2.2.11)$$

is the unique solution to (P2).

Proof: See Appendix A.3.

A more meaningful objective would be to solve:

$$\min_{\tilde{Q} \in \mathcal{Q}} \phi(v^i) \quad \forall i = 1, \dots, n \quad (P3)$$

However a \tilde{Q} that solves (P3) will also solve (P2). Then from Theorem 2.2.2 it follows that if a solution to (P3) exists, it is given by (2.2.11). Factor each of the v^i in the way used in (2.2.7):

$$v^i(z) = v_0^i(z) \hat{v}^i(z) \quad (2.2.14)$$

Define

$$\hat{V} \stackrel{\text{def}}{=} (\hat{v}^1 \quad \hat{v}^2 \quad \dots \quad \hat{v}^n) \quad (2.2.15)$$

Theorem 2.2.3.

- i) If $\hat{V}(z)$ is non-minimum phase (i.e., \hat{V}^{-1} is unstable or improper), then there exists no solution to (P3).
- ii) If $\hat{V}(z)$ is minimum phase, then use of \hat{V} instead of V_M in (2.2.11) yields exactly the same \tilde{Q} , which also solves (P3) and it minimizes $\phi(v)$ for any v that is a linear combination of v^i 's that have the same v_0^i 's.

Proof: See Appendix A.4.

The following corollary to Theorem 2.2.3 holds:

Corollary 2.2.1:

Let

$$V = \text{diag}(v_1, v_2, \dots, v_n) \quad (2.2.16)$$

where $v_1(z), \dots, v_n(z)$ are scalars. Then use of \hat{V} instead of V_M in (2.2.11) yields exactly the same \tilde{Q} , which minimizes $\phi(v)$ for the following n vectors:

$$v = \begin{pmatrix} v_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ v_2 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ v_n \end{pmatrix} \quad (2.2.17)$$

and their multiples, as well as for the linear combinations of those directions that correspond to v_i 's with the same outside the UC zeros in the same degree and the same time delays.

2.2.4. Setpoint Prediction

In the case of setpoint tracking, future values of r are often known and supplied to the computer ahead of time. Then if at time t the setpoint value that is fed to the control algorithm as $Z^{-1}\{r(z)\}$ is the one we wish the plant output to reach at time $t + NT$, when T is the sampling time, the objective function has to be modified to:

$$\phi_N(v) = \|(z^{-N}I - P\tilde{Q})r\|_2^2 \quad (2.2.18)$$

When the above objective function is used in the minimization problems (P1), (P2), (P3), the resulting expressions for the H_2 -optimal controller are the same as in Theorems 2.2.1, 2.2.2, 2.2.3, but with the term z^{-N-1} used instead of z^{-1} inside $\{\cdot\}_*$. All the steps in the proofs remain the same when (2.2.18) is used rather than (2.2.4).

2.3. Intersample Rippling.

The H_2 -optimal controller minimizes the SSE. Therefore it completely disregards the plant output's behavior between the sample points. The result is that

the H_2 -optimal controller may produce an excellent performance at the sample points but suffer from severe intersample rippling. Zafiriou and Morari (1985) examined this type of controller for SISO systems and showed that the problem is caused by zeros of $P(z)$ that are close to the point $(-1,0)$ on the z -plane, which give rise to poles of the H_2 -optimal \tilde{Q} that are close to $(-1,0)$. A modification was introduced to substitute such poles in \tilde{Q} with poles at $z = 0$. The new \tilde{Q} was shown to be free of the problem of intersample rippling and to combine desirable deadbeat type characteristics to those of the H_2 -optimal. This section extends the modification to MIMO systems and general open-loop stable or unstable plants. It should be pointed out that this modification is sufficient only if there are no open-loop oscillatory poles in the continuous plant transfer function, which have become unobservable after sampling.

Let $\tilde{Q}_H(z)$ be an H_2 -optimal \tilde{Q} obtained according to the previous sections. Also let $\delta(z)$ be the least common denominator of $P(z)$, and κ_i , $i = 1, \dots, \rho$ be the roots of $\delta(z)$ close to $(-1,0)$ (or in general with negative real part). Define

$$q_-(z) = z^{-\rho} \prod_{j=1}^{\rho} \frac{(z - \kappa_j)}{(1 - \kappa_j)} \quad (2.3.1)$$

Then \tilde{Q}_H is modified as follows:

$$\tilde{Q}(z) = \tilde{Q}_H(z)q_-(z)b(z) \quad (2.3.2)$$

where the scalar $b(z)$ is selected so that IS1 in (2.1.3) and $(I - P\tilde{Q})V$ remain stable. π_i , $i = 0, 1, \dots, k$ are the unstable roots (including $\pi_0 = 1$) of the least common denominator of $P(z), V(z)$. Let the multiplicity of each of them be m_i . Note that the multiplicity of the ones that are outside the UC is equal to 1, according to A.1, A.3. Remember that according to assumptions A.3, A.4, V has at least as many poles at $z = 1$ as P and each pole of V outside the UC is also a pole of P . Then since \tilde{Q}_H makes IS1 and $(I - P\tilde{Q}_H)V$ stable, it follows that the requirements on

$b(z)$ are:

$$\frac{d^j}{dz^j}(1 - \tilde{q}_-(z)b(z))|_{z=\pi_i} = 0, \quad (j = 0, \dots, m_i - 1), i = 0, 1, \dots, k \quad (2.3.3)$$

We can write

$$b(z) = \sum_{j=0}^{m-1} b_j z^{-j} \quad (2.3.4)$$

where

$$m = \sum_{i=0}^k m_i \quad (2.3.5)$$

and then compute the coefficients $b_j, j = 0, \dots, m - 1$ from (2.3.3). Note that since none of the π_i 's is 0 or ∞ , (2.3.3) is equivalent to

$$\frac{d^j}{d\lambda^j}(1 - q_-(\lambda^{-1})b(\lambda^{-1}))|_{\lambda=\pi_i^{-1}} = 0, \quad (j = 0, \dots, m_i - 1), i = 0, 1, \dots, k \quad (2.3.6)$$

Both $\tilde{q}_-(\lambda^{-1})$ and $b(\lambda^{-1})$ are polynomials in λ and therefore their derivatives with respect to λ can be computed easily. Then (2.3.6) yields a system of m linear equations with m unknowns (b_0, b_1, \dots, b_{m-1}). The resulting controller \tilde{Q} combines the desirable properties of the H_2 -optimal controller and deadbeat type controllers.

2.4. \tilde{Q} for Specific Cases.

This section looks at simplified forms of the general expressions for the H_2 -optimal controller, for specific systems and external inputs.

2.4.1. Stable P.

We have $b_p = b_v = 1$. Then (2.2.11) simplifies to

$$\tilde{Q}_H = zP_M^{-1}\{z^{-1}P_A^{-1}V_M\} * V_M^{-1} \quad (2.4.1)$$

2.4.2. Minimum phase P.

$P(z)$ cannot truly be MP for a physical system. Even if the Laplace transfer matrix representing the continuous plant is MP but strictly proper (as a physical

system), the discretized plant $P(z)$ will still have a delay of one unit because of the sampling. Hence $P_A = z^{-1}I$, $P_M = zP$ and (2.2.11) yields

$$\tilde{Q}_H = P^{-1}(I - b_p b_v^{-1} K V_M^{-1}) \quad (2.4.2)$$

where K is the constant term in a partial fraction expansion of $b_p^{-1} b_v V_M$ or since b_p^{-1}, b_v, V_M are semi-proper, K is the product of the constant terms of the PFE's of b_p^{-1}, b_v, V_M . After some algebra we get

$$K = V_{M,0} \prod_{j=k'+1}^k (-\pi_j)^{m_j} \quad (2.4.3)$$

where k, k', π_j are defined in assumptions A.2, A.3, and $V_{M,0}$ is the first non-zero matrix in the impulse response description of $V(z)$, which can be obtained by long division and is equal to the constant term in the PFE of $V_M(z)$.

2.4.3. Example.

Consider the continuous MP system

$$P_c(s) = \frac{b}{-s + b}, \quad b > 0 \quad (2.4.4)$$

and assume that a step disturbance acts at the process input, i.e., the continuous external input $V_c(s)$ is

$$V_c(s) = d_c(s) = \frac{b}{s(-s + b)} \quad (2.4.5)$$

Then for a sampling time T we have

$$P(z) = \frac{1 - e^{bT}}{z - e^{bT}} \quad (2.4.6)$$

$$V(z) = \frac{(1 - e^{bT})z}{(z - 1)(z - e^{bT})} \quad (2.4.7)$$

$$V_M(z) = zV(z) \quad (2.4.8)$$

Note that $e^{bT} > 1$ since $b > 0$. The H_2 -optimal controller can be obtained from (2.4.2). We have $b_p = b_v$ and so from (2.4.8)

$$K = V_{M,0} = 1 - e^{bT} \quad (2.4.9)$$

Substitution of (2.4.6), (2.4.7), (2.4.8), (2.4.9) into (2.4.3) yields

$$\tilde{Q}_H(z) = \frac{(z - e^{bT})((1 + e^{bT})z - e^{bT})}{(1 - e^{bT})z^2} \quad (2.4.10)$$

From Section 2.3 it follows that in this case we have $\tilde{Q}(z) = \tilde{Q}_H(z)$

2.5. Two-Degree-of-Freedom Structure.

From the discussion of the Internal Stability requirements in Section 2.1, it follows that unstable plant poles limit the possible choices of Q and thus the achievable performance. This however need not be so for setpoint tracking. Consider the general feedback structure of Fig. 4. For the disturbance behavior it is irrelevant if the controller is implemented as one block C as in Fig. 1b, or as two blocks as in Fig. 4. Hence the achievable disturbance rejection is restricted both by the outside the UC zeros and poles of P as the quantitative results of the previous sections indicate.

Let us now proceed from the point where a stabilizing \tilde{Q} and the corresponding C have been found through the results of the previous sections, which produce a satisfactory disturbance response. We can then split C into two blocks C_1 and C_2 such that C_1 is minimum phase and C_2 is stable. Then one can see that the only outside the UC zeros of the stabilized system $PC_1(I + PC_1C_2)^{-1}$ are those of the process P . Thus C_3 can be designed without regard for the unstable poles of P and the achievable setpoint tracking is restricted by the outside the UC zeros of P only.

In summary, the achievable disturbance response of a system is restricted by the presence of the plant zeros and poles that lie outside the UC regardless of how complicated a controller is used. If the Two-Degree-of-Freedom controller shown in Fig. 4 is employed, the achievable setpoint response is restricted only by the zeros. A more rigorous discussion can be found in Vidyasagar (1985).

3. Step 2: Design of F.

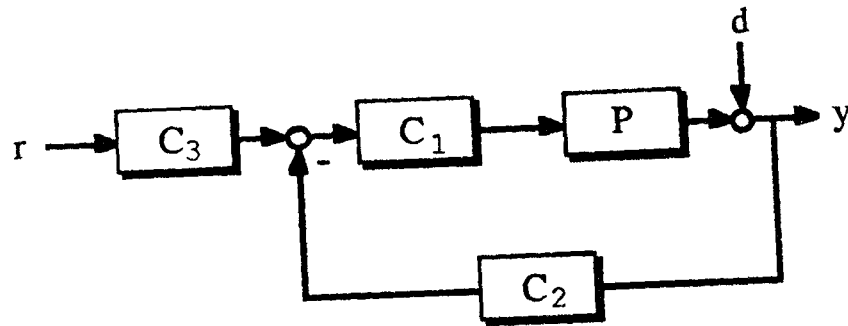


Figure 4. Two-degree-of-freedom feedback structure.

This section deals with the design of the IMC filter $F(z)$ so that the performance characteristics obtained in Step 1 are preserved in the presence of model-plant mismatch.

3.1. Robustness Conditions.

3.1.1. Structured Singular Value.

Potential modeling errors, described as uncertainty associated with the process model, can appear in different forms and places in a multivariable model. This fact makes the derivation of non-conservative conditions that guarantee robustness with respect to model-plant mismatch difficult. The Structured Singular Value (SSV), introduced by Doyle (1982), takes into account the structure of the model uncertainty and it allows the non-conservative quantification of the concept of robust performance.

For a constant complex matrix M the definition of the SSV $\mu_{\Delta}(M)$ depends also on a certain set Δ . Each element Δ of Δ is a block diagonal complex matrix with a specified dimension for each block, i.e.

$$\Delta = \{diag(\Delta_1, \Delta_2, \dots, \Delta_n) | \Delta_j \in \mathbf{C}^{m_j \times m_j}\} \quad (3.1.1)$$

Then

$$\frac{1}{\mu_{\Delta}(M)} = \min_{\Delta \in \Delta} \{\bar{\sigma}(\Delta) | det(I - M\Delta) = 0\} \quad (3.1.2)$$

and $\mu_{\Delta}(M) = 0$ if $det(I - M\Delta) \neq 0 \quad \forall \Delta \in \Delta$. Note that $\bar{\sigma}$ is the maximum singular value of the corresponding matrix.

Details on how the SSV can be used for studying the robustness of a control system can be found in Doyle (1985), where a discussion of the computational problems is also given. For three or fewer blocks in each element of Δ , the SSV can be computed from

$$\mu_{\Delta}(M) = \inf_{D \in \mathbf{D}} \bar{\sigma}(DM D^{-1}) \quad (3.1.3)$$

where

$$\mathbf{D} = \{\text{diag}(d_1 I_{m_1}, d_2 I_{m_2}, \dots, d_n I_{m_n}) \mid d_j \in \mathbf{R}_+\} \quad (3.1.4)$$

and I_{m_j} is the identity matrix of dimension $m_j \times m_j$. For more than three blocks, (3.1.3) still gives an upper bound for the SSV.

3.1.2 Model Uncertainty

In order to effectively use the SSV for designing F , some rearrangement of the block structure is necessary. The IMC structure of Fig. 1a can be written as that of Fig. 5a, where $v = d - r$, $e = y - r$ and

$$G = \begin{pmatrix} 0 & 0 & \tilde{Q} \\ I & I & \tilde{P}\tilde{Q} \\ -I & -I & 0 \end{pmatrix} \quad (3.1.5)$$

where the blocks 0 and I have appropriate dimensions. The block $(P - \tilde{P})$ represents the model-plant mismatch. In order to design a control system that takes into account this modeling error, we need to have some information on how large this mismatch can be. For example we might know a bound $l_a(\omega)$, where ω is the frequency ($z = e^{i\omega T}$), on the additive error:

$$\bar{\sigma}(P - \tilde{P}) \leq l_a \quad (3.1.6)$$

where $\bar{\sigma}(\cdot)$ is the maximum singular value of (\cdot) . However (3.1.6) represents only a very simple way to describe model uncertainty. For multivariable systems, such uncertainty may appear in many different places in the matrix, like specific parameters, elements of P , the inputs or outputs of P , etc. It may then be very conservative to lump this information into (3.1.6). However, provided that we can write P as a linear fractional transformation of its uncertain points, the structure in Fig. 5a can always be transformed into that in Fig. 5b, where Δ is a block diagonal matrix with the additional property that

$$\bar{\sigma}(\Delta) \leq 1 \quad \forall \omega \quad (3.1.7)$$

The superscript u in G^u denotes the dependence on G^u not only on G but also on the specific uncertainty description available for the model \tilde{P} . We shall not demonstrate in detail here, how G^u can be obtained from G . For some common cases of model uncertainty, the expressions can be found in the literature (Zafiriou and Morari, 1986c). For the simple case described by (3.1.6), this can be accomplished by simply multiplying the first row of G with l_a .

Let G^u be partitioned as

$$G^u = \begin{pmatrix} G_{11}^u & G_{12}^u & G_{13}^u \\ G_{21}^u & G_{22}^u & G_{23}^u \\ G_{31}^u & G_{32}^u & G_{33}^u \end{pmatrix} \quad (3.1.8)$$

Then Fig. 5b can be written as Fig. 6 with

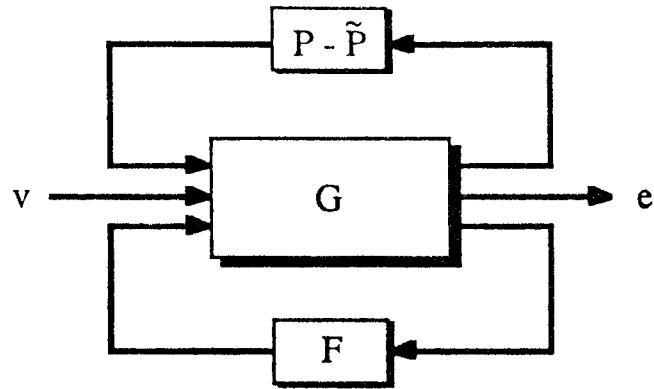
$$\begin{aligned} G^F &= \begin{pmatrix} G_{11}^u & G_{12}^u \\ G_{21}^u & G_{22}^u \end{pmatrix} + \begin{pmatrix} G_{13}^u \\ G_{23}^u \end{pmatrix} (I - FG_{33}^u)^{-1} F \begin{pmatrix} G_{31}^u & G_{32}^u \end{pmatrix} \\ &\stackrel{\text{def}}{=} \begin{pmatrix} G_{11}^F & G_{12}^F \\ G_{21}^F & G_{22}^F \end{pmatrix} \end{aligned} \quad (3.1.9)$$

Note that for P, \tilde{P} , z -transforms and therefore periodic in ω , the block Δ will also be periodic. Hence in this case only the frequencies from 0 to π/T need be considered in (3.1.7). However in Section 3.1.4 it will become apparent that in order to avoid bad intersample behavior, we also have to consider the continuous plant, described by some Laplace transfer function $P_c(s)$ (and $\tilde{P}_c(s)$ for the model). Clearly the modeling error in the description of the discretized plant is related to that in the continuous plant description. For example, let us assume that we have a bound on the additive uncertainty for the continuous plant:

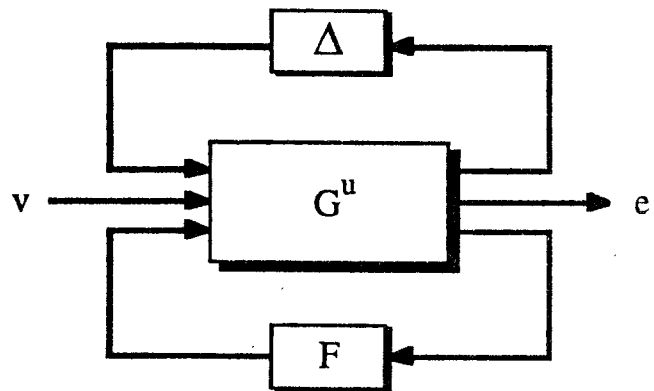
$$\bar{\sigma}(P_c(i\omega) - \tilde{P}_c(i\omega)) \leq l_c(\omega) \quad (3.1.10)$$

Then for the discretized plant we have

$$P(z) - \tilde{P}(z) = Z\mathcal{L}^{-1}\{H(s)(P_c(s) - \tilde{P}_c(s))\} \quad (3.1.11)$$



(a)



(b)

Figure 5. Model uncertainty block diagrams.

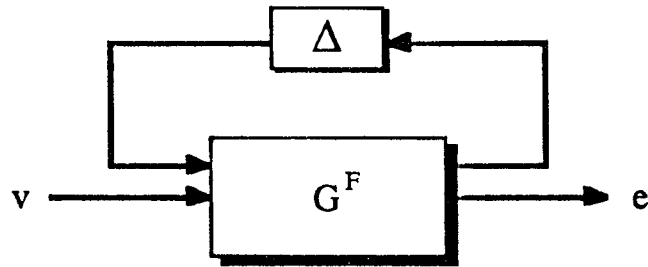


Figure 6. SSV block diagram.

where $H(s)$ is the zero order hold. Then from the property of any z transform $a(z) = Z \mathcal{L}^{-1}\{a_c(s)\}$:

$$a(e^{i\omega T}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} a(i\omega + ik2\pi/T) \quad (3.1.12)$$

the singular value property $\bar{\sigma}(A + B) \leq \bar{\sigma}(A) + \bar{\sigma}(B)$, and (3.1.10), (3.1.11), it follows that

$$\bar{\sigma}(P(e^{i\omega T}) - \tilde{P}(e^{i\omega T})) \leq \frac{1}{T} \sum_{k=-\infty}^{\infty} |H(i\omega + ik2\pi/T)| l_c(i\omega + ik2\pi/T) \quad (3.1.13)$$

$H(s)$ is small at frequencies higher than π/T and goes to 0 as fast as $1/\omega$ as $\omega \rightarrow \infty$. Therefore only a few terms around $k = 0$ are important in the infinite sum. Also note that for a physical system, $l_c(\omega) \rightarrow 0$ at least as fast as $1/\omega$, as $\omega \rightarrow \infty$, and hence the sum converges. Then the bound l_a in (3.1.6) can be set equal to the right hand side of (3.1.13).

However, it is not always possible to obtain in a non-conservative way a mathematical description for the uncertainty in the z -domain, starting from the uncertainty in the s -domain. If first principles models are not available, these descriptions may be the result of experiments conducted with different sampling times, of which one is small enough to approximate the continuous system. A discussion of identification techniques is beyond the scope of this paper. Details on such methods and the resulting modelling error can be found in the literature (e.g., Jenkins and Watts (1969), Astrom and Wittenmark (1984)).

3.1.3. Robust Stability

We now require that the matrix $IS1$ as given by (2.1.2) is stable for all possible plants P . The design of \tilde{Q} according to Section 2 resulted in a stable $IS1$ for $P = \tilde{P}$. In order for $IS1$ to remain stable we need to satisfy the requirements that as we move in a "continuous" way from the model \tilde{P} to the plant P , no closed-loop poles cross the UC and no such poles suddenly appear outside the

UC. The latter requirement is satisfied if we assume that the model and the plant have the same number of poles outside the UC. If this is not the case, another sufficient condition is that G^F is a stable matrix and only stable Δ 's are possible. The SSV can be used to determine if any crossings of the UC occur. Then we can say that the system is stable for any of the plants in the set defined from the bounds on the model uncertainty and which have the same number of outside the UC poles as the model, if and only if (Doyle, 1985)

$$\mu_{\Delta}(G_{11}^F) < 1, \quad 0 \leq \omega \leq \pi/T \quad (3.1.14)$$

3.1.4 Robust Performance

The problem of intersample rippling was addressed for the first step of the design procedure in Section 2.3. There, a simple modification was sufficient because the model $\tilde{P}(z)$ was known exactly. In this section however we have to consider the situation where $P \neq \tilde{P}$ and as a result we have to examine the continuous plant output $y_c(s)$ in order to avoid bad intersample behavior. The obstacle in doing so is the fact that the relation between $y_c(s)$ and $r_c(s)$ or $d_c(s)$ (continuous setpoint and disturbance descriptions) is linear but time varying because of the sampling, and so no transfer function exists that describes this relation. The approach that will be followed in this paper is to obtain a transfer function approximation for the frequencies of interest.

The digital control system is actually implemented as shown in Fig. 7a. The thick lines in the block diagram represent paths along which the signals are described by Laplace transforms, while the thin lines represent digital signals. The block $\gamma(s)$ is an anti-aliasing analog prefilter. Details on the problem of aliasing can be found in the literature (Astrom and Wittenmark, 1980). If the IMC structure is implemented, the block diagram is described in Fig. 7b. The

following notation is used:

$$y(z) = Z \mathcal{L}^{-1}\{y_c(s)\} \quad (3.1.15)$$

$$d(z) = Z \mathcal{L}^{-1}\{d_c(s)\} \quad (3.1.16)$$

$$r(z) = Z \mathcal{L}^{-1}\{r_c(s)\} \quad (3.1.17)$$

$$\tilde{P}_\gamma(z) = Z \mathcal{L}^{-1}\{\gamma(s)H(s)\tilde{P}_c(s)\} \quad (3.1.18)$$

Note that when one wishes to use a $\gamma(s) \neq 1$ in the case of an open-loop unstable plant, the simplest way to avoid any internal stability problems is to use $\tilde{P}_\gamma(z)$ instead of $\tilde{P}(z)$ in Step 1. For the rest of this section we shall assume that $\gamma(s) = 1$ in order to simplify the notation. From Fig. 7b it follows that

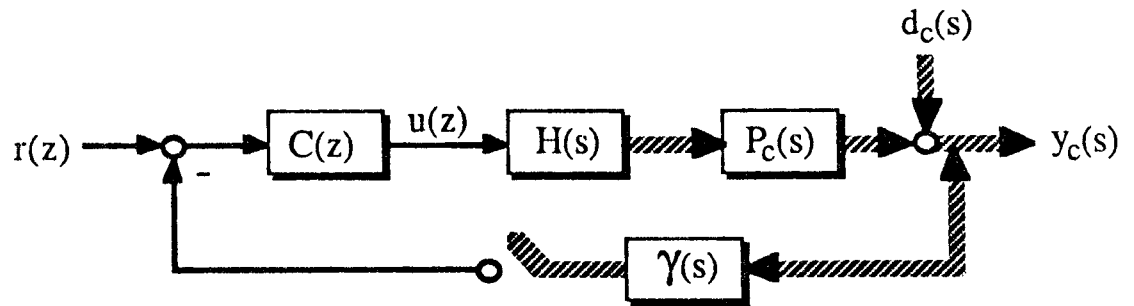
$$\begin{aligned} e_c(s) &= y_c(s) - r_c(s) \\ &= (d_c(s) - r_c(s)) - P_c(s)H(s)\tilde{Q}(e^{sT})F(e^{sT}) \\ &\quad (I - (P(e^{sT}) - \tilde{P}(e^{sT}))\tilde{Q}(e^{sT})F(e^{sT}))^{-1}(d(z) - r(z)) \end{aligned} \quad (3.1.19)$$

We shall now obtain an approximation to (3.1.19) by considering the frequencies $0 \leq \omega \leq \pi/T$. Note that because of the periodicity of $Q(z)$, these are the only frequencies that one can influence independently by using a digital controller. From (3.1.12) it follows that if $a_c(s)$ is small for $\omega > \pi/T$, then

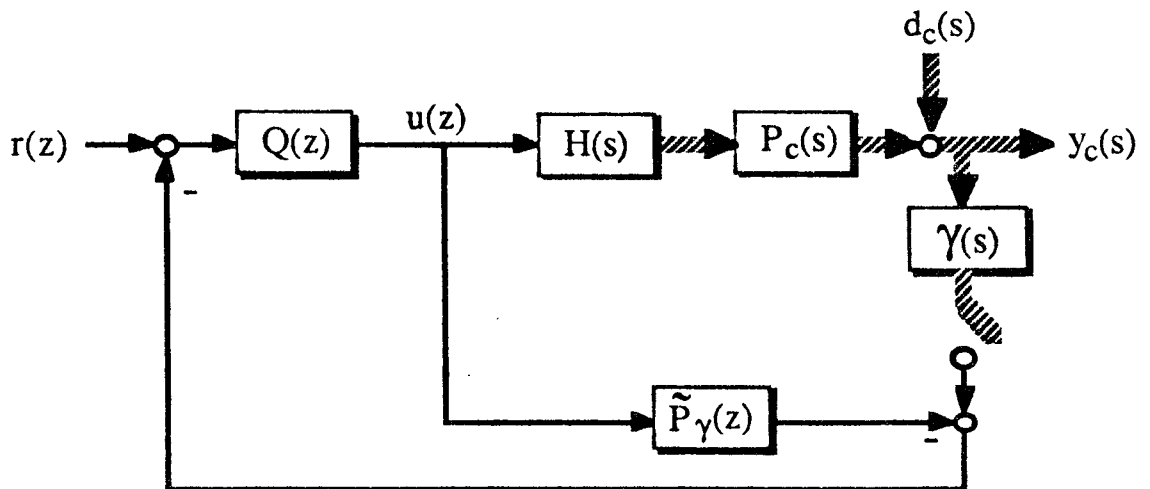
$$a(e^{i\omega T}) \approx \frac{1}{T}a_c(i\omega), \quad 0 \leq \omega \leq \pi/T \quad (3.1.20)$$

Use of (3.1.20) for all the z-transforms in (3.1.19) yields the approximation

$$\begin{aligned} e_c(i\omega) &\approx (I - P_c(i\omega)Q(e^{i\omega T})F(e^{i\omega T})H(i\omega)/T) \\ &\quad (I - (P_c(i\omega) - \tilde{P}_c(i\omega))\tilde{Q}(e^{i\omega T})F(e^{i\omega T})H(i\omega)/T)^{-1}(d_c(i\omega) - r_c(i\omega)) \\ &\stackrel{\text{def}}{=} E_c(i\omega)(d_c(i\omega) - r_c(i\omega)) \end{aligned} \quad (3.1.21)$$



(a)



(b)

Figure 7. Control system implemented on the continuous plant.

(a) Feedback structure.

(b) IMC structure.

Note that the above approximation is valid when the input signals r_c, d_c are small for $\omega > \pi/T$. Clearly, for setpoints one should always select a sampling time small enough to allow tracking of r_c . Note that in reality one does not really have an $r_c(s)$. We can always however assume that $r_c(s)$ represents in the time domain a staircase function that corresponds to the points of $Z^{-1}\{r(z)\}$. For disturbances, if one expects high frequency content at $\omega > \pi/T$ and one cannot reduce T any more, then one should use an anti-aliasing prefilter, whose function is to cutoff frequencies higher than π/T .

Let us use the notation $\tilde{E}_c(i\omega) = E_c(i\omega)$ when $P_c = \tilde{P}_c (\Leftrightarrow P = \tilde{P})$. In the first step of the IMC design procedure, \tilde{Q} is obtained so that it produces satisfactory disturbance rejection and/or setpoint tracking. Since \tilde{E}_c connects the external inputs to the error e_c , a well-designed control system produces a relatively "small" \tilde{E}_c . A measure of the magnitude of the known \tilde{E}_c is its maximum singular value. Let $b(\omega)$ be a frequency function such that

$$\bar{\sigma}(\tilde{E}_c(i\omega)) < b(\omega), \quad 0 \leq \omega \leq \pi/T \quad (3.1.22)$$

When $P \neq \tilde{P}$, the "sensitivity" function E_c is described by (3.1.21). In order for the performance of the control system to remain robust with respect to model-plant mismatch we have to keep e_c small in spite of the modeling error. Similarly to the discussion in Section 3.1.2, we can represent the relation between e_c and $v_c (= d_c - r_c)$ in block diagrams of the form of Figures 5 and 6. The only difference is that we now use $(H(s)\tilde{P}_c(s)/T)$ instead of $\tilde{P}(z) (= \tilde{P}(e^{sT}))$ in G and that the block Δ is obtained from the modeling error in $H(s)(P_c(s) - \tilde{P}_c(s))/T$ and so G^u depends on the continuous plant uncertainty as well. We shall use the subscript c to indicate that.

Then we require:

$$\max_{0 \leq \omega \leq \pi/T} \bar{\sigma}(b(\omega)^{-1} E_c(i\omega)) < 1 \quad \forall \Delta \in \Delta \quad (3.1.23)$$

We can now use the properties of the SSV (Doyle, 1985) to obtain

$$\max_{0 \leq \omega \leq \pi/T} \bar{\sigma}(b(\omega)^{-1} E_c(i\omega)) < 1 \quad \forall \Delta \in \Delta \iff \max_{0 \leq \omega \leq \pi/T} \mu_{\Delta^0}(G_c^b) < 1 \quad (3.1.24)$$

where

$$G_c^b = \begin{pmatrix} I & 0 \\ 0 & b^{-1} \end{pmatrix} G_c^F \quad (3.1.25)$$

$$\Delta^0 = \{diag(\Delta, \Delta^0) | \Delta \in \Delta, \Delta^0 \in C^{n \times n}\} \quad (3.1.26)$$

3.2 Solution to the Filter Synthesis Problem

In this section the filter design problem is formulated and solved as a parameter optimization problem.

3.2.1 Filter Form.

At this point some structure has to be assumed for F , which can be of any general type that the designer wishes. However in order to keep the number of variables in the optimization problem small, a rather simple structure like a diagonal F with first or second-order terms would be recommended. In most cases this is not restrictive because the potentially higher orders of the model \tilde{P} have been included in the controller \tilde{Q} that was designed in the first step of the IMC procedure and which is in general a full matrix. The use of more complex filter structure may be necessary in cases of highly ill-conditioned systems ($\bar{\sigma}(\tilde{P})/\underline{\sigma}(\tilde{P})$ very large). The filter structure for such systems is discussed in detail in Zafriou and Morari (1986c).

Some additional restrictions on the filter exist in the case of an open-loop unstable plant. The filter $F(z)$ is chosen to be a diagonal rational function that satisfies the following requirements.

- a. Internal Stability. IS1 in (2.1.3) must be stable.
- b. Asymptotic setpoint tracking and/or disturbance rejection. $(I - \tilde{P}\tilde{Q}F)v$ must be stable.

Write

$$F(z) = \text{diag}(f_1(z), \dots, f_n(z)) \quad (3.2.1)$$

Then, assumptions A.1, 2, 3, 4, 5 and the facts that $\tilde{Q}(z)$ is designed to make IS1 and $(I - P\tilde{Q})V$ stable, imply that the requirements on an element f_l of F are:

$$\frac{d^j}{dz^j} (1 - f_l(z))|_{z=\pi_i} = 0, \quad j = 0, \dots, m_i - 1 \quad ((m_{ol} - 1) \text{ for } i = 0), \quad i = 0, 1, \dots, k \quad (3.2.2)$$

where $\pi_0 = 1$ and m_{ol} is the highest multiplicity of π_0 as pole of an element of the l^{th} row of V . Note that for $j = 0$, (3.2.2) yields

$$f_l(\pi_i) = 1, \quad i = 0, 1, \dots, k \quad (3.2.3)$$

(3.2.3) clearly shows the limitation that poles outside the UC place on the robustness properties of a control system designed for an open-loop unstable plant. Since because of (3.2.3) one cannot reduce the nominal ($P = \tilde{P}$) closed-loop bandwidth of the system at frequencies corresponding to the unstable poles of the plant, one can only tolerate a relatively small model error at those frequencies.

One can now select for a filter element, the form

$$f(z) = \Phi(z)f_1(z) \quad (3.2.4)$$

where

$$f_1(z) = \frac{(1 - \alpha)z}{z - \alpha} \quad (3.2.5)$$

$$\Phi(z) = \sum_{j=0}^w \beta_j z^{-j} \quad (3.2.6)$$

and the coefficients β_0, \dots, β_w are computed so that (3.2.2) is satisfied for some specified α . The parameter α can be used as a tuning parameter.

Note that for $k = 0, \pi_0 = 1, m_{ol} = 1$, we only need $\Phi(z) = 1$. For the general case, (3.2.2) can be transformed into a system of ν_l linear equations with β_0, \dots, β_w

as unknowns where ν_l is given by

$$\nu_l = m_{ol} + m_1 + \dots + m_k \quad (3.2.7)$$

Since none of the π_i is 0 or ∞ , (3.2.2) is equivalent to

$$\frac{d^j}{d\lambda^j} (1 - f_l(\lambda^{-1}))|_{\lambda=\pi_i^{-1}} = 0,$$

$$j = 0, 1, \dots, m_i - 1 \quad ((m_{ol} - 1) \text{ for } i = 0), \quad i = 0, 1, \dots, k \quad (3.2.8)$$

Then the fact that $\Phi(\lambda^{-1})$ is a polynomial in λ , and the following theorem can help simplify the necessary algebra.

Theorem 3.2.1. (Zafriou and Morari, 1986a).

For the $f_1(z)$ given by (3.2.5) we have

$$\frac{d^j}{d\lambda^j} f_1(\lambda^{-1}) = (1 - \alpha)j! \alpha^j (1 - \alpha\lambda)^{-(j+1)} \quad (3.2.9)$$

One should select $w \geq \nu_l - 1$ so that the system of linear equations has one or more solutions. When $w \geq \nu_l$ we have an underdetermined system and then β_0, \dots, β_w can be obtained as the minimum norm solution. Note that for $\nu_l = 2$ one should select $w \geq 2$ in order to avoid the trivial solution $f(z) = 1$.

Let us now examine the usual situation where $m_i = 1$ for $i = 1, \dots, k$. Then (3.2.2) is equivalent to:

$$\frac{d^j}{dz^j} (1 - f(z))|_{z=\pi_0=1} = 0, \quad j = 0, \dots, m_{ol} - 1 \quad (3.2.10)$$

$$f(\pi_i) = 1, \quad i = 1, \dots, k \quad (3.2.11)$$

Then for this special case, the following theorem holds:

Theorem 3.2.2

When $m_i = 1$ for $i = 1, \dots, k$, the coefficients β_0, \dots, β_w , have to satisfy

$$\begin{pmatrix} \Pi \\ N \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_w \end{pmatrix} = \begin{pmatrix} f_1(\pi_k)^{-1} \\ \vdots \\ f_1(\pi_0)^{-1} \\ -\alpha/(1-\alpha) \\ 0 \\ \vdots \\ 0 \end{pmatrix} \stackrel{\text{def}}{=} \chi \quad (3.2.12)$$

where

$$\Pi = \begin{bmatrix} 1 & \pi_k^{-1} & \dots & \pi_k^{-w} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \pi_0^{-1} & \dots & \pi_0^{-w} \end{bmatrix} \quad (3.2.13)$$

and the elements ν_{ij} of the $(m_{ol} - 1) \times (w + 1)$ matrix N are defined by

$$\nu_{ij} = \begin{cases} 0 & \text{for } i \geq j \\ \frac{(j-1)!}{(j-1-i)!} & \text{for } i < j \end{cases} \quad (3.2.14)$$

Proof: It follows directly from Theorem 3 of Zafiriou and Morari (1986a) and (3.2.11).

In general a $w \geq \nu_l - 1$ should be selected and $\beta_0, \beta_1, \dots, \beta_w$ be obtained as the minimum norm solution to (3.2.12):

$$\begin{pmatrix} \beta_0 \\ \vdots \\ \beta_w \end{pmatrix} = A^T (AA^T)^{-1} \chi \quad (3.2.15)$$

where

$$A \stackrel{\text{def}}{=} \begin{pmatrix} \Pi \\ N \end{pmatrix} \quad (3.2.16)$$

Note that in general it is numerically preferable to compute the pseudo-inverse in (3.2.15) from a singular value decomposition (SVD) of A .

Example 3.2.1: Assume that P and V have one common unstable pole at $\pi \neq 1$, and that $m_{ol} = 0$. Then

$$A = [1 \quad \pi^{-1} \dots \pi^{-w}] \quad (3.2.17)$$

$$\chi = \frac{\pi - \alpha}{(1 - \alpha)\pi} \quad (3.2.18)$$

and (3.2.15) yields

$$\beta_j = \frac{1 - \pi^{-2}}{1 - \pi^{-2}(w+1)} \frac{(\pi - \alpha)\pi^{-j-1}}{1 - \alpha}, \quad j = 0, \dots, w \quad (3.2.19)$$

Note that

$$\sum_{j=0}^w \beta_j^2 = \left(\frac{\pi - \alpha}{(1 - \alpha)\pi} \right)^2 \frac{1 - \pi^{-2}}{1 - \pi^{-2}(w+1)} \rightarrow \frac{(\pi - \alpha)^2(\pi^2 - 1)}{(1 - \alpha)^2\pi^4} \quad (3.2.20)$$

since $\pi > 1$. Hence when π is close to $(1, 0)$, the properties of $f(z)$ in (3.2.4) are similar to those of $f_1(z)$.

Example 3.2.2. (Zafiriou and Morari, 1986a):

Assume that $k = 0$ and that $m_{ol} = 2$. Then the minimum norm solution is

$$\beta_j = -\frac{6j\alpha}{(1 - \alpha)w(w + 1)(2w + 1)}, \quad j = 1, \dots, w \quad (3.2.21)$$

$$\beta_0 = 1 - \sum_{j=1}^w \beta_j \quad (3.2.22)$$

The norm of this solution goes to 0 as $w \rightarrow \infty$ and so the properties of $f(z)$ in (3.2.4) are similar to those of $f_1(z)$ when w is large enough.

3.2.2 Objective

We can write

$$F \stackrel{\text{def}}{=} F(z; \Lambda) \quad (3.2.23)$$

where Λ is an array with the filter parameters.

The problem can now be formulated as a minimization problem over the elements of the array Λ . A constraint is that the elements of Λ corresponding to denominator poles should be such that F is a stable transfer function. Note that if a diagonal filter with elements given by (3.2.4) is used, then each element of Λ corresponds to same α_j , which has to be inside the UC for F to be stable.

Then the problem can be turned into an unconstrained one by using some λ_j as an element of Λ and writing

$$\alpha_j = e^{-T/\lambda_j^2}$$

or

$$\alpha_j = e^{-T/|\lambda_j|} \quad (3.2.24)$$

Hence any λ_j in $(-\infty, \infty)$ produces an α_j in $(0,1]$. Note that if one wishes to use in (3.2.4) a higher order $f_1(z)$ with more parameters, one can write the denominator of each element of F as a product of polynomials of degree 2 and one of degree 1 if the order is odd. Then similarly to (3.2.24) one can write the roots of the polynomials of degree 2 as e^{Tp_1}, e^{Tp_2} , where p_1, p_2 are the roots of $\lambda_2^2 x^2 + \lambda_1^2 x + 1 = 0$ or $|\lambda_2|x^2 + |\lambda_1|x + 1 = 0$ and in doing so turn the problem into an unconstrained one in λ_1, λ_2 .

Our goal is to satisfy (3.1.24). The filter parameters can be obtained by solving

$$\min_{\Lambda} \max_{0 \leq \omega \leq \pi/T} \mu_{\Delta^0}(G_c^b) \quad (P4)$$

It may be however that the optimum values for (P4), still do not manage to satisfy (3.1.24). The reason is usually that the performance requirements set by the selection of $b(\omega)$ in (3.1.22) are too tight to satisfy in the presence of model-plant mismatch. In this case one should choose a less tight bound b and resolve (P4). Note that satisfaction of the Robust Performance condition (3.1.24) does not necessarily imply satisfaction of the Robust Stability condition (3.1.14). Hence when a solution to (P4) is found, one should check if (3.1.14) holds. Note that if the uncertainty description for the continuous plant (used in (3.1.24)) and the discretized plant (used in (3.1.14)) are equivalent, then satisfaction of (3.1.24) is usually sufficient for satisfaction of (3.1.14). If this does not happen then one can always substitute the objective function in (P4) with the μ function given in

(3.1.14) and proceed with the minimization only up to the point where (3.1.14) is satisfied. The computational issues remain the same as those discussed in Section 3.2.3 for (P4). Note however that the result of (P4) may often be a local minimum. To circumvent this problem, (P4) should be solved for a number of starting points. Good initial guesses can often be obtained for the filter parameters by matching them with the frequencies where the peaks of $\mu_{\Delta^0}(G_c^b)$ appears for $F = I$.

3.2.3 Computational Issues

The computational of μ in (P4) is made through (3.1.3); details can be found in Doyle (1982). As it was pointed out in Doyle (1985), the minimization of the Frobenius norm instead of the maximum singular value yields D 's which are very close to the optimal ones for (3.1.3). Note that the minimization of the Frobenius norm is a very simple task. In the computation of the maximum in (P4) only a finite number of frequencies will be considered. Hence (P4) is transformed into

$$\min_{\Lambda} \max_{\omega \in \Omega} \inf_{D \in \mathbf{D}^0} \bar{\sigma}(DG_c^b D^{-1}) \quad (P4')$$

where Ω is a set containing a finite number of frequencies in $[0, \pi/T]$ and \mathbf{D}^0 is the set corresponding to Δ^0 according to (3.1.1) and (3.1.4). Define

$$\Phi_{\infty}(\Lambda) \stackrel{\text{def}}{=} \max_{\omega \in \Omega} \inf_{D \in \mathbf{D}^0} \bar{\sigma}(DG_c^b D^{-1}) \quad (3.2.25)$$

The analytic computation of the gradient of Φ_{∞} with respect to Λ is in general possible. This is not the case when the two or more largest singular values of $DG_c^b D^{-1}$ are equal. However this is quite uncommon and although the computation of a generalized gradient is possible, experience has shown the use of a mean direction to be satisfactory. A similar problem appears when the $\max_{\omega \in \Omega}$ is attained at more than one frequencies, but again the use of a mean direction seems to be sufficient.

Assume that for the value of Λ where the gradient of $\Phi_{\infty}(\Lambda)$ is computed, the $\max_{\omega \in \Omega}$ is attained at $\omega = \omega_0$ and that the $\inf_{D \in \mathbf{D}^0} \bar{\sigma}(DG_c^b(i\omega_0)D^{-1})$ is obtained

at $D = D_0$, where only one singular value σ_1 is equal to $\bar{\sigma}$. Let the singular value decomposition (SVD) be

$$D_0 G_c^b(i\omega_0) D_0^{-1} = (u_1 \quad U) \begin{pmatrix} \sigma_1 & 0 \\ 0 & \Sigma \end{pmatrix} \begin{pmatrix} v_1^* \\ V^* \end{pmatrix} \quad (3.2.26)$$

Then from Zafiriou and Morari (1986 c,d) we have, under the above assumption, that the element of the gradient vector corresponding to the j th element of Λ , λ_j , is given by

$$\begin{aligned} \frac{\partial}{\partial \lambda_j} \Phi_\infty = \operatorname{Re} \left[u_1^* D_0 \begin{pmatrix} G_{c,13}^u \\ b^{-1} G_{c,23}^u \end{pmatrix} (I - F G_{c,33}^u)^{-1} \frac{\partial}{\partial \lambda_j} F \Big|_{\omega=\omega_0} \right. \\ \left. (I - F G_{c,33}^u)^{-1} (G_{c,31}^u \quad G_{c,32}^u) D_0^{-1} v_1 \right] \end{aligned} \quad (3.2.27)$$

where $F, G_{c,i,j}^u, b$ are computed at $\omega = \omega_0$. The derivatives of F with respect to its parameters (elements of Λ) depend on the particular form that the designer selected and they can be easily computed.

4. Illustration.

In this section an example is presented to demonstrate the problem of intersample rippling in the H_2 -optimal controller and the modification that was discussed in Section 2.3. Consider the continuous system

$$P_c(s) = \begin{pmatrix} \frac{0.50}{s+1} & \frac{1.42}{6s+1} \\ \frac{1.00}{2s+1} & \frac{1.00}{4s+1} \end{pmatrix} \quad (4.0.1)$$

The discretized system (zero order hold included) for a sampling time of $T = 1$, is

$$P(z) = \begin{pmatrix} \frac{0.316}{z-0.368} & \frac{0.218}{z-0.846} \\ \frac{0.393}{z-0.607} & \frac{0.221}{z-0.779} \end{pmatrix} \quad (4.0.2)$$

Computation of the roots of $\det[P(z)]$ show that the system in (4.0.2) has two finite zeros, at $a_1 = -0.95$ and $a_2 = 0.75$. The first zero is close to $(-1,0)$ and it is expected to cause intersample rippling when the H_2 -optimal controller is used.

From (4.0.2) it follows that $P_A = z^{-1}I$, $P_M = zP$. We shall consider step setpoint changes as external inputs, i.e.,

$$V(z) = \frac{z}{z-1} I \quad (4.0.3)$$

Then (2.4.1) yields

$$\tilde{Q}_H(z) = z^{-1}P^{-1} \quad (4.0.4)$$

Fig. 8a shows the time response of this control system for a unit step change in the setpoint of output 1:

$$v(z) = \begin{pmatrix} z/(z-1) \\ 0 \end{pmatrix} \quad (4.0.5)$$

The prediction of intersample rippling is verified. Note that at the sample points the outputs are indeed exactly at the setpoints producing the minimum SSE.

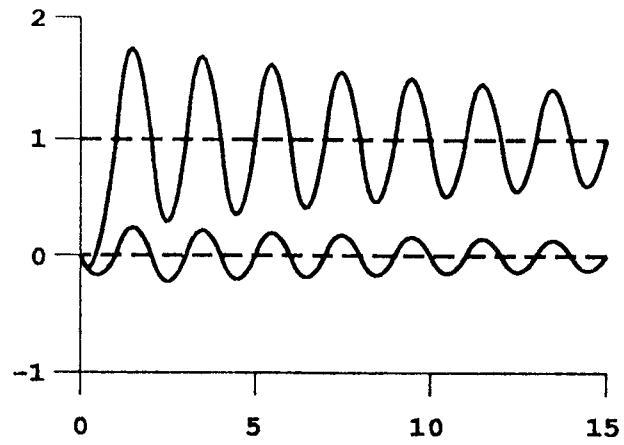
The IMC controller is now obtained from (2.3.2) where $b(z) = 1$ and

$$q_-(z) = \frac{z + 0.95}{1.95z} \quad (4.0.6)$$

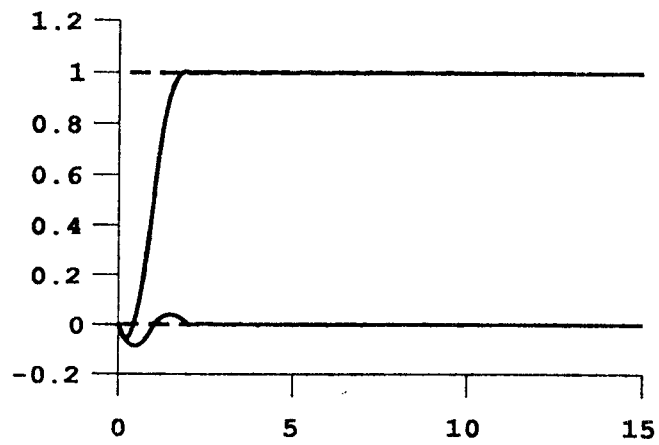
The response for this control system is shown in Fig. 8b. Clearly the problem has disappeared. Finally note that all responses show an inverse response characteristic. This is due to the fact that the continuous system $P_c(s)$ has a right half plane zero.

5. Concluding Remarks.

The results presented in this paper provide a direct synthesis procedure for digital multivariable controllers. The two-step IMC design concept is extended to open-loop unstable systems and the limitations imposed by open-loop unstable poles on achievable performance and robustness are quantified. In the first step the controller is designed for a whole set of external inputs (setpoints or disturbances) and it combines desirable properties of the H_2 -optimal and deadbeat type controllers. In the second step the parameters of the low-pass IMC filter are obtained as the result of the optimization of an SSV based objective function, which reflects the performance of the continuous plant outputs, so that bad intersample behavior is avoided. The use of the SSV allows the treatment of general types of model-plant mismatch.



(a)



(b)

Figure 8. Time response.

- Dashed lines: Setpoints; Solid lines: Outputs.

(a) H_2 -optimal controller \tilde{Q}_H .

(b) IMC controller \tilde{Q} .

APPENDIX A

A.1. Proof of Theorem 2.1.1.

The fact that Q_1 has to be proper in order for Q to be proper and vice versa, follows from the properness of Q_0 and b_p .

i) We shall show that any Q given by (2.1.7) makes IS1 stable. From substitution of (2.1.7) into (2.1.3) it follows that all that is required is that $(Pb_p^2Q_1 \quad b_p^2Q_1P \quad Pb_p^2Q_1P)$ be stable. From the properties of Q_1 , it follows that the third element in the above matrix is stable. Stability of the other two follows by pre- and post-multiplication of that element by P^{-1} , since according to assumptions A.1, A.2, P has no zeros at the location of its unstable poles and these are the only possible unstable poles in the above matrix.

ii) Assume that Q makes IS1 stable. Then the difference matrix

$$IS1(Q) - IS1(Q_0) = ((Q - Q_0 \quad P(Q - Q_0) \quad (Q - Q_0)P \quad P(Q - Q_0)P) \quad (A.1.1)$$

is stable.

The fact that P has no zeros at the location of the unstable poles makes the stability of the matrix in (A.1.1) equivalent to the stability of $(Q - Q_0)$, $P(Q - Q_0)P$. Then, when assumption A.1.c holds, we can write $P = b_p\hat{P}$, where \hat{P} has no zeros at the unstable poles of P and its only unstable poles are at $z = 1$. So, it follows that $(Q - Q_0) = b_p^2Q_1$ with Q_1 stable and such that PQ_1P have no poles at $z = 1$. If A.1.c does not hold, Q_1 should also have the property that it makes PQ_1P stable.

A.2. Proof of Theorem 2.2.1.

We shall assume that a Q_0 exists, which in addition to the properties mentioned in Theorem 2.1.1, it also produces a matrix $(I - PQ_0)V^0$ with no poles at $z = 1$, where V^0 is a diagonal matrix with l_v poles at $z = 1$ in every element, with l_v the maximum number of such poles in any element of v . If assumption

A.1.c does not hold, then each column of V^0 also satisfies A.3.b and Q_0 makes $(I - PQ_0)V^0$ stable. Its existence will be proven by construction. Substitution of (2.1.7) into (2.2.4) and use of the fact that pre- or post-multiplication of a function with an allpass does not change its L_2 -norm, yields:

$$\begin{aligned}\phi(v) &= \|z^{-1}b_p^{-1}b_vP_A^{-1}(I - PQ_0)\hat{v} - z^{-1}b_p b_v P_M Q_1 \hat{v}\|_2^2 \\ &\stackrel{\text{def}}{=} \|f_1 - f_2 Q_1 \hat{v}\|_2^2\end{aligned}\tag{A.2.1}$$

f_1 has no poles at $z = 1$ because $(I - PQ_0)V^0$ has no such poles. Any rational function $f_1(z)$ with no poles on the UC, can be uniquely decomposed into a strictly proper, strictly stable part $\{f_1\}_-$ in H_2^\perp and a strictly unstable part $\{f_1\}_+$ in H_2 :

$$f_1 = \{f_1\}_- + \{f_1\}_+\tag{A.2.2}$$

When A.1.c holds, inspection of (A.2.1) shows that $f_2 Q_1 \hat{v}$ can have no poles on or outside the UC except possibly for some poles at $z = 1$ introduced by \hat{v} . Since f_1 has no poles at $z = 1$, in order for $\phi(v)$ to be finite, $f_2 Q_1 \hat{v}$ should have no poles at $z = 1$. Thus the optimal Q_1 has to cancel any such poles. When A.1.c does not hold, then the fact that $(I - PQ_0)V^0$ is stable and A.1.b imply that an acceptable Q_1 and therefore the optimal Q_1 is such that $f_2 Q_1 v$ is stable. We shall assume that Q_1 has this property. It should be verified at the end however that the solution indeed has the property. Since $f_2 Q_1 \hat{v}$ is strictly proper in addition to being stable, we can write

$$\phi(v) = \|\{f_1\}_+\|_2^2 + \|\{f_1\}_- - f_2 Q_1 \hat{v}\|_2^2\tag{A.2.3}$$

The first term in the right hand side of (A.2.3) does not depend on Q_1 . Hence for solving (P1) we only have to look at the second term. The obvious solution is

$$Q_1 \hat{v} = f_2^{-1} \{f_1\}_-\tag{A.2.4}$$

Clearly such a Q_1 produces a stable $f_2 Q_1 \hat{v}$ as it was assumed. It should now be proved that Q_1 's that satisfy the internal stability requirements exist among those described by (A.2.4), so that the obvious solution is a true solution. For $n = 1$, (A.2.4) yields a unique Q_1 , which can be shown to satisfy the requirements by following the arguments in the proof of Theorem 2.2.2 in Appendix A.3. For $n \geq 2$ write

$$\hat{v} \stackrel{\text{def}}{=} (\hat{v}_1 \quad \hat{v}_2 \quad \dots \quad \hat{v}_n)^T \quad (\text{A.2.5})$$

$$\hat{V}_2 \stackrel{\text{def}}{=} (\hat{v}_2 \quad \dots \quad \hat{v}_n)^T \quad (\text{A.2.6})$$

$$Q_1 \stackrel{\text{def}}{=} (q_1 \quad q_2) \quad (\text{A.2.7})$$

where without loss of generality the first element of v is assumed to be nonzero. Also q_1 is $n \times 1$ and q_2 is $n \times (n - 1)$. Then from (A.2.4) it follows that

$$Q_1 = (\hat{v}_1^{-1}(f_2^{-1}\{f_1\}_- - q_2 \hat{V}_2) \quad q_2) \quad (\text{A.2.8})$$

We now need to show that a proper, stable q_2 exists such that Q_1 is proper, stable and produces a PQ_1P with no poles at $z = 1$ (and no poles outside the UC, when A.1.c does not hold). Select a q_2 of the form:

$$q_2(z) = \hat{q}_2(z)(1 - z^{-1})^{3l} \prod_{i=1}^k (1 - \pi_i z^{-1})^3 \quad (\text{A.2.9})$$

where \hat{q}_2 is proper, stable. Then from (A.2.8) it follows that in order for PQ_1P not to have any poles at $z = 1$ it is sufficient that $P\hat{v}_1^{-1}f_2^{-1}\{f_1\}_- \{P\}_{1^{\text{st}} \text{ row}}$ have no such poles. This holds because the poles at $z = 1$ in the P on the left cancel with the P_M^{-1} in f_2^{-1} and v_1 (and \hat{v}_1) has by assumption A.4 at least as many poles at $z = 1$ as the 1st row of P . When A.1.c does not hold, then the same type of argument and the fact that A.3.b holds imply that PQ_1P has no poles outside the UC either. Let us now examine the stability of Q_1 . The only poles outside

the UC may come from \hat{v}_1^{-1} . Let α be such a pole (zero of v_1). Then for stability we need to find \hat{q}_2 such that

$$\hat{q}_2(\alpha)\hat{V}_2(\alpha) = (1 - \alpha^{-1})^{-3l_v} \prod_{i=1}^k (1 - \pi_i \alpha^{-1})^{-3} f_2^{-1}(\alpha) \{f_1\}_-(\alpha) \quad (\text{A.2.10})$$

The above equation always has a solution because the vector $\hat{V}_2(\alpha)$ is not identically zero since any common outside the UC zeros in v were factored out in v_0 . We now need to examine the properness of Q_1 . Since P_M^{-1} is proper and $\{f_1\}_-$ is strictly proper, $f_2^{-1}\{f_1\}_-$ is proper. Then if \hat{v}_1^{-1} is improper (\hat{v}_1 strictly proper) there exist at least one element in \hat{V}_2 that is semi-proper. Hence by solving a system of linear equations we can always select a $\hat{q}_2(z)$ such that of the first impulse response coefficients of $f_2^{-1}\{f_1\}_- - q_2\hat{V}_2$, as many are zero as we need to make the first element of the matrix in (A.2.8) proper.

We shall now proceed to obtain an expression for $Q\hat{v}$. (2.1.7) and (A.2.8) yield

$$\begin{aligned} Q\hat{v} &= z b_p b_v^{-1} P_M^{-1} [z^{-1} b_p^{-1} b_v P_A^{-1} P Q_0 \hat{v} - \{z^{-1} b_p^{-1} b_v P_A^{-1} P Q_0 \hat{v}\}_- \\ &\quad + \{z^{-1} b_p^{-1} b_v P_A^{-1} \hat{v}\}_-] \\ &= z b_p b_v^{-1} P_M^{-1} [\{z^{-1} b_p^{-1} b_v P_A^{-1} P Q_0 \hat{v}\}_{0+} + \{z^{-1} b_p^{-1} b_v P_A^{-1} \hat{v}\}_-] \quad (\text{A.2.11}) \end{aligned}$$

where $\{\cdot\}_{0+}$ indicates that in the partial fraction expansion all poles on or outside the UC are retained. For (A.2.11), these poles are the poles of $b_p^{-1} b_v \hat{v}$ on or outside the UC; $P_A^{-1} P Q_0 = P_M Q_0$ is strictly stable and proper because Q_0 is a stabilizing controller. When A.1.c holds, the stability of $(I - P Q_0)P$ and the fact that the residues of P at the outside the UC poles are full rank imply that at these poles $I - P Q_0 = 0$. Also the fact that $(I - P Q_0)V^0$ has no poles at $z = 1$ imply that $(I - P Q_0)$ and its derivatives up to the $(l_v - 1)^{th}$ are also equal to zero at $z = 1$. When A.1.c does not hold, the fact that $(I - P Q_0)V^0$ is stable and

that the columns of this diagonal V^0 satisfy A.3.b, imply that $I - PQ_0 = 0$ at $1, \pi_1, \dots, \pi_k$. Thus (A.2.11) simplifies to (2.2.8).

We simply need to establish now that a stabilizing controller Q_0 with the property that $(I - PQ_0)V^0$ has no poles at $z = 1$ exists. The use in (2.2.11) of a V^0 with the properties mentioned at the beginning of this section, instead of V , yields such a controller.

A.3. Proof of Theorem 2.2.2.

The L_2 -norm for a matrix $G(z)$ analytic on the UC is given by

$$\|G\|_2 = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \text{trace}[G^*(e^{i\theta})G(e^{i\theta})] d\theta \right)^{1/2} \quad (\text{A.3.1})$$

Then from (2.2.1), (A.3.1) it follows that

$$\phi(v^1) + \phi(v^2) + \dots + \phi(v^n) = \|(I - P\tilde{Q})V\|_2^2 \stackrel{\text{def}}{=} \phi(V) \quad (\text{A.3.2})$$

The minimization of $\phi(V)$ follows the steps in the proof of Theorem 2.2.1 in Appendix A.2 up to (A.2.4), with V_M used instead of \hat{v} . In this case l_v is the maximum number of poles at $z = 1$ in any element of V . From the equivalent to (A.2.4) equation we obtain

$$Q_1 = f_2^{-1}\{f_1\}_- V_M^{-1} \quad (\text{A.3.3})$$

We now have to establish that Q_1 is stable, proper and produces a PQ_1P with no poles at $z = 1$ (nor outside the UC, when A.1.c does not hold).

In the case where b_p, b_v are not equal to identity, the stability of Q_1 follows from the full rank conditions in A.1.c and A.5.b. In PQ_1P the poles at $z = 1$ of the P on the left cancel with the P_M^{-1} in f_2^{-1} . As for the P on the right, the same follows from assumption A.5.a. When A.1.c does not hold, the same arguments are true for the outside the UC poles as well.

Then in the same way that (2.2.8) follows from (A.2.8), (2.2.11) follows from (A.3.3).

A.4. Proof of Theorem 2.2.3.

A stabilizing controller that solves (P3) has to solve (P1) for all v^i , $i = 1, \dots, n$. Satisfying (2.2.8) for every v^i is equivalent to

$$\tilde{Q} = zb_p b_v^{-1} P_M^{-1} \{z^{-1} b_p^{-1} b_v P_A^{-1} \hat{V}\} \hat{V}^{-1} \quad (\text{A.4.1})$$

Hence the above \tilde{Q} is the only potential solution to (P3). However it is not necessary a stabilizing controller since not only stabilizing \tilde{Q} 's satisfy (2.2.8) for some v . Indeed if \hat{V} is non-minimum phase, \hat{V}^{-1} is unstable and/or improper and this results in an unstable and/or improper \tilde{Q} , which is therefore unacceptable. Hence in such a case, there exists no solution to (P3), which completes the proof of part (i) of the theorem.

In the case where \hat{V}^{-1} is stable and proper (\hat{V} minimum phase), the controller given by (A.4.1) is stable and proper and therefore it is the same as the one given by (2.2.11). This fact can be explained as follows. We have

$$V = \hat{V} V_0 \quad (\text{A.4.2})$$

where

$$V_0 = \text{diag}(v_0^1, v_0^2, \dots, v_0^n) \quad (\text{A.4.3})$$

Since \hat{V}^{-1} is stable and proper, (A.4.2) represents a factorization of V similar to that in (2.2.10). From spectral factorization theory it follows that

$$\hat{V}(z) = V_M(z) A \quad (\text{A.4.4})$$

where A is a constant matrix such that $AA^* = I$. Then from (2.2.11) it follows that use of \hat{V} does not alter \tilde{Q} because A cancels.

Let us now assume without loss of generality that the first j v^i 's have the same v_0^i 's. Consider a v that is a linear combination of v^1, \dots, v^j :

$$v(z) = \alpha_1 v^1(z) + \dots + \alpha_j v^j(z) \quad (\text{A.4.5})$$

Then it follows that

$$v_0(z) = v_0^1(z) = \dots = v_0^j(z) \quad (\text{A.4.6})$$

$$\hat{v}(z) = \alpha_1 \hat{v}^1(z) + \dots + \alpha_j \hat{v}^j(z) \quad (\text{A.4.7})$$

One can easily check that a \tilde{Q} that satisfies (2.2.8) for $\hat{v}^1, \dots, \hat{v}^j$, will also satisfy (2.2.8) for the \hat{v} given by (A.4.7) because of the property

$$\{\alpha_1 f_1(z) + \dots + \alpha_j f_j(z)\}_* = \alpha_1 \{f_1(z)\}_* + \dots + \alpha_j \{f_j(z)\}_* \quad (\text{A.4.8})$$

But then from Theorem 2.2.1 it follows that if a stabilizing controller \tilde{Q} satisfies (2.2.8) for \hat{v} , then it minimizes the L_2 - error $\phi(v)$.

Acknowledgements

Support from the National Science Foundation is gratefully acknowledged.

References.

- ANDERSON, B. D. O., "Recursive Algorithm for Spectral Factorization", *I.E.E.E. Trans. on Circuits and Systems*, CAS-21, 742, 1974.
- ASTROM, K. J. and B. WITTENMARK, *Computer Controlled Systems*, (Englewood Cliffs, NJ: Prentice Hall), 1984.
- CHU, C. C., *H_{∞} -Optimization and Robust Multivariable Control*, Ph.D. Thesis, Dept. of Electr. Eng., Univ. of Minnesota, 1985.
- DESOER, C. A. and J. D. SCHULMAN, "Zeros and Poles of Matrix Transfer Functions and Their Dynamical Interpretation", *I.E.E.E. Trans. on Circuits and Systems*, CAS-21, 3, 1974.
- DOYLE, J. C., "Analysis of Feedback Systems with Structured Uncertainty", *I.E.E. Proc.*, Part D, 129, 242, 1982.
- DOYLE, J. C., *Lecture Notes*, ONR/Honeywell Workshop on Advances on Multivariable Control, 1984.
- DOYLE, J. C., "Structured Uncertainty in Control System Design", *Proc. of the Conf. on Dec. and Control*, Fort Lauderdale, FL, 1985.
- GARCIA, C. E. and M. MORARI, "Internal Model Control. A Review and Some New Results", *Ind. and Eng. Chem., Proc. Des. and Dev.*, 21, 308, 1982.
- GARCIA, C. E. and M. MORARI, "Internal Model Control 2. Design Procedure for Multivariable Systems", *Ind. and Eng. Chem., Proc. Des. and Dev.*, 24, 472, 1985a.
- GARCIA, C. E. and M. MORARI, "Internal Model Control 3. Multivariable Control Law Computation and Tuning Guidelines", *Ind. and Eng. Chem., Proc. Des. and Dev.*, 24, 484, 1985b.
- JENKINS, G. M. and D. G. WATTS, *Spectral Analysis and Its Applications*, Holden-Day, 1969.

- KAILATH, T., *Linear Systems*, (Englewood Cliffs, NJ:Prentice Hall), 1980.
- MORARI, M., E. ZAFIRIOU and C. G. ECONOMOU, *An Introduction to Internal Model Control* (accepted for publication in the series *Lecture Notes in Control and Information Sciences*, Springer-Verlag), 1987.
- MOTYKA, P. R. and J. A. CADZOW, "The Factorization of Discrete-Process Spectral Matrices", *I.E.E.E. Trans. on Autom. Control*, AC-12, 698, 1967.
- SKOGESTAD, S. and M. MORARI, "Control of Ill-conditioned Plants: High Purity Distillation", *Proc. of the A.I.Ch.E. Ann. Meet.*, Miami Beach, FL, 1986.
- VIDYASAGAR, M., *Control System Synthesis*, MIT Press, Cambridge, MA, 1985.
- YOULA, D. C., J. J. BONGIORNO and H. A. JABR, "Modern Wiener-Hopf Design of Optimal Controllers - Part II: The Multivariable Case", *I.E.E.E. Trans. on Autom. Control*, AC-21, 319, 1976.
- ZAFIRIOU, E. and M. MORARI, "Digital Controllers for SISO Systems: A Review and a New Algorithm", *Int. J. of Control*, 42, 855, 1985.
- ZAFIRIOU, E. and M. MORARI, "Design of Robust Digital Controllers and Sampling Time Selection for SISO Systems", *Int. J. of Control*, 44, 711, 1986a.
- ZAFIRIOU, E. and M. MORARI, "Digital Controller Design for Multivariable Systems with Structural Performance Specifications", *submitted to the Int. J. of Control*, 1986b.
- ZAFIRIOU, E. and M. MORARI, "Robust H_2 -Type IMC Controller Design Via the Structured Singular Value", *submitted to Automatica*, 1986c.
- ZAFIRIOU, E. and M. MORARI, "Synthesis of the IMC Filter by Using the Structured Singular Value Approach", *Proc. of the Amer. Control Conf.*, p. 1, Seattle, WA, 1986d.

- ZAMES, G., "Feedback and Optimal Sensitivity: model reference transformations, Multiplicative semi-norms and approximate inverses", *I.E.E.E. Trans. Autom. Control*, AC-26, 301, 1981.
- ZAMES, G. and B. A. FRANCIS, "Feedback, Minimax Sensitivity and Optimal Robustness", *I.E.E.E. Trans. Aut. Control*, AC-28, 585, 1983.