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On the True Cramer-Rao Lower Bound for the DA Joint Estimation of Carrier Phase and Timing Offsets

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***Abstract*—This paper concerns the Cramer-Rao lower bound (CRB) for the data-aided (DA) timing and/or phase recovery, i.e., the synchronization parameter acquisition is aided by a training sequence known to the receiver. For the DA parameter estimation, the CRB typically varies with the training sequence. This indicates that different training sequences offer fundamental different performance. In this manuscript, we derive a closed-form formula of the CRB for timing and phase recovery with respect to any particular training sequence. The bound illustrates the close relation between the training sequence and the fundamental limit on timing and phase synchronization. It provides additional insights on the training sequence design.**

I. Introduction

The Cramer-Rao lower bound (CRB) is a general lower bound on the minimum mean square error (MSE) of any unbiased estimator [1]. The CRB usually serves as a benchmark for the performance of actual estimator. Therefore it receives considerable attention in the literature. In practical systems, synchronization parameters such as timing and carrier phase offsets are usually acquired with the help of training sequence (TS), which is the DA estimation. In the DA case, the CRB generally varies with the TS, which implies that different TS offers fundamental different performance. Therefore it is very important to compute the CRB for any particular TS to understand the fundamental limit that a particular TS has.

However, in the literature ([2] - [6]), the closed-form CRB for DA timing and/or carrier phase recovery for an arbitrary TS is not available. The authors of [2] gave a summary of the CRB's for carrier frequency, phase and timing offsets estimation. The CRB for joint timing and carrier phase recovery was first introduced by Moeneclaey in [3], it was further discussed in his publications [4] and [5]. It is mathematically intractable to evaluate the bound when the TS is arbitrary. Moeneclaey simplified the calculation by the adoption of the strong law of large number and the assumption that the TS is zero mean, i.i.d., and sufficiently long. This method reduces the calculation dramatically, but it also covers the interaction between TS and estimation performance, therefore limits the usage of the CRB. In order to deal with the estimation problem in the presence of nuisance parameters, D'Andrea *et al.* proposed the *modified* CRB (MCRB) in [6]. It is pointed out in [5] that the CRB's derived previously in [3] - [4] are actually MCRB's.

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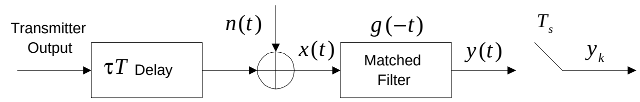

Fig. 1. Modeling of Channel and Matched Filter

In principle, it is possible to use brute-force numerical approach to compute the CRB for any given TS. Such brute-force computation involves the evaluation of derivative numerically and matrix inversion. Besides the computational complexity, brute-force approach does not provide any insight on the interaction between TS and the resultant CRB. In this manuscript, a closed-form CRB (denoted as $\mathrm{CRB_{DA}}$) for the DA joint carrier phase and timing offsets estimation is derived with respect to arbitrary TS. The only assumption is that the derivative of shaping pulse exists (i.e., the pulse is sufficiently smooth). The bound reveals the close relation between TS and performance limit. We found that the $\mathrm{CRB_{DA}}$ for some particular sequence could be significantly lower than that of others. Therefore it provides us insight on the sequence design. Some recent research result on the inverse of Toeplitz matrices [7] is applied in the computation. The frequency domain approach introduced by the Toeplitz matrix expedites the calculation of the bound.

The rest of the paper is organized as follows. In Section II., the problem is formulated mathematically at first. The $\mathrm{CRB_{DA}}$'s for joint timing and phase estimation under the condition of over-sampling (e.g., two samples per symbol for raised-cosine shape) and under-sampling are presented. Section III. evaluates the $\mathrm{CRB_{DA}}$'s. Some comparisons between the bounds proposed here and those derived in [2] [3]-[5] are also discussed. In Section IV., several examples are illustrated. Section V. concludes the paper.

II. The Cramer-Rao lower Bounds

The baseband received signal is modeled as:

$$x(t) = \sqrt{E_s} \sum_{m=-N/2}^{N/2-1} a_m g(t - mT - \tau T) e^{j\phi} + n(t) \tag{1}$$

where $g(t) = g_T(t) \otimes c(t) \otimes f(t)$ (without loss of generality let us assume that $g(t)$ is real), $g_T(t)$ is the transmitter shaping function, $c(t)$ is the channel response, $f(t)$ is the prefilter, $n(t)$ is the additive white Gaussian noise (AWGN) with two-sided power spectral density $N_0/2$, T is the symbol interval, $\{a_m\}$, $m \in \mathcal{Z}$ ($\mathcal{Z}$ the set of integers) is the data sequence drawn from complex plain with $E[a_m] = 0$ and $E[|a_m|^2] = 1$. ϕ is the carrier phase offset, the delay jitter τT models the absence of symbol synchronization between transmitter and receiver, it is assumed that $\tau \in [-0.5, 0.5)$. The received signal $x(t)$ is passed through a matched filter with response $g(-t)$ as shown in Fig. 1. The output $y(t)$ of the matched filter is sampled at the rate of $1/T_s$, typically $T = LT_s$, with L an integer. The TS $\{a_m\}$ ($m = -N/2, \cdots, N/2-1$) is known between the transmitter and receiver. The implicit assumption is that the timing offset τ remains fixed over the duration of observation.

A. Problem Formulation

The sampled output of the matched filter is

$$y_k = \sqrt{E_s} \sum_{m=-N/2}^{N/2-1} a_m r(kT_s - mT - \tau T) e^{j\phi} + N_k \tag{2}$$

where $r(t) = g(t) \otimes g(-t)$, $N(t) = n(t) \otimes g(-t)$ and $N_k = N(kT_s)$ that is a sequence of Gaussian random variables with zero mean and the auto-correlation function

$$R_y[k-l] = E[N_k N_l^*] = \frac{N_0}{2} r((k-l)T_s) \tag{3}$$

We can rewrite Eq. 2 in terms of matrix and vector product. First, let us define the following vectors

$$\begin{aligned} \underline{y} &= [\ y_{-K/2} \ \cdots \ y_0 \ \cdots \ y_{K/2-1}\ ]^T \\ \underline{a} &= [\ a_{-N/2} \ \cdots \ a_0 \ \cdots \ a_{N/2-1}\ ]^T \\ \underline{N} &= [\ N_{-K/2} \ \cdots \ N_0 \ \cdots \ N_{K/2-1}\ ]^T \end{aligned} \tag{4}$$

where $K = L(N+R)$, R models the signal $y(t)$ beyond the TS portion in ideal case in which a shaping pulse $r(t)$ modulated only by the TS $\underline{a}$ is transmitted and used to estimate the parameters. Let us define a K by N matrix $R(\tau)$ with the $\{m,n\}$th element equal to $r((m-K/2)T_s - (n-N/2)T - \tau T)$, for $m = 0, 1, \cdots, K-1$, $n = 0, 1, \cdots, N-1$. With these notations, Eq. 2 can be written as

$$\underline{y} = \sqrt{E_s} R(\tau) \underline{a} e^{j\phi} + \underline{N} \tag{5}$$

The likelihood function of ϕ and τ is formulated as the following. The mean of $\underline{y}$ given $\underline{a}$, ϕ and τ is

$$\underline{m}_y(\underline{a}, \phi, \tau) = E[\underline{y}|\underline{a}, \phi, \tau] = \sqrt{E_s} R(\tau) \underline{a} e^{j\phi} \tag{6}$$

The auto-covariance matrix of vector $\underline{y}$ is

$$\mathrm{cov}[\underline{y}|\underline{a}, \phi, \tau] = \frac{N_0}{2} \Lambda \tag{7}$$

where Λ is a K by K matrix with the $\{k,m\}$th element equal to $r_{km} = r[(k-m)T_s]$, therefore Λ is a Toeplitz matrix (for a stationary random process). The log likelihood function of ϕ, τ given $\underline{a}$ is

$$\begin{aligned} l(\underline{y}|\underline{a}, \phi, \tau) &= -\frac{1}{N_0}[-\underline{y}^H Q \underline{m}_y - \underline{m}_y^H Q \underline{y} + \\ &\quad \underline{m}_y^H Q \underline{m}_y] + C_l \end{aligned} \tag{8}$$

where Q is the inverse matrix of Λ with the assumption that its inverse exists, and C_l is a constant independent of $\underline{m}_y$.

The $\mathrm{CRB_{DA}}$'s are the diagonal elements of the inverse of Fisher information matrix J [1] for the joint estimation $\{\phi, \tau\}$. J is defined as

$$J = \begin{bmatrix} J_{\phi\phi} & J_{\phi\tau} \\ J_{\tau\phi} & J_{\tau\tau} \end{bmatrix} \tag{9}$$

whose element is given by (let $\underline{\theta} = [\theta_1\ \theta_2]$ with $\theta_1 = \phi$ and $\theta_2 = \tau$)

$$J_{\theta_i\theta_j} = E\left[-\frac{\partial^2 l(\underline{y}|\underline{a}, \phi, \tau)}{\partial\theta_i \partial\theta_j}\right] \tag{10}$$

where E denotes the expectation with respect to $\underline{y}$ and τ if τ is random, or it denotes the expectation with respect to $\underline{y}$ if τ is deterministic [1].

B. *The* $\mathrm{CRB_{DA}}$ *for the Over-Sampling Case*

It can be shown that

$$J_{\phi\phi} = \frac{2E_s}{N_0} \underline{a}^H R(\tau)^H Q R(\tau) \underline{a} \tag{11}$$

$$J_{\phi\tau} = \frac{2E_s}{N_0} \Re\left[(-j)\underline{a}^H R(\tau)^H Q \frac{\partial R(\tau)}{\partial\tau} \underline{a}\right] \tag{12}$$

$$J_{\tau\phi} = J_{\phi\tau} \tag{13}$$

$$J_{\tau\tau} = \frac{2E_s}{N_0} \underline{a}^H \frac{\partial R(\tau)^H}{\partial\tau} Q \frac{\partial R(\tau)}{\partial\tau} \underline{a} \tag{14}$$

The CRB_{DA}'s for the DA joint estimation of carrier phase and timing offsets are given by

$$E\left[(\phi-\hat{\phi})^2\right] \geq \text{CRB}_{\text{DA}}(\phi) \triangleq \frac{J_{\tau\tau}}{J_{\phi\phi}J_{\tau\tau}-J_{\phi\tau}^2} \quad (15)$$

$$E\left[(\tau-\hat{\tau})^2\right] \geq \text{CRB}_{\text{DA}}(\tau) \triangleq \frac{J_{\phi\phi}}{J_{\phi\phi}J_{\tau\tau}-J_{\phi\tau}^2} \quad (16)$$

Let us focus our discussion on the band-limited shaping pulse. We first consider the case that L is no less than Nyquist frequency, i.e., $1/T_s \geq 2B$ for B the bandwidth of $r(t)$. In this case, there is no aliasing in the frequency domain, $J_{\phi\phi}$, $J_{\phi\tau}$ and $J_{\tau\tau}$ are independent of τ. It can be shown that they are equal to

$$J_{\phi\phi} = \frac{2E_s}{N_0K}\sum_{m=-K/2}^{K/2-1}\mathcal{RA}_o(m) \quad (17)$$

$$J_{\phi\tau} = -\frac{2E_s}{N_0K}\sum_{m=-K/2}^{K/2-1}\left(\frac{2\pi m}{N+R}\right)\mathcal{RA}_o(m) \quad (18)$$

$$J_{\tau\tau} = \frac{2E_s}{N_0K}\sum_{m=-K/2}^{K/2-1}\left(\frac{2\pi m}{N+R}\right)^2\mathcal{RA}_o(m) \quad (19)$$

where $\mathcal{RA}_o(m)$ (the subscript o refers to *over-sampling*) is defined as

$$\mathcal{RA}_o(m) \triangleq \frac{1}{T_s}\mathcal{R}\left(\frac{2\pi m}{KT_s}\right)\left|\mathcal{A}\left(\frac{2\pi m}{N+R}\right)\right|^2 \quad (20)$$

with $\mathcal{R}(\omega)$ the Fourier transform (FT) of $r(t)$, and $\mathcal{A}(\omega)$ the discrete time Fourier transform (DTFT) of $\underline{a}$, which is defined by $\mathcal{A}(\omega)=\sum_{n=-N/2}^{N/2-1}a_ne^{-j\omega n}$. Basically $\mathcal{RA}_o(\omega)$ is the power spectrum density (PSD) of the signal output from the matched filter.

C. The CRB_{DA} *for the Under-Sampling Case*

In the under-sampling case, i.e., $1/T_s < 2B$, there is aliasing in the frequency domain, then the CRB_{DA} generally depends on the specific value of τ if it is deterministic. In practice, τ can be modeled by a uniformly distributed random variable in the receiver front-end. In this case $J_{\phi\phi}$, $J_{\phi\tau}$ and $J_{\tau\tau}$ should be averaged with respect to both $\underline{y}$ and τ. In a typical communication system, let $L=1$, i.e., one symbol rate sampling, then $K=N+R$, the following integral holds for arbitrary integer k and l

$$\int_{-1/2}^{1/2}e^{j2\pi\tau(k-l)}d\tau = \delta[k-l].$$

After some arithmetic, $J_{\phi\phi}$, $J_{\phi\tau}$ and $J_{\tau\tau}$ become

$$J_{\phi\phi} = \frac{2E_s}{N_0K}\sum_{m=-K/2}^{K/2-1}\frac{\mathcal{RA}_u(m)}{\mathcal{F}(2\pi m/K)} \quad (21)$$

$$J_{\phi\tau} = -\frac{2E_s}{N_0K}\sum_{m=-K/2}^{K/2-1}\left(\frac{2\pi m}{N+R}\right)\frac{\mathcal{RA}_u(m)}{\mathcal{F}(2\pi m/K)} \quad (22)$$

$$J_{\tau\tau} = \frac{2E_s}{N_0K}\sum_{m=-K/2}^{K/2-1}\left(\frac{2\pi m}{N+R}\right)^2\frac{\mathcal{RA}_u(m)}{\mathcal{F}(2\pi m/K)} \quad (23)$$

where $\mathcal{RA}_u(m)$ (the subscript u refers to *under-sampling*) is defined as

$$\mathcal{RA}_u(m) \triangleq \sum_{k=-\infty}^{\infty}\frac{1}{T_s^2}\mathcal{R}\left(\frac{2\pi m}{KT_s}-\frac{2\pi k}{T_s}\right)^2\left|\mathcal{A}\left(\frac{2\pi m}{N+R}\right)\right|^2, \quad (24)$$

and $\mathcal{F}(\omega)$ is the DTFT of $\{r(nT_s)\}$, i.e.,

$$\mathcal{F}(\omega)=\sum_{n=-\infty}^{\infty}r(nT_s)e^{-j\omega n}=\frac{1}{T_s}\sum_{k=-\infty}^{\infty}\mathcal{R}\left(\frac{\omega}{T_s}-\frac{2\pi k}{T_s}\right) \quad (25)$$

In the calculation of Eq. 24, we use the fact that $\mathcal{A}(\omega-2\pi k)=\mathcal{A}(\omega)$, therefore we can separate $\mathcal{A}(\omega)$ and the aliased $\mathcal{R}(\omega)^2$. In practice, a shaping pulse is always band-limited. Typically its effective bandwidth B ranges from $1/2T$ to $1/T$, k in Eq. 25 is usually from -1 to 1.

III. Evaluating the Bounds

Before evaluating the CRB_{DA}'s, we discuss the role of R in our computation first. As we explained in Section II., R is used to model the signal $y(t)$ beyond the TS portion. As discussed in [7], R is determined by the residue error of the approximation (a circular matrix is used to approximate Q under certain condition). It gives us mathematical convenience when we apply the finite boundary strong sense convergence theory related to the inverse of Toeplitz matrix derived in [7] and the discrete Fourier transform (DFT). In numerical evaluation of the bounds, R should be large enough to make the CRB_{DA}'s converge, typically $R \geq 100$ is sufficient in most cases.

A. $J_{\phi\tau}$*: the Cost of Two Unknown Parameters Scenario*

Since $J_{\phi\tau}^2 \geq 0$, from Eq. 15-16 it is clear that

$$\frac{J_{\tau\tau}}{J_{\phi\phi}J_{\tau\tau}-J_{\phi\tau}^2} \geq \frac{J_{\tau\tau}}{J_{\phi\phi}J_{\tau\tau}} = \frac{1}{J_{\phi\phi}} \quad (26)$$

$$\frac{J_{\phi\phi}}{J_{\phi\phi}J_{\tau\tau}-J_{\phi\tau}^2} \geq \frac{J_{\phi\phi}}{J_{\phi\phi}J_{\tau\tau}} = \frac{1}{J_{\tau\tau}} \quad (27)$$

It is easy to verify that the CRB for timing/phase estimation with known phase/timing offset is equal to $1/J_{\tau\tau}$ ($1/J_{\phi\phi}$) respectively, therefore $J_{\phi\tau}$ serves as the *cost* when both phase and timing offsets are unknown. There are two observations:

- The cost could be reduced to zero in the following manner. In the over-sampling case $J_{\phi\tau}$ is given by Eq. 18. According to the assumption that $r(t)$ is real, which means that $\mathcal{R}(\omega)$ is an even function; $(2\pi m/(N+R))$ is an odd function; if $|\mathcal{A}(\omega)|$ is an even function, which is a sufficient condition, $J_{\phi\tau} = 0$. In the under-sampling case, the same result holds.

- As pointed in [2] (p.329), the random data TS could make $J_{\phi\tau} = 0$. A more general sufficient condition is proposed here. In fact, any real TS $\underline{a}$ could make $J_{\phi\tau}$ equal zero.

In the following presentation, we assume that $J_{\phi\tau}$ is equal to zero.

B. The $\mathrm{CRB_{DA}}$ *for Phase Estimation*

The $\mathrm{CRB_{DA}}$ for phase estimation is $\mathrm{CRB_{DA}}(\phi) = 1/J_{\phi\phi}$. According to the Parseval's relation (K-point DFT is an orthogonal transform), in the over-sampling case when $r(t)$ is a Nyquist shaping pulse,

$$\mathrm{CRB_{DA}}(\phi) = \left\{ \frac{2E_s}{N_0} \sum_{n=-N/2}^{N/2-1} |a_n|^2 \right\}^{-1} \tag{28}$$

In PSK type modulation, $|a_n| = 1$, the $\mathrm{CRB_{DA}}$ is independent of rolloff factor (for raised-cosine shape) and TS, which is the same as that in the literature [2] in the over-sampling case.

Worthy of mention is that in the under-sampling case, the bound behaves quite differently. For the Nyquist shaping pulse when the sampling rate $L = 1$ (i.e., $T_s = T$, $K = N + R$), $\mathcal{F}(\omega) = T$. Let us limit our discussion on the raised-cosine pulse because of its popularity. When the rolloff factor α ranges from 0 to 1, the effective bandwidth of $r(t)$ ranges from $1/2T$ to $1/T$.

$$\mathrm{CRB_{DA}}(\phi) = \left\{ \frac{2E_s}{N_0 K T^2} \sum_{m=-K}^{K-1} \mathcal{R}\left(\frac{2\pi m}{KT}\right)^2 \left|\mathcal{A}\left(\frac{2\pi m}{N+R}\right)\right|^2 \right\}^{-1} \tag{29}$$

In the joint estimation, $\mathrm{CRB_{DA}}(\phi)$ is closely related to the TS and shaping pulse, and larger than that in the over-sampling case. For the raised-cosine shape, the $\mathrm{CRB_{DA}}(\phi)$ generally increases as the rolloff factor α increases because increasing α causes more aliasing that hurts the estimation performance.

C. The $\mathrm{CRB_{DA}}$ *for Timing Estimation*

Similarly, in the over-sampling case ($L \geq 2$), the $\mathrm{CRB_{DA}}$ for timing estimation is

$$\mathrm{CRB_{DA}}(\tau) = \left\{ \frac{2E_s}{N_0 K} \sum_{m=-K/2}^{K/2-1} \left(\frac{2\pi m}{N+R}\right)^2 \frac{1}{T_s} \mathcal{R}\left(\frac{2\pi m}{KT_s}\right) \left|\mathcal{A}\left(\frac{2\pi m}{N+R}\right)\right|^2 \right\}^{-1} \tag{30}$$

Unlike $\mathrm{CRB_{DA}}(\phi)$, $\mathrm{CRB_{DA}}(\tau)$ is closely related to the TS and rolloff factor. In the under-sampling case ($L = 1$), the timing bound is

$$\mathrm{CRB_{DA}}(\tau) = \left\{ \frac{2E_s}{N_0 K} \sum_{m=-K/2}^{K/2-1} \left(\frac{2\pi m}{N+R}\right)^2 \frac{1}{T^2} \mathcal{R}\left(\frac{2\pi m}{KT}\right)^2 \left|\mathcal{A}\left(\frac{2\pi m}{N+R}\right)\right|^2 \right\}^{-1} \tag{31}$$

The widely used CRB for timing estimation was derived in the literature [4] for both over and under-sampling cases. As explained before, the assumption that the TS was i.i.d. random data and sufficiently large N was applied to simplify the computation. In the next subsection, we will show that the bounds derived in [4] are the special cases of the $\mathrm{CRB_{DA}}(\tau)$ (Eq. 30 - 31) under the same assumption.

D. Asymptotic Bounds

Let $\Delta f = 1/(N + R)$, because N and R are tied, as either of them goes to ∞, we have the following integral expressions of the bounds. When $L = 1$, we have

$$\mathrm{CRB_{DA}}(\phi) = \left\{ \frac{2E_s}{N_0 T^2} \int_{-\infty}^{\infty} \mathcal{R}\left(\frac{2\pi f}{T}\right)^2 |\mathcal{A}(2\pi f)|^2 \, df \right\}^{-1} \tag{32}$$

and

$$\mathrm{CRB_{DA}}(\tau) = \left\{ \frac{2E_s}{N_0 T^2} \int_{-\infty}^{\infty} 4\pi^2 f^2 \mathcal{R}\left(\frac{2\pi f}{T}\right)^2 |\mathcal{A}(2\pi f)|^2 df \right\}^{-1} \tag{33}$$

For $L \geq 2$, we have

$$\mathrm{CRB_{DA}}(\tau) = \left\{ \frac{2E_s}{N_0 T} \int_{-\infty}^{\infty} 4\pi^2 f^2 \mathcal{R}\left(\frac{2\pi f}{T}\right) |\mathcal{A}(2\pi f)|^2 df \right\}^{-1} \tag{34}$$

For zero mean, i.i.d. TS $\underline{a}$, as N goes to ∞, it can be shown that $|\mathcal{A}(\omega)|^2 \approx N$ using the strong law of large number. It is straightforward to verify that the CRB's derived in [4] are the special cases of our bounds.

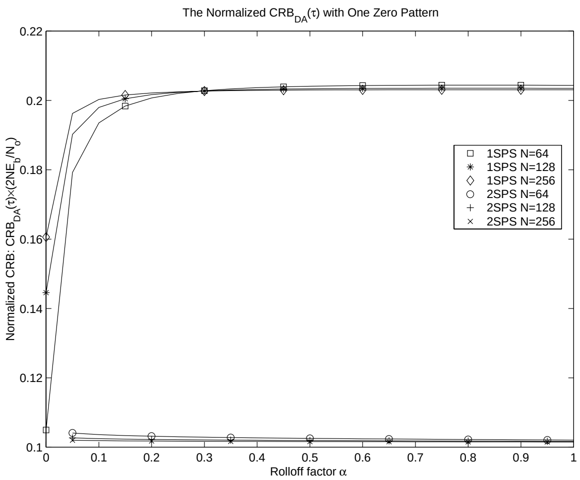

Fig. 2. The Normalized $\mathrm{CRB_{DA}}(\tau)$ for One-Zero Pattern

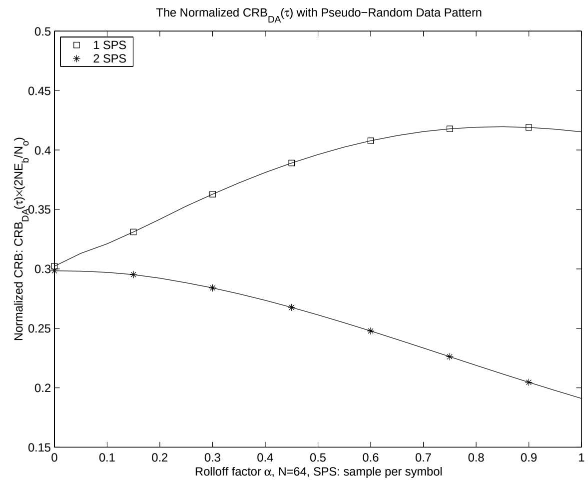

Fig. 3. The Normalized $\mathrm{CRB_{DA}}(\tau)$ for Pseudo-Random Data Pattern

IV. Several Examples

The $\mathrm{CRB_{DA}}$'s give us insight on the effect of training data pattern on the estimation performance limit. We are going to address several data patterns with QPSK modulation to illustrate the application of the bounds.

1) CW Pattern: The continuous wave (CW) pattern is the TS with data pattern $a_k = \sqrt{2}/2(1+j)$, (for $k = -N/2-1, \cdots, N/2$). As heuristically explained in [2] (p.336), the CW pattern is not suitable for timing recovery. The bound $\mathrm{CRB_{DA}}(\tau)$ provides analytical explanation. For large N, the spectrum $|\mathcal{A}(\omega)|$ of $\underline{a}$ is approximately a tone at DC. It is easy to verify that $J_{\tau\tau} \approx 0$, which means $\mathrm{CRB_{DA}}(\tau) \approx \infty$, i.e., there is little timing information in the CW sequence.

2) Alternating One-Zero Pattern: The one zero pattern is the TS with data pattern $a_k = \sqrt{2}/2(1+j)$, k is even, and $a_k = -\sqrt{2}/2(1+j)$, k is odd, which is widely used as preamble in TDMA frame structure for timing recovery. Actually the $\mathrm{CRB_{DA}}(\tau)$ of one-zero pattern is much smaller that of the pseudo-random data pattern. For large N the spectrum $|\mathcal{A}(\omega)|$ of $\underline{a}$ is a tone at $N/2$ in the frequency domain. When $L \geq 2$, we have $\mathrm{CRB_{DA}}(\tau) \approx \{2\pi^2 NE_s/N_0\}^{-1}$; when $L=1$, we have $\mathrm{CRB_{DA}}(\tau) \approx \{\pi^2 NE_s/N_0\}^{-1}$ that is around 3dB worse than the over-sampling bound and $\mathrm{CRB_{DA}}(\phi) \approx \{NE_s/N_0\}^{-1}$. Fig. 2 shows the normalized $\mathrm{CRB_{DA}}(\tau)$ ($\mathrm{CRB_{DA}}(\tau) \times 2NE_s/N_0$) with $L = 1, 2$. As N increases, the normalized bound converges to $2/\pi^2$ and $1/\pi^2$ respectively. The estimation variance in the under-sampling case approaches that in the over-sampling case as $\alpha \approx 0$ because there is little aliasing at that point.

3) Pseudo Random Data Pattern: The pseudo random data pattern (e.g., M-sequence, unique word (UW)) is used to do joint timing and phase estimation in some systems. A 64-symbol UW is selected to evaluate the CRB. The normalized $\mathrm{CRB_{DA}}(\tau)$ is shown in Fig. 3 for $L = 1, 2$. When $L = 1$, increasing α tends to improve the performance, however at the same time causes more aliasing that hurts the performance, therefore there is a certain α which achieves the worst performance; when $L = 2$, $\mathrm{CRB_{DA}}(\tau)$ decreases as α increases because there is no aliasing. We also observe that the bound is larger than that of one-zero pattern in both over and under sampling cases.

V. Conclusions

In this paper, we derive the closed-form formulas of $\mathrm{CRB_{DA}}$'s for carrier phase and timing synchronization with respect to arbitrary training sequence in both under and over sampling cases. These bounds provide additional insights on the sequence design.

References

[1] H. L. Van Trees, *Detection, Estimation and Modulation Theory, Part I*, New York: Wiley, 1968.

[2] H. Meyr, M. Moeneclaey, S. Fechtel, *Digital Communication Receivers, Synchronization, Channel Estimation, and Signal Processing*, New York: Wiley, 1998.

[3] M. Moeneclaey, *"A fundamental lower bound on the performance of practical joint carrier and bit synchronizers"*, *IEEE Trans. Commun.*, vol. COM-32, pp. 1007-1012, Sept. 1984.

[4] C. Georahiades, M. Moeneclaey, *"Sequence estimation and synchronization from nonsynchronized samples"*, *IEEE Trans. Inform. Theory*, vol. IT-37, pp. 1649-1657, Nov. 1991.

[5] M. Moeneclaey *"On the true and the modified Cramer-Rao bounds for the estimation of a scalar parameter in the presence of nuisance parameters"*, *IEEE Trans. Commun.*, vol. COM-46, pp. 1536-1544, Nov. 1998.

[6] A. N. D'Andrea, U. Mengali, R. Reggiannini, *"The modified Cramer-Rao bound and its application to synchronization parameters"*, *IEEE Trans. Commun.*, vol. COM-42, pp. 1391-1399, Feb./Mar./Apr. 1994.

[7] F. W. Sun, Y. Jiang, J. S. Baras, *"On the convergence of the inverse of Toeplitz matrices and its applications"*, to appear, Jan. 2000.