ABSTRACT

Title of Dissertation:	ON THE SMOLUCHOWSKI-KRAMERS APPROXIMATION OF STOCHASTIC DAMPED WAVE EQUATIONS
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In this thesis, we study three problems related to the small mass limit, also known as the Smoluchowski-Kramers diffusion approximation for stochastic damped wave equations. In the first part, we study the validity of a large deviation principle for a class of stochastic nonlinear damped wave equations, including equations of Klein-Gordon type, in the joint small mass and small noise limit. Additionally, we provide a proof of the Smoluchowski-Kramers approximation in the case of variable friction, non-Lipschitz nonlinear term, and unbounded diffusion.

In the second part, we investigate the convergence, in the small mass limit, of the stationary solutions of a class of stochastic damped wave equations, where the friction coefficient depends on the state and the noisy perturbation is of multiplicative type. We demonstrate that the Smoluchowski-Kramers approximation, previously shown to hold for any fixed time interval, remains valid in the long-time regime. Specifically, we prove that the first marginals of any sequence of stationary solutions for the damped wave equation converge to the unique invariant

measure of the limiting stochastic quasilinear parabolic equation.

The final result of this thesis concerns the Smoluchowski-Kramers approximation for a system of stochastic damped wave equations, whose solution is constrained to live on the unitary sphere in the space of square-integrable functions on any fixed interval. The stochastic perturbation is a nonlinear multiplicative Gaussian noise, with the stochastic differential interpreted in Stratonovich sense. Due to its particular structure, this noise not only conserves almost surely the constraint, but also preserves a suitable energy functional. In the small-mass limit, we derive a deterministic system, that remains confined to the unit sphere, but includes additional terms. These terms depend on the reproducing kernel of the noise and account for the interaction between the constraint and the conservative noise.

ON THE SMOLUCHOWSKI-KRAMERS APPROXIMATION OF STOCHASTIC DAMPED WAVE EQUATIONS

by

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Chapter 1: Introduction

1.1 Stochastic damped wave equations with variable friction

We consider the following class of stochastic wave equations with state-dependent damping on a bounded smooth domain $\mathscr{O} \subset \mathbb{R}^d$

$$\begin{cases} \mu \partial_t^2 u_\mu = \Delta u_\mu - \gamma(u_\mu)\partial_t u_\mu + f(u_\mu) + \sigma(u_\mu)\partial_t w^Q, \\ \\ u_\mu(0) = u_0, \quad \partial_t u_\mu(0) = v_0, \qquad u_\mu(t)|_{\partial \mathscr{O}} = 0, \end{cases} \tag{1.1.1}$$

depending on a parameter $0 < \mu << 1$, where the noise $w^Q(t)$ is a cylindrical Q-Wiener process.

By Newton's second law of motion, the solution u_μ of equation (1.1.1) can be interpreted as the displacement field of some particles in a domain $\mathscr{O}$, subject to interaction forces represented by the Laplacian and to nonlinear reactions represented by f, in the presence of a random external forcing $\sigma(u_\mu(t))\partial_t w^Q(t)$ and a state-dependent friction $\gamma(u_\mu(t))\partial_t u_\mu(t)$. Here μ represents the constant density of the particles and we are interested in the regime when $\mu \to 0$, known as the Smoluchowski-Kramers approximation limit (see [45] and [59]).

A series of papers (see [12], [13]) have studied the limiting behavior of u_μ, for a large class of reaction term f, and for both additive and multiplicative noise. For the finite dimensional case, the existing literature is quite broad and we refer in particular to [29], [30], [41], [42] and [60] (see also [14], [21] and [46] for systems subject to a magnetic field, and [44] and [54] for some related multiscaling problems).

In recent years, there has been considerable research into the Smoluchowski-Kramers diffusion approximation for infinite-dimensional systems. The first results in this direction dealt with the case of constant damping term, with smooth noise and regular coefficients (see [12], [13] and [49]). More recently, the case of constant friction has been studied in [34] and [62] for equations perturbed by space-time white noise in dimension $d = 2$, and in [40] for equations with Hölder continuous coefficients in dimension $d = 1$. In all these papers, the fact that the damping coefficient γ is constant leads to a perturbative result, in the sense that, in the small-mass limit, the solution u_μ of the stochastic damped wave equation converges to the solution of the stochastic parabolic problem obtained by taking $\mu = 0$. Specifically, it is proved that in the small mass limit u_μ converges to the solution of the following parabolic problem

$$\begin{cases} \gamma \partial_t u = \Delta u + f(u) + \sigma(u) \partial_t w^Q, \\ \\ u(0) = u_0, \quad u|_{\partial \mathscr{O}} = 0. \end{cases} \tag{1.1.2}$$

More precisely, it is shown that for every $T > 0$ and $\eta > 0$

$$\lim_{\mu \to 0} \mathbb{P}\left(\sup_{t \in [0,T]} \left\| u_\mu(t) - u(t) \right\|_{L^2(\mathscr{O})} > \eta \right) = 0.$$

In fact, in [20] it is proved that when f is Lipschitz continuous, the following stronger convergence holds

$$\lim_{\mu \to 0} \mathbb{E} \sup_{t \in [0,T]} \left\| u_\mu(t) - u(t) \right\|^p_{L^2(\mathscr{O})} = 0,$$

for every $p \geq 1$. Several related problems have been addressed in a variety of both finite and infinite dimensional contexts (see [29] and [60], for the finite dimensional case, and [12], [13], [16], [49], [50], [51], [54] and [56], for the infinite dimensional case).

However, there are relevant situations where this is not the case. For instance, when the constant friction is replaced by a magnetic field, the small-mass limit does not yield the solution to the first-order equation. In fact, even in the case of a constant magnetic field and finite dimension, the small mass limit does not yield the solution of the first-order equation (see [14], [21] and [46] for the finite dimensional case and [20] for the infinite dimensional case). In this case, a possible strategy consists in regularizing the problem by adding a small friction or by smoothing the noise in time and, in the double limit, it is possible to give a meaning to the Smoluchowski-Kramers approximation. In [20], the limiting equation is shown to be an SPDE of hyperbolic type.

In the present thesis, we consider another situation when the small mass limit does not produce a perturbative result. Specifically, we consider a wave equation perturbed by a multiplicative noise, with a friction term whose intensity is state-dependent. This problem has been extensively studied in finite dimension in a series of papers (see [42], and references therein, and also [41]). In these studies, it is demonstrated how the interplay between the non-constant friction coefficient and the noise creates an additional drift in the limiting first order equation, when the mass μ goes to zero. More precisely, the following system is considered

$$\begin{cases} dx_\mu(t) = v_\mu(t)dt, \quad x_\mu(0) = x \in \mathbb{R}^d, \\ \\ \mu dv_\mu(t) = \big[b(x_\mu(t)) - \gamma(x_\mu(t))v_\mu(t)\big]dt + \sigma(x_\mu(t))dW(t), \quad v_\mu(0) = v \in \mathbb{R}^d, \end{cases} \tag{1.1.3}$$

where γ is a matrix valued function defined on $\mathbb{R}^d$, such that for some positive γ_0

$$\inf_{x\in\mathscr{O}} \xi^T \gamma(\xi)\xi \geq \gamma_0|\xi|^2, \quad \xi \in \mathbb{R}^d.$$

It is proved that, as μ goes to zero, x_μ converges in L^2, with respect to the uniform norm in $C([0,T];\mathbb{R}^d)$, to the solution of the first order equation

$$dx(t) = \Big(\frac{b(x(t))}{\gamma(x(t))} + S(x(t))\Big)dt + \frac{\sigma(x(t))}{\gamma(x(t))}dW(t), \quad x(0) = x,$$

where the noise induced drift $S(x)$ is given by

$$S_i(x) = \frac{\partial}{\partial_l}\big[(\gamma^{-1})_{ij}(x)\big]J_{jl}(x),$$

and the matrix valued function J is the solution of the Lyapunov equation

$$J(x)\gamma^*(x) + \gamma(x)J(x) = \sigma(x)\sigma^*(x).$$

The case of SPDEs with state-dependent damping was first considered in [22] for a single equation. In that work, it was shown that if the friction coefficient $\gamma \in C^1(\mathbb{R})$ is strictly positive and bounded, the nonlinear term f is Lipschitz continuous and the diffusion coefficient σ is bounded and Lipschitz continuous, then for any initial condition $(u_0, v_0) \in H_0^1(\mathscr{O}) \times L^2(\mathscr{O})$ and for any $p < \infty$

$$\lim_{\mu\to 0} \mathbb{P}\Big(\|u_\mu - u\|_{L^p(\mathscr{O})} > \eta \Big) = 0, \quad \eta > 0, \tag{1.1.4}$$

where u is the unique solution of the quasilinear stochastic parabolic equation

$$\begin{cases} \partial_t u = \dfrac{1}{\gamma(u)}\Delta u + \dfrac{f(u)}{\gamma(u)} - \dfrac{\gamma'(u)}{2\gamma^3(u)}\displaystyle\sum_{i=1}^{\infty}\big(\sigma(u)Qe_i\big)^2 + \dfrac{\sigma(u)}{\gamma(u)}\partial_t w^Q, \quad t>0, \quad x\in\mathscr{O}, \\ \\ u(0)=u_0, \quad u|_{\partial\mathscr{O}}=0. \end{cases} \tag{1.1.5}$$

It is important to note that if σ is constant, then the noise induced term coincides with the Stratonovich-to-Itô correction, so that equation (1.1.5) can be written as

$$\partial_t u = \frac{1}{\gamma(u)}\Delta u + \frac{f(u)}{\gamma(u)} + \frac{\sigma}{\gamma(u)} \circ \partial_t w^Q.$$

However, if σ is not constant, the Stratonovich-to-Itö correction contains additional terms involving the derivative of σ, which are not present in (1.1.5). This indicates that in the case of an arbitrary state-dependent diffusion coefficient σ the small mass limit does not lead to the perturbative parabolic quasilinear equation, obtained by taking $\mu=0$ and replacing the Itô with the Stratonovich's integral.

Most recently, the same problem for systems of equations has been studied in [11], where $u_\mu(t,x) = \big(u_{1,\mu}(t,x), u_{2,\mu}(t,x), \ldots, u_{r,\mu}(t,x)\big)$ for some $r \geq 2$. The friction coefficient is assumed to be in the form

$$[\gamma(h)k](x) = \mathrm{g}(h)k(x), \quad x\in\mathscr{O},$$

for some differentiable mapping $\mathrm{g}: H^1 \to \mathbb{R}^{r\times r}$, which is bounded together with its derivative, and such that for some $\gamma_0>0$

$$\inf_{h\in H^1}\left\langle \mathrm{g}(h)\xi, \xi\right\rangle_{\mathbb{R}^r} \geq \gamma_0 \|\xi\|^2_{R^r}, \quad \xi\in\mathbb{R}^r,$$

It is proven that under some suitable initial conditions, for every $\rho < 1$, $\vartheta < 2$, $p < 2/(\vartheta - 1)$ and $\eta, T > 0$ it holds

$$\lim_{\mu \to 0} \mathbb{P}\left(\sup_{t \in [0,T]} \left\| u_\mu(t) - u(t) \right\|_{H^\rho} + \int_0^T \left\| u_\mu(t) - u(t) \right\|_{H^\vartheta}^p dt > \eta \right) = 0,$$

where u is the solution of equation

$$\begin{cases} \partial_t u(t) = \mathbf{g}^{-1}(u(t))\Delta u(t) + \mathbf{g}^{-1}(u(t)) f(u(t)) + S(u(t)) + \mathbf{g}^{-1}(u(t))\sigma(u(t))\partial_t w^Q, \\ \\ u(0) = u_0, \quad u|_{\partial \mathscr{O}} = 0. \end{cases} \tag{1.1.6}$$

Here,

$$S(u) := \int_{H^1} \left[D\mathbf{g}^{-1}(u) z \right] z d\nu^u(z), \quad u \in H^1,$$

where ν^u is the unique invariant measure of the Markov transition semigroup associated with equation

$$dy(t) = -\mathbf{g}(u) y(t) dt + \sigma(u) dw^Q(t), \quad y(0) = v.$$

1.1.1 The small noise limit in the Smoluchowski-Kramers approximation

The study of the Smoluchowski-Kramers approximation extends beyond simply proving the limit of the solution u_μ. In many applications, it is important to understand the stability of this approximation with respect to other significant asymptotic features exhibited by the two systems. In this context, [12], [17], and more recently [54], show that the statistically invariant states of the stochastic damped wave equation (in the case of constant friction) converge, in a suitable sense, to the invariant measure of the limiting equation. Similarly, [18] and [19] study the convergence of the quasi-potential, which, as is well-known, describes the asymptotics of the exit time of the solutions from a functional domain and the large deviation principle for the invariant measure.

In Chapter 2 (see [23] for reference), we explore the interplay between the small-mass and small-noise limits for system (1.1.1) with state-dependent damping. Specifically, we focus on the validity of a large deviation principle for the following equation

$$\begin{cases} \mu\partial_t^2 u_\mu(t,x) = \Delta u_\mu(t,x) - \gamma(u_\mu(t,x))\partial_t u_\mu(t,x) + f(x,u_\mu(t,x)) + \sqrt{\mu}\,\sigma(u_\mu(t,\cdot))\partial_t w^Q(t,x), \\ u_\mu(0,x) = u_0(x), \quad \partial_t u_\mu(0,x) = v_0(x), \qquad u_\mu(t,x) = 0,\ \ x \in \partial\mathscr{O}, \end{cases} \tag{1.1.7}$$

where, together with the mass, we are also assuming that the intensity of the noise vanishes. We assume that the friction coefficient $\gamma \in C^1(\mathbb{R})$ and

$$0 < \gamma_0 \leq \gamma(s) \leq \gamma_1, \quad s \in \mathbb{R}.$$

The nonlinearity f is either a Lipschitz-continuous function (in this case we can consider any dimension $d \geq 1$) or a locally Lipschitz-continuous function of the Klein-Gordon type (in this case we can only take $d = 1$). The noise $w^Q(t)$ is a cylindrical Q-Wiener process and σ is a suitable Lipschitz-continuous operator-valued function. Our goal is to prove that, in the joint small mass and small noise limit the family of random variables $\{u_\mu\}_{\mu>0}$ satisfy a large deviation principle in the space $C([0,T];L^p(\mathscr{O}))$, (for some $p > 2$ depending on the dimension d), with respect to the action functional

$$I_T(u) = \inf_{\varphi} \left\{ \frac{1}{2} \int_0^T \|\varphi(t)\|^2_{L^2(\mathscr{O})} \, dt \ : \ u(t) = u^{\varphi}(t), \ t \in [0,T] \right\},$$

where $u^{\varphi}(t)$ denotes the solution of the controlled quasi-linear parabolic equation

$$\begin{cases} \gamma(u(t,x))\partial_t u(t,x) = \Delta u(t,x) + f(x,u(t,x)) + \sigma(u(t,\cdot))\varphi(t,x), \\ \\ u(0,x) = u_0(x), \quad u(t,x) = 0, \quad x \in \partial\mathscr{O}. \end{cases} \tag{1.1.8}$$

In particular, this implies that, despite the presence of a non-constant friction coefficient, the Smoluchowski-Kramers approximation of equation (1.1.1) leads to equation (1.1.5), while the large deviation principle is consistent with equation (1.1.2).

The small parameter we are taking in front of the noise in equation (1.1.7) is $\sqrt{\mu}$. However, we would like to emphasize that this choice is made only for simplicity sake. In fact, $\sqrt{\mu}$ could be replaced by any other positive function $\alpha(\mu)$ such that

$$\lim_{\mu \to 0} \alpha(\mu) = 0, \qquad \lim_{\mu \to 0} \frac{\alpha(\mu)}{\sqrt{\mu}} < \infty,$$

and in this case, the speed of the large deviation principle would be $\alpha^2(\mu)$, instead of μ.

Additionally, in Appendix A we extend the results of [22] and provided a proof of the validity of the Smoluchowski-Kramers approximation for quasi-monotone f having polynomial growth and unbounded diffusion σ (see Hypothesis 2.6 in Chapter 2). This requires proving non-trivial a-priori bounds for the solution u_μ and its time derivative $\partial_t u_\mu$, as well as introducing suitable functional spaces where tightness holds and the small-mass limit can be established.

1.1.2 The small-mass limit for stationary solutions

After establishing the validity of the small mass limits within a fixed time interval, the next step of interest is to compare the long-term dynamics of the second-order system (1.1.1) with those of the first-order system (1.1.5) (to this purpose, see e.g. [17], [18], [19], [21], [23], [44]).

In [12], a comparative analysis of the long-term behavior of equations (1.1.1) (with a constant γ) and (1.1.2) was conducted, assuming both systems to be of gradient type. Notably, in the case where the noise is white in both space and time ($Q = I$) and the dimension is $d = 1$, an explicit expression for the Boltzmann distribution of the process $z_\mu(t) := (u_\mu(t), \partial u_\mu/\partial t(t))$ in the phase space $\mathscr{H} := L^2(0,L) \times H^{-1}(0,L)$ was derived. Since there is no equivalent of the Lebesgue measure in the functional space $\mathscr{H}$, an auxiliary Gaussian measure was introduced, and the density of the Boltzmann distribution was then expressed with respect to such auxiliary Gaussian measure, which itself corresponds to the stationary measure of the linear wave equation associated with problem (1.1.1). In particular, it was shown that the first marginal of the invariant measure linked to the process $z_\mu(t)$ remains independent of $\mu > 0$ and coincides with the invariant measure for the heat equation (1.1.2).

In the case of non-gradient systems, that is when the noise is colored in space and/or of multiplicative type, there is no explicit expression for the invariant measure ν_μ associated with system (1.1.1) and there is no reason to expect that the first marginal of ν_μ does not depend on μ or coincides with the invariant measure ν of system (1.1.2). Nonetheless, in [17] it was proved that, as the mass parameter μ tends to zero, the first marginal of any invariant measure ν_μ associated with the second-order system (1.1.1) converges in a suitable manner to the invariant measure ν of the first-order system (1.1.2). Specifically, the following convergence was established

$$\lim_{\mu \to 0} \mathscr{W}_\alpha \left((\Pi_1 \nu_\mu)', \nu \right) = 0, \tag{1.1.9}$$

where $(\Pi_1 \nu_\mu)'$ denotes the extension of the first marginal of the invariant measure ν_μ to $H := L^2(\mathscr{O})$, and the metric $\mathscr{W}_\alpha$ corresponds to the Wasserstein metric on $\mathscr{P}(H)$ associated with a distance metric α in H, which was determined based on the characteristics of the non-linearity function f under consideration.

We would like to remark that this result about the limiting behavior of stationary solutions of system (1.1.1), in the Smoluchowski-Kramers approximation, is consistent to what proven in [18] and [19] for the limiting behavior of the quasi-potential $V_\mu(u,v)$, that describes the asymptotics of the exit times and the large deviation principle for the invariant measures in equation (1.1.1). Actually, in [18] the quasi-potential $V_\mu(u,v)$ associated with (1.1.1) has been explicitly computed under the assumption that system (1.1.1) is of gradient type and it has been shown that for every fixed $\mu > 0$

$$V_\mu(u) := \inf_{v \in H^{-1}(\mathscr{O})} V_\mu(u,v) = V(u),$$

where $V(u)$ is the quasi-potential associated with (1.1.2). Moreover, in [19] the non-gradient case has been proven that in spite of the fact that there is no explicit expression for $V_\mu(u,v)$, nevertheless

$$\lim_{\mu \to 0} V_\mu(u) = V(u).$$

In Chapter 3 (see [24] for reference), we aim to investigate whether the results established in [17] for the case of constant friction γ can be proven for the case of state-dependent γ, where the Smoluchowski-Kramers approximation gives the more complex stochastic quasi-linear parabolic problem (1.1.5), instead of the simpler parabolic semi-linear problem (1.1.2).

One of the key ingredients used in [17] is the fact that the transition semigroup P_t associated with equation (1.1.2) admits a unique invariant measure $\nu \in \mathscr{P}(H)$. Additionally, the following contraction property holds

$$\mathscr{W}_\alpha\left(P_t^\star \nu_1, P_t^\star \nu_2\right) \leq c\, e^{-\delta t}\, \mathscr{W}_\alpha(\nu_1, \nu_2), \quad t \geq 0, \qquad \nu_1, \nu_2 \in \mathscr{P}(H), \tag{1.1.10}$$

for some $\delta > 0$. For equation (1.1.2), such problems have been extensively studied, and a wide variety of results is available. However, for the quasi-linear problem (1.1.5) the situation is much more intricate, and several fundamental facts are not known, such as whether the semigroup associated is Feller in H or not. In particular, even the use of the Krylov-Bogoliubov theorem for the proof of the existence of an invariant measure in H is not possible. Consequently, we have to employ novel arguments and take a different approach, which in particular brings us to study equations (1.1.1) and (1.1.5) in spaces of lower regularity than $H^1 \times H$ and H, respectively.

In Chapter 3, we prove that for every $\mu > 0$ the Markov transition semigroup associated with the second-order system (1.1.1) admits at least one invariant measure $\nu_\mu^{\mathscr{H}}$ in $\mathscr{H} = H \times H^{-1}$ with support in $\mathscr{H}_1 := H^1 \times H$, and the transition semigroup associated with the first-order quasi-linear equation (1.1.5) has a unique invariant measure ν in H, with support in H^1. Then we show that as the mass parameter μ tends to zero, the first marginal of any invariant measure $\nu_\mu^{\mathscr{H}}$ of (1.1.1) converges weakly in H to the unique invariant measure μ of (1.1.5). The convergence is proved with respect to the Wasserstein distance associated with the H^{-1} norm. Specifically, we show that if we define

$$\alpha(\mathrm{u}_1,\mathrm{u}_2) := \|\mathrm{u}_1-\mathrm{u}_2\|_{H^{-1}}, \qquad \mathrm{u}_1,\mathrm{u}_2 \in H^{-1},$$

then we have

$$\lim_{\mu\to 0} \mathscr{W}_\alpha\left(\Pi_1 \nu_\mu^{\mathscr{H}}, \nu\right) = 0, \tag{1.1.11}$$

where the metric $\mathscr{W}_\alpha$ corresponds to the Wasserstein metric on $\mathscr{P}(H^{-1})$ associated with the distance metric α. This result generalizes the results from [17], where constant friction γ was considered, to the case of the state-dependent γ, and extends the Smoluchowski-Kramers approximation previously validated within a fixed time interval (see [22]) to the long-time regime.

1.2 Stochastic constrained wave equations

As mentioned in Section 1.1 that, the Smoluchowski-Kramers diffusion approximation for SPDEs with state-dependent damping differs from the case with constant friction. Specifically, the non-constant friction introduces a noise-induced term in the small-mass limit. An analogous phenomenon has also been identified in [3], where the stochastic damped wave equations constrained to live on a manifold in the functional space of square-integrable functions $L^2(\mathscr{O})$ was first considered. In fact, in this case, the Smoluchowski-Kramers approximation leads to a stochastic parabolic problem, whose solution is constrained to live on the unitary $L^2(\mathscr{O})$-sphere, and an additional drift term arises.

The study of deterministic and stochastic constrained PDEs is not new. In this context, we highlight the works of Rybka [55] and Caffarelli and Lin [9] where, deterministic heat flows on

Hilbert manifolds were explored as part of a gradient flow approach to a specific minimization problem. Additionally, the constrained version of the deterministic 2-D Navier-Stokes equation was studied in [6] by Brzezniak, Dhariwal and Mariani, as well as in [10] by Caglioti, Pulvirenti, and Rousset. Later, stochastic version of this equation was investigated in [5] and [4] by Brzezniak and Dhariwal.

In [3], well-posedness for a class of abstract stochastic equations in a separable Hilbert space H was first studied. Specifically, they considered the equation

$$u_{tt}(t)+|u_t(t)|_H^2 u(t) = -A_0^2 u(t)+|A_0 u(t)|_H^2 - \gamma u_t(t)+\sigma(u(t),u_t(t))\circ dW(t), \tag{1.2.1}$$

where the solution is subject to the finite-codimension constraint of living on $M = S_H(0,1)$, the unitary sphere of H, with the initial data (u_0, v_0) in TM, the tangent bundle of M. Here A_0 is a non-negative self-adjoint operator on H with domain $D(A_0)$. The well-posedness result for this abstract stochastic equation was then applied to the specific case of the stochastic damped wave equation (1.1.1) constrained to live on the unitary sphere M in H, where the associated Smoluchowski-Kramers diffusion approximation was studied. Namely, they considered the equation

$$\begin{cases} \mu\partial_t^2 u_\mu(t,x)+\mu|\partial_t u_\mu(t)|_{L^2(0,L)}^2 u_\mu(t,x), \\[2em] \qquad = \Delta u_\mu(t,x)+|\nabla u_\mu(t)|_{L^2(0,L)}^2 u_\mu(t,x)-\gamma\partial_t u_\mu(t,x)+\sigma(u_\mu(t))\partial_t w^Q(t,x) \\[2em] u_\mu(0,x)=u_0(x), \quad \partial_t u_\mu(0,x)=v_0(x), \quad u_\mu(t,x)=0, \ \ x\in\partial\mathscr{O}, \end{cases} \tag{1.2.2}$$

where γ is a positive constant, w^Q is a Wiener process on H, with reproducing kernel Hilbert space K and convariance operator Q, and the mapping $\sigma : H_0^1(\mathscr{O}) \to \mathscr{L}(H)$ is such that $\sigma(u)$ projects H onto the tangent space $T_u M$. It was proven in [3] that for any sufficiently smooth initial condition (u_0, v_0) in the tangent bundle of M, the solution u_μ converges in probability, in a suitable functional space, to the unique solution of the limiting equation

$$\begin{cases} \gamma\partial_t u(t,x) = \Delta u(t,x) + |\nabla u(t,x)|_H^2 u(t,x) - \dfrac{1}{2}\|\sigma(u(t))\|^2_{\mathscr{L}_2(K,H)} u(t) + \sigma(u(t))\partial_t w^Q(t,x), \\ \\ u(0,x) = u_0(x), \quad u(t,x) = 0, \;\; x \in \partial\mathscr{O}, \end{cases} \tag{1.2.3}$$

It is important to note that the noise-induced term

$$-\frac{1}{2}\|\sigma(u(t))\|^2_{\mathscr{L}_2(K,H)} u(t),$$

does not coincides with the Stratonovich-to-Itô correction term.

1.2.1 Stochastic constrained wave equations with nonlinear conservative noise

In Chapter 4 (see [25] for reference), we continue the work started in [3] and introduce the following system of stochastic wave equations on the interval $(0,L)$

$$\begin{cases} \mu\partial_t^2 u_\mu(t,x) + \mu|\partial_t u_\mu(t)|^2_{L^2(0,L)} u_\mu(t,x), \\ \\ \qquad = \partial_x^2 u_\mu(t,x) + |\partial_x u_\mu(t)|^2_{L^2(0,L)} u_\mu(t,x) - \gamma\partial_t u_\mu(t,x) + \sqrt{\mu}\left(u_\mu(t)\times\partial_t u_\mu(t)\right)\circ\partial_t w(t,x) \\ \\ u_\mu(0,x) = u_0(x), \quad \partial_t u_\mu(0,x) = v_0(x), \quad u_\mu(t,0) = u_\mu(t,L) = 0, \end{cases} \tag{1.2.4}$$

depending on a parameter $0 < \mu << 1$. Here $u_\mu(t,x) \in \mathbb{R}^3$, for every $(t,x) \in [0,+\infty)\times(0,L)$, the friction coefficient γ is strictly positive, and $w(t)$ is a cylindrical Wiener process, white in time and colored in space, with $\circ$ denoting the Stratonovich stochastic differential. The solution $u_\mu(t)$ is subject to the finite-codimension constraint of living on $M = S_{L^2(0,L)}(0,1)$, the unitary sphere of $L^2(0,L) = L^2(0,L;\mathbb{R}^3)$, with the initial data (u_0,v_0) in $\mathscr{M}$, the tangent bundle of M.

The key and fundamental distinction between the present research and [3] lies in the nature of the random perturbation considered. Actually, unlike any previous work related to the Smoluchowski-Kramers diffusion-approximation, whether in finite or infinite dimensions, the diffusion coefficient σ_μ we considered is nonlinear and includes both the position $u_\mu(t)$ and the velocity $\partial_t u_\mu(t)$, through the vector product $\sqrt{\mu}\,u_\mu(t)\times\partial_t u_\mu(t)$.

The reason why in all previous works the diffusion does not depend on the velocity is that, while one expects a limit for u_μ, there is no limit for $\partial_t u_\mu$, and it is not clear how to make sense of the limit in the equation, especially when it comes to the martingale term. However, as shown in previous studies, the term $\sqrt{\mu}\,\partial_t u_\mu$ can have non trivial limiting behavior, and with the present work we are trying to understand what happens when $\sigma_\mu(u,v) = \sqrt{\mu}\, u \times v$.

In [3], the diffusion coefficient did not include the velocity $\partial_t u_\mu(t)$, so the Itô and Stratonovich interpretations of the stochastic differential yielded the same equation (1.2.2). In contrast, in the current setting, the Itô and Stratonovich interpretations lead to different equations. We choose to interpret the stochastic differential in the Stratonovich sense, which has significant implications. Due to the special structure of the diffusion coefficient, both the Stratonovich and Itô integrals ensure that the solution $(u_\mu(t), \partial_t u_\mu(t))$ remains within the tangent bundle $\mathscr{M}$, for every $t \geq 0$. However, the noise in the Stratonovich sense exhibits more substantial conservative behavior, preserving the energy

$$\mathscr{E}_\mu(t) := |u_\mu(t)|^2_{H^1_0(0,L)} + \mu\, |\partial_t u_\mu(t)|^2_{L^2(0,L)} + \int_0^t |\partial_t u_\mu(s)|^2_{L^2(0,L)}\, ds, \tag{1.2.5}$$

almost surely with respect to $\mathbb{P}$. This phenomenon, well-understood in other contexts, particularly in the parabolic setting, plays a critical role in the scenario considered here, as it provides a key tool for proving the necessary bounds for $u_\mu(t)$ and $\sqrt{\mu}\,\partial_t u_\mu(t)$ in the appropriate functional spaces, uniformly with respect to $\mu \in (0,1)$. And those bounds are fundamental in the proof of the tightness and in the identification of the limit.

After demonstrating that for every fixed $\mu \in (0,1)$ and $p \geq 1$, and any initial condition $(u_0, v_0) \in [H^1_0(0,L) \times L^2(0,L)] \cap \mathscr{M}$ and $p \geq 1$ there exists a unique mild solution of equation (1.2.4)

$$z_\mu = (u_\mu, \partial_t u_\mu) \in L^p(\Omega; C([0,+\infty); [H^1_0(0,L) \times L^2(0,L)] \cap \mathscr{M})),$$

we study the limiting behavior of u_μ, as $\mu \downarrow 0$. Our main result shows that if $(u_0, v_0) \in [H^2(0,L) \times H^1(0,L)] \cap \mathscr{M}$, then, for every $T > 0$ and $\delta < 2$ and for every $\eta > 0$ we have

$$\lim_{\mu \to 0} \mathbb{P}\Big(|u_\mu - u|_{C([0,T];H^\delta)} > \eta \Big) = 0, \tag{1.2.6}$$

where u is the unique solution of the deterministic problem

$$\begin{cases} \gamma \partial_t u(t) + \dfrac{1}{2} \varphi\, \partial_t (|u(t)|^2 u(t)) = \partial_x^2 u(t) + |\partial_x u(t)|_H^2 u(t) \\ \\ \qquad\qquad\qquad + \dfrac{3\varphi}{2\gamma} \left(\left[\partial_x^2 u(t) + |\partial_x u(t)|_H^2 u(t) \right] \cdot u(t) \right) u(t), \\ \\ u(0,x) = u_0(x), \quad u(t,0) = u(t,L) = 0, \end{cases} \tag{1.2.7}$$

where

$$\varphi(x) = \sum_{i=1}^{\infty} \xi_i^2(x), \qquad x \in [0,L],$$

and $\{\xi_i\}_{i \in \mathbb{N}}$ is an orthonormal basis for the reproducing kernel K of the noise $w(t)$.

This result indicates that u_μ converges to a deterministic limit u, which solves a deterministic problem, while preserving the constraint to stay on the unitary sphere of $L^2(0,L)$. Notably, as in the previously mentioned cases - where however only stochastic limits are obtained - in the small-mass limit several noise-induced terms appear in the limiting equation, which depend on the noise present in the second-order problem through the function φ.

It is important to remark that this non-trivial behavior emerges only in the $\sqrt{\mu}$ scaling for the diffusion coefficient, as in the case of μ^α, with $\alpha > 1/2$, the limiting equation (1.2.7) has to be replaced by the constrained parabolic problem

$$\gamma\partial_t u(t) = \partial_x^2 u(t) + |\partial_x u(t)|_H^2 u(t),$$

with the same initial and boundary conditions, and without noise-induced terms. As for the limiting behavior of u_μ when the diffusion is scaled as μ^α, with $\alpha \in [0,1/2)$, the expected limiting equation is not clear. We conjecture that if any limiting point exists, it should satisfy the deterministic equation

$$|u(t)|^2 u(t) = |u_0|^2 u_0 + \frac{3}{\gamma}\int_0^t \big(\partial_x^2 u(s)\cdot u(s)\big)u(s)ds + \frac{3}{\gamma}\int_0^t |\partial_x u(s)|_H^2 |u(s)|^2 u(s)ds. \qquad (1.2.8)$$

However, in order to prove the convergence of any limiting point for the family $\{u_\mu\}_{\mu\in(0,1)}$ to a solution of (1.2.8), tightness in at least $L^2(0,T;H_0^1(0,L))$ would be necessary. Due to the structure of the diffusion coefficient $\sigma(u,v) = u\times v$, the uniform bounds required to establish this tightness appear to be beyond reach.

Chapter 2: On the small noise limit in the Smoluchowski-Kramers approximation of nonlinear wave equations with variable friction

In this chapter, we aim to study the validity of a large deviation principle for the following equation

$$\begin{cases} \mu\partial_t^2 u_\mu(t,x) = \Delta u_\mu(t,x) - \gamma(u_\mu(t,x))\partial_t u_\mu(t,x) + f(x,u_\mu(t,x)) + \sqrt{\mu}\,\sigma(u_\mu(t,\cdot))\partial_t w^Q(t,x), \\ u_\mu(0,x) = u_0(x), \quad \partial_t u_\mu(0,x) = v_0(x), \qquad u_\mu(t,x) = 0, \;\; x \in \partial\mathscr{O}. \end{cases} \tag{2.0.1}$$

The outline of the chapter is as follows. In Section 2.1, we introduce all the assumptions and notations that will be used throughout the chapter. In Section 2.2, we present the main result in this chapter (see Theorem 2.8) and outline the method employed to prove it. In Section 2.3, we study the well-posedness of the controlled version of Eq. (2.0.1) in $\mathscr{H}_1 := H^1 \times H$, assuming that the nonlinearity f is either Lipschitz continuous, or locally Lipschitz-continuous of the Klein-Gordon type. In Section 2.4 we establish uniform bounds for the solution u_μ and $\partial_t u_\mu$ with respect to $\mu \in (0,1)$, and demonstrate the tightness of the corresponding family of probability measures. In Section 2.5, we prove the uniqueness of the limit controlled problem (2.2.2). Finally, in Section 2.6 we prove the main result of this chapter, Theorem 2.8.

2.1 Notations and assumptions

Throughout the present chapter $\mathscr{O}$ is a bounded domain in $\mathbb{R}^d$, with smooth boundary. We denote by H the Hilbert space $L^2(\mathscr{O})$ and by $\langle \cdot, \cdot \rangle_H$ the corresponding inner product. H^1 is the completion of $C_0^\infty(\mathscr{O})$ with respect to norm

$$\|u\|_{H^1}^2 := \|\nabla u\|_H^2 = \int_{\mathscr{O}} |\nabla u(x)|^2 dx,$$

and H^{-1} is the dual space to H^1.

Given the domain $\mathscr{O}$, we denote by $\{e_i\}_{i\in\mathbb{N}} \subset H^1$ the complete orthonormal basis of H which diagonalizes the Laplacian Δ, endowed with Dirichlet boundary conditions on $\partial\mathscr{O}$. Moreover, we denote by $\{-\alpha_i\}_{i\in\mathbb{N}}$ the corresponding sequence of eigenvalues, i.e.

$$\Delta e_i = -\alpha_i e_i, \quad i \in \mathbb{N}.$$

Next, for every $\delta \in \mathbb{R}$, we denote by H^δ the completion of $C_0^\infty(\mathscr{O})$ with respect to the norm

$$\|u\|_{H^\delta}^2 := \sum_{i=1}^\infty \alpha_i^\delta \langle h, e_i \rangle_H^2.$$

Notice that with this definition $H^0 = H$ and, if $\delta_1 < \delta_2$, then $H^{\delta_2} \hookrightarrow H^{\delta_1}$ with compact embedding. We also define

$$\mathscr{H}_\delta := H^\delta \times H^{\delta-1}, \qquad \mathscr{H} := H \times H^{-1}.$$

Next, for every two separable Hilbert spaces E and F, we denote by $\mathscr{L}(E,F)$ the space of all bounded linear operators from E to F, and denote by $\mathscr{L}_2(E,F)$ the space of Hilbert-Schmidt operators from E into F. $\mathscr{L}_2(E,F)$ is a Hilbert space, endowed with the inner product

$$\langle B,C\rangle_{\mathscr{L}_2(E,F)} = \mathrm{Tr}_E\,[B^\star C] = \mathrm{Tr}_F\,[CB^\star],$$

and, as well known, $\mathscr{L}_2(E,F) \subset \mathscr{L}(E,F)$, with

$$\|B\|_{\mathscr{L}(E,F)} \leq \|B\|_{\mathscr{L}_2(E,F)}.$$

2.1.1 The stochastic term

We assume that $w^Q(t)$ is a cylindrical Q-Wiener process, defined on a complete stochastic basis $(\Omega,\mathscr{F},(\mathscr{F}_t)_{t\geq 0},\mathbb{P})$. This means that $w^Q(t)$ can be formally written as

$$w^Q(t) = \sum_{i=1}^{\infty} Qe_i\beta_i(t),$$

where $\{\beta_i\}_{i\in\mathbb{N}}$ is a sequence of independent standard Brownian motions on $(\Omega,\mathscr{F},(\mathscr{F}_t)_{t\geq 0},\mathbb{P})$, $\{e_i\}_{i\in\mathbb{N}}$ is the complete orthonormal system introduced above that diagonalizes the Laplace operator, endowed with Dirichlet boundary conditions, and $Q : H \to H$ is a bounded linear operator,. When $Q = I$, $w^I(t)$ will be denoted by $w(t)$. In particular, we have $w^Q(t) = Qw(t)$.

In what follows we shall denote by H_Q the set $Q(H)$. H_Q is the reproducing kernel of the noise w^Q and is a Hilbert space, endowed with the inner product

$$\langle Qh, Qk\rangle_{H_Q} = \langle h,k\rangle_H, \quad h,k \in H.$$

Note that the sequence $\{Qe_i\}_{i\in\mathbb{N}}$ is a complete orthonormal system in H_Q. Moreover, if U is any Hilbert space containing H_Q such that the embedding of H_Q into U is Hilbert-Schmidt, we have that

$$w^Q \in C([0,T];U). \tag{2.1.1}$$

Hypothesis 2.1. *The mapping $\sigma : H \to \mathscr{L}_2(H_Q,H)$ is defined by*

$$[\sigma(h)Qe_i](x) = \sigma_i(x,h(x)), \quad x \in \mathscr{O} \quad h \in H, \quad i \in \mathbb{N},$$

for some mapping $\sigma_i : \mathscr{O} \times \mathbb{R} \to \mathbb{R}$. We assume that there exists $L > 0$ such that

$$\sup_{x\in\mathscr{O}} \sum_{i=1}^{\infty} |\sigma_i(x,y_1) - \sigma_i(x,y_2)|^2 \leq L|y_1 - y_2|^2, \quad y_1, y_2 \in \mathbb{R}. \tag{2.1.2}$$

Moreover,

$$\sup_{x\in\mathscr{O}} \sum_{i=1}^{\infty} |\sigma_i(x,0)|^2 =: \sigma_0^2 < \infty. \tag{2.1.3}$$

Remark 2.2. 1. Condition (2.1.2) implies that σ is Lipschitz continuous. Namely for any $h_1, h_2 \in H$

$$\|\sigma(h_1) - \sigma(h_2)\|_{\mathscr{L}_2(H_Q,H)} \leq \sqrt{L}\|h_1 - h_2\|_H. \tag{2.1.4}$$

This, together with condition (2.1.3), implies also that σ has linear growth, that is

$$\|\sigma(h)\|_{\mathscr{L}_2(H_Q,H)} \leq \sqrt{L}\|h\|_H + |\mathscr{O}|^{1/2}\sigma_0. \tag{2.1.5}$$

2. If σ is constant, then Hypothesis 2.1 means that σQ is a Hilbert-Schmidt operator in H.

3. If σ is not constant, Hypothesis 2.1 is satisfied for example when

$$[\sigma(h)Qk](x) = s(x,h(x))Qk(x), \quad x \in \mathscr{O}, \quad h,k \in H,$$

for some measurable function $s : \mathscr{O} \times \mathbb{R} \to \mathbb{R}$ such that $s(x,\cdot) : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous, uniformly with respect to $x \in \mathscr{O}$, and for some $Q \in \mathscr{L}(H)$ such that

$$\sum_{i=1}^{\infty} \|Qe_i\|^2_{L^\infty(\mathscr{O})} < \infty. \tag{2.1.6}$$

In case Q is diagonalizable with respect the basis $(e_i)_{i\in\mathbb{N}}$, with $Qe_i = \lambda_i e_i$, condition (2.1.6) reads

$$\sum_{i=1}^{\infty} \lambda_i^2 \|e_i\|^2_{L^\infty(\mathscr{O})} < \infty. \tag{2.1.7}$$

In general (see [36]), we have

$$\|e_i\|_{L^\infty(\mathscr{O})} \leq c\, i^{\alpha},$$

for some $\alpha > 0$, and (2.1.7) becomes

$$\sum_{i=1}^{\infty} \lambda_i^2\, i^{2\alpha} < \infty.$$

In particular, when $d = 1$ or the domain is a hyperrectangle when $d > 1$ the eigenfunctions $(e_i)_{i\in\mathbb{N}}$ are equi-bounded and (2.1.7) becomes $Q \in \mathscr{L}_2(H)$.

2.1.2 The coefficients

Throughout this chapter, we shall assume that the friction coefficient satisfies the following condition

Hypothesis 2.3. *The mapping γ belongs to $C_b^1(\mathbb{R})$ and there exist γ_0 and γ_1 such that*

$$0 < \gamma_0 \leq \gamma(r) \leq \gamma_1, \qquad r \in \mathbb{R}. \tag{2.1.8}$$

In what follows, we shall define

$$g(r) = \int_0^r \gamma(\sigma)\, d\sigma, \qquad r \in \mathbb{R}.$$

Remark 2.4.

1. Clearly $g(0) = 0$ and $g'(r) = \gamma(r)$. In particular, due to (2.1.8), g is uniformly Lipschitz continuous on $\mathbb{R}$.

2. The function g is strictly increasing and

$$|(g(r_1) - g(r_2)|(r_1 - r_2) \geq \gamma_0 |r_1 - r_2|^2, \qquad r_1, r_2 \in \mathbb{R}.$$

As far as the nonlinearity f is concerned, in this chapter we shall consider two situations: f is Lipschitz continuous and $\mathscr{O}$ is a bounded smooth domain in $\mathbb{R}^d$, for any arbitrary $d \geq 1$ (see Hypothesis 2.5), or f is only locally Lipschitz continuous with polynomial growth and $\mathscr{O}$ is a bounded interval in $\mathbb{R}$ (see Hypothesis 2.6).

Hypothesis 2.5. *The mapping $f : \mathscr{O} \times \mathbb{R} \to \mathbb{R}$ is measurable and there exists $c > 0$ such that*

$$\sup_{x \in \mathscr{O}} |f(x,r) - f(x,s)| \leq c\,|r-s|, \quad r,s \in \mathbb{R}.$$

Moreover

$$\sup_{x \in \mathscr{O}} |f(x,0)| < \infty.$$

In what follows, for every function $u : \mathscr{O} \to \mathbb{R}$, we shall denote

$$F(u)(x) = f(x,u(x)), \quad x \in \mathscr{O}.$$

Hypothesis 2.6. *We have $\mathscr{O} = [0,L]$ and the mapping $f : [0,L] \times \mathbb{R} \to \mathbb{R}$ is measurable and satisfies the following conditions.*

1. *There exist $\theta > 1$ and $c_1 > 0$ such that for every $r \in \mathbb{R}$*

$$\sup_{x \in [0,L]} |f(x,r)| \leq c_1 \left(1 + |r|^{\theta}\right), \quad \sup_{x \in [0,L]} |\partial_r f(x,r)| \leq c_1 \left(1 + |r|^{\theta-1}\right). \tag{2.1.9}$$

Moreover, there exists $c_2 > 0$ such that for every $r \in \mathbb{R}$ and $x \in [0,L]$

$$\mathfrak{f}(x,r) := \int_0^r f(x,s)\,ds \leq c_2 \left(1 - |r|^{\theta+1}\right). \tag{2.1.10}$$

2. *For every $x \in [0,L]$, the function $f(x,\cdot) : \mathbb{R} \to \mathbb{R}$ is differentiable and*

$$\sup_{(x,r) \in [0,L] \times \mathbb{R}} \partial_r f(x,r) \leq 0. \tag{2.1.11}$$

3. *For every $r \in \mathbb{R}$, the function $f(\cdot, r) : [0,L] \to \mathbb{R}$ is differentiable and*

$$\sup_{x \in [0,L]} |\partial_x f(x,r)| \leq c\,(1+|r|)\,, \quad r \in \mathbb{R}.$$

Remark 2.7. 1. A typical example of a function f satisfying Hypothesis 2.6 is

$$f(r) = -a\,|r|^{\theta-1} r.$$

2. When $d = 1$, we have that $H^1 \hookrightarrow L^\infty(\mathscr{O})$, and then we get the fundamental fact that $F(u) \in H$, for every $u \in H^1$.

3. In the existing literature the well-posedness of stochastic semi-linear wave equations having polynomial nonlinearities is not restricted to space dimendion $d = 1$. However, here the presence of a state-dependent friction coefficient, and the fact that we are not just interested in the well-posedness of equation (2.0.1), but also in the small mass and small noise limits, makes our analysis more complicated and we can only handle the case of space dimension $d = 1$.

4. We are assuming (2.1.11) just for the sake of simplicity. In fact, our results remain true under the condition

$$\sup_{(x,r) \in [0,L] \times \mathbb{R}} \partial_r f(x,r) < \infty.$$

5. From (2.1.10) and (2.1.11), it is not hard to show that for every $r \in \mathbb{R}$

$$\sup_{x \in [0,L]} r f(x,r) \leq c_2 \Big(1 - |r|^{\theta+1}\Big). \tag{2.1.12}$$

Indeed, if we consider the function

$$G(x,r) := \mathfrak{f}(x,r) - rf(x,r), \quad r \in \mathbb{R},\ x \in [0,L],$$

then for every $x \in [0,L]$, $\partial_r G(x,r) = -r\partial_r f(x,r) \geq 0$ if $r > 0$, and $\partial_r G(x,r) \leq 0$ if $r < 0$. Note that $G(x,0) = 0$, we have $G(x,r) \geq 0$, and thus (2.1.12) follows from (2.1.10).

6. Thanks to (2.1.11) we have

$$\langle F(u) - F(v), u - v\rangle_H \leq 0, \qquad u,v \in H^1.$$

In particular, there exists some $c > 0$ such that

$$\langle F(u), u\rangle_H \leq c\,\|u\|_H, \qquad u \in H^1. \tag{2.1.13}$$

7. Due to (2.1.9), for every $u,v \in H^1$, we have

$$\|F(u) - F(v)\|_H^2 \leq c\int_{\mathscr{O}} \left(1 + |u(x)|^{2(\theta-1)} + |v(x)|^{2(\theta-1)}\right) |u(x) - v(x)|^2\, dx,$$

so that

$$\|F(u) - F(v)\|_H \quad \leq c\left(1 + \|u\|_{H^1}^{\theta-1} + \|v\|_{H^1}^{\theta-1}\right) \|u - v\|_H. \tag{2.1.14}$$

In particular, we have

$$\|F(u)\|_H \leq c\left(1 + \|u\|_{H^1}^{\theta}\right), \quad u \in H^1.$$

8. In the same way

$$\begin{aligned} \|F(u)-F(v)\|_{L^1} \leq\ & c\int_{\mathscr{O}}\left(1+|u(x)|^{\theta-1}+|v(x)|^{\theta-1}\right)|u(x)-v(x)|\,dx \\ & \leq c\left(1+\|u\|^{\theta-1}_{L^{2(\theta-1)}}+\|v\|^{\theta-1}_{L^{2(\theta-1)}}\right)\|u-v\|_H. \end{aligned} \tag{2.1.15}$$

Now, by proceeding as in [52] (see also [13]), for every $n \in \mathbb{N}$ and $x \in \mathscr{O}$ we define

$$f_n(x,r) := \begin{cases} f(x,n)+(r-n)\partial_r f(x,n), & \text{if } r \geq n, \\ f(x,r), & \text{if } r \in [-n,n], \\ f(x,-n)+(r+n)\partial_r f(x,-n), & \text{if } r \leq -n. \end{cases} \tag{2.1.16}$$

Clearly, for every $n \in \mathbb{N}$, the mapping $f_n(x,\cdot) : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous, uniformly with respect to $x \in [0,L]$, and

$$f_n(x,r) = f(x,r), \quad x \in [0,L], \ \ r \in [-n,n].$$

Moreover,

$$\sup_{x\in[0,L]} |f_n(x,r)| \leq c\left(1+|r|^{\theta}\right), \qquad \sup_{x\in[0,L]} |\partial_r f_n(x,r)| \leq c\left(1+|r|^{\theta-1}\right), \tag{2.1.17}$$

for some constant c independent of n, and there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$

$$\mathfrak{f}_n(x,r) := \int_0^r f_n(x,s)\,ds \leq c\left(1 - n^{\theta+1}\right), \quad r \in \mathbb{R}, \quad x \in [0,L]. \tag{2.1.18}$$

2.2 Large deviation principle

Our main result in this chapter is stated as follows.

Theorem 2.8. *Assume Hypotheses 2.1 and 2.3 and either Hypothesis 2.5 or Hypothesis 2.6 and fix $p < \infty$, if $d = 1,2$, and $p < 2d/(d-2)$, if $d > 2$. Then, for every $(u_0, v_0) \in \mathscr{H}_1$ and $T > 0$, the family $\{\mathscr{L}(u_\mu)\}_{\mu>0}$ satisfly a large deviation principle in $C([0,T];L^p(\mathscr{O}))$, as $\mu \downarrow 0$, with action functional*

$$I_T(u) = \inf_{\varphi}\left\{\frac{1}{2}\int_0^T \|\varphi(t)\|_H^2\,dt \;:\; u(t) = u^\varphi(t),\ t \in [0,T]\right\}, \tag{2.2.1}$$

where $u^\varphi(t)$ denotes the unique weak solution to the quasi-linear parabolic equation

$$\begin{cases} \partial_t u(t,x) = \gamma^{-1}(u(t,x))\left[\Delta u(t,x) + f(x,u(t,x)) + \sigma(u(t,\cdot))\varphi(t,x)\right], \\ \\ u(0,x) = u_0(x), \quad u(t,x) = 0, \quad x \in \partial\mathscr{O}. \end{cases} \tag{2.2.2}$$

Theorem 2.8 is proved by following the classical weak convergence approach to large deviations, as developed for SPDEs in [8]. To this purpose, we need first to introduce some notations. For every $T > 0$, we denote by $\mathscr{P}_T$ the set of predictable processes in $L^2(\Omega \times [0,T];H)$, and for every $M > 0$ we introduce the sets

$$\mathscr{S}_{T,M} := \left\{\varphi \in L^2_w(0,T;H) \;:\; \|\varphi\|_{L^2([0,T];H)} \leq M\right\},$$

and

$$\Lambda_{T,M} := \{\varphi \in \mathscr{P}_T \;:\; \varphi \in \mathscr{S}_{T,M},\ \mathbb{P}-\text{a.s.}\}\,.$$

Next, for every $\varphi \in \mathscr{P}_T$ we consider the controlled version of equation (2.0.1)

$$\left\{\begin{array}{l}
\mu\partial_t^2 u_\mu(t,x) = \Delta u_\mu(t,x) - \gamma(u_\mu(t,x))\partial_t u_\mu(t,x) + f(x,u_\mu(t,x)) + \sigma(u_\mu(t,\cdot))Q\varphi(t,x) \\[2em]
\qquad\qquad\qquad + \sqrt{\mu}\,\sigma(u_\mu(t,\cdot))\partial_t w^Q(t,x), \quad t>0, \quad x\in\mathscr{O}, \\[2em]
u_\mu(0,x) = u_0(x), \quad \partial_t u_\mu(0,x) = v_0(x), \qquad u_\mu(t,x) = 0, \quad x \in \partial\mathscr{O},
\end{array}\right. \tag{2.2.3}$$

The well-posedness of the equation above has been proven in [22] in the case the nonlinearity f is Lipschitz-continuous, the diffusion coefficient σ is bounded and the control $\varphi = 0$. In what follows, we will prove that also under the more general conditions we are assuming for f and σ, the following result holds.

Theorem 2.9. *Under Hypotheses 2.1 and 2.3 and either Hypothesis 2.5 or Hypothesis 2.6, for every $T,M>0$ and $\varphi \in \Lambda_{T,M}$ and for every initial conditions $(u_0,v_0)\in\mathscr{H}_1$, there exists a unique adapted process $(u_\mu,v_\mu)\in L^2(\Omega,C([0,T];\mathscr{H}_1))$ which solves the systems of equations*

$$\left\{\begin{array}{l}
u_\mu(t,x) = u_0(x) + \displaystyle\int_0^t v_\mu(s,x)\,ds, \\[2em]
\mu\, v_\mu(t,x) = \mu\, v_0(x) + \displaystyle\int_0^t \left[\Delta u_\mu(s,x) - \gamma(u_\mu(s,x))v_\mu(s,x) + f(x,u_\mu(s,x)) + \sigma(u_\mu(s))Q\varphi(s,x)\right] ds \\[2em]
\qquad\qquad + \sqrt{\mu}\displaystyle\int_0^t \sigma(u_\mu(s))\,dw^Q(s).
\end{array}\right. \tag{2.2.4}$$

Once proved Theorem 2.9, we introduce the following two conditions.

C1. Let $\{\varphi_\mu\}_{\mu>0}$ be an arbitrary family of processe in $\Lambda_{T,M}$ such that

$$\lim_{\mu\to 0}\varphi_\mu = \varphi, \quad \text{in distribution in} \quad L^2_w(0,T;H),$$

where $L^2_w(0,T;H)$ is the space $L^2([0,T];H)$ endowed with the weak topology and $\varphi \in \Lambda_{T,M}$. Then, for every $p<\infty$ we have

$$\lim_{\mu\to 0} u_\mu^{\varphi_\mu} = u^\varphi, \quad \text{weakly in} \quad C([0,T],L^p(\mathscr{O})),$$

where $u_\mu^{\varphi_\mu}$ is the solution to (2.2.3), corresponding to the control φ_μ, and u^φ is the solution to (2.2.2), corresponding to the control φ.

C2. For every $T,R>0$ and $p<\infty$, the level sets $\Phi_{T,R}=\{I_T\leq R\}$ are compact in the space $C([0,T];L^p(\mathscr{O}))$.

As shown in [8], Conditions C1. and C2. imply the validity of a Laplace principle with action functional I_T in the space $C([0,T];L^p(\mathscr{O}))$ for the family $\{u_\mu\}_{\mu>0}$. Due to the compactness of the level sets $\Phi_{T,R}$ stated in C1. this is equivalent to the validity of Theorem 2.8.

Thus, in what follows our strategy will be first proving Theorem 2.9, for every fixed $\mu>0$, and then proving conditions C1. and C2.

2.3 Well-posedness of the controlled second-order system

In Theorem 2.9 the parameter $\mu > 0$ is fixed. This means that in this section, without any loss of generality, we can assume $\mu = 1$. If we denote

$$\eta := v + g(u), \quad z = (u, \eta),$$

then system (2.2.4) can be rewritten as the following abstract stochastic evolution equation

$$dz(t) = \left[A(z(t)) + B_\varphi(t, z(t))\right] dt + \Sigma(z(t))dw^Q(t), \quad z(0) = (u_0, v_0 + g(u_0)), \tag{2.3.1}$$

where

$$A(u, \eta) = (-g(u) + \eta, \Delta u + F(u)), \qquad (u, \eta) \in D(A) = \mathscr{H}_1,$$

$$B_\varphi(t, (u, \eta)) = (0, \sigma(u)Q\varphi(t)), \qquad (u, \eta) \in \mathscr{H}, \quad t \in [0, T],$$

and

$$\Sigma(u, \eta) = (0, \sigma(u)), \qquad (u, \eta) \in \mathscr{H}.$$

This means that the adapted $\mathscr{H}_1$-valued process $z(t) = (u(t), \eta(t))$ is the unique solution to the equation

$$z(t) = (u_0, v_0 + g(u_0)) + \int_0^t [A(z(s)) + B_\varphi(s, z(s))]\, ds + \int_0^t \Sigma(z(s))\, dw^Q(s), \tag{2.3.2}$$

if and only if the adapted $\mathscr{H}_1$-valued process $(u(t),v(t)) = (u(t),-g(u(t))+\eta(t))$ is the unique solution of

$$\begin{cases} u(t,x) = u_0(x) + \displaystyle\int_0^t v(s,x)\,ds, \\[2ex] v(t,x) = \mu\, v_0(x) + \displaystyle\int_0^t \left[\Delta u(s,x) - \gamma(u(s,x))v(s,x) + f(x,u(s,x)) + \sigma(u(s,\cdot))Q\varphi(s,x)\right] ds \\[2ex] \qquad\qquad + \displaystyle\int_0^t \sigma(u(s,\cdot))\,dw^Q(s). \end{cases} \tag{2.3.3}$$

In our proof of Theorem 2.9 we first assume that f satisfies Hypothesis 2.5, and then we consider the case when Hypothesis 2.6 holds.

2.3.1 The case when f satisfies Hypothesis 2.5

In [22, Section 3] an analogous result has been proved, in the case $\varphi = 0$ (and hence $B_\varphi = 0$) and σ is bounded. Here, we extend the arguments used in [22] to consider the case of an arbitrary $\varphi \in \Lambda_{T,M}$, σ having linear growth. As in [22, Section 3], the arguments we are using here are based on classical tools from the theory of monotone non-linear operators and we refer to the monograph [1] for all details.

Since f is assumed to be Lipschitz continuous, we have

$$\|A(z)\|_{\mathscr{H}} \leq c\left(1+\|z\|_{\mathscr{H}_1}\right), \qquad z \in D(A), \tag{2.3.4}$$

and, as shown in [22, Lemma 3.1], there exists $\kappa \in \mathbb{R}$ such that

$$\langle A(z_1) - A(z_2), z_1 - z_2 \rangle_{\mathscr{H}} \leq \kappa \|z_1 - z_2\|^2_{\mathscr{H}}.$$

Moreover, for every $\lambda > 0$ small enough

$$\text{Range}\,(I - \lambda A) = \mathscr{H}.$$

This means that the operator $A : D(A) \subset \mathscr{H} \to \mathscr{H}$ is *quasi m-dissipative*. In particular, this implies that there exists $\lambda_0 > 0$ such that

$$J_\lambda := (I - \lambda A)^{-1}, \quad \lambda \in (0, \lambda_0),$$

is a well-defined Lipschitz-continuous mapping in $\mathscr{H}$ and we can introduce the *Yosida approximation* of A, defined as

$$A_\lambda(z) := A(J_\lambda(z)) = \frac{1}{\lambda}\left(J_\lambda(z) - z\right), \quad z \in \mathscr{H}.$$

Notice that

$$\langle A_\lambda(z_1) - A_\lambda(z_2), z_1 - z_2 \rangle_{\mathscr{H}} \leq \frac{\kappa}{1 - \lambda\kappa} \|z_1 - z_2\|^2_{\mathscr{H}}, \tag{2.3.5}$$

and

$$\|A_\lambda(z_1) - A_\lambda(z_2)\|_{\mathscr{H}} \leq \frac{2}{\lambda(1 - \lambda\kappa)} \|z_1 - z_2\|_{\mathscr{H}}. \tag{2.3.6}$$

Moreover, for every $z \in D(A)$

$$\|A^{\lambda}(z)\|_{\mathscr{H}} \leq \frac{1}{1-\lambda\kappa}\|A(z)\|_{\mathscr{H}},$$

so that for every $z \in D(A)$

$$\|J_{\lambda}(z)-z\|_{\mathscr{H}} = \lambda\|A^{\lambda}(z)\| \leq \frac{\lambda}{1-\lambda\kappa}\|A(z)\|_{\mathscr{H}},$$

and

$$\lim_{\lambda\to 0}\|A_{\lambda}(z)-A(z)\|_{\mathscr{H}} = 0.$$

In [22, Proof of Theorem 3.2], it has been shown that there exists some $\lambda_1 \in (0,\lambda_0)$ such that for every $\lambda \in (0,\lambda_1)$

$$\langle A_{\lambda}(z), z\rangle_{\mathscr{H}_1} \leq -\frac{\gamma_0}{2}\|J_{\lambda}(z)_1\|_{H^1}^2 + c\,\|J_{\lambda}(z)_2\|_{H^{-1}}^2, \qquad z \in \mathscr{H}_1. \tag{2.3.7}$$

Furthermore, for every $\lambda, \nu \in (0,\lambda_0)$ and $z_1, z_2 \in \mathscr{H}_1$ it holds

$$\langle A_{\lambda}(z_1)-A_{\nu}(z_2), z_1-z_2\rangle_{\mathscr{H}} \leq \|z_1-z_2\|_{\mathscr{H}}^2 + c\,(\lambda+\nu)\left(\|z_1\|_{\mathscr{H}_1}^2 + \|z_2\|_{\mathscr{H}_1}^2 + 1\right). \tag{2.3.8}$$

Concerning the random operator B_{φ}, according to Hypothesis 2.1 for every $t \in [0,T]$ we have that $B_{\varphi}(t,\cdot) : \mathscr{H} \to \mathscr{H}_1$ is well defined and, in view of (2.1.4), for any $z_1, z_2 \in \mathscr{H}$

$$\|B_{\varphi}(t,z_1)-B_{\varphi}(t,z_2)\|_{\mathscr{H}_1} = \|(\sigma(u_1)-\sigma(u_2))\,Q\varphi(t)\|_H \leq \sqrt{L}\,\|u_1-u_2\|_H\,\|\varphi(t)\|_H. \tag{2.3.9}$$

Finally, Hypothesis 2.1 implies that the mapping $\Sigma : \mathscr{H} \to \mathscr{L}_2(H_Q, \mathscr{H}_1)$ is well-defined and due to (2.1.4) for any $z_1, z_2 \in \mathscr{H}$

$$\|\Sigma(z_1) - \Sigma(z_2)\|_{\mathscr{L}_2(H_Q,\mathscr{H}_1)} = \|\sigma(u_1) - \sigma(u_2)\|_{\mathscr{L}_2(H_Q,H)} \leq \sqrt{L}\,\|u_1 - u_2\|_H. \tag{2.3.10}$$

Step 1. For every $\lambda \in (0, \lambda_0)$ the approximating problem

$$dz(t) = \left[A_\lambda(z(t)) + B_\varphi(t, z(t))\right] dt + \Sigma(z(t))dw^Q(t), \quad z(0) = (u_0, v_0 + g(u_0)) \tag{2.3.11}$$

admits a unique solution $z_\lambda \in L^2(\Omega; C([0,T]; \mathscr{H}))$.

Proof of Step 1. According to (2.3.6), we have

$$\int_0^T \|A_\lambda(z_1(s)) - A_\lambda(z_2(s))\|^2_{\mathscr{H}}\, ds \leq \frac{c}{\lambda^2} \int_0^T \|z_1(s) - z_2(s)\|^2_{\mathscr{H}}\, ds \leq \frac{c_T}{\lambda^2} \sup_{t\in[0,T]} \|z_1(t) - z_2(t)\|^2_{\mathscr{H}}. \tag{2.3.12}$$

According to (2.3.9), if $\varphi \in \Lambda_{T,M}$, we have

$$\begin{aligned}\int_0^T \|B_\varphi(t, z_1) - B_\varphi(t, z_2)\|^2_{\mathscr{H}_1}\, dt &\leq L \int_0^T \|u_1(t) - u_2(t)\|^2_H \|\varphi(t)\|^2_H\, dt \\ &\leq LM^2 \sup_{t\in[0,T]} \|z_1(t) - z_2(t)\|^2_{\mathscr{H}}, \quad \mathbb{P}-\text{a.s.}\end{aligned} \tag{2.3.13}$$

Finally, according to (2.3.10), we have

$$\mathbb{E}\sup_{t\in[0,T]}\left\|\int_0^t(\Sigma(z_1(s))-\Sigma(z_2(s))dw^Q(s)\right\|^2_{\mathscr{H}_1}\leq c\int_0^T\mathbb{E}\,\|\Sigma(z_1(s))-\Sigma(z_2(s))\|^2_{\mathscr{L}_2(H_Q,\mathscr{H}_1)}\,ds$$

$$\leq TL\mathbb{E}\sup_{t\in[0,T]}\|z_1(t)-z_2(t)\|^2_{\mathscr{H}}. \tag{2.3.14}$$

Therefore, in view of (2.3.12), (2.3.13) and (2.3.14), for every $\lambda\in(0,\lambda_0)$ the mapping

$$\Phi_\lambda(z)(t)=(u_0,v_0+g(u_0))+\int_0^t[A_\lambda(z(s))+B_\varphi(s,z(s))]\,ds+\int_0^t\Sigma(z(s))\,dw^Q(s)$$

is Lipschitz continuous from $L^2(\Omega;C([0,T];\mathscr{H}))$ into itself, and then for every $\lambda\in(0,\lambda_0)$ equation (2.3.11) admits a unique solution $z_\lambda\in L^2(\Omega;C([0,T];\mathscr{H}))$.

Step 2. There exists $c_T>0$ such that

$$\mathbb{E}\sup_{t\in[0,T]}\|z_\lambda(t)\|^2_{\mathscr{H}_1}\leq c_T\left(1+\|z_0\|^2_{\mathscr{H}_1}\right),\qquad\lambda\in(0,\lambda_1). \tag{2.3.15}$$

Proof of Step 2. As a consequence of Itô's formula, we have

$$\|z_\lambda(t)\|^2_{\mathscr{H}_1}=\|z_0\|^2_{\mathscr{H}_1}+2\int_0^t\langle A_\lambda(z_\lambda(s)),z_\lambda(s)\rangle_{\mathscr{H}_1}\,ds+2\int_0^t\langle B_\varphi(s,z_\lambda(s)),z_\lambda(s)\rangle_{\mathscr{H}_1}\,ds$$

$$+\int_0^t\|\Sigma(z_\lambda(s))\|^2_{\mathscr{L}_2(H_Q,\mathscr{H}_1)}ds+2\int_0^t\langle z_\lambda(s),\Sigma(z_\lambda(s))dw^Q(s)\rangle_{\mathscr{H}_1}.$$

Due to (2.3.7), we have

$$\begin{aligned}\int_0^t \langle A_\lambda(z_\lambda(s)), z_\lambda(s)\rangle_{H^{-1}}\, ds &\leq -\frac{\gamma_0}{2}\int_0^t \|J_\lambda(z_\lambda(s))_1\|_{H^1}^2\, ds + c\int_0^t \|J_\lambda(z_\lambda(s))\|_{\mathscr{H}}^2\, ds \\ &\leq -\frac{\gamma_0}{2}\int_0^t \|J_\lambda(z_\lambda(s))_1\|_{H^1}^2\, ds + \int_0^t \|z_\lambda(s)\|_{\mathscr{H}}^2\, ds.\end{aligned} \tag{2.3.16}$$

Next, recalling that $\varphi \in \Lambda_{T,M}$, due to (2.3.9) for every $\delta > 0$ we have

$$\begin{aligned}&\left|\int_0^t \langle B_\varphi(s, z_\lambda(s)), z_\lambda(s)\rangle_{\mathscr{H}_1} ds\right| \leq \delta \int_0^t \|B_\varphi(s, z_\lambda(s))\|_{\mathscr{H}_1}^2\, ds + \frac{c}{\delta}\int_0^t \|z_\lambda(s)\|_{\mathscr{H}_1}^2\, ds \\ &\leq .c_M\left(1 + \sup_{r\in[0,t]} \|z_\lambda(r)\|_{\mathscr{H}}^2\right) + \frac{c}{\delta}\int_0^t \|z_\lambda(s)\|_{\mathscr{H}_1}^2\, ds, \qquad \mathbb{P} - \text{a.s.}\end{aligned} \tag{2.3.17}$$

In the same way, thanks to (2.3.10) for every $\delta > 0$

$$\begin{aligned}&\mathbb{E} \sup_{r\in[0,t]} \left|\int_0^r \langle z_\lambda(s), \Sigma(z_\lambda(s))dw^Q(s)\rangle_{\mathscr{H}_1}\right| \leq c\,\mathbb{E}\left(\sup_{r\in[0,t]} \|z_\lambda(r)\|_{\mathscr{H}_1}^2 \int_0^t \|\Sigma(z_\lambda(s))\|_{\mathscr{L}_2(H_Q,\mathscr{H}_1)}^2 ds\right)^{\frac{1}{2}} \\ &\leq .\mathbb{E} \sup_{r\in[0,t]} \|z_\lambda(r)\|_{\mathscr{H}_1}^2 + \frac{c}{\delta}\int_0^t \mathbb{E}\|z_\lambda(s)\|_{\mathscr{H}_1}^2\, ds + \frac{c_T}{\delta}.\end{aligned} \tag{2.3.18}$$

Finally, due to (2.3.10) we get

$$\int_0^t \|\Sigma(z_\lambda(s))\|_{\mathscr{L}_2(H_Q,\mathscr{H}_1)}^2\, ds \leq c_T\left(1 + \int_0^t \mathbb{E}\|z_\lambda(s)\|_{\mathscr{H}}^2\, ds\right) \tag{2.3.19}$$

Therefore, if we choose $\delta > 0$ small enough, from (2.3.16), (2.3.17), (2.3.18) and (2.3.19) we get

$$\mathbb{E}\sup_{r\in[0,t]}\|z_\lambda(r)\|^2_{\mathscr{H}_1}+\frac{\gamma_0}{2}\int_0^t\mathbb{E}\|J_\lambda(z_\lambda(s))_1\|^2_{H^1}\,ds\leq c_T\int_0^t\mathbb{E}\sup_{r\in[0,s]}\|z_\lambda(r)\|^2_{\mathscr{H}_1}\,ds+c_T.$$

Now, Gronwall's lemma allows to obtain (2.3.15).

Step 3. There exists $z\in L^2(\Omega;C([0,T];\mathscr{H}))$ such that

$$\lim_{\lambda\to 0}\mathbb{E}\sup_{t\in[0,T]}\|z_\lambda(t)-z(t)\|^2_{\mathscr{H}}=0. \tag{2.3.20}$$

Proof of Step 3. For every $\lambda,\nu\in(0,\lambda_1)$, we denote $\rho_{\lambda,\nu}(t):=z_\lambda(t)-z_\nu(t)$. We have

$$\begin{aligned}d\rho_{\lambda,\nu}(t)=\ &\big[A_\lambda(z_\lambda(t))-A_\nu(z_\nu(t))\big]dt\\[2ex]&+\big[B_\varphi(t,z_\lambda(t))-B_\varphi(t,z_\nu(t))\big]dt+\big[\Sigma(z_\lambda(t))-\Sigma(z_\nu(t))\big]dw^Q(t).\end{aligned}$$

We have

$$\begin{aligned}&\|\rho_{\lambda,\nu}(t)\|^2_{\mathscr{H}}=2\int_0^t\langle A_\lambda(z_\lambda(s))-A_\nu(z_\nu(s)),\rho_{\lambda,\nu}(s)\rangle_{\mathscr{H}}ds\\[2ex]&+2\int_0^t\langle B_\varphi(s,z_\lambda(s))-B_\varphi(s,z_\nu(s)),\rho_{\lambda,\nu}(s)\rangle_{\mathscr{H}}ds+\int_0^t\|\Sigma(z_\lambda(s))-\Sigma(z_\nu(s))\|^2_{\mathscr{L}_2(H_Q,\mathscr{H})}ds\\[2ex]&+2\int_0^t\langle\rho_{\lambda,\nu}(s),(\Sigma(z_\lambda(s))-\Sigma(z_\nu(s)))\,dw^Q(s)\rangle_{\mathscr{H}}=:\sum_{k=1}^4 I_k(t).\end{aligned}$$

Due to (2.3.8), we have

$$|I_1(t)| \leq c\int_0^t \|\rho_{\lambda,\nu}(s)\|^2_{\mathscr{H}} ds + c(\lambda+\nu)\int_0^t \left(\|z_\lambda(s)\|^2_{\mathscr{H}_1} + \|z_\nu(s)\|^2_{\mathscr{H}_1} + 1\right) ds, \tag{2.3.21}$$

and due to (2.3.10) we have

$$|I_3(t)| \leq c\int_0^t \|\rho_{\lambda,\nu}(s)\|^2_{\mathscr{H}} ds. \tag{2.3.22}$$

Moreover, by proceeding as in the proof of (2.3.17) and (2.3.18), for every $\delta > 0$ we have

$$\mathbb{E}\sup_{r\in[0,t]} |I_2(r)| + \mathbb{E}\sup_{r\in[0,t]} |I_4(r)| \leq \delta\,\mathbb{E}\sup_{r\in[0,t]} \|\rho_{\lambda,\nu}(r)\|^2_{\mathscr{H}} + \frac{c}{\delta}\int_0^t \mathbb{E}\,\|\rho_{\lambda,\nu}(r)\|^2_{\mathscr{H}}\, dr. \tag{2.3.23}$$

Therefore, if we choose $\delta > 0$ sufficiently small, from (2.3.21), (2.3.22) and (2.3.23), we obtain

$$\begin{aligned}\mathbb{E}\sup_{r\in[0,t]} \|\rho_{\lambda,\nu}(r)\|^2_{\mathscr{H}} \leq \quad & c\int_0^t \mathbb{E}\sup_{r\in[0,s]} \|\rho_{\lambda,\nu}(r)\|^2_{\mathscr{H}}\, ds \\ & +c\,(\lambda+\nu)\int_0^t \left(\|z_\lambda(s)\|^2_{\mathscr{H}_1} + \|z_\nu(s)\|^2_{\mathscr{H}_1} + 1\right) ds,\end{aligned}$$

and the Gronwall lemma, together with (2.3.15), gives

$$\begin{aligned}\mathbb{E}\sup_{r\in[0,T]} \|\rho_{\lambda,\nu}(r)\|^2_{\mathscr{H}} \quad & \leq c_T(\lambda+\nu)\int_0^T \left(\|z_\lambda(s)\|^2_{\mathscr{H}_1} + \|z_\nu(s)\|^2_{\mathscr{H}_1} + 1\right) ds \\ & \leq c_T(\lambda+\nu)\left(1+\|z_0\|^2_{\mathscr{H}}\right).\end{aligned}$$

This implies that

$$\lim_{\lambda,\nu\to 0} \mathbb{E}\sup_{r\in[0,T]} \|\rho_{\lambda,\nu}(r)\|^2_{\mathscr{H}} = 0,$$

so that the family $\{z_\lambda\}_{\lambda\in(0,\lambda_1)}$ is Cauchy and (2.3.20) follows.

Step 4. There exists a unique solution $z \in L^2(\Omega;C([0,T];\mathscr{H}_1))$ for equation (2.3.1).

Proof of step 4. For every $\lambda \in (0,\lambda_0)$ we have that z_λ satisfies equation (2.3.11). Then, by proceeding as in [22, Proof of Theorem 3.2], we take the limit, as λ goes to zero, of both sides of (2.3.11) in $L^2(\Omega;C([0,T];\mathscr{H}_{-1}))$ and thanks to (2.3.20) we obtain that z satisfies the equation

$$z(t) = (u_0, v_0 + g(u_0)) + \int_0^t [A(z(s)) + B_\varphi(s,z(s))]\,ds + \int_0^t \Sigma(z(s))\,dw^Q(s),$$

and $z \in L^2(\Omega;L^\infty(0,T;\mathscr{H}_1))$.

Next, by using again arguments analogous to those used in [22, Proof of Theorem 3.2], we can show that z has continuous trajectories and is the unique solution of equation (2.3.1).

2.3.2 The case when f satisfies Hypothesis 2.6

In view of what we have seen in Subsection 2.3.1, for every $n \in \mathbb{N}$ and for every $\varphi \in \Lambda_{T,M}$ and $(u_0,v_0) \in \mathscr{H}_1$ there exists a unique solution $(u_n,v_n) \in L^2(\Omega;C([0,T];\mathscr{H}_1))$ for the equation

$$\begin{cases} u_n(t,x) = u_0(x) + \displaystyle\int_0^t v_n(s,x)\,ds, \\[2ex] v_n(t,x) = \mu\, v_0(x) + \displaystyle\int_0^t [\Delta u_n(s,x) - \gamma(u_n(s,x))v_n(s,x) + f_n(x,u(s,x)) + \sigma(u_n(s,\cdot))Q\varphi(s,x)]\,ds \\[2ex] \qquad\qquad + \displaystyle\int_0^t \sigma(u(s,\cdot))\,dw^Q(s,x), \end{cases} \tag{2.3.24}$$

where f_n is the function defined in (2.1.16).

For every $n \in \mathbb{N}$, we define

$$\tau_n := \inf \{t \geq 0 \; : \; \|u_n(t)\|_{H^1} \geq n/C\},$$

with $\inf \emptyset = +\infty$, where $C > 0$ is a constant such that $\|\cdot\|_{L^\infty(0,L)} \leq C\|\cdot\|_H$. Clearly $\{\tau_n\}_{n\in\mathbb{N}}$ is an increasing sequence of stopping times.

We denote $\tau := \sup_{n\in\mathbb{N}} \tau_n$, and for every $\omega \in \Omega$ and $t < \tau(\omega) \wedge T$, define

$$z(t)(\omega) := z_n(t)(\omega), \quad \text{if } t < \tau_n(\omega) \leq T.$$

Notice that this is a good definition, as $f_n(r) = f_m(r)$, for every $n \leq m$ and $|r| \leq n$. Moreover, since for $\omega \in \Omega$ and $t \leq \tau_n(\omega) \wedge T$

$$\|u_n(t)(\omega)\|_{L^\infty([0,L])} \leq C\|u_n(t)(\omega)\|_{H^1} \leq n,$$

we have that $f_n(u(s)(\omega)) = f(u(s)(\omega))$. This means $z(t) = z_n(t)$ solves equation (2.3.3) for $t \leq \tau_n \wedge T$.

Step 1. There exists $c_T > 0$ independent of $n \in \mathbb{N}$ such that

$$\mathbb{E} \sup_{t\in[0,T]} \|u(t\wedge\tau_n)\|_H^2 + \int_0^T \mathbb{E}\,\|u(t\wedge\tau_n)\|_{H^1}^2\,dt + \int_0^T \mathbb{E}\,\|u(t\wedge\tau_n)\|_{L^{\theta+1}}^{\theta+1}\,dt$$

$$\leq c_T\left(1+\int_0^T \mathbb{E}\,\|u(t\wedge\tau_n)\|_H^2\,dt + \int_0^T \mathbb{E}\,\|v(t\wedge\tau_n)\|_H^2\,dt + \mathbb{E}\sup_{t\in[0,T]}\|v(t\wedge\tau_n)\|_H^2\right). \tag{2.3.25}$$

Proof of Step 1. Recall that for $t < \tau_n \leq T$, $z(t) = z_n(t)$ is a solution of equation (2.3.3), by proceeding as in [22, Proof of Lemma 4.1], we have

$$\frac{\gamma_0}{4}\|u(t\wedge\tau_n)\|_H^2 + \int_0^{t\wedge\tau_n}\|u(s)\|_{H^1}^2 \leq c + c\,\|v(t\wedge\tau_n)\|_H^2 + \int_0^{t\wedge\tau_n}\|v(s)\|_H^2\,ds$$

$$+\int_0^{t\wedge\tau_n}\langle F(u(s)),u(s)\rangle_H ds + \int_0^{t\wedge\tau_n}\langle u(s),\sigma(u(s))Q\varphi(s)\rangle_H ds + \int_0^{t\wedge\tau_n}\langle u(s),\sigma(u(s))dw^Q(s)\rangle_H. \tag{2.3.26}$$

Thanks to (2.1.12), we have

$$\int_0^{t\wedge\tau_n}\langle F(u(s)),u(s)\rangle_H ds \leq -c_2\int_0^{t\wedge\tau_n}\|u(s)\|_{L^{\theta+1}}^{\theta+1}\,ds + c_2 t \tag{2.3.27}$$

Moreover, by proceeding as in the proof of (2.3.17) and (2.3.18), for every ¿0 we have

$$\mathbb{E}\sup_{r\in[0,t]}\left|\int_0^{r\wedge\tau_n}\langle u(s),\sigma(u(s))Q\varphi(s)\rangle_H ds\right| + \mathbb{E}\sup_{r\in[0,t]}\left|\int_0^{r\wedge\tau_n}\langle u(s),\sigma(u(s))dw^Q(s)\rangle_H\right|$$

$$\leq \delta\mathbb{E}\sup_{r\in[0,t]}\|u(r\wedge\tau_n)\|_H^2 + \frac{c}{\delta}\int_0^t \mathbb{E}\|u(\sigma\wedge\tau_n)\|_H^2\,ds.$$

Therefore, if we choose $\delta > 0$ sufficiently small above, this, together with (2.3.26) and (2.3.27) allows to conclude that (2.3.25) holds true.

Step 2. There exists $c_T > 0$ independent of $n \in \mathbb{N}$ such that

$$\mathbb{E}\sup_{t\in[0,T]}\|(u(t\wedge\tau_n),v(t\wedge\tau_n))\|^2_{\mathscr{H}_1}+\mathbb{E}\sup_{t\in[0,T]}\|u(t\wedge\tau_n)\|^{\theta+1}_{L^{\theta+1}}+\gamma_0\int_0^T\mathbb{E}\|v(s)\|^2_H\,ds\leq c_T\left(1+\|u_0\|^{\theta+1}_{H^1}+\|v_0\|^2_H\right) \tag{2.3.28}$$

Proof of Step 2. From the Itô formula we have

$$\frac{1}{2}\Big[\|u(t\wedge\tau_n)\|^2_{H^1}+\|v(t\wedge\tau_n)\|^2_H\Big]=\frac{1}{2}\Big[\|u_0\|^2_{H^1}+\|v_0\|^2_H\Big]-\int_0^{t\wedge\tau_n}\langle\gamma(u(s))v(s),v(s)\rangle_H\,ds$$

$$+\int_{\mathscr{O}}\mathfrak{f}(x,u_n(t\wedge\tau_n,x))\,dx-\int_{\mathscr{O}}\mathfrak{f}(x,u_0(x))\,dx+\int_0^{t\wedge\tau_n}\langle\sigma(u(s))Q\varphi(s),v(s)\rangle_H\,ds$$

$$+\int_0^{t\wedge\tau_n}\langle\sigma(u(s))dw^Q(s),v(s)\rangle_H+\frac{1}{2}\int_0^{t\wedge\tau_n}\|\sigma(u(s))\|^2_{\mathscr{L}_2(H_Q,H)}\,ds.$$

Thanks to (2.1.10) we have

$$\int_{\mathscr{O}}\mathfrak{f}(x,u(t\wedge\tau_n,x))\,dx\leq c-c_2\,\|u(t\wedge\tau_n)\|^{\theta+1}_{L^{\theta+1}},$$

and

$$\left|\int_{\mathscr{O}}\mathfrak{f}(x,u_0(x))\,dx\right|\leq c\left(1+\int_{\mathscr{O}}|u_0(x)|^{\theta+1}\,dx\right)=c\left(1+\|u_0\|^{\theta+1}_{L^{\theta+1}}\right)\leq c\left(1+\|u_0\|^{\theta+1}_{H^1}\right).$$

Therefore, due to (2.1.5),

$$\sup_{r\in[0,t]}\|u(r\wedge\tau_n)\|_{H^1}^2+\sup_{r\in[0,t]}\|u(r\wedge\tau_n)\|_{L^{\theta+1}}^{\theta+1}+\sup_{r\in[0,t]}\|v(r\wedge\tau_n)\|_H^2+\gamma_0\int_0^{t\wedge\tau_n}\|v(s)\|_H^2\,ds$$

$$\leq c\left(1+\|u_0\|_{H^1}^{\theta+1}+\|v_0\|_H^2\right)+\sup_{r\in[0,t]}\left|\int_0^{r\wedge\tau_n}\langle\sigma(u(s))Q\varphi(s),v(s)\rangle_H\,ds\right|$$

$$+\sup_{r\in[0,t]}\left|\int_0^{r\wedge\tau_n}\langle\sigma(u(s))dw^Q(s),v(s)\rangle_H\right|+c\int_0^{t\wedge\tau_n}\|u(s)\|_H^2\,ds.$$

Since $\varphi\in\Lambda_{T,M}$, by proceeding as in (2.3.17) and (2.3.18), this implies that for every ¿0

$$\mathbb{E}\sup_{r\in[0,t]}\|u(r\wedge\tau_n)\|_{H^1}^2+\mathbb{E}\sup_{r\in[0,t]}\|u(r\wedge\tau_n)\|_{L^{\theta+1}}^{\theta+1}+\mathbb{E}\sup_{r\in[0,t]}\|v(r\wedge\tau_n)\|_H^2+\gamma_0\mathbb{E}\int_0^{t\wedge\tau_n}\|v(s)\|_H^2\,ds$$

$$\leq c\left(1+\|u_0\|_{H^1}^{\theta+1}+\|v_0\|_H^2\right)+\delta\,\mathbb{E}\sup_{r\in[0,t]}\|u(r\wedge\tau_n)\|_H^2+\frac{c}{\delta}\int_0^t\mathbb{E}\|v(s\wedge\tau_n)\|_H^2\,ds+c\int_0^t\mathbb{E}\,\|u(s\wedge\tau_n)\|_H^2\,ds.$$

In particular, if we choose $\delta>0$ small enough we have

$$\begin{aligned}&\mathbb{E}\sup_{r\in[0,t]}\|u(r\wedge\tau_n)\|_{H^1}^2+\mathbb{E}\sup_{r\in[0,t]}\|u(r\wedge\tau_n)\|_{L^{\theta+1}}^{\theta+1}+\mathbb{E}\sup_{r\in[0,t]}\|v(r\wedge\tau_n)\|_H^2\\&\leq c\left(1+\|u_0\|_{H^1}^{\theta+1}+\|v_0\|_H^2\right)+c\int_0^t\mathbb{E}\|v(s\wedge\tau_n)\|_H^2\,ds+c\int_0^t\mathbb{E}\,\|u(s\wedge\tau_n)\|_H^2\,ds,\end{aligned}\tag{2.3.29}$$

and if we choose $\delta > 0$ large enough we have

$$\mathbb{E} \sup_{r \in [0,t]} \|u(r \wedge \tau_n)\|_{L^{\theta+1}}^{\theta+1} + \mathbb{E} \sup_{r \in [0,t]} \|v(r \wedge \tau_n)\|_H^2 + \gamma_0 \mathbb{E} \int_0^{t \wedge \tau_n} \|v(s)\|_H^2 \, ds$$

$$\leq c \left(1 + \|u_0\|_{H^1}^{\theta+1} + \|v_0\|_H^2\right) + c \int_0^t \mathbb{E} \|u(s \wedge \tau_n)\|_H^2 \, ds. \tag{2.3.30}$$

Combining (2.3.30) with (2.3.25) yields that

$$\mathbb{E} \sup_{r \in [0,t]} \|u(r \wedge \tau_n)\|_H^2 \leq c \int_0^t \mathbb{E} \|u(s \wedge \tau_n)\|_H^2 \, ds. \tag{2.3.31}$$

Thanks to Gronwall's lemma, (2.3.28) follows from (2.3.29).

Step 3. There exists $z = (u, v) \in L^2(\Omega; C([0,T]; \mathscr{H}_1))$ solution to problem (2.3.3) such that

$$\mathbb{E} \sup_{t \in [0,T]} \|(u(t), v(t))\|_{\mathscr{H}_1}^2 + \mathbb{E} \sup_{t \in [0,T]} \|u(t)\|_{L^{\theta+1}}^{\theta+1} + \gamma_0 \int_0^T \mathbb{E} \|v(s)\|_H^2 \, ds \leq c_T \left(1 + \|u_0\|_{H^1}^{\theta+1} + \|v_0\|_H^2\right). \tag{2.3.32}$$

.

Proof of Step 3. According to (2.3.28), for every $T > 0$ we have

$$\mathbb{P}(\tau_n \leq T) \leq \frac{C^2}{n^2} \mathbb{E}\big(\|u(\tau_n)\|_{H^1}^2 ; \tau_n \leq T\big) \leq \frac{C^2}{n^2} \mathbb{E} \sup_{t \in [0,T]} \|u(t \wedge \tau_n)\|_{H^1}^2 \leq \frac{c}{n^2},$$

so that

$$\lim_{n \to \infty} \mathbb{P}(\tau_n \leq T) = 0,$$

and hence $\mathbb{P}(\tau = \infty) = 1$. This implies for every $t \in [0,T]$, $z(t \wedge \tau_n) \to z(t)$, $\mathbb{P}$-a.s. as $n \to \infty$, so that z belongs to $L^2(\Omega;C([0,T];\mathscr{H}_1))$ and solves equation (2.3.3). By taking the limit as $n\to\infty$ in (2.3.28), we get (2.3.32).

Step 4. The solution z is unique in $L^2(\Omega;C([0,T];\mathscr{H}_1))$.

Proof of Step 4. Let z_1 and z_2 be two solutions of equation (2.3.1) in $L^2(\Omega;C([0,T];\mathscr{H}_1))$. For every $R>0$, we define

$$\tau_R := \tau_{1,R}\wedge\tau_{2,R},$$

where

$$\tau_{i,R} := \inf\left\{t\geq 0\ :\ \|u_i(t)\|_{H^1}\geq R\right\},\quad i=1,2.$$

Since z_1 and z_2 belong to $L^2(\Omega;C([0,T];\mathscr{H}_1))$, we have

$$\lim_{R\to\infty}\mathbb{P}(\tau_R<T)=0. \tag{2.3.33}$$

Now, if we define $\rho=z_1-z_2$, from the Itô formula we have

$$\|\rho(t\wedge\tau_R)\|_{\mathscr{H}}^2 = 2\int_0^{t\wedge\tau_R}\langle A(z_1(s))-A(z_2(s)),\rho(s)\rangle_{\mathscr{H}}ds$$

$$+2\int_0^{t\wedge\tau_R}\langle B_\varphi(s,z_1(s))-B_\varphi(s,z_2(s)),\rho(s)\rangle_{\mathscr{H}}ds+\int_0^{t\wedge\tau_R}\|\Sigma(z_1(s))-\Sigma(z_2(s))\|^2_{\mathscr{L}_2(H_Q,\mathscr{H})}ds$$

$$+2\int_0^{t\wedge\tau_R}\langle\rho(s),(\Sigma(z_1(s))-\Sigma(z_2(s)))\,dw^Q(s)\rangle_{\mathscr{H}} =: \sum_{k=1}^4 I_k(t\wedge\tau_R).$$

We have

$$\langle A(z_1(s)) - A(z_2(s)), \rho(s)\rangle_{\mathscr{H}} = \quad -\langle (g(u_1(s)) - g(u_2(s)), u_1(s) - u_2(s)\rangle_H$$

$$+\langle F(u_1(s)) - F(u_2(s)), \eta_1(s) - \eta_2(s)\rangle_{H^{-1}},$$

so that, according to (2.1.14)

$$|\langle A(z_1(s)) - A(z_2(s)), \rho(s)\rangle_{\mathscr{H}}| \le c\,\|\rho(s)\|^2_{\mathscr{H}} + c\,\|F(u_1(s)) - F(u_2(s))\|^2_H$$

$$\le c\,\|\rho(s)\|^2_{\mathscr{H}} + c\left(1 + \|u_1(s)\|^{2(\theta-1)}_{H^1} + \|u_2(s)\|^{2(\theta-1)}_{H^1}\right)\|u_1(s) - u_2(s)\|^2_H.$$

This implies that

$$\sup_{r\in[0,t]} |I_1(r\wedge\tau_R)| \le c(R)\int_0^{t\wedge\tau_R} \|\rho(s\wedge\tau_R)\|^2_{\mathscr{H}}\,ds. \tag{2.3.34}$$

Moreover, by proceeding as in (2.3.17) and (2.3.18), for every $\delta > 0$ we have

$$\mathbb{E}\sup_{r\in[0,t]}\left(|I_2(r\wedge\tau_R)| + |I_4(r\wedge\tau_R)|\right) \le \delta\mathbb{E}\sup_{r\in[0,t]}\|\rho(r\wedge\tau_R)\|^2_{\mathscr{H}} + \frac{c}{\delta}\int_0^t \mathbb{E}\|\rho(s\wedge\tau_R)\|^2_{\mathscr{H}}\,ds. \tag{2.3.35}$$

Therefore, since

$$\sup_{r\in[0,t]}|I_3(r\wedge\tau_R)| \le c\int_0^{t\wedge\tau_R}\|\rho(s\wedge\tau_R)\|^2_{\mathscr{H}}\,ds,$$

thanks to (2.3.34) and (2.3.35), if we fix some ¿0 small enough, we get

$$\mathbb{E}\sup_{r\in[0,t]}\|\rho(r\wedge\tau_R)\|^2_{\mathscr{H}} \le c(R)\int_0^t \mathbb{E}\|\rho(r\wedge\tau_R)\|^2_{\mathscr{H}}\,dr.$$

This implies that for every $R > 0$

$$\mathbb{E} \sup_{r\in[0,T]} \|\rho(r\wedge\tau_R)\|^2_{\mathscr{H}} = 0.$$

In view of (2.3.33), by taking the limit as $R\uparrow\infty$ this gives $\mathbb{E}\sup_{r\in[0,T]}\|\rho(r)\|^2_{\mathscr{H}} = 0$ and uniqueness follows.

2.4 A-priori bounds and tightness

In the previous section we have proved that for any $\mu > 0$ and any $T > 0$ there exists a unique solution $(u_\mu, \partial_t u_\mu) \in L^2(\Omega; C([0,T];\mathscr{H}_1))$ to system (2.3.3). Our purpose in this section is proving a bound for $(u_\mu, \partial_t u_\mu)$, which is uniform with respect to μ.

Lemma 2.10. *Under Hypotheses 2.1 and 2.3 and either Hypothesis 2.5 or Hypothesis 2.6, for every $T, M > 0$ and for every initial condition $(u_0, v_0) \in \mathscr{H}_1$, there exist $c_T > 0$ and $\mu_T > 0$ such that for every $\varphi \in \Lambda_{T,M}$ and $\mu \in (0, \mu_T)$*

$$\mathbb{E} \sup_{t\in[0,T]} \|u_\mu(t)\|^2_{H^1} + \mathbb{E} \sup_{t\in[0,T]} \|u_\mu(t)\|^{\theta+1}_{L^{\theta+1}} + \mu\,\mathbb{E} \sup_{t\in[0,T]} \|\partial_t u_\mu(t)\|^2_H + \int_0^T \mathbb{E}\|\partial_t u_\mu(t)\|^2_H\,dt \leq c_T. \tag{2.4.1}$$

Proof. We give our proof in case Hypothesis 2.6 holds and we leave to the reader the proof in case Hypothesis 2.5 holds.

Step 1. There exists $c_T > 0$ such that for all $\mu \in (0,1)$

$$\mathbb{E} \sup_{r\in[0,t]} \|u_\mu(r)\|_H^2 + \int_0^t \mathbb{E}\|u_\mu(s)\|_{H^1}^2\, ds + \int_0^t \mathbb{E}\left\|u_\mu(s)\right\|_{L^{\theta+1}}^{\theta+1} ds$$
$$\leq c_T \left(1 + \int_0^t \mathbb{E}\|u_\mu(s)\|_H^2 ds + \mu \int_0^t \mathbb{E}\|\partial_t u_\mu(s)\|_H^2 ds + \mu^2 \mathbb{E} \sup_{r\in[0,t]} \|\partial_t u_\mu(r)\|_H^2\right). \tag{2.4.2}$$

Proof of Step 1. As shown in [22, Proof of Lemma 4.1], for every $\mu \in (0,1)$ we have

$$\frac{\gamma_0}{4}\|u_\mu(t)\|_H^2 + \int_0^t \|u_\mu(s)\|_{H^1}^2 ds \leq c + c\mu^2 \|\partial_t u_\mu(t)\|_H^2 + \mu \int_0^t \|\partial_t u_\mu(s)\|_H^2 ds$$
$$+ \int_0^t \langle F(u_\mu(s)), u_\mu(s)\rangle_H ds + \int_0^t \langle u_\mu(s), \sigma(u_\mu(s)) Q\varphi_\mu(s)\rangle_H ds + \sqrt{\mu}\int_0^t \langle u_\mu(s), \sigma(u_\mu(s)) dw^Q(s)\rangle_H. \tag{2.4.3}$$

Due to (2.1.12), we have

$$\int_0^t \langle F(u_\mu(s)), u_\mu(s)\rangle_H ds \leq -c_2 \int_0^t \left\|u_\mu(s)\right\|_{L^{\theta+1}}^{\theta+1} ds + c_2 t. \tag{2.4.4}$$

Moreover, by proceeding as in the proof of (2.3.17) and (2.3.18), for every $\delta > 0$ we have

$$\mathbb{E} \sup_{r\in[0,t]} \left|\int_0^r \langle u_\mu(s), \sigma(u_\mu(s)) Q\varphi_\mu(s)\rangle_H ds\right| + \sqrt{\mu}\,\mathbb{E} \sup_{r\in[0,t]} \left|\int_0^t \langle u_\mu(s), \sigma(u_\mu(s)) dw^Q(s)\rangle_H\right|$$
$$\leq \delta \mathbb{E} \sup_{r\in[0,t]} \|u_\mu(r)\|_H^2 + \frac{c}{\delta}\int_0^t \mathbb{E}\|u_\mu(s)\|_H^2 ds.$$

Therefore, if we choose $\delta > 0$ sufficiently small above, this, together with (2.4.4) and (2.4.3) allows to conclude that (2.4.2) holds true.

Step 2. For every $T>0$ there exist $c_T>0$ and $\mu_T>0$ such that

$$\sup_{\mu\in(0,\mu_T)}\mathbb{E}\sup_{t\in[0,T]}\|u_\mu(t)\|_H^2\leq c_T, \tag{2.4.5}$$

and (2.4.1) holds.

Proof of Step 2. As in the previous section, from Itô's formula, we have

$$\|u_\mu(t)\|_{H^1}^2+\mu\|\partial_t u_\mu(t)\|_H^2=\|u_0\|_{H^1}^2+\mu\|v_0\|_H^2-\int_0^t\langle\gamma(u_\mu(s))\partial_t u_\mu(s),\partial_t u_\mu(s)\rangle_H\,ds$$

$$+\int_{\mathscr{O}}\mathfrak{f}(x,u_\mu(t,x))\,dx-\int_{\mathscr{O}}\mathfrak{f}(x,u_0(x))\,dx+\int_0^t\langle\sigma(u_\mu(s))Q\varphi(s),\partial_t u_\mu(s)\rangle_H\,ds$$

$$+\sqrt{\mu}\int_0^t\langle\sigma(u_\mu(s))dw^Q(s),\partial_t u_\mu(s)\rangle_H+\int_0^t\|\sigma(u_\mu(s))\|_{\mathscr{L}_2(H_Q,H)}^2\,ds.$$

Due to (2.1.5), (3.1.4) and (2.1.10), this gives

$$\|u_\mu(t)\|_{H^1}^2+c\|u_\mu(s)\|_{L^{\theta+1}}^{\theta+1}+\mu\|\partial_t u_\mu(t)\|_H^2+\gamma_0\int_0^t\|\partial_t u_\mu(s)\|_H^2\,ds$$

$$\leq c\left(1+\|u_0\|_{H^1}^{\theta+1}+\mu\|v_0\|_H^2\right)+\int_0^t\|\sigma(u_\mu(s))\|_{\mathscr{L}_2(H_Q,H)}^2\,ds$$

$$+\int_0^t\langle\sigma(u_\mu(s))Q\varphi(s),\partial_t u_\mu(s)\rangle_H\,ds+\sqrt{\mu}\int_0^t\langle\sigma(u_\mu(s))dw^Q(s),\partial_t u_\mu(s)\rangle_H.$$

Therefore, by proceeding as in Step 1, for every $\delta > 0$ we have

$$\mathbb{E} \sup_{r\in[0,t]} \|u_\mu(r)\|_{H^1}^2 + \mathbb{E} \sup_{r\in[0,T]} \|u_\mu(r)\|_{L^{\theta+1}}^{\theta+1} + \mu\, \mathbb{E} \sup_{r\in[0,t]} \|\partial_t u_\mu(r)\|_H^2 + \int_0^t \mathbb{E}\|\partial_t u_\mu(s)\|_H^2\, ds$$

$$\leq c\left(1 + \|u_0\|_{H^1}^{\theta+1} + \mu\|v_0\|_H^2\right) + \delta\mathbb{E} \sup_{r\in[0,t]} \|u_\mu(r)\|_H^2 + \frac{c}{\delta}\int_0^t \mathbb{E}\|\partial_t u_\mu(s)\|_H^2\, ds + c\int_0^t \mathbb{E}\|u_\mu(s)\|_H^2\, ds. \tag{2.4.6}$$

If we take $\delta > 0$ small enough in (2.4.6), we get

$$\begin{aligned}&\mathbb{E} \sup_{r\in[0,t]} \|u_\mu(r)\|_{H^1}^2 + \mathbb{E} \sup_{r\in[0,T]} \|u_\mu(r)\|_{L^{\theta+1}}^{\theta+1} + \mu\, \mathbb{E} \sup_{r\in[0,t]} \|\partial_t u_\mu(r)\|_H^2\\ &\leq c\left(1 + \int_0^t \mathbb{E}\|\partial_t u_\mu(s)\|_H^2\, ds\right) + c\int_0^t \mathbb{E}\|u_\mu(s)\|_H^2\, ds,\end{aligned} \tag{2.4.7}$$

and if we take $\delta > 0$ large enough we get

$$\mathbb{E} \sup_{r\in[0,T]} \|u_\mu(r)\|_{L^{\theta+1}}^{\theta+1} + \mu\, \mathbb{E} \sup_{r\in[0,t]} \|\partial_t u_\mu(r)\|_H^2 + \int_0^t \mathbb{E}\|\partial_t u_\mu(s)\|_H^2\, ds \leq c\left(1 + \mathbb{E} \sup_{r\in[0,t]} \|u_\mu(r)\|_H^2\right). \tag{2.4.8}$$

By combining together (2.4.2) and (2.4.8), we can fix $\mu_T > 0$ such that for every $\mu \in (0, \mu_T)$

$$\mathbb{E} \sup_{r\in[0,t]} \|u_\mu(r)\|_H^2 \leq c_T\left(1 + \int_0^t \mathbb{E}\|u_\mu(s)\|_H^2 ds\right),$$

which implies (2.4.5). Thus, from (2.4.5), (2.4.7) and (2.4.8), we obtain (2.4.1).

□

Now, for every $T>0$ and $\mu>0$ we define

$$\rho_\mu(t,x)=g(u_\mu(t,x)),\quad (t,x)\in[0,T]\times\mathscr{O}.$$

According to Hypothesis 2.3, we know that

$$|g(r)|\leq\gamma_1|r|,\qquad |g'(r)|\leq\gamma_1,\qquad r\in\mathbb{R}$$

so that for every $\mu>0$ and $t\in[0,T]$,

$$\|\rho_\mu(t)\|_H\leq\gamma_1\|u_\mu(t)\|_H,\qquad \|\rho_\mu(t)\|_{H^1}\leq\gamma_1\|u_\mu(t)\|_{H^1},\qquad \|\partial_t\rho_\mu(t)\|_H\leq\gamma_1\,\|\partial_t u_\mu(t)\|_H. \tag{2.4.9}$$

Since the function g is strictly increasing, it is invertible and we have

$$u_\mu(t,x)=g^{-1}(\rho_\mu(t,x)),\quad (t,x)\in[0,T]\times\mathscr{O},$$

which implies that

$$\Delta u_\mu(t,x)=\operatorname{div}\left[\nabla g^{-1}(\rho_\mu(t))\right]=\operatorname{div}\left[\frac{1}{\gamma(g^{-1}(\rho_\mu(t)))}\nabla\rho_\mu(t)\right].$$

Moreover, by the definition of ρ_μ,

$$\nabla\rho_\mu(t)=\gamma(u_\mu(t))\nabla u_\mu(t),\qquad \partial_t\rho_\mu(t)=\gamma(u_\mu(t))\partial_t u_\mu(t).$$

This means that if we integrate equation (2.2.3) (with φ_μ) with respect to t we have

$$\begin{aligned}\rho_\mu(t)+\mu\partial_t u_\mu(t) = g(u_0)+\mu v_0+\int_0^t \operatorname{div}\left[b(\rho_\mu(s))\nabla\rho_\mu(s)\right]ds+\int_0^t F_g(\rho_\mu(s))ds\\ +\int_0^t \sigma_g(\rho_\mu(s))Q\varphi_\mu(s)ds+\sqrt{\mu}\int_0^t \sigma_g(\rho_\mu(s))dw^Q(s),\end{aligned} \tag{2.4.10}$$

where for every $r\in\mathbb{R}$ and $x\in\mathscr{O}$

$$b(r) := \frac{1}{\gamma(g^{-1}(r))}, \qquad f_g(x,r) := f(x,g^{-1}(r)), \tag{2.4.11}$$

and for every $u\in H^1$

$$F_g(u) = f_g\circ u, \qquad \sigma_g(u) := \sigma(g^{-1}\circ u). \tag{2.4.12}$$

Theorem 2.11. *Assume Hypotheses 2.1 and 2.3 and either Hypothesis 2.5 or Hypothesis 2.6 and fix an arbitrary $T>0$ and $(u_0,v_0)\in\mathscr{H}_1$. Then, for any family of predictable controls $\{\varphi_\mu\}_{\mu\in(0,\mu_T)}\subset\Lambda_{T,M}$, the family of probabilities $\{\mathscr{L}(\rho_\mu)\}_{\mu\in(0,\mu_T)}$ is tight in $C([0,T];H^\delta)$, for every $\delta<1$.*

Proof. According to (2.4.1) and (2.4.9), we have that

$$\mathbb{E}\sup_{r\in[0,t]}\|\rho_\mu(r)\|_H^2+\int_0^t \mathbb{E}\|\partial_t\rho_\mu(s)\|_{H^1}^2\,ds\le c_T, \qquad \mu\in(0,\mu_T).$$

This means that for every $\varepsilon>0$ there exists $L_\varepsilon>0$ such that if we denote by K_ε the ball of radius L_ε in $C([0,T];H^1)\cap W^{1,2}([0,T];H)$, then

$$\inf_{\mu\in(0,\mu_T)}\mathbb{P}(\rho_\mu\in K_\varepsilon)\ge 1-\varepsilon.$$

This allows to conclude as, due to the Aubin-Lions lemma, the set K_ε is compact in $C([0,T];H^\delta)$, for every $\delta<1$. $\square$

2.5 The limit controlled problem

In order to prove conditions C1 and C2, we need first to understand better the controlled quasi-linear parabolic problem

$$\left\{\begin{array}{l}\partial_t\rho(t,x)=\operatorname{div}\left[b(\rho(t,x))\nabla\rho(t,x)\right]+f_g(x,\rho(t,x))+\sigma_g(\rho(t,\cdot))\varphi(t,x),\quad t>0,\quad x\in\mathscr{O},\\[10pt] \rho(0,x)=g(u_0(x)),\qquad \rho(t,x)=0,\quad x\in\partial\mathscr{O}.\end{array}\right. \tag{2.5.1}$$

In view of Hypothesis 2.3, we have

$$\frac{1}{\gamma_1}|r|\leq|g^{-1}(r)|\leq\frac{1}{\gamma_0},\quad r\in\mathbb{R}.$$

When Hypothesis 2.6 holds, thanks to (2.1.9), this means that for every $u\in L^{\theta+1}([0,L])$

$$\|F_g(u)\|_{H^{-1}}\leq c\int_0^L(1+|g^{-1}(u(x))|^\theta)\,dx\leq c\left(1+\|u\|_H^{\frac{2}{\theta-1}}\|u\|_{L^{\theta+1}}^{\frac{(\theta+1)(\theta-2)}{\theta-1}}\right). \tag{2.5.2}$$

Next, again due to Hypothesis 2.3 we have

$$\frac{1}{\gamma_1}\leq\frac{g^{-1}(r)}{r}\leq\frac{1}{\gamma_0},\qquad r\in\mathbb{R},$$

so that, thanks to Hypothesis 2.6, we get,

$$f_g(r)r = f(g^{-1}(r))g^{-1}(r)\cdot\frac{r}{g^{-1}(r)} \leq c\left(1-|r|^{\theta+1}\right), \qquad r\in\mathbb{R},$$

(here we define $g^{-1}(0)/0 = 1/\gamma(0)$). In particular, we have

$$\langle F_g(u),u\rangle_H \leq c\left(1-\|u\|_{L^{\theta+1}}^{\theta+1}\right), \qquad u\in H^1. \tag{2.5.3}$$

Moreover, since g^{-1} is increasing and f is decreasing, we have that f_g is decreasing, so that

$$\langle F_g(u_1)-F_g(u_2),u_1-u_2\rangle_H \leq 0. \tag{2.5.4}$$

Finally, thanks to Hypotheses 2.3 and 2.1,

$$|b(r)|\leq\frac{1}{\gamma_0}, \qquad b(r)\geq\frac{1}{\gamma_1}, \qquad |b(r)-b(s)|\leq c\,|r-s|, \qquad r,s\in\mathbb{R}, \tag{2.5.5}$$

and

$$\|\sigma_g(u_1)-\sigma_g(u_2)\|_{\mathscr{L}(H_Q,H)}\leq c\,\|u_1-u_2\|_H, \qquad u_1,u_2\in H. \tag{2.5.6}$$

Definition 2.12. *A function $\rho\in L^2([0,T];H^1)$ is a weak solution to equation (2.5.1) if for every test function $\psi\in C_0^\infty(\mathscr{O})$ and $t\in[0,T]$*

$$\langle\rho(t),\psi\rangle_H = \langle g(u_0),\psi\rangle_H - \int_0^t\langle b(\rho(s))\nabla\rho(s),\nabla\psi\rangle_H ds + \int_0^t\langle F_g(\rho(s))+\sigma_g(\rho(s))Q\varphi(s),\psi\rangle_H\,ds. \tag{2.5.7}$$

Proposition 2.13. *Assume Hypotheses 2.1 and 2.3 and either Hypothesis 2.5 or Hypothesis 2.6 and fix any $T > 0$ and $\varphi \in L^2([0,T];H)$. Then, for every $u_0 \in H^1$ there is at most one weak solution $\rho \in L^2([0,T];H^1)$ to equation (2.5.1).*

Proof. We adapt here the method introduced in the proof of [22, Theorem 6.2]. To this purpose, let $(\phi_n)_{n\in\mathbb{N}}$ be the sequence of twice differentiable functions constructed in [43, Theorem 3.1] such that

$$\phi_n'(0) = 0, \quad |\phi_n'(r)| \leq 1, \quad 0 \leq \phi_n''(r) \leq \frac{2}{n|r|}, \quad r \in \mathbb{R}, \tag{2.5.8}$$

and

$$\lim_{n\to\infty} \sup_{r\in\mathbb{R}} \big|\phi_n(r) - |r|\big| = 0. \tag{2.5.9}$$

Now, suppose $\rho_1, \rho_2 \in L^2([0,T];H^1)$ are both solutions to (2.5.1). Using integration by parts, for any fixed positive superharmonic function $\psi \in C_0^\infty(\mathscr{O})$, we have

$$\begin{aligned}
\big\langle \phi_n(\rho_1(t) - \rho_2(t)), \psi \big\rangle_H &= \int_0^t \big\langle \phi_n'(\rho_1(s) - \rho_2(s))(F_g(\rho_1(s)) - F_g(\rho_2(s))), \psi \big\rangle_H ds \\
&+ \int_0^t \big\langle \phi_n'(\rho_1(s) - \rho_2(s))\Big(\sigma_g(\rho_1(s)) - \sigma_g(\rho_2(s))\Big)Q\varphi(s), \psi \big\rangle_H ds \\
&- \int_0^t \big\langle \phi_n''(\rho_1(s) - \rho_2(s))\nabla(\rho_1(s) - \rho_2(s))\Big(b(\rho_1(s))\nabla\rho_1(s) - b(\rho_2(s))\nabla\rho_2(s)\Big), \psi \big\rangle_H ds \\
&- \int_0^t \big\langle \phi_n'(\rho_1(s) - \rho_2(s))\Big(b(\rho_1(s))\nabla\rho_1(s) - b(\rho_2(s))\nabla\rho_2(s)\Big), \nabla\psi \big\rangle_H ds \\
&=: \sum_{k=1}^4 I_{k,n}(t).
\end{aligned} \tag{2.5.10}$$

In the case when f satisfies Hypothesis 2.5, by the Lipschitz continuity of F_g, we have

$$I_{1,n}(t) \leq c \int_0^t \big\langle |\rho_1(s) - \rho_2(s)|, \psi \big\rangle_H ds.$$

In the case when f satisfies Hypothesis 2.6, thanks to the fact that $F_g' \leq 0$, $\phi_n'' \geq 0$ and $\psi \geq 0$, we have $I_{1,n}(t) \leq 0$.

Next, by proceeding as in the proof of [22, Theorem 6.2], we know that

$$I_{3,n}(t) + I_{4,n}(t) \leq \frac{c_T}{n} \big(\|\psi\|_{H^1} + \|\psi\|_{L^\infty} \big) \int_0^t \Big(\|\rho_1(s)\|_{H^1}^2 + \|\rho_2(s)\|_{H^1}^2 \Big) ds.$$

And moreover, it follows from (2.1.2) that

$$\begin{aligned}
I_{2,n}(t) \quad &= \int_0^t \int_{\mathscr{O}} \phi_n'(\rho_1(s) - \rho_2(s)) \Big[\big(\sigma_g(\rho_1(s)) - \sigma_g(\rho_2(s))\big) Q\varphi(s) \Big] \psi(x) dx ds \\
&\leq \int_0^t \int_{\mathscr{O}} \sum_{i=1}^{\infty} \Big| \varphi_i(s) \Big(\sigma_i(x, u_1(s)) - \sigma_i(x, u_2(s)) \Big) \Big| \psi(x) dx ds \\
&\leq c \int_0^t \|\varphi(s)\|_H \int_{\mathscr{O}} \Big(\sum_{i=1}^{\infty} \Big| \sigma_i(x, u_1(s)) - \sigma_i(x, u_2(s)) \Big|^2 \Big)^{\frac{1}{2}} \psi(x) dx ds \\
&\leq C \int_0^t \|\varphi(s)\|_H \big\langle |\rho_1(s) - \rho_2(s)|, \psi \big\rangle_H ds,
\end{aligned}$$

where $\varphi_i(t) := \langle \varphi(t), e_i \rangle_H$.

Therefore, we have for every $t \geq 0$

$$\left\langle \phi_n(\rho_1(t)-\rho_2(t)),\psi\right\rangle_H$$

$$\leq \frac{c_T\left(\|\psi\|_{H^1}+\|\psi\|_{L^\infty}\right)}{n}\int_0^t\left(\|\rho_1(s)\|_{H^1}^2+\|\rho_2(s)\|_{H^1}^2\right)ds+C\int_0^t\left(1+\|\varphi(s)\|_H\right)\left\langle|\rho_1(s)-\rho_2(s)|,\psi\right\rangle_H ds.$$

Taking $n\to\infty$, we obtain

$$\left\langle|\rho_1(t)-\rho_2(t)|,\psi\right\rangle_H \leq C\int_0^t\left(1+\|\varphi(s)\|_H\right)\left\langle|\rho_1(s)-\rho_2(s)|,\psi\right\rangle_H ds,$$

and then by the Gronwall lemma, we conclude

$$\left\langle|\rho_1(t)-\rho_2(t)|,\psi\right\rangle_H = 0,\quad t\geq 0.$$

Finally, due to the arbitrariness of positive superharmonic test function $\psi\in C_0^\infty(\mathscr{O})$, we have $\rho_1=\rho_2$. This completes the proof.

□

Proposition 2.14. *Assume Hypotheses 2.1 and 2.3 and either Hypothesis 2.5 or Hypothesis 2.6 and fix any $T>0$, $u_0\in H^1$ and $\varphi\in L^2([0,T];H)$. Suppose that ρ is the a weak solution of equation (2.5.1). Then, if we define $u:=g^{-1}(\rho)$, we have that $u\in L^2([0,T];H^1)$ is a weak solution to equation (2.2.2).*

Proof. Due to the fact that $\rho\in L^2([0,T];H^1)$ and g^{-1} is differentiable with bounded derivative, we have that $u \in L^2([0,T];H^1)$ and

$$\nabla(g(u(t))) = g'(u(t))\nabla u(t) = \gamma(u(t))\nabla u(t).$$

Recalling how b, F_g and σ_g were defined, this implies

$$S_\varphi(t,\rho(t)) := \operatorname{div}\left[b(\rho(t))\nabla\rho(t)\right] + F_g(\rho(t)) + \sigma_g(\rho(t))Q\varphi(t)$$

$$= \operatorname{div}\left[\frac{1}{\gamma(u(t))}\nabla(g(u(t))\right] + f(u(t)) + \sigma(u(t))Q\varphi(t) = \Delta u(t) + f(u(t)) + \sigma(u(t))Q\varphi(t)$$

in H^{-1} sense. Moreover, by mollfyng ρ with respect to t and x and then by taking the limit, we have that

$$\partial_t u \in L^2([0,T];H^{-1})$$

and

$$\partial_t\rho(t) = g'(u(t))\partial_t u(t) = \gamma(u(t))\partial_t u(t),$$

in H^{-1} sense. We can now conclude, as we know that $\partial_t\rho(t) = \mathscr{S}_\varphi(t,\rho(t))$, in H^{-1} sense. □

2.6 Proof of Theorem 2.8

In order to prove Theorem 2.8, we will show that the conditions C1 and C2 that we introduced in Section 2.2 are both satisfied. We will consider here the case the nonlinearity f satisfies Hypothesis 2.6 and we leave to the reader to adapt out proof to the case f satisfies Hypothesis 2.5.

Theorem 2.15. *Under Hypotheses 2.1 and 2.3 and either Hypothesis 2.5 or Hypothesis 2.6, condition C1 holds.*

Proof. Let us fix $T, M > 0$ and let $\{\varphi_\mu\}$ be a family of processes in $\Lambda_{T,M}$ such that

$$\lim_{\mu \to 0} \varphi_\mu = \varphi, \quad \text{in distribution in } L^2_w(0,T;H),$$

where $L^2_w(0,T;H)$ is the space $L^2([0,T];H)$ endowed with the weak topology and $\varphi \in \Lambda_{T,M}$.

For every sequence $\{\mu_k\}_{k\in\mathbb{N}}$ converging to 0, as $k \uparrow \infty$, we denote $\rho_k = g(u_k)$, where $u_k := u^{\varphi_{\mu_k}}_{\mu_k}$ is the solution of equation (2.2.3), corresponding to the control φ_{μ_k}. Thanks to Theorem 2.11 and Lemma 2.10, for an arbitrary $\delta < 1$ we have that the family

$$\left\{\mathscr{L}(\rho_k,\ \mu_k \partial_t u_k, \varphi_{\mu_k})\right\}_{k\in\mathbb{N}} \subset \mathscr{P}\left(C([0,T];H^\delta) \times C([0,T];H) \times \mathscr{S}_{T,M}\right)$$

is tight. We denote by ρ a weak limit point for the sequence $\{\rho_k\}_{k\in\mathbb{N}}$ and we denote

$$\Lambda := C([0,T];H^\delta) \times C([0,T];H) \times \mathscr{S}_{T,M} \times C([0,T],U)),$$

where U is the Hilbert space such that (2.1.1) holds. By the Skorokhod Theorem there exist random variables

$$\mathscr{Y} = (\hat{\rho}, 0, \hat{\varphi}, \hat{w}^Q), \qquad \mathscr{Y}_k = \left(\hat{\rho}_k, \hat{\theta}_k, \hat{\varphi}_k, \hat{w}^Q_k\right), \quad k \in \mathbb{N},$$

defined on a probability space $(\hat{\Omega}, \hat{\mathscr{F}}, \{\hat{\mathscr{F}}_t\}_{t\in[0,T]}, \hat{\mathbb{P}})$, such that

$$\mathscr{L}(\mathscr{Y}) = \mathscr{L}(\rho, 0, \varphi, w^Q), \qquad \mathscr{L}(\mathscr{Y}_k) = \mathscr{L}(\rho_k,\ \mu_k \partial_t u_k, \varphi_{\mu_k}, w^Q), \quad k \in \mathbb{N},$$

and such that

$$\lim_{k\to\infty} \mathscr{Y}_k = \mathscr{Y} \quad \text{in } \Lambda, \qquad \hat{\mathbb{P}} - \text{a.s.} \tag{2.6.1}$$

For every $k \in \mathbb{N}$ and $\psi \in H^2$, we have

$$\langle \hat{\rho}_k(t) + \hat{\theta}_k(t), \psi \rangle_H = \langle g(u_0) + \mu_k v_0, \psi \rangle_H + \int_0^t \langle \operatorname{div} [b(\hat{\rho}_k(s)) \nabla \hat{\rho}_k(s)], \psi \rangle_H \, ds$$

$$+ \int_0^t \langle F_g(\hat{\rho}_k(s)), \psi \rangle_H \, ds + \int_0^t \langle \sigma_g(\hat{\rho}_k(s)) Q \, \hat{\varphi}_k(s), \psi \rangle_H \, ds + \sqrt{\mu_k} \int_0^t \langle \sigma_g(\hat{\rho}_{\mu_k}(s)) d\hat{w}^Q_k(s), \psi \rangle_H.$$

Thanks to (2.6.1), for every $t \in [0, T]$ we have

$$\lim_{k\to\infty} \langle \hat{\rho}_k(t) + \hat{\theta}_k(t), \psi \rangle_H = \langle \hat{\rho}(t), \psi \rangle_H, \qquad \hat{\mathbb{P}} - \text{a.s.} \tag{2.6.2}$$

Next, if we define $\hat{u}_k := g^{-1}(\hat{\rho}_k)$ and $\hat{u} := g^{-1}(\hat{\rho})$, we have

$$\int_0^t \langle b(\hat{\rho}_k(s)) \nabla \hat{\rho}_k(s), \nabla \psi \rangle_H ds - \int_0^t \langle b(\hat{\rho}(s)) \nabla \hat{\rho}(s), \nabla \psi \rangle_H \, ds$$

$$= \int_0^t \langle \nabla \hat{u}_k(s), \nabla \psi \rangle_H ds - \int_0^t \langle \nabla \hat{u}(s), \nabla \psi \rangle_H ds = - \int_0^t \langle (\hat{u}_k(s) - \hat{u}(s)), \Delta \psi \rangle_H ds.$$

In particular, since (2.6.1) implies the $\mathbb{P}$-a.s. convergence of $\hat{u}_k$ to $\hat{u}$ in $C([0,T];H)$, we get that

$$\lim_{k\to\infty}\int_0^t \langle b(\hat{\rho}_k(s))\nabla\hat{\rho}_k(s),\nabla\psi\rangle_H\,ds = \int_0^t \langle b(\hat{\rho}(s))\nabla\hat{\rho}(s),\nabla\psi\rangle_H\,ds,\quad \hat{\mathbb{P}}-\text{a.s.} \tag{2.6.3}$$

It is immediate to check that estimate (2.1.15) extends to F_g. Thus we have

$$\left|\int_0^t \langle F_g(\hat{\rho}_k(s)),\psi\rangle_H\,ds - \int_0^t \langle F_g(\hat{\rho}(s)),\psi\rangle_H\,ds\right| \leq \int_0^t \|F_g(\hat{\rho}_k(s)) - F_g(\hat{\rho}(s))\|_{L^1}\,ds\,\|\psi\|_{H^1}$$

$$\leq c\int_0^t \left(1+\|\hat{\rho}_k(s)\|^{\theta-1}_{L^{2(\theta-1)}}+\|\hat{\rho}(s)\|^{\theta-1}_{L^{2(\theta-1)}}\right)\|\hat{\rho}_k(s)-\hat{\rho}(s)\|_H\,ds\,\|\psi\|_{H^1},$$

so that, thanks to (2.6.1),

$$\lim_{k\to\infty}\int_0^t \langle F_g(\hat{\rho}_k(s)),\psi\rangle_H\,ds = \int_0^t \langle F_g(\hat{\rho}(s)),\psi\rangle_H\,ds,\qquad \hat{\mathbb{P}}-\text{a.s.} \tag{2.6.4}$$

Now, for any $h\in L^2(0,T;H)$, due to (2.5.6)

$$\left|\int_0^t \langle \sigma_g(\hat{\rho}(s))Qh(s),\psi\rangle_H ds\right| \leq c\|\psi\|_H\left(\int_0^T (1+\|\hat{\rho}(s)\|_H^2)ds\right)^{\frac{1}{2}}\left(\int_0^T \|h(s)\|_H^2 ds\right)^{\frac{1}{2}}$$

and this implies that the mapping

$$h\in L^2([0,T];H)\mapsto \int_0^t \langle \sigma_g(\hat{\rho}(s)))Qh(s),\psi\rangle_H\,ds\in\mathbb{R}$$

is a linear functional so that, thanks to (2.6.1), for every fixed $t\geq 0$

$$\lim_{k\to\infty}\int_0^t \langle \sigma_g(\hat{\rho}(s))Q\hat{\varphi}_k(s),\psi\rangle_H ds = \int_0^t \langle \sigma_g(\hat{\rho}(s))Q\hat{\varphi}(s),\psi\rangle_H\,ds,\qquad \hat{\mathbb{P}}-\text{a.s.} \tag{2.6.5}$$

Moreover, we have

$$\left|\int_0^t \langle(\sigma_g(\hat{\rho}_k(s)) - \sigma_g(\hat{\rho}(s)))Q\hat{\varphi}_k(s), \psi\rangle_H ds\right|$$

$$\leq \|\psi\|_H \int_0^t \|\sigma_g(\hat{\rho}_k(s)) - \sigma_g(\hat{\rho}(s))\|_{\mathscr{L}(H_Q,H)}\|\hat{\varphi}_k(s)\|_H\, ds \leq c\,\|\psi\|_H\|\hat{\rho}_k - \hat{\rho}\|_{L^2([0,T];H)}\|\hat{\varphi}_k\|_{L^2([0,T];H)},$$

and by using again (2.6.1) we get

$$\lim_{k\to\infty}\int_0^t \langle(\sigma_g(\hat{\rho}_k(s)) - \sigma_g(\hat{\rho}(s)))Q\hat{\varphi}_k(s), \psi\rangle_H ds = 0, \qquad \hat{\mathbb{P}} - \text{a.s.}$$

This, together with (2.6.5), implies

$$\lim_{k\to\infty}\int_0^t \langle\sigma_g(\hat{\rho}_k(s))Q\hat{\varphi}_k(s), \psi\rangle_H ds = \int_0^t \langle\sigma_g(\hat{\rho}(s))Q\hat{\varphi}(s), \psi\rangle_H ds, \qquad \hat{\mathbb{P}} - \text{a.s.} \tag{2.6.6}$$

Finally, since

$$\sup_{k\in\mathbb{N}} \hat{\mathbb{E}} \sup_{t\in[0,T]}\left|\int_0^t \langle\sigma_g(\hat{\rho}_{\mu_k}(s))d\hat{w}_k^Q(s), \psi\rangle_H\right|^2 \leq c \sup_{k\in\mathbb{N}} \|\psi\|_H^2 \int_0^T \left(1+\hat{\mathbb{E}}\|\hat{\rho}_{\mu_k}(s)\|_H^2\right) ds < \infty,$$

we conclude that

$$\lim_{k\to\infty}\sqrt{\mu_k}\sup_{t\in[0,T]}\hat{\mathbb{E}}\left|\int_0^t \langle\sigma_g(\hat{\rho}_{\mu_k}(s))d\hat{w}_k^Q(s), \psi\rangle_H\right|^2 = 0.$$

This, together with (2.6.2), (2.6.3), (2.6.4) and (2.6.6) implies that

$$\langle \hat{\rho}(t), \psi \rangle_H = \quad \langle g(u_0), \psi \rangle_H - \int_0^t \langle b(\hat{\rho}(s)) \nabla \hat{\rho}(s), \nabla \psi \rangle_H ds$$

$$\int_0^t \left[\langle F_g(\hat{\rho}(s)), \psi \rangle_H + \langle \sigma_g(\hat{\rho}(s)) Q \hat{\varphi}(s), \psi \rangle_H \right] ds.$$

As proven in Proposition 2.13, the equation above has at most one solution. Thus, for every sequence $\{\mu_k\}_{k\in\mathbb{N}} \downarrow 0$ the sequence $\{\rho_k\}_{k\in\mathbb{N}}$ converges in distribution to the solution ρ of equation (2.5.1) with respect to the strong topology of $C([0,T];H^\delta)$, for any arbitrary $\delta < 1$, and hence, due to the embedding of H^1 into $L^p(\mathscr{O})$, with respect to the strong topology of $C([0,T];L^p(\mathscr{O}))$, for every $p < \infty$, if $d = 1,2$ and $p < 2d/(d-2)$, if $d > 2$. In particular, this implies that the sequence $\{u_k\}_{k\in\mathbb{N}}$ converges in distribution to the solution u^φ of equation (2.2.2) with respect to the strong topology of $C([0,T];L^p(\mathscr{O}))$ and condition C1 holds.

As a consequence of the arguments used above to prove condition C1, we have that the mapping

$$\varphi \in L^2_w(0,T;H) \mapsto u^\varphi \in C([0,T];L^p(\mathscr{O}))$$

is continuous. Therefore, since $\Lambda_{T,M}$ is compact in $L^2_w(0,T;H)$, for every $M > 0$, we have that

$$\Phi_{T,R} = \{I_T \leq R\} = \{u^\varphi \,:\, \varphi \in \Lambda_{T,2R^2}\}$$

is compact, and condition C2 follows. □

Chapter 3: The small-mass limit for stationary solutions of stochastic wave equations with state dependent friction

In this chapter, we compare the long-term dynamics of the second-order system with state-dependent damping

$$\begin{cases} \mu\partial_t^2 u_\mu(t,x) = \Delta u_\mu(t,x) - \gamma(u_\mu(t,x))\partial_t u_\mu(t,x) + f(x,u_\mu(t,x)) + \sqrt{\mu}\,\sigma(u_\mu(t,\cdot))\partial_t w^Q(t,x), \\ u_\mu(0,x) = u_0(x), \quad \partial_t u_\mu(0,x) = v_0(x), \qquad u_\mu(t,x) = 0, \;\; x \in \partial\mathscr{O}, \end{cases} \tag{3.0.1}$$

with that of the first-order system

$$\begin{cases} \gamma(u(t,x))\partial_t u(t,x) = \Delta u(t,x) + f(u(t,x)) - \dfrac{\gamma'(u(t,x))}{2\gamma^2(u(t,x))}\displaystyle\sum_{i=1}^{\infty}|[\sigma(u(t,\cdot)Qe_i](x)|^2 \\ \qquad\qquad\qquad\qquad +\sigma(u(t,\cdot))\partial_t w^Q(t,x), \\ u(0,x) = u_0(x), \qquad u(t,x) = 0, \;\; x \in \partial\mathscr{O}, \end{cases} \tag{3.0.2}$$

which represents the small-mass limit of system (3.0.1) within fixed time interval.

The structure of this chapter is as follows. In Section 3.1 we introduce the notations and assumptions that will be used throughout the chapter. In Section 3.2 we state the main result (see Theorem 3.9). In Section 3.3, we study the well-posedness of system (3.2.1) in $\mathscr{H}$. Specifically, we prove that for every initial condition $(u_0, v_0) \in \mathscr{H} := H \times H^{-1}$ system (3.2.1) admits a unique generalized solution in $\mathscr{H}$. This result enables the introduction of the transition semigroup $P_t^{\mu,\mathscr{H}}$, $t \geq 0$, for every $\mu > 0$, and we show that it admits an invariant measure $\nu_\mu^{\mathscr{H}}$ supported in $\mathscr{H}_1 := H^1 \times H$.

In Section 3.4, we derive uniform a-priori bounds for the solutions of system (3.0.1), including uniform bounds for the moment of the invariant measure $\nu_\mu^{\mathscr{H}}$. In Section 3.5 we move our analysis to the limiting equation (3.0.2) in $\mathscr{H}$. We introduce the auxiliary problem (3.5.1) and first prove its well-posedness in H. We then extend these results to problem (3.0.2). Additionally, we study the well-posedness for equation (3.5.1) in H^{-1}.

In Section 3.6, we investigate the ergodic behavior of the limiting equation (3.5.1) in H^{-1}. We prove that the corresponding semigroup has a contractive property in H^{-1} and admits a unique invariant measure in this space. Furthermore, we show that such invariant measure is supported in H, and its restriction to H is the unique invariant measure for the semigroup associated with equation (3.5.1). Finally, we establish that this implies the semigroup associated with equation (3.0.2) admits a unique invariant measure in H.

In Section 3.7, we complete the proof of the main result in this chapter, Theorem 3.9 .

3.1 Notations and assumptions

Throughout the present chapter $\mathscr{O}$ is a bounded domain in $\mathbb{R}^d$, with $d \geq 1$, having a smooth boundary. We denote by H the Hilbert space $L^2(\mathscr{O})$ and by $\|\cdot\|_H$ and $\langle \cdot, \cdot \rangle_H$ the corresponding norm and inner product.

We denote by A the realization of the Laplace operator Δ, endowed with Dirichlet boundary conditions. As known there exists a complete orthonormal basis $\{e_i\}_{i\in\mathbb{N}}$ of H which diagonalizes A. Using the same notations as in Chapter 2, we denote by $\{-\alpha_i\}_{i\in\mathbb{N}}$ the corresponding sequence of eigenvalues, and we define H^δ (see Section 2.1 for definition), for every $\delta \in \mathbb{R}$, and

$$\mathscr{H}_\delta := H^\delta \times H^{\delta-1}, \qquad \mathscr{H} := H \times H^{-1}.$$

Next, if X is any Polish space, we denote by $B_b(X)$ the space of Borel bounded functions $\varphi : X \to \mathbb{R}$, endowed with the sup-norm

$$\|\varphi\|_\infty := \sup_{h\in X} |\varphi(h)|.$$

Moreover, we denote by $C_b(X)$ the subspace of uniformly continuous and bounded functions.

3.1.1 Assumptions

As in Chapter 2, we assume that $w^Q(t)$ is a cylindrical Q-Wiener process (see Section 2.1), defined on a complete stochastic basis $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$. $Q : H \to H$ is a bounded linear operator. When $Q = I$, the process $w^I(t)$ will be denoted by $w(t)$. In particular, we have $w^Q(t) = Qw(t)$.

Moreover, we shall denote by H_Q the set $Q(H)$, the reproducing kernel of the noise w^Q (also see Section 2.1 for more details).

Hypothesis 3.1. *The mapping $\sigma : H \to \mathscr{L}_2(H_Q, H)$ is defined by*

$$[\sigma(h)Qe_i](x) = \sigma_i(x, h(x)), \quad x \in \mathscr{O} \quad h \in H, \quad i \in \mathbb{N},$$

for some measurable mappings $\sigma_i : \mathscr{O} \times \mathbb{R} \to \mathbb{R}$. We assume that there exists $L_\sigma > 0$ such that

$$\sup_{x \in \mathscr{O}} \sum_{i=1}^{\infty} |\sigma_i(x,y_1) - \sigma_i(x,y_2)|^2 \leq L_\sigma |y_1 - y_2|^2, \quad y_1, y_2 \in \mathbb{R}. \tag{3.1.1}$$

Moreover, we assume σ is bounded, that is,

$$\sigma_\infty := \sup_{h \in H} \|\sigma(h)\|_{\mathscr{L}_2(H_Q,H)} < \infty. \tag{3.1.2}$$

Remark 3.2. 1. Condition (3.1.1) implies that $\sigma : H \to \mathscr{L}_2(H_Q, H)$ is Lipschitz continuous. Namely, for any $h_1, h_2 \in H$

$$\|\sigma(h_1) - \sigma(h_2)\|_{\mathscr{L}_2(H_Q,H)} \leq \sqrt{L_\sigma} \|h_1 - h_2\|_H. \tag{3.1.3}$$

2. If the noise is additive, then Hypothesis 3.1 is satisfied when $\mathrm{Tr}Q^2 < +\infty$.

Hypothesis 3.3. *The mapping γ belongs to $C_b^1(\mathbb{R})$ and there exist γ_0 and γ_1 such that*

$$0 < \gamma_0 \leq \gamma(r) \leq \gamma_1, \qquad r \in \mathbb{R}. \tag{3.1.4}$$

If we define

$$g(r) := \int_0^r \gamma(\sigma)\, d\sigma, \quad r \in \mathbb{R}, \tag{3.1.5}$$

the function $g : \mathbb{R} \to \mathbb{R}$ is differentiable, strictly increasing and invertible so that its inverse $g^{-1} : \mathbb{R} \to \mathbb{R}$ is differentiable, with

$$\sup_{r \in \mathbb{R}} (g^{-1})'(r) \leq \frac{1}{\gamma_0}. \tag{3.1.6}$$

Hypothesis 3.4. *The mapping $f : \mathscr{O} \times \mathbb{R} \to \mathbb{R}$ is measurable and there exists a positive constant L_f such that*

$$L_f < \frac{\alpha_1 \gamma_0}{\gamma_1}, \tag{3.1.7}$$

and

$$\sup_{x \in \mathscr{O}} |f(x,r) - f(x,s)| \leq L_f |r - s|, \quad r, s \in \mathbb{R}. \tag{3.1.8}$$

Moreover,

$$\sup_{x \in \mathscr{O}} |f(x,0)| < \infty.$$

In what follows, for every $x \in \mathscr{O}$ and $r \in \mathbb{R}$ we denote

$$\mathfrak{f}(x,r) := \int_0^r f(x,s)ds,$$

and for every function $h : \mathscr{O} \to \mathbb{R}$, we denote

$$F(h)(x) := f(x,h(x)), \qquad x \in \mathscr{O}.$$

Remark 3.5. 1. Condition (3.1.8) implies that $F : H \to H$ is Lipschitz continuous. Namely for any $h_1, h_2 \in H$

$$\|F(h_1) - F(h_2)\|_H \leq L_f \|h_1 - h_2\|_H .$$

Moreover, there exists $c > 0$ such that

$$\|F(h)\|_H \leq L_f \|h\|_H + c. \tag{3.1.9}$$

2. If the friction coefficient γ is constant, then $\gamma_0 = \gamma_1$, and condition (3.1.7) becomes

$$L_f < \alpha_1.$$

3. It is immediate to check that if for every $h \in H$ we define

$$\Lambda(h) := \int_{\mathscr{O}} f(x, h(x))\, dx,$$

then

$$[D\Lambda(h)](x) = f(x, h(x)), \quad x \in \mathscr{O}. \tag{3.1.10}$$

Hypothesis 3.6. *We assume*

$$L_f + \frac{L_\sigma}{2\gamma_0} < \frac{\alpha_1 \gamma_0}{\gamma_1}. \tag{3.1.11}$$

Remark 3.7. If σ is constant, then $L_\sigma = 0$ and Hypothesis 3.6 reduces to condition (3.1.7) in Hypothesis 3.4.

3.2 Main results

For every $\mu > 0$, we denote $v_\mu := \partial_t u_\mu$, and rewrite equation (3.0.1) as the following system

$$\begin{cases} du_\mu(t) = v_\mu(t)dt, \\ \\ \mu dv_\mu(t) = \big[Au_\mu(t) - \gamma(u_\mu(t))v_\mu(t) + F(u_\mu(t))\big]dt + \sigma(u_\mu(t))dw^Q(t), \\ \\ u_\mu(0) = \mathfrak{u}_0, \qquad v_\mu(0) = \mathfrak{v}_0, \end{cases} \tag{3.2.1}$$

where A is the realization in H of the Laplacian Δ, endowed with Dirichlet boundary conditions. In [22] it has been proven that, under Hypotheses 3.1, 3.3 and 3.4 (without condition (3.1.7)), for every $(u_0, v_0) \in L^2(\Omega; \mathscr{H}_1)$ and for every $\mu, T > 0$, there exists a unique solution $\zeta_\mu \in L^2(\Omega; C([0,T]; \mathscr{H}_1))$ which solves system (3.2.1), in the sense that

$$\begin{cases} u_\mu(t) = u_0 + \displaystyle\int_0^t v_\mu(s)ds, \\ \mu v_\mu(t) = \mu v_0 + \displaystyle\int_0^t \left[A u_\mu(s) - \gamma(u_\mu(s)) v_\mu(s) + F(u_\mu(s))\right] ds \\ \qquad + \displaystyle\int_0^t \sigma(u_\mu(s)) dw^Q(s). \end{cases} \tag{3.2.2}$$

In particular, we can introduce the transition semigroup $P_t^{\mu,\mathscr{H}_1}$ associated with equation (3.2.1) in $\mathscr{H}_1$, defined by

$$P_t^{\mu,\mathscr{H}_1} \varphi(\mathfrak{z}) = \mathbb{E}\, \varphi\big(z_\mu^{\mathfrak{z}}(t)\big), \quad \mathfrak{z} \in \mathscr{H}_1, \ \ t \geq 0,$$

for every $\varphi \in B_b(\mathscr{H}_1)$.

In what follows, we will need to study system (3.2.1) also in the space of lower regularity $\mathscr{H}$, and for this reason we introduce the following notion of *generalized solution*.

Definition 3.8. *For every $\mu, T > 0$ and every $(\mathfrak{u}_0, \mathfrak{v}_0) \in \mathscr{H}$, we say that the process $z_\mu \in L^2(\Omega; C([0,T]; \mathscr{H}))$ is a generalized solution of system (3.2.1) if there exists a sequence $\{\mathfrak{u}_{0,n}, \mathfrak{v}_{0,n}\}_{n\in\mathbb{N}} \subset \mathscr{H}_1$ converging to $(\mathfrak{u}_0, \mathfrak{v}_0)$ in $\mathscr{H}$, as $n \to +\infty$, such that*

$$\lim_{n\to+\infty} z_{\mu,n} = z_\mu \ \ in \ \ L^2(\Omega; C([0,T]; \mathscr{H})),$$

where $z_{\mu,n} \in L^2(\Omega; C([0,T]; \mathscr{H}_1))$ is the unique solution of equation (3.2.1) with initial conditions

$(\mathfrak{u}_{0,n}, \mathfrak{v}_{0,n})$.

Notice that if $(u_0, v_0) \in \mathscr{H}_1$, then the generalized solution coincides with the solution defined above in $\mathscr{H}_1$ in the sense of (3.2.2).

In Section 3.3 we will prove that, under Hypotheses 3.1, 3.3 and 3.4, for every $\mu, T > 0$ there exists a unique generalized solution $z_\mu \in L^2(\Omega; C([0,T]; \mathscr{H}))$ for system (3.2.1). This will allow us to introduce the transition semigroup associated with (3.2.1) in $\mathscr{H}$, which will denoted by $P_t^{\mu,\mathscr{H}}$. Clearly, if $\varphi \in B_b(\mathscr{H})$, for every $\mu > 0$ and $\mathfrak{z} \in \mathscr{H}_1$ we have

$$P_t^{\mu,\mathscr{H}_1} \varphi(\mathfrak{z}) = P_t^{\mu,\mathscr{H}} \varphi(\mathfrak{z}), \quad t \geq 0.$$

In Section 3.3, we will also show that under the same Hypotheses, for every $\mu > 0$, the semigroup $P_t^{\mu,\mathscr{H}}$ admits an invariant measure $\nu_\mu^{\mathscr{H}}$ in $\mathscr{H}$, with $\text{supp}(\nu_\mu^{\mathscr{H}}) \subset \mathscr{H}_1$. In particular, since $\text{supp}(\nu_\mu^{\mathscr{H}}) \subset \mathscr{H}_1$ and $\mathscr{B}(\mathscr{H}_1) \subset \mathscr{B}(\mathscr{H})$, we will have that $\nu_\mu^{\mathscr{H}}$ is also a probability measure on $\mathscr{H}_1$. In what follows, it will be convenient to denote the restriction of $\nu_\mu^{\mathscr{H}}$ to $\mathscr{H}_1$ by $\nu_\mu^{\mathscr{H}_1}$.

Given a lower semicontinuous metric α on H^{-1}, it is possible to introduce the distance $\mathscr{W}_\alpha : \mathscr{P}(H^{-1}) \times \mathscr{P}(H^{-1}) \to [0, +\infty]$ defined by

$$\mathscr{W}_\alpha(\nu_1, \nu_2) = \sup_{[\varphi]_{\text{Lip}^\alpha_{H^{-1}}} \leq 1} \left| \int_{H^{-1}} \varphi(\mathfrak{r}) \nu_1(d\mathfrak{r}) - \int_{H^{-1}} \varphi(\mathfrak{r}) \nu_2(d\mathfrak{r}) \right|,$$

where

$$[\varphi]_{\text{Lip}^\alpha_{H^{-1}}} = \sup_{\substack{\mathfrak{r}_1, \mathfrak{r}_2 \in H \\ \mathfrak{r}_1 \neq \mathfrak{r}_2}} \frac{|\varphi(\mathfrak{r}_1) - \varphi(\mathfrak{r}_2)|}{\alpha(\mathfrak{r}_1, \mathfrak{r}_2)}.$$

Notice that the following Kantorovich-Rubinstein identity holds

$$\mathscr{W}_\alpha(\nu_1,\nu_2) = \inf_{\lambda\in\mathscr{C}(\nu_1,\nu_2)} \int\int \alpha(\mathfrak{r}_1,\mathfrak{r}_2)\,\lambda(d\mathfrak{r}_1,d\mathfrak{r}_2), \tag{3.2.3}$$

where $\mathscr{C}(\nu_1,\nu_2)$ is the set of all couplings of (ν_1,ν_2). Notice that it is possible to prove that the infimum above is attained at some $\bar{\lambda}$.

The main result in this chapter is stated as follows.

Theorem 3.9. *Assume Hypotheses 3.1 to 3.6, and define*

$$\alpha(\mathfrak{u}_1,\mathfrak{u}_2) := \|\mathfrak{u}_1-\mathfrak{u}_2\|_{H^{-1}}, \qquad \mathfrak{u}_1,\mathfrak{u}_2\in H^{-1}.$$

Then we have

$$\lim_{\mu\to 0}\mathscr{W}_\alpha\left(\Pi_1\nu_\mu^{\mathscr{H}},\nu\right) = 0,$$

where ν is the unique invariant measure for P_t^H, the transition semigroup associated to the limiting equation (3.0.2)*. Moreover,*

$$\lim_{\mu\to 0}\Pi_1\nu_\mu^{\mathscr{H}} = \nu, \qquad \textit{weakly in } H.$$

3.3 Generalized solutions for second-order systems and invariant measures

We have seen in Section 3.2 that Eq. (3.0.1) is equivalent to system (3.2.1). In fact, we can give another equivalent formulation for (3.2.1). Actually, if g is the function introduced in (3.1.5) and we define

$$\eta := \frac{1}{\sqrt{\mu}}\big(\mu\partial_t u + g(u)\big), \quad \zeta = (u,\eta),$$

system (3.2.1) can be rewritten as

$$d\zeta_\mu(t) = \mathscr{A}_\mu(\zeta_\mu(t))dt + \Sigma_\mu(\zeta_\mu(t))dw^Q(t), \quad \zeta_\mu(0) = \Big(\mathfrak{u}_0, \sqrt{\mu}\mathfrak{v}_0 + \frac{g(\mathfrak{u}_0)}{\sqrt{\mu}}\Big), \tag{3.3.1}$$

where we denoted

$$\mathscr{A}_\mu(\zeta) := \frac{1}{\sqrt{\mu}}\Big(\eta - \frac{g(u)}{\sqrt{\mu}}, Au + F(u)\Big), \quad \zeta = (u,\eta) \in D(\mathscr{A}_\mu) = \mathscr{H}_1,$$

and

$$\Sigma_\mu(\zeta) := \frac{1}{\sqrt{\mu}}\big(0, \sigma(u)\big), \quad \zeta = (u,\eta) \in \mathscr{H}.$$

This means that, for every $\mu > 0$ and every $(\mathfrak{u}_0, \mathfrak{v}_0) \in \mathscr{H}_1$, the adapted $\mathscr{H}_1$-valued process $\zeta_\mu = (u_\mu, \eta_\mu)$ is the unique solution of equation (2.3.1), with $\zeta_\mu(0) = \big(\mathfrak{u}_0, \sqrt{\mu}\mathfrak{v}_0 + g(\mathfrak{u}_0)/\sqrt{\mu}\big)$, if and only if the adapted $\mathscr{H}_1$-valued process

$$z_\mu(t) := (u_\mu(t), v_\mu(t)) = (u_\mu(t), \eta_\mu(t)/\sqrt{\mu} - g(u_\mu(t))/\mu), \qquad t \geq 0,$$

is the unique solution of system (3.2.1), with $z_\mu(0) = \mathfrak{z}_0 := (\mathfrak{u}_0, \mathfrak{v}_0)$.

In [22] it has been proven that, under Hypotheses 3.1, 3.3 and 3.4, without condition (3.1.7), for every $\zeta^0 \in L^2(\Omega; \mathscr{H}_1)$ and for every $\mu, T > 0$, there exists a unique solution $\zeta_\mu \in L^2(\Omega; C([0,T]; \mathscr{H}_1))$ for equation (2.3.1), with $\zeta_\mu(0) = \zeta^0$. In particular, this implies that for every $(\mathfrak{u}_0, \mathfrak{v}_0) \in L^2(\Omega; \mathscr{H}_1)$, and for every $\mu, T > 0$, there exists a unique solution $z_\mu \in L^2(\Omega; C([0,T]; \mathscr{H}_1))$

to equation (3.2.1) with $z_\mu(0) = (\mathfrak{u}_0, \mathfrak{v}_0)$.

3.3.1 Generalized solutions for system (3.3.1)

In order to study the well-posedness of generalized solutions for system (3.2.1), we study the analogous problem for system (3.3.1). We start with the following definition of generalized solution for system (3.3.1).

Definition 3.10. *For every $\mu, T > 0$ and every $\zeta^0 \in \mathscr{H}$, we say that $\zeta_\mu \in L^2(\Omega; C([0,T]; \mathscr{H}))$ is a generalized solution of problem (3.3.1), with initial condition ζ^0, if there exists a sequence $\{\zeta_n^0\}_{n\in\mathbb{N}} \subset \mathscr{H}_1$ converging to ζ^0 in $\mathscr{H}$, as $n \to +\infty$, such that*

$$\lim_{n\to+\infty} \zeta_{\mu,n} = \zeta_\mu \ \text{ in } \ L^2(\Omega; C([0,T]; \mathscr{H})),$$

where $\zeta_{\mu,n} \in L^2(\Omega; C([0,T]; \mathscr{H}_1))$ is the unique solution of equation (3.3.1) with initial condition ζ_n^0.

The existence and uniqueness of generalized solutions of system (3.3.1) in the space $\mathscr{H}$ are given as follows.

Lemma 3.11. *Under Hypotheses 3.1, 3.3 and 3.4, for every $\mu, T > 0$ and every $\zeta^0 \in \mathscr{H}$, there exists a unique generalized solution $\zeta_\mu \in L^2(\Omega; C([0,T]; \mathscr{H}))$ for equation (3.3.1), and such solution coincides with the classical solution when $\zeta^0 \in \mathscr{H}_1$. Moreover, if ζ_μ^1, ζ_μ^2 are two generalized solutions of (3.3.1), with initial conditions $\zeta^1, \zeta^2 \in \mathscr{H}$, respectively, then*

$$\mathbb{E} \sup_{t\in[0,T]} \|\zeta_\mu^1(t) - \zeta_\mu^2(t)\|_{\mathscr{H}}^2 \le e^{c_\mu T} \|\zeta^1 - \zeta_2\|_{\mathscr{H}}^2, \tag{3.3.2}$$

for some constant c_μ.

Proof. Without any loss of generality, we assume $\mu = 1$, and for simplicity of notation, we denote $\mathscr{A}_1$ and Σ_1 by $\mathscr{A}$ and Σ, respectively. In [22], it is proved that the operator $\mathscr{A}$ is quasi-m-dissipative in $\mathscr{H}$. Namely, there exists $\omega \geq 0$ such that for every $\zeta, \theta \in D(\mathscr{A})$

$$\big\langle \mathscr{A}(\zeta) - \mathscr{A}(\theta), \zeta - \theta \big\rangle_{\mathscr{H}} \leq \omega \, \|\zeta - \theta\|_{\mathscr{H}}^2, \tag{3.3.3}$$

and there exists $\lambda_0 > 0$ such that

$$\mathrm{Range}(I - \lambda \mathscr{A}) = \mathscr{H}, \quad \lambda \in (0, \lambda_0).$$

Now, let $\zeta^0 \in \mathscr{H}$ and let $\{\zeta_n^0\}_{n \in \mathbb{N}} \subset \mathscr{H}_1$ be any sequence converging to ζ^0 in $\mathscr{H}$. For every $n \in \mathbb{N}$, we denote by ζ_n the unique solution of equation (3.3.1) with initial condition $\zeta_n(0) = \zeta_n^0$. By applying Itô's formula, thanks to (3.1.1) and (3.3.3), we get

$$\begin{aligned}
&\frac{1}{2} d \, \|\zeta_n(t) - \zeta_m(t)\|_{\mathscr{H}}^2 \\
&\quad = \big\langle \mathscr{A}(\zeta_n(t)) - \mathscr{A}(\zeta_m(t)), \zeta_n(t) - \zeta_m(t) \big\rangle_{\mathscr{H}} dt + \frac{1}{2} \, \|\Sigma(\zeta_n(t)) - \Sigma(\zeta_m(t))\|_{\mathscr{L}_2(H_Q, \mathscr{H})}^2 \, dt \\
&\qquad\qquad + \big\langle \zeta_n(t) - \zeta_m(t), \big[\Sigma(\zeta_n(t)) - \Sigma(\zeta_m(t))\big] dw^Q(t) \big\rangle_{\mathscr{H}} \\
&\quad \leq c \, \|\zeta_n(t) - \zeta_m(t)\|_{\mathscr{H}}^2 \, dt + \big\langle \zeta_n(t) - \zeta_m(t), \big[\Sigma(\zeta_n(t)) - \Sigma(\zeta_m(t))\big] dw^Q(t) \big\rangle_{\mathscr{H}}.
\end{aligned} \tag{3.3.4}$$

Due to (3.1.3) we have

$$\mathbb{E} \sup_{s\in[0,t]} \left| \int_0^s \left\langle \zeta_n(r) - \zeta_m(r), \left[\Sigma(\zeta_n(r)) - \Sigma(\zeta_m(r))\right] dw^Q(r) \right\rangle_{\mathscr{H}} \right|$$

$$\leq c\,\mathbb{E} \left(\int_0^t \|\zeta_n(s) - \zeta_m(s)\|_{\mathscr{H}}^4 \, ds \right)^{\frac{1}{2}} \leq \frac{1}{4} \mathbb{E} \sup_{s\in[0,t]} \|\zeta_n(s) - \zeta_m(s)\|_{\mathscr{H}}^2 + c \int_0^t \mathbb{E}\|\zeta_n(s) - \zeta_m(s)\|_{\mathscr{H}}^2 \, ds.$$

Thus, if we first integrate both sides in (3.3.4) with respect to time, and then take the supremum and the expectation, we get

$$\mathbb{E} \sup_{s\in[0,t]} \|\zeta_n(s) - \zeta_m(s)\|_{\mathscr{H}}^2 \leq \left\|\zeta_n^0 - \zeta_m^0\right\|_{\mathscr{H}}^2 + c \int_0^t \mathbb{E}\,\|\zeta_n(s) - \zeta_m(s)\|_{\mathscr{H}}^2 \, ds,$$

and the Gronwall's lemma gives

$$\mathbb{E} \sup_{s\in[0,t]} \|\zeta_n(s) - \zeta_m(s)\|_{\mathscr{H}}^2 \leq e^{ct} \left\|\zeta_n^0 - \zeta_m^0\right\|_{\mathscr{H}}^2, \qquad t \geq 0,$$

for some constant c. In particular, this implies that the sequence $\{\zeta_n\}_{n\in\mathbb{N}}$ is Cauchy in the space $L^2(\Omega; C([0,T];\mathscr{H}))$, so there exists a limit $\zeta \in L^2(\Omega; C([0,T];\mathscr{H}))$. It is easy to see that the limit ζ does not depend on the choice of the sequence $\{\zeta_n^0\} \subset \mathscr{H}_1$, which implies the uniqueness of generalized solutions. Finally, by using a similar argument as above, we obtain (3.3.2). □

Remark 3.12. When $\zeta^0 \in \mathscr{H}_1$, the unique generalized solution ζ_μ of equation (3.3.1) coincides with its unique classical solution.

3.3.2 Generalized solutions for system (3.2.1)

As we have seen above, due to Hypothesis 3.3, $z_\mu = (u_\mu, v_\mu)$ is a generalized solution to (3.2.1) with initial condition $\mathfrak{z}_0 = (\mathfrak{u}_0, \mathfrak{v}_0)$ if and only if $\zeta_\mu = \left(u_\mu, \sqrt{\mu} v_\mu + g(u_\mu)/\sqrt{\mu}\right)$ is a generalized solution for system (3.3.1) with initial condition $\zeta_\mu(0) = (\mathfrak{u}_0, \sqrt{\mu}\mathfrak{v}_0 + g(\mathfrak{u}_0)/\sqrt{\mu})$. In this case, we have

$$u_\mu = \Pi_1 \zeta_\mu, \quad v_\mu = \frac{1}{\mu}\Big(-g(u_\mu) + \sqrt{\mu}\,\Pi_2 \zeta_\mu\Big), \quad \mu > 0. \tag{3.3.5}$$

Thus, as a consequence of Lemma 3.11 and Remark 3.12, we have the following result.

Proposition 3.13. *Fix* $(\mathfrak{u}_0, \mathfrak{v}_0) \in \mathscr{H}$ *and assume Hypotheses 3.1, 3.3 and 3.4. Then, for every* $\mu, T > 0$ *there exists a unique generalized solution* $z_\mu \in L^2(\Omega; C([0,T]; \mathscr{H}))$ *for system (3.2.1). In particular, when* $(\mathfrak{u}_0, \mathfrak{v}_0) \in \mathscr{H}_1$*, the unique generalized solution coincides with the unique classical solution.*

3.3.3 Invariant measures for system (3.2.1)

Now, we shall prove that, for every fixed $\mu > 0$, system (3.2.1) admits an invariant measure $\nu_\mu^{\mathscr{H}}$ in $\mathscr{H}$, which is supported in $\mathscr{H}_1$.

Proposition 3.14. *Assume Hypotheses 3.1, 3.3 and 3.4. Then, for every* $\mu > 0$*, the semigroup* $P_t^{\mu,\mathscr{H}}$ *admits an invariant measure* $\nu_\mu^{\mathscr{H}}$ *in* $\mathscr{H}$*, with* $\operatorname{supp}(\nu_\mu^{\mathscr{H}}) \subset \mathscr{H}_1$.

Proof. First, if $z_\mu^{\mathfrak{z}_1}$ and $z_\mu^{\mathfrak{z}_2}$ are generalized solutions to system (3.2.1), with initial conditions

$\mathfrak{z}_1, \mathfrak{z}_2 \in \mathscr{H}$, respectively, then due to (3.3.2) and (3.3.5), it is easy to see that for every $t \geq 0$

$$\mathbb{E}\left\|z_\mu^{\mathfrak{z}_1}(t) - z_\mu^{\mathfrak{z}_2}(t))\right\|_{\mathscr{H}} \leq c_\mu(t)\,\|\mathfrak{z}_1 - \mathfrak{z}_2\|_{\mathscr{H}},$$

for some $c_\mu(t) > 0$. This means that the transition semigroup $P_t^{\mu,\mathscr{H}}$ is Feller on $\mathscr{H}$.

Now, for every $\mathfrak{z} \in \mathscr{H}$ we introduce the following family of measures on $\mathscr{H}$

$$\Gamma_t^\mu(\mathfrak{z},\cdot) := \frac{1}{t}\int_0^t (P_t^{\mu,\mathscr{H}})^\star \delta_{\mathfrak{z}}\,dt, \qquad t > 0,$$

and for every $R > 0$ we define the set

$$B_R := \Big\{\mathfrak{z} \in \mathscr{H}_1 : \|\mathfrak{z}\|_{\mathscr{H}_1} \leq R\Big\}.$$

Then, from (3.4.1) and (3.4.2) with $\mu = 1$, we have

$$\Gamma_t^\mu(0, B_R^c) = \frac{1}{t}\int_0^t \mathbb{P}\big(\left\|z_\mu^0(s)\right\|_{\mathscr{H}_1} > R\big)ds \leq \frac{c}{R^2}, \quad t > 0, \qquad R > 0,$$

and, due to the compactness of the embedding of $\mathscr{H}_1$ into $\mathscr{H}$, this implies that the family of measures $\{\Gamma_t^\mu(0,\cdot)\}_{t\geq 0}$ is tight in $\mathscr{H}$. By the Prokhorov theorem, there exists some sequence $t_n \uparrow \infty$ such that $\Gamma_{t_n}^\mu(0,\cdot)$ converges weakly, as $n \to +\infty$, to a probability measure $\nu_\mu^{\mathscr{H}}$ that is invariant for $P_t^{\mu,\mathscr{H}}$. Moreover, since

$$\nu_\mu^{\mathscr{H}}(B_R^c) \leq \frac{c}{R^2}, \quad R > 0,$$

it follows that $\operatorname{supp}(\nu_\mu^{\mathscr{H}}) \subset \mathscr{H}_1$.

$\square$

Remark 3.15. Since $\operatorname{supp}(\nu_\mu^{\mathscr{H}}) \subset \mathscr{H}_1$ and $\mathscr{B}(\mathscr{H}_1) \subset \mathscr{B}(\mathscr{H})$, we have that $\nu_\mu^{\mathscr{H}}$ is also a probability measure on $\mathscr{H}_1$. In what follows, it will be convenient to denote the restriction of $\nu_\mu^{\mathscr{H}}$ to $\mathscr{H}_1$ by $\nu_\mu^{\mathscr{H}_1}$.

3.4 Some uniform bounds

In the previous section, we have seen that for every $\mu > 0$, system (3.2.1) has an invariant measure. In this section, we will prove some uniform bounds for the moments of such family of invariant measures. To this purpose, we need to start with suitable uniform bounds for the solution (u_μ, v_μ) of system (3.2.1). Some of them have been already proved in [22, Proposition 4.2, Remark 4.3]. In what follows, we show how those bounds depend on time and on random initial conditions in $L^2(\Omega;\mathscr{H}_1)$.

Lemma 3.16. *Assume Hypotheses 3.1, 3.3 and 3.4, and fix $(\xi,\eta) \in L^2(\Omega;\mathscr{H}_1)$. For every $\mu, T > 0$, let $(u_\mu, v_\mu) \in L^2(\Omega; C([0,T];\mathscr{H}_1))$ be the unique solution to system (3.2.1) with initial conditions (ξ,η). Then there exist two constants $\mu_0 \in (0,1)$ and $c > 0$, independent of $T > 0$, such that for every $\mu \in (0,\mu_0)$ and $t \in [0,T]$*

$$\begin{aligned} &\mathbb{E}\sup_{s\in[0,t]} \left\|u_\mu(s)\right\|_{H^1}^2 + \mu\,\mathbb{E}\sup_{s\in[0,t]} \left\|v_\mu(s)\right\|_H^2 + \int_0^t \mathbb{E}\left\|v_\mu(s)\right\|_H^2 ds \\ &\qquad \leq c\left(\frac{t}{\mu}+1\right) + c\left(\mathbb{E}\|\xi\|_{H^1}^2 + \mu\,\mathbb{E}\|\eta\|_H^2\right), \end{aligned} \tag{3.4.1}$$

and

$$\mathbb{E}\sup_{s\in[0,t]}\left\|u_\mu(s)\right\|_H^2+\int_0^t\mathbb{E}\left\|u_\mu(s)\right\|_{H^1}^2ds\le c\left(1+t+\mathbb{E}\left\|\xi\right\|_H^2+\mu\,\mathbb{E}\left\|\xi\right\|_{H^1}^2+\mu^2\,\mathbb{E}\left\|\eta\right\|_H^2\right).\tag{3.4.2}$$

Proof. Let $(\mathfrak{u}_0,\mathfrak{v}_0)\in L^2(\Omega;\mathscr{H}_1)$ and let $(u_\mu,v_\mu)\in L^2(\Omega;C([0,T];\mathscr{H}_1))$ be the unique solution to system (3.2.1). By proceeding as in the proof of [22, Lemma 4.2], we have for every $\mu\in(0,1)$

$$\begin{aligned}&\mathbb{E}\sup_{s\in[0,t]}\left\|u_\mu(s)\right\|_{H^1}^2+\mu\,\mathbb{E}\sup_{s\in[0,t]}\left\|v_\mu(s)\right\|_H^2+\int_0^t\mathbb{E}\left\|v_\mu(s)\right\|_H^2ds\\&\qquad\le c\left(\frac{t}{\mu}+\left(1+\mathbb{E}\left\|\xi\right\|_{H^1}^2+\mu\mathbb{E}\left\|\eta\right\|_H^2\right)+\int_0^t\mathbb{E}\left\|u_\mu(s)\right\|_H^2ds\right).\end{aligned}\tag{3.4.3}$$

Moreover, by proceeding as in the proof of [22, Lemma 4.1], we have $\mathbb{P}$-a.s.

$$\frac{\gamma_0}{4}\left\|u_\mu(t)\right\|_H^2\le c\left(\left\|\xi\right\|_H^2+\mu^2\left\|\eta\right\|_H^2\right)+c\,\mu^2\left\|v_\mu(t)\right\|_H^2+\mu\int_0^t\left\|v_\mu(s)\right\|_H^2ds$$

$$-\int_0^t\left\|u_\mu(s)\right\|_{H^1}^2ds+\int_0^t\left\langle F(u_\mu(s)),u_\mu(s)\right\rangle_Hds+\int_0^t\left\langle u_\mu(s),\sigma(u_\mu(s))dw^Q(s)\right\rangle_H.$$

Due to (3.1.9), for every $\delta>0$ we have

$$\mathbb{E}\sup_{s\in[0,t]}\left|\int_0^s\left\langle F(u_\mu(r)),u_\mu(r)\right\rangle_Hdr\right|\le\frac{1}{\alpha_1}(L_f+\delta)\int_0^t\mathbb{E}\left\|u_\mu(s)\right\|_{H^1}^2ds+c_\delta t.$$

Moreover, due to (3.1.2) we have

$$
\begin{aligned}
&\mathbb{E}\sup_{s\in[0,t]}\left|\int_0^s \left\langle u_\mu(r),\sigma(u_\mu(r))dw^Q(r)\right\rangle_H\right| \\
&\qquad \leq c\left(\mathbb{E}\int_0^t \|u_\mu(s)\|_H^2\,ds\right)^{\frac{1}{2}} \leq \frac{\delta}{\alpha_1}\int_0^t \mathbb{E}\left\|u_\mu(t)\right\|_{H^1}^2 dt + c_\delta.
\end{aligned}
$$

According to (3.1.7), we can fix $\delta > 0$ such that

$$
\frac{1}{\alpha_1}\left(L_f + 2\,\delta\right) < 1,
$$

and this yields

$$
\begin{aligned}
&\mathbb{E}\sup_{s\in[0,t]}\left\|u_\mu(s)\right\|_H^2 + \int_0^t \mathbb{E}\left\|u_\mu(s)\right\|_{H^1}^2 ds \\
&\qquad \leq c\left(1+t+\mathbb{E}\|\xi\|_H^2+\mu^2\mathbb{E}\|\eta\|_H^2\right) + c\,\mu^2\,\mathbb{E}\sup_{s\in[0,t]}\left\|v_\mu(s)\right\|_H^2 + c\,\mu\int_0^t \mathbb{E}\left\|v_\mu(s)\right\|_H^2 ds.
\end{aligned}
\tag{3.4.4}
$$

Thus (3.4.2) holds by combining (3.4.4) with (3.4.3). Finally, by combining (3.4.2) with (3.4.3), we complete the proof of (3.4.1).

□

Lemma 3.17. *Let $\{(\xi_\mu,\eta_\mu)\}_{\mu\in(0,1)} \subset L^2(\Omega;\mathscr{H}_1)$ be a family of random variables such that*

$$
\sup_{\mu\in(0,1)}\mathbb{E}\Big(\left\|\xi_\mu\right\|_{H^1}^2 + \mu\left\|\eta_\mu\right\|_H^2\Big) < \infty. \tag{3.4.5}
$$

If $(u_\mu,v_\mu) \in L^2(\Omega;C([0,T];\mathscr{H}_1))$ is the solution to system (3.2.1) with initial condition (ξ_μ,η_μ), *then there exist $\mu_T \in (0,\mu_0)$ and $c_T > 0$ such that for every $\mu \in (0,\mu_T)$*

$$\mathbb{E} \sup_{t\in[0,T]} \left(\|u_\mu(t)\|_{H^1}^2 + \mu \|v_\mu(t)\|_H^2 \right) \leq \frac{c_T}{\sqrt{\mu}} + \left(\mathbb{E}\|\xi_\mu\|_{H^1}^2 + \mu\,\mathbb{E}\|\eta_\mu\|_H^2 \right). \tag{3.4.6}$$

Proof. If for every $\mu \in (0,\mu_0)$ and $t \in [0,T]$, we define

$$L_\mu(t) := \|u_\mu(t)\|_{H^1}^2 + \mu \|v_\mu(t)\|_H^2 - \left(\|\xi_\mu\|_{H^1}^2 + \mu \|\eta_\mu\|_H^2 \right),$$

then (3.4.6) is equivalent to

$$\sqrt{\mu}\,\mathbb{E} \sup_{t\in[0,T]} L_\mu(t) \leq c_T, \qquad \mu \in (0,\mu_T), \tag{3.4.7}$$

for some constants $\mu_T \in (0,1)$ and $c_T > 0$.

Now, if we assume (3.4.7) is not true, there exists a sequence $(\mu_k)_{k\in\mathbb{N}} \subset (0,\mu_0)$ converging to 0, as $k \to \infty$, such that

$$\lim_{k\to\infty} \sqrt{\mu_k}\,\mathbb{E} \sup_{t\in[0,T]} L_{\mu_k}(t) = +\infty. \tag{3.4.8}$$

For every $k \in \mathbb{N}$, the mapping $t \mapsto L_{\mu_k}(t)$ is continuous $\mathbb{P}$-a.s., so that there exists a random time $t_k \in [0,T]$ such that

$$L_{\mu_k}(t_k) = \sup_{t\in[0,T]} L_{\mu_k}(t).$$

As a consequence of Itô's formula, we have

$$\frac{1}{2}d\left(\|u_\mu(t)\|_{H^1}^2+\mu\,\|_\mu(t)\|_H^2\right)$$

$$=\left(\langle F(u_\mu(t),v_\mu(t)\rangle_H-\langle\gamma(u_\mu(t))v_\mu(t),v_\mu(t)\rangle_H+\frac{1}{2\mu}\|\sigma(u_\mu(t))\|_{\mathscr{L}_2(H_Q,H)}^2\right)dt$$

$$+\langle v_\mu(t),\sigma(u_\mu(t))dw^Q(t)\rangle_H$$

$$\leq\left(c\left(\|u_\mu(t)\|_H^2+1\right)-\frac{\gamma_0}{2}\|v_\mu(t)\|_H^2+\frac{\sigma_\infty^2}{2\mu}\right)dt+\langle v_\mu(t),\sigma(u_\mu(t))dw^Q(t)\rangle_H.$$

Hence, if s is any random time such that $\mathbb{P}(s\leq t_k)=1$, we have

$$L_{\mu_k}(t_k)-L_{\mu_k}(s)\leq\frac{\sigma_\infty^2}{\mu_k}(t_k-s)+c\int_s^{t_k}\left(1+\left\|u_{\mu_k}(r)\right\|_H^2\right)dr+2\big(M_k(t_k)-M_k(s)\big),$$

where

$$M_k(t):=\int_0^t\left\langle v_{\mu_k}(r),\sigma(u_{\mu_k}(r))dw^Q(r)\right\rangle_H.$$

If we define

$$U_k:=c\int_0^T\left\|u_{\mu_k}(t)\right\|_H^2dt,\qquad M_k:=\sup_{t\in[0,T]}|M_k(t)|,$$

this implies that there exists some constant $\lambda>0$, independent of k, such that

$$L_{\mu_k}(t_k)-L_{\mu_k}(s)\leq\frac{\lambda}{\mu_k}(t_k-s)+U_k+4M_k,$$

and since $L_{\mu_k}(s) = 0$, if we take $s = 0$ we get

$$t_k \geq \frac{\mu_k}{\lambda}\Big(L_{\mu_k}(t_k) - U_k - 4M_k\Big) =: \frac{\mu_k \theta_k}{\lambda}.$$

Now, on the set $E_k := \{\theta_k > 0\}$, we fix an arbitrary $s \in [t_k - \mu_k\theta_k/(2\lambda), t_k]$ and we have

$$L_{\mu_k}(s) \geq L_{\mu_k}(t_k) - \frac{1}{2}\theta_k - U_k - 4M_k = \frac{1}{2}\theta_k > 0. \tag{3.4.9}$$

Hence, if we define

$$I_k := \int_0^T L^+_{\mu_k}(s)\, ds,$$

due to (3.4.9) we have

$$I_k \geq \int_{t_k - \frac{\mu_k \theta_k}{2\lambda}}^{t_k} L_{\mu_k}(s) ds \geq \frac{\mu_k}{4\lambda}\, \theta_k^2, \qquad \text{on } E_k,$$

so that

$$\mathbb{E}(I_k; E_k) \geq \mathbb{E}\left(\frac{\mu_k}{4\lambda}\, \theta_k^2; E_k\right). \tag{3.4.10}$$

Now, according to (3.4.1), (3.4.2) and (3.4.5)

$$\mathbb{E}\, U_k \leq c\Big(1 + T + \mathbb{E}\, \|\xi_{\mu_k}\|_H^2 + \mu_k\, \mathbb{E}\, \|\xi_{\mu_k}\|_{H^1}^2 + \mu_k^2\, \mathbb{E}\, \|\eta_{\mu_k}\|_H^2\Big) \leq c_T,$$

and

$$\mathbb{E} M_k \leq c\Big(\int_0^T \mathbb{E}\, \|v_{\mu_k}(t)\|_H^2\, dt\Big)^{\frac{1}{2}} \leq c\left(1 + \frac{T}{\mu_k} + \mathbb{E}\, \|\xi_{\mu_k}\|_{H^1}^2 + \mu_k\, \mathbb{E}\, \|\eta_{\mu_k}\|_H^2\right)^{\frac{1}{2}} \leq c_T \left(1 + \frac{1}{\mu_k}\right)^{\frac{1}{2}},$$

so that

$$\limsup_{k\to\infty} \sqrt{\mu_k}\,(\mathbb{E}\,U_k + 4\,\mathbb{E}\,M_k) < +\infty.$$

Thanks to (3.4.8) this gives

$$\lim_{k\to\infty} \sqrt{\mu_k}\,\mathbb{E}(\theta_k) = +\infty,$$

and hence

$$\lim_{k\to\infty} \sqrt{\mu_k}\,\mathbb{E}(\theta_k; E_k) = +\infty. \tag{3.4.11}$$

Now, according to (3.4.10), we have

$$\mathbb{E}(I_k; E_k) \geq \frac{\mu_k}{4\lambda}\mathbb{E}\left(\theta_k^2; E_k\right) \geq \frac{\mu_k}{4\lambda}\left(\mathbb{E}(\theta_k; E_k)\right)^2,$$

and due to (3.4.11), this implies

$$\lim_{k\to\infty} \mathbb{E}(I_k; E_k) = +\infty.$$

However, as a consequence of (3.4.1), (3.4.2) and (3.4.5), we have

$$\sup_{k\in\mathbb{N}} \mathbb{E}\,I_k \leq \sup_{k\in\mathbb{N}} \int_0^T \mathbb{E}\,|L_{\mu_k}(s)|ds \leq c_T \sup_{k\in\mathbb{N}} \left(1 + T + \mathbb{E}\left\|\xi_{\mu_k}\right\|_{H^1}^2 + \mu_k\mathbb{E}\left\|\eta_{\mu_k}\right\|_H^2\right) < +\infty,$$

and this gives a contradiction, since $\mathbb{E}(I_k; E_k) \leq \mathbb{E}(I_k)$ for every $k \in \mathbb{N}$. In particular, this means that claim (3.4.7) is true, and (3.4.6) holds.

□

Lemma 3.18. *For every $\mu > 0$, if $\nu_\mu^{\mathscr{H}} \in \mathscr{P}(\mathscr{H})$ is any invariant measure for $P_t^{\mu,\mathscr{H}}$ supported in*

$\mathscr{H}_1$*, then* $\nu_\mu^{\mathscr{H}_1} \in \mathscr{P}(\mathscr{H}_1)$ *is invariant for* $P_t^{\mu,\mathscr{H}_1}$*. Moreover,*

$$\sup_{\mu\in(0,1)} \int_{\mathscr{H}_1} \left(\|\mathfrak{u}\|_{H^1}^2 + \mu \|\mathfrak{v}\|_H^2 \right) \nu_\mu^{\mathscr{H}_1}(d\mathfrak{u}, d\mathfrak{v}) < \infty. \tag{3.4.12}$$

Proof. First, we show the invariance of $\nu_\mu^{\mathscr{H}_1}$ for $P_t^{\mu,\mathscr{H}_1}$. Due to the invariance of $\nu_\mu^{\mathscr{H}}$ in $\mathscr{H}$, for every $\varphi \in C_b(\mathscr{H})$ we have

$$\int_{\mathscr{H}} P_t^{\mu,\mathscr{H}} \varphi(\mathfrak{z}) \nu_\mu^{\mathscr{H}}(d\mathfrak{z}) = \int_{\mathscr{H}} \varphi(\mathfrak{z}) \nu_\mu^{\mathscr{H}}(d\mathfrak{z}).$$

Thus, since $\operatorname{supp}(\nu_\mu^{\mathscr{H}}) \subset \mathscr{H}_1$ and $\mathscr{B}(\mathscr{H}_1) \subset \mathscr{B}(\mathscr{H})$, for every $\varphi \in C_b(\mathscr{H})$ we get

$$\int_{\mathscr{H}_1} P_t^{\mu,\mathscr{H}} \varphi(\mathfrak{z}) \nu_\mu^{\mathscr{H}_1}(d\mathfrak{z}) = \int_{\mathscr{H}_1} \varphi(\mathfrak{z}) \nu_\mu^{\mathscr{H}_1}(d\mathfrak{z}).$$

If $(\hat{e}_i)_{i\in\mathbb{N}} \subset \mathscr{H}_1$ is an orthonormal basis of $\mathscr{H}$, for every $n \in \mathbb{N}$ we denote by Π_n the projection of $\mathscr{H}$ onto $\mathscr{H}(n) := \operatorname{span}(\hat{e}_1, \dots, \hat{e}_n)$. We have that $\Pi_n : \mathscr{H} \to \mathscr{H}_1$ is continuous and

$$\left|\Pi_n h\right|_{\mathscr{H}_1} \leq c_n |h|_{\mathscr{H}}, \quad h \in \mathscr{H}, \qquad \lim_{n\to\infty} \left|\Pi_n h - h\right|_{\mathscr{H}_1} = 0, \quad h \in \mathscr{H}_1.$$

Hence, if for any $\varphi \in C_b(\mathscr{H}_1)$ and $n \in \mathbb{N}$, we define $\varphi_n := \varphi \circ \Pi_n$, we have $\varphi_n \in C_b(\mathscr{H})$ and

$$\lim_{n\to\infty} |\varphi_n(h) - \varphi(h)| = 0, \quad h \in \mathscr{H}_1.$$

For every $n \in \mathbb{N}$, we have $\sup_{n\in\mathbb{N}} \|\varphi_n\|_\infty \leq \|\varphi\|_\infty$, and the dominated convergence theorem implies that for any given $\mu > 0$ and $\varphi \in C_b(\mathscr{H}_1)$

$$\lim_{n\to\infty} P_t^{\mu,\mathscr{H}} \varphi_n(\mathfrak{z}) = \lim_{n\to\infty} \mathbb{E}\, \varphi_n\big(z_\mu^{\mathfrak{z}}(t)\big) = \mathbb{E}\, \varphi\big(z_\mu^{\mathfrak{z}}(t)\big) = P_t^{\mu,\mathscr{H}_1} \varphi(\mathfrak{z}), \quad \mathfrak{z} \in \mathscr{H}_1, \qquad t \geq 0.$$

In particular, by taking the limit as n goes to infinity in both sides of

$$\int_{\mathscr{H}_1} P_t^{\mu,\mathscr{H}} \varphi_n(\mathfrak{z}) \nu_\mu^{\mathscr{H}_1}(d\mathfrak{z}) = \int_{\mathscr{H}_1} \varphi_n(\mathfrak{z}) \nu_\mu^{\mathscr{H}_1}(d\mathfrak{z}), \quad \varphi \in C_b(\mathscr{H}_1),$$

we conclude that

$$\int_{\mathscr{H}_1} P_t^{\mu,\mathscr{H}_1} \varphi(\mathfrak{z}) \nu_\mu^{\mathscr{H}_1}(d\mathfrak{z}) = \int_{\mathscr{H}_1} \varphi(\mathfrak{z}) \nu_\mu^{\mathscr{H}_1}(d\mathfrak{z}), \qquad \varphi \in C_b(\mathscr{H}_1),$$

and this implies the invariance of $\nu_\mu^{\mathscr{H}_1}$.

Next, in order to prove (3.4.12), we consider the Komolgorov operator associated to $P_t^{\mu,\mathscr{H}_1}$ in $\mathscr{H}_1$

$$\mathscr{N}_\mu \varphi(\mathfrak{u},\mathfrak{v}) \;\; = \frac{1}{2\mu^2} \mathrm{Tr}_H \Big[\big(\sigma(\mathfrak{u})Q\big)(\sigma(\mathfrak{u})Q)^* D_{\mathfrak{v}}^2 \varphi(\mathfrak{u},\mathfrak{v}) \Big] + \big\langle \mathfrak{v}, D_{\mathfrak{u}} \varphi(\mathfrak{u},\mathfrak{v}) \big\rangle_{H^1}$$

$$+ \frac{1}{\mu} \big\langle A\mathfrak{u} - \gamma(\mathfrak{u}) v + F(\mathfrak{u}), D_{\mathfrak{v}} \varphi(\mathfrak{u},\mathfrak{v}) \big\rangle_H.$$

If, with the notations of Section 3.1, we define

$$\varphi_\mu(\mathfrak{u},\mathfrak{v}) := \frac{1}{2}\Big(\|\mathfrak{u}\|_{H^1}^2 + \mu\, \|\mathfrak{v}\|_H^2 \Big) - \int_{\mathscr{O}} \mathfrak{f}(x,\mathfrak{u}(x))dx = \frac{1}{2}\Big(\|\mathfrak{u}\|_{H^1}^2 + \mu\, \|\mathfrak{v}\|_H^2 \Big) - \Lambda(\mathfrak{u}),$$

due to (3.1.10) we have

$$D_{\mathfrak{u}}\varphi_\mu(\mathfrak{u},\mathfrak{v}) = (-A)\mathfrak{u} - f(\cdot,\mathfrak{u}), \quad D_{\mathfrak{v}}\varphi_\mu(\mathfrak{u},\mathfrak{v}) = \mu\,\mathfrak{v}, \quad D^2_{\mathfrak{v}}\varphi_\mu(\mathfrak{u},\mathfrak{v}) = \mu\, I_H.$$

Then, we have

$$\begin{aligned}
&\mathscr{N}_\mu\varphi_\mu(\mathfrak{u},\mathfrak{v})\\
&= \frac{1}{2\mu}\mathrm{Tr}_H\Big[\big(\sigma(\mathfrak{u})Q\big)\big(\sigma(\mathfrak{u})Q\big)^*\Big] + \big\langle \mathfrak{v}, \mathfrak{u} - (-A)^{-1}F(\mathfrak{u})\big\rangle_{H^1} + \frac{1}{\mu}\big\langle A\mathfrak{u} - \gamma(\mathfrak{u})\mathfrak{v} + F(\mathfrak{u}), \mu\mathfrak{v}\big\rangle_H\\
&= \frac{1}{2\mu}\,\|\sigma(\mathfrak{u})\|^2_{\mathscr{L}_2(H_Q,H)} + \big\langle \mathfrak{v}, -A\mathfrak{u} - F(\mathfrak{u})\big\rangle_H + \frac{1}{\mu}\big\langle A\mathfrak{u} - \gamma(\mathfrak{u})\mathfrak{v} + F(\mathfrak{u}), \mu\mathfrak{v}\big\rangle_H\\
&= \frac{1}{2\mu}\,\|\sigma(\mathfrak{u})\|^2_{\mathscr{L}_2(H_Q,H)} - \big\langle \gamma(\mathfrak{u})\mathfrak{v}, \mathfrak{v}\big\rangle_H \leq \frac{1}{2\mu}\,\|\sigma(\mathfrak{u})\|^2_{\mathscr{L}_2(H_Q,H)} - \gamma_0\,\|\mathfrak{v}\|^2_H.
\end{aligned} \tag{3.4.13}$$

By the invariance of $\nu_\mu^{\mathscr{H}_1}$ in $\mathscr{H}_1$, we have

$$\int_{\mathscr{H}_1} \mathscr{N}_\mu\varphi_\mu(\mathfrak{u},\mathfrak{v})\,\nu_\mu^{\mathscr{H}_1}(d\mathfrak{u},d\mathfrak{v}) = 0,$$

and thus, due to (3.4.13) and (3.1.2),

$$\sup_{\mu\in(0,1)} \mu\int_{\mathscr{H}_1} \|\mathfrak{v}\|^2_H\,\nu_\mu^{\mathscr{H}_1}(d\mathfrak{u},d\mathfrak{v}) < \infty. \tag{3.4.14}$$

Next, we consider the function

$$\psi_\mu(\mathfrak{u},\mathfrak{v}) := \frac{1}{2}\Big(\mu\,\|\mathfrak{u}\|^2_{H^1} + \|g(\mathfrak{u}) + \mu\mathfrak{v}\|^2_H\Big).$$

We have

$$D_{\mathfrak{u}}\psi(\mathfrak{u},\mathfrak{v})=\mu\,(-A)\mathfrak{u}+\gamma(\mathfrak{u})\,(g(\mathfrak{u})+\mu\,\mathfrak{v})\,,\quad D_{\mathfrak{v}}\psi(\mathfrak{u},\mathfrak{v})=\mu\,(g(\mathfrak{u})+\mu\,\mathfrak{v})\,,\quad D^2_{\mathfrak{v}}\psi(\mathfrak{u},\mathfrak{v})=\mu^2 I_H,$$

so that

$$\begin{aligned}
\mathscr{N}_\mu\,\psi_\mu(\mathfrak{u},\mathfrak{v}) &= \frac{1}{2}\mathrm{Tr}_H\Big[\big(\sigma(\mathfrak{u})Q\big)(\sigma(\mathfrak{u})Q)^*\Big]+\big\langle\mathfrak{v},\mu\mathfrak{u}+(-A)^{-1}\gamma(\mathfrak{u})g(\mathfrak{u})+\mu(-A)^{-1}\gamma(\mathfrak{u})\mathfrak{v}\big\rangle_{H^1}\\
&\quad+\frac{1}{\mu}\big\langle A\mathfrak{u}-\gamma(\mathfrak{u})\mathfrak{v}+F(\mathfrak{u}),\mu^2\mathfrak{v}+\mu g(\mathfrak{u})\big\rangle_H\\
&=\frac{1}{2}\,\|\sigma(\mathfrak{u})\|^2_{\mathscr{L}_2(H_Q,H)}+\mu\big\langle\mathfrak{v},-A\mathfrak{u}+\frac{\gamma(\mathfrak{u})}{\mu}g(\mathfrak{u})+\gamma(\mathfrak{u})\mathfrak{v}\big\rangle_H+\big\langle A\mathfrak{u}-\gamma(\mathfrak{u})\mathfrak{v}+F(\mathfrak{u}),\mu\mathfrak{v}+g(\mathfrak{u})\big\rangle_H\\
&=\frac{1}{2}\,\|\sigma(\mathfrak{u})\|^2_{\mathscr{L}_2(H_Q,H)}-\big\langle\gamma(\mathfrak{u})\nabla\mathfrak{u},\nabla\mathfrak{u}\big\rangle_H+\mu\big\langle F(\mathfrak{u}),\mathfrak{v}\big\rangle_H+\big\langle F(\mathfrak{u}),g(\mathfrak{u})\big\rangle_H.
\end{aligned}$$

Note that for every $\delta>0$ and $\mu\in(0,1)$

$$\mu\big|\big\langle F(\mathfrak{u}),\mathfrak{v}\big\rangle_H\big|\leq\mu\,\|F(\mathfrak{u})\|_H\,\|\mathfrak{v}\|_H\leq\delta\big(1+\|\mathfrak{u}\|^2_{H^1}\big)+c_\delta\,\mu^2\,\|\mathfrak{v}\|^2_H\,,$$

and

$$\big|\big\langle F(\mathfrak{u}),g(\mathfrak{u})\big\rangle_H\big|\leq\big(L_f\gamma_1+\delta\big)\,\|\mathfrak{u}\|^2_H+c_\delta\leq\frac{1}{\alpha_1}\big(L_f\gamma_1+\delta\big)\,\|\mathfrak{u}\|^2_{H^1}+c_\delta,$$

so that, due to the invariance of $\nu_\mu^{\mathscr{H}}$, we have

$$\begin{aligned}
&\gamma_0\int_{\mathscr{H}_1}\|\mathfrak{u}\|^2_{H^1}\,\nu_\mu^{\mathscr{H}}(d\mathfrak{u},d\mathfrak{v})\\
&\quad\leq\frac{\sigma_\infty^2}{2}+\left(\frac{L_f\,\gamma_1}{\alpha_1}+2\,\delta\right)\int_{\mathscr{H}_1}\|\mathfrak{u}\|^2_{H^1}\,\nu_\mu^{\mathscr{H}}(d\mathfrak{u},d\mathfrak{v})+c_\delta+c_\delta\,\mu^2\int_{\mathscr{H}_1}\|\mathfrak{v}\|^2_H\,\nu_\mu^{\mathscr{H}}(d\mathfrak{u},d\mathfrak{v}).
\end{aligned}$$

Thanks to (3.1.7), this implies that we can take $\delta > 0$ sufficiently small so that

$$\frac{L_f \gamma_1}{\alpha_1} + 2\delta < \gamma_0,$$

and then

$$\int_{\mathscr{H}_1} \|\mathfrak{u}\|_{H^1}^2 \, \nu_\mu^{\mathscr{H}_1}(d\mathfrak{u}, d\mathfrak{v}) \leq c \left(1 + \mu^2 \int_{\mathscr{H}_1} \|\mathfrak{v}\|_H^2 \, \nu_\mu^{\mathscr{H}_1}(d\mathfrak{u}, d\mathfrak{v})\right), \quad \mu \in (0,1).$$

By combining this with (3.4.14), we complete the proof of (3.4.12). □

Remark 3.19. 1. In Proposition 3.14, we have seen that for every $\mu > 0$ the semigroup $P_t^{\mu,\mathscr{H}}$ admits an invariant measure. Thanks to Lemma 3.18, this implies that for every $\mu > 0$ the transition semigroup $P_t^{\mu,\mathscr{H}_1}$ admits an invariant measure in $\mathscr{H}_1$.

2. As a consequence of (3.4.12), we have

$$\sup_{\mu \in (0,1)} \int_{\mathscr{H}} \left(\|\mathfrak{u}\|_{H^1}^2 + \mu \, \|\mathfrak{v}\|_H^2\right) \nu_\mu^{\mathscr{H}}(d\mathfrak{u}, d\mathfrak{v}) < \infty. \tag{3.4.15}$$

3.5 The limiting problem

In order to study the limiting problem (3.0.2), we consider first the following quasilinear stochastic parabolic equation

$$\begin{cases} \partial_t \rho(t,x) = \operatorname{div}\big(b(\rho(t,x))\nabla\rho(t,x)\big) + f_g(x,\rho(t,x)) + \sigma_g(\rho(t,\cdot))\partial_t w^Q(t,x), \\[2ex] \rho(0,x) = \mathfrak{r}_0(x), \qquad \rho(t,\cdot)|_{\partial\mathscr{O}} = 0, \end{cases} \tag{3.5.1}$$

where for every $r \in \mathbb{R}$ and $x \in \mathscr{O}$

$$b(r) := \frac{1}{\gamma(g^{-1}(r))}, \quad f_g(x,r) := f(x, g^{-1}(r)),$$

and for every $h \in H$

$$\sigma_g(h) := \sigma\left(g^{-1} \circ h\right).$$

As we explained in [22], due to a generalized Itô's formula, the solutions u and ρ of equations (3.0.2) and (3.5.1), respectively, are related by

$$\rho^{\mathfrak{r}_0}(t) = u^{\mathfrak{u}_0}(t), \quad t \geq 0, \quad \mathfrak{r}_0 := g(\mathfrak{u}_0). \tag{3.5.2}$$

From Hypothesis 3.3, we know

$$\frac{1}{\gamma_1} \leq b(r) \leq \frac{1}{\gamma_0}, \quad r \in \mathbb{R}.$$

Moreover, if we define

$$F_g(h)(x) := f_g(x, h(x)), \quad x \in \mathscr{O},$$

due to Hypotheses 3.1, 3.3 and 3.4, and due to (3.4.13), for every $h_1, h_2 \in H$ we have

$$\left\|F_g(h_1) - F_g(h_2)\right\|_{H^{-1}} \leq \frac{1}{\sqrt{\alpha_1}} \left\|F_g(h_1) - F_g(h_2)\right\|_H \leq \frac{L_f}{\sqrt{\alpha_1}\gamma_0} \|h_1 - h_2\|_H,$$

and

$$\left\|\sigma_g(h_1)-\sigma_g(h_2)\right\|_{\mathscr{L}_2(H_Q,H^{-1})} \leq \frac{1}{\sqrt{\alpha_1}}\left\|\sigma_g(h_1)-\sigma_g(h_2)\right\|_{\mathscr{L}_2(H_Q,H)} \leq \frac{\sqrt{L_\sigma}}{\sqrt{\alpha_1}\gamma_0}\left\|h_1-h_2\right\|_H.$$

Moreover, for every $\delta>0$

$$\left|\left\langle F_g(h),h\right\rangle_H\right| \leq \left(\frac{L_f}{\gamma_0}+\delta\right)\|h\|_H^2+c_\delta \leq \frac{1}{\alpha_1}\left(\frac{L_f}{\gamma_0}+\delta\right)\|h\|_{H^1}^2+c_\delta, \quad h\in H^1, \tag{3.5.3}$$

and

$$\left|\left\langle F_g(h),h\right\rangle_{H^{-1}}\right| \leq \frac{1}{\alpha_1}\left\|F_g(h)\right\|_H\|h\|_H \leq \frac{1}{\alpha_1}\left(\frac{L_f}{\gamma_0}+\delta\right)\|h\|_H^2+c_\delta, \quad h\in H. \tag{3.5.4}$$

Finally, thanks to (3.1.2) we have

$$\left\|\sigma_g(h)\right\|_{\mathscr{L}_2(H_Q,H)} \leq \sigma_\infty, \qquad h\in H. \tag{3.5.5}$$

3.5.1 Well-posedness of Eq. (3.5.1) in H

Throughout this section we will not need to assume condition (3.1.7) in Hypothesis 3.4. Namely, we will just assume that the mapping $f:\mathscr{O}\times\mathbb{R}\to\mathbb{R}$ is measurable, with

$$\sup_{x\in\mathscr{O}}|f(x,0)|<\infty, \qquad \sup_{x\in\mathscr{O}}|f(x,r)-f(x,s)|\leq c|r-s|, \quad r,s\in\mathbb{R}. \tag{3.5.6}$$

As a consequence of the limiting result proved in [22], the well-posedness of equation (3.5.1) has been established when the initial condition $\mathfrak{r}_0\in H^1$. Here we want to prove the existence and uniqueness of solutions of (3.5.1) when $\mathfrak{r}_0 \in L^2(\Omega; H)$.

Definition 3.20. *Let $\mathfrak{r}_0 \in L^2(\Omega; H)$. An adapted process $\rho \in L^2(\Omega; C([0,T]; H) \cap L^2(0,T; H^1))$ is a solution of equation (3.5.1) if for every $\varphi \in C_0^\infty(\mathscr{O})$*

$$\begin{aligned} \langle \rho(t), \varphi \rangle_H &= \langle \mathfrak{r}_0, \varphi \rangle_H - \int_0^t \langle b(\rho(s)) \nabla \rho(s), \nabla \varphi \rangle_H ds \\ &\quad + \int_0^t \langle F_g(\rho(s)), \varphi \rangle_H ds + \int_0^t \langle \varphi, \sigma_g(\rho(s)) dw^Q(s) \rangle_H, \quad \mathbb{P}\text{-a.s.} \end{aligned} \tag{3.5.7}$$

In order to study equation (3.5.1), we first consider the following approximating problem

$$\begin{cases} \partial_t \rho^\varepsilon(t,x) = \operatorname{div}\big(b(\rho^\varepsilon(t,x)) \nabla \rho^\varepsilon(t,x)\big) - \varepsilon \Delta^2 \rho^\varepsilon(t,x) + f(x, \rho(t,x)) + \sigma_g(\rho(s,\cdot)) \partial_t w^Q(t,x), \\ \rho^\varepsilon(0,x) = \mathfrak{r}_0, \quad \rho^\varepsilon(t,\cdot)|_{\partial \mathscr{O}} = 0, \end{cases} \tag{3.5.8}$$

with $0 < \varepsilon << 1$ (for a similar approach see e.g. [27]).

Lemma 3.21. *Assume Hypotheses 3.1 and 3.3 and condition (3.5.6). Then, for every $\varepsilon, T > 0$ and every $\rho_0 \in L^2(\Omega; H)$, equation (3.5.8) admits a unique solution*

$$\rho_\varepsilon \in L^2(\Omega; C([0,T]; H) \cap L^2(0,T; H^2)).$$

Moreover, there exists some $c_T > 0$ such that for every $\varepsilon > 0$

$$\mathbb{E} \sup_{t \in [0,T]} \|\rho^\varepsilon(t)\|_H^2 + \frac{2}{\gamma_1} \int_0^t \mathbb{E} \|\nabla \rho^\varepsilon(s)\|_H^2 ds + 2\varepsilon \int_0^t \mathbb{E} \|\Delta \rho^\varepsilon(s)\|_H^2 ds \leq c_T \left(1 + \mathbb{E}\, \|\mathfrak{r}_0\|_H^2\right). \tag{3.5.9}$$

Proof. The uniqueness and the existence of solutions for equation (3.5.8) can be proven by proceeding as in the proof of [31, Theorem 5.1].

In order to prove the energy estimate (3.5.9), we apply Itô's formula and we get

$$\frac{1}{2}\|\rho^\varepsilon(t)\|_H^2 = \frac{1}{2}\|\mathfrak{r}_0\|_H^2 + \int_0^t \left\langle \operatorname{div}\left(b(\rho^\varepsilon(s))\nabla\rho^\varepsilon(s)\right), \rho^\varepsilon(s)\right\rangle_H ds - \varepsilon\int_0^t \left\langle \Delta^2\rho^\varepsilon(s), \rho^\varepsilon(s)\right\rangle_H ds$$

$$+\int_0^t \left\langle F_g(\rho^\varepsilon(s)), \rho^\varepsilon(s)\right\rangle_H ds + \frac{1}{2}\int_0^t \|\sigma_g(\rho^\varepsilon(s))\|^2_{\mathscr{L}_2(H_Q,H)} ds + \int_0^t \left\langle \rho^\varepsilon(s), \sigma_g(\rho^\varepsilon(s)) dw^Q(s)\right\rangle_H$$

$$\leq \frac{1}{2}\|\mathfrak{r}_0\|_H^2 - \frac{1}{\gamma_1}\int_0^t \|\nabla\rho^\varepsilon(s)\|_H^2 ds - \varepsilon\int_0^t \|\Delta\rho^\varepsilon(s)\|_H^2 ds$$

$$+c\int_0^t \left(1+\|\rho^\varepsilon(s)\|_H^2\right) ds + 2\int_0^t \left\langle \rho^\varepsilon(s), \sigma_g(\rho^\varepsilon(s)) dw^Q(s)\right\rangle_H.$$

Note that

$$\mathbb{E}\sup_{s\in[0,t]}\left|\int_0^s \left\langle \rho^\varepsilon(r), \sigma_g(\rho^\varepsilon(r)) dw^Q(r)\right\rangle_H\right| \leq c\int_0^t \mathbb{E}\|\rho^\varepsilon(s)\|_H^2 ds + c_T,$$

and hence

$$\mathbb{E}\sup_{s\in[0,t]}\|\rho^\varepsilon(s)\|_H^2 + \frac{2}{\gamma_1}\mathbb{E}\int_0^t \|\nabla\rho^\varepsilon(s)\|_H^2 ds + 2\varepsilon\,\mathbb{E}\int_0^t \|\Delta\rho^\varepsilon(s)\|_H^2 ds$$

$$\leq \mathbb{E}\|\mathfrak{r}_0\|_H^2 + c\int_0^t \mathbb{E}\|\rho^\varepsilon(s)\|_H^2 ds + c_T.$$

Therefore, the Gronwall lemma gives (3.5.9).

□

Proposition 3.22. *Assume Hypotheses 3.1, 3.3 and condition* (3.5.6)*, and fix* $\mathfrak{r}_0 \in L^2(\Omega;H)$*. Then, for every* $T>0$*, there exists a unique solution*

$$\rho \in L^2(\Omega;C([0,T];H)) \cap L^2(0,T;H^1))$$

of equation (3.5.1)*. Moreover, there exists some constant* $c_T>0$ *such that*

$$\mathbb{E} \sup_{t\in[0,T]} \|\rho(t)\|_H^2 + \mathbb{E}\int_0^T \|\nabla\rho(s)\|_H^2 ds \leq c_T \left(1+\mathbb{E}\|\mathfrak{r}_0\|_H^2\right). \tag{3.5.10}$$

Proof. By proceeding as in [23, Theorem 6.2], we can show that equation (3.5.1) admits at most one solution in $L^2(\Omega;C([0,T];H))\cap L^2(0,T;H^1))$. Hence, if we show that there exists a probabilistically weak solution

$$(\hat{\Omega},\hat{\mathscr{F}},\{\hat{\mathscr{F}}\}_t,\hat{\mathbb{P}},\hat{w}^Q,\hat{\rho})$$

such that $\hat{\rho}\in L^2(\hat{\Omega};C([0,T];H))\cap L^2(0,T;H^1))$, the existence and uniqueness of a probabilistically strong solution for equation (3.5.1) follows.

Step 1. There exists a filtered probability space $(\hat{\Omega},\hat{\mathscr{F}},\{\hat{\mathscr{F}}\}_t,\hat{\mathbb{P}})$, a cylindrical Wiener process $\hat{w}^Q$ associated with $\{\hat{\mathscr{F}}\}_t$ and a process $\hat{\rho}\in L^2(\Omega;L^\infty(0,T;H))\cap L^2(0,T;H^1))$ such that

$$\begin{aligned}\left\langle\hat{\rho}(t),\varphi\right\rangle_H &= \left\langle\mathfrak{r}_0,\varphi\right\rangle_H - \int_0^t \left\langle b(\hat{\rho}(s))\nabla\hat{\rho}(s),\nabla\varphi\right\rangle_H ds \\ &\quad + \int_0^t \left\langle F_g(\hat{\rho}(s)),\varphi\right\rangle_H ds + \int_0^t \left\langle \varphi,\sigma_g(\hat{\rho}(s))d\hat{w}^Q(s)\right\rangle_H, \quad \hat{\mathbb{P}}\text{-a.s.}\end{aligned}$$

for every $\varphi\in C_0^\infty(\mathscr{O})$.

Proof of Step 1. According to Proposition 3.21, we know that for every $\varepsilon > 0$ there exists a unique solution ρ_ε to equation (3.5.8), and

$$\sup_{\varepsilon\in(0,1)}\left(\mathbb{E}\sup_{t\in[0,T]}\|\rho^\varepsilon(t)\|_H^2+\mathbb{E}\int_0^T\|\rho^\varepsilon(t)\|_{H^1}^2dt\right)<\infty. \tag{3.5.11}$$

For every $h \in (0,T)$ and $t \in [0,T-h]$ we have

$$\begin{aligned}\rho^\varepsilon(t+h)-\rho^\varepsilon(t)&=\int_t^{t+h}\operatorname{div}\left(b(\rho^\varepsilon(s))\nabla\rho^\varepsilon(s)\right)ds-\varepsilon\int_t^{t+h}\Delta^2\rho^\varepsilon(s)\,ds\\&\quad+\int_t^{t+h}F_g(\rho^\varepsilon(s))\,ds+\int_t^{t+h}\sigma_g(\rho^\varepsilon(s))dw^Q(s)=:\sum_{k=1}^4 I_k^\varepsilon(t,h).\end{aligned}$$

We have

$$\sup_{t\in[0,T-h]}\|I_1^\varepsilon(t,h)\|_{H^{-1}}\leq c\sup_{t\in[0,T]}\int_t^{t+h}\|\rho^\varepsilon(s)\|_{H^1}\,ds\leq c\left(\int_0^T\|\rho^\varepsilon(s)\|_{H^1}^2\,ds\right)^{1/2}h^{1/2}. \tag{3.5.12}$$

For $I_2^\varepsilon(t,h)$, if $\varepsilon \in (0,1)$ we have

$$\sup_{t\in[0,T-h]}\|I_2^\varepsilon(t,h)\|_{H^{-3}}\leq\sup_{t\in[0,T-h]}\int_t^{t+h}\|\rho^\varepsilon(s)\|_{H^1}\,ds\leq c\left(\int_0^T\|\rho^\varepsilon(s)\|_{H^1}^2\,ds\right)^{1/2}h^{1/2}, \tag{3.5.13}$$

and for $I_3^\varepsilon(t,h)$ we have

$$\sup_{t\in[0,T-h]}\|I_3^\varepsilon(t,h)\|_H\leq c\sup_{t\in[0,T-h]}\int_t^{t+h}\left(1+\|\rho^\varepsilon(s)\|_H\right)ds\leq c_T\left(1+\sup_{t\in[0,T]}\|\rho^\varepsilon(s)\|_H\right)h, \tag{3.5.14}$$

Finally, for $I_4^\varepsilon(t,h)$, by using a factorization argument as in [26, Theorem 5.11 and Theorem

5.15], due to the boundedness of σ_g in $\mathscr{L}_2(H_Q,H)$ we obtain that for some $\theta \in (0,1)$

$$\sup_{\varepsilon\in(0,1)} \mathbb{E}\,\|I_4^\varepsilon(t,h)\|_{C^\theta([0,T];H)} < \infty. \tag{3.5.15}$$

Therefore, by putting together (3.5.12), (3.5.13), (3.5.14) and (3.5.15), thanks to (3.5.11) we conclude

$$\sup_{t\in[0,T-h]} \|\rho^\varepsilon(t+h)-\rho^\varepsilon(t)\|_{H^{-3}} \leq c\,h^{\frac{1}{2}\wedge\theta}, \qquad h\in[0,T),$$

and together with the bound

$$\sup_{\varepsilon\in(0,1)} \mathbb{E} \sup_{t\in[0,T]} \|\rho^\varepsilon(t)\|_H < \infty,$$

due to [58, Theorem 7] this implies that $\{\rho^\varepsilon\}_{\varepsilon\in(0,1)}$ is tight in $L^\infty(0,T;H^{-\alpha})$, for every $\alpha>0$. Moreover, since for every $\beta\in(-3,1)$ we have

$$\|u\|_{H^\beta} \leq \|u\|_{H^1}^{\frac{3+\beta}{4}}\,\|u\|_{H^{-3}}^{\frac{1-\beta}{4}},$$

and the bound

$$\sup_{\varepsilon\in(0,1)} \int_0^T \mathbb{E}\,\|\rho^\varepsilon(s)\|_{H^1}^2\,ds < \infty$$

holds, thanks again to [58, Theorem 7], we have that the family $\{\rho^\varepsilon\}_{\varepsilon\in(0,1)}$ is tight also in the space $L^{8/(3+\beta)}(0,T;H^\beta)$, for every $\beta\in(-3,1)$.

In what follows, for every $\alpha>0$ and $\beta\in(-3,1)$ we denote

$$\mathscr{X}_{\alpha,\beta}(T) := \Big[L^\infty(0,T;H^{-\alpha})\cap L^{8/(3+\beta)}(0,T;H^\beta)\Big]\times C([0,T];U),$$

where U is any Hilbert space such that the embedding $H_Q \hookrightarrow U$ is Hilbert-Schmidt. Due to the tightness of $\{\rho^{\varepsilon}, w^{Q}\}_{\varepsilon\in(0,1)}$ in $\mathscr{X}_{\alpha,\beta}(T)$, there exists a sequence $\varepsilon_n \downarrow 0$ such that $\mathscr{L}(\rho^{\varepsilon_n}, w^{Q})$ is weakly convergent in $\mathscr{X}_{\alpha,\beta}(T)$. Due to Skorohod's Theorem this implies that there exists a probability space $(\hat{\Omega}, \hat{\mathscr{F}}, \hat{P})$, a sequence of $\mathscr{X}_{\alpha,\beta}(T)$-valued random variables $\mathscr{Y}_n = (\hat{\rho}_n, \hat{w}_n^Q)$ and a $\mathscr{X}_{\alpha,\beta}(T)$-valued random variable $\mathscr{Y} = (\hat{\rho}, \hat{w}^Q)$, all defined on the probability space $(\hat{\Omega}, \hat{\mathscr{F}}, \hat{P})$, such that

$$\mathscr{L}(\mathscr{Y}_n) = \mathscr{L}(\rho^{\varepsilon_n}, w^Q), \tag{3.5.16}$$

and

$$\lim_{n\to\infty}\Big(\|\hat{\rho}_n - \hat{\rho}\|_{L^\infty(0,T;H^{-\alpha})} + \|\hat{\rho}_n - \hat{\rho}\|_{L^{8/(3+\beta)}(0,T;H^\beta)} + \|\hat{w}_n^Q - \hat{w}^Q\|_{C([0,T];U)}\Big) = 0, \quad \hat{\mathbb{P}} - \text{a.s.} \tag{3.5.17}$$

Now, we have

$$\int_0^t \big\langle \operatorname{div}\left(b(\hat{\rho}_n(s))\nabla\hat{\rho}_n(s)\right), \varphi\big\rangle_H ds = -\int_0^t \big\langle b(\hat{\rho}_n(s))\nabla\hat{\rho}_n(s), \nabla\varphi\big\rangle_H ds$$

$$= -\int_0^t \big\langle \nabla(B(\hat{\rho}_n(s)), \nabla\varphi\big\rangle_H ds = \int_0^t \big\langle B(\hat{\rho}_n(s)), \Delta\varphi\big\rangle_H ds,$$

and thanks to (3.5.16) and (3.5.17), this gives for every $\varphi \in C_0^\infty(\mathscr{O})$,

$$\begin{aligned}\big\langle \hat{\rho}_n(t), \varphi\big\rangle_H &= \big\langle \mathfrak{r}_0, \varphi\big\rangle_H + \int_0^t \big\langle B(\hat{\rho}_n(s)), \Delta\varphi\big\rangle_H ds - \varepsilon_n \int_0^t \big\langle \hat{\rho}_n(s), \Delta^2\varphi\big\rangle_H ds \\ &\quad + \int_0^t \big\langle F_g(\hat{\rho}_n(s)), \varphi\big\rangle_H ds + \int_0^t \big\langle \varphi, \sigma_g(\hat{\rho}_n(s)) d\hat{w}_n^Q(s)\big\rangle_H .\end{aligned}$$

Thus, by using the general argument introduced in [27, proof of Theorem 4.1], thanks to (3.5.17) we can take the limit as $n \to \infty$ of both sides in the equality above, and we obtain that $\hat{\rho}$ satisfies (3.5.7), with w^Q replaced by $\hat{w}^Q$. Moreover, $\hat{\rho} \in L^2(\hat{\Omega}; L^\infty(0,T;H)) \cap L^2(0,T;H^1))$ and satisfies (3.5.10), with $\mathbb{E}$ replaced by $\hat{\mathbb{E}}$.

Step 2. We have that there exists a unique solution $\rho \in L^2(\Omega; C([0,T];H) \cap L^2(0,T;H^1))$ that satisfies (3.5.10).

Proof of Step 2. Due to what we have seen above, there exists a unique solution

$$\rho \in L^2(\Omega; L^\infty(0,T;H) \cap L^2(0,T;H^1))$$

that satisfies (3.5.10). It only remains to prove that $\rho \in C([0,T];H)$, $\mathbb{P}$-a.s. By proceeding as in [27, Section 4.3] we consider the problem

$$\begin{cases} \partial_t \xi(t,x) = \Delta \xi(t,x) + \sigma_g(\rho(t,\cdot)) \partial_t w^Q(t,x), \\ \\ \xi(0,x) = \mathfrak{r}_0(x), \qquad \xi(t,\cdot)|_{\partial \mathscr{O}} = 0, \end{cases}$$

whose unique solution ξ belongs to $L^2(\Omega; C([0,T];H) \cap L^2(0,T;H^1))$. Then, if we denote $\eta(t) := \rho(t) - \xi(t)$, we have that $\eta \in L^\infty(0,T;H) \cap L^2(0,T;H^1)$, $\mathbb{P}$-a.s., and solves

$$\begin{cases} \partial_t \eta(t,x) = \operatorname{div}\big(b(\rho(t,x)) \nabla \eta(t,x)\big) + \operatorname{div}\big[(b(\rho(t,x)) - I) \nabla \xi(t,x)\big] + f_g(x, \rho(t,x)), \\ \\ \eta(0,x) = 0, \qquad \eta(t,\cdot)|_{\partial \mathscr{O}} = 0. \end{cases} \tag{3.5.18}$$

Now, if we denote by $U(t,s)$ the evolution family associated with the time-dependent differential operator

$$\mathscr{L}_t \varphi(x) = \operatorname{div}\left[b(\rho(t,x)) \nabla \varphi(x)\right], \quad x \in \mathscr{O},$$

we have that

$$\eta(t,x) = \int_0^t U(t,s)\left[\operatorname{div}\left[b(\rho(t,\cdot)) - I\right]\nabla\xi(s,\cdot) + f_g(\cdot,\rho(s,\cdot))\right](x)\,ds,$$

and since $\xi \in L^2(0,T;H^1)$ and $\rho \in L^\infty(0,T;H)$, $\mathbb{P}$-a.s., we get that $\eta \in C([0,T];H)$, $\mathbb{P}$-a.s. In particular, $\rho = \eta + \xi$ belongs to $C([0,T];H)$, $\mathbb{P}$-a.s.

□

3.5.2 Well-posedness of Eq. (3.0.2) in H

From the well-posedness of the quasilinear stochastic parabolic equation (3.5.1), we get the well-poseness of equation (3.0.2) in H.

By proceeding as in the proof of [22, Theorem 7.1], we can show that $u \in L^2(\Omega; C([0,T];H) \cap L^2([0,T];H^1))$ is a solution to equation (3.0.2) with initial condition $\mathfrak{u}_0 \in L^2(\Omega;H)$ if and only if $\rho := g(u)$ is a weak solution to equation (3.5.1) with initial value $\mathfrak{r}_0 = g(\mathfrak{u}_0) \in L^2(\Omega;H)$. Moreover, as a consequence of the Lipschitz continuity of g and g^{-1} on $\mathbb{R}$, we have the following result:

Proposition 3.23. *Assume Hypotheses 3.1, 3.3 and condition (3.5.6). For every $T > 0$ and every $\mathfrak{u}_0 \in L^2(\Omega;H)$, there exists a unique weak solution $u \in L^2(\Omega; C([0,T];H) \cap L^2(0,T;H^1))$, to equation (3.0.2) such that*

$$\mathbb{E} \sup_{t\in[0,T]} \|u(t)\|_H^2 + \mathbb{E}\int_0^T \|u(t)\|_{H^1}^2\,dt \leq c_T\left(1 + \mathbb{E}\|\mathfrak{u}_0\|_H^2\right).$$

In what follows, we shall denote

$$P_t^H \varphi(\mathrm{u}) := \mathbb{E}\varphi(u^{\mathrm{u}}(t)), \quad \mathrm{u} \in H, \quad t \geq 0,$$

for every $\varphi \in B_b(H)$.

3.5.3 Energy estimates

Lemma 3.24. *Under Hypotheses 3.1, 3.3 and 3.4, there exist some $\lambda > 0$ and $c > 0$, such that for every $t \geq 0$*

$$\mathbb{E}\,\|\rho(t)\|_H^2 \leq c\left(1 + e^{-\lambda t}\mathbb{E}\,\|\mathfrak{r}_0\|_H^2\right), \qquad \mathbb{E}\int_0^t \|\rho(s)\|_{H^1}^2\,ds \leq c\left(t + \mathbb{E}\,\|\mathfrak{r}_0\|_H^2\right). \tag{3.5.19}$$

Proof. We apply Itô's formula and we get

$$\frac{1}{2}d\,\|\rho(t)\|_H^2 \leq -\gamma_1^{-1}\,\|\rho(t)\|_{H^1}^2\,dt + \left\langle F_g(\rho(t)), \rho(t)\right\rangle_H dt + \frac{1}{2}\left\|\sigma_g(\rho(t))\right\|_{\mathscr{L}_2(H_Q,H)}^2 dt$$
$$+ \langle \rho(t), \sigma_g(\rho(t))dw^Q(t)\rangle_H.$$

Then thanks to (3.5.3) and (3.1.7), together with 3.5.5, we can find some constant $\lambda > 0$ such that

$$\frac{d}{dt}\mathbb{E}\,\|\rho(t)\|_H^2 + \lambda\mathbb{E}\,\|\rho(t)\|_{H^1}^2 \leq c,$$

and this allows we complete the proof. □

Due to estimates (3.5.19) and the Lipschitz continuity of g and g^{-1} on $\mathbb{R}$, estimates analogous to (3.5.19) holds for the solution u.

Proposition 3.25. *Assume Hypotheses 3.1, 3.3 and 3.4. For every $T > 0$ and every $\mathfrak{u}_0 \in L^2(\Omega; H)$, there exists a unique $u \in L^2(\Omega; C([0,T]; H) \cap L^2(0,T; H^1))$ which solves equation (3.0.2) in the following sense*

$$\langle u(t), \psi \rangle_H = \langle \mathfrak{u}_0, \psi \rangle_H - \int_0^t \left\langle \frac{\nabla u(s)}{\gamma(u(s))}, \nabla \psi \right\rangle_H ds - \int_0^t \left\langle \nabla \left(\frac{1}{\gamma(u(s))} \right) \cdot \nabla u(s), \psi \right\rangle_H ds$$

$$+ \int_0^t \left\langle \frac{f(u(s))}{\gamma(u(s))}, \psi \right\rangle_H ds - \int_0^t \left\langle \frac{\gamma'(u(s))}{2\gamma(u(s))^3} \sum_{i=1}^{\infty} (\sigma(u(s)) Q e_i)^2, \psi \right\rangle_H ds$$

$$+ \int_0^t \left\langle \frac{\sigma(u(s))}{\gamma(u(s))} dw^Q(s), \psi \right\rangle_H,$$

for any $\varphi \in C_0^\infty(\mathscr{O})$. Moreover, for every $t \geq 0$

$$\mathbb{E} \|u(t)\|_H^2 \leq c \left(1 + e^{-\lambda t} \mathbb{E} \|\mathfrak{u}_0\|_H^2 \right), \qquad \mathbb{E} \int_0^t \|u(s)\|_{H^1}^2 ds \leq c \left(t + \mathbb{E} \|\mathfrak{u}_0\|_H^2 \right).$$

3.5.4 Well-posedness of Eq. (3.5.1) in H^{-1}

Here, we will use the results we have just mentioned about the well-posedness of equation (3.5.1) in H, to study its well-posedness in H^{-1}.

Definition 3.26. *For every fixed $\mathfrak{r}_0 \in H^{-1}$ and $T > 0$, an adapted process $\rho \in L^2(\Omega; L^2(0,T; H))$ is a solution of equation (3.5.1) with initial condition ρ_0 if for every $\varphi \in C_0^\infty(\mathscr{O})$*

$$\langle \rho(t), \varphi \rangle_H = \langle \mathfrak{r}_0, \varphi \rangle_H + \int_0^t \langle B(\rho(s)), \Delta \varphi \rangle_H ds + \int_0^t \langle F_g(\rho(s)), \varphi \rangle_H ds + \int_0^t \langle \varphi, \sigma_g(\rho(s)) dw^Q(s) \rangle_H,$$

$\mathbb{P}$*-a.s., where*

$$B(r) := \int_0^r b(s)ds, \quad r \in \mathbb{R}.$$

Proposition 3.27. *Assume Hypotheses 3.1, 3.3 3.4 and 3.6. Then, for every $\mathfrak{r}_0 \in H^{-1}$ and every $T > 0$, there exists a unique solution*

$$\rho^{\mathfrak{r}_0} \in L^2(\Omega; C([0,T]; H^{-1}) \cap L^2(0,T;H))$$

to equation (3.5.1). Moreover, there exist $c, \lambda > 0$ independent of $T > 0$ such that for every $t \in [0,T]$

$$\mathbb{E}\, \|\rho^{\mathfrak{r}_0}(t)\|_{H^{-1}}^2 \leq c \left(1 + e^{-\lambda t}\, \mathbb{E}\, \|\mathfrak{r}_0\|_{H^{-1}}^2\right), \qquad \mathbb{E} \int_0^t \|\rho^{\mathfrak{r}_0}(s)\|_H^2\, ds \leq c \left(t + \mathbb{E}\, \|\mathfrak{r}_0\|_{H^{-1}}^2\right). \quad (3.5.20)$$

Proof. We fix an arbitrary sequence $\{\mathfrak{r}_\varepsilon\}_{\varepsilon>0} \subset H$ converging to $\mathfrak{r}_0$ strongly in H^{-1}, as $\varepsilon \to 0$. Thanks to Proposition 3.22, for each $\varepsilon > 0$ there exists a solution $\rho_\varepsilon \in L^2(\Omega; C([0,T]; H) \cap L^2([0,T]; H^1))$ for problem (3.5.1) with initial condition $\mathfrak{r}_\varepsilon$. If for every $\varepsilon, \delta > 0$ we define

$$\vartheta_{\varepsilon,\delta}(t) := \rho_\varepsilon(t) - \rho_\delta(t), \qquad t \in [0,T],$$

we have

$$\frac{1}{2} d \left\| \vartheta_{\varepsilon,\delta}(t) \right\|_{H^{-1}}^2$$

$$= -\left\langle B(\rho_\varepsilon(t)) - B(\rho_\delta(t)), \vartheta_{\varepsilon,\delta}(t) \right\rangle_H dt + \left\langle F_g(\rho_\varepsilon(t)) - F_g(\rho_\delta(t)), \vartheta_{\varepsilon,\delta}(t) \right\rangle_{H^{-1}} dt$$

$$+\frac{1}{2}\left\|\sigma_g(\rho_\varepsilon(t))-\sigma_g(\rho_\delta(t))\right\|^2_{\mathscr{L}_2(H_Q,H^{-1})}dt+\left\langle\vartheta_{\varepsilon,\delta}(t),\left[\sigma_g(\rho_\varepsilon(t))-\sigma_g(\rho_\delta(t))\right]dw^Q(t)\right\rangle_{H^{-1}}.$$

Since

$$B(r)=\int_0^r b(s)\,ds=\int_0^r\frac{1}{\gamma(g^{-1}(s))}\,ds,$$

we have

$$(B(r_1)-B(r_2))(r_1-r_2)\geq\frac{1}{\gamma_1}|r_1-r_2|^2,\qquad r_1,r_2\in\mathbb{R},$$

so that

$$\frac{1}{2}d\left\|\vartheta_{\varepsilon,\delta}(t)\right\|^2_{H^{-1}}\leq -c_0\left\|\vartheta_{\varepsilon,\delta}(t)\right\|^2_H dt+\left\langle\vartheta_{\varepsilon,\delta}(t),\left[\sigma_g(\rho_\varepsilon(t))-\sigma_g(\rho_\delta(t))\right]dw^Q(t)\right\rangle_{H^{-1}},\tag{3.5.21}$$

where

$$c_0:=\left(\frac{1}{\gamma_1}-\frac{L_\sigma}{2\alpha_1\gamma_0^2}-\frac{L_f}{\alpha_1\gamma_0}\right)>0,$$

last inequality following from (3.1.11).

Hence, if we first integrate both sides in (3.5.21) with respect to time and then take the expectation, we get

$$\sup_{s\in[0,T]}\mathbb{E}\left\|\vartheta_{\varepsilon,\delta}(s)\right\|^2_{H^{-1}}+2\,c_0\int_0^T\mathbb{E}\left\|\vartheta_{\varepsilon,\delta}(s)\right\|^2_H ds\leq\mathbb{E}\left\|\mathfrak{r}_\varepsilon-\mathfrak{r}_\delta\right\|^2_{H^{-1}}$$

and this implies that the sequence (ρ_ε) is Cauchy in $C([0,T];L^2(\Omega;H^{-1}))\cap L^2(\Omega;L^2(0,T;H)))$. In particular, it converges to some ρ in $C([0,T];L^2(\Omega;H^{-1}))\cap L^2(\Omega;L^2(0,T;H)))$, as $\varepsilon\to 0$.

For every $\varepsilon > 0$ and $\varphi \in C_0^\infty(\mathscr{O})$ we have

$$\left\langle \rho_\varepsilon(t), \varphi \right\rangle_H = \left\langle \mathfrak{r}_\varepsilon, \varphi \right\rangle_H + \int_0^t \left(\left\langle B(\rho_\varepsilon(s)), \Delta\varphi \right\rangle_H + \left\langle F_g(\rho_\varepsilon(s)), \varphi \right\rangle_H \right) ds + \int_0^t \left\langle \varphi, \sigma_g(\rho_\varepsilon(s)) dw^Q(s) \right\rangle_H,$$

Then, due to the Lipschitz continuity of B, F_g and σ_g, we can take the limit in both sides of the identity above, as $\varepsilon \to 0$, and we get that ρ is a solution for (3.5.1).

To prove the uniqueness, assume that ρ_1, ρ_2 are two solutions to (3.5.1). By proceeding as above, we have

$$\sup_{t \in [0,T]} \mathbb{E} \left\| \rho_1(t) - \rho_2(t) \right\|_{H^{-1}}^2 + c_0 \int_0^T \mathbb{E} \left\| \rho_1(t) - \rho_2(t) \right\|_H^2 dt \leq 0,$$

which gives $\rho_1 = \rho_2$.

Next, we prove that $\rho \in L^2(\Omega; C([0,T]; H^{-1}) \cap L^2(0,T;H)))$. We apply Itô's formula to $\|\rho\|_{H^{-1}}^2$ and we get

$$\begin{aligned}
\frac{1}{2} d \left\| \rho(t) \right\|_{H^{-1}}^2 = & \frac{1}{2} \left\| \sigma_g(\rho(t)) \right\|_{\mathscr{L}_2(H_Q, H^{-1})}^2 dt - \langle B(\rho(t)), \rho(t) \rangle_H dt \\
& + \left\langle F_g(\rho(t)), \rho(t) \right\rangle_{H^{-1}} dt + \langle \rho(t), \sigma_g(\rho(t)) dw^Q(t) \rangle_{H^{-1}} \\
& \leq c - \gamma_1^{-1} \left\| \rho(t) \right\|_H^2 dt + \left\langle F_g(\rho(t)), \rho(t) \right\rangle_{H^{-1}} dt + \langle \rho(t), \sigma_g(\rho(t)) dw^Q(t) \rangle_{H^{-1}}.
\end{aligned}$$

Due to (3.1.7) there exists $\bar{\delta} > 0$ such that

$$c_1 := \frac{1}{\gamma_1} - \frac{L_f + \bar{\delta}}{\alpha_1 \gamma_0} > 0,$$

so that, thanks to (3.5.4) we have

$$\frac{1}{2} d\,\|\rho(t)\|_{H^{-1}}^2 \leq c - c_1\,\|\rho(t)\|_H^2\,dt + \langle \rho(t), \sigma_g(\rho(t))dw^Q(t)\rangle_{H^{-1}}. \tag{3.5.22}$$

Now, since we have

$$\mathbb{E} \sup_{s\in[0,t]} \left| \int_0^s \langle \rho(t), \sigma_g(\rho(t))dw^Q(t)\rangle_{H^{-1}} \right| \leq \frac{1}{4}\mathbb{E} \sup_{s\in[0,t]} \|\rho(s)\|_{H^{-1}}^2 + c,$$

if we integrate both sides in (3.5.22) and then take the supremum with respect to time and the expectation, we get

$$\mathbb{E} \sup_{t\in[0,T]} \|\rho(t)\|_{H^{-1}}^2 + \int_0^T \mathbb{E}\,\|\rho(s)\|_H^2\,ds \leq c_T\left(1 + \|\mathfrak{r}_0\|_{H^{-1}}^2\right),$$

which, in particular implies that $\rho \in L^2(\Omega; L^\infty(0,T;H^{-1}) \cap L^2(0,T;H)))$. Moreover, since ρ solves equation (3.5.1), it belongs to $C([0,T];H)$, $\mathbb{P}$-a.s.

Finally, in order to prove (3.5.20), we take the expectation of both sides of (3.5.22) and we get

$$\frac{d}{dt}\mathbb{E}\,\|\rho(t)\|_{H^{-1}}^2 + 2c_1\mathbb{E}\,\|\rho(s)\|_H^2\,ds \leq c$$

and this implies that there exist some $c, \lambda > 0$ such that (3.5.20) holds. $\square$

3.6 Ergodic behavior of the limiting equation

In this section, we first study the existence of a unique invariant measure for R_t^H and $R_t^{H^{-1}}$, and then we show how this implies the existence of a unique invariant measure for P_t^H.

3.6.1 Ergodicity of Eq. (3.5.1)

In what follows, we denote by $R_t^{H^{-1}}$ the transition semigroup associated to equation (3.5.1) on H^{-1}

$$R_t^{H^{-1}}\varphi(\mathfrak{r}) := \mathbb{E}\varphi(\rho^{\mathfrak{r}}(t)), \qquad \mathfrak{r} \in H^{-1}, \qquad t \geq 0,$$

for every $\varphi \in B_b(H^{-1})$. Similarly, we denote by R_t^H the transition semigroup associated to equation (3.5.1) on H,

$$R_t^H\varphi(\mathfrak{r}) := \mathbb{E}\varphi(\rho^{\mathfrak{r}}(t)), \quad \mathfrak{r} \in H, \quad t \geq 0,$$

for every $\varphi \in B_b(H)$. Clearly, if $\mathfrak{r} \in H$ and $\varphi \in B_b(H^{-1})$, then

$$R_t^H\varphi(\mathfrak{r}) = R_t^{H^{-1}}\varphi(\mathfrak{r}), \qquad t \geq 0.$$

For every $A \in \mathscr{B}(H^{-1})$ we have that $A \cap H \in \mathscr{B}(H)$. Thus, if $\nu \in \mathscr{P}(H)$, we can define its extension $\nu' \in \mathscr{P}(H^{-1})$ by setting

$$\nu'(A) = \nu(A \cap H), \qquad A \in \mathscr{B}(H^{-1}).$$

With this definition, $\operatorname{supp}(\nu') \subset H$. Indeed, if we denote by $B_H(\mathfrak{r}, R)$ the closed ball in H centered at $\mathfrak{r} \in H$ with radius $R > 0$, then $B_H^c(\mathfrak{r}, R) \in \mathscr{B}(H)$, so that

$$\lim_{R\to+\infty} \nu'(B_H^c(0,R)) = \lim_{R\to+\infty} \nu(B_H^c(0,R)) = 0,$$

which implies that $\operatorname{supp}(\nu') \subset H$.

Proposition 3.28. *Assume Hypotheses 3.1, 3.3, 3.4 and 3.6, and define* $\alpha(\mathfrak{r},\mathfrak{s}) := |\mathfrak{r}-\mathfrak{s}|_{H^{-1}}$. *Then, there exist some positive constant* λ_0, t_0 *and* c *such that*

$$\mathscr{W}_\alpha\left((R_t^{H^{-1}})^*\nu_1,(R_t^{H^{-1}})^*\nu_2\right) \leq c\,e^{-\lambda_0 t}\,\mathscr{W}_\alpha(\nu_1,\nu_2), \qquad t > t_0. \tag{3.6.1}$$

Moreover, $R_t^{H^{-1}}$ *has a unique invariant measure* $\nu^{H^{-1}}$ *such that* $\operatorname{supp}(\nu^{H^{-1}}) \subset H^1$ *and*

$$\mathscr{W}_\alpha\left((R_t^{H^{-1}})^*\delta_{\mathfrak{r}},\,\nu^{H^{-1}}\right) \leq c\,(1+\|\mathfrak{r}\|_{H^{-1}})\,e^{-\lambda_0 t}, \qquad t \geq 0, \quad \mathfrak{r} \in H^{-1}. \tag{3.6.2}$$

Proof. Let $\rho^{\mathfrak{r}_1}, \rho^{\mathfrak{r}_2}$ be two solutions of (3.5.1), with initial conditions $\mathfrak{r}_1, \mathfrak{r}_2 \in H^{-1}$, respectively. By proceeding as in the proof of Proposition 3.27, we have

$$\mathbb{E}\,\|\rho^{\mathfrak{r}_1}(t)-\rho^{\mathfrak{r}_1}(t)\|_{H^{-1}}^2 \leq e^{-\lambda t}\,\|\mathfrak{r}_1-\mathfrak{r}_2\|_{H^{-1}}^2\,, \qquad t \geq 0,$$

for some constant $\lambda > 0$. In particular, the semigroup $R_t^{H^{-1}}$ is Feller in H^{-1} and for every $\varphi \in \mathrm{Lip}_b(H^{-1})$ and $\mathfrak{r}_1, \mathfrak{r}_2 \in H^{-1}$

$$\left|R_t^{H^{-1}}\varphi(\mathfrak{r}_1) - R_t^{H^{-1}}\varphi(\mathfrak{r}_2)\right| \leq [\varphi]_{\mathrm{Lip}_{H^{-1},\alpha}} e^{-\lambda t/2}\,\|\mathfrak{r}_1-\mathfrak{r}_2\|_{H^{-1}}\,, \qquad t \geq 0. \tag{3.6.3}$$

As shown e.g. in [39, Theorem 2.5], (3.6.3) implies (3.6.1). Moreover, it implies that $R_t^{H^{-1}}$ has at most one invariant measure.

If for every $R > 0$ and $t > 0$ we denote

$$B_R := \left\{\mathfrak{r} \in H^{-1} : \|\mathfrak{r}\|_{H^1} \leq R\right\}, \qquad \Gamma_t := \frac{1}{t}\int_0^t (R_t^{H^{-1}})^*\,\delta_0\,dt.$$

Then, thanks to (3.5.19), for every $R > 0$ and $t > 0$ we have

$$R_t(B_R^c) = \frac{1}{t}\int_0^t \mathbb{P}\left(\left\|\rho^0(s)\right\|_{H^1} > R\right) ds \leq \frac{c}{R^2}. \tag{3.6.4}$$

Since B_R is compactly embedded in H^{-1}, this implies that the family of measures $\{\Gamma_t\}_{t>0}$, is tight in H^{-1}. Then, by Prokhorov's Theorem, there exists $t_n \uparrow \infty$ such that Γ_{t_n} converges weakly to some probability measure in $\mathscr{P}(H^{-1})$ which is invariant for $R_t^{H^{-1}}$ and, due to what we have seen above, such measure is the unique invariant measure $\nu^{H^{-1}}$ of $R_t^{H^{-1}}$. Moreover, (3.6.4) gives

$$\nu^{H^{-1}}(B_R^c) \leq \liminf_{n\to\infty} \Gamma_{t_n}(B_R^c) \leq \frac{c}{R^2}, \qquad R > 0,$$

so that $\operatorname{supp}(\nu^{H^{-1}}) \subset H^1$.

Finally, in order to prove (3.6.2), we first notice that due to the invariance of $\nu^{H^{-1}}$ and (3.5.20)

$$\begin{aligned}
\int_{H^{-1}} \|\mathfrak{r}\|_{H^{-1}}^2 \, \nu^{H^{-1}}(d\mathfrak{r}) &\leq \liminf_{R\to\infty} \int_{H^{-1}} \left(\|\mathfrak{r}\|_{H^{-1}}^2 \wedge R\right) \nu^{H^{-1}}(d\mathfrak{r}) \\
&= \liminf_{R\to\infty} \int_{H^{-1}} \left(\mathbb{E}\,\|\rho^{\mathfrak{r}}(t)\|_{H^{-1}}^2 \wedge R\right) \nu^{H^{-1}}(d\mathfrak{r}) \leq c\left(1 + e^{-\lambda t}\int_{H^{-1}} \|\mathfrak{r}\|_{H^{-1}}^2 \nu^{H^{-1}}(d\mathfrak{r})\right).
\end{aligned}$$

Thus, if we take $\bar{t} > 0$ such that $ce^{-\lambda\bar{t}} = 1/2$, we get

$$\int_{H^{-1}} \|\mathfrak{r}\|_{H^{-1}}^2 \, \nu^{H^{-1}}(d\mathfrak{r}) \leq c. \tag{3.6.5}$$

Then, in view of (3.6.3), for every $\varphi \in \mathrm{Lip}_b(H^{-1})$ we have

$$\mathscr{W}_\alpha\left((R_t^{H^{-1}})^*\delta_{\mathfrak{r}}, \nu^{H^{-1}}\right) \leq \left|R_t^{H^{-1}}\varphi(\mathfrak{r}) - \int_{H^{-1}} \varphi(\mathfrak{s})\, \nu^{H^{-1}}(d\mathfrak{s})\right|$$

$$\leq \int_{H^{-1}} \left|R_t^{H^{-1}}\varphi(\mathfrak{r}) - R_t^{H^{-1}}\varphi(\mathfrak{s})\right| \nu^{H^{-1}}(d\mathfrak{s}) \leq [\varphi]_{\mathrm{Lip}_{H^{-1}},\alpha}\, e^{-\lambda t/2} \int_{H^{-1}} \|\mathfrak{r}-\mathfrak{s}\|_{H^{-1}} \nu^{H^{-1}}(d\mathfrak{s}),$$

and (3.6.5) allows to obtain (3.6.2), with $\lambda_0 = \lambda/2$.

□

Remark 3.29. Based on the fact that $\mathscr{B}(H) \subset \mathscr{B}(H^{-1})$ and the fact that $\mathrm{supp}\,(\nu^{H^{-1}}) \subset H^1$, we have that $\nu^{H^{-1}} \in \mathscr{P}(H^{-1})$ is also a probability measure on H. In what follows, it will be convenient to distinguish the restriction of $\nu^{H^{-1}}$ to H from $\nu^{H^{-1}}$ itself and for this reason we will denote it by ν^H.

Proposition 3.30. *The probability measure ν^H is the unique invariant measure for the transition semigroup R_t^H. Moreover,* $\mathrm{supp}\,(\nu^H) \subset H^1$ *and*

$$\int_H \|\mathfrak{r}\|_{H^1}^2\, \nu^H(d\mathfrak{r}) < \infty. \tag{3.6.6}$$

Proof. By proceeding as in the proof of Lemma 3.18, it is possible to show that ν^H is invariant for R_t^H, and from Proposition 3.28 we get that $\mathrm{supp}(\nu^H) \subset H^1$.

To prove its uniqueness, we notice that if $\nu \in \mathscr{P}(H)$ is any invariant measure for R_t^H, then its extension $\nu' \in \mathscr{P}(H^{-1})$, with the support in H, is invariant for $R_t^{H^{-1}}$. From Proposition 3.28, we have $\nu' = \nu^{H^{-1}}$, and hence

$$\nu(A) = \nu'(A) = \nu^{H^{-1}}(A) = \nu^H(A), \qquad A \in \mathscr{B}(H),$$

which implies that $\nu = \nu^H$.

Finally, in order to prove (3.6.6), we consider the Komolgorov operator associated to R_t^H

$$N\varphi(\mathfrak{r}) = \frac{1}{2}\mathrm{Tr}_H\Big[\big(\sigma_g(\mathfrak{r})Q\big)\big(\sigma_g(\mathfrak{r})Q\big)^* D^2\varphi(\mathfrak{r})\Big] + \big\langle \mathrm{div}\big(b(\mathfrak{r})\nabla\mathfrak{r}\big) + F_g(\mathfrak{r}), D\varphi(\mathfrak{r})\big\rangle_H,$$

We consider the function $\varphi(\mathfrak{r}) := \|\mathfrak{r}\|_H^2/2$, then

$$N\varphi(\mathfrak{r}) = \frac{1}{2}\left\|\sigma_g(\mathfrak{r})\right\|^2_{\mathscr{L}_2(H_Q,H)} - \big\langle b(\mathfrak{r})\nabla\mathfrak{r}, \nabla\mathfrak{r}\big\rangle_H + \big\langle F_g(\mathfrak{r}), \mathfrak{r}\big\rangle_H \leq \frac{1}{2}\sigma_\infty^2 - \gamma_1^{-1}\|\nabla\mathfrak{r}\|_H^2 + \big\langle F_g(\mathfrak{r}), \mathfrak{r}\big\rangle_H,$$

so thanks to (3.5.3) and (3.1.7), by the invariance of ν^H on H we have

$$\int_H \|\mathfrak{r}\|^2_{H^1}\, \nu^H(d\mathfrak{r}) < \infty.$$

$\square$

Remark 3.31. As a direct consequence of (3.6.6), we have

$$\int_{H^{-1}} \|\mathfrak{r}\|^2_{H^1}\, \nu^{H^{-1}}(d\mathfrak{r}) < \infty. \tag{3.6.7}$$

3.6.2 Ergodicity of Eq. (3.0.2)

Due to Hypothesis 3.3, with an abuse of notation in this section we will look at g and g^{-1} as mappings on H

$$[g(h)](x) := g(h(x)), \quad [g^{-1}(h)](x) := g^{-1}(h(x)), \quad x \in \mathscr{O}, \qquad h \in H.$$

For every probability measure $\nu \in \mathscr{P}(H)$, we define probability measures $\nu \circ g$ and $\nu \circ g^{-1}$ $\in \mathscr{P}(H)$ by

$$\big(\nu \circ g\big)(A) := \nu(g(A)), \quad \big(\nu \circ g^{-1}\big)(A) := \nu(g^{-1}(A)), \quad A \in \mathscr{B}(H).$$

Clearly, we have

$$(\nu \circ g) \circ g^{-1} = (\nu \circ g^{-1}) \circ g = \nu. \tag{3.6.8}$$

Now, we recall that we denoted by P_t^H the transition semigroup associated to the limiting problem (3.0.2)

$$P_t^H \varphi(\mathfrak{u}) := \mathbb{E}\, \varphi(u^{\mathfrak{u}}(t)), \qquad \mathfrak{u} \in H, \qquad t \geq 0,$$

for every $\varphi \in B_b(H)$. For every $\mathfrak{r}, \mathfrak{u} \in H$ and $t \geq 0$ we have

$$g^{-1}(\rho^{\mathfrak{r}}(t)) = u^{g^{-1}(\mathfrak{r})}(t), \qquad \rho^{g(\mathfrak{u})}(t) = u^{\mathfrak{u}}(t).$$

Hence, if we define the operator $T_g : C_b(H) \to C_b(H)$ by

$$[T_g \varphi](\mathfrak{u}) = \varphi(g(\mathfrak{u})), \quad \mathfrak{u} \in H,$$

we have $T_g^{-1} := T_{g^{-1}}$,

$$\int_H [T_g \varphi](\mathfrak{u})\, (\nu \circ g)(d\mathfrak{u}) = \int_H \varphi(\mathfrak{r})\, \nu(d\mathfrak{r}), \tag{3.6.9}$$

and for every $\varphi \in C_b(H)$

$$R_t^H \varphi(\mathfrak{r}) = \mathbb{E}\,\varphi(\rho^{\mathfrak{r}}(t)) = \mathbb{E}\,[T_g\varphi](u^{g^{-1}(\mathfrak{r})}(t)) = P_t^H[T_g\varphi](g^{-1}(\mathfrak{r})), \qquad t \geq 0. \tag{3.6.10}$$

Lemma 3.32. *$\nu \in \mathscr{P}(H)$ is invariant for P_t^H if and only $\nu \circ g^{-1}$ is invariant for R_t^H. In particular, $\nu^H \circ g$ is the unique invariant measure for P_t^H.*

Proof. Assume $\nu \in \mathscr{P}(H)$ is invariant for P_t^H. Then, thanks to (3.6.10) and (3.6.9), for every $\varphi \in C_b(H)$ and $t \geq 0$ we have

$$\int_H R_t^H \varphi(\mathfrak{r})\,(\nu \circ g^{-1})(d\mathfrak{r}) = \int_H R_t^H \varphi(g(\mathfrak{u}))\,\nu(d\mathfrak{u}) = \int_H P_t^H[T_g\varphi](\mathfrak{u})\,\nu(d\mathfrak{u})$$

$$= \int_H [T_g\varphi](\mathfrak{u})\,\nu(d\mathfrak{u}) = \int_H \varphi(\mathfrak{r})\,(\nu \circ g^{-1})(d\mathfrak{r}).$$

This implies that $\nu \circ g^{-1}$ is invariant for R_t^H. In the same way, if $\lambda \in \mathscr{P}(H)$ is invariant for R_t^H, then $\lambda \circ g$ is invariant for P_t^H. Hence, we can conclude due to (3.6.8).

Our statement can be rephrased by saying that there exists a unique invariant measure for R_t^H if and only is there exists a unique invariant measure for P_t^H. Therefore, since we have shown in Corollary 3.30 that ν^H is the unique invariant measure for R_t^H, we obtain that $\nu^H \circ g$ is the unique invariant measure for P_t^H.

□

3.7 Proof of Theorem 3.9

We notice that Theorem 3.9 is proved once we prove that if $(\nu_\mu^{\mathscr{H}})_{\mu>0} \subset \mathscr{P}(\mathscr{H})$ is a family of invariant measures for the transition semigroups $P_t^{\mu,\mathscr{H}}$, such that $\mathrm{supp}(\nu_\mu^{\mathscr{H}}) \subset \mathscr{H}_1$, then we have

$$\lim_{\mu\to 0} \mathscr{W}_\alpha\left(\left[\left(\Pi_1 \nu_\mu^{\mathscr{H}}\right)\circ g^{-1}\right]', \nu^{H^{-1}}\right) = 0, \tag{3.7.1}$$

where $\nu^{H^{-1}}$ is the unique invariant measure for $R_t^{H^{-1}}$ in H^{-1}.

Actually, in view of (3.4.15), the family of probability measures $(\Pi_1 \nu_\mu^{\mathscr{H}})_{\mu\in(0,1)}$ is tight in H^δ, for every $\delta < 1$. If ν is any weak limit of $\Pi_1 \nu_\mu^{\mathscr{H}}$ in H, as $\mu \to 0$, we have $(\Pi_1 \nu_\mu^{\mathscr{H}})\circ g^{-1}$ converges weakly to $\nu \circ g^{-1}$ on H. Due to the continuity of the embedding of H^{-1} into H,

$$\left[(\Pi_1 \nu_\mu^{\mathscr{H}})\circ g^{-1}\right]' \rightharpoonup \left(\nu\circ g^{-1}\right)', \quad \text{as } \mu \to 0,$$

as measures on H^{-1}. On the other hand, according to (3.7.1) we have that $\left[(\Pi_1 \nu_\mu^{\mathscr{H}})\circ g^{-1}\right]'$ converges weakly to $\nu^{H^{-1}}$ in H^{-1}, so that $(\nu\circ g^{-1})' = \nu^{H^{-1}}$ in H^{-1}. This implies that $\nu \circ g^{-1} = \nu^H \in \mathscr{P}(H)$, and thus $\nu = \nu^H \circ g \in \mathscr{P}(H)$. Since this holds for every weak limit ν of $\Pi_1 \nu_\mu^{\mathscr{H}}$, we conclude that $\Pi_1 \nu_\mu^{\mathscr{H}}$ converges weakly to $\nu^H \circ g$ in H, as $\mu \to 0$, and, due to Lemma 3.32, $\nu^H \circ g$ is the unique invariant measure for P_t^H.

3.7.1 Proof of (3.7.1)

Due to the invariance of $\nu_\mu^{\mathscr{H}}$ and $\nu^{H^{-1}}$, we have

$$\begin{aligned}\mathscr{W}_\alpha\left(\left[\left(\Pi_1 \nu_\mu^{\mathscr{H}}\right)\circ g^{-1}\right]', \nu^{H^{-1}}\right) &\leq \mathscr{W}_\alpha\left(\left[\Pi_1((P_t^{\mu,\mathscr{H}})^\star \nu_\mu^{\mathscr{H}})\circ g^{-1}\right]', (R_t^{H^{-1}})^*\left[\left(\Pi_1 \nu_\mu^{\mathscr{H}}\right)\circ g^{-1}\right]'\right)\\ &\quad + \mathscr{W}_\alpha\left((R_t^{H^{-1}})^*\left[\left(\Pi_1 \nu_\mu^{\mathscr{H}}\right)\circ g^{-1}\right]', (R_t^{H^{-1}})^* \nu^{H^{-1}}\right).\end{aligned}$$

According to (3.6.1), we have

$$\mathscr{W}_\alpha\left((R_t^{H^{-1}})^*\left[\left(\Pi_1 \nu_\mu^{\mathscr{H}}\right)\circ g^{-1}\right]^\prime,(R_t^{H^{-1}})^*\nu^{H^{-1}}\right)\leq c\,e^{-\lambda_0 t}\mathscr{W}_\alpha\left(\left[\left(\Pi_1 \nu_\mu^{\mathscr{H}}\right)\circ g^{-1}\right]^\prime,\nu^{H^{-1}}\right),$$

and then, if we pick $\bar{t}>0$ such that $ce^{-\lambda_0\bar{t}}\leq 1/2$, we obtain

$$\mathscr{W}_\alpha\left(\left[\left(\Pi_1 \nu_\mu^{\mathscr{H}}\right)\circ g^{-1}\right]^\prime,\nu^{H^{-1}}\right)\leq 2\,\mathscr{W}_\alpha\left(\left[(\Pi_1((P_{\bar{t}}^{\mu,\mathscr{H}})^\star \nu_\mu^{\mathscr{H}})\circ g^{-1}\right]^\prime,(R_{\bar{t}}^{H^{-1}})^*\left[\left(\Pi_1 \nu_\mu^{\mathscr{H}}\right)\circ g^{-1}\right]^\prime\right).$$

Now, if we fix a $\mathscr{F}_0$-measurable $\mathscr{H}_1$-valued random variable $\vartheta_\mu:=(\xi_\mu,\eta_\mu)$, distributed as the invariant measure $\nu_\mu^{\mathscr{H}}$, the Kantorovich-Rubinstein identity (3.2.3) gives for every $t\geq 0$

$$\mathscr{W}_\alpha\left(\left[(\Pi_1((P_t^{\mu,\mathscr{H}})^\star \nu_\mu^{\mathscr{H}})\circ g^{-1}\right]^\prime,(R_t^{H^{-1}})^*\left[\left(\Pi_1 \nu_\mu^{\mathscr{H}}\right)\circ g^{-1}\right]^\prime\right)\leq \mathbb{E}\,\alpha(g(u_\mu^{\vartheta_\mu}(t)),\rho^{g(\xi_\mu)}(t)).$$

Thus, (3.7.1) follows once we prove that for every $t\geq 0$ large enough

$$\lim_{\mu\to 0}\mathbb{E}\,\alpha(g(u_\mu^{\vartheta_\mu}(t)),\rho^{g(\xi_\mu)}(t))=\lim_{\mu\to 0}\mathbb{E}\,\|g(u_\mu^{\vartheta_\mu}(t))-\rho^{g(\xi_\mu)}(t)\|_{H^{-1}}=0. \tag{3.7.2}$$

According to (3.4.15) we have that $\vartheta_\mu\in L^2(\Omega;\mathscr{H}_1)$, for every $\mu\in(0,1)$. Hence, if we denote $\rho_\mu(t):=g(u_\mu^{\vartheta_\mu}(t))$, by proceeding as in [22, Section 5], we can rewrite equation (3.2.1) in the following way

$$\rho_\mu(t)+\mu\,v_\mu^{\vartheta_\mu}(t)=g(\xi_\mu)+\mu\,\eta_\mu+\int_0^t\Delta[B(\rho_\mu(s))]ds+\int_0^t F_g(\rho_\mu(s))ds+\int_0^t\sigma_g(\rho_\mu(s))dw^Q(s),$$

where the identity holds in H^{-1} sense. Since $\rho^{g(\xi_\mu)}$ solves equation (3.5.1) with initial condition

$g(\xi_\mu) \in L^2(\Omega; H)$ in H^{-1} sense, we have

$$\rho_\mu(t) - \rho^{g(\xi_\mu)}(t) + \mu\, v_\mu^{\vartheta_\mu}(t) = \mu\, \eta_\mu + \int_0^t \Delta\left[B(\rho_\mu(s)) - B(\rho^{g(\xi_\mu)}(s))\right] ds$$

$$+ \int_0^t \left(F_g(\rho_\mu(s)) - F_g(\rho^{g(\xi_\mu)}(s))\right) ds + \int_0^t \left(\sigma_g(\rho_\mu(s)) - \sigma_g(\rho^{g(\xi_\mu)}(s))\right) dw^Q(s).$$

If we define $\vartheta_\mu(t) := \rho_\mu(t) - \rho^{g(\xi_\mu)}(t)$, as a consequence of Itô's formula, we have

$$\begin{aligned}
&\frac{1}{2}\mathbb{E}\,\|\vartheta_\mu(t) + \mu\, v_\mu^{\vartheta_\mu}(t)\|_{H^{-1}}^2 \\
&= \frac{1}{2}\mu^2\,\mathbb{E}\,\|\eta_\mu\|_{H^{-1}}^2 - \mathbb{E}\int_0^t \left\langle B\big(\rho_\mu(s)\big) - B\big(\rho^{g(\xi_\mu)}(s)\big), \vartheta_\mu(s) + \mu\, v_\mu^{\vartheta_\mu}(s)\right\rangle_H ds \\
&\quad + \mathbb{E}\int_0^t \left\langle F_g\big(\rho_\mu(s)\big) - F_g\big(\rho^{g(\xi_\mu)}(s)\big), \vartheta_\mu(s) + \mu\, v_\mu^{\vartheta_\mu}(s)\right\rangle_{H^{-1}} ds \\
&\quad + \frac{1}{2}\mathbb{E}\int_0^t \|\sigma_g(\rho_\mu(s)) - \sigma_g(\rho^{g(\xi_\mu)}(s))\|_{\mathscr{L}_2(H_Q, H^{-1})}^2 ds
\end{aligned}$$

so that

$$\mathbb{E}\,\|\vartheta_\mu(t) + \mu\, v_\mu^{\vartheta_\mu}(t)\|_{H^{-1}}^2 \leq \mu^2\mathbb{E}\left\|\eta_\mu\right\|_{H^{-1}}^2 - 2\mu\,\mathbb{E}\int_0^t \left\langle B\big(\rho_\mu(s)\big) - B\big(\rho^{g(\xi_\mu)}(s)\big), v_\mu^{\vartheta_\mu}(s)\right\rangle_H ds$$

$$+2\mu\,\mathbb{E}\int_0^t \left\langle F_g\big(\rho_\mu(s)\big) - F_g\big(\rho^{g(\xi_\mu)}(s)\big), v_\mu^{\vartheta_\mu}(s)\right\rangle_{H^{-1}} ds - c_0\,\mathbb{E}\int_0^t \|\vartheta_\mu(s)\|_H^2 ds,$$

where

$$c_0 := 2\Big(\frac{1}{\gamma_1} - \frac{L_\sigma}{2\alpha_1\gamma_0^2} - \frac{L_f}{\alpha_1\gamma_0}\Big) > 0.$$

Since B has linear growth, thanks to (3.4.1) and (3.4.2) for every $\mu \in (0, \mu_0)$ we have

$$\mu \cdot \mathbb{E}\left|\int_0^t \left\langle B(\rho_\mu(s)) - B\big(\rho^{g(\xi_\mu)}(s)\big), v_\mu^{\vartheta_\mu}(s)\right\rangle_H ds\right|$$

$$\leq c\left(\int_0^t \left(1+\mathbb{E}\left\|\rho_\mu(t)\right\|_H^2+\mathbb{E}\left\|\rho^{g(\xi_\mu)}(t)\right\|_H^2\right)dt\right)^{\frac{1}{2}}\left(\int_0^t \mu^2\mathbb{E}\left\|v_\mu^{\vartheta_\mu}(t)\right\|_H^2 dt\right)^{\frac{1}{2}}$$

$$\leq c\left(1+t+\mathbb{E}\left\|\xi_\mu\right\|_{H^1}^2+\mu^2\mathbb{E}\|\eta_\mu\|_H^2\right)^{\frac{1}{2}}\left(\mu\, t+\mu^2+\mu^2\mathbb{E}\|\xi_\mu\|_{H^1}^2+\mu^3\mathbb{E}\left\|\eta_\mu\right\|_H^2\right)^{\frac{1}{2}}$$

$$\leq c_t\left(1+\int_{\mathscr{H}}\left(\|u\|_{H^1}^2+\mu^2\left\|v\right\|_H^2\right)\nu_\mu^{\mathscr{H}}(du,dv)\right)^{\frac{1}{2}}\left(\mu+\mu^2\int_{\mathscr{H}}\left(\|u\|_{H^1}^2+\mu\left\|v\right\|_H^2\right)\nu_\mu^{\mathscr{H}}(du,dv)\right)^{\frac{1}{2}}$$

$$\leq c_t\sqrt{\mu}\left(1+\int_{\mathscr{H}}\left(\|u\|_{H^1}^2+\mu\left\|v\right\|_H^2\right)\nu_\mu^{\mathscr{H}}(du,dv)\right).$$

Similarly, thanks to the linear growth of F_g, we have for every $\mu\in(0,\mu_0)$

$$\mu\cdot\mathbb{E}\left|\int_0^t\left\langle F_g(\rho_\mu(s))-F_g\left(\rho^{g(\xi_\mu)}(s)\right),v_\mu^{\vartheta_\mu}(s)\right\rangle_{H^{-1}}ds\right|$$

$$\leq c_t\sqrt{\mu}\left(1+\int_{\mathscr{H}}\left(\|u\|_{H^1}^2+\mu\left\|v\right\|_H^2\right)\nu_\mu^{\mathscr{H}}(du,dv)\right).$$

Moreover, due to (3.4.15) we know the family of random variable ϑ_μ satisfies (3.4.5). Then, from (3.4.6) we obtain that for every $\mu\in(0,\mu_t)$

$$\mu^2\mathbb{E}\left\|v_\mu^{\vartheta_\mu}(t)\right\|_{H^{-1}}^2\leq c_t\sqrt{\mu}+c\,\mu\left(\mathbb{E}\left\|\xi_\mu\right\|_{H^1}^2+\mu\,\mathbb{E}\left\|\eta_\mu\right\|_H^2\right)$$

$$=c_t\sqrt{\mu}+c\,\mu\int_{\mathscr{H}}\left(\|u\|_{H^1}^2+\mu\left\|v\right\|_H^2\right)\nu_\mu^{\mathscr{H}}(du,dv),$$

Therefore, from (3.4.15) and (3.1.11) we conclude that for every $\mu \in (0, \mu_t)$

$$\frac{1}{2}\mathbb{E}\,\|\rho_\mu(t)-\rho^{g(\xi_\mu)}(t)\|^2_{H^{-1}} \leq \left(\mathbb{E}\,\|\rho_\mu(t)-\rho^{g(\xi_\mu)}(t)+\mu v_\mu^{\vartheta_\mu}(t)\|^2_{H^{-1}}+\mu^2\mathbb{E}\,\|v_\mu^{\vartheta_\mu}(t)\|^2_{H^{-1}}\right)$$

$$\leq c_t\sqrt{\mu}\left(1+\int_{\mathscr{H}}\left(\|u\|^2_{H^1}+\mu\,\|v\|^2_H\right)\nu_\mu^{\mathscr{H}}(du,dv)\right),$$

and (3.7.2) follows.

Chapter 4: The small-mass limit for constrained wave equations with nonlinear conservative noise

In this chapter, we study the small-mass limit of the following system of stochastic constrained wave equations on the interval $(0,L)$

$$\begin{cases} \mu\partial_t^2 u_\mu(t,x) + \mu|\partial_t u_\mu(t)|^2_{L^2(0,L)} u_\mu(t,x), \\ \qquad = \partial_x^2 u_\mu(t,x) + |\partial_x u_\mu(t)|^2_{L^2(0,L)} u_\mu(t,x) - \gamma\partial_t u_\mu(t,x) + \sqrt{\mu}\left(u_\mu(t)\times\partial_t u_\mu(t)\right)\circ\partial_t w(t,x) \\ u_\mu(0,x) = u_0(x), \quad \partial_t u_\mu(0,x) = v_0(x), \quad u_\mu(t,0) = u_\mu(t,L) = 0, \end{cases} \tag{4.0.1}$$

where the solution $u_\mu(t)$ is constrained to live on $M = S_{L^2(0,L)}(0,1)$, the unitary sphere of $L^2(0,L)$, with initial data $(u_0,v_0)\in\mathscr{M}$, the tangent bundle of M.

Let us outline the structure of the chapter. In Section 4.1, we introduce the assumptions and notations used throughout the chapter. Section 4.2 presents the main results (see Theorem 4.4). In Section 4.3, we investigate the well-posedness of equation (4.0.1). Section 4.4 establishes uniform bounds for the solution u_μ and $\sqrt{\mu}\,\partial_t u_\mu$ with respect to $\mu\in(0,1)$. In Section 4.5, we focus on the limiting equation (4.5.1), introducing an equivalent formulation and proving the uniqueness of the solution in a suitable functional space. Finally, in Section 4.6, we demonstrate the validity of the small-mass limit. This is achieved by first integrating (4.0.1) with respect to time, rearranging all terms appropriately, and then by proving tightness and identifying the weak limit as the unique solution to equation (4.5.1).

4.1 Notations and assumptions

Let H denote the Hilbert space $L^2(0,L;\mathbb{R}^3)$, for some fixed $L>0$, endowed with the inner product

$$\langle u,v\rangle_H = \sum_{i=1}^{3}\langle u_i,v_i\rangle_{L^2(0,L)} = \int_0^L (u(x)\cdot v(x))\,dx,$$

and the corresponding norm $|\cdot|_H$. Notice that here and in what follows, we shall denote the scalar product of two vectors $h,k\in\mathbb{R}^3$ by $(h\cdot k)$. Moreover we shall denote the norm of a vector h in $\mathbb{R}^3$ by $|h|_{\mathbb{R}^3}$, or just by $|h|$, when there is no risk of confusion.

Next, for every $k\in\mathbb{N}$ we shall denote by H^k the closure of $C_0^\infty([0,L])$ in $W^{k,2}(0,L)$, where $W^{k,2}(0,L)$ is the space of all functions $u\in H$ such that $D^h u$ exists in the weak sense, for every $h\le k$, and $D^h u\in H$. Due to the Poincaré inequality, we can endow H^k with the norm

$$|u|_{H^k} := |D^k u|_H.$$

Moreover, we shall set $\mathscr{H}_k := H^{k+1}\times H^k$. When $k=0$, we will simply denote $\mathscr{H}_0$ by $\mathscr{H}$. Finally, for every function $u:(0,L)\to\mathbb{R}^3$ we shall denote

$$|u|_\infty = \sup_{x\in[0,L]} |u(x)|_{\mathbb{R}^3}.$$

Notice that since $[0,L]\subset\mathbb{R}$, we have $H^1\hookrightarrow L^\infty(0,L)$.

If E and F are Banach spaces, the class of all bounded linear operators from E to F will be denoted by $\mathscr{L}(E,F)$. We will use a shortcut notation $\mathscr{L}(E)$ for $\mathscr{L}(E,E)$. It is known that $\mathscr{L}(E,F)$ is also a Banach space. By $\mathscr{L}_2(E,E;F)$ we will denote the Banach space of all bounded bilinear operators from $E\times E =: E^2$ to F. If K is another Hilbert space, by $\mathscr{L}_2(K,F)$, we will denote the Hilbert space of all Hilbert-Schmidt operators from K to F, endowed with the natural inner product and norm. It is known that $\mathscr{L}_2(K,F)\hookrightarrow \mathscr{L}(K,F)$ continuously. If $\{e_j\}_{j\in\mathbb{N}}$ is an orthonormal basis of a separable Hilbert space K which is continuously embedded into a Banach space E and

$$\sum_{j=1}^{\infty} |e_j|_E^2 < \infty,$$

then for every $\Lambda \in \mathscr{L}_2(E\times E;F)$ we put

$$\mathrm{tr}_K(\Lambda) = \sum_{i=1}^{\infty} \Lambda(e_j,e_j). \tag{4.1.1}$$

In what follows, we denote by M the unit sphere in H

$$M = \big\{u\in H : |u|_H = 1\big\},$$

and by $\mathscr{M}$ the corresponding tangent bundle

$$\mathscr{M} = \big\{(u,v)\in M\times H : \big\langle u,v\big\rangle_H = 0\big\}.$$

Now, we rewrite equation (4.0.1) as the following system

$$\begin{cases} du_\mu(t) = v_\mu(t)dt, \\ \\ dv_\mu(t) = \dfrac{1}{\mu}\left[\partial_x^2 u_\mu(t) + |\partial_x u_\mu(t)|_H^2 u_\mu(t) - \mu |v_\mu(t)|_H^2 u_\mu(t) - \gamma v_\mu(t)\right] dt \\ \\ \qquad\qquad\qquad + \dfrac{1}{\sqrt{\mu}}\big(u_\mu(t) \times v_\mu(t)\big) \circ dw(t), \\ \\ u_\mu(0) = u_0, \quad v_\mu(0) = v_0, \qquad\quad u_\mu(t,0) = u_\mu(t,L) = 0. \end{cases} \tag{4.1.2}$$

Here $w(t)$ is a cylindrical Wiener process in H. Thus, if we denote by K its reproducing kernel Hilbert space, we have

$$w(t,x) = \sum_{i=1}^{\infty} \xi_i(x)\beta_i(t), \qquad\quad (t,x) \in [0,+\infty) \times (0,L),$$

where $\{\beta_i(t)\}_{i\in\mathbb{N}}$ is a sequence of independent standard Brownian motions, all defined on the stochastic basis $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$, and $\{\xi_i\}_{i\in\mathbb{N}}$ is an orthonormal basis of K. In what follows we shall denote by$E$ a Banach space containing K, such that the embedding of K in E is Hilbert-Schmidt. In particular

$$\sum_{i=1}^{\infty} |\xi_i|_E^2 < \infty.$$

Moreover, we shall assume the following conditions are satisfied.

Hypothesis 4.1. *All functions ξ_i belong to $C^1([0,L])$. Moreover, if we denote*

$$\varphi(x) := \sum_{i=1}^{\infty} |\xi_i(x)|^2, \qquad \varphi_1(x) := \sum_{i=1}^{\infty} |\xi_i'(x)|^2, \qquad x \in (0,L),$$

we have that φ and φ_1 belong to $L^\infty(0,L)$.

Remark 4.2. Since for every $x \in (0,L)$, we have

$$\Big|\sum_{i=1}^{\infty} \xi_i(x)\xi_i'(x)\Big| \leq \sqrt{\varphi(x)\varphi_1(x)},$$

thanks to Hypothesis 4.1 we have that φ is weakly differentiable and

$$\varphi'(x) = 2\sum_{i=1}^{\infty} \xi_i(x)\xi_i'(x), \qquad x \in (0,L).$$

In particular, this allows to conclude that $\varphi \in W^{1,\infty}(0,L)$ with

$$|\varphi'|_\infty \leq c\,(|\varphi|_\infty + |\varphi_1|_\infty).$$

$\square$

4.2 Main results

In what follows, we denote by A the realization in H of the second derivative operator, endowed with Dirichlet boundary conditions, and for all $\mu > 0$ define

$$\mathscr{A}_\mu z := \big(v, \mu^{-1}Au\big), \quad z = (u,v) \in D(\mathscr{A}_\mu) = \mathscr{H}_1.$$

Moreover, for every $\mu > 0$ we define

$$F_\mu(z) := -\mu\, |v|_H^2 u + |\partial_x u|_H^2 u, \qquad z = (u,v) \in \mathscr{H}.$$

With these notations, system (4.0.1) can be rewritten as

$$dz_\mu(t) = \mathscr{A}_\mu z_\mu(t)dt + \frac{1}{\mu}\left(0, F_\mu(z_\mu(t)) - \gamma v_\mu(t)\right)dt + \frac{1}{\sqrt{\mu}}\left(0, u_\mu(t) \times v_\mu(t)\right) \circ dw(t). \quad (4.2.1)$$

with $z_\mu(0) = z_0 = (u_0, v_0)$.

The first result we will prove in this paper is the following well-posedness result for system (4.0.1) in $\mathscr{H}$.

Theorem 4.3. *For every $\mu > 0$ and $z_0 = (u_0, v_0) \in \mathscr{H} \cap \mathscr{M}$, there exists a unique mild solution to the stochastic constrained wave equation (4.0.1). Namely, there exists a unique $\mathscr{H} \cap \mathscr{M}$-valued continuous and adapted process $z_\mu(t) = (u_\mu(t), v_\mu(t))$, $t \geq 0$, such that the following hold.*

1. *The process $u_\mu(t)$ has M-valued trajectories of class C^1 and*

$$v_\mu(t) = \partial_t u_\mu(t), \quad t \geq 0.$$

2. *The process $z_\mu(t)$ satisfies the equation*

$$z_\mu(t) = \mathscr{S}_\mu(t) z_0 + \frac{1}{\mu}\int_0^t \mathscr{S}_\mu(t-s)\big(0, F_\mu(z_\mu(s)) - \gamma v_\mu(s)\big)ds$$

$$+\frac{1}{\sqrt{\mu}}\int_0^t \mathscr{S}_\mu(t-s)\big(0, (u_\mu(s) \times v_\mu(s)) \circ dw(s)\big),$$

for every $t \geq 0$, $\mathbb{P}$-almost surely.

3. *The following identity holds for every $t \geq 0$*

$$|u_\mu(t)|^2_{H^1} + \mu\,|v_\mu(t)|^2_H + 2\gamma\int_0^t |v_\mu(s)|^2_H\,ds = |u_0|^2_{H^1} + \mu\,|v_0|^2_H, \quad \mathbb{P}-a.s. \tag{4.2.2}$$

The second result, which represents the main goal of this paper, concerns the limiting behavior of the process $u_\mu(t)$, as $\mu \downarrow 0$.

Theorem 4.4. *Fix $(u_0, v_0) \in \mathscr{H}_1 \cap \mathscr{M}$. Then, for every $T > 0$ and $\delta < 2$ and for every $\eta > 0$ we have*

$$\lim_{\mu\to 0} \mathbb{P}\Big(|u_\mu - u|_{C([0,T];H^\delta)} > \eta\Big) = 0,$$

where u is the unique solution of the following deterministic problem

$$\begin{cases} \partial_t\Big[\Big(\gamma + \dfrac{1}{2}\varphi|u(t)|^2\Big)u(t)\Big] = \partial_x^2 u(t) + |\partial_x u(t)|^2_H u(t) + \dfrac{3\varphi}{2\gamma}\left(\left[\partial_x^2 u(t) + |\partial_x u(t)|^2_H u(t)\right]\cdot u(t)\right)u(t), \\[2em] u(0,x) = u_0(x), \quad u(t,0) = u(t,L) = 0. \end{cases} \tag{4.2.3}$$

4.3 Well-posedness of stochastic constrained wave equations

As known (see e.g. [7, Definition 3.1]), for an arbitrary function $G : F \to \mathscr{L}(E,F)$ of class C^1

$$\int_0^t G(z(s)) \circ dW(s) := \int_0^t G(z(s))\,dW(s) + \frac{1}{2}\int_0^t \mathrm{tr}_K\left[G'(z(s))G(z(s))\right]ds, \tag{4.3.1}$$

with tr_K defined as in (4.1.1). Note that

$$G'(z)\,G(z) \in \mathscr{L}(E;\mathscr{L}(E,F)) \equiv \mathscr{L}_2(E\times E,F), \qquad z\in F,$$

and

$$G'(z)G(z)(e_1,e_2) = [G'(z)\big(G(z)e_1)]e_2, \qquad (e_1,e_2)\in E\times E, \quad z\in F.$$

This means that $\mathrm{tr}_K\big[G'(z)G(z)\big]$ is a well defined element of F and satisfies

$$\mathrm{tr}_K[G'(z)G(z)] = \sum_{i=1}^{\infty}[G'(z)\big(G(z)e_j\big)]e_j,$$

where $\{e_j\}_{j\in\mathbb{N}}$ is an orthonormal basis of K. In particular, if we take

$$G(u,v)k := (0,\sigma(u,v)k), \quad (u,v)\in\mathscr{H}, \quad k\in E,$$

for some $\sigma:\mathscr{H}\to\mathscr{L}(E,H^1)$, we have

$$[G'(u,v)G(u,v)](k,h) = (0,\partial_v\sigma(u,v)[\sigma(u,v)(k)](h))\,,$$

so that

$$\mathrm{tr}_K[G'(z)G(z)] = (0,\mathrm{tr}_K[\partial_v\sigma(u,v)\sigma(u,v)])\,. \tag{4.3.2}$$

In what follows, we will take

$$\sigma(u,v)k = (u\times v)k.$$

The following result holds.

Lemma 4.5. *The map $\sigma : \mathscr{H}_k \to \mathscr{T}_2(K,H^k)$ is Lipschitz-continuous on balls and has polynomial growth, for $k = 0,1$. Moreover, its Fréchet derivative along any direction $v \in H^k$ is given by*

$$\partial_v \sigma(z)y = u \times y, \quad z = (u,v) \in \mathscr{H}_k, \quad y \in H^k,$$

and the function

$$\mathscr{H}_k \ni z = (u,v) \mapsto \operatorname{tr}_K [\partial_v \sigma(z)\sigma(z)] = \operatorname{tr}_K [u \times (u \times v)] = \varphi\, [u \times (u \times v)] \in H^k, \tag{4.3.3}$$

is Lipschitz-continuous on balls with polynomial growth.

Proof. *Case $k = 0$.* For every $z_1 = (u_1, v_1)$ and $z_2 = (u_2, v_2)$ in $\mathscr{H}$ we have

$$\begin{aligned}
\|\sigma(z_1) - \sigma(z_2)\|^2_{\mathscr{T}_2(K,H)} &= \sum_{i=1}^{\infty} |(\sigma(z_1) - \sigma(z_2))\xi_i|^2_H \\
&= \int_0^L |u_1(x) \times v_1(x) - u_2(x) \times v_2(x)|^2 \sum_{i=1}^{\infty} |\xi_i(x)|^2\, dx \leq c\, |\varphi|_\infty |u_1 \times v_1 - u_2 \times v_2|^2_H \\
&\leq c\, |\varphi|_\infty \Big(|u_1 - u_2|^2_{H^1} |v_1|^2_H + |u_2|^2_{H^1} |v_1 - v_2|^2_H \Big) \leq c\, |\varphi|_\infty \big(|z_1|^2_{\mathscr{H}} + |z_2|^2_{\mathscr{H}} \big) |z_1 - z_2|^2_{\mathscr{H}}.
\end{aligned}$$

Since $\sigma(0) = 0$, this implies that $\sigma : \mathscr{H} \to \mathscr{T}_2(K,H)$ is well defined and locally Lipschitz continuous, with quadratic growth.

For every $u \in H^1$ fixed, the mapping $\sigma(u,\cdot) : H \to \mathscr{L}(E,H)$ is linear and its derivative $\partial_v \sigma(u,\cdot)$ is given by

$$\partial_v \sigma(z)\, y = u \times y \in \mathscr{L}(E,H), \qquad z = (u,v) \in \mathscr{H}, \quad y \in H.$$

In particular $\partial_v \sigma(z)\,\sigma(z) \in \mathscr{L}(E \times E, H)$ and

$$\mathrm{tr}_K\left[\partial_v \sigma(z)\sigma(z)\right] = \mathrm{tr}_K\left[u \times (u \times v)\right] = \sum_{i=1}^{\infty} \left(u \times (u \times v)\xi_i\right)\xi_i = \varphi\left(u \times (u \times v)\right).$$

Now, for every $h, k \in \mathbb{R}^3$ we have

$$h \times (h \times k) = -|h|^2 k + (h \cdot k)h, \tag{4.3.4}$$

so that we can write

$$\mathrm{tr}_K\left[\partial_v \sigma(z)\sigma(z)\right] = \varphi(-|u|^2 v + (u \cdot v)u),$$

and

$$|\mathrm{tr}_K\left[u \times (u \times v)\right]|_H^2 = \int_0^L \varphi(x)^2 \left|-|u(x)|^2 v(x) + (u(x) \cdot v(x))u(x)\right|^2 dx$$

$$= \int_0^L \varphi(x)^2 \left(|u(x)|^4 |v(x)|^2 - |u(x)|^2 (u(x) \cdot v(x))^2\right) dx \leq c\,|\varphi|_\infty^2 |u|_{H^1}^4 |v|_H^2 \leq c|\varphi|_\infty^2 |z|_{\mathscr{H}}^6.$$

Moreover,

$$\left|\mathrm{tr}_K(u_1 \times (u_1 \times v_1)) - \mathrm{tr}_K(u_2 \times (u_2 \times v_2))\right|_H^2$$

$$\leq c \int_0^L \varphi(x)^2 \left(\left(|u_1(x)|^2 + |u_2(x)|^2\right)\left(|v_1(x)|^2 + |v_2(x)|^2\right)|u_1(x) - u_2(x)|^2\right.$$

$$\left. + \left(|u_1(x)|^4 + |u_2(x)|^4\right)|v_1(x) - v_2(x)|^2\right) dx$$

$$\leq c|\varphi|_\infty^2\Big(\big(|u_1|_{H^1}^2+|u_2|_{H^1}^2\big)\big(|v_1|_H^2+|v_2|_H^2\big)|u_1-u_2|_{H^1}^2+\big(|u_1|_{H^1}^4+|u_2|_{H^1}^4\big)|v_1-v_2|_H^2\Big)$$

$$\leq c|\varphi|_\infty^2\big(|z_1|_{\mathscr{H}}^4+|z_2|_{\mathscr{H}}^4\big)|z_1-z_2|_{\mathscr{H}}^2.$$

This implies that the mapping $z\in\mathscr{H}\mapsto \mathrm{tr}_K\,[\partial_v\sigma(z)\sigma(z)]\in H$ is well-defined and locally Lischitz-continuous, with cubic growth.

Case $k=1$. For any $z_1=(u_1,v_1)$ and $z_2=(u_2,v_2)$ in $\mathscr{H}_1$, we have

$$\|\sigma(z_1)-\sigma(z_2)\|_{\mathscr{T}_2(K,H^1)}^2=\sum_{i=1}^\infty|(\sigma(z_1)-\sigma(z_2))\xi_i|_{H^1}^2$$

$$\leq 2\sum_{i=1}^\infty\left|\xi_i\,(u_1\times v_1-u_2\times v_2)'\right|_H^2+2\sum_{i=1}^\infty\left|\xi_i'\,(u_1\times v_1-u_2\times v_2)\right|_H^2$$

$$\leq c|\varphi|_\infty\Big(|(u_1-u_2)'\times v_1|_H^2+|(u_1-u_2)\times v_1'|_H^2+|u_2'\times(v_1-v_2)|_H^2+|u_2\times(v_1-v_2)'|_H^2\Big)$$

$$+c|\varphi_1|_\infty\Big(|(u_1-u_2)\times v_1|_H^2+|u_2\times(v_1-v_2)|_H^2\Big)$$

$$\leq c\big(|\varphi|_\infty+|\varphi_1|_\infty\big)\big(|u_1-u_2|_{H^1}^2|v_1|_{H^1}^2+|u_2|_{H^1}^2|v_1-v_2|_{H^1}^2\big)$$

$$\leq c\big(|\varphi|_\infty+|\varphi_1|_\infty\big)\big(|z_1|_{\mathscr{H}_1}^2+|z_2|_{\mathscr{H}_1}^2\big)|z_1-z_2|_{\mathscr{H}_1}^2.$$

This implies the local Lipschitz-continuity and polynomial growth of the mapping $\sigma : \mathscr{H}_1 \to \mathscr{T}_2(K,H^1)$. Finally, for every $z_1 = (u_1,v_1)$ and $z_2 = (u_2,v_2)$ in $\mathscr{H}_1$

$$\left|\mathrm{tr}_K(u_1 \times (u_1 \times v_1)) - \mathrm{tr}_K(u_2 \times (u_2 \times v_2))\right|^2_{H^1}$$

$$\leq c\,|\varphi|^2_\infty \Big(|(u_1-u_2)' \times (u_1 \times v_1)|^2_H + |(u_1-u_2) \times (u_1 \times v_1)'|^2_H \Big)$$

$$+c\,|\varphi|^2_\infty \Big(|u_2' \times (u_1 \times v_1 - u_2 \times v_2)|^2_H + |u_2 \times (u_1 \times v_1 - u_2 \times v_2)'|^2_H \Big)$$

$$+c\,|\varphi'|^2_\infty |u_1 \times (u_1 \times v_1) - u_2 \times (u_2 \times v_2)|^2_H$$

$$\leq c\left(|\varphi|^2_\infty + |\varphi'|^2_\infty\right)\left(|z_1|^2_{\mathscr{H}_1} + |z_2|^2_{\mathscr{H}_1}\right)^2 |z_1 - z_2|^2_{\mathscr{H}_1},$$

and this implies that the mapping

$$\mathscr{H}_1 \ni \mapsto \mathrm{tr}_K\,[u \times (u \times v)] \in H^1,$$

is locally Lipschitz continuous and has cubic growth.

$\square$

As a consequence of (4.3.1), (4.3.2) and (4.3.3), we can rewrite system (4.0.1) as

$$
\begin{cases}
du_\mu(t) = v_\mu(t)dt, \\[2ex]
dv_\mu(t) = \dfrac{1}{\mu}\big[\partial_x^2 u_\mu(t) + |\partial_x u_\mu(t)|_H^2 u_\mu(t) - \mu |v_\mu(t)|_H^2 u_\mu(t) - \gamma v_\mu(t) \\[2ex]
\qquad\qquad + \dfrac{1}{2\mu}\operatorname{tr}_K(u_\mu(t)\times(u_\mu(t)\times v_\mu(t)))\big]\,dt + \dfrac{1}{\sqrt{\mu}}\big(u_\mu(t)\times v_\mu(t)\big)dw(t), \\[2ex]
u_\mu(0) = u_0, \quad v_\mu(0) = v_0, \qquad\qquad u_\mu(t,0) = u_\mu(t,L) = 0.
\end{cases}
\tag{4.3.5}
$$

In particular, equation (4.3.5) can be rewritten as

$$
dz_\mu(t) = \mathscr{A}_\mu z_\mu(t)dt + \frac{1}{\mu}\big(0, F_\mu(z_\mu(t)) - \gamma v_\mu(t)\big)\,dt
$$

$$
+\frac{1}{2\mu}\big(0, \operatorname{tr}_K\big(u_\mu \times (u_\mu(t)\times v_\mu(t))\big)\big)\,dt + \frac{1}{\sqrt{\mu}}\big(0, u_\mu(t)\times v_\mu(t)\big)\,dw(t),
$$

with $z_\mu(0) = z_0 = (u_0, v_0)$. This allows to say that if a process $u_\mu(t)$ has M-valued trajectories of class C^1 and

$$
v_\mu(t) = \partial_t u_\mu(t), \quad t \geq 0,
$$

and $z_\mu(t) = (u_\mu(t), v_\mu(t))$, then the process z_μ is a mild solution of equation (4.1.2), with initial condition z_0, if for every $t \geq 0$, $\mathbb{P}$-almost surely,

$$z_\mu(t) = \mathscr{S}_\mu(t)z_0 + \frac{1}{\mu}\int_0^t \mathscr{S}_\mu(t-s)\big(0, F_\mu(z_\mu(s)) - \gamma v_\mu(s)\big)ds$$

$$+\frac{1}{2\mu}\int_0^t \mathscr{S}_\mu(t-s)\big(0, tr_K\left(u_\mu(s)\times(u_\mu(s)\times v_\mu(s))\right)\big)ds$$

$$+\frac{1}{\sqrt{\mu}}\int_0^t \mathscr{S}_\mu(t-s)\big(0, (u_\mu(s)\times v_\mu(s))\,dw(s)\big).$$

4.3.1 Proof of Theorem 4.3

It is immediate to check that $F_\mu : \mathscr{H}_k \to H^k$ is Lipschitz continuous when restricted to balls, for all $k \geq 0$, and has cubic growth. Thus, in view of Lemma 4.5, the proof follows from a modification of the arguments introduced in [3, proof of Theorems 2.9 and 2.10].

Due to the local Lipschitz continuity of all coefficients in $\mathscr{H}$, equation (4.3.5) admits a unique maximal local mild solution $z_\mu \in C([0,\tau_\mu);\mathscr{H})$, defined up to a certain stopping time τ_μ. Our purpose is showing that $z_\mu(t) \in \mathscr{M}$, for all $t \in [0,\tau_\mu)$, and

$$\mathbb{P}(\tau_\mu = \infty) = 1. \tag{4.3.6}$$

In this way, we get the existence and uniqueness of a global mild solution $z_\mu \in C([0,+\infty);\mathscr{H} \cap \mathscr{M})$.

In order to prove the invariance of the tangent bundle $\mathscr{M}$, we introduce the following processes

$$\vartheta_\mu(t) := \frac{1}{2}\left(|u_\mu(t)|_H^2 - 1\right), \qquad \eta_\mu(t) := \langle u_\mu(t), v_\mu(t)\rangle_H, \quad t \in [0, \tau_\mu).$$

If we show that they satisfy the linear system

$$\begin{cases} d\vartheta_\mu(t) = \eta_\mu(t)\,dt \\ d\eta_\mu(t) + \dfrac{\gamma}{\mu}\,d\vartheta_\mu(t) = \left(\dfrac{1}{\mu}|u_\mu(t)|_{H^1}^2 - |v_\mu(t)|_H^2\right)\vartheta_\mu(t), \qquad t \in [0, \tau_\mu), \end{cases} \tag{4.3.7}$$

since $\vartheta_\mu(0) = \eta_\mu(0) = 0$, we obtain that $\vartheta_\mu(t) = \eta_\mu(t) = 0$, for every $t \in [0, \tau_\mu)$, $\mathbb{P}$-a.s., and this implies that $z_\mu(t) \in \mathscr{M}$, for every $t \in [0, \tau_\mu)$, $\mathbb{P}$-a.s.

As in [3], it can be shown the following fact.

Lemma 4.6. *Assume that a local process $z_\mu(t) = (u_\mu(t), v_\mu(t))$, $t \in [0, \sigma_\mu)$ is a solution to*

$$z_\mu(t) = \mathscr{S}_\mu(t)z_0 + \int_0^t \mathscr{S}_\mu(t-s)\big(0, f(s)\big)\,ds + \int_0^t \mathscr{S}_\mu(t-s)\big(0, g(s)\big)\,dw(s), \qquad t \in [0, \sigma_\mu). \tag{4.3.8}$$

where all processes are progressively measurable, f is H-valued and g is $\mathscr{T}_2(K,H)$-valued. Then, for every $t \in [0, \sigma_\mu)$, $\mathbb{P}$-almost surely,

$$\begin{aligned} \langle u_\mu(t), v_\mu(t)\rangle_H = \langle u_\mu(0), v_\mu(0)\rangle_H - \frac{1}{\mu}\int_0^t |\partial_x u_\mu(s)|_H^2\,ds + \int_0^t \langle u_\mu(s), f(s)\rangle_H\,ds \\ + \int_0^t |v_\mu(s)|_H^2\,ds + \int_0^t \langle u_\mu(s), g(s)\,dw(s)\rangle_H, \end{aligned} \tag{4.3.9}$$

and

$$|v_\mu(t)|_H^2 + \frac{2}{\mu}|\partial_x u_\mu(t)|_H^2 = |v_\mu(0)|_H^2 + \frac{2}{\mu}|\partial_x u_\mu(0)|_H^2 + 2\int_0^t \langle v_\mu(s), f(s)\rangle_H\, ds$$

$$+2\int_0^t \langle v_\mu(s), g(s)dw(s)\rangle_H + \int_0^t \|g(s)\|^2_{\mathscr{T}_2(K,H)}\, ds. \tag{4.3.10}$$

□

Now, the local solution $z_\mu(t) = (u_\mu(t), v_\mu(t))$ of equation (4.3.5) satisfies equation (4.3.8), with

$$f(s) := -|v_\mu(s)|_H^2 u_\mu(s) + \frac{1}{\mu}|u_\mu(s)|^2_{H^1} u_\mu(s) - \frac{\gamma}{\mu} v_\mu(s) + \frac{1}{2\mu}\mathrm{tr}_K\left[u_\mu(s) \times (u_\mu(s) \times v_\mu(s))\right],$$

and

$$g(s) := \frac{1}{\sqrt{\mu}}\sigma(z_\mu(s)) = \frac{1}{\sqrt{\mu}}\, u_\mu(s) \times v_\mu(s).$$

Notice that

$$\langle u_\mu(s), f(s)\rangle_H = -|v_\mu(s)|_H^2 |u_\mu(s)|_H^2 + \frac{1}{\mu}|u_\mu(s)|^2_{H^1}|u_\mu(s)|_H^2 - \frac{\gamma}{\mu}\eta_\mu(s),$$

and for every $\xi \in K$

$$\langle u_\mu(s), g(s)\xi\rangle_H = 0.$$

Thus, thanks to identity (4.3.9) in Lemma 4.6, we have

$$\eta_\mu(t)-\eta_\mu(0)$$

$$=\frac{1}{\mu}\int_0^t |u_\mu(s)|^2_{H^1}\left(|u_\mu(s)|^2_H-1\right)ds-\int_0^t |v_\mu(s)|^2_H\left(|u_\mu(s)|^2_H-1\right)ds-\frac{\gamma}{\mu}\int_0^t \langle u_\mu(s),v_\mu(s)\rangle_H\,ds$$

$$=\frac{1}{\mu}\int_0^t |u_\mu(s)|^2_{H^1}\,\vartheta_\mu(s)\,ds-\int_0^t |v_\mu(s)|^2_H\,\vartheta_\mu(s)\,ds-\frac{\gamma}{\mu}\int_0^t \eta_\mu(s)\,ds.$$

In particular, the processes $\vartheta_\mu(t)$ and $\eta_\mu(t)$ satisfy equation (4.3.7), and, as explained above, this implies that $z_\mu(t)\in\mathscr{M}$, for every $t\in[0,\tau_\mu)$, $\mathbb{P}$-a.s.

Next, let us prove (4.3.6). Since $(u_\mu(t),v_\mu(t))\in\mathscr{M}$, we have

$$\langle v_\mu(t),f(t)\rangle_H=-\frac{\gamma}{\mu}|v_\mu(t)|^2_H+\frac{1}{2\mu}\langle v_\mu(t),\mathrm{tr}_K\left[u_\mu(s)\times(u_\mu(s)\times v_\mu(s))\right]\rangle_H.$$

As

$$\langle v_\mu(t),g(t)\xi\rangle_H=0,\qquad \xi\in K,$$

for every $t<\tau_\mu$ this gives

$$d|v_\mu(t)|^2_H=\frac{2}{\mu}\left\langle v_\mu(t),\partial_x^2 u_\mu(t)\right\rangle_H dt-\frac{2\gamma}{\mu}|v_\mu(t)|^2_H dt+\frac{1}{\mu}\,\|u(t)\times v(t)\|^2_{\mathscr{T}_2(K,H)}\,dt$$

$$+\frac{1}{\mu}\langle v_\mu(t),\mathrm{tr}_K\left[u_\mu(s)\times(u_\mu(s)\times v_\mu(s))\right]\rangle_H\,dt.$$

In view of (4.3.4), for every $h,k \in \mathbb{R}^3$ we have

$$|h\times k|^2 + (k\cdot[h\times(h\times k)]) = |h\times k|^2 + |(h\cdot k)|^2 - |h|^2|k|^2 = 0.$$

Therefore, we obtain

$$d|v_\mu(t)|_H^2 = -\frac{1}{\mu}\,d|u_\mu(t)|_{H^1}^2 - \frac{2\gamma}{\mu}|v_\mu(t)|_H^2 dt,$$

and this implies that for every $t < \tau_\mu$

$$|u_\mu(t)|_{H^1}^2 + \mu\,|v_\mu(t)|_H^2 + 2\gamma\int_0^t |v_\mu(s)|_H^2 ds = |u_0|_{H^1}^2 + \mu\,|v_0|_H^2, \quad \mathbb{P}\text{-a.s.} \tag{4.3.11}$$

In particular, we can conclude that for every $\mu > 0$ there exists some deterministic constant $\kappa_\mu > 0$ such that

$$\sup_{t\in[0,\tau_\mu)} |z_\mu(t)|_{\mathscr{H}} \leq \kappa_\mu, \qquad \mathbb{P}-\text{a.s.},$$

and (4.3.6) follows. Finally, by combining together (4.3.6) and (4.3.11), we obtain (4.2.2).

4.4 Energy estimates

In Theorem 4.3 we have seen that for every $(u_0,v_0) \in \mathscr{H}\cap\mathscr{M}$ and $\mu > 0$, equation (4.0.1) (and, equivalently, equation (4.3.5)) admits a unique mild solution $z_\mu = (u_\mu,v_\mu) \in C([0,+\infty);\mathscr{H})$. Moreover, for every $\mu > 0$ and $t > 0$ the following identity holds

$$|u_\mu(t)|_{H^1}^2 + \mu|v_\mu(t)|_H^2 + 2\gamma\int_0^t |v_\mu(s)|_H^2 ds = |u_0|_{H^1}^2 + \mu|v_0|_H^2, \quad \mathbb{P}\text{-a.s.} \tag{4.4.1}$$

In this section, our purpose is proving that for every $(u_0,v_0)\in\mathscr{H}_1\cap\mathscr{M}$, there exists a constant $c>0$ such that for every $\mu\in(0,1)$ and $t\geq 0$

$$\mathbb{E}\sup_{r\in[0,t]}\Big(|u_\mu(r)|^2_{H^2}+\mu|v_\mu(r)|^2_{H^1}\Big)+\mathbb{E}\int_0^t|v_\mu(s)|^2_{H^1}ds\leq c. \tag{4.4.2}$$

The inequality above is a consequence of the following Lemma.

Lemma 4.7. *For every $(u_0,v_0)\in\mathscr{H}_1\cap\mathscr{M}$, there exist two constants $c_1,c_2>0$ depending only on $|(u_0,\sqrt{\mu}\,v_0)|_{\mathscr{H}_1}$, φ and φ_1, such that for every $\mu\in(0,1)$ and $t\geq 0$*

$$\begin{aligned}&\mathbb{E}\sup_{r\in[0,t]}\Big(|u_\mu(r)|^2_{H^2}+\mu|v_\mu(r)|^2_{H^1}+\mu|u_\mu(r)|^2_{H^1}|v_\mu(r)|^2_{H}\Big)\\&+\mathbb{E}\int_0^t\Big(|v_\mu(s)|^2_{H^1}+|u_\mu(s)|^2_{H^2}|v_\mu(s)|^2_H+\mu|v_\mu(s)|^2_{H^1}|v_\mu(s)|^2_H+\mu|u_\mu(s)|^2_{H^1}|v_\mu(s)|^4_H\Big)ds\\&\leq c_1\Big(|u_0|^2_{H^2}+\mu|v_0|^2_{H^1}+\mu|u_0|^2_{H^1}|v_0|^2_H\Big)\exp\Big(c_2\big(|u_0|^2_{H^1}+\mu|v_0|^2_H\big)\Big).\end{aligned} \tag{4.4.3}$$

Proof. By applying Itô's formula to $|v_\mu(t)|^2_{H^1}$, we get

$$\begin{aligned}d|v_\mu(t)|^2_{H^1}=\frac{2}{\mu}\Big(&\langle v_\mu(t),\partial_x^2u_\mu(t)\rangle_{H^1}+|u_\mu(t)|^2_{H^1}\langle v_\mu(t),u_\mu(t)\rangle_{H^1}-\mu|v_\mu(t)|^2_H\langle v_\mu(t),u_\mu(t)\rangle_{H^1}\\&-\gamma|v_\mu(t)|^2_{H^1}+\frac{1}{2}\langle v_\mu(t),\mathrm{tr}_K\big(u_\mu(t)\times(u_\mu(t)\times v_\mu(t))\big)\rangle_{H^1}+\frac{1}{2}\left\|u_\mu(t)\times v_\mu(t)\right\|^2_{\mathscr{T}_2(K,H^1)}\Big)\,dt\\&+\frac{2}{\sqrt{\mu}}\langle v_\mu(t),(u_\mu(t)\times v_\mu(t))dw(t)\rangle_{H^1}\end{aligned}$$

$$= -\frac{1}{\mu} d|u_\mu(t)|^2_{H^2} + |u_\mu(t)|^2_{H^1} \left(\frac{1}{\mu} d|u_\mu(t)|^2_{H^1} + d|v_\mu(t)|^2_H \right) - d\left(|u_\mu(t)|^2_{H^1} |v_\mu(t)|^2_H \right)$$

$$+ \frac{1}{\mu} \Big(-2\gamma |v_\mu(t)|^2_{H^1} + \big\langle v_\mu(t), \mathrm{tr}_K \big(u_\mu(t) \times (u_\mu(t) \times v_\mu(t)) \big) \big\rangle_{H^1} + \big\| u_\mu(t) \times v_\mu(t) \big\|^2_{\mathscr{T}_2(K,H^1)} \Big) dt$$

$$+ \frac{2}{\sqrt{\mu}} \big\langle v_\mu(t), (u_\mu(t) \times v_\mu(t)) dw(t) \big\rangle_{H^1}. \tag{4.4.4}$$

According to (4.4.1), this implies that

$$d\Big(|u_\mu(t)|^2_{H^2} + \mu |v_\mu(t)|^2_{H^1} + \mu |u_\mu(t)|^2_{H^1} |v_\mu(t)|^2_H \Big)$$

$$= -2\gamma |u_\mu(t)|^2_{H^1} |v_\mu(t)|^2_H dt - 2\gamma |v_\mu(t)|^2_{H^1} dt + \big\langle v_\mu(t), \mathrm{tr}_K \big(u_\mu(t) \times (u_\mu(t) \times v_\mu(t)) \big) \big\rangle_{H^1} dt$$

$$+ \big\| u_\mu(t) \times v_\mu(t) \big\|^2_{\mathscr{T}_2(K,H^1)} dt + 2\sqrt{\mu} \big\langle v_\mu(t), (u_\mu(t) \times v_\mu(t)) dw(t) \big\rangle_{H^1}.$$

Now, let us define

$$Y_{\mu,a}(t) := \exp\left(-a \int_0^t |v_\mu(s)|^2_H ds \right), \quad t \geq 0, \quad \mu > 0, \tag{4.4.5}$$

for some constant $a > 0$ to be determined later. We have

$$d\Big(Y_{\mu,a}(t) \Big(|u_\mu(t)|^2_{H^2} + \mu |v_\mu(t)|^2_{H^1} + \mu |u_\mu(t)|^2_{H^1} |v_\mu(t)|^2_H \Big) \Big)$$

$$
\begin{aligned}
&= Y_{\mu,a}(t)\Big(-a|u_\mu(t)|^2_{H^2}|v_\mu(t)|^2_H - a\mu|v_\mu(t)|^2_{H^1}|v_\mu(t)|^2_H - a\mu|u_\mu(t)|^2_{H^1}|v_\mu(t)|^4_H \\
&\quad -2\gamma|u_\mu(t)|^2_{H^1}|v_\mu(t)|^2_H - 2\gamma|v_\mu(t)|^2_{H^1} + \big\langle v_\mu(t), \mathrm{tr}_K\big(u_\mu(t)\times(u_\mu(t)\times v_\mu(t))\big)\big\rangle_{H^1} \\
&\quad + \big\|u_\mu(t)\times v_\mu(t)\big\|^2_{\mathscr{T}_2(K,H^1)}\Big)dt + 2\sqrt{\mu}\, Y_{\mu,a}(t)\big\langle v_\mu(t), (u_\mu(t)\times v_\mu(t))dw(t)\big\rangle_{H^1}.
\end{aligned} \tag{4.4.6}
$$

Thus, if we integrate with respect to $t \geq 0$, we get

$$
\begin{aligned}
&Y_{\mu,a}(t)\Big(|u_\mu(t)|^2_{H^2} + \mu|v_\mu(t)|^2_{H^1} + \mu|u_\mu(t)|^2_{H^1}|v_\mu(t)|^2_H\Big) + \int_0^t Y_{\mu,a}(s)\Big(a|u_\mu(s)|^2_{H^2}|v_\mu(s)|^2_H \\
&\quad + a\mu|v_\mu(s)|^2_{H^1}|v_\mu(s)|^2_H + a\mu|u_\mu(s)|^2_{H^1}|v_\mu(s)|^4_H + 2\gamma|u_\mu(s)|^2_{H^1}|v_\mu(s)|^2_H + 2\gamma|v_\mu(s)|^2_{H^1}\Big)ds \\
&= |u_0|^2_{H^2} + \mu|v_0|^2_{H^1} + \mu|u_0|^2_{H^1}|v_0|^2_H \\
&\quad + \int_0^t Y_{\mu,a}(s)\, J(u_\mu(s), v_\mu(s))ds + 2\sqrt{\mu}\int_0^t Y_{\mu,a}(s)\, G(u_\mu(s), v_\mu(s))dw(s),
\end{aligned} \tag{4.4.7}
$$

where

$$
J(u,v) := \big\langle v, \mathrm{tr}_K(u\times(u\times v))\big\rangle_{H^1} + \|u\times v\|^2_{\mathscr{T}_2(K,H^1)}, \quad (u,v)\in\mathscr{H}_1,
$$

and

$$
G(u,v)\xi := \big\langle v, (u\times v)\xi\big\rangle_{H^1}, \quad (u,v)\in\mathscr{H}_1, \quad \xi\in K.
$$

Note that for every $(u,v) \in \mathscr{H}_1$

$$D\big(\mathrm{tr}_K\big(u \times (u \times v)\big)\big) = \big(-|u|^2 v + (u \cdot v)u\big)\,\varphi'$$

$$+\Big(-2(Du \cdot u)v - |u|^2 Dv + (Du \cdot v)u + (u \cdot Dv)u + (u \cdot v)Du\Big)\varphi,$$

so that we have

$$\big\langle v, \mathrm{tr}_K\big(u \times (u \times v)\big)\big\rangle_{H^1} = \int_0^L \Big(-|u|^2(v \cdot Dv) + (u \cdot v)(u \cdot Dv)\Big)\varphi' dx$$

$$+\int_0^L \Big(-2(Du \cdot u)(Dv \cdot v) - |u|^2|Dv|^2 + (Du \cdot v)(u \cdot Dv) + (u \cdot Dv)^2 + (u \cdot v)(Du \cdot Dv)\Big)\varphi dx. \tag{4.4.8}$$

Moreover,

$$\|u \times v\|^2_{\mathscr{T}_2(K,H^1)} = \sum_{i=1}^{\infty} \int_0^L \big|(u \times v)\xi_i' + D(u \times v)\xi_i\big|^2 dx = \int_0^L \sum_{i=1}^{\infty} \big|(u \times v)\xi_i' + D(u \times v)\xi_i\big|^2 dx$$

$$= \int_0^L |u \times v|^2 \varphi_1 dx + \int_0^L \big|D(u \times v)\big|^2 \varphi dx + 2\int_0^L (u \times v \cdot D(u \times v)) \sum_i \xi_i \xi_i' dx$$

$$= \int_0^L |u \times v|^2 \varphi_1 dx + \int_0^L \big|Du \times v + u \times Dv\big|^2 \varphi dx$$

$$+2\int_0^L (u \times v) \cdot \big(Du \times v + u \times Dv\big) \sum_i \xi_i \xi_i' dx. \tag{4.4.9}$$

Due to the well-known identity

$$((a\times b)\cdot(c\times d))=(a\cdot c)(b\cdot d)-(a\cdot d)(b\cdot c),\qquad a,b,c,d\in\mathbb{R}^3,$$

from (4.4.8) and (4.4.9) we get

$$\begin{aligned}J(u,v)=&\int_0^L (u\times v)^2\varphi_1 dx+2\int_0^L \big((u\times v)\cdot(Du\times v)\big)\sum_i \xi_i\xi_i'\,dx\\&+\int_0^L \big[(Du\times v)^2-\big((Du\times u)\cdot(Dv\times v)\big)\big]\varphi\,dx.\end{aligned}\tag{4.4.10}$$

In particular, this implies that for any $\varepsilon>0$

$$\begin{aligned}|J(u,v)|\leq\, & c|\varphi_1|_\infty|u|_{H^1}^2|v|_H^2+c\sqrt{|\varphi|_\infty|\varphi_1|_\infty}|u|_{H^1}|u|_{H^2}|v|_H^2\\&+c|\varphi|_\infty|u|_{H^2}|u|_{H^1}|v|_{H^1}|v|_H\leq\varepsilon\,|u|_{H^1}^2|v|_{H^1}^2+c(\varepsilon)|u|_{H^2}^2|v|_H^2,\end{aligned}$$

for some constant $c(\varepsilon)=c(\varepsilon,|\varphi|_\infty,|\varphi_1|_\infty)>0$. Hence, according to (4.4.1), we obtain

$$\begin{aligned}&\mathbb{E}\sup_{r\in[0,t]}\left|\int_0^r Y_{\mu,a}(s)J(u_\mu(s),v_\mu(s))ds\right|\\&\leq\varepsilon\left(|u_0|_{H^1}^2+\mu|v_0|_H^2\right)\mathbb{E}\int_0^t Y_{\mu,a}(s)|v_\mu(s)|_{H^1}^2\,ds+c(\varepsilon)\,\mathbb{E}\int_0^t Y_{\mu,a}(s)|u_\mu(s)|_{H^2}^2|v_\mu(s)|_H^2\,ds.\end{aligned}\tag{4.4.11}$$

Next, for every $(u,v) \in \mathscr{H}_1$ and $k \in K$ we have

$$G(u,v)k = \int_0^L \big(Dv \cdot (u \times v)k' + [Du \times v + u \times Dv]\,k\big)\, dx$$

$$= \int_0^L \big(Dv \cdot (u \times v)k' + (Du \times v)k\big)\, dx,$$

so that

$$\|G(u,v)\|^2_{\mathscr{T}_2(K,\mathbb{R})} = \sum_{i=1}^{\infty} \Big| \int_0^L \left[(Dv \cdot (u \times v))\, \xi_i' + (Dv \cdot (Du \times v))\, \xi_i\right]\, dx \Big|^2$$

$$\leq |v|^2_{H^1} \sum_{i=1}^{\infty} \Big(|(u \times v)\xi_i|^2_H + |(Du \times v)\xi_i|^2_H \Big) \leq c\big(|\varphi|_\infty + |\varphi_1|_\infty\big)|u|^2_{H^2}|v|^2_{H^1}|v|^2_H. \tag{4.4.12}$$

This implies that for every $\varepsilon > 0$ we can fix some $c(\varepsilon) = c(\varepsilon, |\varphi|_\infty, |\varphi_1|_\infty) > 0$ such that,

$$\sqrt{\mu}\; \mathbb{E} \sup_{r \in [0,t]} \Big| \int_0^r Y_{\mu,a}(s) G(u_\mu(s), v_\mu(s)) dw(s) \Big| \leq c\, \mathbb{E} \left(\int_0^t \mu Y^2_{\mu,a}(s) \left\| G(u_\mu(s), v_\mu(s)) \right\|^2_{\mathscr{T}_2(K,\mathbb{R})} ds \right)^{\frac{1}{2}}$$

$$\leq c \left(|\varphi|_\infty + |\varphi_1|_\infty\right)^{\frac{1}{2}} \mathbb{E} \left(\int_0^t \mu Y^2_{\mu,a}(s) |u_\mu(s)|^2_{H^2} |v_\mu(s)|^2_{H^1} |v_\mu(s)|^2_H\, ds \right)^{\frac{1}{2}}$$

$$\leq \varepsilon\mu\; \mathbb{E} \sup_{r \in [0,t]} Y_{\mu,a}(r)|v_\mu(r)|^2_{H^1} + c(\varepsilon)\mathbb{E} \int_0^t Y_{\mu,a}(s)|u_\mu(s)|^2_{H^2}|v_\mu(s)|^2_H ds. \tag{4.4.13}$$

Therefore, if we pick

$$\bar{\varepsilon} := \frac{1}{2} \wedge \gamma \left(|u_0|^2_{H^1} + \mu |v_0|^2_H\right)^{-1},$$

and $\overline{a} = a(\bar{\varepsilon}) > 0$ large enough in (4.4.11) and (4.4.13), from (4.4.7) we get

$$\mathbb{E} \sup_{r\in[0,t]} \left(Y_{\mu,\overline{a}}(r) \Big(|u_\mu(r)|^2_{H^2} + \mu |v_\mu(r)|^2_{H^1} + \mu |u_\mu(r)|^2_{H^1} |v_\mu(r)|^2_H \Big) \right)$$

$$+c\,\mathbb{E} \int_0^t Y_{\mu,\overline{a}}(s) \Big(|v_\mu(s)|^2_{H^1} + |u_\mu(s)|^2_{H^2} |v_\mu(s)|^2_H + \mu |v_\mu(s)|^2_{H^1} |v_\mu(s)|^2_H + \mu |u_\mu(s)|^2_{H^1} |v_\mu(s)|^4_H \Big) ds$$

$$\leq |u_0|^2_{H^2} + \mu |v_0|^2_{H^1} + \mu |u_0|^2_{H^1} |v_0|^4_H.$$

Finally, since (4.4.1) gives for every $t \geq 0$

$$\int_0^t |v_\mu(s)|^2_H ds \leq \frac{1}{2\gamma} \big(|u_0|^2_{H^1} + \mu |v_0|^2_H \big), \quad \mathbb{P}-\text{a.s.},$$

we conclude

$$\mathbb{E} \sup_{r\in[0,t]} \Big(|u_\mu(r)|^2_{H^2} + \mu |v_\mu(r)|^2_{H^1} + \mu |u_\mu(r)|^2_{H^1} |v_\mu(r)|^2_H \Big)$$

$$+c\mathbb{E} \int_0^t \Big(|v_\mu(s)|^2_{H^1} + |u_\mu(s)|^2_{H^2} |v_\mu(s)|^2_H + \mu |v_\mu(s)|^2_{H^1} |v_\mu(s)|^2_H + \mu |u_\mu(s)|^2_{H^1} |v_\mu(s)|^4_H \Big) ds$$

$$\leq \Big(|u_0|^2_{H^2} + \mu |v_0|^2_{H^1} + \mu |u_0|^2_{H^1} |v_0|^2_H \Big) \exp \Big(\frac{\overline{a}}{2\gamma} \big(|u_0|^2_{H^1} + \mu |v_0|^2_H \big) \Big),$$

and this implies (4.4.3).

$\square$

4.5 The limiting problem

We consider the following deterministic equation

$$\begin{cases} \partial_t\Big[\Big(\gamma+\dfrac{1}{2}\varphi|u(t)|^2\Big)u(t)\Big] = \partial_x^2 u(t)+|\partial_x u(t)|_H^2 u(t)+\dfrac{3\varphi}{2\gamma}\left(\big[\partial_x^2 u(t)+|\partial_x u(t)|_H^2 u(t)\big]\cdot u(t)\right)u(t), \\ \\ u(0,x)=u_0(x), \quad u(t,0)=u(t,L)=0, \end{cases} \tag{4.5.1}$$

where, we recall, $\varphi(x) := \sum_{i=1}^{\infty}|\xi_i(x)|^2$, for $x\in(0,L)$.

Definition 4.8. *Let $u_0\in H^1\cap M$. We say that u is a solution to equation (4.5.1) in $[0,T]$ if*

$$u\in C([0,T];H^1)\cap L^2(0,T;H^2), \qquad \partial_t u\in L^2(0,T;H),$$

and the identity

$$\Big(\gamma+\frac{1}{2}\varphi|u(t)|^2\Big)u(t) = \Big(\gamma+\frac{1}{2}\varphi|u_0|^2\Big)u_0$$

$$+\int_0^t\left(\partial_x^2 u(s)+|\partial_x u(s)|_H^2 u(s)+\frac{3\varphi}{2\gamma}\left(\big[\partial_x^2 u(s)+|\partial_x u(s)|_H^2 u(s)\big]\cdot u(s)\right)u(s)\right)ds,$$

holds in H, for a.e. $t\in[0,T]$.

In the following lemma we show that equation (4.5.1) has an equivalent formulation.

Lemma 4.9. *Let $u_0\in H^1\cap M$. Then any function $u\in C([0,T];H^1)\cap L^2(0,T;H^2)$, with $\partial_t u\in$*

$L^2(0,T;H)$, satisfies equation (4.5.1) if and only if satisfies the following equation

$$\begin{cases} \gamma\partial_t u(t) = \partial_x^2 u(t) + |\partial_x u(t)|_H^2 u(t) + \dfrac{1}{2}\operatorname{tr}_K\big(u(t)\times(u(t)\times\partial_t u(t))\big), \quad t>0, \\ \\ u(0,x) = u_0(x), \quad u(t,0) = u(t,L) = 0. \end{cases} \tag{4.5.2}$$

Proof. If u satisfies equation (4.5.2), then we have

$$\gamma\partial_t u(t) = \partial_x^2 u(t) + |\partial_x u(t)|_H^2 u + \frac{1}{2}\varphi\big(-|u(t)|^2\partial_t u(t) + (u(t)\cdot\partial_t u(t))u(t)\big)$$

$$= \partial_x^2 u(t) + |\partial_x u(t)|_H^2 u(t) + \frac{1}{2}\varphi\Big(-\partial_t(|u(t)|^2 u(t)) + 3(u(t)\cdot\partial_t u(t))u(t)\Big),$$

the identity holding $\mathbb{P}$-a.s. in $L^2(0,T;H)$. Then, since

$$\gamma u(t)\cdot\partial_t u(t) = \partial_x^2 u(t)\cdot u(t) + |\partial_x u(t)|_H^2|u(t)|^2,$$

we have

$$\gamma(u(t)\cdot\partial_t u(t))u(t) = (\partial_x^2 u(t)\cdot u(t))u(t) + |\partial_x u(t)|_H^2|u(t)|^2 u(t).$$

This implies that

$$\gamma\partial_t u(t) + \frac{1}{2}\varphi\partial_t(|u(t)|^2 u(t))$$

$$= \partial_x^2 u(t) + |\partial_x u(t)|_H^2 u(t) + \frac{3\varphi}{2\gamma}(\partial_x^2 u(t)\cdot u(t))u(t) + \frac{3\varphi}{2\gamma}|\partial_x u(t)|_H^2|u(t)|^2 u(t).$$

On the other hand, if u is a solution of equation (4.5.1), in order to prove that it is also a solution to (4.5.2) it suffices to show that

$$\gamma u(t)\cdot\partial_t u(t) = \partial_x^2 u(t)\cdot u(t) + |\partial_x u(t)|_H^2 |u(t)|^2.$$

Indeed, note that

$$\gamma\partial_t u(t) + \frac{1}{2}\varphi\Big(2(u(t)\cdot\partial_t u(t))u(t) + |u(t)|^2\partial_t u(t)\Big)$$

$$= \partial_x^2 u(t) + |\partial_x u(t)|_H^2 u(t) + \frac{3\varphi}{2\gamma}(\partial_x^2 u(t)\cdot u(t))u(t) + \frac{3\varphi}{2\gamma}|\partial_x u(t)|_H^2 |u(t)|^2 u(t),$$

so that if we take the scalar product by u of both sides, we get

$$\Big(1+\frac{3}{2\gamma}\varphi|u(t)|^2\Big)\Big((\gamma u(t)\cdot\partial_t u(t)) - (\partial_x^2 u(t)\cdot u(t)) - |\partial_x u(t)|_H^2|u(t)|^2\Big) = 0,$$

which completes the proof.

□

Remark 4.10. *By using the* (4.5.2) *formulation of equation* (4.5.1)*, it is immediate to check that if $u_0\in M$ then $u(t)\in M$, for every $t\in[0,T]$. Actually, for any $t>0$ we have*

$$\frac{1}{2}\frac{d}{dt}\Big(|u(t)|_H^2 - 1\Big) = \big\langle\partial_x^2 u(t), u(t)\big\rangle_H + |\partial_x u(t)|_H^2|u(t)|_H^2 + \frac{1}{2}\big\langle\varphi(u(t)\times(u(t)\times\partial_t u(t))), u(t)\big\rangle_H$$

$$= |\partial_x u(t)|_H^2\Big(|u(t)|_H^2 - 1\Big).$$

Combined with the fact that $|u_0|_H - 1 = 0$ and $\partial_x u\in C([0,T];H)$, this implies that $|u(t)|_H = 1$, for any $t\geq 0$.

Lemma 4.11. *Let u be a solution to equation (4.5.1), with $u_0 \in H^1 \cap M$. Then for every $t \geq 0$ we have*

$$|u(t)|_{H^1}^2 + 2\gamma \int_0^t |\partial_t u(s)|_H^2 \, ds \leq |u_0|_{H^1}^2. \tag{4.5.3}$$

Proof. If we use the (4.5.2) formulation of equation (4.5.1), we get

$$\gamma|\partial_t u(t)|_H^2 = \left\langle \partial_x^2 u(t), \partial_t u(t) \right\rangle_H + \left\langle |\partial_x u(t)|_H^2 u(t), \partial_t u(t) \right\rangle_H$$

$$-\frac{1}{2}\left\langle \varphi |u(t)|^2 \partial_t u(t), \partial_t u(t) \right\rangle_H + \frac{1}{2}\left\langle \varphi (u(t) \cdot \partial_t u(t)) u(t), \partial_t u(t) \right\rangle_H.$$

Recalling that $|u(t)|_H = 1$, this gives

$$\gamma|\partial_t u(t)|_H^2 = -\frac{1}{2}\frac{d}{dt}|u(t)|_{H^1}^2 - \frac{1}{2}\left\langle \varphi |u(t)|^2 \partial_t u(t), \partial_t u(t) \right\rangle_H + \frac{1}{2}\left\langle \varphi (u(t) \cdot \partial_t u(t)) u(t), \partial_t u(t) \right\rangle_H$$

$$\leq -\frac{1}{2}\frac{d}{dt}|u(t)|_{H^1}^2,$$

and (4.5.3) follows once we integrate both sides in time.

□

Proposition 4.12. *Let u_1 and u_2 be any two solutions of equation (4.5.1), with initial conditions $u_{1,0}, u_{2,0} \in H^2 \cap M$, respectively. Then there exist some constants $c_1, c_2 > 0$, depending only on $u_{1,0}$, $u_{2,0}$ and φ, such that for every $t \geq 0$*

$$|u_1(t) - u_2(t)|_{H^1}^2 + \int_0^t |\partial_t u_1(s) - \partial_t u_2(s)|_H^2 ds \leq c_1 |u_{1,0} - u_{2,0}|_{H^1}^2 e^{c_2 t}. \tag{4.5.4}$$

In particular, there is at most one solution to equation (4.5.1) in $C([0,T];H^1) \cap L^2(0,T;H^2)$, with

$\partial_t u \in L^2(0,T;H)$.

Proof. Let us fix $u_{1,0}, u_{2,0} \in H^1 \cap M$ and let u_1, u_2 be solutions of equation (4.5.1) with initial conditions $u_{1,0}, u_{2,0}$, respectively. If we denote $v_1 = \partial_t u_1$ and $v_2 = \partial_t u_2$, then, recalling (4.4.5)

$$\gamma|(v_1 - v_2)(t)|_H^2 = \big\langle \partial_x^2 (u_1 - u_2)(t), (v_1 - v_2)(t) \big\rangle_H$$

$$+ \big\langle |\partial_x u_1(t)|_H^2 u_1(t) - |\partial_x u_2(t)|_H^2 u_2(t), (v_1 - v_2)(t) \big\rangle_H$$

$$- \frac{1}{2} \big\langle \varphi\big(|u_1(t)|^2 v_1(t) - |u_2(t)|^2 v_2(t)\big), (v_1 - v_2)(t) \big\rangle_H$$

$$+ \frac{1}{2} \big\langle \varphi\big((u_1(t) \cdot v_1(t)) u_1(t) - (u_2(t) \cdot v_2(t)) u_2(t)\big), (v_1 - v_2)(t) \big\rangle_H.$$

Hence, if a is an arbitrary positive constant and we denote

$$Y_a(t) := \exp\left(-a \int_0^t \big(1 + |v_2(s)|_H^2\big) ds \right), \quad t \geq 0, \tag{4.5.5}$$

we get

$$\frac{d}{dt}\Big(Y_a(t)|(u_1 - u_2)(t)|_{H^1}^2\Big) = Y_a(t)\Big(\frac{d}{dt}|(u_1 - u_2)(t)|_{H^1}^2 - a\big(1 + |v_2(t)|_H^2\big)|(u_1 - u_2)(t)|_{H^1}^2\Big)$$

$$= Y_a(t)\Big(- a\big(1 + |v_2(t)|_H^2\big)|(u_1 - u_2)(t)|_{H^1}^2 + 2\big\langle |u_1(t)|_{H^1}^2 u_1(t) - |u_2(t)|_{H^1}^2 u_2(t), (v_1 - v_2)(t) \big\rangle_H$$

$$-2\gamma|(v_1-v_2)(t)|_H^2-\left\langle\varphi\big(|u_1(t)|^2v_1(t)-|u_2(t)|^2v_2(t)\big),(v_1-v_2)(t)\right\rangle_H$$

$$+\left\langle\varphi\big((u_1(t)\cdot v_1(t))u_1(t)-(u_2(t)\cdot v_2(t))u_2(t)\big),(v_1-v_2)(t)\right\rangle_H\Big). \qquad (4.5.6)$$

Now, for any $\varepsilon>0$ we can find $c(\varepsilon)>0$ such that

$$\left\langle|u_1(t)|_{H^1}^2u_1(t)-|u_2(t)|_{H^1}^2u_2(t),(v_1-v_2)(t)\right\rangle_H$$

$$=|u_1(t)|_{H^1}^2\left\langle(u_1-u_2)(t),(v_1-v_2)(t)\right\rangle_H+\left\langle\big(|u_1(t)|_{H^1}^2-|u_2(t)|_{H^1}^2\big)u_2(t),(v_1-v_2)(t)\right\rangle_H$$

$$\leq|u_1(t)|_{H^1}^2\Big(\varepsilon|(v_1-v_2)(t)|_H^2+c(\varepsilon)|(u_1-u_2)(t)|_H^2\Big)$$

$$+c|u_2(t)|_H\big(|u_1(t)|_{H^1}+|u_2(t)|_{H^1}\big)\Big(\varepsilon|(v_1-v_2)(t)|_H^2+c(\varepsilon)|(u_1-u_2)(t)|_{H^1}^2\Big). \qquad (4.5.7)$$

Moreover

$$-\left\langle\varphi\big(|u_1(t)|^2v_1(t)-|u_2(t)|^2v_2(t)\big),(v_1-v_2)(t)\right\rangle_H$$

$$=-\left\langle\varphi|u_1(t)|^2(v_1-v_2)(t),(v_1-v_2)(t)\right\rangle_H-\left\langle\varphi\big(|u_1(t)|^2-|u_2(t)|^2\big)v_2(t),(v_1-v_2)(t)\right\rangle_H$$

$$\leq-\left\langle\varphi|u_1(t)|^2(v_1-v_2)(t),(v_1-v_2)(t)\right\rangle_H$$

$$+|\varphi|_\infty\big(|u_1(t)|_{H^1}+|u_2(t)|_{H^1}\big)|(u_1-u_2)(t)|_{H^1}|v_2(t)|_H|(v_1-v_2)(t)|_H$$

$$\leq -\big\langle \varphi|u_1(t)|^2(v_1-v_2)(t),(v_1-v_2)(t)\big\rangle_H$$

$$+c|\varphi|_\infty\big(|u_1(t)|_{H^1}+|u_2(t)|_{H^1}\big)\Big(\varepsilon|(v_1-v_2)(t)|_H^2+c(\varepsilon)|v_2(t)|_H^2|(u_1-u_2)(t)|_{H^1}^2\Big), \tag{4.5.8}$$

and

$$\big\langle \varphi\big((u_1(t)\cdot v_1(t))u_1(t)-(u_2(t)\cdot v_2(t))u_2(t)\big),(v_1-v_2)(t)\big\rangle_H$$

$$=\big\langle \varphi\big(u_1(t)\cdot(v_1-v_2)(t)\big)u_1(t),(v_1-v_2)(t)\big\rangle_H$$

$$+\big\langle \varphi\big((u_1-u_2)(t)\cdot v_2(t)\big)u_1(t),(v_1-v_2)(t)\big\rangle_H+\big\langle \varphi(u_2(t)\cdot v_2(t))(u_1-u_2)(t),(v_1-v_2)(t)\big\rangle_H$$

$$\leq \big\langle \varphi\big(u_1(t)\cdot(v_1-v_2)(t)\big)u_1(t),(v_1-v_2)(t)\big\rangle_H$$

$$+c|\varphi|_\infty\big(|u_1(t)|_{H^1}+|u_2(t)|_{H^1}\big)\Big(\varepsilon|(v_1-v_2)(t)|_H^2+c(\varepsilon)|v_2(t)|_H^2|(u_1-u_2)(t)|_{H^1}^2\Big). \tag{4.5.9}$$

Since $\varphi \geq 0$, we have

$$-\big\langle \varphi |u_1|^2 (v_1 - v_2), v_1 - v_2 \big\rangle_H + \big\langle \varphi \big(u_1 \cdot (v_1 - v_2)\big) u_1, v_1 - v_2 \big\rangle_H \leq 0.$$

Hence, thanks to (4.5.3) we can take $\bar{\varepsilon} > 0$ sufficiently small and $\bar{a} = \bar{a}(\bar{\varepsilon}) > 0$ sufficiently large so that if we replace (4.5.7), (4.5.8) and (4.5.9) into (4.5.6), we get

$$\begin{aligned} Y_{\bar{a}}(t)|u_1(t) - u_2(t)|_{H^1}^2 + c \int_0^t Y_{\bar{a}}(s)\big(1 + |v_2(s)|_H^2\big)|u_1(s) - u_2(s)|_{H^1}^2 ds \\ + c \int_0^t Y_{\bar{a}}(s)|v_1(s) - v_2(s)|_H^2 ds \leq |u_{1,0} - u_{2,0}|_{H^1}^2, \end{aligned} \tag{4.5.10}$$

for some constant $c = c(u_{1,0}, u_{2,0}, \varphi) > 0$. Finally, since from (4.5.3)

$$Y_{\bar{a}}(t) \geq \exp\Big(-\bar{a}\Big(t + |u_{2,0}|_{H^1}^2 / 2\gamma\Big)\Big), \quad t > 0,$$

we can complete the proof of (4.5.4).

□

4.6 Proof of the validity of the small-mass limit

In this final section, we conclude the proof of Theorem 4.4. We first prove some identities, then we investigate tightness and finally we proceed with the proof of the theorem.

4.6.1 An identity

Lemma 4.13. *For every $\mu > 0$ and $(u_0, v_0) \in \mathscr{H}_1 \cap \mathscr{M}$ the solution $(u_\mu(t), v_\mu(t))$ of system* (4.0.1) *(or, equivalently system* (4.3.5)*) satisfies the following identity, for every $t \geq 0$*

$$\gamma u_\mu(t) + \frac{1}{2}\varphi|u_\mu(t)|^2 u_\mu(t) + \mu v_\mu(t)$$

$$= \gamma u_0 + \frac{1}{2}\varphi|u_0|^2 u_0 + \mu v_0 + \int_0^t \partial_x^2 u_\mu(s)ds + \int_0^t |\partial_x u_\mu(s)|_H^2 u_\mu(s)ds$$

$$+\frac{3}{2\gamma}\varphi \int_0^t \big(\partial_x^2 u_\mu(s)\cdot u_\mu(s)\big)u_\mu(s)ds + \frac{3}{2\gamma}\varphi\int_0^t |\partial_x u_\mu(s)|_H^2 |u_\mu(s)|^2 u_\mu(s)ds + R_\mu(t), \tag{4.6.1}$$

where

$$R_\mu(t) = \frac{3\mu}{2\gamma}\varphi(u_0\cdot v_0)u_0 - \frac{3\mu}{2\gamma}\varphi(u_\mu(t)\cdot v_\mu(t))u_\mu(t) - \mu\int_0^t |v_\mu(s)|_H^2 u_\mu(s)ds$$

$$+\frac{3\mu}{2\gamma}\varphi\int_0^t (u_\mu(s)\cdot v_\mu(t))v_\mu(s)ds + \frac{3\mu}{2\gamma}\varphi\int_0^t |v_\mu(s)|^2 u_\mu(s)ds$$

$$-\frac{3\mu}{2\gamma}\varphi\int_0^t |v_\mu(s)|_H^2|u_\mu(s)|^2 u_\mu(s)ds + \sqrt{\mu}\int_0^t (u_\mu(s)\times v_\mu(s))dw(s)$$

$$=: \frac{3\mu}{2\gamma}\varphi(u_0\cdot v_0)u_0 + \sum_{i=1}^{6} J_{\mu,i}(t). \tag{4.6.2}$$

Proof. In view of (4.3.5), we have

$$d(u_\mu(t)\cdot v_\mu(t)) = |v_\mu(t)|^2dt + \frac{1}{\mu}\big(\partial_x^2 u_\mu(t)\cdot u_\mu(t)\big)dt + \frac{1}{\mu}|\partial_x u_\mu(t)|_H^2|u_\mu(t)|^2dt$$

$$-|v_\mu(t)|_H^2|u_\mu(t)|^2dt - \frac{\gamma}{\mu}(u_\mu(t)\cdot v_\mu(t))dt + \frac{1}{2\mu}\varphi\Big(-|u_\mu(t)|^2v_\mu(t) + (u_\mu(t)\cdot v_\mu(t))u_\mu(t)\Big)\cdot u_\mu(t)dt$$

$$+\frac{1}{\sqrt{\mu}}u_\mu(t)\cdot(u_\mu(t)\times v_\mu(t))dw(t) = |v_\mu(t)|^2dt + \frac{1}{\mu}\big(\partial_x^2 u_\mu(t)\cdot u_\mu(t)\big)dt$$

$$+\frac{1}{\mu}|\partial_x u_\mu(t)|_H^2|u_\mu(t)|^2dt - |v_\mu(t)|_H^2|u_\mu(t)|^2dt - \frac{\gamma}{\mu}(u_\mu(t)\cdot v_\mu(t))dt.$$

This implies that

$$d\Big((u_\mu(t)\cdot v_\mu(t))u_\mu(t)\Big)$$

$$= (u_\mu(t)\cdot v_\mu(t))v_\mu(t)dt + |v_\mu(t)|^2u_\mu(t)dt + \frac{1}{\mu}\big(\partial_x^2 u_\mu(t)\cdot u_\mu(t)\big)u_\mu(t)dt$$

$$+\frac{1}{\mu}|\partial_x u_\mu(t)|_H^2|u_\mu(t)|^2u_\mu(t)dt - |v_\mu(t)|_H^2|u_\mu(t)|^2u_\mu(t)dt - \frac{\gamma}{\mu}(u_\mu(t)\cdot v_\mu(t))u_\mu(t)dt.$$

Hence, for every $t\geq 0$, we obtain

$$\gamma\int_0^t(u_\mu(s)\cdot v_\mu(s))u_\mu(s)ds = \int_0^t\big(\partial_x^2 u_\mu(s)\cdot u_\mu(s)\big)u_\mu(s)ds + \int_0^t|\partial_x u_\mu(s)|_H^2|u_\mu(s)|^2u_\mu(s)ds$$

$$-\mu(u_\mu(t)\cdot v_\mu(t))u_\mu(t)+\mu(u_0\cdot v_0)u_0+\mu\int_0^t(u_\mu(s)\cdot v_\mu(s))v_\mu(s)ds+\mu\int_0^t|v_\mu(s)|^2u_\mu(s)ds$$

$$-\mu\int_0^t|v_\mu(s)|_H^2|u_\mu(s)|^2u_\mu(s)ds.$$

By using the fact that

$$d(|u(t)|^2u(t))=2(u(t)\cdot v(t))u(t)+|u(t)|^2v(t),$$

this implies

$$\int_0^t\Big(-|u_\mu(s)|^2v_\mu(s)+(u_\mu(s)\cdot v_\mu(s))u_\mu(s)\Big)ds$$

$$=-|u_\mu(t)|^2u_\mu(t)+|u_0|^2u_0+3\int_0^t(u_\mu(s)\cdot v_\mu(s))u_\mu(s)ds=-|u_\mu(t)|^2u_\mu(t)+|u_0|^2u_0$$

$$+\frac{3}{\gamma}\int_0^t\big(\partial_x^2u_\mu(s)\cdot u_\mu(s)\big)u_\mu(s)ds+\frac{3}{\gamma}\int_0^t|\partial_xu_\mu(s)|_H^2|u_\mu(s)|^2u_\mu(s)ds$$

$$-\frac{3\mu}{\gamma}(u_\mu(t)\cdot v_\mu(t))u_\mu(t)+\frac{3\mu}{\gamma}(u_0\cdot v_0)u_0+\frac{3\mu}{\gamma}\int_0^t(u_\mu(s)\cdot v_\mu(s))v_\mu(s)ds$$

$$+\frac{3\mu}{\gamma}\int_0^t|v_\mu(s)|^2u_\mu(s)ds-\frac{3\mu}{\gamma}\int_0^t|v_\mu(s)|_H^2|u_\mu(s)|^2u_\mu(s)ds. \tag{4.6.3}$$

Finally, we rewrite system (4.3.5) in the following form

$$\gamma u_\mu(t)+\mu v_\mu(t)=\gamma u_0+\mu v_0+\int_0^t \partial_x^2 u_\mu(s)ds+\int_0^t |\partial_x u_\mu(s)|_H^2 u_\mu(s)ds-\mu\int_0^t |v_\mu(s)|_H^2 u_\mu(s)ds$$

$$+\frac{1}{2}\varphi\int_0^t \Big(-|u_\mu(s)|^2 v_\mu(s)+(u_\mu(s)\cdot v_\mu(s))u_\mu(s)\Big)ds+\sqrt{\mu}\int_0^t (u_\mu(s)\times v_\mu(s))dw(s). \tag{4.6.4}$$

Therefore, if we replace (4.6.3) into (4.6.4), we get (4.6.1), with $R_\mu(t)$ defined by (4.6.2). □

Lemma 4.14. *For every* $(u_0,v_0)\in\mathscr{H}_1\cap\mathscr{M}$ *and* $t>0$ *we have*

$$\lim_{\mu\to 0}\mathbb{E}\sup_{r\in[0,t]}|R_\mu(r)|_H^2=0. \tag{4.6.5}$$

Proof. First, note that, since

$$|J_{\mu,1}(t)|_H\leq c\,|\varphi|_\infty \mu |u_\mu(t)|_{H^1}^2|v_\mu(t)|_H,$$

due to (4.4.1) we have

$$\lim_{\mu\to 0}\mathbb{E}\sup_{r\in[0,t]}|J_{\mu,1}(r)|_H^2=0, \tag{4.6.6}$$

and since

$$|J_{\mu,2}(r)|_H^2\leq ct\mu^2\int_0^t |v_\mu(s)|_H^4 ds\leq ct\mu^2\int_0^t |v_\mu(s)|_H^4|u_\mu(s)|_{H^1}^2\,ds,\qquad r\in[0,t],$$

due to (4.4.3) we have

$$\lim_{\mu\to 0}\mathbb{E}\sup_{r\in[0,t]}|J_{\mu,2}(r)|_H^2=0. \tag{4.6.7}$$

Moreover, thanks to (4.4.1), (4.4.3) and

$$|J_{\mu,3}(r)|_H^2 + |J_{\mu,4}(r)|_H^2 \leq ct|\varphi|_\infty^2 \mu^2 \int_0^t |u_\mu(s)|_{H^1}^2 |v_\mu(s)|_{H^1}^2 |v_\mu(s)|_H^2 ds, \quad r \in [0,t],$$

we have

$$\lim_{\mu \to 0} \mathbb{E} \sup_{r \in [0,t]} \left(|J_{\mu,3}(r)|_H^2 + |J_{\mu,4}(r)|_H^2 \right) = 0. \tag{4.6.8}$$

From (4.4.1), (4.4.3), and

$$|J_{\mu,5}|_H^2 \leq ct|\varphi|_\infty^2 \mu^2 \mathbb{E} \int_0^t |u_\mu(s)|_{H^1}^6 |v_\mu(s)|_H^4 ds, \quad r \in [0,t],$$

it follows that

$$\lim_{\mu \to 0} \mathbb{E} \sup_{r \in [0,t]} |J_{\mu,5}(r)|_H^2 = 0. \tag{4.6.9}$$

Furthermore, we have

$$\mathbb{E} \sup_{r \in [0,t]} |J_{\mu,6}(r)|_H^2 \leq c\mu \mathbb{E} \int_0^t \left\| u_\mu(s) \times v_\mu(s) \right\|_{\mathscr{T}_2(K,H)}^2 ds \leq c|\varphi|_\infty \mu \mathbb{E} \int_0^t |u_\mu(s)|_{H^1}^2 |v_\mu(s)|_H^2 ds,$$

so that, thanks again to (4.4.1),

$$\lim_{\mu \to 0} \mathbb{E} \sup_{r \in [0,t]} |J_{\mu,6}(r)|_H^2 = 0. \tag{4.6.10}$$

Finally, combining (4.6.6)-(4.6.10), we complete our proof. □

4.6.2 Tightness

Proposition 4.15. *For every $(u_0, v_0) \in \mathscr{H}_1 \cap \mathscr{M}$ and $T > 0$, the family of probability measures $\left\{Ł(u_\mu)\right\}_{\mu\in(0,1)}$ is tight in $C([0,T];H^\delta)$, for every $\delta < 2$.*

Proof. According to (4.4.2), we have that

$$\sup_{\mu\in(0,1)} \mathbb{E}\left(\sup_{t\in[0,T]} |u_\mu(t)|^2_{H^2} + \int_0^T |\partial_t u_\mu(s)|^2_{H^1} ds\right) < +\infty. \tag{4.6.11}$$

This, in particular, implies that for every $\varepsilon > 0$ there exists $L_\varepsilon > 0$ such that if we denote by K_ε the ball of radius L_ε in $C([0,T];H^2) \cap W^{1,2}(0,T;H^1)$, then

$$\inf_{\mu\in(0,1)} \mathbb{P}(u_\mu \in K_\varepsilon) \geq 1 - \varepsilon.$$

Due to the Aubin-Lions lemma, we know that the set K_ε is compact in $C([0,T];H^\delta)$, for every $\delta < 2$. □

4.6.3 Proof of Theorem 4.4

Thanks to Proposition 4.15 and (4.4.2), we have the family $\left\{Ł(u_\mu, \mu\,\partial_t u_\mu)\right\}_{\mu\in(0,1)}$ is tight in $C([0,T];H^\delta) \times C([0,T];H^1)$, for any $\delta < 2$. If for every $T > 0$ and $\delta < 2$ we define

$$\Gamma_{T,\delta} := C([0,T];H^\delta) \times C([0,T];H^1) \times C([0,T];E),$$

where E is any Banach space such that the embedding $K \subset E$ is Hilbert-Schmidt, then as a consequence of Skorokhod's theorem, for any sequence $\{\mu_k\}_{k\in\mathbb{N}}$, converging to zero, there exists a subsequence, still denoted by $\{\mu_k\}_{k\in\mathbb{N}}$, and some $\Gamma_{T,\delta}$-valued random variables

$$\mathscr{Y}_k := \big(\rho_k, \mu_k\vartheta_k, \hat{w}_k\big), \quad \mathscr{Y} := \big(\rho, \vartheta, \hat{w}\big), \quad k\in\mathbb{N},$$

all defined on some probability space $\big(\hat{\Omega}, \hat{\mathscr{F}}, \{\hat{\mathscr{F}}_t\}_{t\in[0,T]}, \hat{\mathbb{P}}\big)$, such that

$$Ł(\mathscr{Y}_k) = Ł\big(u_{\mu_k}, \mu_k\partial_t u_{\mu_k}, w\big), \quad k\in\mathbb{N}, \tag{4.6.12}$$

and

$$\lim_{k\to\infty}\Big(|\rho_k-\rho|_{C([0,T];H^\delta)} + \mu_k|\vartheta_k|_{C([0,T];H^1)} + |\hat{w}_k-\hat{w}|_{C([0,T];E)}\Big) = 0, \quad \hat{\mathbb{P}}-\text{a.s..} \tag{4.6.13}$$

In particular, due to (4.4.1), (4.4.2), (4.6.11) and (4.6.12), we have

$$\sup_{k\in\mathbb{N}}\left(\hat{\mathbb{E}}\sup_{t\in[0,T]}\Big(|\rho_k(t)|^2_{H^2} + \mu_k|\vartheta_k(t)|^2_{H^1}\Big) + \hat{\mathbb{E}}\int_0^T|\vartheta_k(s)|^2_{H^1}ds\right) < +\infty, \tag{4.6.14}$$

and there exists a deterministic $c>0$ such that

$$\sup_{k\in\mathbb{N}}\left(\sup_{t\in[0,T]}|\rho_k(t)|_{H^1} + \int_0^T|\partial_t\rho_k(s)|^2_H\,ds\right) \leq c, \quad \hat{\mathbb{P}}\text{-a.s.} \tag{4.6.15}$$

Thanks to (4.6.13) and (4.6.15), it follows that $\rho\in L^2(\hat{\Omega};L^\infty(0,T;H^\delta))$, for every $\delta<2$, and

$$\sup_{t\in[0,T]}|\rho(t)|_{H^1} \leq c, \quad \hat{\mathbb{P}}\text{-a.s..} \tag{4.6.16}$$

Moreover, from (4.6.14) and (4.6.15) we get that ρ is weakly differentiable in time, with $\partial_t \rho \in L^2(0,T;H)$ and

$$\int_0^T |\partial_t \rho(s)|_H^2\, ds \leq c, \qquad \hat{\mathbb{P}}\text{-a.s.} \tag{4.6.17}$$

Finally, as a consequence of (4.6.13) and (4.6.14), we have

$$\int_0^T \hat{\mathbb{E}} |\rho(s)|_{H^2}^2\, ds < \infty. \tag{4.6.18}$$

Now, if we can show that ρ solves equation (4.5.1), then by the uniqueness of solutions for equation (4.5.1), due to the classical argument by Gyongy and Krylov (see [38]), we can conclude that u_{μ_k} converges to u in $C([0,T];H^\delta)$, for every sequence $\mu_k \downarrow 0$, and the convergence is in probability.

Due to (4.6.12) and identities (4.6.1) and (4.6.2), we have that for every $\psi \in C_0^\infty([0,L])$

$$\langle \gamma\rho_k(t) + \frac{1}{2}\varphi|\rho_k(t)|^2\rho_k(t) + \mu_k \vartheta_k(t), \psi\rangle_H = \langle \gamma u_0 + \frac{1}{2}\varphi|u_0|^2 u_0 + \mu_k v_0, \psi\rangle_H + \int_0^t \langle \partial_x^2 \rho_k(s), \psi\rangle_H ds$$

$$+ \int_0^t \langle |\rho_k(s)|_{H^1}^2 \rho_k(s), \psi\rangle_H ds + \frac{3}{2\gamma}\int_0^t \langle \varphi\big(\partial_x^2\rho_k(s)\cdot\rho_k(s)\big)\rho_k(s), \psi\rangle_H ds$$

$$+ \frac{3}{2\gamma}\int_0^t \langle \varphi|\rho_k(s)|_{H^1}^2 |\rho_k(s)|^2 \rho_k(s), \psi\rangle_H ds + \langle \hat{R}_k(t), \psi\rangle_H,$$

where

$$\widehat{R}_k(t) = \frac{3\mu_k}{2\gamma}\varphi(u_0\cdot v_0)u_0 - \frac{3\varphi\mu_k}{2\gamma}(\rho_k(t)\cdot\vartheta_k(t))\rho_k(t) - \mu_k\int_0^t |\vartheta_k(s)|_H^2 \rho_k(s) ds$$

$$+\frac{3\varphi\mu_k}{2\gamma}\int_0^t(\rho_k(s)\cdot\vartheta_k(t))\vartheta_k(s)ds+\frac{3\varphi\mu_k}{2\gamma}\int_0^t|\vartheta_k(s)|^2\rho_k(s)ds$$

$$-\frac{3\mu_k}{2\gamma}\int_0^t|\vartheta_k(s)|_H^2|\rho_k(s)|^2\rho_k(s)ds+\sqrt{\mu_k}\int_0^t(\rho_k(s)\times\vartheta_k(s))d\hat{w}_k(s).$$

We have

$$\sup_{t\in[0,T]}\left|\varphi\Big(|\rho_k(t)|^2\rho_k(t)-|\rho(t)|^2\rho(t)\Big)\right|_H$$

$$\leq c|\varphi|_\infty\sup_{t\in[0,T]}\Big(\big|(|\rho_k(t)|^2-|\rho(t)|^2)\rho_k(t)\big|_H+\big||\rho(t)|^2(\rho_k(t)-\rho(t))\big|_H\Big)$$

$$\leq c|\varphi|_\infty\sup_{t\in[0,T]}\Big((|\rho_k(t)|_{H^1}^2+|\rho(t)|_{H^1}^2)|\rho_k(t)-\rho(t)|_H\Big).$$

Then, in view of (4.6.13), (4.6.15) and (4.6.16) we conclude

$$\lim_{k\to\infty}\sup_{t\in[0,T]}\left|\langle\gamma\rho_k(t)+\frac{1}{2}\varphi|\rho_k(t)|^2\rho_k(t)+\mu_k\vartheta_k(t),\psi\rangle_H-\langle\gamma\rho(t)+\frac{1}{2}\varphi|\rho(t)|^2\rho(t),\psi\rangle_H\right|=0,\quad\hat{\mathbb{P}}\text{-a.s.}\tag{4.6.19}$$

Since

$$\sup_{t\in[0,T]}\left|\int_0^t\langle\partial_x^2(\rho_k-\rho)(s),\psi\rangle_H ds\right|\leq|\psi|_{H^1}\int_0^T|\rho_k(s)-\rho(s)|_{H^1}ds,$$

due to (4.6.13) we have

$$\lim_{k\to\infty}\sup_{t\in[0,T]}\left|\int_0^t\langle\partial_x^2\rho_k(s),\psi\rangle_H ds-\int_0^t\langle\partial_x^2\rho(s),\psi\rangle_H\,ds\right|=0,\qquad\widehat{\mathbb{P}}\text{-a.s.},\tag{4.6.20}$$

and since

$$\sup_{t\in[0,T]}\Big|\int_0^t\Big(|\rho_k(s)|^2_{H^1}\rho_k(s)-|\rho(s)|^2_{H^1}\rho(s)\Big)ds\Big|_H$$

$$\leq\int_0^T\big||\rho_k(s)|^2_{H^1}-|\rho(s)|^2_{H^1}\big|ds+\int_0^T|\rho(s)|^2_{H^1}|\rho_k(s)-\rho(s)|_H ds,$$

we get

$$\lim_{k\to\infty}\sup_{t\in[0,T]}\left|\int_0^t\langle|\rho_k(s)|^2_{H^1}\rho_k(s),\psi\rangle_H ds-\int_0^t\langle|\rho(s)|^2_{H^1}\rho(s),\psi\rangle_H\,ds\right|=0,\qquad\hat{\mathbb{P}}\text{-a.s.}\quad(4.6.21)$$

Moreover, for every $\eta\in C([0,T];H^1)\cap L^2(0,T;H^2)$ and $s\in[0,T]$ we have

$$\langle\big(\partial_x^2\eta(s)\cdot\eta(s)\big)\eta(s)\varphi,\psi\rangle_H=-\langle\big(\partial_x\eta(s),\eta(s)\big)\eta(s)\varphi,\psi'\rangle_H$$

$$-\langle\big(\partial_x\eta(s),\eta(s)\big)\big(\partial_x\eta(s)\varphi+\eta(s)\varphi'\big)+|\partial_x\eta(s)|^2\eta(s)\varphi,\psi\rangle_H.$$

This implies that

$$\int_0^t\langle\big(\partial_x^2\rho_k(s)\cdot\rho_k(s)\big)\rho_k(s)\varphi-\big(\partial_x^2\rho(s)\cdot\rho(s)\big)\rho(s)\varphi,\psi\rangle_H\,ds$$

$$=-\int_0^t\langle\big[\big(\partial_x\rho_k(s),\rho_k(s)\big)\rho_k(s)-\big(\partial_x\rho(s),\rho(s)\big)\rho(s)\big]\varphi,\psi'\rangle_H\,ds$$

$$-\int_0^t\langle\big[\big(\partial_x\rho_k(s),\rho_k(s)\big)\big(\partial_x\rho_k(s)\varphi+\rho_k(s)\varphi'\big)-\big(\partial_x\rho(s),\rho(s)\big)\big(\partial_x\rho(s)\varphi+\rho(s)\varphi'\big)\big],\psi\rangle_H\,ds$$

$$-\int_0^t \langle \left[|\partial_x \rho_k(s)|^2 \rho_k(s) - |\partial_x \rho(s)|^2 \rho(s)\right] \varphi, \psi\rangle_H \, ds =: \sum_{i=1}^{3} J_{i,k}(t).$$

We have

$$|J_{1,k}(t)| \leq c\, |\varphi|_\infty |\psi'|_\infty \int_0^T |\rho_k(s) - \rho(s)|_{H^1} |\rho_k(s)|^2_{H^1} \, ds$$

$$+ c\, |\varphi|_\infty |\psi'|_\infty \int_0^T |\rho_k(s) - \rho(s)|_{H^1} |\rho(s)|_{H^1} \left(|\rho_k(s)|_{H^1} + |\rho(s)|_{H^1}\right) ds,$$

and, thanks to (4.6.13), (4.6.15) and (4.6.16), we conclude that

$$\lim_{k\to\infty} \sup_{t\in[0,T]} |J_{1,k}(t)| = 0, \qquad \hat{\mathbb{P}} - \text{a.s.}. \tag{4.6.22}$$

Next, we have

$$|J_{2,k}(t)| \leq c\, |\psi|_\infty \int_0^T |\rho_k(s) - \rho(s)|_{H^1} \left(|\rho_k(s)|_{H^1} + |\rho(s)|_{H^1}\right) \left(|\rho_k(s)|_{H^2} |\varphi|_\infty + |\rho_k(s)|_{H^1} |\varphi'|_\infty\right) ds$$

$$+ c\, |\psi|_\infty \left(|\varphi|_\infty + |\varphi'|_\infty\right) \int_0^T |\rho_k(s) - \rho(s)|_{H^1} |\rho(s)|_{H^2} |\rho(s)|_{H^1} \, ds.$$

Thus, as a consequence of (4.6.13) and bounds (4.6.14), (4.6.15), (4.6.16) and (4.6.18), we can conclude that

$$\lim_{k\to\infty} \hat{\mathbb{E}} \sup_{t\in[0,T]} |J_{2,k}(t)| = 0. \tag{4.6.23}$$

Finally, we have

$$|J_{3,k}(t)| \leq c\, |\varphi|_\infty |\psi|_\infty \int_0^T |\rho_k(s) - \rho(s)|_{H^1} \left[|\rho_k(s)|_{H^1} \left(|\rho_k(s)|_{H^2} + |\rho(s)|_{H^2}\right) + |\rho(s)|^2_{H^1}\right] ds,$$

and, due again to (4.6.13), (4.6.14) and(4.6.18), we conclude

$$\lim_{k\to\infty} \hat{\mathbb{E}} \sup_{t\in[0,T]} |J_{3,k}(t)| = 0. \tag{4.6.24}$$

Therefore, as a consequence of (4.6.22), (4.6.23) and (4.6.24), we conclude that

$$\lim_{k\to\infty} \hat{\mathbb{E}} \sup_{t\in[0,T]} \left| \int_0^t \langle \big(\partial_x^2 \rho_k(s)\cdot\rho_k(s)\big)\rho_k(s)\varphi - \big(\partial_x^2 \rho(s)\cdot\rho(s)\big)\rho(s)\varphi, \psi\rangle_H \, ds \right| = 0. \tag{4.6.25}$$

Next, for every $t \in [0,T]$ we have

$$\left|\varphi \int_0^t \Big(|\rho_k(s)|_{H^1}^2 |\rho_k(s)|^2 \rho_k(s) - |\rho(s)|_{H^1}^2 |\rho(s)|^2 \rho(s) \Big) ds \right|_H$$

$$\leq c|\varphi|_\infty \left(\int_0^T \big| |\rho_k(s)|_{H^1}^2 - |\rho(s)|_{H^1}^2 \big| \cdot \big| |\rho_k(s)|^2 \rho_k(s) \big|_H ds \right.$$

$$\left. + \int_0^T |\rho(s)|_{H^1}^2 \big| (|\rho_k(s)|^2 - |\rho(s)|^2)\rho_k(s) \big|_H ds + \int_0^T |\rho(s)|_{H^1}^2 \big| |\rho(s)|^2 (\rho_k(s) - \rho(s)) \big|_H ds \right)$$

$$\leq c|\varphi|_\infty \Big(\int_0^T |\rho_k(s)|_{H^1}^2 \big(|\rho_k(s)|_{H^1} + |\rho(s)|_{H^1} \big) |\rho_k(s) - \rho(s)|_{H^1} ds$$

$$+ \int_0^T \big(|\rho_k(s)|_{H^1}^4 + |\rho(s)|_{H^1}^4 \big) |\rho_k(s) - \rho(s)|_H ds \Big).$$

Thus, thanks again to (4.6.15) and (4.6.16), we get

$$\lim_{k\to\infty} \sup_{t\in[0,T]} \left| \int_0^t \langle \varphi |\rho_k(s)|_{H^1}^2 |\rho_k(s)|^2 \rho_k(s), \psi \rangle_H \, ds - \int_0^t \langle \varphi |\rho(s)|_{H^1}^2 |\rho(s)|^2 \rho(s), \psi \rangle_H \, ds \right| = 0, \quad \hat{\mathbb{P}}\text{-a.s.} \tag{4.6.26}$$

By using the same arguments as in the proof of Lemma 4.14, we conclude that

$$\lim_{k\to\infty} \hat{\mathbb{E}} \sup_{t\in[0,T]} \left| \langle \widehat{R}_k(t), \psi \rangle_H \right|^2 = 0. \tag{4.6.27}$$

Finally, combining (4.6.19), (4.6.20), (4.6.21), (4.6.25) and (4.6.26) together with (4.6.27), we can conclude that ρ satisfies equation (4.5.1). As we have seen above, this allows to conclude the proof of Theorem 4.4.

Appendix A: The Smoluchowski-Kramers approximation of SWEs with variable friction

The Smoluchowski-Kramers approximation for system (1.1.1) has been studied in [22], under the assumptions that f is Lipschitz-continuous and σ is bounded. In this appendix, we extend this result to the case when σ is unbounded and f exhibits polynomial growth (see Hypothesis 2.6). Specifically, we analyze the small-mass limit for the following system

$$\begin{cases} \mu\partial_t^2 u_\mu(t,x) = \Delta u_\mu(t,x) - \gamma(u_\mu(t,x))\partial_t u_\mu(t,x) + f(x,u_\mu(t,x)) + \sigma(u_\mu(t,\cdot))\partial_t w^Q(t,x), \\ u_\mu(0,x) = u_0(x), \quad \partial_t u_\mu(0,x) = v_0(x), \qquad u_\mu(t,x) = 0, \;\; x \in \partial\mathscr{O}. \end{cases} \tag{A.0.1}$$

We assume that Hypotheses 2.1, 2.3 and Hypothesis 2.6 hold. By applying Theorem 2.9 with the control $\varphi = 0$, we know that for every $T > 0$ and every $(u_0, v_0) \in \mathscr{H}_1$, there exists a unique adapted solution u_μ to equation (A.0.1), such that $(u_\mu, \partial_t u_\mu) \in L^2(\Omega; C([0,T]; \mathscr{H}_1))$.

For every $a \in [0,1)$ we denote

$$X(a) := \bigcap_{q<q(a)} L^q(0,T;H^a), \qquad Y(a) := \bigcap_{p<2/a} L^p(0,T;H^a),$$

where

$$q(a) := \frac{2(\theta+1)}{2+(\theta-1)a}.$$

The main result of this appendix is as follows.

Theorem A.1. *Assume Hypotheses 2.1, 2.3 and 2.6 hold, and for every $\mu > 0$, let u_μ denote the unique solution to equation (A.0.1), with the initial conditions $(u_0, v_0) \in \mathscr{H}_1$.*

1. *If $\theta \in (1,3)$, then for every $a \in [0,1)$ and $\eta > 0$ we have*

$$\lim_{\mu\to 0} \mathbb{P}\Big(\|u_\mu - u\|_{X(a)} > \eta \Big) = 0,$$

where $u \in L^2(\Omega; L^2([0,T]; H^1))$ is the unique solution to equation

$$\begin{cases} \gamma(u(t,x))\partial_t u(t,x) = \Delta u(t,x) + f(u(t,x)) - \dfrac{\gamma'(u(t,x))}{2\gamma^2(u(t,x))} \displaystyle\sum_{i=1}^{\infty} |[\sigma(u(t,\cdot)Qe_i](x)|^2 \\[10pt] \qquad\qquad\qquad + \sigma(u(t,\cdot))\partial_t w^Q(t,x), \\[10pt] u(0,x) = u_0(x), \qquad u(t,0) = u(t,L) = 0. \end{cases} \tag{A.0.2}$$

2. *If we assume*

$$\begin{cases} \exists \rho \in [0, (\theta+1)/4) \ \text{ such that } \ \|\sigma(h)\|_{\mathscr{L}_2(H_Q,H)} \le c\big(1 + \|h\|_H^\rho\big), \qquad h \in H, \\[10pt] \theta \in (1,3], \quad \text{if } \rho > 0, \qquad \theta \in (1,5), \quad \text{if } \rho = 0, \end{cases} \tag{A.0.3}$$

then for every $a \in [0,1)$ *and* $\eta > 0$ *we have*

$$\lim_{\mu \to 0} \mathbb{P}\Big(\left\|u_\mu - u\right\|_{Y(a)} > \eta \Big) = 0.$$

A.1 Energy estimates

One of the key ingredients in our proof of Theorem A.1 is the tightness of $\{u_\mu\}_{\mu \in (0,1)}$ in suitable functional spaces. This will require the following a-priori bounds.

Lemma A.2. *Assume Hypotheses 2.1, 2.3 and 2.6 hold, and fix* $T > 0$ *and* $(u_0, v_0) \in \mathscr{H}_1$.

1. *There exist* $\mu_T \in (0,1)$ *and* $c_T > 0$ *such that for every* $\mu \in (0, \mu_T)$,

$$\mathbb{E} \sup_{t \in [0,T]} \left\|u_\mu(t)\right\|_H^2 + \mathbb{E} \int_0^T \left\|u_\mu(t)\right\|_{H^1}^2 dt + \mathbb{E} \int_0^T \left\|u_\mu(t)\right\|_{L^{\theta+1}}^{\theta+1} dt \leq c_T, \tag{A.1.1}$$

and

$$\mathbb{E} \sup_{t \in [0,T]} \left\|u_\mu(t)\right\|_{H^1}^2 + \mathbb{E} \sup_{t \in [0,T]} \left\|u_\mu(t)\right\|_{L^{\theta+1}}^{\theta+1} + \mu \mathbb{E} \sup_{t \in [0,T]} \left\|\partial_t u_\mu(t)\right\|_H^2 + \mathbb{E} \int_0^T \left\|\partial_t u_\mu(t)\right\|_H^2 dt \leq \frac{c_T}{\mu}. \tag{A.1.2}$$

2. *If, in addition, condition (A.0.3) holds, then for every* $\mu \in (0, \mu_T)$

$$\mathbb{E} \sup_{t \in [0,T]} \left\|u_\mu(t)\right\|_{H^1}^2 + \mathbb{E} \sup_{t \in [0,T]} \left\|u_\mu(t)\right\|_{L^{\theta+1}}^{\theta+1} + \mu \mathbb{E} \sup_{t \in [0,T]} \left\|\partial_t u_\mu(t)\right\|_H^2 \leq \frac{c_T}{\mu^\beta}, \tag{A.1.3}$$

where

$$\beta = \beta(\rho) := \frac{\theta + 1}{2(\theta + 1 - 2\rho)}.$$

Remark A.3. 1. If the mapping σ is bounded, then condition (A.0.3) is satisfied for $\rho = 0$, in which case $\beta = 1/2$, so that for every $\mu \in (0, \mu_T)$

$$\sqrt{\mu}\,\mathbb{E}\sup_{t\in[0,T]}\left(\left\|u_\mu(t)\right\|_{H^1}^2 + \left\|u_\mu(t)\right\|_{L^{\theta+1}}^{\theta+1} + \mu\left\|\partial_t u_\mu(t)\right\|_H^2\right) \leq c_T.$$

2. When condition (A.0.3) holds with $\rho = 0$, in order to prove (A.1.3) we can take any $\theta > 1$.

3. If $\rho \in [0, (\theta+1)/4)$, then $\beta < 1$. Therefore, if condition (A.0.3) holds, due to (A.1.3) we have

$$\lim_{\mu\to 0}\mu^2\mathbb{E}\sup_{t\in[0,T]}\left\|\partial_t u_\mu(t)\right\|_H^2 = 0, \tag{A.1.4}$$

which is the same bound proven in [22].

Proof. Estimates (A.1.1) and (A.1.2) can be proved by proceeding as in the proof of Lemma 2.10. Thus, we will only prove (A.1.3) under condition (A.0.3).

For every $\mu \in (0,1)$ and $t \in [0,T]$, define

$$L_\mu(t) := \left\|u_\mu(t)\right\|_{H^1}^2 + \int_0^L \left(c_2 - \mathfrak{f}(x, u_\mu(t,x))\right)dx + \mu\left\|\partial_t u_\mu(t)\right\|_H^2,$$

where the function $\mathfrak{f}$ and the constant c_2 have been introduced in Hypothesis 2.6. Due to (2.1.10), we have that

$$L_\mu(t) \geq \|u_\mu(t)\|_{H^1}^2 + \|u_\mu(t)\|_{L^{\theta+1}}^{\theta+1} + \mu\,\|\partial_t u_\mu(t)\|_H^2, \qquad \mathbb{P}-\text{a.s.}$$

Thus, we obtain (A.1.3) once we have proved that

$$\mu^\beta\mathbb{E}\sup_{t\in[0,T]}L_\mu(t) \leq c_T. \tag{A.1.5}$$

Assume (A.1.5) is not true. Then there is a sequence $(\mu_k)_{k\in\mathbb{N}} \subset (0,1)$ converging to 0, as $k \to \infty$, such that

$$\lim_{k\to\infty} \mu_k^\beta\, \mathbb{E} \sup_{t\in[0,T]} L_{\mu_k}(t) = +\infty. \tag{A.1.6}$$

For every $k \in \mathbb{N}$, the mapping $t \mapsto L_{\mu_k}(t)$ is continuous $\mathbb{P}$-a.s., so that there exists a random time $t_k \in [0,T]$ such that

$$L_{\mu_k}(t_k) = \sup_{t\in[0,T]} L_{\mu_k}(t).$$

As a consequence of the Itô formula, if s is any random time such that $\mathbb{P}(s \le t_k) = 1$, we have

$$L_{\mu_k}(t_k) - L_{\mu_k}(s) \le \frac{1}{\mu_k}\int_s^{t_k} \left\|\sigma(u_{\mu_k}(r))\right\|^2_{\mathscr{L}_2(H_Q,H)}\,dr + 2\big(M_k(t_k) - M_k(s)\big),$$

where

$$M_k(t) := \int_0^t \left\langle \partial_t u_{\mu_k}(r), \sigma(u_{\mu_k}(r))dw^Q(r)\right\rangle_H.$$

Thanks to Young's inequality, since $2\beta \ge 1$ and $\mu_k < 1$, we have

$$\frac{1}{\mu_k}\int_s^{t_k} \left\|\sigma(u_{\mu_k}(r))\right\|^2_{\mathscr{L}_2(H_Q,H)}\,dr \le \frac{c}{\mu_k}\int_s^{t_k}\left(1 + \left\|u_{\mu_k}(r)\right\|_H^{2\rho}\right)dr$$

$$\le c\left(\frac{t_k - s}{\mu_k} + \frac{t_k - s}{\mu_k^{\frac{\theta+1}{\theta+1-2\rho}}} + \int_0^T \left\|u_{\mu_k}(t)\right\|_H^{\theta+1}dt\right) \le c\left(\frac{t_k - s}{\mu_k^{2\beta}} + \int_0^T \left\|u_{\mu_k}(t)\right\|_H^{\theta+1}dt\right).$$

Therefore, if we define

$$U_k := \int_0^T \left\|u_{\mu_k}(t)\right\|^{\theta+1}_{L^{\theta+1}}dt, \quad \text{and} \quad M_k := \sup_{t\in[0,T]} |M_k(t)|,$$

we can fix a constant κ_T independent of k, with $L_{\mu_k}(0) \le \kappa_T$, such that

$$L_{\mu_k}(t_k) - L_{\mu_k}(s) \le \kappa_T \left(\frac{t_k - s}{\mu_k^{2\beta}} + U_k \right) + 4M_k.$$

In particular, if we take $s = 0$, we have

$$t_k \ge \frac{\mu_k^{2\beta}}{\kappa_T} \Big(L_{\mu_k}(t_k) - 4M_k - \kappa_T (U_k + 1) \Big) =: \frac{\mu_k^{2\beta}}{\kappa_T} \delta_k. \tag{A.1.7}$$

On the set $E_k := \{\delta_k > 0\}$, for any $s \in \left[t_k - \frac{\mu_k^{2\beta}}{2\kappa_T}\delta_k, t_k\right]$ we have

$$L_{\mu_k}(s) \ge L_{\mu_k}(t_k) - \frac{1}{2}\delta_k - \kappa_T U_k - 4M_k = \frac{1}{2}\Big[L_{\mu_k}(t_k) - 4M_k - \kappa_T (U_k - 1) \Big].$$

Hence, if we define

$$I_k := \int_0^T L_{\mu_k}(s) ds,$$

recalling how δ_k was defined in (A.1.7), we have

$$I_k \ge \int_{t_k - \frac{\mu_k^{2\beta}}{2\kappa_T}\delta_k}^{t_k} L_{\mu_k}(s) ds \ge \frac{\mu_k^{2\beta}}{4\kappa_T} \left[\Big(L_{\mu_k}(t_k) - 4M_k - \kappa_T U_k \Big)^2 - \kappa_T^2 \right],$$

so that

$$\mathbb{E}(I_k) \ge \mathbb{E}(I_k; E_k) \ge \mathbb{E}\left(\frac{\mu_k^{2\beta}}{4C_T} \Big(L_{\mu_k}(t_k) - 4M_k - C_T U_k \Big)^2 ; E_k \right) - \frac{\mu_k^{2\beta}}{4} C_T. \tag{A.1.8}$$

Now, thanks to condition (A.0.3), and estimates (A.1.1) and (A.1.2) we have

$$\mathbb{E}(M_k) \leq \mathbb{E}\left(\int_0^T \left\|\partial_t u_{\mu_k}(t)\right\|_H^2 \left\|\sigma(u_{\mu_k}(t))\right\|_{\mathscr{L}_2(H_Q,H)}^2 dt\right)^{\frac{1}{2}}$$

$$\leq \mathbb{E} \sup_{t\in[0,T]} \left\|\sigma(u_{\mu_k}(t))\right\|_{\mathscr{L}_2(H_Q,H)} \left(\int_0^T \left\|\partial_t u_{\mu_k}(t)\right\|_H^2 dt\right)^{\frac{1}{2}}$$

$$\leq c\,\mathbb{E} \sup_{t\in[0,T]} \left\|\sigma(u_{\mu_k}(t))\right\|_{\mathscr{L}_2(H_Q,H)}^{\frac{2}{\rho}} + c\,\mathbb{E}\left(\int_0^T \left\|\partial_t u_{\mu_k}(t)\right\|_H^2 dt\right)^{\frac{1}{2-\rho}}$$

$$\leq c\left(1+\mathbb{E} \sup_{t\in[0,T]} \left\|u_{\mu_k}(t)\right\|_H^2\right) + c\left(\int_0^T \mathbb{E}\left\|\partial_t u_{\mu_k}(t)\right\|_H^2 dt\right)^{\frac{1}{2-\rho}} \leq \frac{C_T}{\mu^{\frac{1}{2-\rho}}}.$$

In particular, since (A.0.3) implies

$$\beta = \frac{\theta+1}{2(\theta+1-2\rho)} \geq \frac{1}{2-\rho},$$

we have that

$$\limsup_{k\to\infty} \mu_k^{\beta}\,\mathbb{E}(M_k) < +\infty. \tag{A.1.9}$$

Moreover, by (A.1.1) and (A.1.2), we have

$$\sup_k \mathbb{E}(U_k) < \infty. \tag{A.1.10}$$

Therefore, due to (A.1.7), (A.1.9) and (A.1.10), as a consequence of (A.1.6) we have

$$\lim_{k\to\infty} \mu_k^\beta \mathbb{E}(\delta_k) = +\infty. \tag{A.1.11}$$

Now, we have

$$\mu_k^\beta \mathbb{E}(\delta_k) = \mu_k^\beta \mathbb{E}(\delta_k; E_k) \leq \mathbb{E}\big(\mu_k^\beta(\delta_k + \kappa_T); E_k\big) \leq \Big[\mathbb{E}\big(\mu_k^{2\beta}(\delta_k + \kappa_T)^2; E_k\big)\Big]^{\frac{1}{2}},$$

so that, thanks to (A.1.8) we have

$$\mathbb{E}(I_k) \geq \frac{1}{4\kappa_T}\mathbb{E}\big(\mu_k^{2\beta}(\delta_k + \kappa_T)^2; E_k\big) - \frac{\mu_k^{2\beta}\kappa_T}{4} \geq \frac{1}{4\kappa_T}\big[\mu_k^\beta \mathbb{E}(\delta_k)\big]^2 - \frac{\mu_k^{2\beta}\kappa_T}{4}.$$

Due to (A.1.11), this implies

$$\lim_{k\to\infty} \mathbb{E}(I_k) = +\infty.$$

However, the limit above is not possible. Actually, since we

$$L_{\mu_k}(t) \leq \left\|u_{\mu_k}(t)\right\|_{H^1}^2 + \mu_k \left\|\partial_t u_{\mu_k}(t)\right\|_H^2 + Lc\Big(1 + \left\|u_{\mu_k}(t)\right\|_{L^{\theta+1}}^{\theta+1}\Big), \ \ \mathbb{P}\text{-a.s.}$$

as a consequence of (A.1.1) and (A.1.2) we have

$$\sup_{k\in\mathbb{N}} \mathbb{E}(I_k) < +\infty,$$

and this gives a contradiction. In particular, this means that claim (A.1.5) is true, and (A.1.3) holds.

□

A.2 Tightness

As in Section 2.5, for every $T>0$ and $\mu>0$ we have defined

$$\rho_\mu(t,x)=g(u_\mu(t,x)),\quad (t,x)\in[0,T]\times[0,L],$$

and, by integrating equation (A.0.1) with respect to t, we got

$$\begin{aligned}\rho_\mu(t)+\mu\partial_t u_\mu(t)=g(u_0)+\mu v_0+\int_0^t \operatorname{div}\big[b(\rho_\mu(s))\nabla\rho_\mu(s)\big]ds+\int_0^t F_g(\rho_\mu(s))ds\\ +\int_0^t \sigma_g(\rho_\mu(s))dw^Q(s),\end{aligned}\tag{A.2.1}$$

where we recall that b, f_g, F_g and σ_g were defined in (2.4.11) and (2.4.12).

Definition A.4. *Let E be a Banach space with norm $\|\cdot\|_E$. Given $r>1$ and $\lambda\in(0,1)$, we denote by $W^{\lambda,r}(0,T;E)$ be the Banach space of all $u\in L^p(0,T;E)$ such that*

$$[u]_{W^{\lambda,r}(0,T;E)}:=\int_0^T\int_0^T\frac{\|u(t)-u(s)\|_E^r}{|t-s|^{1+\lambda r}}dtds<\infty,$$

endowed with the norm

$$\|u\|_{W^{\lambda,r}(0,T;E)}^r=\int_0^T\|u(t)\|_E^r\,dt+[u]_{W^{\lambda,r}(0,T;E)}.$$

It is possible to prove that if $\lambda r<1$, $p\leq r/(1-\lambda r)$ and $1\leq r\leq p$, then $W^{\lambda,r}(0,T;E)\subset$

$L^p(0,T;E)$ and there exists some $c>0$ such that for all $u \in W^{\lambda,r}(0,T;E)$

$$\|\tau_h(u)-u\|_{L^p(0,T-h;E)} \leq c\,h^{\lambda} T^{1/p-1/r}\,[u]_{W^{\lambda,r}(0,T;E)}, \quad h>0, \tag{A.2.2}$$

where

$$\tau_h(u)(t) = u(t+h), \quad t \in [-h, T-h],$$

(see [58, Lemma5]).

Proposition A.5. *Assume Hypotheses 2.1, 2.3 and 2.6 hold, and fix any $T>0$ and $(u_0,v_0) \in \mathscr{H}_1$, and an arbitrary sequence $(\mu_k)_{k\in\mathbb{N}} \subset (0,1)$ that converges to 0.*

1. *The family of probability measures $\left(\mathscr{L}(\rho_{\mu_k})\right)_{k\in\mathbb{N}}$ is tight in $X(a)$, for every $a \in [0,1)$.*
2. *If condition (A.0.3) holds, then the family of probability measures $\left(\mathscr{L}(\rho_{\mu_k})\right)_{k\in\mathbb{N}}$ is tight in $Y(a)$, for every $a \in [0,1)$.*

Remark A.6. 1. By taking $a=0$, we have

$$\left(\mathscr{L}(\rho_{\mu_k})\right)_{k\in\mathbb{N}} \text{ is tight in } \bigcap_{q<\theta+1} L^q(0,T;H).$$

and thanks to the embedding $H^a \hookrightarrow C([0,L])$, for $a \in (1/2,1)$, we have

$$\left(\mathscr{L}(\rho_{\mu_k})\right)_{k\in\mathbb{N}} \text{ is tight in } \bigcap_{q<4(\theta+1)/(\theta+3)} L^q(0,T;C([0,L])).$$

2. When condition (A.0.3) holds, we have that

$$\left(\mathscr{L}(\rho_{\mu_k})\right)_{k\in\mathbb{N}} \text{ is tight in } \bigcap_{p<\infty} L^p(0,T;H).$$

Proof. For every $0 \le t_1 \le t_2 \le T$, we have

$$\begin{aligned}
\mathbb{E}\left\|\int_{t_1}^{t_2} \operatorname{div}\left[b(\rho_\mu(s))\nabla\rho_\mu(s)\right]ds\right\|_{H^{-1}}^{(\theta+1)/\theta} &\le \left(\mathbb{E}\left\|\int_{t_1}^{t_2} \operatorname{div}\left[b(\rho_\mu(s))\nabla\rho_\mu(s)\right]ds\right\|_{H^{-1}}^{2}\right)^{(\theta+1)/2\theta}\\
&\le C(t_2-t_1)^{(\theta+1)/2\theta}\left(\mathbb{E}\int_0^T \left\|\rho_\mu(s)\right\|_{H^1}^2 ds\right)^{(\theta+1)/2\theta},
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}\left\|\int_{t_1}^{t_2} F_g(\rho_{\mu_k}(s))ds\right\|_{H^{-1}}^{(\theta+1)/\theta} &\le C\mathbb{E}\left(\int_{t_1}^{t_2}\left(1+\left\|u_\mu(s)\right\|_{\theta+1}^{\theta}\right)ds\right)^{(\theta+1)/\theta}\\
&\le C(t_2-t_1)^{1/\theta}\mathbb{E}\int_0^T \left(1+\left\|u_\mu(s)\right\|_{\theta+1}^{\theta+1}\right)ds,
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}\left\|\int_{t_1}^{t_2} \sigma_g(\rho_\mu(s))dw^Q(s)\right\|_{H^{-1}}^{(\theta+1)/\theta} &\le \left(\mathbb{E}\left\|\int_{t_1}^{t_2} \sigma_g(\rho_\mu(s))dw^Q(s)\right\|_{H^{-1}}^{2}\right)^{(\theta+1)/2\theta}\\
&\le C(t_2-t_1)^{(\theta+1)/2\theta}\left(1+\mathbb{E}\sup_{t\in[0,T]}\left\|u_\mu(t)\right\|_H^2\right)^{(\theta+1)/2\theta}.
\end{aligned}$$

In view of (A.1.1) and (A.2.1), it is not difficult to show that for every $\lambda \in (0, 1/(\theta+1))$,

$$\sup_{\mu\in(0,\mu_T)} \mathbb{E}[\rho_\mu + \mu\partial_t u_\mu]_{W^{\lambda,\theta_0}(0,T;H^{-1})}^{\theta_0} < \infty, \tag{A.2.3}$$

where

$$\theta_0 := \frac{\theta+1}{\theta} \in (1,2).$$

Moreover, by (A.1.1) and (A.1.2) we have

$$\sup_{\mu\in(0,\mu_T)} \mathbb{E}\left\|\rho_\mu+\mu\partial_t u_\mu\right\|^2_{L^\infty([0,T];H)}<\infty. \tag{A.2.4}$$

Therefore, from (A.2.3) and (A.2.4) we conclude that for every $\varepsilon>0$, there exists $L_1(\varepsilon)>0$ such that, if we define

$$K_1^\varepsilon=\left\{f:[0,T]\times\mathbb{R}\to\mathbb{R}:[f]_{W^{\lambda,\theta_0}(0,T;H^{-1})}+\|f\|_{L^\infty(0,T;H)}\leq L_1(\varepsilon)\right\},$$

then

$$\inf_{\mu\in(0,\mu_T)}\mathbb{P}\left(\rho_\mu+\mu\partial_t u_\mu\in K_1^\varepsilon\right)>1-\varepsilon/4.$$

According to (A.2.2), we have that for every $p<(\theta+1)/\theta$

$$\lim_{h\to 0}\|\tau_h f-f\|_{L^p(0,T-h;H^{-1})}=0,\quad f\in K_1^\varepsilon.$$

Hence, in view of [58, Theorem 6], we have that K_1^ε is relatively compact in $L^q(0,T,H^{-\delta})$, for every $q<\infty$ and $\delta>0$.

Next, due to (A.1.2), we have

$$\lim_{\mu\to 0}\mathbb{E}\left\|\mu\partial_t u_\mu\right\|^2_{L^2([0,T];H)}=0,$$

hence for every sequence $(\mu_k)_{k\in\mathbb{N}}\subset(0,\mu_T)$ converging to zero, there exists a compact K_2^ε in

$L^2([0,T];H)$ such that

$$\mathbb{P}\big(\mu_k \partial_t u_{\mu_k} \in K_2^\varepsilon\big) > 1 - \varepsilon/4, \quad k \in \mathbb{N}.$$

Since $L^2([0,T];H) \subset L^{\theta_0}([0,T];H^{-\delta})$, for $\delta > 0$, we have that K_2^ε is also compact in $L^{\theta_0}([0,T];H^{-\delta})$, which implies that $K_1^\varepsilon + K_2^\varepsilon$ is relatively compact in $L^{\theta_0}([0,T];H^{-\delta})$, and for every $k \in \mathbb{N}$,

$$\mathbb{P}\big(\rho_{\mu_k} \in K_1^\varepsilon + K_2^\varepsilon\big) \geq 1 - \varepsilon/2.$$

Moreover, thanks to estimate (A.1.1) there exists $L_2(\varepsilon) > 0$ such that, if we define

$$K_3^\varepsilon = \Big\{ f : [0,T] \times \mathbb{R} \to \mathbb{R} : \|f\|_{L^{\theta+1}([0,T];H)} \leq L_2(\varepsilon) \Big\},$$

then

$$\inf_{\mu \in (0,\mu_T)} \mathbb{P}\big(\rho_\mu \in K_3^\varepsilon\big) \geq 1 - \varepsilon/4,$$

and thus

$$\inf_{k \in \mathbb{N}} \mathbb{P}\Big(\rho_{\mu_k} \in \big(K_1^\varepsilon + K_2^\varepsilon\big) \cap K_3^\varepsilon\Big) \geq 1 - 3\varepsilon/4.$$

By using again [58, Theorem 6], $\big(K_1^\varepsilon + K_2^\varepsilon\big) \cap K_3^\varepsilon$ is relatively compact in $L^p([0,T];H^{-\delta})$ for every $\delta > 0$ and $p < \theta + 1$. This implies that the family of probability measures $\big(\mathscr{L}(\rho_{\mu_k})\big)_{k \in \mathbb{N}}$ is tight in $L^p([0,T];H^{-\delta})$, for every $\delta > 0$ and $p < \theta + 1$.

Now, according to [58, Theorem 1], we have

$$\lim_{h \to 0} \sup_{f \in \left(K_1^\varepsilon + K_2^\varepsilon\right) \cap K_3^\varepsilon} \|\tau_h f - f\|_{L^p([0,T];H^{-\delta})} = 0.$$

Furthermore, since

$$\sup_{\mu\in[0,\mu_T]} \mathbb{E}\left\|\rho_\mu\right\|^2_{L^2([0,T];H^1)} < \infty,$$

there exists $L_3(\varepsilon) > 0$ such that, if we define

$$K_4^\varepsilon = \Big\{ f : [0,T]\times\mathbb{R}\to\mathbb{R} \ : \ \|f\|_{L^2([0,T];H^1)} \le L_3(\varepsilon) \Big\},$$

then

$$\inf_{\mu\in(0,\mu_T)} \mathbb{P}\big(\rho_\mu \in K_4^\varepsilon\big) \ge 1-\varepsilon/4.$$

Thus, if we define

$$K^\varepsilon = \big(K_1^\varepsilon + K_2^\varepsilon\big) \cap K_3^\varepsilon \cap K_4^\varepsilon,$$

we have

$$\inf_{k\in\mathbb{N}} \mathbb{P}(\rho_{\mu_k} \in K^\varepsilon) \ge 1-\varepsilon.$$

For every $a \in [0,1)$ and for any $\delta > 0$, by interpolation

$$\|u\|_{H^a} \le C(a,\delta)\, \|u\|_{H^{-\delta}}^{\frac{1-a}{1+\delta}}\, \|u\|_{H^1}^{\frac{a+\delta}{1+\delta}}.$$

Thus, according to [58, Theorem 7], we have K^ε is relatively compact in $L^q(0,T;H^a)$, where $q = q(a,\delta,p)$ satisfies

$$\frac{1}{q} = \frac{1-a}{p(1+\delta)} + \frac{a+\delta}{2(1+\delta)}, \quad \delta > 0, \ \ p < \theta+1.$$

This means that K^{ε} is relatively compact in $L^q(0,T;H^a)$, for every $q < q(a)$, where

$$q(a) = \frac{2(\theta+1)}{2+(\theta-1)a}, \quad a \in [0,1),$$

so that $\big(\mathscr{L}(\rho_{\mu_k})\big)_{k\in\mathbb{N}}$ is tight in $X(a)$.

Now let us assume that the condition (A.0.3) holds. Thanks to (A.1.4) we know that

$$\lim_{\mu\to 0} \mathbb{E}\left\|\mu\partial_t u_\mu\right\|_{L^\infty([0,T];H)} = 0.$$

Since $L^\infty(0,T;H) \subset L^q(0,T;H^{-\delta})$, for every $q<\infty$ and $\delta>0$, we can proceed as in the proof of part (1), and we have that $\big(\mathscr{L}(\rho_{\mu_k})\big)_{k\in\mathbb{N}}$ is tight in $L^p([0,T];H^{-\delta})$ for every $p<\infty$ and $\delta>0$. Finally, since

$$\sup_{\mu\in[0,\mu_T]} \mathbb{E}\left\|\rho_\mu\right\|^2_{L^2([0,T];H^1)} < \infty, \tag{A.2.5}$$

by using the same argument as in the proof of part (1), we have that for every $a\in[0,1)$, $\big(\mathscr{L}(\rho_{\mu_k})\big)_{k\in\mathbb{N}}$ is tight in $L^q([0,T];H^a)$, where $q=q(a,\delta,p)$ satisfies

$$\frac{1}{q} = \frac{1-a}{p(1+\delta)} + \frac{a+\delta}{2(1+\delta)}, \quad \delta>0, \;\; p<\infty.$$

This implies that $\big(\mathscr{L}(\rho_{\mu_k})\big)_{k\in\mathbb{N}}$ is tight in $L^q([0,T];H^a)$, for every $q<2/a$, and the proof of part (2) follows.

$\square$

A.3 The limiting problem

In this section we will prove the uniqueness of solutions for equation (A.0.2). To this purpose, we shall first study the following quasilinear parabolic equation

$$\begin{cases} \partial_t \rho(t,x) = \operatorname{div}\big[b(\rho(t,x))\nabla\rho(t,x)\big] + f_g(x,\rho(t,x)) + \sigma_g(\rho(t,\cdot))dw^Q(t,x) \\ \\ \rho(0,x) = g(u_0(x)), \quad \rho(t,0) = \rho(t,L) = 0. \end{cases} \tag{A.3.1}$$

Definition A.7. *An $(\mathscr{F}_t)_{t\geq 0}$ adapted process $\rho \in L^2(\Omega; L^2(0,T;H^1))$ is a solution of equation (A.3.1) if for every test function $\psi \in C_0^\infty([0,L])$*

$$\begin{aligned} \langle \rho(t), \psi\rangle_H = \langle g(u_0), \psi\rangle_H + \int_0^t \langle b(\rho(s))\nabla\rho(s), \nabla\psi\rangle_H ds \\ + \int_0^t \langle F_g(\rho(s)), \psi\rangle_H ds + \int_0^t \langle \sigma_g(\rho(s))dw^Q(s), \psi\rangle_H. \end{aligned} \tag{A.3.2}$$

By proceeding as in the proof of [22, Theorem 6.2], combined with similar arguments as in the proof of Proposition 2.13, we have the following result.

Proposition A.8. *Under Hypotheses 2.1, 2.3 and 2.6, there is at most one solution $\rho \in L^2(\Omega; L^2(0,T;H^1))$ to equation* (A.3.1).

Moreover, by proceeding as in the proof of [22, Theorem 7.1], we have the following result.

Proposition A.9. *Assume Hypotheses 2.1 and 2.3 and either Hypothesis 2.5 or Hypothesis 2.6 and fix any $T > 0$, $u_0 \in H^1$. Suppose that $\rho \in L^2(\Omega; L^2(0,T;H^1))$ is the a solution of equation* (A.3.1). *Then, if we define $u := g^{-1}(\rho)$, we have that u belongs to $L^2(\Omega; L^2(0,T;H^1))$ and is a solution to equation* (A.0.2). *Furthermore, equation* (A.0.2) *admits at most one solution*

$u \in L^2(\Omega; L^2(0,T;H^1))$.

A.4 Proof of Theorem A.1

For every sequence $\{\mu_k\}_{k\in\mathbb{N}} \subset (0, \mu_T)$ converging to 0 as $k \to \infty$, we denote

$$u_k := u_{\mu_k} \quad \text{and} \quad \rho_k := g(u_k), \quad k \in \mathbb{N}.$$

In view of the first part of Proposition A.5, if we define

$$X := \bigcap_{0 \le a < 1} X(a),$$

we have that the family

$$\{\mathscr{L}(\rho_k,\ \mu_k \partial_t u_k)\}_{k\in\mathbb{N}} \subset \mathscr{P}\Big(X \times L^2(0,T;H)\Big)$$

is tight.

We denote by ρ a weak limit point for the sequence $\{\rho_k\}_{k\in\mathbb{N}}$ and we denote

$$\mathscr{K} := X \times L^2(0,T;H) \times C([0,T],U),$$

where U is the Hilbert space such that the embedding $H_Q \subset U$ is Hilbert-Schmidt. According to the Skorokhod Theorem there exist random variables

$$\mathscr{Y} = (\hat{\rho}, 0, \hat{w}^Q), \qquad \mathscr{Y}_k = \left(\hat{\rho}_k, \hat{\theta}_k, \hat{w}_k^Q\right), \quad k \in \mathbb{N},$$

defined on a probability space $(\hat{\Omega}, \hat{\mathscr{F}}, \{\hat{\mathscr{F}}_t\}_{t\in[0,T]}, \hat{\mathbb{P}})$, such that

$$\mathscr{L}(\mathscr{Y}) = \mathscr{L}(\rho, 0, w^Q), \qquad \mathscr{L}(\mathscr{Y}_k) = \mathscr{L}(\rho_k,\ \mu_k \partial_t u_k, w^Q), \quad k \in \mathbb{N},$$

and such that

$$\lim_{k\to\infty} \mathscr{Y}_k = \mathscr{Y} \ \text{ in } \ \mathscr{K}, \quad \hat{\mathbb{P}}\text{-a.s.} \tag{A.4.1}$$

In particular,

$$\lim_{k\to\infty} \Big(\|\hat{\rho}_k - \hat{\rho}\|_{L^p([0,T];H)} + \|\hat{\rho}_k - \hat{\rho}\|_{L^q([0,T];C([0,L]))} \Big) = 0, \quad \hat{\mathbb{P}}\text{-a.s.} \tag{A.4.2}$$

for every $p < \theta + 1$ and every $q < 4(\theta+1)/(\theta+3)$. Moreover, thanks to (A.2.5), we have that $\hat{\rho} \in L^2(\Omega; L^2([0,T];H^1))$, and, taking possibly a subsequence,

$$\lim_{k\to\infty} \hat{\rho}_k = \hat{\rho}, \quad \text{in } \ L^2_w([0,T];H^1), \quad \hat{\mathbb{P}}\text{-a.s.}$$

where $L^2_w([0,T];H^1)$ is the space $L^2([0,T];H^1)$ endowed with the weak topology.

By proceeding as in the proof of [22, Theorem 7.1], thanks to Proposition A.9, in order to prove Theorem A.1, it is sufficient to show that $\hat{\rho}$ solves the parabolic equation (**??**).

For every $k \in \mathbb{N}$ and $\psi \in C_0^\infty([0,L])$, we have

$$\begin{aligned} \langle \hat{\rho}_k(t) + \hat{\theta}_k(t), \psi \rangle_H = \langle g(u_0) + \mu_k v_0, \psi \rangle_H - \int_0^t \langle b(\hat{\rho}_k(s)) \nabla \hat{\rho}_k(s), \nabla \psi \rangle_H ds \\ + \int_0^t \langle F_g(\hat{\rho}_k(s)), \psi \rangle_H ds + \int_0^t \langle \sigma_g(\hat{\rho}_k(s)) d\hat{w}_k^Q(s), \psi \rangle_H, \ \hat{\mathbb{P}}\text{-a.s.} \end{aligned} \tag{A.4.3}$$

Since $\hat{\rho}_k + \hat{\theta}_k$ converges to $\hat{\rho}$ in $L^2(0,T;H)$, $\hat{\mathbb{P}}$-a.s., we have

$$\lim_{k\to\infty}\int_0^t \langle \hat{\rho}_k(s)+\hat{\theta}_k(s),\psi\rangle_H\,ds = \int_0^t \langle \hat{\rho}(s),\psi\rangle_H\,ds, \qquad t\in[0,T], \quad \hat{\mathbb{P}}\text{-a.s.} \tag{A.4.4}$$

As in the proof of [22, Theorem 7.1], we have that

$$\lim_{k\to\infty}\sup_{t\in[0,T]}\left|\int_0^t \langle b(\hat{\rho}_k(s))\nabla\hat{\rho}_k(s),\nabla\psi\rangle_H ds - \int_0^t \langle b(\hat{\rho}(s))\nabla\hat{\rho}(s),\nabla\psi\rangle_H ds\right| = 0, \ \ \hat{\mathbb{P}}\text{-a.s.} \tag{A.4.5}$$

Next, as (2.1.15) extends to F_g, for each $k\in\mathbb{N}$,

$$\begin{aligned}
&\left|\int_0^t \langle F_g(\hat{\rho}_k(s)) - F_g(\hat{\rho}(s)),\psi\rangle_H\right| \\
&\le c\,\|\psi\|_{H^1}\int_0^t \left(1+\|\hat{\rho}_k(s)\|^{\theta-1}_{L^{2(\theta-1)}} + \|\hat{\rho}(s)\|^{\theta-1}_{L^{2(\theta-1)}}\right)\|\hat{\rho}_k(s)-\hat{\rho}(s)\|_H\,ds \\
&\le c\,\|\psi\|_{H^1}\left(1+\|\hat{\rho}_k\|^{\theta-1}_{L^{q(\theta-1)}([0,T];C([0,L]))} + \|\hat{\rho}\|^{\theta-1}_{L^{q(\theta-1)}([0,T];C([0,L]))}\right)\|\hat{\rho}_k-\hat{\rho}\|_{L^p([0,T];H)},
\end{aligned}$$

for any p, q satisfying $p^{-1}+q^{-1}=1$. Now, if $\theta\in(1,3)$, we can fix

$$\frac{4(\theta+1)}{7+2\theta-\theta^2} < p < \theta+1,$$

so that $q(\theta-1) < 4(\theta+1)/(\theta+3)$. Then thanks to (A.4.2), we have

$$\lim_{k\to\infty}\sup_{t\in[0,T]}\left|\int_0^t \langle F_g(\hat{\rho}_k(s)),\psi\rangle_H ds - \int_0^t \langle F_g(\hat{\rho}(s)),\psi\rangle_H ds\right| = 0, \quad \hat{\mathbb{P}}\text{-a.s.} \tag{A.4.6}$$

Finally, since

$$\lim_{k\to\infty}\sup_{t\in[0,T]}\left\|\hat{w}_k^Q(t)-\hat{w}^Q(t)\right\|_U = 0, \quad \mathbb{P}\text{-a.s.}$$

and

$$\lim_{k\to\infty} \|\hat{\rho}_k - \hat{\rho}\|_{L^2([0,T];H)} = 0, \quad \mathbb{P}\text{-a.s.}$$

with the uniform estimate

$$\sup_{k\in\mathbb{N}} \mathbb{E} \sup_{t\in[0,T]} \|\hat{\rho}_k(t)\|_H^2 < \infty,$$

by [37, Corollary 4.5] we have that

$$\lim_{k\to\infty} \sup_{t\in[0,T]} \left| \int_0^t \langle \sigma_g(\hat{\rho}_k(s)) d\hat{w}_k^Q(s), \psi\rangle_H - \int_0^t \langle \sigma_g(\hat{\rho}(s)) d\hat{w}^Q(s), \psi\rangle_H \right| = 0, \quad \text{in probability.} \tag{A.4.7}$$

Therefore, combining (A.4.4)–(A.4.7), if we integrate with respect to time both sides of equation (A.4.3) and take the limit as $k\to\infty$, it follows that for every $\psi \in C_0^\infty([0,L])$ and $t\in[0,T]$,

$$\begin{aligned}\int_0^t \langle \hat{\rho}(s), \psi\rangle_H \, ds = \int_0^t \Bigg[\langle g(u_0), \psi\rangle_H - \int_0^s \langle b(\hat{\rho}(r))\nabla\hat{\rho}(r), \nabla\psi\rangle_H dr \\ + \int_0^s \langle F_g(\hat{\rho}(r)), \psi\rangle_H dr + \int_0^s \langle \sigma_g(\hat{\rho}(r)) d\hat{w}^Q(r), \psi\rangle_H \Bigg] \, ds, \;\; \hat{\mathbb{P}}\text{-a.s.}\end{aligned}$$

Due to the arbitrariness of $t\in[0,T]$, this means that $\hat{\rho} \in L^2(\Omega; X\cap L^2(0,T;H^1))$ solves equation (A.3.1) with initial data u_0, and the first part of the theorem is proved.

We omit the proof of the second part as it is analogous to the one we have just seen. We only notice that in order to prove (A.4.6) we need that $q(\theta-1) < 4$. In particular, we need $4/(\theta-1) > 1$, and this is satisfied if $\theta \in (1,5)$.

Bibliography

[1] V. Barbu, NONLINEAR DIFFERENTIAL EQUATIONS OF MONOTONE TYPES IN BANACH SPACES, Springer Monographs in Mathematics, 2010.

[2] J. Birrell, S. Hottovy, G. Volpe, J. Wehr, *Small mass limit of a Langevin equation on a manifold* 18 (2017), pp. 707–755.

[3] Z. Brzezniak, S. Cerrai, *Stochastic wave equations with constraints: well-posedness and Smoluchowski-Kramers diffusion approximation*, arXiv: 2303.09717.

[4] Z. Brzeźniak and G. Dhariwal, *Stochastic constrained Navier-Stokes equations on* $\mathbb{T}^2$. J. Differential Equations 285 (2021), pp. 128–174.

[5] Z. Brzeźniak, G. Dhariwal, J. Hussain and M. Mariani, *Stochastic and deterministic constrained partial differential equations* in STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS AND RELATED FIELDS, Springer Proc. Math. Stat. 229 (2018), pp. 133-146.

[6] Z. Brzeźniak, G. Dhariwal and M. Mariani, *2D constrained Navier-Stokes equations* J. Differential Equations 264 (2018), pp. 2833–2864.

[7] Z. Brzeźniak and K.D. Elworthy, *Stochastic differential equations on Banach manifolds.* Methods Funct. Anal. Topology 6 (2000), pp. 43–84.

[8] A. Budhiraja, P. Dupuis, V. Maroulas, *Large deviations for infinite dimensional stochastic dynamical systems*, The Annals of Probability 36 (2008), pp. 1390-1420.

[9] L. Caffarelli, F. Lin, *Nonlocal heat flows preserving the* L^2*-energy*, Discrete and Continuous Dynamical Systems 32 (2009), pp. 49–64.

[10] E. Caglioti, M. Pulvirenti and F. Rousset, *On a constrained 2-D Navier-Stokes equation* Communications in Mathematical Physics 290 (2009), pp. 651–677.

[11] S. Cerrai, A. Debussche, *Smoluchowski-Kramers diffusion approximation for systems of stochastic damped wave equations with non-constant friction*, arXiv:2312.08925.

[12] S. Cerrai, M. Freidlin, *On the Smoluchowski-Kramers approximation for a system with an infinite number of degrees of freedom*, Probability Theory and Related Fields 135 (2006), pp. 363-394.

[13] S. Cerrai, M. Freidlin, *Smoluchowski-Kramers approximation for a general class of SPDE's*, Journal of Evolution Equations 6 (2006), pp. 657-689.

[14] S. Cerrai, M. Freidlin, *Small mass asymptotics for a charged particle in magnetic field and long-time influence of small perturbations*, Journal of Statistical Physics 144 (2011), pp. 101-123.

[15] S. Cerrai, M. Freidlin, *Large deviations for the Langevin equation with strong damping*, Journal of Statistical Physics 161 (2015), pp. 859–875.

[16] S. Cerrai, M. Freidlin, M. Salins, *On the Smoluchowski-Kramers approximation for SPDEs and its interplay with large deviations and long time behavior*, Discrete and Continuous Dynamical Systems, Series A, 37 (2017), pp. 33–76.

[17] S. Cerrai, N. Glatt-Holtz, *On the convergence of stationary solutions in the Smoluchowski-Kramers approximation of infinite dimensional systems*, Journal of Functional Analysis 278 (2020), pp. 1–38.

[18] S. Cerrai, M. Salins, *Smoluchowski-Kramers approximation and large deviations for infinite dimensional gradient systems*, Asymptotics Analysis 88 (2014), pp. 201-215.

[19] S. Cerrai, M. Salins, *Smoluchowski-Kramers approximation and large deviations for infinite dimensional non-gradient systems with applications to the exit problem*, Annals of Probability 44 (2016), pp. 2591–2642.

[20] S. Cerrai, M. Salins, *On the Smoluchowski-Kramers approximation for a system with an infinite number of degrees of freedom subject to a magnetic field*, Stochastic Processes and Their Applications 127 (2017) pp. 273–303.

[21] S. Cerrai, J. Wehr, Y. Zhu, *An averaging approach to the Smoluchowski-Kramers approximation in the presence of a varying magnetic field*, Journal of Statistical Physics 181 (2020), pp. 132–148.

[22] S. Cerrai, G. Xi, *A Smoluchowski-Kramers approximation for an infinite dimensional system with state-dependent damping*, Annals of Probability, 50 (2022), pp. 874–904.

[23] S. Cerrai, M. Xie, *On the small noise limit in the Smoluchowski-Kramers approximation of nonlinear wave equations with variable friction*, Transaction of the American Mathematical Society, 372 (2023), pp. 7657–7689.

[24] S. Cerrai, M. Xie, *On the small-mass limit for stationary solutions of stochastic wave equations with state dependent friction*, Applied Mathematics & Optimization, 90(1) (2024), p.7.

[25] S. Cerrai, M. Xie, *The small-mass limit for some constrained wave equations with nonlinear conservative noise*, arXiv preprint arXiv:2409.08021.

[26] G. Da Prato and J. Zabczyk, STOCHASTIC EQUATIONS IN INFINITE DIMENSIONS, Cambridge University Press, Second Edition, 2014.

[27] A. Debussche, M. Hofmanovà, J. Vovelle, *Degenerate parabolic stochastic partial differential equations: Quasilinear case*, Annals of Probability 44 (2016), pp. 1916–1955.

[28] J. Földes, N. Glatt-Holtz, G. Richards, E. Thomann, *Ergodic and mixing properties of the Boussinesq equations with a degenerate random forcing*, Journal of Functional Analysis 269 (2015) pp. 2427–2504.

[29] M. Freidlin, *Some remarks on the Smoluchowski-Kramers approximation*, J. Statist. Phys. 117 (2004), pp. 617–634.

[30] M. Freidlin, W. Hu, *Smoluchowski–Kramers approximation in the case of variable friction*, Journal of Mathematical Sciences 179 (2011), pp. 184–207.

[31] H. Frid, Y. Li, D. Marroquin, J. F. Nariyoshi, Z. Zeng, *The strong trace property and the Neumann problem for stochastic conservation laws*, Stochastics and Partial Differential Equations: Analysis and Computations 10 (2022), pp. 1–59.

[32] H. Frid, Y. Li, D. Marroquin, J. F. Nariyoshi, Z. Zeng, *The Dirichlet problem for stochastic degenerate parabolic-hyperbolic equations*, Communications in Mathematical Analysis and Applications 1 (2022), pp. 1–71.

[33] P. Fritz, P. Gassiat, T. Lyons, *Physical Brownian motion in magnetic field as a rough path*, Transaction or the American Mathematical Society 367 (2015), pp. 7939–7955.

[34] R. Fukuizumi, M. Hoshino, T. Inui, *Non relativistic and ultra relativistic limits in 2D stochastic nonlinear damped Klein-Gordon equation*, Nonlinearity 35 (2022), pp. 2878–2919.

[35] B. Gess, M. Hofmanovà, *Well-posedness and regularity for quasilinear degenerate parabolic-hyperbolic SPDE*, Annals of Probability 46 (2018), pp. 2495–2544.

[36] D. Greiser, *Uniform bounds for eigenfunctions of the Laplacian on manifolds with boundary*, Communications in Partial Differential equations 27 (2002), pp. 1283–1299.

[37] I. Gyöngy, *Existence and uniqueness results for semilinear stochastic partial differential equations*, Stochastic Processes and Their Applications 73 (1998), pp. 271–299.

[38] I. Gyöngy, N.V. Krylov, *Existence of strong solutions for Itô's stochastic equations via approximations*, Probability Theory and Related Fields 103 (1996), pp. 143-158

[39] M. Hairer, J. Mattingly, *Spectral gaps in Wasserstein distances and the 2D stochastic Navier-Stokes equations*, Annals of Probability 36 (2008), pp. 2050–2091.

[40] Y. Han, *Stochastic wave equation with Hölder noise coefficient: well-posedness and small mass limit*, arXiv:2305.04068.

[41] D. Herzog, S. Hottovy, G. Volpe, *The small-mass limit for Langevin dynamics with unbounded coefficients and positive friction*, Journal of Statistical Physics 163 (2016), pp. 659–673.

[42] S. Hottovy, A. McDaniel, G. Volpe, J. Wehr, *The Smoluchowski-Kramers limit of stochastic differential equations with arbitrary state-dependent friction*, Communications in Mathematical Physics 336 (2015), pp. 1259–1283.

[43] Hofmanová. M, Zhang. T., *Quasilinear parabolic stochastic partial differential equations: existence, uniqueness*, Stochastic Processes and their Applications, 127.10 (2017), pp.3354-3371.

[44] W. Hu, K. Spiliopoulos, *Hypoelliptic multiscale Langevin diffusions: large deviations, invariant measures and small mass asymptotics*, Electronic Journal of Probability 22 (2017).

[45] H. Kramers, *Brownian motion in a field of force and the diffusion model of chemical reactions*, Physica 7 (1940), pp. 284–304.

[46] J. J. Lee, *Small mass asymptotics of a charged particle in a variable magnetic field*, Asymptotic Analysis 86 (2014), pp. 99–121.

[47] W. Liu, *Existence and uniqueness of solutions to nonlinear evolution equations with locally monotone operators* Nonlinear Analysis 74 (2011), pp. 7543–7561.

[48] W. Liu, M. Röckner. *SPDE in Hilbert space with locally monotone coefficients*, Journal of Functional Analysis 259.11 (2010): 2902-2922.

[49] Y. Lv, A. Roberts, *Averaging approximation to singularly perturbed nonlinear stochastic wave equations*, Journal of Mathematical Physics 53 (2012), pp. 1–11.

[50] Y. Lv, A. Roberts, *Large deviation principle for singularly perturbed stochastic damped wave equations*, Stochastic Analysis and Applications 32 (2014), pp. 50-60.

[51] Y. Lv, A. Roberts, W. Wang, *Approximation of the random inertial manifold of singularly perturbed stochastic wave equations*, Stochastics and Dynamics 32, (2014) 1370018, 21 pp.

[52] A. Millet, P. Morien, *On a nonlinear stochastic wave equation in the plane: existence and uniqueness of solution*, Annals of Probability 11 (2001), pp. 211–236.

[53] E. Nelson, DYNAMICAL THEORIES OF BROWNIAN MOTION., Princeton University Press, Princeton, N.J., 1967

[54] H. Nguyen, *The small-mass limit and white-noise limit of an infinite dimensional generalized Langevin equation*, Journal of Statistical Physics 173 (2018), pp. 411–437.

[55] P. Rybka, *Convergence Of Heat Flow On a Hilbert Manifold*, Proceedings of the Royal Society of Edinburgh 136 (2006), pp. 851–862.

[56] M. Salins, *Smoluchowski-Kramers approximation for the damped stochastic wave equation with multiplicative noise in any spatial dimension*, Stochastic Partial Differential Equations: Analysis and Computation 7 (2019), pp. 86–122.

[57] C. Shi, W. Wang, *Small mass limit and diffusion approximation for a generalized langevin equation with infinite number degrees of freedom*, Journal of Differential Equations 286 (2021), pp. 645–675.

[58] J. Simon, *Compact sets in the space* $L^p(0,T;B)$, Annali di Matematica pura ed applicata 146.1 (1986): 65-96.

[59] M. Smoluchowski, *Drei Vortage über Diffusion Brownsche Bewegung und Koagulation von Kolloidteilchen*, Physik Zeit. 17 (1916), pp. 557-585.

[60] K. Spiliopoulos, *A note on the Smoluchowski-Kramers approximation for the Langevin equation with reflection*, Stochastics and Dynamics 7 (2007), pp. 141–152.

[61] L. Xie, L. Yang, *The Smoluchowski-Kramers limits of stochastic differential equations with irregular coefficients*, Stochastic Processes and their Applications 150 (2022), pp. 91–115.

[62] Y. Zine, *Smoluchowski-Kramers approximation for the singular stochastic wave equations in two dimensions*, arXiv:2206.08717.