THEOREMS ON STABILITY AND CONVERGENCE IN NUMERICAL SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS

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HISTORICAL BACKGROUND

While great strides have been made in the solution of ordinary differential equations by numerical methods, much less has been done with numerical solutions of partial differential equations. That the latter are more difficult to handle is true, but surely much of the lethargy in the development of the theory behind the numerical solution of partial differential equations was due to the fact that such methods were seldom, if ever, used. This was due to the tremendous labor involved in using such methods to solve even a simple practical problem. Undoubtedly the recent increase in interest in this field is due, to a large extent, to the development of the modern high-speed calculators and to the increased number of them in everyday use.

It was not until 1910 that one of the first important papers was written on this subject (ref. 6). The author of the article, L. F. Richardson, numerically approximated the solution of the one dimensional heat flow equation

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} \tag{1.1}$$

with simple initial and boundary conditions, by means of the difference equation

$$U(x,t+\Delta t) - U(x,t-\Delta t) = 2r\left[U(x+\Delta x,t) - 2U(x,t) + U(x-\Delta x,t)\right] \quad (1.2)$$

where r, called the "ratio", equals $\frac{\Delta t}{(\Delta x)^2}$. Using time and space increments of 0.001 and 0.1, respectively, Richardson computed, by means of (1.2), the temperature on the mesh points from $t = 0$ to $t = 0.005$, for certain values of x. Comparing his result with that from solving (1.1) by means of a Fourier series (with the same initial and boundary conditions) he found them to be quite good.

Perhaps the outstanding early paper on the subject was one by Courant, Friedrichs, and Lewy (ref. 2). A difference equation with given auxiliary conditions was defined as convergent provided the solution of the difference equation approached the solution of the differential equation when the distance between mesh points approached zero. In this paper it was first shown that the ratio of the increments affected the ability of a difference equation to converge and in one particular hyperbolic problem it was shown that, to be sure of convergence, the ratio $\frac{\Delta t}{\Delta x}$ must be taken less than one.[1]

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[1] In a hyperbolic partial differential equation the ratio is $\frac{\Delta t}{\Delta x}$.

Later von Neumann, in an entirely different manner but using the same problem as Courant, Friedrichs, and Lewy, again showed that $\frac{\Delta t}{\Delta x}$ must be less than 1 to make convergence certain. He recognized that "stability" is a separate factor, and defined a difference equation as being stable when the numerical solution of the difference equation can be made arbitrarily close to the exact solution of the difference equation. He showed that the value of the ratio was the determining factor in stability (ref. 5).[2]

By using von Neumann's method Kaplan and O'Brien recently showed that Richardson's results, previously mentioned, were entirely in error (ref. 4). It was shown that Richardson was using an unstable representation and that his results, if carried a little further in the time direction, would oscillate with ever increasing amplitude. His method was used by them to actually demonstrate this showing that the results soon alternately exceeded the initial temperature and fell below the final temperature. A stable solution was then used, Δt and Δx unchanged in size, and the result was a close approximation to the

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[2] Von Neumann himself did not publish these results.

solution of the differential equation (this in spite of the fact that the difference equation chosen actually possessed a larger truncation error).

In 1949 O'Brien, Hyman, and Kaplan showed that in a simple parabolic partial differential equation the difference in value between the exact solution of the partial differential equation and a numerical solution of a corresponding partial difference equation, when a stable ratio was used, was due almost entirely to the difference in the exact solution of the differential equation and an exact solution of the difference equation rather than in the difference between the exact solution of the difference equation and its numerical solution caused by the accumulation of round-off errors. Even in an unstable case it was shown that lack of convergence more than lack of stability was responsible for the large error in the numerical solution in this particular case (ref. 5).

PRELIMINARY MATERIAL

Consider a partial differential equation of the parabolic or hyperbolic type with suitable auxiliary conditions. Assume that a solution exists, perhaps in the form of a converging infinite series; such a solution will henceforth be called the exact solution of the differential equation and will be represented by the letter T (see equation 3.5). Let a suitable difference equation be used to approximate the differential equation and choose auxiliary conditions which will suitably approximate the auxiliary conditions of the differential equation. Its solution will be designated an exact solution of the difference equation and it will be represented by the letter U (see equation 3.9). There may be more than one exact solution for a given difference equation depending on the chosen auxiliary conditions. Let the numerical solution derived from the difference representation be represented by the letter N.[1]

It will be most convenient to use the term "convergence" in this

[1] Henceforth when we speak of the solution of a difference or differential equation it will be understood that suitable auxiliary conditions are also included.

paper in the usual way although I believe a better definition is given in the footnote below.[2]

The term "stability" has two different definitions and, in reference 5, they are distinguished between and called weak stability and strong stability. We will use the terms with the following meanings:

A numerical representation is weakly stable provided a single error, introduced at any point in the numerical computation, will not increase with succeeding steps.

A numerical representation is strongly stable provided the accumulation of round-off errors, introduced at each point and carried on

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[2] A difference equation is called convergent when, by properly letting x and t go to zero, the value of $|U - T|$ can be made arbitrarily small for any set of values of x and t that lie on the mesh. That this is the most logical definition follows from the fact that we would, in practice, use a difference equation to approximate the solution of a differential equation. Whether we then used a numerical solution or not we would have the solution only at points of the mesh. No matter how small we chose Δx and Δt there would be an infinite (continuum) number of values of (x,t) within our range that would never be, in fact could never be, covered. In the problem considered in this paper we see that all of the mesh points will be rational sets of values which, even in the limit as $\Delta x, \Delta t \to 0$, covers only a relatively small part of all the possible points in our interval.

At first this might cause some alarm and misgivings as to the value of a difference solution but a little consideration shows such concern to be groundless. For even if it were possible for us to get a ridiculous answer from our difference equation for points off the mesh it would not matter. Our values on the mesh can be made as close to the true result, at these points, as desired. If we should find it necessary to determine the value of the solution at an inaccessible point we could do so by one of the many interpolation methods available.

by the numerical process, will not increase beyond a certain bound with succeeding steps.

In reference 5 it is assumed that weak stability implies strong stability provided the round-off errors are of a random nature. This assumption is justified in each of the problems actually computed. In reference 3 it is pointed out that although round-off errors may be non-random in certain regions of integration, the randomness may be regained by carrying extra figures in calculating those regions.

The theorems dealt with in this paper concern weak stability and since it is convenient (although not necessary) it will be assumed that weak stability implies strong stability.

Convergence has to do with T and U while strong stability has to do with U and N. If a numerical solution lacks either convergence or strong stability N will not generally be a good approximation to T. The important question is "will a certain numerical solution properly approximate the solution of the given differential equation?" That this is also a difficult question is immediately evident for generally one uses a numerical solution because no other one is available. It is the principal aim of this paper to show that, in certain parabolic cases at least, convergence is closely related to weak stability and, in particular, that one may insure convergence (as well as strong stability) by proper choice of r. Then, whether T and U are obtainable or not, one need only choose r within the proper range and not only weak stability, but convergence, is insured and the numerical solution will approximate the true solution to any desired degree of accuracy.

As stated, strong stability has to do with U and N. We call $|U - N|$ the numerical error: this error arises in various ways, round-off errors being of prime interest to us. N is called strongly stable when $|U - N|$ can be kept arbitrarily small throughout the entire region of solution by properly choosing the size of Δx and Δt and by properly deciding on the number of significant figures to retain in the steps of the computation.

THE PROBLEM

As was already stated, the principle aim of this paper is to give a practical means of determining conditions under which a numerical solution will suitably approximate the true solution of a parabolic partial differential equation. In this section we will limit ourselves to a simple initial condition.

Before proving the theorems (3.1 and 3.2) some groundwork is necessary and three lemmas, which will be used in establishing theorem 3.2, will be proved.

Let us turn our attention now to a specific problem. Assume a bar of unit width and unspecified length be heated uniformly to a temperature 1°. Let each side be suddenly cooled to, and kept at, 0° and assume the heat flow in one dimension (namely across the width of the bar). Let its constants be such that

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} \qquad (3.1)$$

represents the heat flow, where $T = T(x,t)$ represents the temperature, t the time, and x the distance, (measured across the bar).

The partial differential equation (3.1) is readily solved for the particular solution

$$T(x,t) = c_1 e^{kt} \left(c_2 \cos \sqrt{-k}\, x + c_3 \sin \sqrt{-k}\, x\right) \tag{3.2}$$

Imposing the boundary conditions

$$T(0,t) = 0, \quad T(1,t) = 0 \quad (t>0) \tag{3.3}$$

and the initial condition

$$T(x,0) = 1 \tag{3.4}$$

one finds the solution to be

$$T(x,t) = \frac{4}{\pi} \sum_{k=1,3,5,\dots}^{\infty} \frac{1}{k}\, e^{-\pi^2 k^2 t} \sin \pi k x. \tag{3.5}$$

Equation (3.5) is what has been referred to as the exact solution of the partial differential equation (3.1) with the initial and boundary conditions (3.3) and (3.4).

Next approximate (3.1) by the difference equation

$$U(x,t+\Delta t) = U(x,t) + r\left[U(x+\Delta x,t) - 2U(x,t) + U(x-\Delta x,t)\right], \tag{3.6}$$

where Δt and Δx are the time and space increments respectively, and r, the ratio, equals $\Delta t/(\Delta x)^2$. [1]

A solution of (3.6) satisfying (3.3) is seen to be

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[1] There are other approximations of (3.1) different from (3.6) (e.g. see reference 5).

$$U(x,t) = A_k e^{a_k t} \sin \pi k x,$$

A_k arbitrary, k an integer greater than 0, and

$$a_k = \frac{1}{\Delta t} \ln \left[1 - 4r \sin^2 \frac{(\pi k \Delta x)}{2}\right]. \tag{3.7}$$

In place of the initial condition $T(x,0) \equiv 1$, $x \in (0,1)$, use a trigonometric polynomial approximation in the sense of least square error. To satisfy this, choose A_k so that

$$\int_0^1 \left[1 - \sum_{K=1}^{M-1} A_k \sin \pi k x\right]^2 dx \tag{3.8}$$

is a minimum giving the suitable approximation

$$U(x,t) = \frac{4}{\pi} \sum_{K=1,3,\cdots}^{M-1} \frac{1}{k} \sin \pi k x \left[1 - 4r \sin^2 \frac{(\pi k)}{2M}\right]^{\frac{tM^2}{r}}, \tag{3.9}$$

where M, the number of subdivisions of the width of the bar, is given by $M \Delta x = 1$.

Equation (3.9) is what has been referred to as an exact solution of the difference equation.

To determine the conditions under which (3.6) will be stable assume $e(x,t)$ is the total error at the point (x,t). Then $e(x,t)$ satisfies the "variational equation"

$$\frac{e(x,t + \Delta t) - e(x,t)}{\Delta t} = \frac{e(x + \Delta x,t) - 2e(x,t) + e(x - \Delta x,t)}{(\Delta x)^2} \tag{3.10}$$

That the form of (3.10) is the same as that of (3.6) is due to the fact that (3.6) is linear.

The errors $e(x,t)$ at each of the $M - 1$ interior mesh points on the line $t = 0$ are arbitrary quantities and small. They may be represented by a Fourier series of complex exponentials of the form

$$\sum_{n=1}^{M-1} A_n e^{ib_n x} . \tag{3.11}$$

Choose a_n the same as in (3.7) and

$$\sum_{n=1}^{M-1} A_n e^{a_n t} e^{ib_n x} \tag{3.12}$$

satisfies (3.10) and reduces to (3.11) when $t = 0$. Any term in the sum (3.12) is a solution of (3.10); let such a term be represented by

$$Ae^{at}e^{ibx}$$

The value of b depends on M and on the fact that the width of the bar is taken as 1; thus it is rational. Here $a = a(b)$ is generally complex. The ratio of error growth is $e^{a\Delta t}$. A sufficient condition for stability is therefore

$$\left| e^{a\Delta t} \right| < 1. \tag{3.13}$$

To determine values in (3.10) such that (3.13) holds (i.e. errors diminish giving stability) put $e^{at + ibx}$ into (3.10). This gives

$$\frac{e^{a(t+\Delta t)+ibx} - e^{at+ibx}}{\Delta t} = \frac{e^{at+ib(x+\Delta x)} - 2e^{at+ibx} + e^{at+ib(x-\Delta x)}}{(\Delta x)^2} \qquad (3.14)$$

Let $y = e^{a\Delta t}$ and (3.14) becomes, with slight simplification,

$$y - 1 = r\left[e^{ib\Delta x} - 2 + e^{-ib\Delta x}\right] = -4r\sin^2\frac{(b\Delta x)}{2}.$$

Then to satisfy (3.13) take

$$-1 < 1 - 4r\sin^2\frac{(b\Delta x)}{2} < 1.$$

The right side of the inequality is evidently true; the left inequality holds for all b if and only if $r < \frac{1}{2}$. This must be made to hold for all b because in a numerical solution all frequencies are possible due to the small errors always present. Thus, in this problem, the value $r = \frac{1}{2}$ separates the region of stability, where errors damp out, from the region of instability, where some errors grow. The value of r, the "ratio", is the determining factor in the stability of a difference representation.

The effect of the ratio upon convergence will now be considered.

Theorem 3.1. In the problem under discussion, namely equations (3.1) with conditions (3.3) and (3.4), and (3.6) with conditions (3.3) and (3.8), stability is necessary to insure convergence.

Proof. By applying Weierstrass' M - test one sees that (3.5) converges uniformly with respect to x, $0 \leqq x \leqq 1$, $t \geqq t_o > 0$. It will now be shown that (3.9), the solution under consideration of the corresponding difference equation, diverges to infinity as the number of intervals M increases without bound, $x \neq 0, 1$, $t > 0$, provided r is chosen greater than $\frac{1}{2}$.

For each value of M (3.9) yields a different series; consider these series as successive values of M are used. Keep

$$M - M_o \leqq k \leqq M - 1, \tag{3.15}$$

where M_o is a fixed arbitrary positive constant greater than 1, and note the result as M increases without bound. The value of

$$1 - 4r \sin^2 \frac{(\pi k)}{2M}$$

eventually exceeds, and remains greater than, one. Choose one term from each series, subject to the condition (3.15), and there follows a sequence of terms that diverge to infinity in spite of the tendency of $\frac{\sin \pi kx}{k}$ toward zero. Q. E. D.

Three lemmas will be found useful in the work that is to follow.

<u>Lemma 3.1</u>. For $r \in (0,\frac{1}{4}]$, $z \in [0,V]$, $V < \frac{\pi}{2}$, M an integer $\geqq 1$,

$$\left[1 - 4r \sin^2 z\right]^{\frac{tM^2}{r}}$$

is non-increasing in r.

<u>Proof</u>. If $\begin{Bmatrix} t = 0 \\ z = 0 \end{Bmatrix}$ the lemma is true. Let $z > 0$ and let us choose a fixed value for $t > 0$. Choose any S and s as long as they satisfy

$$\tfrac{1}{2} \geqq 2s \geqq S > s > 0.$$

$$\left[1 - 4s \sin^2 z\right]^{\frac{S}{s}} \geqq 1 - 4S \sin^2 z + 8S(S - s)\sin^4 z + \frac{32S}{3}(S - s)(2s - S)\sin^6 z$$

$$\geqq 1 - 4S \sin^2 z.$$

Thus

$$\frac{\left[1 - 4s \sin^2 z\right]^{\frac{S}{s}}}{1 - 4S \sin^2 z} \geqq 1$$

or

$$\frac{\left[1 - 4s \sin^2 z\right]^{\frac{tM^2}{s}}}{\left[1 - 4S \sin^2 z\right]^{\frac{tM^2}{S}}} \geqq 1$$

Q. E. D.

<u>Lemma 3.2</u>. With the same restrictions on r, z, t and M as in Lemma (3.1), for any $\varepsilon > 0$ there exists an M_o so large that

$$\left[1 - 4r \sin^2 z\right]^{\frac{tM^2}{r}} - e^{-4M^2 z^2 t} \leqq \frac{\varepsilon}{M}$$

whenever $M > M_o$.

<u>Proof</u>. <u>Case i</u>. Let $z \in (0, M^{-2/3}]$.

$$\lim_{r \to 0} \left[1 - 4r \sin^2 z \right]^{\frac{tM^2}{r}} = e^{-4tM^2 \sin^2 z}$$

Hence, from Lemma (3.1), $e^{-4tM^2 \sin^2 z}$ serves as an upper bound for

$$\left[1 - 4r \sin^2 z \right]^{\frac{tM^2}{r}} .$$

Then

$$\left[1 - 4r \sin^2 z \right]^{\frac{tM^2}{r}} - e^{-4M^2 z^2 t} \leqq e^{-4tM^2 \sin^2 z} - e^{-4M^2 z^2 t}.$$

$$e^{-4tM^2 \sin^2 z} \leqq e^{-4tM^2 z^2 + (4/3)\, tM^2 z^4}.$$

Let $$J(z,M) \equiv e^{-4tM^2 z^2 + (4/3) tM^2 z^4} - e^{-4tM^2 z^2}$$

$$J(0,M) = 0$$

$$J_z(z,M) = 8tM^2 z e^{-4tM^2 z^2} \left[\left(\frac{2}{3} z^2 - 1 \right) e^{(4/3) tM^2 z^4} + 1 \right] .$$

Let $$K(z,M) \equiv \left(\frac{2}{3} z^2 - 1 \right) e^{(4/3) tM^2 z^4} + 1$$

$$K(0,M) = 0$$

$$K_z(z,M) = \frac{4z}{3} e^{(4/3) tM^2 z^4} \left[4tM^2 z^2 \left(\frac{2}{3} z^2 - 1 \right) + 1 \right]$$

Let $$L(z,M) \equiv 4tM^2 z^2 \left(\frac{2}{3} z^2 - 1 \right) + 1.$$

Since $J(z,M) \gtreqq 0$, $L(z,M) \gtreqq 0$; in fact

$$0 \leqq L(z,M) \leqq 1.$$

Since this is true for any fixed t,

$$0 \leqq z < \frac{1}{M}.$$

Hence

$$K_z(z,M) \leqq \frac{4}{3M} e^{\frac{4t}{3M^2}} \leqq \frac{4}{3M},$$

$$K(z,M) \leqq \frac{4}{3M^{5/3}},$$

$$J_z(z,M) \leqq \frac{32tM^{1/3}ze^{-4tM^2z^2}}{3}.$$

Then $$\text{Max } J_z(z,M) \leqq \frac{16\sqrt{t}}{3\sqrt{2}\, e^{\frac{1}{2}}M^{2/3}},$$

$$J(z,M) \leqq \frac{16\sqrt{t}}{3\sqrt{2}\, e^{\frac{1}{2}}M^{4/3}} \leqq \frac{\varepsilon}{M}.$$

Hence $$\left[1 - 4r\sin^2 z\right]^{\frac{tM^2}{r}} - e^{-4M^2z^2t} \leqq \frac{\varepsilon}{M}.$$

<u>Case</u> <u>ii</u>. Let $z \in (M^{-2/3}, \frac{\pi}{2})$

$$\left[1 - 4r\sin^2 z\right]^{\frac{tM^2}{r}},$$

considered as a function of z, is monotone decreasing in $z \in (M^{-2/3}, \frac{\pi}{2})$

when $r \in (0, \frac{1}{4}]$ having

$$\left[1 - 4r\sin^2 (M^{-2/3})\right]^{\frac{tM^2}{r}}$$

for upper bound. But for any $\varepsilon > 0$ there exists an $M_0 > 0$ so large that, for $M > M_o$

$$\left[1 - 4r\sin^2 (M^{-2/3})\right]^{\frac{tM^2}{r}} < \frac{\varepsilon}{M} \qquad \text{Q. E. D.}$$

<u>Lemma</u> <u>3.3</u>. Under the conditions of lemma (3.2)

$$e^{-4M^2z^2t} - \left[1 - 4r\sin^2 z\right]^{\frac{tM^2}{r}} \leqq \frac{\varepsilon}{M}$$

<u>Proof</u>. <u>Case</u> <u>i</u>. Let $z \in (0, M^{-2/3}]$.

Since $$\left[1 - 4r\sin^2 z\right]^{\frac{tM^2}{r}}$$

is monotone decreasing in r, it has for lower bound

$$(\cos z)^{8tM^2}.$$

$$e^{-4M^2z^2t} - \left[1 - 4r\sin^2 z\right]^{\frac{tM^2}{r}} \leqq e^{-4M^2z^2t} - (\cos z)^{8tM^2} \equiv G(z,M).$$

$$G(0,M) = 0$$

$$G_z(z,M) = 8M^2tz\left[- e^{-rM^2z^2t} + \frac{\tan z}{z}(\cos z)^{8tM^2}\right].$$

Call

$$H(z,M) = \left[- e^{-4M^2z^2t} + \frac{\tan z}{z}(\cos z)^{8tM^2}\right].$$

If $$(\cos z)^{8tM^2} \geqq e^{-4M^2z^2t}$$

case i is proved. Consider

$$(\cos z)^{8tM^2} \leqq e^{-4M^2z^2t}$$

$$H(z,M) \leqq e^{-4M^2z^2t}\left[\frac{\tan z}{z} - 1\right]$$

$$\operatorname{Tan} z \leqq z + \frac{z^3}{3} + z^5.$$

$$H(z,M) \leqq e^{-4M^2z^2t}\left(\frac{z^2}{3} + z^4\right) \leqq z^2e^{-4M^2z^2t}$$

$$z^2e^{-4M^2z^2t} \leqq \frac{e^{-1}}{4tM^2}$$

$$G_z(z,M) \leqq 2e^{-1}M^{-2/3}$$

$$G(z,M) \leqq 2e^{-1}M^{-4/3} \leqq \frac{\varepsilon}{M}.$$

Thus $$e^{-4M^2z^2t} - \left[1 - 4r\sin^2 z\right]^{\frac{tM^2}{r}} \leqq \frac{\varepsilon}{M}$$

<u>Case</u> <u>ii</u>. Let $z \in (M^{-2/3}, \frac{\pi}{2})$

$$e^{-4M^2z^2t} \leqq e^{-4tM^{2/3}} \leqq \frac{\varepsilon}{M} \qquad \text{for any } \varepsilon > 0,$$

if M is chosen large enough.

Hence $$e^{-4M^2z^2t} - \left[1 - 4r\sin^2 z\right]^{\frac{tM^2}{r}} \leqq \frac{\varepsilon}{M} \text{ for } z \in (M^{-2/3}, \tfrac{\pi}{2})$$

since $$\left[1 - 4r\sin^2 z\right]^{\frac{tM^2}{r}} \geqq 0. \qquad \text{Q. E. D.}$$

The following theorem can now readily be proved.

<u>Theorem</u> <u>3.2</u>. In the problem under discussion, (see theorem 3.1) $r \in (0,\frac{1}{4}]$ is a sufficient condition for convergence.

Proof. It is necessary to show that, for any $d > 0$ there exists M_0 such that, for $M > M_0$,

$$\left| \frac{4}{\pi} \sum_{k=1,3,\dots}^{\infty} \frac{\sin \pi kx}{k} e^{-\pi^2 k^2 t} - \frac{4}{\pi} \sum_{k=1,3,\dots}^{M-1} \frac{\sin \pi kx}{k} \left[1 - 4r \sin^2 \frac{\pi k}{2M} \right]^{\frac{tM^2}{r}} \right| < d \qquad (3.16)$$

whenever $0 < r \leqq \frac{1}{4}$ and $t \geqq 0$ are fixed, and uniformly with respect to $x \in [0,1]$.

It is clear that (3.16) is true provided $t = 0$, $x = 0$, or $x = 1$. Assume then that $t > 0$ and $x \in (0,1)$.

One easily sees that

$$\frac{4}{\pi} \sum_{k=1,3,\dots}^{\infty} \frac{\sin \pi kx}{k} e^{-\pi^2 k^2 t} \qquad (3.17)$$

is uniformly convergent with respect to our values of x.

Hence for any $d_1 > 0$ there exists an M_1 such that $M > M_1$ implies

$$\frac{4}{\pi} \left| \sum_{k=1,3,\dots}^{\infty} \frac{\sin \pi kx}{k} e^{-\pi^2 k^2 t} - \sum_{k=1,3,\dots}^{M-1} \frac{\sin \pi kx}{k} e^{-\pi^2 k^2 t} \right| < d_1 \qquad (3.18)$$

Now consider

$$\frac{4}{\pi} \left| \sum_{k=1,3,\dots}^{M-1} \frac{\sin \pi kx}{k} e^{-\pi^2 k^2 t} - \sum_{k=1,3,\dots}^{M-1} \frac{\sin \pi kx}{k} \left[1 - 4r \sin^2 \frac{\pi k}{2M} \right]^{\frac{tM^2}{r}} \right|$$

which may be written

$$\frac{4}{\pi}\left|\sum_{k=1,3,\cdots}^{M-1}\frac{\sin\pi kx}{k}\left\{e^{-\pi^2k^2t}-\left[1-4r\sin^2\frac{\pi k}{2M}\right]^{\frac{tM^2}{r}}\right\}\right| \tag{3.19}$$

By means of lemmas (3.2) and (3.3) the above can be made, for any chosen $d_2 > 0$,

$$\leq \frac{4}{\pi}\left|\sum_{k=1,3,\cdots}^{M-1}\frac{\sin\pi kx}{k}\right|\frac{d_2}{2} < d_2 \tag{3.20}$$

by choosing $M >$ some fixed value, say M_2.

Now for any $d > 0$ we can choose M_0 so large that, when $M > M_0$, $d_1 < \frac{d}{2}$, $d_2 < \frac{d}{2}$ so that (3.16) is satisfied.

Q. E. D.

A GENERALIZATION OF THE INITIAL CONDITION

The previous problem and the theorem stating that choosing $r \in (0,\frac{1}{4}]$ insures convergence poses the question as to the possibility of using some such criterion in more general parabolic problems and in hyperbolic problems. In this section it is proved that such a criterion does exist for the previous problem with a generalized initial condition.

In particular consider again the differential equation (3.1) subject to the boundary conditions (3.3). Let the initial condition be

$$T(x,0^+) = f(x) \tag{4.1}$$

where $f(x)$ is assumed piecewise continuous and has one sided derivatives in the interval $0 \leqq x \leqq 1$.

In solving the differential equation one can use the previous result

$$T(x,t) = \sum_{n=1}^{\infty} C_n e^{-n^2\pi^2 t} \sin n\pi x$$

together with the condition (4.1) and get as a solution of the new system

$$T(x,t) = \sum_{n=1}^{\infty} C_n e^{-n^2\pi^2 t} \sin n\pi x \tag{4.2}$$

$$C_n = 2\int_0^1 f(x') \sin n\pi x' \ dx'$$

Now consider the difference equation (3.6) subject to the boundary conditions (3.3). In place of the initial condition (4.1) again use a trigonometric polynomial in the sense of least square error so that

$$\int_0^1 \left[f(x) - \sum_{n=1}^{\infty} A_n \sin n\pi x\right]^2 dx \tag{4.3}$$

is a minimum. This condition is satisfied by taking

$$A_n = 0, \quad n > M$$

$$A_n = C_n, \quad n \leq M$$

giving

$$\left.\begin{aligned} U(x,t) &= \sum_{n=1}^{M} A_n \sin \pi n x \left[1 - 4r \sin^2\left(\frac{\pi n}{2M}\right)\right]^{\frac{tM^2}{r}} \\ A_n &= 2\int_0^1 f(x') \sin \pi n x' \ dx' \end{aligned}\right\} \tag{4.4}$$

The change in the initial condition in no way affects the ratio so that $\frac{1}{2}$ still separates the region of stability from that of instability.

<u>Lemma 4.1</u>. Let $t > 0$, $x \in [0,1]$ and the series

$$\sum_{n=1}^{\infty} C_n e^{-n^2\pi^2 t} \sin n\pi x$$

converges uniformly in x.

Proof. The series $\sum_{n=1}^{\infty} C_n \sin n\pi x$ converges to $f(x)$ by hypothesis. The sequence $\left\{e^{-n^2\pi^2 t}\right\}$ is independent of x, monotone decreasing in n and the first term is $e^{-\pi^2 t}$. Hence $\sum_{n=1}^{\infty} C_n e^{-n^2\pi^2 t} \sin n\pi x$ is uniformly converging in x (ref. 1). Q. E. D.

Theorem 4.1. Consider the partial differential equation (3.1) with the conditions (3.3) and (4.1), and the partial difference equation (3.6) with the conditions (3.3) and (4.3).

Stability is a necessary condition to insure convergence of the difference solution to the solution of the differential equation.

Proof. The proof of theorem (3.1), without important modifications, will serve here. Q. E. D.

Theorem 4.2. Consider the partial differential equation (3.1) and the difference equation (3.6) with the auxiliary conditions the same as in theorem (4.1). A sufficient condition for convergence of the difference solution to that of the solution of the differential equation is that $r \in (0, \frac{1}{4}]$.

Proof. It is necessary and sufficient to show that, for $0 < r \leqq \frac{1}{4}$ and for any $d > 0$ that there exists an M_0 so large that, for all $M > M_0$,

$$\left| \sum_{n=1}^{\infty} C_n e^{-n^2\pi^2 t} \sin n\pi x - \sum_{n=1}^{M} A_n \sin n\pi x \left[1 - 4r \sin^2\left(\frac{\pi n}{2M}\right)\right]^{\frac{tM^2}{r}} \right| < d, \qquad (4.5)$$

where C_n and A_n are the same as previously defined.

If $t = 0$ (4.5) reduces to

$$\left|\sum_{n=1}^{\infty} (C_n - A_n) \sin n\pi x\right| = \left|\sum_{n=M+1}^{\infty} C_n \sin n\pi x.\right|$$

This expression can be made arbitrarily small by choosing M large enough.

Consider t fixed but greater than 0.

$$\sum_{n=1}^{\infty} C_n e^{-n^2\pi^2 t} \sin n\pi x$$

is uniformly converging so for any $d_1 > 0$ there exists an M_1 so large

that, when $M > M_1$

$$\left| \sum_{n=M+1}^{\infty} C_n e^{-n^2\pi^2 t} \sin n\pi x \right| < d_1 \qquad (4.6)$$

Next consider

$$\left| \sum_{n=1}^{M} C_n e^{-n^2\pi^2 t} \sin n\pi x - \sum_{n=1}^{M} A_n \sin n\pi x \left[1 - 4r \sin^2 \frac{\pi n}{2M}\right]^{\frac{tM^2}{r}} \right|$$

$$= \left| \sum_{n=1}^{M} C_n \sin n\pi x \left\{ e^{-4M^2 z^2 t} - \left[1 - 4r \sin^2 z\right]^{\frac{tM^2}{r}} \right\} \right| \qquad (4.7)$$

where $z = \frac{\pi n}{2M}$. The expression

$$(4.7) \leqq \sum_{n=1}^{M} \left|C_n \sin n\pi x\right| \quad \frac{d_2}{M} \qquad (4.8)$$

where $d_2 > 0$ can be chosen as small as we wish.

Let $$d_3 = \max \left| C_n \sin n\pi x \right|$$

and $$(4.8) \leqq M \frac{d_3 d_2}{M} = d_4, \tag{4.9}$$

where $d_4 = d_3 d_2$.

Finally one can choose d_2 so small, and M_0 so large, that when $M > M_0$

$$d < d_1 \not< d_4$$

and equation (4.5) will be satisfied.

Q. E. D.

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