MARYLAND

TECHNICAL
RESEARCH
REPORT

SRC TR 88-91

On Robust Stability of Linear State Space Models

by

M. K. H. Fan
J. C. Doyle
A. L. Tits

SYSTEMS RESEARCH CENTER
UNIVERSITY OF MARYLAND
COLLEGE PARK, MARYLAND 20742

ON ROBUST STABILITY OF LINEAR STATE SPACE MODELS

Michael K.H. Fan
University of Maryland

John C. Doyle
California Institute of Technology

André L. Tits
University of Maryland

Introduction. The structured singular value (μ), introduced by Doyle [1] allows to analyze robust stability and performance of linear systems affected by parametric as well as dynamic uncertainty. While exact computation of μ can be prohibitively complex, an efficiently computable upper bound was obtained in [2], yielding a practical sufficient condition for robust stability and performance.

In this note, the results of [2] are used to study the case of state space models of the form

$$\dot{x} = (A_0 + \sum_{i=1}^{m} \delta_i A_i)x \tag{1}$$

where the A_i's are $n \times n$ real matrices and the δ_i's are uncertain real parameters. The case where the A_i's have low rank is given special attention. When the A_i's all have rank one, (1) is equivalent to the model used by Qiu and Davison [3], which itself generalizes that used by Yedavalli [4]. By means of two examples, we compare our bound to those proposed in [3] and [4].

Preliminaries. Throughout the note, given any square complex matrix M, we denote by $\overline{\sigma}(M)$ its largest singular value and by M^H its complex conjugate transpose. Given any Hermitian matrix A, we denote by $\overline{\lambda}(A)$ its largest eigenvalue. Given any integer k, I_k denotes the $k \times k$ identity matrix and O_k the $k \times k$ zero matrix. Finally, j will denote $\sqrt{-1}$.

Given a $p \times p$ complex matrix M and positive integers $k_1, \ldots, k_m$, with $\sum_{q=1}^{m} k_q = p$, consider the family of block diagonal $p \times p$ matrices (In this note we consider only parametric perturbations)

$$\mathcal{X} = \{\text{block diag } (\delta_1 I_{k_1}, \ldots, \delta_m I_{k_m}) : \delta_q \in \mathbb{R}\} \ .$$

Definition 1. [1] The *structured singular value* $\mu_{\mathcal{X}}(M)$ of a complex $p \times p$ matrix M with respect to $\mathcal{X}$ is 0 if there is no Δ in $\mathcal{X}$ such that $\det(I - \Delta M) = 0$, and

$$\mu_{\mathcal{X}}(M) = \left(\min_{\Delta \in \mathcal{X}} \{\overline{\sigma}(\Delta) : \det(I - \Delta M) = 0\} \right)^{-1}$$

otherwise. □

Exact computation of the structured singular value is generally intractable. In [2], the following computable upper bound was obtained.

Fact 1. [2] For any matrix M and $\mathcal{X}$,

$$\mu_{\mathcal{X}}(M) \leq \hat{\mu}_{\mathcal{X}}(M) := \inf_{D \in \mathcal{D}_{\mathcal{X}}} \nu_{\mathcal{X}}(DMD^{-1})$$

where

$$\mathcal{D}_{\mathcal{X}} = \left\{ \text{block diag}(D_1, \ldots, D_m) : 0 < D_q = D_q^H \in \mathbb{C}^{k_q \times k_q} \right\}$$

and where, for any matrix A and $\mathcal{X}$, $\nu_{\mathcal{X}}(A)$ is defined by

$$\nu_{\mathcal{X}}(A) = \sqrt{\max \left\{ 0, \inf_{G \in \mathcal{G}_{\mathcal{X}}} \overline{\lambda}[A^H A + j(GA - A^H G)] \right\}}$$

with

$$\mathcal{G}_{\mathcal{X}} = \left\{ \text{block diag}(G_1, \ldots, G_m) : G_q = G_q^H \in \mathbb{C}^{k_q \times k_q} \right\} .$$

□

An efficient algorithm for computing $\hat{\mu}_{\mathcal{X}}(M)$ is described in [5,6].

<u>Main result.</u> Following [7], one can easily show that system (1) is asymptotically stable for all $|\delta_i| \leq \delta$ if, and only if,

$$\delta < \left(\sup_{\omega \geq 0} \mu_{\mathcal{X}}(H_1(j\omega)) \right)^{-1}$$

where $H_1(s)$ is the transfer matrix defined by

$$H_1(s) = \begin{bmatrix} I \\ I \\ \vdots \\ I \end{bmatrix} (sI - A_0)^{-1} \left[A_1 | A_2 | \cdots | A_m \right]$$

and where

$$\mathcal{X} = \{\text{block diag } (\delta_1 I_n, \ldots, \delta_m I_n) : \delta_q \in \mathbb{R}\} .$$

However, whenever some A_i's are not of full rank, one can obtain a necessary and sufficient condition for robust stability involving a transfer matrix of lower size than H_1. To see this, decompose each A_i as

$$A_i = b_i c_i^T$$

where $b_i, c_i \in \mathbb{R}^{n \times r_i}$, with r_i the rank of A_i, and define

$$B = [\, b_1 | \cdots | b_m \,]$$

and

$$C = \begin{bmatrix} c_1^T \\ \vdots \\ c_m^T \end{bmatrix} .$$

The following is then easily proven using [7].

Proposition 1. The system in (1) is asymptotically stable if, and only if,

$$\delta < \left(\sup_{\omega \geq 0} \mu_{\mathcal{X}}(H_2(j\omega)) \right)^{-1}$$

where $H_2(s)$ is the transfer matrix defined by

$$H_2(s) = C(sI - A)^{-1} B$$

and where

$$\mathcal{X} = \{\text{block diag } (\delta_1 I_{r_1}, \ldots, \delta_m I_{r_m}) : \delta_q \in \mathbb{R}\} .$$

□

Substituting for $\mu_{\mathcal{X}}$ its upper bound $\hat{\mu}_{\mathcal{X}}$ gives the following sufficient condition

Corollary 1. The system in (1) is asymptotically stable if

$$\delta < \left(\sup_{\omega \geq 0} \hat{\mu}_{\mathcal{X}}(H_2(j\omega)) \right)^{-1}$$

where $H_2(s)$ and $\mathcal{X}$ are defined as in Proposition 1. □

Models of the type (1) for which the A_i's have low rank are of practical importance. The case where all A_i's have rank one corresponds to the model used by Qiu and Davison,

$$\dot{x} = (A + B\Delta C)x \tag{2}$$

where Δ is uncertain. Yedavalli considered the model

$$\dot{x} = (A + \Delta)x \ ,$$

with Δ uncertain, which is clearly a special case of (2).

Numerical examples. We conclude this note by comparing on two examples the results obtained using Corollary 1 above to those obtained using the techniques of [3] and [4]. The examples are borrowed from [3,4,8].

Example 1. The following matrices were considered in [3,4].

$$A + B\Delta C = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -3+\delta_1 & -2+\delta_2 \\ 1 & 0 \end{bmatrix} ,$$

$$A + B\Delta C = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -3+\delta_1 & -2 \\ 1+\delta_2 & 0 \end{bmatrix} ,$$

$$A + B\Delta C = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -2+\delta_2 \\ 1+\delta_1 & 0 \end{bmatrix} ,$$

$$A + B\Delta C = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \delta_1 & 0 & 0 \\ 0 & \delta_2 & 0 \\ 0 & 0 & \delta_3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -3+\delta_1 & -2+\delta_3 \\ 1+\delta_2 & 0 \end{bmatrix} .$$

Bounds of δ given in [4] which guarantees robust stability were 1.0, 0.48, 0.5 and 0.317, respectively. Bounds given in [3] were 1.5201, 0.9150, 0.8108 and 0.6848, respectively. Our bounds are 2,1,1 and 1, respectively. In these cases, our bounds are also exact.

Example 2. Consider the following system [3,8].

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} 7 & 8 \\ 12 & 14 \end{bmatrix} u, \quad y = \begin{bmatrix} 7 & -8 \\ -6 & 7 \end{bmatrix} x$$

with output feedback control

$$u = \begin{bmatrix} -k_1 & 0 \\ 0 & -k_2 \end{bmatrix} y \ .$$

The controller gains are subject to uncertainty such that $|\Delta k_1| \leq \delta/2$ and $|\Delta k_2| \leq \delta$. The nominal value of controller gains are $k_1 = k_2 = 1$. Corollary 1 implies that the closed loop system is stable if

$$\delta < \hat{\delta} := \left(\sup_{\omega \geq 0} \hat{\mu}_{\mathcal{X}}(H(j\omega)) \right)^{-1} \tag{3}$$

where $H(s) = C(sI - A)^{-1}B$, with

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} + \begin{bmatrix} 7 & 8 \\ 12 & 14 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 7 & -8 \\ -6 & 7 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix}$$

$$B = \begin{bmatrix} 3.5 & 8 \\ 6 & 14 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 7 & -8 \\ -6 & 7 \end{bmatrix} .$$

Solution of the optimization problem in (3) yields $\hat{\delta} = 0.0816$. This result agrees with that in [3]. It turns out to be an exact bound.

Acknowledgements. This research was supported in part by the National Science Foundation's Engineering Research Centers Program: NSF CDR 8803012 and by the National Science Foundation under Grant DMC-84-51515.

Authors' Addresses. Michael K.H. Fan is with the Systems Research Center, University of Maryland, College Park, MD 20742. John C. Doyle is with the Electrical Engineering Department, California Institute of Technology, Pasadena, CA 91125. André L. Tits is with the Systems Research Center and the Electrical Engineering Department, University of Maryland, College Park, MD 20742.

References.

[1] J.C. Doyle, "Analysis of Feedback Systems with Structured Uncertainties," *Proc. IEE-D* 129 (1982), 242–250.

[2] M.K.H. Fan, A.L. Tits & J.C. Doyle, "Robustness in the Presence of Joint Parametric Uncertainty and Unmodeled Dynamics," *Proc. of the 1988 American Control Conference*, Atlanta, Georgia (June 1988).

[3] L. Qiu & E.J. Davison, "New Pertubation Bounds for the Robust Stability of Linear State Space Models," *Proceedings of the 25th Conference on Decision and Control*, Athens, Greece (December 1986).

[4] R.K. Yedavalli, "Perturbation Bounds for Robust Stability in Linear State Space Models," *Internat. J. Control* 42 (1985), 1507–1517.

[5] M.K.H. Fan, "On Computing the Structured Singular Value of a Square Transfer Matrix," *(in preparation)* (1988).

[6] M.K.H. Fan, "User's Guide for MUSOL2: A Package for Computing the Structured Singular Value or Its Upper Bound," Systems Research Center, University of Maryland, Technical Research Report, SRC TR 87-204, 1988.

[7] J.C. Doyle, J.E. Wall & G. Stein, "Performance and Robustness Analysis for Structured Uncertainty," *Proc. 21st IEEE Conf. on Decision and Control*, Orlando, Florida (December 1982).

[8] W.H. Lee, "Robust Analysis for State Space Models," Alphatech Inc., Report TP-151, 1982.