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Recent Advances in the Use of the Internal Model Control Structure for the Synthesis of Robust Multivariable Controllers

by

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Abstract.

This paper presents the following recent theoretical developments in the IMC methodology:

- Multivariable controller design for the minimization of the Integral Squared error (ISE) for every input direction in a set and their linear combinations.
- Treatment of open-loop unstable plants; use of the two-degree-of- freedom controller.
- Minimization of the Structured Singular Value (SSV) for robust performance over the IMC Filter parameters; unconstrained problem; analytic computation of the gradients.
- Computation of the worst (over all possible plants) ISE for a particular setpoint or disturbance input.

The paper deals with comtinuous systems. Extension to sampled-data systems is straightforward but not included here for lack of space.

† This paper is based on the PhD research of the author. The work was carried out at the California Institute of Technology with Prof. M. Morari as PhD advisor.

1. Preliminaries

1.1. Internal Model Control

The Internal Model Control (IMC) structure, introduced by Garcia and Morari (1982), has been widely recognized as very useful in clarifying the issues related to the mismatch between the model used for controller design and the actual process. The IMC structure (Fig.1a), is mathematically equivalent to the classical feedback structure (Fig.1b). The IMC controller Q and the feedback C are related through

$$Q = C(I + \tilde{P}C)^{-1} \tag{1.1.1}$$

$$C = Q(I - \tilde{P}Q)^{-1} \tag{1.1.2}$$

where $\tilde{P}$ is the process model.

$\underline{P = \tilde{P}.}$

In this case the overall transfer function connecting the set-points r and disturbances d to the errors $e = y - r$, where y are the process outputs, is

$$e = y - r = (I - PQ)(d - r) \stackrel{\text{def}}{=} \tilde{E} \quad (d - r) \tag{1.1.3}$$

Hence the IMC stucture becomes effectively open-loop (Fig.2a) and the design of Q is simplified. Note that the IMC controller is identical to the parameter of the Q-parametrization (Zames, 1981). Also the addition of a diagonal filter F by writing

$$Q = \tilde{Q}F \tag{1.1.4}$$

introduces parameters (the filter time constants) which can be used for adjusting on-line the speed of response for each process output.

$\underline{P \neq \tilde{P}.}$

The model-plant mismatch generates a feedback signal in the IMC stucture which can cause performance deterioration or even instability. Since the relative modeling error is larger at higher frequencies, the addition of the low-pass filter F (Fig.2b) adds robustness characteristics into the control system. In this case the closed-loop transfer function is

$$e = y - r = (I - P\tilde{Q}F)(I - (P - \tilde{P})\tilde{Q}F)^{-1}(d - r) \stackrel{\text{def}}{=} E \quad (d - r) \tag{1.1.5}$$

Hence the IMC structure gives rise rather naturally to a two step design procedure:

Step 1: Design $\tilde{Q}$, assuming $P = \tilde{P}$.

Step 2: Design F so that the closed-loop characteristics that $\tilde{Q}$ produces in Step 1, are preserved in the presence of model-plant mismatch ($P \neq \tilde{P}$).

1.2. Internal Stability

A linear time invariant control system is internally stable if the transfer functions between any two points of the control system are stable. A more detailed discussion of the concept of internal stability can be found in the literature (e.g Morari et al., 1987).

Examination of the feedback structure of Fig. 1b results in the requirement that all elements in the matrix $IS1$ in (1.2.1) are stable.

$$IS1 = (\, C(I+PC)^{-1} \quad PC(I+PC)^{-1} \quad CP(I+CP)^{-1} \quad (I+PC)^{-1}P\,) \tag{1.2.1}$$

For the remainder of this section we shall assume that $P = \tilde{P}$. The additional requirements to take care of modeling error are discussed in section 3.3. Use of (1.1.1) or (1.1.2) in (1.2.1) yields

$$IS1 = (\, Q \quad PQ \quad QP \quad (I-PQ)P\,) \tag{1.2.2}$$

Note that stability of each element in (1.2.2) implies internal stability when the control system is implemented as the feedback structure in Fig. 1b, where C is obtained from the Q used in (1.2.2) through (1.1.2).

In order for the control system to be stable when implemented in the IMC stucture of Fig.1a, internal stability arguments (Morari et al.,1987) lead to the requirement that all elements of IS2 are stable.

$$IS2 = (\, Q \quad PQ \quad QP \quad (I-PQ)P \quad PQP \quad P\,) \tag{1.2.3}$$

Hence if the process P is open-loop unstable, $IS2$ will also be unstable and the control system has to be imlemented in the feedback stucture of Fig.1b. Still, the two step IMC design procedure can be used for the design of Q, as described in the following sections. C can then be obtained from (1.1.2) and the structure in Fig.1b implemented.

Note that when the process is open-loop stable, it follows from (1.2.2) that the only requirement for internal stability is that Q is stable.

2. Step 1: Design of $\tilde{Q}$

Throughout this section the assumption is made that $P = \tilde{P}$.

2.1. Objective

The performance objective adopted in this paper is to minimize the Integral Squared Error (ISE) for the error signal e given by (1.1.3). This is an H_2−type objective. Other objectives like an H_∞−type can be used (Zafiriou and Morari, 1986) but they will not be discussed here.

For a specified external system input v ($v = d$ for $r = 0$; $v = -r$ for $d = 0$), the ISE is given by the square of the L_2−norm of e:

$$\Phi(v) \stackrel{\text{def}}{=} ||e||_2^2 = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^*(i\omega)e(i\omega) \quad d\omega \tag{2.1.1}$$

From (1.1.3) we get

$$\Phi(v) = ||e||_2^2 = ||\tilde{E}v||_2^2 = ||(I - P\tilde{Q})v||_2^2 \tag{2.1.2}$$

Hence one objective could be

$$\min_{\tilde{Q}} \Phi(v) \tag{2.1.3}$$

for a particular input $v = (v_1 \quad v_2 \quad \dots \quad v_n)^T$, where $\tilde{Q}$ satisfies the internal stability requirements of section 1.2.

Minimizing the ISE just for one vector v however is not very meaningful, because of the different directions in which the disturbances enter the process or the setpoints are changed. What is desirable is to find a $\tilde{Q}$, that minimizes $\Phi(v)$ for every single v in a set of external inputs v of interest for the particular process. This set can be defined as

$$\mathcal{V} = \{v(s) | v(s) = diag(v_1(s), ..., v_n(s))x, \quad x \in \mathbf{R}^n\} \tag{2.1.4}$$

where $v_1(s), ..., v_n(s)$, describe the frequency content of the external system inputs, e.g. steps, ramps or other types of inputs.

The objective can then be written as

$$\min_{\tilde{Q}} \Phi(v) \qquad \forall v \in \mathcal{V} \tag{2.1.5}$$

under the constraint that $\tilde{Q}$ satisfies the internal stability requirements. It should be noted however that a linear time invariant $\tilde{Q}$ that solves (2.1.5) does not necessarily exist. In section 2.3, it will be shown that this is the case for some $\mathcal{V}$'s.

2.2. Parametrization of all stabilizing $\tilde{Q}$'s.

The process P can in general be open-loop unstable. The following assumption simplifies the solution of the optimization problem:

Assumption A.1. If π is a pole of the model $\tilde{P}$ in the open RHP, then:

a. Its order is equal to 1.

b. $\tilde{P}$ has no zeros at π.

c. The residual matrix corresponding to π is full rank.

Assumption A.1.a, is made to simplify the notation and it is the usual case. The results can be extended to higher order poles. A.1.b is always true for SISO systems. MIMO systems however can have zeros at the location of a pole (Kailath, 1980). This requires an exact cancellation in $det[\tilde{P}(s)]$ and therefore the assumption that this does not happen is not restrictive because such a cancellation will usually not happen anymore when a slight perturbation in the coefficients of $\tilde{P}$ is introduced. A.1.c is also always true for SISO systems, but it can be quite restrictive for MIMO systems. Instead of A.1.c however, an additional assumption can be made on the input for which the optimal controller is designed. This is discussed in section 2.3.

Assumption A.1 is not made for poles at the origin because more than one such poles may appear in an element of $\tilde{P}$, introduced by capacitances that are present in the process. The following assumption true for all practical process control problems is made:

Assumption A.2. Any poles of $\tilde{P}$ or P on the imaginary axis are at $s = 0$. Also $\tilde{P}$ has no finite zeros on the imaginary axis.

Let $\pi_1, ..., \pi_q$ be the poles of $\tilde{P}$ in the open RHP. Define the allpass

$$b_p(s) = \prod_{i=1}^{q} \frac{-s + \pi_i}{s + \pi_i^*} \tag{2.2.1}$$

where the superscript * denotes complex conjugate (and transpose when applied to a matrix).

If A.1.c does not hold then define

$$b_p(s) = 1 \tag{2.2.1'}$$

The following Theorem holds:

Theorem 2.2.1.

Assume that $Q_0(s)$ satisfies the internal stability requirements of section 1.2, i.e. it produces a matrix $IS1$ with stable elements. Then all Q's that make $IS1$ stable are given by

$$Q(s) = Q_0(s) + b_p(s)^2 Q_1(s) \tag{2.2.2}$$

where Q_1 is any stable transfer matrix such that

i) If A.1.c holds, then PQ_1P has no poles at $s = 0$.

ii) If A.1.c does not hold, then PQ_1P has no poles in the closed RHP.

Proof: See Appendix A.1.

2.3. Solution to (2.1.3)

This is the first step towards obtaining a solution to (2.1.5), if such a solution exists. In this section we only consider one particular input v. The plant P can be factored into an allpass portion P_A and a minimum phase (MP) portion P_M such that

$$P = P_A P_M \tag{2.3.1}$$

Hence P_A is stable and such that $P_A^*(i\omega)P_A(i\omega) = I$. Also P_M^{-1} is stable. This inner-outer coprime factorization can be accomplished through the spectral factorization of $P(-s)^T P(s)$, where 'T' denotes transpose. Details on these problems can be found in the literature (Anderson, 1967; Chu, 1985; Doyle et al., 1984).

Let $v_0(s)$ be the scalar allpass that includes the common RHP zeros of the elements of v. Factor v as follows:

$$v(s) = v_0(s) \begin{pmatrix} \hat{v}_1(s) & \dots & \hat{v}_n(s) \end{pmatrix}^T \stackrel{\text{def}}{=} v_0(s)\hat{v}(s) \tag{2.3.2}$$

Without loss of generality make the following assumption for the input v for which $\tilde{Q}$ is designed:

Assumption A.3.

a. The poles of each nonzero element of v (or $\hat{v}$) in the open RHP (if any) are the first q' poles π_i of the plant in the open RHP.

b. If A.1.c does not hold, then every nonzero element of v (or $\hat{v}$) includes all the open RHP poles of $\tilde{P}$ each with degree 1.

To simplify the arguments in the paper, we shall assume that if A.3.b is satisfied, then A.1.c is not. In this way the proper choices in the definitions and the proofs will be made on the basis of A.1.c. If both A.1.c and A.3.b hold, then the results that apply to the case where A.1.c does not hold but A.3.b does, are still correct.

Define

$$b_v(s) = \prod_{i=1}^{q'} \frac{-s+\pi_i}{s+\pi_i^*} \tag{2.3.3}$$

If A.1.c does not hold define

$$b_v(s) = 1 \tag{2.3.3$'$}$$

An different assumption is made for the poles of v at $s = 0$:

Assumption A.4. Let l_i be the maximum number of poles at $s = 0$ that an element of the i^{th} row of P has. Then $v_i(s)$ has at least l_i poles at $s = 0$. Also v has no other poles or any zeros on the imaginary axis.

The above assumptions are not restrictive in the case where v is a output disturbance d, because in a practical situation we want to design for an output disturbance produced by a disturbance that has passed through the process and therefore includes the unstable process poles (e.g., an output disturbance produced by a disturbance on the manipulated variables). Note that the control system will still reject with no steady-state offset, other disturbances with fewer unstable poles. The assumption is different for poles at $s = 0$, because their number in each row of $\tilde{P}$ can be different, since capacitances may be associated with only certain process outputs. Also the output disturbance may have more poles at $s = 0$ than the process (e.g., a persistent disturbance in the manipulated variables).

The assumptions might be restrictive in the case of setpoints though. However for setpoint tracking the use of the Two-Degree-of-Freedom structure, which will be discussed briefly in section 2.5, allows us to disregard the existence of any unstable poles of P and therefore this assumption need not be made for setpoints.

The following theorem holds:

Theorem 2.3.1.

The set of controllers $\tilde{Q}$ that solve (2.1.3) satisfy

$$\tilde{Q}\hat{v} = b_p b_v^{-1} P_M^{-1} \{b_p^{-1} b_v P_A^{-1} \hat{v}\}_* \tag{2.3.4}$$

where the operator $\{.\}_*$ denotes that after a partial fraction expancion of the operand all terms involving the poles of P_A^{-1} are omitted. Furthermore, for $n \geq 2$ the number of stabilizing controllers that satisfy (2.3.4) is infinite. Guidelines for the construction of such a controller are given in the proof.

Proof: See Appendix A.2

2.4. Solution to (2.1.5)

Write

$$V(s) \stackrel{\text{def}}{=} diag(v_1(s), ..., v_n(s)) \tag{2.4.1}$$

$$\hat{V}(s) \stackrel{\text{def}}{=} diag(\hat{v}_1(s), ..., \hat{v}_n(s)) \tag{2.4.2}$$

The following Theorem holds:

Theorem 2.4.1.

i) If all the RHP zeros of V appear in every element of V with the same degree, then the controller $\tilde{Q}$ that solves (2.1.5) is given by

$$\tilde{Q} = b_p b_v^{-1} P_M^{-1} \{b_p^{-1} b_v P_A^{-1} \hat{V}\}_* \hat{V}^{-1} \tag{2.4.3}$$

ii) If an element of V has a RHP zero that does not appear in all the other elements with the same degree, then there exists no stabilizing $\tilde{Q}$ that solves (2.1.5), unless P is stable and minimum phase in which case $\tilde{Q} = P^{-1}$.

Proof: See Appendix A.3.

The case descibed by Thm.2.4.1.ii, where no optimal solution exists, is not necessarily rare. Since v can be an output disturbence d, the designer might want to specify it as some common input, e.g. a step, going through some transfer matrix. For such a v, its elements may very well include different RHP zeros. When this happens, a solution to an alternative problem exists. Factor each element v_i of V into a stable allpass part v_{Ai} and a minimum phase v_{Mi}:

$$v_i(s) = v_{Ai}(s) v_{Mi}(s) \tag{2.4.4}$$

The following theorem holds.

Theorem 2.4.2.

The controller

$$\tilde{Q} = b_p b_v^{-1} P_M^{-1} \{b_p^{-1} b_v P_A^{-1} V_M\}_* V_M^{-1} \tag{2.4.5}$$

minimizes $\Phi(v)$ for the following n directions x:

$$x = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad , \quad \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} \quad , \quad \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \tag{2.4.6}$$

and their multiples, as well as for the linear combinations of those directions that correspond to elements of V with the same RHP zeros in the same degree.

Proof: See Appendix A.4.

2.5. Two-Degree-of-Freedom Structure

From the discussion of the Internal Stability requirements in section 1.2, it follows that RHP poles in the plant limit the possible choices of Q and thus the achievable performance. This however need not be so for setpoint tracking. Consider the general feedback structure of Fig.5. For the disturbance behavior it is irrelevant if the controller is implemented as one block C as in Fig. 1b, or as two blocks as in Fig. 5. Hence the achievable disturbance rejection is restricted both by the RHP zeros and poles of P as the quantitative results of the previous sections indicate.

Let us now proceed from the point where a stabilizing $\tilde{Q}$ and the corresponding C have been found through the results of the previous sections, which produce a satisfactory disturbance response. We can then split C into two blocks C_1 and C_2 such that C_1 is minimum phase and C_2 is stable. Then one can easily see that the only RHP zeros of the stabilized system $PC_1(I + PC_1C_2)^{-1}$ are those of the process P. Thus C_3 can be designed without regard for the RHP poles of P and the achievable setpoint tracking is restricted by the RHP zeros of P only.

In summary, the achievable disturbance response of a system is restricted by the presence of the plant RHP zeros and poles regardless of how complicated a controller is used. If the Two-Degree-of-Freedom controller shown in Fig.5 is employed, the achievable setpoint response is restricted by the RHP zeros only. A more rigorous discussion can be found in Vidyasagar (1985).

3. Model Uncertainty

3.1. Structured Singular Value

Potential modeling errors, described as uncertainty associated with the process model, can appear in different forms and places in a multivariable model. This fact makes the derivation of non-conservative conditions that guarantee robustness with respect to model-plant mismatch difficult. The Structured Singular Value (SSV), introduced by Doyle (1982), takes into account the structure of the model uncertainty and it allows the non-conservative quantification of the concept of robust performance.

For a constant complex matrix M the definition of the SSV $\mu_{\boldsymbol{\Delta}}(M)$ depends also on a certain set $\boldsymbol{\Delta}$. Each element Δ of $\boldsymbol{\Delta}$ is a block diagonal complex matrix with a specified dimension for each block, i.e.

$$\mathbf{\Delta} = \left\{diag(\Delta_1, \Delta_2, ..., \Delta_n) | \Delta_j \in \mathbf{C}^{m_j \times m_j}\right\} \tag{3.1.1}$$

Then

$$\frac{1}{\mu_{\mathbf{\Delta}}(M)} = \min_{\Delta \in \mathbf{\Delta}} \{\bar{\sigma}(\Delta) | det(I - M\Delta) = 0\} \tag{3.1.2}$$

and $\mu_{\mathbf{\Delta}}(M) = 0$ if $det(I - M\Delta) \neq 0 \quad \forall \Delta \in \mathbf{\Delta}$. Note that $\bar{\sigma}$ is the maximum singular value of the corresponding matrix.

Details on how the SSV can be used for studying the robustness of a control system can be found in Doyle (1985), where a discussion of the computational problems is also given. For three or fewer blocks in each element of $\mathbf{\Delta}$, the SSV can be computed from

$$\mu_{\mathbf{\Delta}}(M) = \inf_{D \in \mathbf{D}} \bar{\sigma}(DMD^{-1}) \tag{3.1.3}$$

where

$$\mathbf{D} = \{diag(d_1 I_{m_1}, d_2 I_{m_2}, ..., d_n I_{m_n}) | d_j \in \mathbf{R}_+\} \tag{3.1.4}$$

and I_{m_j} is the identity matrix of dimension $m_j \times m_j$. For more than three blocks, (3.1.3) still gives an upper bound for the SSV.

3.2. Block Structure

In order to effectively use the SSV for designing F, some rearrangement of the block structure is necessary. The IMC structure of Fig.1a can be written as that of Fig.3a, where $v = d - r$, $e = y - r$ and

$$G = \begin{pmatrix} 0 & 0 & \tilde{Q} \\ I & I & \tilde{P}\tilde{Q} \\ -I & -I & 0 \end{pmatrix} \tag{3.2.1}$$

where the blocks 0 and I have appropriate dimensions.

The structure in Fig.3a can always be transformed into that in Fig.3b, where Δ is a block diagonal matrix with the additional property that

$$\bar{\sigma}(\Delta) \leq 1 \qquad \forall \omega \tag{3.2.2}$$

The superscript u in G^u denotes the dependance of G^u not only on G but also on the specific uncertainty description available for the model $\tilde{P}$. Only some of the more common types will be covered here to demonstrate how this is done, but it is straightforward to apply the same concepts to other types of uncertainty descriptions, like parametric uncertainty.

i) Multivariable Additive Uncertainty.

The information on the model uncertainty is of the form

$$\bar{\sigma}(P - \tilde{P}) \le l_a(\omega) \tag{3.2.3}$$

where l_a is a known function of frequency. In this case we can easily write $P - \tilde{P} = l_a\Delta$ where $\bar{\sigma}(\Delta) \le 1$ and so obtain

$$G^u = G^a = \begin{pmatrix} l_a I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} G \tag{3.2.4}$$

ii) Multivariable Input Multiplicative Uncertainty.

$$\bar{\sigma}(\tilde{P}^{-1}(P - \tilde{P})) \le l_i(\omega) \tag{3.2.5}$$

where l_i is known. Then

$$G^u = G^i = G \begin{pmatrix} l_i\tilde{P} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \tag{3.2.6}$$

iii) Multivariable Output Multiplicative Uncertainty.

$$\bar{\sigma}((P - \tilde{P})\tilde{P}^{-1}) \le l_o(\omega) \tag{3.2.7}$$

$$G^u = G^o = \begin{pmatrix} l_o\tilde{P} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} G \tag{3.2.8}$$

iv) Element by Element Additive Uncertainty.

For each element p_{ij} of P we have

$$|p_{ij} - \tilde{p}_{ij}| \le l_{ij}(\omega), \qquad i = 1, ..., n; \quad j = 1, ..., n \tag{3.2.9}$$

Then

$$P - \tilde{P} = J_1 \Delta L J_2 \tag{3.2.10}$$

where

$$L = diag(l_{11}, l_{12}, ..., l_{1n}, l_{21}, ..., l_{nn}) \tag{3.2.11}$$

$$J_1 = \begin{pmatrix} 1 & \dots & 1 & 0 & \dots & 0 & \dots & \dots & 0 & \dots & 0 \\ 0 & \dots & 0 & 1 & \dots & 1 & \dots & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 & \dots & \dots & 1 & \dots & 1 \end{pmatrix} \tag{3.2.12}$$

$$J_2 = \begin{pmatrix} I_n \\ I_n \\ \vdots \\ I_n \end{pmatrix} \tag{3.2.13}$$

From (3.2.10) it follows that

$$G^u = G^{ebe} = \begin{pmatrix} LJ_2 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} G \begin{pmatrix} J_1 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \tag{3.2.14}$$

Note that all the above relations yield a G^u already partitioned as

$$G^u = \begin{pmatrix} G^u_{11} & G^u_{12} & G^u_{13} \\ G^u_{21} & G^u_{22} & G^u_{23} \\ G^u_{31} & G^u_{32} & G^u_{33} \end{pmatrix} \tag{3.2.15}$$

Then Fig.3b can be written as Fig.4 with

$$\begin{aligned} G^F &= \begin{pmatrix} G^u_{11} & G^u_{12} \\ G^u_{21} & G^u_{22} \end{pmatrix} + \begin{pmatrix} G^u_{13} \\ G^u_{23} \end{pmatrix} (I - FG^u_{33})^{-1} F \begin{pmatrix} G^u_{31} & G^u_{32} \end{pmatrix} \\ &\stackrel{\text{def}}{=} \begin{pmatrix} G^F_{11} & G^F_{12} \\ G^F_{21} & G^F_{22} \end{pmatrix} \end{aligned} \tag{3.2.16}$$

3.3. Robust Stability

We now require that the matrix $IS1$ as given by (1.2.1) is stable for all possible plants P. The design of $\tilde{Q}$ according to section 2 resulted in a stable $IS1$ for $P = \tilde{P}$. In order for $IS1$ to remain stable we need to satisfy the requirements that as we move in a "continuous" way from the model $\tilde{P}$ to the plant P, no closed-loop RHP poles cross the imaginary axis and no such poles suddenly appear in the RHP. The latter requirement is satisfied if we assume that the model and the plant have the same number of RHP poles. The SSV can be used to determine if any crossings of the imaginary axis occur. Then we can say that the system is stable for any of the plants in the set defined from the bounds on the model uncertainty and which have the same number of RHP poles as the model, if and only if (Doyle, 1985)

$$\mu_\Delta(G^F_{11}) < 1 \qquad \forall \omega \tag{3.3.1}$$

3.4. Robust Performance

In the first step of the IMC design procedure a controller $\tilde{Q}$ is obtained, which produces satisfactory disturbance rejection and/or setpoint tracking. This response is described by the "sensitivity" function $\tilde{E}$ given by (1.1.3). Since $\tilde{E}$ connects the external inputs to the error e, a well-designed control system produces a relatively "small" $\tilde{E}$. A measure of the magnitude of the known $\tilde{E}$ is its maximum singular value. Let $b(\omega)$ be a frequency function such that

$$\bar{\sigma}(\tilde{E}(i\omega)) < b(\omega) \qquad \forall\omega \tag{3.4.1}$$

When $P \neq \tilde{P}$, the sensitivity function E is described by (1.1.5). Note that $E = \tilde{E}$ when $P = \tilde{P}$. In order for the performance of the control system to remain robust with respect to model-plant mismatch we have to keep E small in spite of the modeling error. Hence we require that

$$\sup_{\omega} \bar{\sigma}(b(\omega)^{-1}E(i\omega)) < 1 \qquad \forall\Delta \in \mathbf{\Delta} \tag{3.4.2}$$

We can now use the properties of the SSV (Doyle,1985) to obtain

$$\sup_{\omega} \bar{\sigma}(b(\omega)^{-1}E(i\omega)) < 1 \qquad \forall\Delta \in \mathbf{\Delta} \iff \sup_{\omega} \mu_{\mathbf{\Delta}^0}(G^b) < 1 \tag{3.4.3}$$

where

$$G^b = \begin{pmatrix} I & 0 \\ 0 & b^{-1} \end{pmatrix} G^F \tag{3.4.4}$$

$$\mathbf{\Delta}^0 = \left\{ diag(\Delta, \Delta^0) | \Delta \in \mathbf{\Delta}, \Delta^0 \in \mathbf{C}^{n\times n} \right\} \tag{3.4.5}$$

The worst possible ISE that any plant within the uncertainty bounds can produce for a particular input v is given by the following theorem.

Theorem 3.4.1:

For a specified v define

$$G^x \stackrel{\text{def}}{=} \begin{pmatrix} I & 0 \\ 0 & x \end{pmatrix} G^F \begin{pmatrix} I & 0 \\ 0 & v \end{pmatrix} \tag{3.4.6}$$

where x is a scalar function of ω and the blocks 0 have the appropriate dimensions (in general non-square). Augment G^x, which is in general a "tall" matrix, to obtain a square matrix:

$$G^x_{full} = \begin{pmatrix} G^x & 0 \end{pmatrix} \tag{3.4.7}$$

Then

$$\mu_{\Delta^o}(G^x_{full}(i\omega)) = 1 \iff x(\omega) = x_0(\omega) \qquad \forall\omega \tag{3.4.8}$$

defines a function x_0 of frequency and

$$\sup_{\Delta\in\boldsymbol{\Delta}} ||Ev||_2 = ||x_0^{-1}||_2 \tag{3.4.9}$$

Proof: See Appendix B.1.

Note that as it turned out $x_0^{-1} = \sup_{\Delta\in\boldsymbol{\Delta}} \bar\sigma(Ev)$, but the only way to compute it is through (3.4.8). Also without loss of generality x can be assumed to be positive since the value of $\mu_{\Delta^o}(G^x_{full})$ depends only on $|x|$. The following theorem simplifies the problem of computing x_0.

Theorem 3.4.2:

Let

$$M^x = \begin{pmatrix} M_{11} & M_{12} \\ xM_{21} & xM_{22} \end{pmatrix} \tag{3.4.10}$$

where x a positive scalar.

Then $\inf_{D\in\mathbf{D}} \bar\sigma(DM^xD^{-1})$ is a non-decreasing function of x, where $\mathbf{D} = \{diag(D_1, D_2)\}$.

Proof: See Appendix B.2.

Note that G^x_{full} is a special case of M in the Theorem and so Theorem 3.4.2 applies to (3.4.11).

4. Step 2: Design of F

4.1. Filter Structure

The filter parameters can now be computed so that the robustness conditions that were discussed in section 3 are satisfied. To do so, some structure will have to be assumed for F, which can be of any general type that the designer wishes. However in order to keep the number of variables in the optimization problem small, a rather simple structure like a diagonal F with first or second order terms would be recommended. In most cases this is not restrictive because the potentially higher orders of the model $\tilde P$ have been included in the controller $\tilde Q$ that was designed in the first step of the IMC procedure and which is in general a full matrix. Some additional restrictions on the filter exist in the case of an open-loop unstable plant. Also the use of more complex filter structure may be necessary in cases of highly ill-conditioned systems ($\bar\sigma(\tilde P)/\underline{\sigma}(\tilde P)$ very large).

i) Open-loop unstable plants.

The IMC filter $F(s)$ is chosen to be a diagonal rational function that satisfies the following requirements.

a. Pole-zero excess. The controller $Q = \tilde{Q}F$ must be proper. Assume that the designer has specified a pole-zero excess of m for the filter $F(s)$.

b. Internal stability. $IS1$ in (1.2.2) must be stable.

c. Asymptotic tracking of disturbances. $(I - \tilde{P}\tilde{Q}F)v$ must be stable.

Write

$$F(s) = diag(f_1(s), ..., f_n(s)) \tag{4.1.1}$$

Under assumptions A.1,2,3,4, (b),(c) are equivalent to the following conditions. Let π_i $(i = 1, q)$ be an open RHP pole of $\tilde{P}$ (with order 1 according to A.1.a) and $\pi_0 = 0$ and l_{vk} the multiplicity of such a pole in the k^{th} element of V. Then the k^{th} element, f_k of the filter F must satisfy:

$$f_k(\pi_i) = 1, \qquad i = 0, 1, ..., q \tag{4.1.2}$$

$$\frac{d^j}{ds^j} f_k(s)|_{s=\pi_0} = 0, \qquad j = 1, ..., l_{vk} - 1 \tag{4.1.3}$$

(4.1.2) clearly shows the limitation that RHP poles place on the robustness properties of a control system designed for an open-loop unstable plant. Since because of (4.1.2) one cannot reduce the nominal $(P = \tilde{P})$ closed-loop bandwidth of the system at frequencies corresponding to the RHP poles of the plant, one can only tolerate a relatively small model error at those frequencies.

One can write for a filter element $f_k(s)$:

$$f_k(s) = \frac{a_{n_k-1,k}s^{n_k-1} + ... + a_{1,k}s + a_{0,k}}{(\lambda s + 1)^{m+n_k-1}} \tag{4.1.4}$$

where

$$n_k = l_{vk} + q \tag{4.1.5}$$

and then compute the numerator coefficients for a specific tuning parameter λ from (4.1.2), (4.1.3).

In the simple case where $l_{vk} = 1$, one can develop an explicit formula for a filter element $f(s)$:

$$f(s) = \frac{1}{(\lambda s + 1)^{m+q}} \sum_{j=0}^{q} (\lambda \pi_j + 1)^{m+q} \prod_{i=0, i \neq j}^{q} \frac{s - \pi_i}{\pi_j - \pi_i} \tag{4.1.6}$$

Example 4.1.1. Assume that we have a pole-zero excess of m and there is only one pole π. Then from (4.1.6)

$$f(s) = \frac{(\lambda\pi + 1)^m}{(\lambda s + 1)^m} \tag{4.1.7}$$

If $\pi = 0$, (4.1.7) reduces to the standard filter for stable systems $f(s) = (\lambda s + 1)^{-m}$.

Example 4.1.2. Assume that $m = 2$ and the only pole is a double pole at $s = 0$. Then from (4.1.2), (4.1.3) for $q = 0$

$$f(s) = \frac{3\lambda s + 1}{(\lambda s + 1)^3} \tag{4.1.8}$$

ii) Ill-conditioned plants.

The problems arise because the optimal controller $\tilde{Q}$ designed for $\tilde{P}$ tends to be an approximate inverse of $\tilde{P}$ and as a result $\tilde{Q}$ is ill-conditioned as well, which means that a lot of detuning action will be required in a diagonal F to guarantee robust stability. The result is that although stability is maintained, the response is very sluggish and therefore the robust performance condition is very difficult to satisfy. A way to address this problem is to try to use a filter that acts directly on the singular values of $\tilde{Q}$, at the frequency where the condition number of $\tilde{Q}$ is highest, say ω^*. Let

$$\tilde{Q}(i\omega^*) = U_Q \Sigma_Q V_Q^* \tag{4.1.9}$$

be the SVD of $\tilde{Q}$ at ω^* and let R_u, R_v, be real matrices that solve the pseudo-diagonalization problems:

$$U_Q^* R_u \approx I \tag{4.1.10}$$

$$V_Q^* R_v \approx I \tag{4.1.11}$$

Then for the IMC controller Q that includes the filter, use the expression

$$Q(s) = R_u F_1(s) R_u^{-1} \tilde{Q}(s) F_2(s) \tag{4.1.12}$$

or

$$Q(s) = \tilde{Q}(s) R_v F_1(s) R_v^{-1} F_2(s) \tag{4.1.13}$$

where $F_1(s)$, $F_2(s)$ are diagonal filters, such that $F_1(0) = F_2(0) = I$. Nota that when $\tilde{P}$ has poles at $s = 0$, every element of $F_1(s)$ must satisfy (4.1.3) for $j = 1, \dots, l_v$, where

$$l_v = \max_{k=1,\dots,q} l_{vk} \tag{4.1.14}$$

It should be pointed out that the success of this approach depends on how good any of the pseudo-diagonalizations (4.1.10) or (4.1.11) is. The diagonalization will be perfect if U_Q or V_Q is real. This will happen if $\omega^* = 0$, which is the case when the problems arise because the plant is ill-conditioned at steady-state, as for example high purity distillation columns are.

One can put this control structure in the form of Fig.3, as follows. Define

$$F(s) = diag(F_1(s), F_2(s)) \tag{4.1.15}$$

$$\tilde{Q}_A(s) = R_u \qquad or \qquad \tilde{Q}(s)R_v \tag{4.1.16}$$

$$A(s) = R_u^{-1}\tilde{Q}(s) \qquad or \qquad R_v^{-1} \tag{4.1.17}$$

depending on whether (4.1.12) or (4.1.13) is used. Obtain G^u by substituting $\tilde{Q}$ with $\tilde{Q}_A$ in (3.2.1). Then in Fig.3 use instead of G^u, $G^{u,ill}$, where

$$G^{u,ill} = \begin{pmatrix} G^u_{11} & G^u_{12} & G^u_{13} & 0 \\ G^u_{21} & G^u_{22} & G^u_{23} & 0 \\ 0 & 0 & 0 & A \\ G^u_{31} & G^u_{32} & G^u_{33} & 0 \end{pmatrix} \tag{4.1.18}$$

4.2. Objective

We can write

$$F \stackrel{\text{def}}{=} F(s; \Lambda) \tag{4.2.1}$$

where Λ is an array with the filter parameters.

The problem can now be formulated as a minimization problem over the elements of the array Λ. A constraint is that the part of Λ corresponding to denominator time constants should be such that F is a stable transfer function. However the problem can be turned into an unconstrained one by writing the denominator of each element of F as a product of polynomials of degree 2 and one of degree 1 if the order is odd, with the constant terms of the polynomials equal to 1. Then the stability requirement translates into the requirement that the coefficients (elements of Λ) are positive, which is a constraint that can be eliminated by writing λ_k^2 or $|\lambda_k|$ instead of λ_k for the corresponding filter parameters.

Our goal is to satisfy (3.4.3). The filter parameters can be obtained by solving

$$\min_{\Lambda} \sup_{\omega} \mu_{\Delta^o}(G^b) \tag{4.2.2}$$

It may be however that the optimum values for (4.2.2), still do not manage to satisfy (3.4.3). The reason may be that an F with more parameters is required, but more often that the performance requirements set by the selection of $b(\omega)$ in (3.4.1) are too tight to satisfy in the presence of model-plant mismatch. In this case one should choose a less tight bound b and resolve (4.2.2). Note that satisfaction of the Robust Performance condition (3.4.3) implies satisfaction of the Robust Stability condition (3.3.1) as well.

A different objective can be set in the case where the ISE for a particular external input direction v is of special interest to the designer. The objective is then to minimize (3.4.9) for a specified v (set-point or disturbance). Hence the filter parameters are obtained by solving

$$\min_{\Lambda} \|x_0^{-1}\|_2 \tag{4.2.3}$$

It should be pointed that contrary to the problems addressed in section 2, where a minimization for a set of v's could be carried out, (4.2.3) cannot be solved for a set of v's. The reason is the presence of modeling error in the problem definition.

4.3. Computational Issues

i) Solution of (4.2.2).

The computation of μ in (4.2.2) is made through (3.1.3); details can be found in Doyle (1982). As it was pointed out in Doyle (1985), the minimization of the Frobenious norm instead of the maximum singular value yields D's which are very close to the optimal ones for (3.1.3). Note that the minimization of the Frobenious norm is a very simple task. In the computation of the supremum in (4.2.2) only a finite number of frequencies is considered. Hence (4.2.2) is transformed into

$$\min_{\Lambda} \max_{\omega\in\Omega} \inf_{D\in\mathbf{D}^0} \bar{\sigma}(DG^bD^{-1}) \tag{4.3.1}$$

where Ω is a set containing a finite number of frequencies and $\mathbf{D}^0$ is the set corresponding to $\mathbf{\Delta}^0$ according to (3.1.1) and (3.1.4). Define

$$\Phi_\infty(\Lambda) \stackrel{\text{def}}{=} \max_{\omega\in\Omega} \inf_{D\in\mathbf{D}^0} \bar{\sigma}(DG^bD^{-1}) \tag{4.3.2}$$

The analytic computation of the gradient of Φ_∞ with respect to Λ is in general possible. This is not the case when the two or more largest singular values of DG^bD^{-1} are equal. However this is quite uncommon and although the computation of a generalized gradient is possible, experience has shown the use of a mean direction to be satisfactory. A similar problem appears when the $\max_{\omega \in \Omega}$ is attained at more than one frequencies, but again the use of a mean direction seems to be sufficient. We shall now proceed to obtain the expression for the gradient of $\Phi_\infty(\Lambda)$ in the general case.

Assume that for the value of Λ where the gradient of $\Phi_\infty(\Lambda)$ is computed, the $\max_{\omega \in \Omega}$ is attained at $\omega = \omega_0$ and that the $\inf_{D \in \mathbf{D}^o} \bar{\sigma}(DG^b(i\omega_0)D^{-1})$ is obtained at $D = D_0$, where only one singular value σ_1 is equal to $\bar{\sigma}$. Let the singular value decomposition (SVD) be

$$D_0 G^b(i\omega_0) D_0^{-1} = \begin{pmatrix} u_1 & U \end{pmatrix} \begin{pmatrix} \sigma_1 & 0 \\ 0 & \Sigma \end{pmatrix} \begin{pmatrix} v_1^* \\ V^* \end{pmatrix} \tag{4.3.3}$$

Then for the element of the gradient vector corresponding to the filter parameter λ_k we have under the above assumptions:

$$\frac{\partial}{\partial \lambda_k} \Phi_\infty = \frac{\partial}{\partial \lambda_k} \sigma_1 (D_0 G^b(i\omega_0) D_0^{-1}) \tag{4.3.4}$$

because $\nabla_{D_0}(\sigma_1) = 0$ since we are at an optimum with respect to the D's. To simplify the notation use

$$A = D_0 G^b(i\omega_0) D_0^{-1} = U_A \Sigma_A V_A^* \tag{4.3.5}$$

By using the properties of the SVD we obtain from (4.3.3)

$$\begin{aligned}
AA^* = U_A \Sigma_A^2 U_A^* &\Rightarrow u_1^* \frac{\partial}{\partial \lambda_k}(AA^*) u_1 = u_1^* U_A \frac{\partial}{\partial \lambda_k}(\Sigma_A^2) U_A^* u_1 \\
&\Rightarrow u_1^* \left(\frac{\partial}{\partial \lambda_k}(A) A^* + A \frac{\partial}{\partial \lambda_k}(A^*)\right) u_1 = u_1^* U_A \left(2\Sigma_A \frac{\partial}{\partial \lambda_k}(\Sigma_A)\right) U_A^* u_1 \\
&\Rightarrow u_1^* \frac{\partial}{\partial \lambda_k}(A) v_1 \sigma_1 + \sigma_1 v_1^* \frac{\partial}{\partial \lambda_k}(A^*) u_1 = 2\sigma_1 \frac{\partial}{\partial \lambda_k}(\sigma_1) \\
&\Rightarrow \frac{\partial}{\partial \lambda_k}(\sigma_1) = \mathrm{Re}\left[u_1^* \frac{\partial}{\partial \lambda_k}(D_0 G^b(i\omega_0) D_0^{-1}) v_1 \right]
\end{aligned} \tag{4.3.6}$$

Use of (4.3.4), (3.2.16), (3.4.4), (4.3.6), and of the property

$$\frac{d}{dz}(M(z)^{-1}) = -M(z)^{-1} \frac{d}{dz}(M(z)) M(z)^{-1} \tag{4.3.7}$$

where M(z) is a matrix, yields after some algebra

$$\begin{aligned}
\frac{\partial}{\partial \lambda_k} \Phi_\infty = \mathrm{Re}\Bigg[& u_1^* D_0 \begin{pmatrix} G_{13}^u \\ b^{-1} w G_{23}^u \end{pmatrix} (I - F G_{33}^u)^{-1} \frac{\partial}{\partial \lambda_k}(F(i\omega_0)) \\
& (I - F G_{33}^u)^{-1} \begin{pmatrix} G_{31}^u & G_{32}^u \end{pmatrix} D_0^{-1} v_1 \Bigg]
\end{aligned} \tag{4.3.8}$$

where F, G^u_{ij}, b, w are computed at $\omega = \omega_0$. The derivatives of F with respect to its parameters (elements of Λ) depend on the particular form that the designer selected and they can be easily computed.

ii) Solution of (4.2.3).

• The first issue in this case is the computation of x_0. Note that this computation has to be made at every frequency ω. In practice only a set Ω with a finite number of frequencies is used, from which $||x_0^{-1}||_2$ can be computed approximately. Theorem 2 indicates that any basic descent method should be sufficient. The fact that it is possible to obtain an analytic expression for the gradient of $\mu_{\Delta^0}(G^x_{full}(i\omega))$ with respect to x, simplifies the problem even further. This is possible when (1.2.3) is used for the computation of μ and the two largest singular values of $DG^x_{full}D^{-1}$ for the optimal D's at the value of x where the gradient is computed, are not equal to each other. If this not the case a mean direction can be used as mentioned in the H_∞ case above.

Let the $\inf_{D \in \mathbf{D}^0} \bar{\sigma}(DG^x_{full}(i\omega)D^{-1})$ be attained for $D_0 = D_0(\omega; x)$ and let σ_1 be the maximum singular value and u_1, v_1 the corresponding singular vectors. Then the same steps for obtaining (4.3.6) are valid. Hence by using (3.4.6) and (3.4.7) we get after some algebra

$$\frac{\partial}{\partial x}\left(\mu_{\Delta^0}(G^x_{full}(i\omega))\right) = \mathrm{Re}\left[u_1^* D_0 \begin{pmatrix} 0 & 0 & 0 \\ WG^F_{21} & WG^F_{22}v & 0 \end{pmatrix} D_0^{-1} v_1\right] \tag{4.3.9}$$

• The second computational issue is the solution of (4.2.3). To obtain the gradient of $||x_0^{-1}||_2$ with respect to the filter parameters, we need to compute the gradient of $x_0(\omega)$ with respect to these parameters for every frequency $\omega \in \Omega$. From the definition of x_0 in (3.4.8) we see that as some filter parameter λ_k changes, $x_0(\omega)$ will also change so that $\mu_{\Delta^0}(G^x_{full}(i\omega))$ remains constantly equal to 1. Hence we can write

$$\frac{\partial \mu}{\partial x_0} \frac{\partial x_0}{\partial \lambda_k} + \frac{\partial \mu}{\partial \lambda_k} = 0 \Longrightarrow \quad \frac{\partial x_0}{\partial \lambda_k} = -\frac{\partial \mu}{\partial \lambda_k} / \frac{\partial \mu}{x_0} \tag{4.3.10}$$

where μ is computed through (3.1.3). The denominator in the right hand side of (4.3.10) is given from (4.3.9). As for the numerator, it can be computed in the same way as (4.3.6) and (4.3.8) but with G^x_{full} instead of G^b:

$$\frac{\partial}{\partial \lambda_k}\left(\mu_{\Delta^0}(G^x_{full}(i\omega))\right) = \mathrm{Re}\left[u_1^* D_0 \begin{pmatrix} G^u_{13} \\ xWG^u_{23} \end{pmatrix} (I - FG^u_{33})^{-1}\right.$$

$$\frac{\partial}{\partial \lambda_k}(F(i\omega))(I - FG^u_{33})^{-1}\left(G^u_{31} \quad G^u_{32}v \quad 0 \right) D_0^{-1}v_1 \Big] \tag{4.3.11}$$

Hence $\partial x_0/\partial \lambda_k$ can be computed from (4.3.9), (4.3.10), (4.3.11).

Appendix A

A.1. Proof of Thm. 2.2.1.

i) We shall show that any Q given by (2.2.2) makes $IS1$ stable. From substitution of (2.2.2) into (1.2.2) it follows that all that is required is that $\left(Pb_p^2Q_1 \quad b_p^2Q_1P \quad Pb_p^2Q_1P \right)$ be stable, which is true because of Assumptions A.1, A.2 and the properties of Q_1.

ii) Assume that Q makes $IS1$ stable. Then the difference matrix

$$IS1(Q) - IS1(Q_0) = \left((Q - Q_0) \quad P(Q - Q_0) \quad (Q - Q_0)P \quad P(Q - Q_0)P \right) \tag{A.1.1}$$

is stable. The fact that P has no zeros at the location of the unstable poles makes the stability of the matrix in (A.1.1) equivalent to the stability of $(Q - Q_0)$, $P(Q - Q_0)P$. Then, when assumption A.1.c holds, we have $P = b_p\hat{P}$, where $\hat{P}$ has no zeros at the open RHP poles of P and its only only unstable poles are at $s = 0$, from which it follows that $(Q - Q_0) = b_p^2Q_1$ with Q_1 stable and such that PQ_1P has no poles at $s = 0$. If A.1.c does not hold, Q_1 should also have the property that it makes PQ_1P stable.

A.2. Proof of Thm. 2.3.1.

We shall assume that a Q_0 exists, which in addition to the properties mentioned in Thm. 2.1.1, it also produces a matrix $(I - PQ_0)V^0$ with no poles at $s = 0$, where V^0 is a diagonal matrix with l_v poles at $s = 0$ in every element with l_v the maximum number of such poles in any element of v. If assumption A.1.c does not hold, then each column of V^0 also satisfies A.3.b and Q_0 makes $(I - PQ_0)V^0$ stable. Its existence will be proven by finding an optimal solution that has such properties. Substitution of (2.2.2) into (2.1.2) and use of the fact that pre- or postmultiplication of a function with an allpass does not change its L_2-norm, yields:

$$\begin{aligned} \Phi(v) &= \| b_p^{-1}b_vP_A^{-1}(I - PQ_0)\hat{v} - b_pb_vP_MQ_1\hat{v}\|_2^2 \\ &\stackrel{\text{def}}{=} \| f_1 - f_2Q_1\hat{v}\|_2^2 \end{aligned} \tag{A.2.1}$$

L_2, the space of functions square integrable on the imaginary axis, can be decomposed into two subspaces, H_2 the subspace of functions analytic in the RHP (stable functions) and its orthogonal complement $H_2^\perp$ that includes any strictly unstable functions. Then f_1 can be uniquely decomposed into two orthogonal functions $\{f_1\}_- \in H_2$ and $\{f_1\}_+ \in H_2^\perp$:

$$f_1 = \{f_1\}_- + \{f_1\}_+ \tag{A.2.2}$$

From (A.2.1) one can see that if improper Q's are allowed, then f_1 may not be an L_2 function. However, in order for $\Phi(v)$ to be finite, the optimal Q_1 has to make $f_1 - f_2 Q_1 \hat{v}$ strictly proper. The assumption will be made that is the case and it will be verified at the solution has this property. Hence to proceed we shall use the convention that when a decomposition as in (A.2.2) of a function is obtained through a partial fraction expansion, all improper and the constant terms are included in $\{.\}_-$.

When A.1.c holds, inspection of (A.2.1) shows that $f_2 Q_1 \hat{v}$ can have no poles in the closed RHP except possibly for some poles at $s = 0$ introduced by $\hat{v}$. f_1 however has no poles at $s = 0$ because $(I - PQ_0)V^0$ has no such poles. Hence for $\Phi(v)$ to be finite, $f_2 Q_1 \hat{v}$ should have no poles at $s = 0$. Hence the optimal Q_1 has to be such that these poles are cancelled. When A.1.c does not hold, then the fact that $(I - PQ_0)V^0$ is stable impies that an acceptable Q_1 (and therefore the optimal Q_1 as well) makes $PQ_1 v$ stable and therefore the optimal Q_1 is such that $f_2 Q_1 v$ is stable. We shall assume that Q_1 has this property. It should be verified at the end however that the solution indeed has the property. We can then write

$$\Phi(v) = ||\{f_1\}_+||_2^2 + ||\{f_1\}_- - f_2 Q_1 \hat{v}||_2^2 \tag{A.2.3}$$

The first term in the right hand side of (A.2.3) does not depend on Q_1. Hence for solving (2.1.3) we only have to look at the second term. The obvious solution is

$$Q_1 \hat{v} = f_2^{-1} \{f_1\}_- \tag{A.2.4}$$

Clearly such a Q_1 produces a stable $f_2 Q_1 \hat{v}$ as it was assumed. Also $f_1 - f_2 Q_1 \hat{v} = \{f_1\}_+$, which has no improper or constant terms.

It should now be proved that Q_1's that satisfy the internal stability requirements exist among those described by (A.2.4) so that the obvious solution is a true solution. For $n = 1$, (A.2.4) yields a unique Q_1, which can be shown to satisfy the requirements by following the arguments in the Proof of Thm 2.4.1 in Appendix A.3. For $n \geq 2$ write

$$Q_1 \stackrel{\text{def}}{=} (\, q_1 \quad q_2 \,) \tag{A.2.5}$$

$$\hat{V}_2 \stackrel{\text{def}}{=} (\, \hat{v}_2 \quad \dots \quad \hat{v}_n \,)^T \tag{A.2.6}$$

where q_1 is $n \times 1$ and q_2 is $n \times (n-1)$. Then from (A.2.4) it follows that

$$Q_1 = (\, \hat{v}_1^{-1}(f_2^{-1}\{f_1\}_- - q_2\hat{V}_2) \quad q_2 \,) \tag{A.2.7}$$

We now need to show that a stable q_2 exists such that Q_1 is stable and produces a PQ_1P with no poles at $s = 0$ (or in the closed RHP when A.1.c does not hold). Write

$$q_2 = s^{3l_v} \prod_{i=1}^{q} (s - \pi_i)^3 \hat{q}_2 \tag{A.2.8}$$

where $\hat{q}_2$ is stable. Then from (A.2.7) it follows that in order for PQ_1P not to have any poles at $s = 0$ it is sufficient that $P\hat{v}_1^{-1}f_2^{-1}\{f_1\}_-\{P\}_{1st row}$ have no such poles. This holds because the poles in the P on the left cancel with the P_M^{-1} in f_2^{-1} and v_1 has by assumption A.4 at least as many poles at $s = 0$ as the 1st row of P. When A.1.c does not hold, then the same type of argument and the fact that A.3.b holds, imply that PQ_1P has no poles in the open RHP either. Let us now examine the stability of Q_1. The only poles in the open RHP may come from $\hat{v}_1^{-1}$. Let α be such a pole (zero of v_1). Then for stability we need to find $\hat{q}_2$ such that

$$\hat{q}_2(\alpha)\hat{V}_2(\alpha) = \alpha^{-3l_v} \prod_{i=1}^{q} (\alpha - \pi_i)^{-3} f_2^{-1}(\alpha)\{f_1\}_-(\alpha) \tag{A.2.9}$$

The above equation always has a solution because the vector $\hat{V}_2(\alpha)$ is not identically zero since any common RHP zeros in v were factored out in v_0.

We shall now proceed to obtain an expression for $Q\hat{v}$. (2.2.2) and (A.2.7) yield

$$\begin{aligned} Q\hat{v} &= b_p b_v^{-1} P_M^{-1} \left[b_p^{-1} b_v P_A^{-1} P Q_0 \hat{v} - \{b_p^{-1} b_v P_A^{-1} P Q_0 \hat{v}\}_- + \{b_p^{-1} b_v P_A^{-1} \hat{v}\}_- \right] \\ &= b_p b_v^{-1} P_M^{-1} \left[\{b_p^{-1} b_v P_A^{-1} P Q_0 \hat{v}\}_{0+} + \{b_p^{-1} b_v P_A^{-1} \hat{v}\}_- \right] \end{aligned} \tag{A.2.10}$$

where $\{.\}_{0+}$ indicates that in the partial fraction expansion all poles in the closed RHP are retained. For (A.2.10), these poles are the poles of $b_p^{-1} b_v \hat{v}$ in the closed RHP; $P_A^{-1} P Q_0 = P_M Q_0$ is strictly stable because Q_0 is a stabilizing controller.

When A.1.c holds, the stability of $(I - PQ_0)P$ and the fact that the residues of P at the open RHP poles are full rank imply that at these poles $I - PQ_0 = 0$. Also the fact that $(I - PQ_0)V^0$ has no poles at $s = 0$ imply that $(I - PQ_0)$ and its derivatives up to the $(l_v - 1)^{th}$ are also equal to zero at $s = 0$. When A.1.c does not hold, the fact that $(I - PQ_0)V^0$ is stable and that the columns of the diagonal V^0 satisfy A.3.b, imply that $(I - PQ_0) = 0$ at $\pi_1, ..., \pi_q$. Thus (A.2.10) simplifies to (2.3.4).

We simply need to establish that a stabilizing controller Q_0 with the property that $(I - PQ_0)V^0$ has no unstable poles exists. The selection of a V^0 with the properties mentioned in the beginning of this section and no RHP zeros and its use instead of V in (2.4.3) yields such a controller as it follows from the proof of Thm. 2.4.1 in Appendix A.3.

<u>A.3.</u> Proof of Thm. 2.4.1.

A stabilizing controller that solves (2.1.5) has to solve (2.1.3) for all $x \in \mathbf{R}^n$. Hence it has to satisfy (2.3.4) for all $v = Vx, \quad x \in \mathbf{R}^n$. For each of the n linearly independent directions $(2,1,...,1)$, $(1,2,...,1)$, $(1,1,...,2)$, the factor $v_0(s)$ containing the common RHP zeros of its elements is the same as the one for the direction $(1,1,...,1)$. Therefore for each of them we can substitute in (2.3.4) $\hat{v} = \hat{V}x$, where $\hat{V}$ is defined through (2.2.1),(2.3.2),(2.4.2). Then from Linear Algebra it follows that there is only one Q with this property:

$$Q = b_p b_v^{-1} P_M^{-1} \{b_p^{-1} b_v P_A^{-1} \hat{V}\}_* \hat{V}^{-1} \tag{2.4.1}$$

This solution however is not necessary stabilizing because not every Q that satisfies (2.3.4) for some x, is. To start with, Q is not stable if $\hat{V}$ has RHP zeros (unless of course P is stable and minimum phase). This will be the case when there RHP zeros in V that are not present in every element of V. In this case, there exists no solution to (2.1.5), which is part (ii) of the Theorem. When $\hat{V}^{-1}$ is stable, we still have to establish that the internal stability matrix $IS1$ is stable. Careful inspection shows that both Q and PQ are stable. We also have

$$(I - PQ)P = b_p b_v^{-1} P_A \{b_p^{-1} b_v P_A^{-1} \hat{V}\}_A \hat{V}^{-1} P \tag{A.3.1}$$

where $\{.\}_A$ indicates that after a partial fraction expansion, ony the terms corresponding to poles of P_A^{-1} are retained. These poles are cancelled in (A.3.1) by P_A. Then from assumptions A.3, A.4, it follows that $(I - PQ)P$ is stable.

A final, but very important point is to show that the above Q minimizes the ISE for any $x \in \mathbf{R}^n$, when of course all the RHP zeros of V appear in every element with the same degree. But then $v_0(s)$ is the same for any direction $x \in \mathbf{R}^n$ and therefore for any x it suffices that (2.3.4) is satisfied, a property which the above controller has.

A.4. Proof of Thm. 2.4.2.

The proof follows that of Thm. 2.4.1 in A.3, with V_M used instead of $\hat{V}$. V_M appears because the directions in (2.4.6) are used and as a result for each direction the corresponding v_0 includes all the RHP zeros of the corresponding element v_i of V.

Appendix B

B.1. Proof of Theorem 3.4.1.

For a matrix K partitioned as

$$K = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \tag{B.1.1}$$

define

$$R(K,\Delta) \stackrel{\text{def}}{=} K_{22} + K_{21}\Delta(I - K_{11}\Delta)^{-1}K_{12} \tag{B.1.2}$$

Then the transfer function relating v to e in Fig.4 is $R(G^F, \Delta)$ and since Fig.1a and Fig.4 are equivalent, we get by using (1.1.5)

$$E = R(G^F, \Delta) \tag{B.1.3}$$

The properties of the SSV and (3.4.8) imply (Doyle,1985) that

$$\sup_{\Delta \in \mathbf{\Delta}} \bar{\sigma}(R(G^{x_0}_{full}, \Delta)) = 1 \tag{B.1.4}$$

From (3.4.6), (3.4.7), (B.1.2), (B.1.3), it follows after some algebra that

$$R(G^{x_0}_{full}, \Delta) = (\, x_0 Ev \quad 0\,) \tag{B.1.5}$$

Then from (B.1.4),(B.1.5) and the definition of the singular values, it follows, since $x_0 Ev$ is a vector:

$$\sup_{\Delta \in \mathbf{\Delta}} (x_0^2 v^* E^* E v) = 1 \qquad \forall \omega$$

$$\Longrightarrow \sup_{\Delta \in \mathbf{\Delta}} \int_{-\infty}^{+\infty} v^* E^* E v \quad d\omega \quad = \int_{-\infty}^{+\infty} x_0^{-2} \quad d\omega$$

$\Longleftrightarrow$ (3.4.9) *QED*

B.2. Proof of theorem 3.4.2.

Let $0 < x_2 \leq x_1$. Then we can write $x_2 = x_1\beta$, where $0 < \beta \leq 1$. From (3.4.10) we have

$$DM^{x_2}D^{-1} = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & \beta I \end{pmatrix} M^{x_1}D^{-1}$$
$$= \begin{pmatrix} I & 0 \\ 0 & \beta I \end{pmatrix} DM^{x_1}D^{-1} \qquad (B.2.1)$$

Then the properties of the singular values yield

$$(B.2.1) \Longrightarrow \bar{\sigma}(DM^{x_2}D^{-1}) \leq \bar{\sigma}\begin{pmatrix} I & 0 \\ 0 & \beta I \end{pmatrix} \quad \bar{\sigma}(DM^{x_1}D^{-1})$$
$$\Longrightarrow \bar{\sigma}(DM^{x_2}D^{-1}) \leq \bar{\sigma}(DM^{x_1}D^{-1}) \qquad \forall D \in \mathbf{D}$$
$$\Longrightarrow \inf_{D\in\mathbf{D}} \bar{\sigma}(DM^{x_2}D^{-1}) \leq \inf_{D\in\mathbf{D}} \bar{\sigma}(DM^{x_1}D^{-1}) \qquad QED$$

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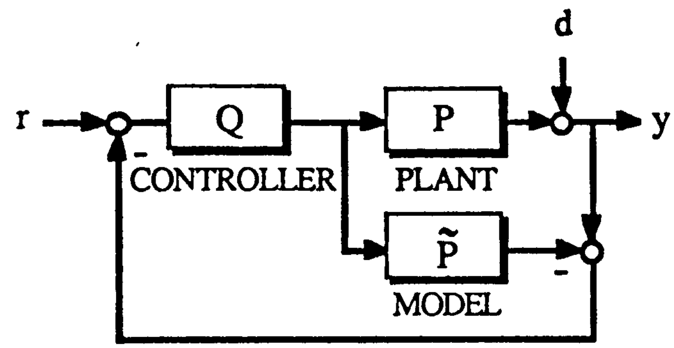
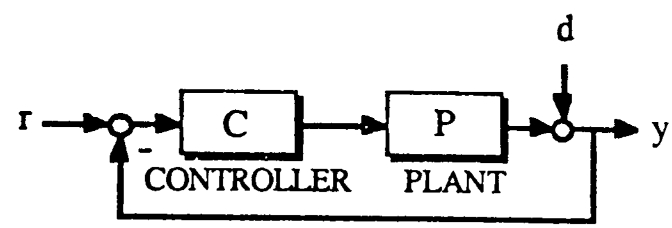


(a)



(b)

Figure 1.

(a) Internal Model Control structure.

(b) Feedback Control structure.

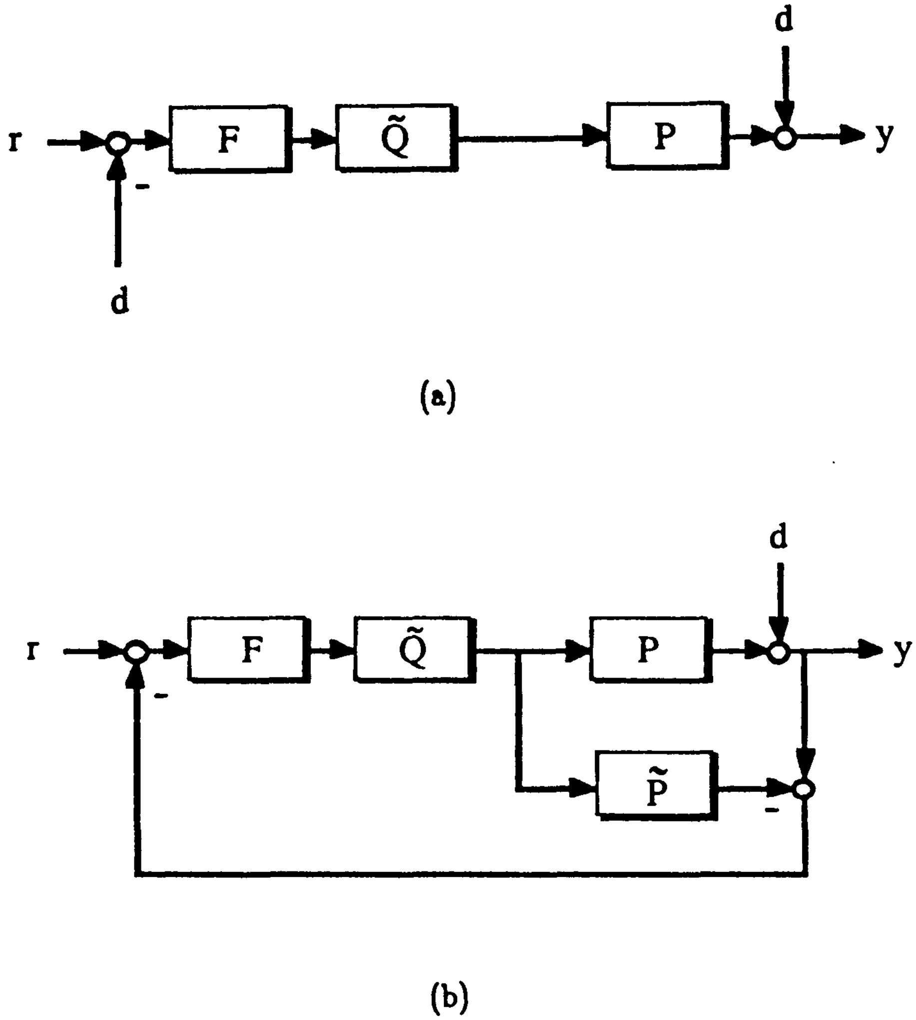
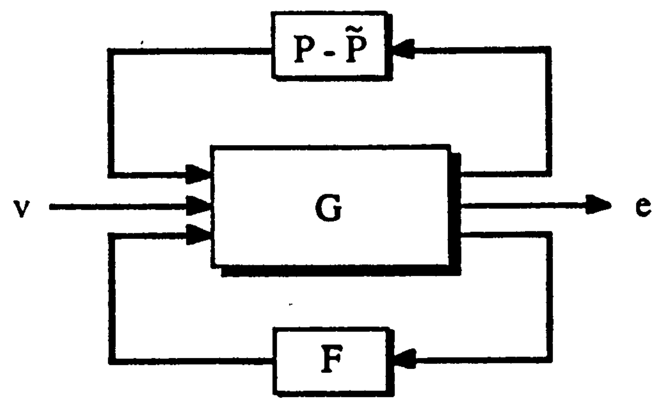


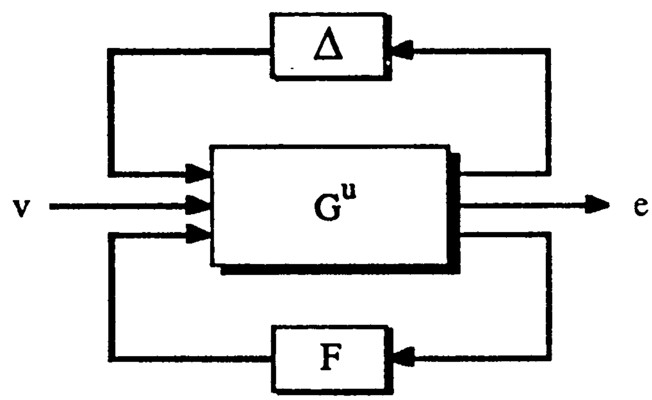
Figure 2. IMC structure with the filter F.

(a) $P = \tilde{P}$.

(b) $P \neq \tilde{P}$.



(a)



(b)

Figure 3. Model uncertainty block diagrams.

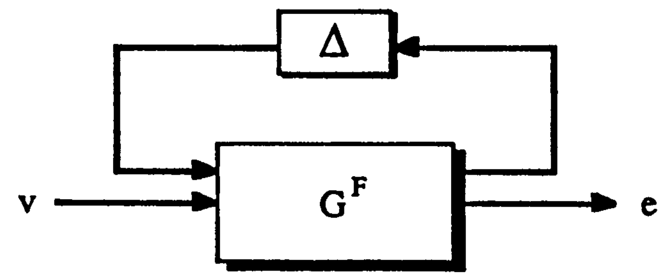


Figure 4. SSV block diagram.

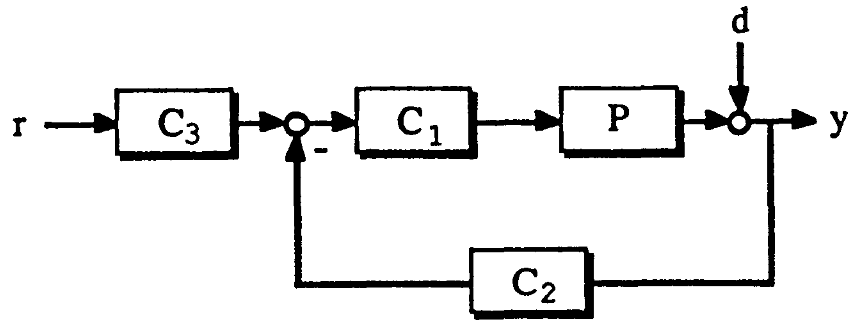


Figure 5. Two-degree-of-freedom feedback structure.