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1 Introduction

1. Pareto Nash Reply
2. Existence
3. Characterization of Pareto Nash Reply using Multicriteria Linear Programming.
4. Supported Strategies

We observe the notions of Pareto Nash strategies and their relations to multi-parametric linear programming has made repeated appearances the game theory literature. . **What we contribute** . We feel that the theory of MOG has significant potential to model several practical scenarios especially in the context of networked communications. For examples we refer the reader to [1].

This paper is organised as follows.

2 Multi-Objective Games

In conventional game theoretic problems, it is usually assumed that the decision makers usually make their decisions based on a *scalar payoff*. But in many practical problems in economics and engineering, decision makers usually deal with multiple objectives or payoffs. In these problems, one needs to consider a *vector payoff* function. The notion of vector payoffs was originally introduced by *Blackwell* [2] and later by *Contini* [3]. A more rigorous model for *zero-sum* games and *games against nature* was studied by *Zeleny* [12]. In [12], *Zeleny* solves the multiple objective zero-sum game using multi-parametric criteria linear programming. In the preceding work by *Contini*, a *non-zero-sum* version of the Multi-Objective Game (MOG) is introduced. A general version concerning the *n-person* MOG in the non-cooperative setting is introduced in [4]. A further extension to cooperative games and hybrid games is introduced by *Zhao* in [9]. Then notions of Pareto Nash strategies was reborn in the works by *Zelikovsky* [5] where the authors provide algorithms to obtain the Pareto Nash Equilibria.

We observe the notions of Pareto Nash strategies and their relations to multi-parametric linear programming has made repeated appearances the game theory literature. . **What we contribute** . We feel that the theory of MOG has

significant potential to model several practical scenarios especially in the context of networked communications. For examples we refer the reader to [1].

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3 Mathematical Notation

In this section, we introduce the notations which we will use hereafter. We follow the standard notions used in the text by *Weibull* [10]. We shall consider *finite games in normal form.* Let $I = \{1, 2, \cdots, n\}$ be the set of *players* where $n \geq 2$. For each player $i \in I$, let S_i be her finite set of *pure strategies.* The pure strategies set of player $i \in I$ is written as $S_i = e_i^1, e_i^2, \cdots, e_i^{m_i}$, for some $m_i \geq 2$. The vector s of pure strategies, $s = (s_1, s_2, \cdots, s_n)$, where s_i is the pure strategy of any player i, is called the *pure strategy profile.* The pure strategies profiles live in in the cartesian product space $S = \times_i S_i$.

For any strategy profile $s \in S$ and player $i \in I$, let $\pi_i(s) \in \mathcal{R}^{l_i}$ represent the l_i dimensional vector payoff function for player i when all the players play a strategy profile s. The combined payoff function $\pi : S \to \times_i R^{l_i}$ is the collection $\pi(s) = (\pi_1(s), \pi_2(s), \cdots, \pi_n(s))$.

A mixed strategy for player i is a probability distribution over her set S_i of pure strategies. Let x_i denote the mixed strategy for player i. Thus x_i lives in the $m_i - 1$ dimensional *unit simplex* Δ_i. Where,

$$\Delta_i = \{x_i \in R_+^{m_i} \quad : \quad \sum_{h=1}^{m_i} x_{ih} = 1$$

Since these a probability distributions over the pure strategies, vertices of Δ_i correspond the mixed strategies $e_i^1 = (1, 0, \cdots, 0), e_i^2 = (0, 1, \cdots, 0), \cdots, e_i^{m_i} = (0, 0, \cdots, 1)$. And thus the mixed strategies can be alternatively represented as $x_i = \sum_{h=1}^{m_i} x_{ih} e_i^h$. A *mixed strategy profile* is a vector $x = (x1, x2, \cdots, x_n)$, which lives in $\Theta = \times_i \Delta_i$. Let $C(x_i)$ be the support of the vector x_i. i.e $C_i = \{h \quad | \quad x_{ih} > 0\}$.

The mixed strategies payoff functions are given by $u_i(x) = \sum_{s \in S} x(s)\pi_i(s)$. Where $x(s) = \prod_i x_{is_i}$. If player j strategy in the strategy profile x is replaced by another strategy y_j we denote the replaced profile by (y_j, x_{-j}). Then the payoff function can be expressed as

$$u_i(x) = \sum_{k=1}^{m_j} u_i(e_j^k, x_{-j}) x_{jk}$$

It should be noted here that $u_i \in \mathcal{R}^{l_i}$. Each component function $u_i^k \quad k \in \{1, 2, \cdots, l_i\}$ is a multi-linear mapping that is linear in each vector component $x_j \in \Delta_j$. The collection of payoff vectors for each $i \in I$ is called the *combined mixed strategy payoff function* and is denoted by $u(x) = (u_1(x), u_2(x), \cdots, u_n(x)$.

3.1 Multiple Matrix Games

Since we consider the two player games ($|I| = n = 2$) in greater detail in this work, we introduce the notion of *Multiple Matrix Games* (MMG) which is a nat-

Notation	Definition	Name
$x \geq y$	$x_i \geq y_i \quad i = 1, 2, .., l$	Weak component-wise order
$x > y$	$x_i \geq y_i \quad i = 1, 2, .., l$ and $x \neq y$	Component-wise order
$x \gg y$	$x_i > y_i \quad i = 1, 2, .., l$	Strict component-wise order

Table 1: Table of Orders in $\mathcal{R}^l$

ural extension to the *Bi-Matrix Games* in the single objective setting. The same matrix notation has also been considered by [12]. Associated with each player is a sequence of matrix payoff functions $(A_1, A_2, \cdots, A_{l_1})$ and $(B_1, B_2, \cdots, B_{l_2})$ (for *player I* and *II* respectively). We assume that *player I* is the row-player and *player II* is column player. Let us suppose that *player I* chooses strategy $\xi \in \Delta_1$ and *player II* chooses strategy $\eta \in \Delta_2$. Then the expected scalar payoff with respect to the p^{th} and q^{th} objective of the MOG for player *I* and *II* is given by $\xi^T A_p \eta$ and $\xi^T B_q \eta$ respectively, where $p \in \{1, 2, \cdots, l_1\}$ and $q \in \{1, 2, \cdots, l_2\}$.

4 Pareto Nash Equilibrium

As in the case of Single Objective Games, the notion of equilibrium can be defined in terms of *unfruitful deviation* from the equilibrium strategies. In the MOG setting, this can be interpreted as follows.

Deviations from the equilibrium strategies do not offer any gains to any of the pay-off functions for any of the players.

In order to compare vector payoffs we introduce the notions of orders in $\mathcal{R}^l$ in Table 1. These orders are commonly used in multi-criteria optimization literature. In order to rigorously define the *Pareto Nash Equilibrium*, we introduce the notion of a *Pareto Reply*.

Definition The *Pareto Reply* of player $i \in I$ for the strategy profile x_{-i} of the rest of the players is defined as that strategy $x_i \in \Delta_i$ such that the strategy profile (x_i, x_{-i}) is pareto optimal with respect to the vector payoff function $u_i(., x_{-i})$. Let us denote the Pareto Response correspondence as $\beta_i^P : \times_{k \neq i} \Delta_k \rightarrow 2^{\Delta_i}$.

i.e. suppose x_{-i} is the strategy profile of all the players $k \neq i$ then

$$\beta_i^P(x_{-i}) = \{x_i \in \Delta_i \quad | \quad \nexists z \in \Delta_i \quad \textit{such that } u_i(z, x_{-i}) > u_i(x_i, x_{-i}\}$$

where $>$ is the *component-wise* order defined in Table 1.

The *combined pareto reply* correspondence of the strategy profile x is then given by $\beta^P(x) = (\beta_1^P(x_{-1}), \beta_2^P(x_{-2}), \cdots, \beta_n^P(x_{-n}))$.

Definition A strategy profile $x^* = (x_1^*, x_2^*, \cdots, x_n^*)$ is called a *Pareto Nash Equilibrium* for the MOG if $x^* \in \beta^P(x^*)$. The set of all Pareto Nash Equilibria is denoted by Θ^{PNE}.

In other words a *Pareto Equilibrium Strategy* is a fixed point of the *Pareto Reply Correspondence*. It should be mentioned here that the *Kakutani Fixed Point* theorem cannot be invoked to prove the existence of such an equilibrium here because the pareto replies do not necessarily form a convex set. For detailed reference on these fixed point problems we refer the reader to [6]. We note that in the current game theoretic literature, there are several complicated proofs for existence of equilibrium in various topological spaces. We however give an alternative proof for the existence of the Pareto Nash which is solely motivated from Nash's original work and Pareto Optimality.

4.1 Pareto Dominating Replies

Nash in his seminal work on Non-Cooperative Games ([7], [8]) introduces the notion of *better responses* which has motivated the study of several similar dynamics in the literature [10]. However it should be mentioned that *Nash* used the *better response maps* to prove the existence of a non-cooperative equilibrium. In the same spirit we define the so called *Pareto Dominating Reply*. Let suppose we can express the payoff vector as $u_i(x) = (u_i^1(x), u_i^2, \cdots, u_i^{l_i})$. Choose an arbitrarily small $\epsilon > 0$.

Then let

$$\begin{aligned} c_i^h(x) = \quad & \min((u_i^1(e_i^h, x_{-i}) - u_i^1(x) + \epsilon)^+, (u_i^2(e_i^h, x_{-i}) - u_i^2(x) + \epsilon)^+, \cdots, (u_i^{l_i}(e_i^h, x_{-i}) - u_i^{l_i}(x) + \epsilon)^+) \\ & \max((u_i^1(e_i^h, x_{-i}) - u_i^1(x))^+, (u_i^2(e_i^h, x_{-i}) - u_i^2(x))^+, \cdots, (u_i^{l_i}(e_i^h, x_{-i}) - u_i^{l_i}(x))^+) \end{aligned}$$

The *Pareto Dominating Reply* map $D^P : \Theta \to \Theta$ is defined as $D^P : x \mapsto x'$

$$x'_{ih} = \frac{x_{ih} + c_i^h(x)}{1 + \sum_{k \in S_i} c_i^k(x)} \quad \forall h \in S_i \quad \forall i \in I$$

Theorem 4.1 *The map $D^P : \Theta \to \Theta$ has a fixed point x^f.*

Proof The map D^P is continuous and the Θ is a compact convex set. Then by Brouwer's fixed point theorem([6]) there exists a fixed point x^f for D^P in Θ.

Lemma 4.2 *For any strategy profile x and $\forall i \in I \quad \exists p \in \{1, 2, \cdots, l_i\}$ and $h \in C(x_i)$ such that $u_i^p(e_i^h, x_{-i}) \leq u_i^p(x)$.*

Proof Suppose there $\nexists h \in C(x_i)$ such that $u_i^p(e_i^h, x_{-i}) \leq u_i^p(x)$. Then for all $h \in C(x_i)$ we have

$$\begin{aligned} u_i^p(e_i^h, x_{-i}) &> u_i^p(x) \\ \Rightarrow \sum_{h \in C(x_i)} x_{ih} u_i^p(e_i^h, x_{-i}) &> \sum_{h \in C(x_i)} x_{ih} u_i^p(x) \\ \Rightarrow u_i^p(x) &> u_i^p(x) \end{aligned}$$

Which is not possible. Hence proved.

Theorem 4.3 *x is the fixed point of D^P if and only if it is a Pareto Nash Equilibrium.*

Proof We prove the first implication that if x is a fixed point then x is a Pareto Nash Equilibrium.
If x is a fixed point then

$$x_{ih} = \frac{x_{ih} + c_i^h(x)}{1 + \sum_{k \in S_i} c_i^k(x)} \quad \forall h \in S_i \quad \forall i \in I$$

By using Lemma 4.2 we know that $\exists h \in C(x_i)$ such that $u_i^p(e_i^h, x_{-i}) leu_i^p(x)$. For such an h, the above equation reduces to

$$x_{ih} = \frac{x_{ih}}{1 + \sum_{k \in S_i} c_i^k(x)}$$

Since $x_{ih} > 0$ we have $\sum_{k \in S_i} c_i^k(x) = 0$. Since $c_i^k(x) \geq 0 \quad \forall k \in S_i$, we have $c_i^k(x) = 0 \quad \forall k \in S_i \quad \forall i \in I$.

Next we show that any $i \in I \; c_i^k(x) = 0 \quad k \in S_i \quad \Rightarrow x_i \in \beta_i^P(x_{-i})$. Since this is true $\forall i \in I \Rightarrow x$ is a Pareto Nash Equilibrium.

$$\begin{aligned}
&c_i^k(x) = 0 \quad \forall k \in S_i \\
&\Rightarrow \nexists e_i^k \textit{ such that } u_i(e_i^k, x_{-i}) > u_i(x) \\
&\Rightarrow \nexists z \in \Delta_i \textit{ such that } u_i(z, x_{-i}) > u_i(x) (\textit{Since } \Delta_i = Co(e_i^k | k \in S_i))
\end{aligned}$$

where $Co(S)$ stands for the Convex Hull of the set S.

Next we prove the other implication. If x is Pareto Nash Equilibrium, then x is a fixed point of D^P.
Since x is a Pareto Nash Equilibrium, $\forall i \in I, x_i \in \beta_i^P(x_{-i})$.

$$\begin{aligned}
&\Rightarrow \nexists z \in \Delta_i \textit{ such that } u_i(z, x_{-i}) > u_i(x) \\
&\Rightarrow \textit{In particular } \; \nexists h \in S_i \textit{ such that } u_i(e_i^h, x_{-i}) > u_i(x) \\
&\Rightarrow c_i^h = 0 \quad \forall h \in S_i \quad \forall i \in I
\end{aligned}$$

Therefore x is a fixed point of the D^P.

We have shown that $\Theta^{PNE} \neq \emptyset$. We trust that this proof method would inspire some dynamics to achieve the equilibria in the MOG setting. We next characterize the equilibrium set Θ^{PNE}.

5 Pareto Nash Reply, A Multi-criteria Linear Programming problem

In this section we characterize the geometry of the Pareto Nash Reply set. To aid us, we introduce some further notations. Throughout this section the notation for the orders on vectors is consistent with that in Table 1. We first introduce the **incomplete**.

5.1 Pareto Optimality in Games

Given a vector payoff function $u_i(x) = (u_i^1(x), u_i^2(x), \cdots, u_i^{l_i}(x))$ the decision problem for player i is to choose a $x_i \in \Delta_i$ according to the Pareto class $./id/R^{l_i}$. *i.e.*

$$\max_{x_i \in \Delta_i}{}^{P} u_i(x_i, x_{-i}) \tag{1}$$

where the superscript P stands for Pareto maximization. Let us denote the payoff space $\mathcal{U}_i(x_{-i}) := \{u_i(x_i, x_{-i}) | x_i \in \Delta_i\}$.

Definition A feasible solution $\hat{x}_i \in \Delta_i$ is called *efficient* or *Pareto Optimal* for the profile x_{-i} of the other players, if there is no other $x_i \in \Delta_i$ such that $u_i(x_i, x_{-i}) > u_i(\hat{x}_i, x_{-i})$. If $\hat{x}_i$ is efficient, $u_i(\hat{x}_i, x_{-i})$ is called *non-dominated point* in $\mathcal{U}_i$. If $x_i^1, x_i^2 \in \Delta_i$ and $u_i(x_i^1, x_{-i}) > u_i(x_i^2, x_{-i})$ we say x_i^1 *dominates* x_i^2 and $u_i(x_i^1, x_{-i})$ *dominates* $u_i(x_i^2, x_{-i})$. The set of all inefficients solutions is denoted by $\mathcal{X}_{iE}$. The set of all non-dominated points is denoted by $\mathcal{U}_{iN}$.

Next we proceed to characterize the non-dominated set $\mathcal{U}_{iN}$ for the problem at hand. The objective function for (1) is indeed a linear objective in x_i. Since $x_i \in \Delta^i$ lives in a finite dimensional vector space, we can associate an *objective matrix* U_i, which is a $l_i \times |S_i|$ matrix. U_i will be a function of the strategy profile x_{-i} of the other players. But in a game setting we assume that this is know to the optimization problem(1). Thus the Pareto problem can be reformulated as

$$\max_{x_i \in \Delta_i}{}^P U_i x_i \tag{2}$$

Definition Let $x_i \in \mathcal{X}_{iE}$. If there is some $\lambda_i \in \mathcal{R}^{l_i}_{>0}$ such that $x_i \in \mathcal{X}_{iE}$ is an optimal solution of $\max\limits_{x_i \in \Delta_i} \lambda_i^T u_i(x_i, x_{-i})$ then x_i is called a *supported efficient strategy* and $u_i(x_i, x_{-i})$ is called *supported non-dominated payoff* for player i. The set of all supported efficient strategies and supported non-dominated payoffs are denoted by $\mathcal{X}_{isE}$ and $\mathcal{U}_{isN}$.

We next present a sequence of lemmas which aid in characterizing the efficient strategies and non-dominated points. For detailed proofs we refer to the reader to [11].

Lemma 5.1 $\mathcal{X}_{iE} \neq \emptyset$

Proof Δ_i is a closed set. The payoff set for player $i, \mathcal{U}_i = \{U_i x_i | x_i \in \Delta_i\}$ is also thus a closed set (By continuity of the bounded linear operator). Hence $\mathcal{X}_{iE} \neq \emptyset$

Lemma 5.2 $\mathcal{X}_{iE} = \mathcal{X}_{isE}$ *and* $\mathcal{U}_{isN} = \mathcal{U}_{isN}$.

Proof Δ_i is a convex set.
$\Rightarrow \{U_i x_i | x_i \in \Delta_i\}$ is a convex set. $\Rightarrow \mathcal{U}_{isN} = \mathcal{U}_{isN}$. (Theorem 3.5 [11]).

Lemma 5.2 suggests that the Pareto reply can be solved by *weighted sum scalarization*. For reference on this technique we suggest (Chapter 3 [11]). It should be mentioned here that this exactly the approach used to develop algorithms to obtain the Pareto Nash Equilibrium in [5]. However the algorithm that their work suggest is only an existential arguement. It does not provide any means to construct the weights for the scalarization. However for our problem, we identify that there is more structure to construct these weights λ_i and obtain a constructive algorithm to obtain the Pareto Nash Strategies. To justify our algorithm we present some principles from the *simplex methods* of linear programming. For a detailed reference on this topic we refer the reader to Chapter 2 of [13] and Chapters 7 and 8 of [11]. Next we present a series of lemmas that help to characterize the geometry of the Pareto reply set. *INcomplete*

Definition The weighted sum linear program corresponding to the pareto reply problem (2) is . $\lambda_i \in \mathcal{R}^{l_i}$

$$\max_{x_i \in \Delta_i} \lambda_i^T U_i x_i$$

And let us denote the set of solutions for $\lambda_i \in \mathcal{R}^{l_i}_{>0}$ as $LP(\lambda_i)$.

From Lemma 5.2 we know that all the Pareto replies for player i can be obtained from $\{LP(\lambda_i) \quad \forall \lambda_i \in \mathcal{R}^{l_i}_{>0}\}$. Hence,

Lemma 5.3 $\cup_{\lambda_i \in \mathcal{R}^{l_i}{}_{i>0}} LP(\lambda_i) = \beta_i^P(x_{-i})$

Lemma 5.4 *The payoff space $\mathcal{U}_i(x_{-i})$ is supported by a finite set of hyperplanes.*

Proof Δ_i is a convex compact set. If u_i is a bounded function for every objective, then $\mathcal{U}_i(x_{-i})$ is a *convex polytope*. A convex polytope has a finite number of faces and each face is supported by a hyperplane (Chapter 2 of [13]).

Lemma 5.5 $dim(\mathcal{U}_i(x_{-i})) = \min(rank(U_i), |S_i| - 1)$

Proof $U_i : \Delta_i \rightarrow \mathcal{R}^{l_i}$ is a linear transformation of the $|S_i| - 1$ dimensional simplex to $\mathcal{R}^{l_i}$.

Theorem 5.6 *There exists a finite set $\Lambda_i = \{\lambda_{ik} \in \mathcal{R}^{l_i}_{>0}\}$ such that $\cup_{\lambda_i \in \Lambda_i} LP(\lambda_i) = \beta_i^P(x_{-i})$*

Proof The **existential proof** is a direct implication from Lemma 5.4. We provide a **constructive proof** using the method of *multicriteria linear programming*. Algorithm 1 is an abstraction of the Algorithms 6.2 and 7.1 of [11]. By transversing through the *connected efficient bases*, we obtain the kissing planes of the $\mathcal{U}_i(x_{-i})$. A detailed discussion on this algorithm we refer the reader to Chapter 7 of [11]. The number of efficient pivots is finite and hence $\Lambda_i = \mathcal{L}$.

Algorithm 1 Compute Weights

List $\mathcal{L} = \{\}$
Find a `initial basic feasible solution`.
Determine an `initial efficient basis` and corresponding `weights` λ_{i1}.
$\mathcal{L} \leftarrow \mathcal{L} \cup \lambda_{i1}$
$k \leftarrow 1$
while `Efficient Pivot` $\neq \emptyset$ **do**
  Perform `Efficient Pivot` operation.
  Determine `efficient basis` and `weights` λ_{ik}.
  $\mathcal{L} \leftarrow \mathcal{L} \cup \lambda_{ik}$.
  $k \leftarrow k + 1$
end while

Theorem 5.6 establishes through construction a finite set of weights which give all the Pareto best replies of player i. At the Pareto Nash equilibrium each player's Pareto reply is thus supported by a finite weight vector set Λ_i^{PNE}.

Theorem 5.7 *At the Pareto Nash equilibrium every player solves finite sequence of single objective games.*

Proof At the Pareto Nash equilibrium $x^{PNE} \in \Theta^{PNE}$, $x_i^{PNE} \in \beta_i^P(x_{-i}^{PNE})$.
$\Rightarrow x_i^{PNE} \in \cup_{\lambda_i \in \Lambda_i^{PNE}} LP(\lambda_i)$ (Theorem 5.6).
Thus every player i plays a corresponding a finite sequence of single objective auxiliary games $\forall \lambda_{ik} \in \Lambda_i^{PNE}$

$$\max_{x_i \in \Delta_i} \lambda_{ik}^T U_i x_i$$

This simple algorithmic interpretation of the Pareto Nash equilibria yields insights on the structure of the convex polyhedron which acts as the decision space for every player. The payoff space $\mathcal{U}_i(x_{-i})$ is a convex polyhedron with the corner points being $u(e_i^h, x_{-i}), \quad h \in S_i$. This suffices to characterize the Pareto Nash Equilibrium set.

Structure of the Pareto Nash Equilibria

Lemma 5.8
$\Theta^{PNE} = \{x \in \Theta| \quad \cap_q u_i^q(x) - u_i^q(e_i^h, x_{-i} \geq 0 \text{ } or \text{ } \cup_q u_i^q(x) - u_i^q(e_i^h, x_{-i} > 0$
$\forall i \quad \forall h \in S_i\}$

Proof The convex polyhedron interpretation of $\mathcal{U}$ shows that the Pareto Nash equilibria like Nash Equilibria can be characterized by *unilateral deviations.*
$\forall i \in \{1, 2, \cdots, n\}$

$$\begin{aligned} \{h \in S_i| \quad \forall q \in \{1, 2, \cdots, l_i\} \quad u_i^q(e_i^h, x_{-i}) \geq u_i^q(x) \text{ } and \\ \exists q \in \{1, 2, \cdots, l_i\} \quad u_i^q(e_i^h, x_{-i}) > u_i^q(x)\} = \quad & \emptyset \\ \{h \in S_i| \quad \exists q \in \{1, 2, \cdots, l_i\} \quad u_i^q(e_i^h, x_{-i}) < u_i^q(x) \text{ } or \\ \exists q \in \{1, 2, \cdots, l_i\} \quad u_i^q(e_i^h, x_{-i}) \leq u_i^q(x)\} = \quad & S_i \end{aligned}$$

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