WEYL-HEISENBERG WAVELET EXPANSIONS: EXISTENCE AND STABILITY IN WEIGHTED SPACES

by

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Dissertation submitted to the Faculty of the Graduate School
of The University of Maryland in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
1989

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ABSTRACT

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David Francis Walnut, Doctor of Philosophy, 1989

Dissertation directed by: Professor John J. Benedetto, Department of Mathematics.

The theory of wavelets can be used to obtain expansions of vectors in certain spaces. These expansions are like Fourier series in that each vector can be written in terms of a fixed collection of vectors in the Banach space and the coefficients satisfy a "Plancherel Theorem" with respect to some sequence space. In Weyl-Heisenberg expansions, the expansion vectors (wavelets) are translates and modulates of a single vector (the analyzing vector).

The thesis addresses the problem of the existence and stability of Weyl-Heisenberg expansions in the space of functions square-integrable with respect to the measure $w(x)\,dx$ for a certain class of weights w. While the question of the existence of such expansions is contained in more general theories, the techniques used here enable one to obtain more general and explicit results.

In Chapter 1, the class of weights of interest is defined and properties of these weights proven.

In Chapter 2, it is shown that Weyl-Heisenberg expansions exist if the analyzing vector is locally bounded and satisfies a certain global decay condition.

In Chapter 3, it is shown that these expansions persist if the translations and modulations are not taken at regular intervals but are perturbed by a small amount. Also, the expansions are stable if the analyzing vector is perturbed. It is also shown here that under more general assumptions, expansions exist if translations and modulations are taken at any sufficiently dense lattice of points.

Like orthonormal bases, the coefficients in Weyl-Heisenberg expansions can be formed by the inner product of the vector being expanded with a collection of wavelets generated by a transformed version of the analyzing vector. In Chapter 4, it is shown that this transformation preserves certain decay and smoothness conditions and a formula for this transformation is given.

In Chapter 5, results on Weyl-Heisenberg expansions in the space of square-integrable functions are presented.

ACKNOWLEDGMENT

I wish to thank my advisor John Benedetto who suggested wavelet theory as a dissertation topic, and who has been a source of mathematical support and inspiration throughout.

I also wish to express my appreciation to my friend Chris Heil with whom I learned wavelet theory and with whom I have had countless enlightening conversations concerning wavelets over the past two years.

I wish also to thank Dr. Karlheinz Gröchenig of the University of Connecticut for suggesting many productive avenues of inquiry, and Dr. George Benke and Dr. Ward Evans of the MITRE Corporation for giving me the opportunity to learn and write about wavelet theory while employed at MITRE.

Finally, I owe thanks to the Sloan Foundation for their financial support during this past year.

TABLE OF CONTENTS

INTRODUCTION

Anyone who has ever heard a sound such as a siren or a piece of music has registered the impression that the signal consists of a combination of different frequencies at different times. However, this common intuition is not reflected in classical Fourier analysis techniques. Specifically, expanding a compactly supported function, thought of as a signal of finite duration, in an ordinary Fourier series can be interpreted as viewing the signal as the superposition of signals of constant pitch and fixed amplitude which persist for the entire duration of the original signal.

The first attempt to define a transformation which reflected the way the ear percieves sound was made in [Ga]. Gabor was inspired by some techniques in use by quantum physicists for decades and was the first to apply them to signal analysis. A version of Gabor's transform, known as the Short-time Fourier Transform, has been in use by signal processing engineers for many years. The idea behind this transform is the following. In order to obtain a picture of how the frequencies present in a signal change over time, one divides the signal into distinct time intervals, then takes the Fourier transform of each piece.

These ideas were given a rigorous mathematical foundation in [DGM] and [D1], where the theory of frames in Hilbert space (cf. Section 0.5) was used to define a short-time Fourier transform in which there is more freedom in the choice of a window function, thereby obtaining a generalization of Fourier series to the Hilbert space $L^2(\mathbb{R})$. These expansions were referred to as Weyl-Heisenberg (W-H) wavelet expansions because of their relationship to the so-called wavelet transform (or affine wavelet transform) defined in [GM], and to the wavelet orthonormal basis of Daubechies and Meyer.

In [F2], Feichtinger obtains W-H expansions of distributions on $\mathbb{R}^k$ lying in certain Banach spaces called modulation spaces, which are defined by smoothness and decay conditions. These include the space $L^2(\mathbb{R})$. Here, Feichtinger used the theory of Wiener-type spaces (cf. [F1]) to obtain W-H wavelet expansions by a method analogous to that used in [FJ1] to obtain affine wavelet expansions in Besov spaces. This theory was superseded by the general theory found in [FG1] and [FG2], which proved the existence of W-H and affine wavelet expansions of distributions in a large class of spaces, the coorbit spaces. This theory is quite abstract and relies on the theory of group representations to obtain its results.

Thus, there are three methods for obtaining W-H expansions of distributions in Banach spaces. Daubechies' method (cf. [DGM], [D1]) is essentially restricted to the space $L^2(\mathbb{R}^k)$, but provides very general conditions on the mother wavelets and the frame parameters which guarantee the existence of frames. Also, these methods provide good estimates on the critical values of frame parameters and frame bounds for a given mother wavelet.

The method of Feichtinger in [F2] shows the existence of W-H wavelet expansions in a larger class of spaces than just $L^2(\mathbb{R}^k)$. This method is a great deal more abstract than Daubechies' method but still gives a very general, easily checkable condition on a mother wavelet guaranteeing that it generate a set of W-H atoms for a given modulation space. Estimates on the values of the lattice parameters and on the atomic bounds are difficult in this case.

The method of Feichtinger and Gröchenig (cf. [FG1], [FG2]) is a very beautiful and general theory which shows the existence of wavelet expansions by means of both W-H and affine wavelets for a very large class of Banach spaces. This method gives specific results concerning stability of the wavelet expansions under perturbations. This is not done in either of the first two methods and makes explicit an advantage which expansions in terms on non-orthogonal sets of vectors possess over orthogonal expansions. On the other hand, this method is very abstract and does not give a transparent condition on a vector guaranteeing that it generate a set of W-H atoms, nor does it provide a simple means of obtaining estimates on valid parameter values or atomic bounds.

This paper presents a method for finding sets of W-H atoms for spaces other than the Hilbert space $L^2(\mathbb{R}^k)$, namely the spaces

$L^2_w(\mathbb{R}^k)$, with very explicit conditions on mother wavelets which guarantee that they generate sets of atoms. Although $L^2_w(\mathbb{R}^k)$ is a Hilbert space with respect to the inner product $\langle\cdot,\cdot\rangle_w$ (cf. Section 0.3), it is viewed as a Banach space and no attention is paid to its Hilbert space structure. The question of Hilbert space frames for $L^2_w(\mathbb{R}^k)$ is dealt with in Section 2.5. It will turn out that the crucial property of $L^2_w(\mathbb{R}^k)$ that enables this method to work is the fact that functions in $L^2_w(\mathbb{R}^k)$ are locally in $L^2(\mathbb{R}^k)$, specifically that $L^2_w(\mathbb{R}^k) = W(L^2(\mathbb{R}^k), L^2_w(\mathbb{R}^k))$ (cf. [F1]). Also, this method enables one to prove more general and different stability results than [FG1] in this case.

The paper is organized as follows. Chapter 0 contains notations and definitions used throughout the paper. Chapter 1 contains the definition and properties of the types of weights for which the results of the paper hold.

Chapter 2 contains the proofs of the existence of sets of W-H atoms for $L^2_w(\mathbb{R}^k)$. Section 2.1 shows that the spaces $L^2_w(\mathbb{R}^k)$ are actually coorbit spaces in the sense of [FG1] so that the existence of such expansions for some mother wavelets can be inferred from [FG1]. Section 2.2 gives basic results on the Wiener-type space which will turn out to be the reservoir of mother wavelets generating sets of W-H atoms for $L^2_w(\mathbb{R}^k)$. Section 2.3 gives the proof of the existence of these atoms for appropriate mother wavelets, and Section 2.4 does the same in certain Sobolev spaces. Section 2.5 compares the concepts of

Banach frames and Hilbert space frames of W-H wavelets for the Banach space $L^2_w(\mathbb{R}^k)$ and the Hilbert space $L^2_w(\mathbb{R}^k)$, respectively, and shows that they are not equivalent (cf. Section 0.5). Section 2.6 compares the notions of a Banach frame and a set of atoms for the space $L^2_w(\mathbb{R}^k)$.

Chapter 3 presents some of the stability results yielded by the methods of the previous chapter. Section 3.1 gives results on stability of sets of W-H atoms for $L^2_w(\mathbb{R}^k)$ under perturbation of the mother wavelet and of the lattice points. Section 3.2 says that, under a reasonable but strictly more general condition on the mother wavelets, the results of Section 2.3 can be modified to show that sets of W-H atoms exist for a "rectangle" of parameter values. This can be thought of as a result on stability under perturbation of the lattice parameters.

Chapter 4 presents stability results of a different kind. Given a set of W-H atoms, $\{E_{mb}T_{na}g\}$, for some $g \in L^2_w(\mathbb{R}^k)$ and a, b > 0, we can define the operator U formally by

$$Uf = \sum_n \sum_m \langle f, E_{mb}T_{na}g\rangle E_{mb}T_{na}g.$$

If U defined in this way makes sense, and is invertible, we can write

$$f = U(U^{-1}f) = \sum_n \sum_m \langle f, E_{mb}T_{na}U^{-1}g\rangle E_{mb}T_{na}g.$$

Operators of the form

$$Sf = \sum_n \sum_m \langle f, E_{mb}T_{na}\varphi\rangle E_{mb}T_{na}\psi$$

for certain functions φ and ψ, are studied in Chapter 4, where is is shown that, for a large class of φ and ψ, and appropriate values of a and b, the operator S makes sense, is continuous, and is continuously invertible on many Banach spaces. The reason these can be thought of as stability results is that on certain spaces the operator S defined above is the identity operator when $\varphi = U^{-1}g$ and $\psi = g$. To say that U (or more generally S) is a continuously invertible operator on many Banach spaces is to say that in the W-H expansion of a function, the function "inside" the inner product has many of the same properties (decay, smoothness, etc.) as the function "outside" the inner product. Also, this shows that one can obtain W-H expansions of distributions in a large variety of Banach spaces, though in most cases, one cannot conclude that $\{E_{mb}T_{na}g\}$ is a set of atoms for those spaces.

In Section 4.1, Banach spaces of functions on $\mathbb{R}^k$ defined by decay conditions, including the L^p spaces and the weighted L^p spaces are considered. In Section 4.2, the same is done for spaces defined by smoothness conditions, specifically Banach spaces of distributions defined by decay conditions on their Fourier transforms. Also, this section gives formulas for the derivative of the function Sf for appropriate f, and it is shown that if f, φ, and ψ are in $\mathcal{S}(\mathbb{R}^k)$ then so is Sf and that in fact S is a continuous operator from $\mathcal{S}(\mathbb{R}^k)$ into itself. Section 4.3 gives a formula for computing the operator S^{-1}, and Section 4.4 generalizes a result of Benedetto in [B], and show that on the spaces $L^p_w(\mathbb{R}^k)$, the continuously defined analogue of the operator S can be inverted.

Chapter 5 presents some results in the special case when w = 1, i.e., for the Hilbert space $L^2(\mathbb{R}^k)$. Section 5.1 presents two results on the general theory of frames in Hilbert spaces due to Gröchenig and Heil. These are included because they are used elsewhere in the paper. The remaining two sections give a closer examination of results in [D1], specifically a result on existence of frames in $L^2(\mathbb{R}^k)$, which is generalized slightly, and a result on phase-space localization, for which a more transparent hypothesis is given.

CHAPTER 0

NOTATION AND DEFINITIONS

Section 0.1. Basic symbols.

1. $\mathbb{C}$ denotes the complex numbers.

   If $z\in\mathbb{C}$, the modulus or absolute value of z is denoted by $|z|$.

   The complex conjugate of z is denoted by $\bar{z}$.

   $\mathbb{T}$ denotes the torus, the set of complex numbers of modulus 1.

2. If $k \geq 1$ is an integer, then

   $\mathbb{R}^k$ denotes k-dimensional Euclidean space,

   $\hat{\mathbb{R}}^k$ denotes the dual group of $\mathbb{R}^k$,

   $\mathbb{Z}^k$ denotes the set of k-tuples of integers.

   If $x\in\mathbb{R}^k$ and $x = (x_1, x_2, \ldots, x_k)$ then

$$|x|^2 = x_1^2 + x_2^2 + \cdots + x_k^2,$$

and

$$|x|_{max} = \max\{|x_j|: \quad j = 1, 2, \ldots, k\}.$$

Note that $|x|_{max} \leq |x| \leq k|x|_{max}$.

Given x and y in $\mathbb{R}^k$ with $x = (x_1, x_2, \ldots, x_k)$ and $y = (y_1, y_2, \ldots, y_k)$, the inner product of x and y is given by

$$\langle x, y\rangle = \sum_{j=1}^{k} x_j \overline{y_j}.$$

If $n\in\mathbb{Z}^k$ and $n = (n_1, n_2, \ldots, n_k)$ then

$$|n| = \max\{|n_j|: \quad j = 1, 2, \ldots, k\}.$$

A *multi-index* is an n-tuple of non-negative integers.

If α is a multi-index and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ then

$$|\alpha| = \sum_{j=1}^{k} \alpha_j,$$

$$\alpha! = \alpha_1! \alpha_2! \cdots \alpha_k!,$$

and if $x \in \mathbb{R}^k$,

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k}.$$

Given a differentiable function f, α a multiindex,

$$D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_{\alpha_1} \partial x_{\alpha_2} \cdots \partial x_{\alpha_k}}.$$

A rectangle $R \subset \mathbb{R}^k$ is a set of the form

$$[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_k, b_k]$$

where for each i, $b_i > a_i$. A rectangle is a cube if for all i and j, $|a_i - b_i| = |a_j - b_j|$. Given $a > 0$, the cube $[-a/2, a/2]^k$ is denoted Q_a.

Section 0.2. Summation and integration.

1. Lebesgue measure on $\mathbb{R}^k$ is denoted by dx.

Given a measurable set $E \subset \mathbb{R}^k$, the Lebesgue measure of E is denoted by $|E|$.

Given a measurable set $E \subset \mathbb{R}^k$, the characteristic function of E is denoted by 1_E.

Unless otherwise specified, all integrals will be over $\mathbb{R}^k$.

2. Unless otherwise specified, a series of the form

$$\sum_n c_n$$

with $c_n \in \mathbb{C}$ will mean the sum over $n \in \mathbb{Z}^k$.

3. Given $1 \le p < \infty$, and a sequence (ω_n) of positive numbers, we define the Banach space $\ell^p_\omega(\mathbb{Z}^k)$ as the space of all sequences (c_n) of numbers such that

$$\sum_n |c_n|^p \omega_n < \infty$$

with norm given by

$$\|(c_n)\|_{p,\omega} = \left[\sum_n |c_n|^p \omega_n\right]^{1/p}.$$

4. A series of numbers,

$$\sum_n c_n,$$

is said to converge to a number c if given $\varepsilon > 0$, there is a finite set $F \subset \mathbb{Z}^k$ such that for all finite sets G containing F,

$$\left|c - \sum_{n \in G} c_n\right| < \varepsilon.$$

5. A series of numbers is said to be Cauchy if given $\varepsilon > 0$, there is a finite set $F \subset \mathbb{Z}^k$ such that for any finite set $G \subset \mathbb{Z}^k$,

$$\left| \sum_{n \in G \setminus F} c_n \right| < \varepsilon.$$

6. Unless otherwise specified, a sequence of the form (c_n) with $c_n \in \mathbb{C}$ will mean a sequence over $n \in \mathbb{Z}^k$.

7. Given a Banach space B with norm $\|\cdot\|_B$, a series of functions,

$$\sum_n f_n,$$

is said to converge to f in B if given $\varepsilon > 0$, there is a finite set $F \subset \mathbb{Z}^k$ such that for any finite set $G \subset \mathbb{Z}^k$ containing F,

$$\left\| f - \sum_{n \in G \setminus F} f_n \right\|_B < \varepsilon.$$

8. Given a Banach space B with norm $\|\cdot\|_B$, a series of functions is said to be Cauchy in B if given $\varepsilon > 0$, there is a finite set F $\subset \mathbb{Z}^k$ such that for all finite sets $G \subset \mathbb{Z}^k$,

$$\left\| \sum_{n \in G \setminus F} f_n \right\|_B < \varepsilon.$$

Section 0.3. Function spaces.

1. Given a measurable function f on $\mathbb{R}^k$, we define $f_-(x) = f(-x)$ for all $x \in \mathbb{R}^k$.

2. Given $1 \le p < \infty$, and a positive, locally integrable function w, we define the Banach space $L^p_w(\mathbb{R}^k)$ as the space of all measurable functions f such that

$$\int |f(x)|^p \, w(x) \, dx < \infty$$

with norm given by

$$\|f\|_{p,w} = \left(\int |f(x)|^p \, w(x) \, dx\right)^{1/p}.$$

If $p = \infty$, we define the Banach space $L^\infty_w(\mathbb{R}^k)$ as the space of all measurable functions f such that

$$\|f\|_{\infty,w} = \text{ess sup}\{|f(x)|w(x): \ x \in \mathbb{R}^k\} < \infty.$$

3. Given two measurable functions f and g, we define

$$\langle f, g \rangle = \int f(x)\bar{g}(x) \, dx$$

whenever the integral makes sense and if w is a locally integrable function, we define

$$\langle f, g \rangle_w = \int f(x)\bar{g}(x)w(x) \, dx$$

whenever the integral makes sense.

4. $C_0(\mathbb{R}^k)$ is the Banach space of continuous functions vanishing at infinity, equipped with the sup-norm topology.

5. $C_b(\mathbb{R}^k)$ is the Banach space of bounded, continuous functions equipped with the sup-norm topology.

6. Given a non-negative integer r, we define the space $C^r(\mathbb{R}^k)$ as the space of functions such that for every multiindex α, with $|\alpha| \le r$, $D^\alpha f$ exists and is continuous. We denote by $C_c^r(\mathbb{R}^k)$ the subspace of $C^r(\mathbb{R}^k)$ consisting of those elements of $C^r(\mathbb{R}^k)$ which have compact support.

7. We define the space $C^\infty(\mathbb{R}^k)$ as the space of functions possessing arbitrarily many continuous derivatives, and the space $C_c^\infty(\mathbb{R}^k)$ as the subspace of $C^\infty(\mathbb{R}^k)$ consisting of those elements of $C^\infty(\mathbb{R}^k)$ which have compact support.

8. $\mathcal{S}(\mathbb{R}^k)$ is the space of $C^\infty(\mathbb{R}^k)$ functions such that for every multiindex α and integer $n \ge 0$,

$$\sup\{|D^\alpha f(x)(1+|x|)^n|: \ x\in\mathbb{R}^k\} < \infty.$$

The dual of $\mathcal{S}(\mathbb{R}^k)$, the space of tempered distributions, is denoted $\mathcal{S}'(\mathbb{R}^k)$.

Section 0.4. The Fourier transform.

1. The Fourier transform of a function $f \in L^1(\mathbb{R}^k)$ is

$$\hat{f}(\gamma) = \int f(x)e^{-2\pi i\langle x,\gamma\rangle}\,dx.$$

for $\gamma \in \hat{\mathbb{R}}^k$.

2. The Fourier transform of a function $f \in L^2(\mathbb{R}^k)$ is

$$\hat{f}(\gamma) = \lim_{n\to\infty} \int_{Q_n} f(x)e^{-2\pi i\langle x,\gamma\rangle}\,dx$$

in $L^2(\mathbb{R}^k)$.

3. Given $a > 0$, the Fourier transform of an a-periodic function $f \in L^2(Q_a)$ is

$$\hat{f}(n) = \int_{Q_a} f(x)e^{-2\pi i\langle n,x\rangle}\,dx$$

for $n \in \mathbb{Z}^k$.

Section 0.5. Atoms and frames.

DEFINITION 0.5.1. Let B be a Banach space and denote by B_d an appropriate sequence space associated to B. A collection of vectors $\{g_i: i \in I\}$ in B is a *set of atoms* for B if there is a collection of linear functionals on B, called λ_i, such that

(1) each $f \in B$ can be written $f = \sum \lambda_i(f)\, g_i$, where the sum converges in B-norm, and

(2) there exist constants c_1, $c_2 > 0$ such that for all $f \in B$,

$$c_1 \|f\|_B \leq \|(\lambda_i(f))\|_{B_d} \leq c_2 \|f\|_B.$$

The smallest value of c_2 and the largest value of c_1 which work are the *atomic bounds* for $\{g_i\}$, c_1 being the *lower bound* and c_2, the *upper bound*.

DEFINITION 0.5.2. Let B, B_d be as in Definition 0.1. A collection of vectors $\{e_i: i \in I\}$ in B', the dual of B, is called a *Banach frame* for B if there exist constants d_1, $d_2 > 0$, such that for all $f \in B$,

$$d_1 \|f\|_B \leq \|(\langle f, e_i \rangle)\|_{B_d} \leq d_2 \|f\|_B.$$

The smallest value of d_2 and the largest value of d_1 which work are the *frame bounds* of $\{e_i\}$, d_1 being the *lower bound* and d_2, the *upper bound*.

DEFINITION 0.5.3 Let $\mathbf{H}$ be a Hilbert space and $\{x_n: n \in I\}$ a collection of vectors in $\mathbf{H}$. Then $\{x_n\}$ is a *frame* for $\mathbf{H}$ if there exist constants A, B > 0 such that for all $x \in \mathbf{H}$,

$$A\|x\|^2 \le \sum_n |\langle x, x_n \rangle|^2 \le B\|x\|^2.$$

The smallest value of A and the largest value of B which work are called the *frame bounds* of $\{x_n\}$.

DEFINITION 0.5.4. Let $\mathbf{H}$ be a Hilbert space, and $\{x_n: n \in I\}$ a collection of vectors in $\mathbf{H}$. Then $\{x_n\}$ is a *set of atoms* for $\mathbf{H}$ if the following conditions hold.

(1) there exist linear functionals $a_n: \mathbf{H} \longrightarrow \mathbb{C}$ for each $n \in I$ such that for each $x \in \mathbf{H}$,

$$x = \sum_n a_n(x) x_n$$

where the sum converges strongly in $\mathbf{H}$, and

(2) there exist constants A, B > 0 such that for all $x \in \mathbf{H}$,

$$A\|x\|^2 \le \sum_n |a_n(x)|^2 \le B\|x\|^2$$

The smallest value of A and the largest value of B which work are called the *atomic bounds* of $\{x_n\}$.

DEFINITION 0.5.5. Let $\{x_n: n \in I\}$ be a frame for a Hilbert space $\mathbf{H}$. Let $\{e_n\}$ be a collection of vectors such that for all $x \in \mathbf{H}$,

$$x = \sum_n \langle x, e_n \rangle x_n$$

where the sum converges strongly in $\mathbf{H}$. Then $\{e_n\}$ is said to be *dual* to $\{x_n\}$.

DEFINITION 0.5.6. Given u, $v \in \mathbb{R}^k$, and f a measurable function, we define the functions $E_u f$ and $T_v f$ on $\mathbb{R}^k$ by

$$E_u f(x) = e^{2\pi i \langle u,x\rangle} f(x),$$

$$T_v f(x) = f(x-v).$$

Also, given a locally integrable function w on $\mathbb{R}^k$ such that for each $v \in \mathbb{R}^k$ the operator T_v is bounded on $L^p_w(\mathbb{R}^k)$ for $1 \le p < \infty$, we define the function $\tilde{w}(v)$ by

$$\tilde{w}(v) = \|T_v\|_{L^2_w \to L^2_w}.$$

DEFINITION 0.5.7. Given a Banach space B of functions on $\mathbb{R}^k$, $g \in B$, and a, b > 0, we say that (g, a, b) *generates a Weyl-Heisenberg (W-H) frame* for B if the collection $\{E_{mb}T_{na}g\}$ is a Banach frame for B. Similarly, we say that (g, a, b) *generates a set of Weyl-Heisenberg (W-H) atoms* for B if $\{E_{mb}T_{na}g\}$ is a set of atoms for B. The numbers a and b are the *frame parameters*, a being the *translation parameter*, and b the *modulation parameter*. The collection $\{(na, mb): n, m \in \mathbb{Z}^k\}$ is the *translation-modulation lattice*. The vector g is the *analyzing vector* or *mother wavelet*.

REMARK 0.5.8. In the remainder of this paper, we will often use the terms "sets of atoms" and "atomic decompositions" to refer to expansions in the sense of Definition 0.5.1. It is important to distinguish between this type of decomposition and the type encountered for example in the theory of H^p spaces (cf. [C],

[La]). Atomic decompositions in the sense of Definition 0.5.1 are more like Fourier series expansions or some other kind of orthogonal expansion than those given in [C] and [La].

The important characteristics of atomic decompositions in the sense of Definition 0.5.1 are the following.

1. A set of atoms is a fixed collection of vectors in terms of which every other vector in the space can be expanded. That is, the atoms are independent of the vector being expanded.

2. The expansion coefficients depend linearly on the vector being expanded.

3. The sum expanding a given vector is required to converge in norm to the vector.

The corresponding characteristics of atomic decompositions in the sense of [C] and [La] are the following.

1. The atoms by means of which a distribution is expanded depend on the distribution.

2. The expansion coefficients do not necessarily depend linearly on the distrubution being expanded.

3. The expansion sum is only required to converge in the sense of distributions.

CHAPTER 1

Moderate Weights

The notion of a moderate weight was first defined by Feichtinger and Gröchenig in [FG1]. In this chapter, we prove some important and useful properties of these weights and of the submultiplicative functions associated to them.

In particular, we show that these weights are well suited for W-H expansions of functions in $L^2_w(\mathbb{R}^k)$ in that they lend themselves naturally to discretization, thereby allowing one to define the growth condition characterizing the expansion coefficients. The relevant property of the weights, property (4) of Theorem 1.1.6, can be stated as follows: The values of a moderate weight at any two points are comparable, with the constants of comparability depending not on the location of the points but only on the distance between them. Thus, given a partition of $\mathbb{R}^k$ into cubes of fixed size, w can be replaced by an equivalent weight which is constant on each element of the partition. Such a discrete-valued version of w is given in Definition 1.1.10.

The characterization of moderate weights as those weights for which $L^p_w(\mathbb{R}^k)$ is translation-invariant (cf. Theorem 1.1.6) is well-known to Feichtinger and Gröchenig (cf. [F2], [FG1]) but has not appeared in print.

Section 1.1. Properties of Moderate Weights.

DEFINITION 1.1.1. A function $w: \mathbb{R}^k \longrightarrow \mathbb{R}^+$ is called a *submultiplicative weight* provided that the following hold.

(1) $w(0) = 1$, and

(2) $w(x+y) \le w(x)w(y)$ for all $x, y \in \mathbb{R}^k$.

DEFINITION 1.1.2. A locally integrable function $w: \mathbb{R}^k \longrightarrow \mathbb{R}^+$ is called a *moderate weight* provided that there exists a submultiplicative weight m such that for all x, $y \in \mathbb{R}^k$,

$$w(x+y) \le m(x)w(y).$$

DEFINITION 1.1.3. Given a locally integrable function $w: \mathbb{R}^k \longrightarrow \mathbb{R}^+$ and $1 \le p < \infty$, we say that $L^p_w(\mathbb{R}^k)$ is *translation-invariant* if for each $a \in \mathbb{R}^k$, $T_a(L^p_w(\mathbb{R}^k)) \subset L^p_w(\mathbb{R}^k)$.

LEMMA 1.1.4. (cf. [K]) Let m be a submultiplicative weight. Then m is locally bounded.

PROOF. Suppose not then I claim that m would be unbounded in every neighborhood of 0. That is, suppose that for some $\varepsilon > 0$, $m(x) \le M$ on Q_ε for some $M < \infty$. Then by submultiplicativity, $m(x) \le M^2$ on $Q_\varepsilon + Q_\varepsilon = Q_{2\varepsilon}$, and in general, $m(x) \le M^{2^n}$ on $Q_{2^n\varepsilon}$. Since every compact set is contained in $Q_{2^n\varepsilon}$ for some n, m is locally bounded. Consequently, if m is not locally bounded, then m is unbounded on Q_ε for every $\varepsilon > 0$.

This implies that on a set of positive measure, m takes the value $+\infty$. To see this, fix $\varepsilon > 0$ so small that for every $y \in Q_\varepsilon$, $|Q_1 \Delta (Q_1 + y)| < 1/4$. Then given $N \in \mathbb{N}$, there exists $x_N \in Q_\varepsilon$ such that $m(x_N) \geq N^2$. Now, for each $x \in \mathbb{R}^k$, $N^2 \leq m(x_N) \leq m(x)m(x_N - x)$ which implies that for all $x \in \mathbb{R}^k$, either $m(x) \geq N$ or $m(x_N - x) \geq N$ (or both).

Let $A_N = \{x \in Q_1 : \ m(x) \geq N\}$. I claim that $|A_N| \geq 1/4$ for all N. Suppose not, then $|Q_1 \backslash A_N| \geq 3/4$. But if $x \in Q_1 \backslash A_N$ then $x \notin A_N$ so that $m(x) < N$. Thus, $m(x_N - x) \geq N$ and so $x_N - x \in A_N$ as long as $x_N - x \in Q_1$ so that $|A_N| \geq |[x_N - (Q_1 \backslash A_N)] \cap Q_1| \geq |Q_1 \backslash A_N| - |(Q_1 + x_N) \backslash Q_1|$ $\geq 3/4 - 1/4 = 1/2 > 1/4$. Now, the sequence $\{A_N\}_{N=1}^{\infty}$ is nested and each is contained in Q_1. Thus, if $A = \bigcap A_N$ then $|A| \geq 1/4$ and $m(x) = \infty$ on A. This is clearly impossible since m was assumed to be real-valued. Thus, m is bounded on compact sets.∎

PROPOSITION 1.1.5. Let w be a moderate weight, then w is locally bounded.

PROOF. This proof is almost identical to that of Lemma 1.1.4. Let $\varepsilon > 0$, and let $m(x) \leq \tilde{M}$ on Q_ε. Suppose that w is bounded by M on Q_ε. Then we have, as in the proof of Lemma 1.1.4, that for any $n \in \mathbb{N}$, $m(x) \leq \tilde{M}^{2^n}$ on $Q_{2^n\varepsilon}$ and hence that $w(x) \leq \tilde{M}^{2^{n-1}} (\tilde{M}M)^{2^{n-2}}$ on

$Q_{2^n\varepsilon}$. Thus, if w is not locally bounded, it must be unbounded in every neighborhood of 0.

Suppose this is the case. Let $\varepsilon > 0$ and let $m(x) \le M_0$ on $Q_{1+\varepsilon}$. Then given $B > 0$ there is x_0 with $|x_0| < \varepsilon$ and $w(x_0) \ge BM_0$. For each $x \in Q_1$, we have that $BM_0 \le w(x_0) \le m(x_0-x)w(x) \le M_0 w(x)$ since $x \in Q_1$ implies that $x_0 - x \in Q_{1+\varepsilon}$. Thus, $w(x) \ge B$ on Q_1. But since B was arbitrary, this means $w(x) = \infty$ on Q_1, clearly an impossibility. Thus, w must be locally bounded.∎

THEOREM 1.1.6. Let $w: \mathbb{R}^k \longrightarrow \mathbb{R}^+$ be a locally integrable function and let $1 \le p < \infty$. Then the following are equivalent.

(1) w is a moderate weight.

(2) $L^p_w(\mathbb{R}^k)$ is translation-invariant.

(3) For every compact set $K \subset \mathbb{R}^k$ there is a constant $C(K)$ such that

$$\sup_x \left[\sup_{y \in K} \frac{w(x+y)}{w(x)} \right] \le C(K) < \infty.$$

(4) For every $v > 0$ there exists a constant $B(v)$ such that

$$\sup_{x \in Q} w(x) \le B(v) \inf_{x \in Q} w(x)$$

for every cube $Q \subset \mathbb{R}^k$ with $|Q| = v$.

(5) For every cube $Q \subset \mathbb{R}^k$ centered at the origin, there exist constants $A_1(Q) > 0$ and $A_2(Q) < \infty$ such that for every $x \in \mathbb{R}^k$,

$$A_1(Q) \le \left[\frac{1}{w(x)} \int_{x+Q} w(t)\, dt \right] \le A_2(Q).$$

PROOF.

(1)$\Longrightarrow$(2) Suppose w is moderate. Then if $1 \le p < \infty$, $a \in \mathbb{R}^k$, and $f \in L^p_w(\mathbb{R}^k)$,

$$\|T_a f\|^p_{w,p} = \int |f(x-a)|^p w(x)\,dx = \int |f(x)|^p w(x+y)\,dx$$

$$\le m(y)\int |f(x)|^p w(x)\,dx = m(y)\|f\|^p_{w,p}.$$

Since m is finite-valued, $L^p_w(\mathbb{R}^k)$ is translation invariant.

(2)$\Longrightarrow$(1) First we show that if $L^p_w(\mathbb{R}^k)$ is translation-invariant, then $\|T_a\|_{L^p_w \to L^p_w}$ is finite for each $a \in \mathbb{R}^k$. To do this, we use the Closed Graph Theorem.

Let $a \in \mathbb{R}^k$, and suppose that $f_n \longrightarrow f$ in $L^p_w(\mathbb{R}^k)$ and that $T_a f_n \longrightarrow g$ in $L^p_w(\mathbb{R}^k)$. We will show that $T_a f = g$. Suppose not, then there exists a set E with $0 < |E| < \infty$, $E \subset K$ where K is some compact set in $\mathbb{R}^k$, and $m > 0$ such that $|g(x)-f(x-a)|^p > m$ for all $x \in E$. Since $f_n \longrightarrow f$ in $L^p_w(\mathbb{R}^k)$, $f_n w^{1/p} \longrightarrow f w^{1/p}$ in $L^p(\mathbb{R}^k)$ so that by passing to a subsequence, we may assume without loss of generality that $f_n w^{1/p} \longrightarrow f w^{1/p}$ almost everywhere and hence that $T_a f_n \longrightarrow T_a f$ almost everywhere.

Since $|E| < \infty$, Egoroff's Theorem implies that there is a set $A \subset E$ with $0 < |A|$ and such that $T_a f_n \longrightarrow T_a f$ uniformly on A. Now we have that

$$m\left[\int_A w(x)\,dx\right]^{1/p} \le \left[\int_A |g(x)-f(x-a)|^p w(x)\,dx\right]^{1/p}$$

$$\le \|g-T_a f_n\|_{L^p_w(A)} + \|T_a f_n - T_a f\|_{L^p_w(A)}$$

$$\leq \|g-T_af_n\|_{p,w} + \left[\int_A |T_af_n(x)-T_af(x)|^p w(x)\ dx\right]^{1/p}$$

$$\leq \|g-T_af_n\|_{p,w} + \sup_A |T_af_n(x)-T_af(x)| \left[\int_A w(x)\ dx\right]^{1/p} \longrightarrow 0$$

as $n\to\infty$. Since w is locally integrable, this contradicts the assumption that $m > 0$.

Now let $m(a) = \|T_a\|^p_{L^p_w\to L^p_w}$. Then

$$\int |f(x-a)|^p w(x)\ dx = \int |f(x)|^p w(x+a)\ dx \leq \int |f(x)|^p m(a)w(x)\ dx$$

for all $f\in L^p_w(\mathbb{R}^k)$. Thus, $w(x+a) \leq m(a)w(x)$ for all $a\in\mathbb{R}^k$ and almost every $x\in\mathbb{R}^k$. Also, since $T_0 = Id$ and since $T_{a+b} = T_aT_b$, m is a submultiplicative weight. Thus, w is moderate.

It should be noted here that in the above paragraphs, m was defined in a p-dependent way, when certainly m is independent of p. I claim that given $1 \leq p,q < \infty$, $\|T_a\|^p_{L^p_w\to L^p_w} = \|T_a\|^q_{L^q_w\to L^q_w}$. To see this, note that for all $f\in L^p_w(\mathbb{R}^k)$, $\|T_af\|^p_{p,w} \leq \|T_a\|^p_{L^p_w\to L^p_w}\|f\|^p_{p,w}$ and that $\|T_a\|^p_{L^p_w\to L^p_w}$ is the smallest such constant for which this holds uniformly in f. We also know that $\|T_af\|^p_{p,w} = \|T_af^{p/q}\|^q_{q,w} \leq$ $\|T_a\|^q_{L^q_w\to L^q_w}\|f^{p/q}\|^q_{q,w} = \|T_a\|^q_{L^q_w\to L^q_w}\|f\|^p_{p,w}$. Thus, $\|T_a\|^p_{L^p_w\to L^p_w} \leq$ $\|T_a\|^q_{L^q_w\to L^q_w}$. The same argument with p and q reversed gives finally that $\|T_a\|^p_{L^p_w\to L^p_w} = \|T_a\|^q_{L^q_w\to L^q_w}$.

(1)$\Longrightarrow$(3) First, by Lemma 1.1.4, we know that m is locally bounded.

Put $C(K) = \sup_K m(x)$. Then for all $y\in K$,

$$\frac{w(x+y)}{w(x)} \leq m(y) \leq C(K)$$

independently of $x\in\mathbb{R}^k$. Thus,

$$\sup_x\left\{\sup_{y\in K}\frac{w(x+y)}{w(x)}\right\} \le C(K) < \infty.$$

(3)$\Longrightarrow$(2) Let $a\in\mathbb{R}^k$ be contained in some compact set K. Then by (3), $w(x+a) \le C(K)w(x)$ for every $x\in\mathbb{R}^k$. Thus,

$$\int|f(x)|^p w(x+a)\,dx \le C(K)\int|f(x)|^p w(x)\,dx$$

and $L^p_w(\mathbb{R}^k)$ is translation invariant.

(3)$\Longrightarrow$(4) Suppose that w is a moderate weight and let $Q \subset \mathbb{R}^k$ be a cube with $|Q|=v$. Then $Q=x'+Q'$ where x' is some point in $\mathbb{R}^k$ and Q' is that unique cube of volume v such that $Q'=-Q'$, i.e. such that Q' is centered at the origin.

Since w is a moderate weight, there exists a constant $C(Q'')$ such that for every $x\in\mathbb{R}^k$, $w(x+y) \le C(Q'')w(x)$ for all $y\in Q''$. Also, we have that $w(x)=w(y+(x-y)) \le C(Q'')w(x-y)$ for all $y\in Q''$. Since $Q''=-Q''$, we may write the above as $w(x) \le C(Q'')w(x+y)$.

Certainly the above holds when $x=x'$. Fix y_1 and y_2 in Q''. Then

$$\frac{1}{w(x'+y_1)}\,w(x'+y_2) \le \frac{C(Q'')}{w(x')}\,C(Q'')w(x') = C(Q'')^2 < \infty.$$

Since $Q=x'+Q''$, we have that

$$\frac{\sup_{Q} w(x)}{\inf_{Q} w(x)} = \frac{\sup_{x'+Q''} w(x)}{\inf_{x'+Q''} w(x)} \le C(Q'')^2 = B(v)$$

since Q″ depends only on the volume of Q.

(4)$\Longrightarrow$(3) Let $K \subset \mathbb{R}^k$ be a fixed compact set. Then $K \subset Q$ for some cube Q and

$$\sup_x \left[\sup_{y\in K} \frac{w(x+y)}{w(x)}\right] \le \sup_x \left[\sup_{y\in Q} \frac{w(x+y)}{w(x)}\right]$$

$$= \sup_x \left[\frac{1}{w(x)} \sup_{y\in Q} w(x+y)\right] \le \sup_x \left[\sup_{t\in x+Q} \frac{1}{w(t)} \sup_{t\in x+Q} w(t)\right]$$

$$\le B(|Q|)$$

since $w(x) > 0$ for all $x\in\mathbb{R}^k$ implies that $\sup_{t\in E} \frac{1}{w(t)} = \frac{1}{\inf_{t\in E} w(t)}$ for any set E.

(4)$\Longrightarrow$(5) Let $Q \subset \mathbb{R}^k$ be a cube. Then

$$\int_{x+Q} w(t)\,dt \le |Q| \sup_{x+Q} w(t)$$

$$\le B(|Q|)|Q| \inf_{x+Q} w(t) \le B(|Q|)|Q|w(x)$$

for almost every $x\in\mathbb{R}^k$. Similarly,

$$\int_{x+Q} w(t)\,dt \ge |Q| \inf_{x+Q} w(t)$$

$$\ge |Q|B(|Q|)^{-1} \sup_{x+Q} w(t) \ge |Q|B(|Q|)^{-1} w(x)$$

for almost every $x\in\mathbb{R}^k$.

(5)$\Longrightarrow$(4) Let Q be a cube in $\mathbb{R}^k$ centered at the origin and let $x \in \mathbb{R}^k$ be fixed. Let $Q' = Q+Q$, and $Q'' = Q'+Q'$. Then, given $y \in x+Q$, $x+Q \subset y+Q' \subset x+Q''$ and $y+Q \subset x+Q' \subset y+Q''$. Now,

$$\frac{1}{w(y)}\int_{x+Q'} w(t)\,dt \le \frac{1}{w(y)}\int_{y+Q''} w(t)\,dt \le \frac{A_2(Q'')}{A_1(Q')}\,A_1(Q')$$

$$\le \frac{A_2(Q'')}{A_1(Q')}\,\frac{1}{w(x)}\int_{x+Q'} w(t)\,dt.$$

Thus, for all $x \in \mathbb{R}^k$ and $y \in x+Q$,

$$(*) \qquad \frac{w(x)}{w(y)} \le \frac{A_2(Q'')}{A_1(Q')}.$$

Also,

$$\frac{1}{w(y)}\int_{x+Q'} w(t)\,dt \ge \frac{1}{w(y)}\int_{y+Q} w(t)\,dt \ge \frac{A_1(Q)}{A_2(Q')}\,A_2(Q')$$

$$\ge \frac{A_1(Q)}{A_2(Q')}\,\frac{1}{w(x)}\int_{x+Q'} w(t)\,dt.$$

Thus, for all $x \in \mathbb{R}^k$ and $y \in x+Q$,

$$(**) \qquad \frac{w(x)}{w(y)} \ge \frac{A_1(Q)}{A_2(Q')}.$$

Therefore, for all $y \in x+Q$,

$$w(x) \le \frac{A_2(Q'')}{A_1(Q')}\,w(y)$$

and so

$$w(x) \le \frac{A_2(Q'')}{A_1(Q')}\,\inf_{t \in x+Q} w(t)$$

by (*). Similarly, by (**),

$$w(x) \ge \frac{A_1(Q)}{A_2(Q')}\,\sup_{t \in x+Q} w(t)$$

which gives finally

$$\sup_{t \in x+Q} w(t) \le \frac{A_2(Q'')A_2(Q')}{A_1(Q')A_1(Q)}\,\inf_{t \in x+Q} w(t).$$

Since x was arbitrary, and the constant does not depend on x, we are done.∎

PROPOSITION 1.1.7. Let w be a moderate weight. Then there exist a continuous function w_0 and positive constants c_1 and c_2 such that

$$c_1 w_0(x) \le w(x) \le c_2 w_0(x)$$

for all $x \in \mathbb{R}^k$. Thus, $L^p_w(\mathbb{R}^k) = L^p_{w_0}(\mathbb{R}^k)$ for $1 \le p < \infty$.

PROOF. Let $k \in C_c(\mathbb{R}^k)$ with $\mathrm{supp}(k) \subset K$, for some cube K. Suppose that $\int k(x)\,dx = 1$ and that $k \ge 0$. Let

$$w_0(x) = \int w(y)k(x-y)\,dy = \int_{x+K} w(y)k(x-y).$$

Certainly, w_0 is continuous and

$$\inf_{y \in x+K} w(y) \le w_0(x) \le \sup_{y \in x+K} w(y).$$

By property (4) of moderate weights given in Theorem 1.1.6, we have that

$$1/B(|K|) \sup_{y \in x+K} w(y) \le w_0(x) \le B(|K|) \inf_{y \in x+K} w(y)$$

and finally that for every $x \in \mathbb{R}^k$,

$$1/B(|K|) w_0(x) \le w(x) \le B(|K|) w_0(x). \blacksquare$$

REMARK 1.1.8. Proposition 1.1.7 says that given a moderate weight w, there is a continuous function which defines the same weighted-L^p space as w.

PROPOSITION 1.1.9. Let w be moderate with associated function m. Then

(1) $m(x) \geq 1/w(0)\ w(x)$ for all $x \in \mathbb{R}^k$,

(2) $m_-(x) \geq w(0)\ 1/w(x)$ for all $x \in \mathbb{R}^k$.

PROOF. (1) follows immediately from the fact that $w(x+y) \leq m(y)w(x)$ and by putting $x = 0$. (2) follows from the fact that $w((x+y)-y) \leq m(-y)w(x+y)$ which implies that

$$\frac{1}{w(x+y)} \leq m(-y)\frac{1}{w(x)}$$

which gives the result when $x = 0$.■

DEFINITION 1.1.10. Given a cube $Q \subset \mathbb{R}^k$, $a > 0$, $n \in \mathbb{Z}^k$, we define $\omega(n;Q,a) = \inf_{x \in Q} w(x-na)$. We will denote $\omega(n;Q_1,a)$ by $\omega(n;a)$.

PROPOSITION 1.1.11. Let $a \in \mathbb{R}^k$, $Q_0 \subset Q \subset \mathbb{R}^k$ be cubes. Then there are constants c_1 and c_2 independent of n such that for all $n \in \mathbb{Z}^k$,

$$c_1\ \omega(n;Q,a) \leq \omega(n;Q_0,a) \leq c_2\ \omega(n;Q,a).$$

Moreover if Q and Q_0 are any two cubes in $\mathbb{R}^k$, then there are constants d_1 and d_2 such that for all $n \in \mathbb{Z}^k$,

$$d_1\ \omega(n;Q,a) \leq \omega(n;Q_0,a) \leq d_2\ \omega(n;Q,a).$$

PROOF. $\omega(n;Q,a) = \inf_{x \in Q} w(x-na) \leq \inf_{x \in Q_0} w(x-na) = \omega(n;Q_0,a)$, so that $c_1 = 1$. Now,

$$\begin{aligned} \inf_{x \in Q_0} w(x-na) &\leq \sup_{x \in Q} w(x-na) \\ &= \sup_{x \in Q+na} w(x) \\ &\leq B(|Q|) \inf_{x \in Q+na} w(x) = B(|Q|) \inf_{x \in Q} w(x-na). \end{aligned}$$

Thus, $c_2 = B(|Q|)$ and the first part of the conclusion holds.

Now, given arbitrary cubes Q, Q_0 in $\mathbb{R}^k$, we can certainly find a cube Q' such that $Q \subset Q'$ and $Q_0 \subset Q'$. From this the result follows with $d_1 = B(|Q'|)^{-1}$ and $d_2 = B(|Q'|)$.■

EXAMPLES 1.1.12.

(1) Any finite-valued, submultiplicative function is a moderate weight. In particular, a Beurling weight, i.e., a continuous submultiplicative function, is a moderate weight. For example, if $n \geq 0$, then $(1+|x|^2)^{n/2}$ is a Beurling weight and hence a moderate weight.

(2) For $x \in \mathbb{R}$, let $w(x) = e^x$. Then $w(0) = 1$, and $w(x+y) \leq w(x)w(y)$ for all $x, y \in \mathbb{R}$. Thus, w is moderate with itself as the associated submultiplicative function. This example shows that if m is a submultiplicative function associated to a moderate weight, then m need not be bounded away from zero.

(3) It is easy to see that condition (4) of Theorem 1.1.6 is symmetric in w and $1/w$. That is, if w is moderate then so is $1/w$. If m is a submultiplicative function associated to w then m_- is a submultiplicative function associated to $1/w$. To see this, note that for every x and y, $w(x) = w(-y+(x+y)) \leq m(-y)w(x+y)$, so that $1/w(x+y) \leq m_-(y)[1/w(x)]$.

We can now say that for every $n \in \mathbb{R}$, $(1+|x|^2)^{n/2}$ is moderate. In fact, the reciprocal of any finite valued (and non-zero) submultiplicative function is moderate.

(4) A finite-valued submultiplicative function need not be continuous. For example, let $w(0) = 1$, and $w(x) = 2$ if $x \neq 2$. Then w is submultiplicative, but discontinuous at 0. Consequently, a moderate weight need not be continuous.

CHAPTER 2

Sets of Atoms for $L^2_w(R^k)$.

This chapter is devoted to showing the existence of W-H atoms for $L^2_w(\mathbb{R}^k)$. That such atoms exist is shown in [F2] and [FG1], using different techniques in each case. The theory of Feichtinger and Gröchenig ([FG1], [FG2]) demonstrates in part the existence of such wavelet expansions in a large collection of Banach spaces. Also, Feichtinger's theory of Gabor-type decompositions of modulation spaces ([F2]), while only done for polynomial weights, goes through without modification for moderate weights.

The method used here is adapted from [F2] and exploits the local character of $L^2_w(\mathbb{R}^k)$, i.e., that a function in $L^2_w(\mathbb{R}^k)$ is locally in $L^2(\mathbb{R}^k)$, to obtain a larger class of mother wavelets for $L^2_w(\mathbb{R}^k)$ and also to give a more computationally explicit means of obtaining appropriate translation and modulation parameters for generating the atoms.

Also, as we shall see in Chapter 3, this technique can be used to prove stability results for W-H atoms which are stronger than similar results in [FG1].

In Section 2.1, we show directly that [FG1] can be applied to the case of $L^2_w(\mathbb{R}^k)$ by showing that $L^2_w(\mathbb{R}^k)$ is the coorbit space associated to a function space on the Heisenberg group.

The techniques of [F2] rely heavily on the theory of Wiener-type spaces developed in [F1], especially on the convolution relations between these spaces. This is necessary because most of the work there is done in the frequency domain. Since the spaces $L^2_w(\mathbb{R}^k)$ are defined by a local $L^2(\mathbb{R}^k)$ condition, we can do all of our work in the time domain. Consequently, we do not require all of the power of the theory of Wiener-type spaces. We do, however, require at least some definitions and basic properties of these spaces. These are given in Section 2.2.

Section 2.3 gives conditions on a function in a certain Wiener-type space which guarantee that, as a mother wavelet, it generates a set of W-H atoms for $L^2_w(\mathbb{R}^k)$. Section 2.4 shows that such expansions exist for functions in certain Sobolev spaces.

Related to the notion of a set of atoms for a Banach space is that of a Banach frame (cf. Section 0.5). In a Hilbert space, these notions are equivalent but the proof of the result breaks down for general Banach spaces (cf. [Gr1], Theorem 5.1.1). Also, in a Hilbert space, the coefficient functionals are unique (cf. [H], Theorem 5.1.6). Here, too, it is not clear how the result can be extended to general Banach spaces.

Of course, $L^2_w(\mathbb{R}^k)$ is a Hilbert space with respect to a weighted inner product. However, the notions of a set of atoms for the Hilbert space $L^2_w(\mathbb{R}^k)$ and that of a set of atoms for the Banach space $L^2_w(\mathbb{R}^k)$ are not the same. This is also true of the notions of a Hilbert frame and a Banach frame for $L^2_w(\mathbb{R}^k)$. The latter is the subject of Section 2.5.

In Section 2.6, we examine the relationship between sets of atoms for the Banach space $L^2_w(\mathbb{R}^k)$ and Banach frames for the same space.

Section 2.1. Coorbit spaces

In this section, we show that $L^2_w(\mathbb{R}^k)$ is a coorbit space in the sense of [FG1] and [FG2] whenever w is moderate. This implies the existence of W-H expansions of functions in $L^2_w(\mathbb{R}^k)$ by means of the Feichtinger/Gröchenig theory.

DEFINITION 2.1.1. The Heisenberg group, $\mathbb{H}$, is the set $\mathbb{T}\times\mathbb{R}^k\times\hat{\mathbb{R}}^k$ with the following group operation. Given (t_1,a_1,b_1), $(t_2,a_2,b_2)\in\mathbb{H}$,

$$(t_1,a_1,b_1)\cdot(t_2,a_2,b_2) = (t_1t_2e^{2\pi i b_1 a_2},a_1+a_2,b_1+b_2).$$

Also,

$$(t,a,b)^{-1} = (t^{-1}e^{2\pi i ab},-a,-b).$$

The identity element in $\mathbb{H}$ is $(1,0,0)$. $\mathbb{H}$ is topologized by the product topology on $\mathbb{T}\times\mathbb{R}^k\times\hat{\mathbb{R}}^k$.

The left-invariant Haar measure on $\mathbb{H}$ is denoted $d\mu$. The measure $d\mu$ is also given by the product measure on $\mathbb{T}\times\mathbb{R}^k\times\hat{\mathbb{R}}^k$, dt da db, where dt is normalized so that $\int_{\mathbb{T}}dt = 1$.

Given functions F and G on $\mathbb{H}$, we define the convolution of F and G by

$$F*G(x) = \int F(y^{-1}x)G(y)\, d\mu(y).$$

Given f, g in appropriate spaces, we define the function $V_g(f)$ on $\mathbb{H}$ by

$$V_g(f)(t,a,b) = t\langle f, T_bE_ag\rangle = te^{2\pi i\langle a,b\rangle}\int f(s)\bar{g}(s-b)e^{-2\pi i\langle b,s\rangle}\, ds.$$

The function $V_g(f)$ is referred to as the *voice transform* of f with respect to g (cf. [FG1]).

DEFINITION 2.1.2. Given a moderate weight, w, we define the corresponding weight w_3 on $\mathbb{H}$ by $w_3(t,a,b) = w(b)$.

Define $\tilde{w}_3(t,a,b) = \|T_{(t,a,b)}\|$ where T is the left-translation operator on the space $L^2_{w_3}(\mathbb{H})$. That is, if $x,y\in\mathbb{H}$, then $T_xF(y) = F(x^{-1}y)$.

DEFINITION 2.1.3. Given w moderate, we define $W(L^2(\mathbb{R}^k), L^2_w(\mathbb{R}^k))$ as the Banach space of functions, f, on $\mathbb{R}^k$ such that, for some fixed $k\in C_c(\mathbb{R}^k)$,

$$\|f\|_{W(L^2,L^2_w)} = \left(\int \|fT_bk\|^2_{\mathcal{F}L^2}\, w(b)\, db\right)^{1/2} < \infty$$

where $\|g\|_{\mathcal{F}L^2} = \|\hat{g}\|_2 = \|g\|_2$ (see [F1] for details).

LEMMA 2.1.4. Let w be a moderate weight. Then $\tilde{w}_3(t,a,b) = \tilde{w}(b)$ for almost every $b\in\mathbb{R}^k$ ($\tilde{w}$ is defined in Definition 0.5.7).

PROOF. Note first that for every $x \in \mathbb{R}^k$, $w(x+y) \le \tilde{w}^2(x)w(y)$. Let $F \in L^2_{w_3}(\mathbb{H})$ and let $(t_0, a_0, b_0) \in \mathbb{H}$. Then

$$\iiint |F(t-t_0, a-a_0, b-b_0)|^2 \; w_3(t,a,b) \; dt \; da \; db$$

$$= \iiint |F(t,a,b-b_0)|^2 \; w(b) \; dt \; da \; db$$

$$= \iiint |F(t,a,b)|^2 \; w(b+b_0) \; dt \; da \; db$$

$$\le \tilde{w}^2(b_0) \; \|F\|^2_{L^2_{w_3}(\mathbb{H})}.$$

Thus $\tilde{w}_3(t,a,b) \le \tilde{w}(b)$ almost everywhere. To see equality, let $\varepsilon > 0$ and $f \in L^2_w(\mathbb{R}^k)$ be such that $\|T_b f\|_{2,w} \ge (\tilde{w}(b)-\varepsilon) \; \|f\|_{2,w}$. Let $g \in L^2(\mathbb{R}^k)$ be such that $\|g\|_2 = 1$. Then defining $F(t,a,b) = f(b)g(a)$ gives that $\|F\|_{L^2_w(\mathbb{H})} = \|f\|_{2,w}$. Now,

$$\iiint |F(t-t_0, a-a_0, b-b_0)|^2 \; w(b) \; dt \; da \; db$$

$$= \int |g(a-a_0)|^2 \; da \int |f(b-b_0)|^2 \; w(b) \; db \ge (\tilde{w}(b)-\varepsilon)^2 \; \|F\|^2_{L^2_w(\mathbb{H})}.$$

Thus, $\tilde{w}_2(t,a,b) \ge \tilde{w}(b)-\varepsilon$ for almost every $b \in \mathbb{R}^k$ and all $\varepsilon > 0$. Hence, the lemma is proved.■

LEMMA 2.1.5. The space $L^1_{\tilde{w}_3}(\mathbb{H})$ is a Banach module over $L^2_{w_3}(\mathbb{H})$ with respect to convolution.

PROOF. Let $H \in L^2_{1/\tilde{w}_3}(\mathbb{H})$. Then

$$|\langle F*G, H\rangle| = \left|\iint F(y^{-1}x)G(y)H(x) \; d\mu(y) \; d\mu(x)\right|$$

$$\le \int |G(y)| \int |F(y^{-1}x)| \, |H(x)| \; d\mu(x) \; d\mu(y)$$

$$\le \|H\|_{2,1/\tilde{w}_3}\int |G(y)|\,\|T_yF\|_{2,\tilde{w}_3}\,d\mu(y)$$

$$\le \|H\|_{2,1/\tilde{w}_3}\|F\|_{2,\tilde{w}_3}\|G\|_{1,\tilde{w}_3}.$$

Since $\|F*G\|_{2,\tilde{w}_3} = \sup\{|<F*G,H>|: \ \|H\|_{2,1/\tilde{w}_3} = 1\}$, we are done.■

LEMMA 2.1.6. $W(\mathcal{F}L^2(\mathbb{R}^k), L^2_w(\mathbb{R}^k)) = Co(Y)$ where $Y = L^2_{w_3}(\mathbb{H})$. That is, given g such that $V_g(g)\in L^1_{\tilde{w}_3}(\mathbb{H})$, $f\in W(\mathcal{F}L^2(\mathbb{R}^k), L^2_w(\mathbb{R}^k))$ if and only if $V_g(f)\in Y$.

PROOF. First observe that

$$V_g(f)(t,a,b) = t<f, T_aE_bg> = te^{2\pi i<a,b>}\int f(s)\bar{g}(s-b)\ e^{-2\pi i<a,s>}ds$$

$$= te^{2\pi i<a,b>}(fT_b\bar{g})^\wedge(a).$$

Now, let $\bar{g}\in C_c(\mathbb{R}^k)$. Then $f\in W(\mathcal{F}L^2, L^2_w)$ if and only if

$$\|f\|_{W(L^2,L^2_w)} = \int \|fT_b\bar{g}\|^2_{\mathcal{F}L^2}\ w(b)\ db = \iint |(fT_b\bar{g})^\wedge(a)|^2\ w(b)\ da\ db$$

$$= \iiint |V_g(f)(t,a,b)|^2\ w(b)\ dt\ da\ db < \infty.$$

The Feichtinger/Gröchenig theory asserts that if the above holds for some g with $V_g(g)\in L^1_{\tilde{w}_3}(\mathbb{H})$, then it holds for all such g. Thus, we can extend to all g such that $V_g(g)\in L^1_{\tilde{w}_3}(\mathbb{H})$ since certainly, if $g\in C_c(\mathbb{R}^k)$ then $V_g(g)\in L^1_{\tilde{w}_3}(\mathbb{H})$.■

THEOREM 2.1.7. (Feichtinger/Gröchenig)
$L^2_w(\mathbb{R}^k) = Co(Y)$ where $Y = L^2_{w_3}(\mathbb{H})$.

PROOF. This is true since, by Plancherel's Theorem,

$$W(\mathcal{F}L^2(\mathbb{R}^k), L^2_w(\mathbb{R}^k)) = W(L^2(\mathbb{R}^k), L^2_w(\mathbb{R}^k)) = L^2_w(\mathbb{R}^k).■$$

Section 2.2. Lemmas on Wiener-type spaces

This section contains results on the compatibility of norms in Wiener-type spaces of the form $W(L^{\infty}, L^{1}_{w})$ where w is a moderate weight. The purpose of this section is to provide specific estimates for constants whose existence can be inferred directly from [F1]. These constants will play a role in the results which follow in this chapter.

DEFINITION 2.2.1. A partition, **P**, of $\mathbb{R}^k$ into a countable collection of closed rectangles with disjoint interiors, $\mathbf{P} = \{I_v\}_{v \in A}$, where A is some index set, is called a *bounded partition* if there exist numbers, r, R > 0 such that $0 < r \le \ell(I_v) \le m(I_v) \le R < \infty$, for all $v \in A$, where $\ell(I_v)$ is the length of the smallest side of I_v, and $m(I_v)$ is the length of the largest side of I_v. The numbers r and R are the *bounds* of **P**, r being the *lower bound* and R the *upper bound* of **P**. In particular, observe that $r^k \le |I_v| \le R^k$ for all $v \in A$.

DEFINITION 2.2.2. Given a function g, a moderate weight w, and a partition **P** of $\mathbb{R}^k$, we define the *Wiener space norm corresponding to w and* **P**, or just the *Wiener space norm corresponding to* **P** when w is understood, by

$$\|g\|_{w,P} = \sum_{v\in A} \|g1_{I_v}\|_\infty \ \omega(P,v)$$

where $\omega(P,v) = \inf_{x\in I_v} w(x)$.

If $d > 0$, and $P = \{Q_d+dn: n\in\mathbb{Z}^k\}$, then $\|\cdot\|_{w,P}$ is denoted $\|\cdot\|_{w,d}$. If $w = 1$, then $\|\cdot\|_{w,P}$ is denoted $\|\cdot\|_{\infty,1,d}$.

We define

$$W(L^\infty(\mathbb{R}^k), L^1_w(\mathbb{R}^k)) = \{f: \|f\|_{w,1} < \infty\}.$$

Since $\mathbb{R}^k$ is understood, we will write simply $W(L^\infty, L^1_w)$, and if $w = 1$, $W(L^\infty, L^1)$.

REMARK 2.2.3. Let us define, for P a bounded partition with bounds r and R, the norm

$$\|g\|^*_{w,P} = \sum_{v\in A} \|g1_{I_v}\|_\infty \ \omega^*(P,v)$$

where $\omega^*(P,v) = \sup_{x\in I_v} w(x)$. Since P is bounded, for each $v\in A$, I_v is contained in a cube Q_v such that $|Q_v| \le R^k$. Hence by Theorem 1.1.6(4), for all $v\in A$,

$$\omega(P,v) = \inf_{x\in I_v} w(x) \le \sup_{x\in I_v} w(x) = \omega^*(P,v)$$

and

$$\begin{aligned}\omega^*(P,v) \le \sup_{x\in Q_v} w(x) &\le B(R^k) \inf_{x\in Q_v} w(x) \\ &\le B(R^k) \inf_{x\in I_v} w(x) = B(R^k)\ \omega(P,v)\end{aligned}$$

and hence $\|\cdot\|_{w,P}$ is equivalent to $\|\cdot\|^*_{w,P}$. Actually, all that was required was the upper bound on P. It is clear from the above that we could replace $\omega(P,v)$ by $w(x_v)$ for any $x_v\in I_v$ and still define an equivalent norm.

LEMMA 2.2.4. Let $P_1 = \{I_v: v \in A\}$, $P_2 = \{L_m: m \in B\}$ be bounded partitions of $\mathbb{R}^k$ with bounds r_1, R_1, and r_2, R_2 respectively. Suppose that P_2 refines P_1. Then there exists a number $M > 0$ such that for all $j \in A$,

$$\#\{m \in B: I_v \cap L_m \neq \emptyset\} = \#\{m \in B: L_m \subset I_v\} \leq M < \infty.$$

PROOF. Since P_1 and P_2 are bounded partitions with the given bounds, $|I_v| \leq R_1^k$ and $|L_m| \leq r_2^k$ for all $v \in A$ and $m \in B$. Thus if $I_v = \bigcup_{i=1}^{N_v} L_{m_i}$ then since the sets L_{m_i} are pairwise disjoint almost everywhere, we have that

$$1 \leq N_v \leq (R_1/r_2)^k.$$

Putting $M = (R_1/r_2)^k$, we are done.∎

LEMMA 2.2.5. Let $P_1 = \{I_v: v \in A\}$, $P_2 = \{L_m: m \in B\}$ be two partitions of $\mathbb{R}^k$ into non-empty, closed rectangles with disjoint interiors which are not necessarily bounded and let w be a moderate weight. Suppose that P_2 refines P_1 and that for each $v \in A$, there exists a number M_v such that

$$\#\{m \in B: L_m \subset I_v\} \leq M_v < \infty$$

and such that

$$\sup_{v \in A} M_v = M < \infty.$$

Finally, suppose that P_1 has an upper bound in the sense of Definition 2.2.1, that bound being R. Then the two norms $\|\cdot\|_{w,P_1}$ and $\|\cdot\|_{w,P_2}$ are equivalent.

PROOF. Note first that, as in Remark 2.2.3, if $L_m \subset I_v$ then,

$$\omega(P_2, m) = \inf_{x \in L_m} w(x) \le \sup_{x \in I_v} w(x)$$
$$\le B(R^k) \inf_{x \in I_v} w(x) = B(R^k)\ \omega(P_1, v).$$

Now, given j, $I_v = \bigcup_{i=1}^{N_v} L_{m_i}$ where the collection of L_{m_i} are pairwise disjoint almost everywhere and $1 \le N_v \le M$. Now

$$\|g1_{I_v}\|_\infty\ \omega(P_1, v) = \|g1_{L_{m_i(v)}}\|_\infty\ \omega(P_1, v) \le \|g1_{L_{m_i(v)}}\|_\infty\ \omega(P_2, m_i(v))$$

for some $1 \le i \le N_v$. Also, because P_2 refines P_1, there is a one-to-one correspondence between v and $m_i(v)$. Thus

$$\sum_v \|g1_{I_v}\|_\infty\ \omega(P_1, v) \le \sum_v \|g1_{L_{m_i(v)}}\|_\infty\ \omega(P_2, m_i(v))$$
$$\le \sum_v \|g1_{L_m}\|_\infty\ \omega(P_2, m).$$

That is, $\|g\|_{w,P_1} \le \|g\|_{w,P_2}$.

Now,

$$\sum_m \|g1_{L_m}\|_\infty\ \omega(P_2, m) = \sum_v \sum_{\{m: L_m \subseteq I_v\}} \|g1_{L_m}\|_\infty\ \omega(P_2, m)$$
$$\le \sum_v \sum_{\{m: L_m \subseteq I_v\}} \|g1_{I_v}\|_\infty\ \omega(P_2, m) \qquad (\text{since } L_m \subset I_v)$$
$$\le \sum_v B(R^k) \sum_{\{m: L_m \subseteq I_v\}} \|g1_{I_v}\|_\infty\ \omega(P_1, v)$$
$$\le M\ B(R^k) \sum_v \|g1_{I_v}\|_\infty\ \omega(P_1, v).$$

That is, $\|g\|_{w,P_2} \le M\ B(R^k)\ \|g\|_{w,P_1}$. ■

LEMMA 2.2.6. Let $P_1 = \{I_v: v \in A\}$, $P_2 = \{L_m: m \in B\}$ be two bounded partitions of $\mathbb{R}^k$ with bounds r_1, R_1 and r_2, R_2 respectively and let w be a moderate weight. Then $\|\cdot\|_{w,P_1}$ is equivalent to $\|\cdot\|_{w,P_2}$.

PROOF. Consider the partition of $\mathbb{R}^k$ defined by $\mathbf{P}_3 = \{I_v \cap L_m: v\in A,\ m\in B\}$. Relabel the sets in $\mathbf{P}_3$ so that we may write $\mathbf{P}_3 = \{Q_s: s\in C\}$. Clearly, $\mathbf{P}_3$ refines both $\mathbf{P}_1$ and $\mathbf{P}_2$ and I claim that there is a number M such that for all $v\in A$ and $m\in B$,

$$\#\{s\in C: Q_s \subset I_v\} \leq M$$

and

$$\#\{s\in C: Q_s \subset L_m\} \leq M.$$

To see why this is true, let

$$I_v = \bigcup_{i=1}^{N_v} Q_{s_i} = \bigcup_{i=1}^{N_v} L_{m_i} \cap I_v.$$

Since $\ell(I_v) \leq R_1$ and $m(L_m) \geq r_2 > 0$ for all $v\in A$ and $m\in B$ any edge of I_v can pass through at most $\lfloor R_1/r_2\rfloor+2$ of the L_{m_i}. Thus

$$\#\{s\in C: Q_s \subset I_v\} = \#\{m\in B: L_m \cap I_v \neq \emptyset\} \leq \left(\lfloor R_1/r_2\rfloor+2\right)^k$$

independent of j. A similar calculation shows the same result for the L_m where the upper bound is $\left(\lfloor R_2/r_1\rfloor+2\right)^k$. Thus we let

$$M = \max\left\{\left(\lfloor R_1/r_2\rfloor+2\right)^k, \left(\lfloor R_2/r_1\rfloor+2\right)^k\right\}.$$

Also, since $\mathbf{P}_1$ and $\mathbf{P}_2$ are bounded, Lemma 2.2.5 implies that

$$\begin{aligned}\|g\|_{w,P_1} \leq \|g\|_{w,P_3} &\leq M\ B(R_2^k)\|g\|_{w,P_2} \\ &\leq M\ B(R_2^k)\|g\|_{w,P_3} \leq M^2 B(R_1^k)B(R_2^k)\|g\|_{w,P_1}.\end{aligned}$$

Hence the two norms are equivalent. ■

COROLLARY 2.2.7. Let c, d be positive numbers, and assume that d > c. Then for any function g,

$$\left(\lfloor d/c\rfloor+2\right)^{-k}\|g\|_{\infty,1,c} \le \|g\|_{\infty,1,d} \le 2^{k}\|g\|_{\infty,1,c}.$$

and

$$\left(\lfloor d/c\rfloor+2\right)^{-k}B(d^{k})^{-1}\|g\|_{\tilde{w},c} \le \|g\|_{\tilde{w},d} \le 2^{k}B(c^{k})\|g\|_{\tilde{w},c}.$$

PROOF. Consider the two partitions $P_1 = \{Q_c+nc\colon n\in\mathbb{Z}^k\}$ and $P_2 = \{Q_d+nd\colon n\in\mathbb{Z}^k\}$. It is easy to see that the largest number of elements of P_1 intersecting a given element of P_2 is $\left(\lfloor d/c\rfloor+2\right)^{k}$ and the largest number of elements of P_2 intersecting a given element of P_1 is 2^k. Thus the results follows from the arguments of Lemma 2.2.6.■

COROLLARY 2.2.8. Let d > 0, $a\in\mathbb{R}^k$, $g\in W(L^\infty,L^1)$, and $g\in W(L^\infty,L^1_w)$ for w a moderate weight. Then

$$\|T_a g\|_{\infty,1,d} \le 2^{k}\|g\|_{\infty,1,d}$$

and

$$\|T_a g\|_{w,d} \le 2^{k}B(d^{k})\|g\|_{w,d}.$$

PROOF. We are comparing the Wiener space norms corresponding to the partitions $P = \{Q_d+nd\colon n\in\mathbb{Z}^k\}$ and $P_a = \{Q_d+a+nd\colon n\in\mathbb{Z}^k\}$. It is easy to see that the largest number of elements of P intersecting a given element of P_a is 2^k and the largest number of elements of P_a intersecting a given element of P is also 2^k. Thus the results follow from the arguments of Lemma 2.2.6.■

PROPOSITION 2.2.9. Let $g_{-}\in W(L^{\infty}, L^{1}_{\tilde{w}})$. Then $g\in L^{2}_{1/w}(\mathbb{R}^{k})$.

PROOF. Note first that any function $h\in W(L^{\infty}, L^{1}_{\tilde{w}})$ if and only if $h\tilde{w}\in W(L^{\infty}, L^{1})$. Since $w(x+y) \leq \tilde{w}(x)^{2}w(y)$ for all x, $y\in\mathbb{R}^{k}$, Proposition 1.1.9 says that

$$\frac{1}{w(x)} \leq w(0)\tilde{w}_{-}^{2}(x).$$

Thus, $L^{2}_{\tilde{w}^{2}_{-}}(\mathbb{R}^{k}) \subset L^{2}_{1/w}(\mathbb{R}^{k})$.

I claim that $g\in L^{2}_{\tilde{w}^{2}_{-}}(\mathbb{R}^{k})$. Note that

$$\left(\int |g(x)|^{2}\tilde{w}_{-}^{2}(x)\ dx\right)^{1/2} = \left(\int |g_{-}\tilde{w}(x)|^{2}dx\right)^{1/2}$$

$$= \left(\int \sum_{n} |g_{-}\tilde{w}1_{Q_{1}+n}(x)|^{2}dx\right)^{1/2} \leq \sum_{n} \left(\int_{Q_{1}+n} |g_{-}\tilde{w}(x)|^{2}dx\right)^{1/2}$$

$$\leq \sum_{n} \|g_{-}\tilde{w}1_{Q_{1}+n}\|_{\infty} < \infty. \blacksquare$$

EXAMPLE 2.2.10.

(1) If there is a $C > 0$ such that for almost all $x\in\mathbb{R}^{k}$,

$$|g(x)| \leq C(1+|x|)^{k+1}$$

then $g\in W(L^{\infty}(\mathbb{R}^{k}), L^{1}(\mathbb{R}^{k}))$, $k \geq 1$.

(2) Let $w(x) = (1+|x|)^{n}$ for some integer $n \geq 0$. Let $g\in\mathcal{S}(\mathbb{R}^{k})$. Then for all multiindices α, $D^{\alpha}g\in W(L^{\infty}, L^{1}_{w})$. This is true because

$$\|D^{\alpha}g(x)(1+|x|)^{n}(1+|x|)^{k+1}\|_{\infty} < \infty,$$

so that

$$D^{\alpha}g(x)(1+|x|)^{n} \leq C(1+|x|)^{k+1}.$$

Thus, $(D^{\alpha}g)w\in W(L^{\infty}, L^{1})$, so that $D^{\alpha}g\in W(L^{\infty}, L^{1}_{w})$.

Since w is a Beurling weight, we can take

$$\tilde{w}(x) = (1+|x|)^{n}.$$

Thus, $D^{\alpha}g\in W(L^{\infty}, L^{1}_{\tilde{w}})$ for all α and n.

Section 2.3. Existence of atoms for $L^2_w(R^k)$

In this section, we present a two-step method for obtaining sets of W-H atoms for the space $L^2_w(\mathbb{R}^k)$, when w is moderate. First, we assume that the analyzing vector is bounded and compactly supported, and compute explicitly the coefficient functionals. Next, we extend the collection of possible analyzing vectors to a large class of functions which do not necessarily have compact support. We show how to determine the coefficient functionals in this case also.

The following two lemmas establish the existence of appropriate decompositions of $L^2_w(\mathbb{R}^k)$ when the analyzing vector has compact support.

LEMMA 2.3.1. Let φ be a compactly supported function, Q a cube with side b_0 and $\mathrm{supp}(\varphi) \subset Q$. Suppose that for some $a > 0$, there exist numbers A, B > 0 such that

$$A \le \sum_n |\varphi(x-na)|^2 \le B$$

for almost every $x \in \mathbb{R}^k$. If $0 < b \le b_0$ then there exist constants C_1, $C_2 > 0$, independent of b, such that for all $f \in L^2_w(\mathbb{R}^k)$,

$$b^{-k}C_1\|f\|^2_{2,w} \le \sum_n \sum_m |\langle f, E_{mb}T_{na}\varphi\rangle|^2 \, \omega(n;Q,a) \le b^{-k}C_2\|f\|^2_{2,w}.$$

PROOF.

$$\sum_n \sum_m |\langle f, E_{mb}T_{na}\varphi\rangle|^2 \, \omega(n;Q,a)$$

$$= \sum_n \omega(n;Q,a) \sum_m |\langle f\cdot T_{na}\varphi, E_{mb}\rangle|^2$$

$$= b^{-k} \sum_n \omega(n;Q,a) \int_{Q+na} |f(x)|^2 |\varphi(x-na)|^2 dx$$

$$= (*).$$

Now,

$$(*) \le b^{-k} \sum_n \int_{Q+na} |f(x)|^2 w(x) |\varphi(x-na)|^2 \, dx$$

$$= b^{-k} \int |f(x)|^2 w(x) \sum_n |\varphi(x-na)|^2 \, dx \le b^{-k} B \|f\|_{2,w}^2.$$

Since w is a moderate weight, by Theorem 1.1.6(4), we have that

$$(*) \ge b^{-k} B(|Q|)^{-1} \sum_n \int_{Q+na} |f(x)|^2 |\varphi(x-na)|^2 w(x) \, dx$$

$$\ge b^{-k} B(|Q|)^{-1} A \|f\|_{2,w}^2.$$

Since $B(|Q|)$ is independent of b, we are done.■

LEMMA 2.3.2. Let φ, g be bounded, compactly supported functions such that

(1) $\text{supp}(\varphi) \subset Q$, and $\text{supp}(g) \subset Q_0$ where Q and Q_0 are cubes, $Q \subset Q_0$, and Q_0 has side b_0, and

(2) for some $a > 0$, there exist numbers A_0, $B_0 > 0$ such that

$$A_0 \le \left| \sum_n g(x-na)\bar{\varphi}(x-na) \right| \le B_0.$$

Define the operator, S, by

$$Sf = \sum_n \sum_m \langle f, E_{mb}T_{na}\varphi\rangle \, E_{mb}T_{na}g.$$

Then the sum defining S converges strongly in $L_w^2(\mathbb{R}^k)$ and moreover,

$$Sf(x) = f(x) b^{-k} \sum_n g(x-na)\bar{\varphi}(x-na)$$

for all $0 < b < b_0$, S is a bijective homeomorphism of $L^2_w(\mathbb{R}^k)$ onto itself, and for all $f \in L^2_w(\mathbb{R}^k)$

$$f = \sum_n \sum_m \langle S^{-1}f, E_{mb}T_{na}\varphi\rangle\, E_{mb}T_{na}g.$$

PROOF. We will see in Lemma 2.3.7 that the sum defining S converges strongly since $g \in W(L^\infty, L^1_\omega)$ and

$$\sum_n \sum_m |\langle f, E_{mb}T_{na}\varphi\rangle|^2\, \omega(n;Q,a) < \infty,$$

where $\mathrm{supp}(\varphi) \subset Q$. Since the sum defining S converges strongly, it converges as an iterated sum in $L^2_w(\mathbb{R}^k)$. Since g is bounded and compactly supported, $T_{na}g\cdot\sum_m \langle fT_{na}\bar{\varphi}, E_{mb}\rangle E_{mb}$ converges strongly in $L^2_w(\mathbb{R}^k)$, provided $0 < b \le b_0$, to $b^{-k}fT_{na}(g\bar{\varphi})$. Specifically, since $f \in L^2_w(\mathbb{R}^k)$, $fT_{na}\bar{\varphi} \in L^2(Q_0)$. Thus, $\sum_m \langle fT_{na}\bar{\varphi}, E_{mb}\rangle E_{mb}$ converges in $L^2(Q_0)$ to $b^{-k}fT_{na}\bar{\varphi}$. Since $\mathrm{supp}(T_{na}g)$ is compact,

$$T_{na}g \sum_m \langle fT_{na}\bar{\varphi}, E_{mb}\rangle E_{mb}$$

converges in $L^2(\mathbb{R}^k)$ and $L^2_w(\mathbb{R}^k)$ to $b^{-k}fT_{na}(g\bar{\varphi})$.

Since the series $\sum g(x-na)\bar{\varphi}(x-na)$ converges uniformly on compact sets, we have that $\sum fT_{na}(g\bar{\varphi})$ converges strongly in $L^2_w(\mathbb{R}^k)$. Specifically, since $f \in L^2_w(\mathbb{R}^k)$, for all $\varepsilon > 0$, there exists $R > 0$ such that

$$\int_{|x|\ge R} |f(x)|^2 w(x)\, dx < (\varepsilon/B_0)^2.$$

Also, there exists $N > 0$ such that if $|n| \ge N$ then

$$\mathrm{supp}\Big[g(x-na)\bar{\varphi}(x-na)\Big] \subset \Big\{x:\ |x| > R\Big\}.$$

Thus,

$$\left\| \sum_{|n|\geq N} f(x)g(x-na)\bar{\varphi}(x-na) \right\|_{2,w}^2$$

$$= \int |f(x)|^2 w(x) \left| \sum_{|n|\geq N} g(x-na)\bar{\varphi}(x-na) \right| dx$$

$$\leq \left\| \sum_{|n|\geq N} g(x-na)\bar{\varphi}(x-na) \right\|_{\infty}^2 \int_{|x|\geq R} |f(x)|^2 w(x)\ dx < \varepsilon^2$$

Therefore, $\sum_n fT_{na}gT_{na}\bar{\varphi}$ converges strongly in $L^2_w(\mathbb{R}^k)$.

Since S is given by multiplication by a function bounded above and below, it is a continuous map from $L^2_w(\mathbb{R}^k)$ onto itself, and has a continuous inverse. Therefore, we have that

$$f = S(S^{-1}f) = \sum_n \sum_m \langle S^{-1}f, E_{mb}T_{na}\varphi\rangle\ E_{mb}T_{na}g. \blacksquare$$

COROLLARY 2.3.3. Let φ, g, satisfy the hypotheses of Lemma 2.3.1 for some $a > 0$. Then for any $0 < b \leq b_0$, there exists a collection of continuous linear functionals, $a_{n,m}$: $L^2_w(\mathbb{R}^k) \longrightarrow \mathbb{C}$ such that for all $f \in L^2_w(\mathbb{R}^k)$,

$$f = \sum_n \sum_m a_{n,m}(f)\ E_{mb}T_{na}g$$

strongly in $L^2_w(\mathbb{R}^k)$ and there exist constants C_1, $C_2 > 0$, independent of b, such that

$$b^k C_1 \|f\|_{2,w}^2 \leq \sum_n \sum_m |a_{n,m}(f)|^2 \omega(n;Q,a) \leq b^k C_2 \|f\|_{2,w}^2.$$

PROOF. By Lemma 2.3.1 we can let $a_{n,m}(f) = \langle S^{-1}f, E_{mb}T_{na}\varphi\rangle$ and get that

$$f = \sum_n \sum_m a_{n,m}(f)\ E_{mb}T_{na}g$$

strongly. Letting $C=B(|Q|)$, Lemma 2.3.1 says that

$$b^{-k}AC^{-1}\|S^{-1}f\|_{2,w}^2$$

$$\leq \sum_n \sum_m |\langle S^{-1}f, E_{mb}T_{na}\varphi\rangle|^2 \ \omega(n;Q,a) \ \leq \ b^{-k}B\|S^{-1}f\|_{2,w}^2.$$

Now,

$$\|S^{-1}\| \ \leq \ (b^k/A) \text{ and } \|S^{-1}\| \ \geq \ (b^k/B),$$

so,

$$b^k(A/CB^2) \ \|f\|_{2,w}^2 \ \leq \ \sum_n \sum_m |a_{n,m}(f)|^2 \ \omega(n;Q,a) \ \leq \ b^k(B/A^2) \ \|f\|_{2,w}^2. \blacksquare$$

REMARK 2.3.4. The reason for using two functions, φ and g to define the decompositions is to get atomic decomposition constants, called C_1 and C_2 in Corollary 2.3.3, which do not depend on b.

The only requirement of g, besides that it be bounded and compactly supported, is that $\sum_n g(x-na)\overline{\varphi}(x-na)$ be bounded above and below. This condition depends only on the values of g on the support of φ. Thus, we can alter g arbitrarily off the support of φ, provided we keep it compactly supported and bounded, and still infer the existence of appropriate coefficient functionals as in Corollary 2.3.3 (cf. Example 5.1.3).

The following results establish the existence of a large class of mother wavelets for $L^2_w(\mathbb{R}^k)$.

LEMMA 2.3.5. Let g be such that g, $g_{\sim} \in W(L^\infty, L^1_{\tilde{\omega}})$. Suppose that for some a > 0, there exist constants A, B > 0 such that for almost every $x \in \mathbb{R}^k$,

$$A \leq \sum_n |g(x-na)|^2 \leq B.$$

Then there exists a cube Q and constants A', B' > 0, depending on Q, such that for every cube Q_0 containing Q, and almost every $x \in \mathbb{R}^k$,

$$A' \leq \left| \sum_n \bar{g}1_Q(x-na) g1_{Q_0}(x-na) \right| \leq B'.$$

In particular,

$$A' \leq \sum_n |g1_Q(x-na)|^2 \leq B'.$$

PROOF. Observe first that $g \in W(L^\infty, L^1_{\tilde{w}})$ if and only if $g\tilde{w} \in W(L^\infty, L^1)$.

Claim: Let h be any function. Then

$$\left(\sum_n |h(x-na)|^2 \right)^{1/2} \leq \|h\|_{\infty,1,a}$$

for almost every x in $\mathbb{R}^k$.

Proof of claim:

$$\operatorname{ess\,sup}_x \left(\sum_n |h(x-na)|^2 \right)^{1/2} \leq \left(\sum_n \operatorname{ess\,sup}_{x \in Q_a} |h(x-na)|^2 \right)^{1/2}$$

$$\leq \sum_n \|h1_{Q_a+na}\|_\infty = \|h\|_{\infty,1,a}. \square$$

If we now write $g(x) = g_1(x)+h_1(x) = g_2(x)+h_2(x)$, then

$$\sum_n |g(x-na)| = \left| \sum_n g(x-na)\bar{g}(x-na) \right|$$

$$= \left| \sum_n [g_1(x-na)+h_1(x-na)][\overline{g_2}(x-na)+\overline{h_2}(x-na)] \right|$$

$$= \left| \sum_n g_1(x-na)\overline{g_2}(x-na) + \sum_n h_1(x-na)\overline{g_2}(x-na) \right.$$

$$\left. + \sum_n g_2(x-na)\overline{h_2}(x-na) + \sum_n h_1(x-na)\overline{h_2}(x-na) \right|$$

$$= (*).$$

Now, since for all x, $1 = \widetilde{w}(0) \le \widetilde{w}(x)\widetilde{w}(-x)$, we have that

$$(*) \le \left|\sum_n g_1(x-na)\overline{g_2}(x-na)\right|$$

$$+\left(\left|\sum_n h_1(x-na)\overline{g_2}(x-na)\right|+\left|\sum_n g_1(x-na)\overline{h_2}(x-na)\right|\right.$$

$$\left.+\left|\sum_n h_1(x-na)\overline{h_2}(x-na)\right|\right)$$

$$\le \left|\sum_n g_1(x-na)\overline{g_2}(x-na)\right|$$

$$+\sum_n |h_1(x-na)|\widetilde{w}(x-na)|g_{2-}(na-x)|\widetilde{w}(na-x)$$

$$+\sum_n |g_1(x-na)|\widetilde{w}(x-na)|h_{2-}(na-x)|\widetilde{w}(na-x)$$

$$+\sum_n |h_1(x-na)|\widetilde{w}(x-na)|h_{2-}(na-x)|\widetilde{w}(na-x)$$

$$= \left|\sum_n g_1(x-na)\overline{g_2}(x-na)\right| + \sum_n |h_1\widetilde{w}(x-na)|\,|g_{2-}\widetilde{w}(na-x)|$$

$$+\sum_n |g_1\widetilde{w}(x-na)|\,|h_{2-}\widetilde{w}(na-x)| + \sum_n |h_1\widetilde{w}(x-na)|\,|h_{2-}\widetilde{w}(na-x)|$$

$$\le \left|\sum_n g_1(x-na)\overline{g_2}(x-na)\right| + \Big(\|h_1\widetilde{w}\|_{\infty,1,a}\|g_{2-}\widetilde{w}\|_{\infty,1,a}$$

$$+\|g_1\widetilde{w}\|_{\infty,1,a}\|h_{2-}\widetilde{w}\|_{\infty,1,a}+\|h_1\widetilde{w}\|_{\infty,1,a}\|h_{2-}\widetilde{w}\|_{\setminus,1,a}\Big).$$

Similarly,

$$(*) \ge \left|\sum_n g_1(x-na)\overline{g_2}(x-na)\right| - \Big(\|h_1\widetilde{w}\|_{\infty,1,a}\|g_{2-}\widetilde{w}\|_{\infty,1,a}$$

$$+\|g_1\widetilde{w}\|_{\infty,1,a}\|h_{2-}\widetilde{w}\|_{\infty,1,a}+\|h_1\widetilde{w}\|_{\infty,1,a}\|h_{2-}\widetilde{w}\|_{\infty,1,a}\Big).$$

Now, given $\varepsilon > 0$, there exists a cube Q such that

$$\left\|g\widetilde{w}-g1_{Q_0}\widetilde{w}\right\|_{\infty,1,a} < \varepsilon, \text{ and } \left\|g_-\widetilde{w}-g_-1_{Q_0}\widetilde{w}\right\|_{\infty,1,a} < \varepsilon$$

for all cubes Q_0 containing Q. Let ε be so small that $\varepsilon(\|g\widetilde{w}\|_{\infty,1,a}+\|g_-\widetilde{w}\|_{\infty,1,a})+\varepsilon^2 < A/2$ and choose Q corresponding to this ε. Let Q_0 be any cube such that $Q \subset Q_0$. Let $g_1 = g1_Q$, $h_1 =$

$g - g1_Q$, $g_2 = g1_{Q_0}$, and $h_2 = g - g1_{Q_0}$. Then $\|h_1\tilde{w}\|_{\infty,1,a} < \varepsilon$, $\|h_{2}\tilde{w}\|_{\infty,1,a} < \varepsilon$, $\|g_1\tilde{w}\|_{\infty,1,a} \le \|g\tilde{w}\|_{\infty,1,a}$, and $\|g_{2}\tilde{w}\|_{\infty,1,a} \le \|g\tilde{w}\|_{\infty,1,a}$. Combining this with the above inequalities gives that for almost every $x\in\mathbb{R}^k$,

$$0 < A/2 \le \left| \sum_n \bar{g}1_Q(x-na)g1_{Q_0}(x-na)\right| \le B+A/2 < \infty. \blacksquare$$

The functions $g1_Q$ and $g1_{Q_0}$ will play the roles of φ and g respectively in Lemmas 2.3.1 and 2.3.2.

LEMMA 2.3.6. Let $\{h_n: n\in\mathbb{Z}^k\}$ be such that $h_n\in L^2(\mathbb{R}^k)$, and $\mathrm{supp}(h_n) \subset Q+nc$ for each $n\in\mathbb{Z}^k$ where Q is a cube of side $d > 0$. If $c \ge d > 0$, then

$$\left\| \sum_n h_n\right\|_{2,w} \le B(d^k)^{1/2}\left(\sum_n \|h_n\|_2^2\ \omega(n;Q,c)\right)^{1/2}.$$

If $0 < c < d$, then

$$\left\| \sum_n h_n\right\|_{2,w} \le [B(d^k)3^kc^{-k}d^k]^{1/2}\left(\sum_n \|h_n\|_2^2\ \omega(n;Q,c)\right)^{1/2}.$$

PROOF. If $c \ge d > 0$, then the supports of the h_n are all disjoint and

$$\int\left| \sum_n h_n(x)\right|^2 w(x)\ dx = \sum_n \int |h_n(x)|^2\ w(x)\ dx$$

$$\le B(|Q_d|) \sum_n \int |h_n(x)|^2\ dx\ \omega(n;Q,c)$$

$$= B(d^k) \sum_n \|h_n\|_2^2\ \omega(n;Q,c).$$

If $c < d$ then let $R = \lfloor d/c\rfloor+2$. then given $n\in\mathbb{Z}^k$, $n \ne 0$,

$$Q \cap (Q+cRn) = \varnothing.$$

Thus we may partition $\mathbb{Z}^k$ into R^k disjoint pieces. That is, we may write $\mathbb{Z}^k = \bigcup_{i=1}^{R^k} I_i$ where

$$n, j \in I_i \Rightarrow (Q+cn) \cap (Q+jn) = \emptyset \text{ if } n \neq j.$$

More specifically, for each multi-index $j = (j_1, \ldots, j_k)$ with $j_m \in \mathbb{Z}$ and $0 \leq j_m \leq R-1$ for $m=1,2,\ldots,k$, set $I_j = \{j+Rl\colon\ l \in \mathbb{Z}^k\}$.

Relabeling the I_j gives us the partition we want. Since for any numbers $a_i \in \mathbb{C}$,

$$\left(\sum_{i=1}^{n} |a_i|\right)^2 \leq n \sum_{i=1}^{n} |a_i|^2,$$

we have that

$$\begin{aligned}
\int \Big| \sum_n h_n(x) \Big|^2 w(x)\, dx &\leq \int \left(\sum_{r=1}^{R^k} \sum_{n \in I_r} |h_n(x)| \right)^2 w(x)\, dx \\
&\leq R^k \int \sum_{r=1}^{R^k} \left(\sum_{n \in I_r} |h_n(x)| \right)^2 w(x)\, dx = R^k \int \sum_{r=1}^{R^k} \sum_{n \in I_r} |h_n(x)|^2 w(x)\, dx \\
&= R^k \sum_n \int |h_n(x)|^2 w(x)\, dx \leq R^k B(|Q_d|) \sum_n \int |h_n(x)|^2\, dx\ \omega(n;Q,c) \\
&\leq 3^k c^{-k} d^k B(d^k) \sum_n \|h_n\|_2^2\ \omega(n;Q,c). \blacksquare
\end{aligned}$$

LEMMA 2.3.7. Let $(a_{n,m})$ be a sequence of numbers only finitely many of which are non-zero, $a,\ b > 0$, and $g \in W(L^\infty, L^1_{\tilde{w}})$. Then for any $0 < d \leq 1/b$, and cube Q of side d,

$$\begin{aligned}
&\Big\| \sum_n \sum_m a_{n,m}\, T_{na} E_{mb} g \Big\|_{2,w} \\
&\leq [B(d^k) 3^k a^{-k} d^{-k}]^{1/2} \left(\sum_\nu \|g 1_{Q+d\nu}\|_\infty\ \tilde{w}(d\nu) \right) \\
&\qquad\qquad b^{-k/2} \left(\sum_n \sum_m |a_{n,m}|^2\ \omega(n;Q,a) \right)^{1/2}
\end{aligned}$$

where $B(d^k)$ is the constant defined in Theorem 1.1.6(4) and hence is independent of b.

PROOF. We can write

$$g = \sum_{\nu} T_{d\nu}(T_{-d\nu} g 1_{Q+d\nu}) = \sum_{\nu} T_{d\nu} g_{\nu}$$

where g_ν is supported in Q for all ν and where $\|g_\nu\|_\infty = \|g 1_{Q+d\nu}\|_\infty$.

Thus,

$$\Big\| \sum_n \sum_m a_{n,m} \; T_{na}E_{mb} \Big\|_{2,w}$$

$$= \Big\| \sum_\nu T_{d\nu}\Big[\sum_n T_{na}\Big(\sum_m a_{n,m} \; E_{mb} g_\nu \Big)\Big] \Big\|_{2,w}$$

$$\le \sum_\nu \tilde{w}(d\nu) \Big\| \sum_n T_{na}\Big(\sum_m a_{n,m} \; E_{mb} g_\nu \Big) \Big\|_{2,w}$$

$$\le \sum_\nu \tilde{w}(d\nu) \; 3^{k/2} a^{-k/2} d^{k/2} B(d^k) \Big[\sum_n \Big\| \sum_m a_{n,m} \; E_{mb} g_\nu \Big\|_2^2 \; \omega(n;Q,a) \Big]^{1/2}$$

$= (*)$

by Lemma 2.3.6 since

$$\operatorname{supp}\Big(\sum_m a_{n,m} E_{mb} g_\nu \Big) \subset Q$$

for all n in $\mathbb{Z}^k$. Now, since the side length of Q is $\le 1/b$ and $\operatorname{supp}(g_\nu) \subset Q$,

$$\Big\| \sum_m a_{n,m} E_{mb} g_\nu \Big\|_2^2 = \Big\| g_\nu \sum_m a_{n,m} E_{mb} \Big\|_{L^2(Q)}^2 \le \|g_\nu\|_\infty^2 \; b^{-k} \sum_m |a_{n,m}|^2.$$

Thus,

$$(*) \le \sum_\nu \tilde{w}(d\nu) \|g_\nu\|_\infty \; 3^{k/2} a^{-k/2} d^{k/2} B(d^k)$$

$$b^{-k/2} \Big[\sum_n \sum_m |a_{n,m}|^2 \; \omega(n;Q,a) \Big]^{1/2}$$

$$\le [B(d^k) 3^k a^{-k} d^{-k}]^{1/2} \Big(\sum_\nu \|g 1_{Q+d\nu}\|_\infty \; \tilde{w}(d\nu) \Big)$$

$$b^{-k/2} \Big[\sum_n \sum_m |a_{n,m}|^2 \; \omega(n;Q,a) \Big]^{1/2} . \blacksquare$$

COROLLARY 2.3.8. Let $\omega(n;a) \equiv \omega(n;Q_1,a)$ and denote by $\tilde{B}(v)$ the constant corresponding to the moderate weight $\tilde{w}$ as in Theorem 1.1.6(4). Then for any sequence of numbers $(a_{n,m})$, with the property that

$$\sum_{n,m} |a_{n,m}|^2 \ \omega(n;a) \le \infty,$$

there is a constant C independent of g and b, for $b \le 1$, such that

$$\Big\| \sum_n \sum_m a_{n,m} \ T_{na}E_{mb}g \Big\|_{2,w} \le C\|g\|_{\tilde{w},1} b^{-k/2}\Big[\sum_n \sum_m |a_{n,m}|^2 \ \omega(n;a) \Big]^{1/2}$$

and the sum on the left side converges strongly in $L^2_w(\mathbb{R}^k)$.

PROOF. Suppose first that only finitely many of the $a_{n,m}$ are non-zero. By Proposition 1.1.11, if we let $C_1 = B(|Q_\alpha|)$ where $\alpha = \max\{1,d\}$ then $\omega(n;Q,a) \le C_1\omega(n;a)$. Now, by definition,

$$\sum_\nu \|g1_{Q+d\nu}\|_\infty \ \tilde{w}(d\nu) \le \tilde{B}(d^k)\|g\|_{\tilde{w},d}$$

and by Corollary 2.2.6, if $d > 1$ then $\|g\|_{\tilde{w},d} \le 2^kB(d^k)\|g\|_{\tilde{w},1}$ and if $d < 1$, $\|g\|_{\tilde{w},d} \le (\lfloor d \rfloor+2)^kB(1)\|g\|_{\tilde{w},1}$. Combining the above with Lemma 2.3.7 gives the conclusion with

$$C = C_1(\lfloor d \rfloor+2)^kB(1)\tilde{B}(d^k)[B(d^k)3^ka^{-k}d^{-k}]^{1/2}.$$

Since the only requirement of d was that $0 < d \le 1/b$, the inequality follows in this case. For arbitrary $(a_{n,m})$, the fact that

$$\sum_{n,m} |a_{n,m}|^2 \ \omega(n;a) < \infty$$

enables us to show that the series

$$\sum_n \sum_m a_{n,m} \ T_{na}E_{mb}g$$

is Cauchy in $L^2_w(\mathbb{R}^k)$. The conclusion follows from completeness of $L^2_w(\mathbb{R}^k)$.■

THEOREM 2.3.9. Let g be such that g, $g_{\sim}\in W(L^\infty,L^1_{\tilde{w}})$ and for some a > 0 there exist constants A, B > 0 such that for almost every $x\in\mathbb{R}^k$,

$$(*) \qquad A \le \sum_n |g(x-na)|^2 \le B.$$

Then there exist a cube Q and $b_0 > 0$ such that for all $0 < b \le b_0$, there exist linear functionals $a_{n,m}$: $L^2_w(\mathbb{R}^k) \longrightarrow \mathbb{C}$ such that for all $f\in L^2_w(\mathbb{R}^k)$,

$$f = \sum_n \sum_m a_{n,m}(f)\ T_{na}E_{mb}g$$

strongly and there exist constants τ_1, $\tau_2 > 0$ such that for all $f\in L^2_w(\mathbb{R}^k)$,

$$\tau_1\|f\|^2_{2,w} \le \sum_n \sum_m |a_{n,m}(f)|^2\ \omega(\underset{\cdot}{n};Q,a) \le \tau_2\|f\|^2_{2,w}.$$

PROOF. By Lemma 2.3.7, there exists a cube Q and constants A', B' > 0 such that Q has side d and

$$A' \le \left|\sum_n \bar{g}1_Q(x-na)g1_{Q_0}(x-na)\right| \le B'$$

for every cube Q_0 containing Q and almost every $x\in\mathbb{R}^k$. Now fix Q_0 such that $Q \subset Q_0$, Q_0 has side b_0 and

$$\sum_\nu \left\|(g1_{Q_0}-g)1_{Q+d\nu}\right\|_\infty\ \tilde{w}(d\nu) < \lambda\left(3^{k/2}a^{-k/2}d^{k/2}B(|Q|)B'^{\,1/2}\right)^{-1}$$

for some $0 < \lambda < 1$. Now by Lemmas 2.3.1 and 2.3.2, for all $0 < b \le b_0$, there exist linear functionals $c_{n,m}$: $L^2_w(\mathbb{R}^k) \longrightarrow \mathbb{C}$ such that

$$f = \sum_n \sum_m c_{n,m}(f)\ T_{na}E_{mb}(g1_{Q_0})$$

and

$$C_1^{1/2}\|f\|_{2,w} \le b^{-k/2}\left[\sum_n \sum_m |c_{n,m}(f)|^2\ \omega(n;Q,a)\right]^{1/2} \le C_2^{1/2}\|f\|_{2,w}$$

where C_2 is independent of b. By Lemma 2.3.7,

$$\left\|f-\sum_n \sum_m c_{n,m}(f)\ T_{na}E_{mb}g\right\|_{2,w}$$

$$= \left\|\sum_n \sum_m c_{n,m}(f)\ T_{na}E_{mb}(g1_{Q_0}-g)\right\|_{2,w}$$

$$\le 3^{k/2}a^{-k/2}d^{k/2}B(|Q|)\sum_\nu \left\|(g1_{Q_0}-g)1_{Q+d\nu}\right\|_\infty \tilde{w}(d\nu)$$

$$b^{-k/2}\left[\sum_n \sum_m |c_{n,m}(f)|^2\ \omega(n;Q,a)\right]^{1/2}$$

$$< \lambda\|f\|_{2,w}.$$

Hence if we define the operator U: $L^2_w(\mathbb{R}^k) \longrightarrow L^2_w(\mathbb{R}^k)$ by

$$U(f) = \sum_n \sum_m c_{n,m}(f)\ E_{mb}T_{na}g,$$

we have that $\|I-U\| \le \lambda < 1$ so that U is continuously invertible. Defining

$$a_{n,m}(f) = c_{n,m}(U^{-1}f)$$

we have that

$$f = U(U^{-1}f) = \sum_n \sum_m a_{n,m}(f)\ T_{na}E_{mb}g$$

where the coefficient functionals, $a_{n,m}$, satisfy the appropriate estimates.■

Section 2.4. Existence of atoms in Sobolev Spaces

Because the translation and modulation operators exchange roles under the action of the Fourier transform, the results of Section 2.3 give decomposition theorems for functions whose Fourier transforms lie in $L^2_w(\hat{\mathbb{R}}^k)$. In particular, if the weight w is a polynomial, then we have decompositions for certain of the Bessel potential spaces, or Sobolev spaces (cf. [F2]).

DEFINITION 2.4.1. Let w be a moderate weight on $\hat{\mathbb{R}}^k$, then we define the spaces $\mathcal{L}^2_w(\mathbb{R}^k)$ by

$$\mathcal{L}^2_w(\mathbb{R}^k) = \left\{f: \int |\hat{f}(\gamma)|^2 w(\gamma)\, d\gamma = \|f\|^2_{\mathcal{L}^2_w} < \infty\right\}$$

If $w(\gamma) = (1+|\gamma|^2)^{\alpha/2}$ for some $\alpha > 0$, we denote $\mathcal{L}^2_w(\mathbb{R}^k)$ by $\mathcal{L}^2_\alpha(\mathbb{R}^k)$. This is a Sobolev space of order α.

THEOREM 2.4.2. Let $g \in \mathcal{L}^2_w(\mathbb{R}^k)$ be such that $\hat{g}$, $\hat{g}_- \in W(L^\infty, L^1_{\tilde{w}})$ and suppose that for some $a > 0$, there exist constants A, $B > 0$ such that

$$A \le \sum_n |\hat{g}(\gamma - na)|^2 \le B$$

for almost all $\gamma \in \hat{\mathbb{R}}^k$. Then there exist a cube Q and $b_0 > 0$ such that for every $0 < b \le b_0$, there exists a collection of linear functionals $a_{n,m}$: $\mathcal{L}^2_w(\mathbb{R}^k) \longrightarrow \mathbb{C}$ and constants τ_1, $\tau_2 > 0$ such that for every $f \in \mathcal{L}^2_w(\mathbb{R}^k)$,

$$f = \sum_n \sum_m a_{n,m}(f)\ T_{mb}E_{na}g$$

where the sum converges strongly in $\mathcal{L}_w^2(\mathbb{R}^k)$ and

$$\tau_1 \|f\|_{\mathcal{L}_w^2} \le \left[\sum_n \sum_m |a_{n,m}(f)|^2\ \omega(n;Q,a)\right]^{1/2} \le \tau_2 \|f\|_{\mathcal{L}_w^2}.$$

PROOF. If $f \in \mathcal{L}_w^2(\mathbb{R}^k)$ then $\hat{f} \in L_w^2(\hat{\mathbb{R}}^k)$. Thus, it follows from Theorem 2.3.9 that there exists a cube Q, and $b_0 > 0$ such that for every $0 < b < b_0$, there exists a collection of linear functionals $b_{n,m}$: $L_w^2(\hat{\mathbb{R}}^k) \longrightarrow \mathbb{C}$ such that

$$\hat{f} = \sum_n \sum_m b_{n,m}(\hat{f})\ E_{mb}T_{na}\hat{g}$$

where the sum converges strongly in $L_w^2(\hat{\mathbb{R}}^k)$. Thus we have that

$$f = \sum_n \sum_m b_{n,m}(\hat{f})\ T_{mb}E_{-na}g$$

where the sum converges strongly in $\mathcal{L}_w^2(\mathbb{R}^k)$. Also by Theorem 2.3.9, there exist constants τ_1 and τ_2 such that

$$\tau_1 \|\hat{f}\|_{2,w} \le \left[\sum_n \sum_m |b_{n,m}(\hat{f})|^2\ \omega(n;Q,a)\right]^{1/2} \le \tau_2 \|\hat{f}\|_{2,w}.$$

Putting $a_{n,m}(f) = b_{-n,m}(\hat{f})$ we are done.∎

REMARK 2.4.3. A comparison of Theorems 2.3.9 and 2.4.2 reveal how the properties of a function are reflected in the coefficients in a Weyl-Heisenberg decomposition. Roughly speaking, $f \in \mathcal{L}_w^2(\mathbb{R}^k)$ is characterized by a smoothness condition. For example, if the weight being considered is $w(x)=(1+|x|)^n$ then to say that $f \in L_w^2(\mathbb{R}^k)$ says that the n^{th} order distributional derivatives of f are in $L^2(\mathbb{R}^k)$. The smoothness of f is reflected in the local behavior of the coefficients of the W-H decomposition. If we suppose that f =

$\sum\sum a_{n,m}\, T_{mb}E_{na}g$ for some g, one can think of the sequence $\{a_{n,m}\}$, m fixed, as reflecting the behavior of f in a neighborhood of the point mb. The smoothness of f appears in the rapid decay of the sequence $\{a_{n,m}\}$, m fixed, specifically that

$$\sum_n |a_{n,m}|^2\ \omega(n;Q,a) < \infty$$

for each m.

If we are decomposing $L^2_w(\mathbb{R}^k)$, the situation is reversed. A function $f \in L^2_w(\mathbb{R}^k)$ is characterized by a decay condition which is global in nature. If we suppose that $f = \sum\sum b_{n,m}\, E_{mb}T_{na}g$, then the sequence $\{b_{n,m}\}$, n fixed, reflects the behavior of f in a neighborhood of the point na. The most we can say about this sequence is that it is square summable for each n. That is, the global structure of f is not present in the local coefficients. The global properties of f are reflected in the behavior of the sequence $\{b_{n,m}\}$, m fixed, specifically that

$$\sum_n |b_{n,m}|^2\ \omega(n;Q,a) < \infty$$

for each m.

Section 2.5. Banach frames and Hilbert frames for $L^2_w(R^k)$

What we have shown in the previous sections of this chapter is that if the number a and the function $g \in W(L^\infty, L^1_{\tilde{w}})$ satisfy certain conditions then for all sufficiently small b, the collection of functions $\{E_{mb}T_{na}g\}$ is a set of atoms for $L^2_w(\mathbb{R}^k)$. Dual to the notion of a set of atoms for a Banach space is the notion of a Banach frame (cf. [Gr1]). In a Hilbert space, these notions are equivalent.

Since $L^2_w(\mathbb{R}^k)$ is a Hilbert space with respect to a weighted inner product (cf. Section 0.3), it is natural to investigate the relationship between Banach frames of W-H wavelets for the Banach space $L^2_w(\mathbb{R}^k)$ and frames of such wavelets for the Hilbert space $L^2_w(\mathbb{R}^k)$. In this section, we show that the two notions are not equivalent in the simple case of a compactly supported, continuous mother wavelet.

THEOREM 2.5.1. Let g be a bounded function supported in a compact set, say in a cube Q with side length at most 1/b. If for some a > 0, there are constants A, B > 0 such that for almost every $x \in \mathbb{R}^k$,

$$(1) \qquad A \le \sum_n \omega(n;Q,a)|g(x-na)|^2 \le B$$

then $\{E_{mb}T_{na}g\}$ is a frame for the Hilbert space $L^2_w(\mathbb{R}^k)$. That is, there are constants C_1, $C_2 > 0$ such that for all $f \in L^2_w(\mathbb{R}^k)$,

$$C_1\|f\|_{2,w}^2 \le \sum_n \sum_m |\langle f, E_{mb}T_{na}g\rangle_w|^2 \le C_2\|f\|_{2,w}^2.$$

Moreover, condition (1) is necessary for the conclusion to hold.

PROOF.

$$\sum_n \sum_m |\langle f, E_{mb}T_{na}g\rangle_w|^2$$

$$= \sum_n \sum_m \left|\int f(x)w(x)\bar{g}(x-na)\ e^{2\pi ib\langle m,x\rangle}\ dx\right|^2$$

$$= b^{-k}\sum_n \int_{Q+na} |f(x)|^2 w(x)^2 |g(x-na)|^2\ dx = (*).$$

Now, since w is a moderate weight,

$$(*) \le B(|Q|)d_2b^{-k}\sum_n \omega(n;Q,a)\int |f(x)|^2 w(x)|g(x-na)|^2\ dx$$

$$= B(|Q|)d_2b^{-k}\int |f(x)|^2 w(x) \sum_n \omega(n;Q,a)|g(x-na)|^2\ dx$$

$$\le B(|Q|)d_2Bb^{-k}\|f\|_{2,w}^2$$

where d_2 is the constant given in Proposition 1.1.11. Also,

$$(*) \ge d_1b^{-k}\sum_n \omega(n;Q,a)\int |f(x)|^2 w(x)|g(x-na)|^2\ dx$$

$$\ge Ad_1b^{-k}\|f\|_{2,w}^2$$

with d_1 the constant of Proposition 1.1.11.

To see that the condition (1) is necessary, suppose for example that $\sum \omega(n;Q,a)|g(x-na)|^2$ is unbounded above. Then given $M > 0$ there is a set E with $0 < |E| < \infty$ such that

$$\sum_n \omega(n;Q,a)|g(x-na)|^2 > M$$

for all $x \in E$. Let $f(x) = w(x)^{-1/2}1_E(x)|E|^{-1/2}$ (this can be done since w is positive and finite-valued). Then $\|f\|_{2,w} = 1$ and by the above calculations,

$$\sum_n \sum_m |\langle f, E_{mb}T_{na}g\rangle_w|^2$$

$$\geq d_1 b^{-k} \int |f(x)|^2 w(x) \sum_n \omega(n;Q,a)|g(x-na)|^2 \, dx$$

$$= d_1 b^{-k} |E| \int_E \sum_n \omega(n;Q,a)|g(x-na)|^2 \, dx > d_1 b^{-k} M.$$

Thus, there is no upper frame bound. A similar calculation shows that if $\sum \omega(n;Q,a)|g(x-na)|^2$ were not bounded below, there would be no lower frame bound.■

THEOREM 2.5.2. Let g be a bounded function, supported in a compact set, say a cube Q with side length at most 1/b. If for some a > 0, there are constants A, B > 0 such that for almost every $x \in \mathbb{R}^k$,

$$A \leq \sum_n |g(x-na)|^2 \leq B, \tag{2}$$

then $\{E_{mb}T_{na}g\}$ is a Banach frame for $L^2_w(\mathbb{R}^k)$. That is, there are constants C_1, $C_2 > 0$ such that for all $f \in L^2_w(\mathbb{R}^k)$,

$$C_1 \|f\|^2_{2,w} \leq \sum_n \sum_m |\langle f, E_{mb}T_{na}g\rangle|^2 \, \omega(n;Q,a) \leq C_2 \|f\|^2_{2,w}.$$

Moreover, condition (2) is necessary for the conclusion to hold.

PROOF.

$$\sum_n \sum_m |\langle f, E_{mb}T_{na}g\rangle|^2 \, \omega(n;Q,a)$$

$$= \sum_n \sum_m \left| \int f(x)\bar{g}(x-na) \, e^{-2\pi i b\langle m,x\rangle} \, dx \right|^2 \omega(n;Q,a)$$

$$= b^{-k} \sum_n \int_{Q+na} |f(x)|^2 |g(x-na)|^2 \, dx \; \omega(n;Q,a) = (*).$$

Now, since w is a moderate weight, there is a constant d_1 (cf. Proposition 1.1.11) such that $w(x) \geq d_1 \omega(n;Q,a)$ for every $x \in Q+na$. Thus,

$$(*) \le d_1 b^{-k} \sum_n \int_{Q+na} |f(x)|^2 w(x) |g(x-na)|^2 \, dx$$
$$= d_1 b^{-k} \int |f(x)|^2 w(x) \sum_n |g(x-na)|^2 \, dx$$
$$\le d_1 B b^{-k} \|f\|_{2,w}^2.$$

Also, there is a constant d_2 (cf. Proposition 1.1.11) such that $w(x) \le B(|Q|) d_2 \omega(n;Q,a)$ for every $x \in Q+na$. Thus,

$$(*) \ge [B(|Q|)d_2]^{-1} b^{-k} \sum_n \int_{Q+na} |f(x)|^2 w(x) |g(x-na)|^2 \, dx$$
$$= [B(|Q|)d_2]^{-1} b^{-k} \int |f(x)|^2 w(x) \sum_n |g(x-na)|^2 \, dx$$
$$\ge [B(|Q|)d_2]^{-1} A b^{-k} \|f\|_{2,w}^2.$$

The necessity of condition (2) follows as in the previous theorem. Specifically, suppose that $\sum |g(x-na)|^2$ was not bounded below, then given $\delta > 0$ there would be a set E such that $0 < |E| < \infty$ and such that $\sum |g(x-na)|^2 < \delta$ for all $x \in E$. Let $f(x) = w(x)^{-1/2} 1_E(x) |E|^{-1/2}$ which can be done since w is positive and finite-valued. Thus, $\|f\|_{2,w} = 1$ and

$$(*) \ge d_1 b^{-k} \int |f(x)|^2 w(x) \sum_n |g(x-na)|^2 \, dx$$
$$= d_1 b^{-k} |E|^{-1} \int_E \sum_n |g(x-na)|^2 \, dx < d_1 b^{-k} B \delta \|f\|_{2,w}^2.$$

Since δ was arbitrary, there is no lower frame bound. Similarly, if $\sum |g(x-na)|^2$ were unbounded above, there would be no upper frame bound.∎

REMARK 2.5.3. An examination of Theorem 2.5.1 reveals that unless w is bounded above and away from zero, condition (1) is vacuous.

That is, if g is any function, compactly supported or not, which satisfies (1), then $g = 0$ almost everywhere.

To see this, suppose not. Then there is a set $E \subset \mathbb{R}^k$ with $0 < |E| < \infty$ such that for some $\alpha > 0$, $|g(x)| \geq \alpha$ on E. If $w(x)$ is unbounded above, then given $M > 0$ and any cube Q, there is an $n_0 \in \mathbb{Z}^k$ such that $\omega(n_0; Q, a) > M$. If $x \in E + n_0 a$, then

$$\sum_n \omega(n; Q, a) |g(x-na)|^2 \geq \omega(n_0; Q, a) |g(x-n_0)|^2 \geq \alpha M.$$

Since M was arbitrary, (1) fails. If $w(x)$ were not bounded away from zero, we could in a similar fashion show that $\{E_{mb}T_{na}g\}$ failed to have a lower frame bound. Thus no compactly supported function can generate a Hilbert space frame for $L^2_w(\mathbb{R}^k)$ for all b in a neighborhood of zero.

Now, let g be a continuous function with compact support which does not vanish in the interior of its support. Then for some small $a > 0$, condition (2) of Theorem 2.5.2 is satisfied so that for all b sufficiently small, (g,a,b) generates a Banach frame for $L^2_w(\mathbb{R}^k)$. Obviously, then, the notions of a Banach frame and a Hilbert space frame for $L^2_w(\mathbb{R}^k)$ are not equivalent.

Section 2.6. Banach frames and sets of atoms in $L^2_w(\mathbb{R}^k)$

In this section, we examine the relationship between sets of atoms and Banach frames for $L^2_w(\mathbb{R}^k)$. It is well-known (cf. [Gr1]) that, in a Hilbert space, the notion of a set of atoms and a frame are equivalent. Also, Chris Heil has proven a remarkable result which says that in a Hilbert space, the dual frame associated to any frame is unique. We will prove slightly weaker analogues of these results.

In what follows, $\omega(n;a)$ is taken to mean $\omega(n;Q_1,a)$ for $a > 0$.

THEOREM 2.6.1. Let $a,\ b > 0$ and $g \in W(L^\infty, L^1_\omega)$. If $\{E_{mb}T_{na}g\}$ is a set of atoms for $L^2_w(\mathbb{R}^k)$ with atomic bounds A, B and coefficient functionals $a_{m,n}$, then it is a Banach frame for $L^2_{1/w}(\mathbb{R}^k)$. PROOF. Suppose that $\{E_{mb}T_{na}g\}$ is a set of atoms for $L^2_w(\mathbb{R}^k)$. Note that for any $h \in L^2_w(\mathbb{R}^k)$,

$$|\langle f,h\rangle| = \left|\langle f, \sum_n \sum_m a_{n,m}(h)\ E_{mb}T_{na}g\rangle\right|$$

$$= \left|\sum_n \sum_m \langle f, E_{mb}T_{na}g\rangle\ \overline{a_{n,m}(h)}\right|$$

$$\le \left[\sum_n \sum_m |a_{n,m}(h)|^2\ \omega(n;a)\right]^{1/2} \left[\sum_n \sum_m |\langle f, E_{mb}T_{na}g\rangle|^2\ 1/\omega(n;a)\right]^{1/2}.$$

Since $\|f\|^2_{2,1/w} = \sup\{\,|\langle f,h\rangle| : \|h\|_{2,w} = 1\}$, we have that

$$\|f\|^2_{2,1/w} \le B \sum_n \sum_m |\langle f, E_{mb}T_{na}g\rangle|^2\ 1/\omega(n;a).$$

Now, let $a_{n,m} = \langle f, E_{mb}T_{na}g\rangle$, and observe that for any sequence $\{b_{n,m}\}$ we have that

$$|\langle a_{n,m}, b_{n,m}\rangle| = \left|\sum_n \sum_m a_{n,m}\,\overline{b_{n,m}}\right| = \left|\sum_n \sum_m \langle f, E_{mb}T_{na}g\rangle\,\overline{b_{n,m}}\right|$$
$$= \left|\langle f, \sum_n \sum_m b_{n,m}\, E_{mb}T_{na}g\rangle\right| \le \|f\|_{2,1/w} \left\|\sum_n \sum_m b_{n,m}\, E_{mb}T_{na}g\right\|_{2,w}.$$

By Corollary 2.3.8, we know that there is a constant, C, depending only on k, a, g, and b such that

$$\left\|\sum_n \sum_m b_{n,m}\, E_{mb}T_{na}g\right\|_{2,w}^2 \le C\sum_n \sum_m |b_{n,m}|^2\, \omega(n;a).$$

Since the dual space of ℓ^2_ω is $\ell^2_{1/\omega}$, we have that

$$\sum_n \sum_m |\langle f, E_{mb}T_{na}g\rangle|^2\; 1/\omega(n;a)$$
$$= \sup\left\{\left|\sum_n \sum_m \langle f, E_{mb}T_{na}g\rangle\,\overline{b_{n,m}}\right| : \|(b_{n,m})\|_{\ell^2_\omega} = 1\right\}$$

and hence that

$$\sum_n \sum_m |\langle f, E_{mb}T_{na}g\rangle|^2\; 1/\omega(n;a) \le C\|f\|^2_{2,1/w}. \blacksquare$$

Here we state the result due to Heil mentioned in the introduction to this section. Its proof can be found in [H] and is reproduced in Section 5.1.

THEOREM 5.1.6. (Heil) Let **H** be a Hilbert space, and $\{x_n\}$ a set of atoms for **H**. Let a_n be the collection of coefficient functionals associated to $\{x_n\}$. Then $a_n(f) = \langle f, S^{-1}x_n\rangle$ where

$$Sf = \sum_n \langle f, x_n\rangle x_n.$$

THEOREM 2.6.2. let $w_0(x) = \max\{1, \tilde{w}(x)\}$, and let g be such that g, $g_- \in W(L^\infty, L^1_{w_0})$. Suppose that for some a > 0, there exist constants A, B > 0 such that for almost every $x \in \mathbb{R}^k$,

$$A \le \sum_n |g(x-na)|^2 \le B.$$

Then there is a $b_0 > 0$ such that for all $0 < b \le b_0$, $\{E_{mb}T_{na}g\}$ is a Banach frame for $L^2_w(\mathbb{R}^k)$, that is, there exist constants c_1, $c_2 > 0$ such that for all $f \in L^2_w(\mathbb{R}^k)$,

$$c_1\|f\|^2_{2,w} \le \sum_n \sum_m |\langle f, E_{mb}T_{na}g\rangle|^2 \, \omega(n;a) \le c_2\|f\|_{2,w}.$$

PROOF. Consider the coefficient functionals, $a_{n,m}$, defined in Theorem 2.3.9. I claim that if b is sufficiently small and if $f \in L^2(\mathbb{R}^k) \cap L^2_w(\mathbb{R}^k)$, then $\sum\sum |a_{n,m}(f)|^2$ is equivalent to $\|f\|^2_2$, and $\sum\sum |a_{n,m}(f)|^2 \, \omega(n;a)$ is equivalent to $\|f\|^2_{2,w}$.

Choose a cube Q_0 so large that there are constants A_0, $B_0 > 0$ such that for almost every $x \in \mathbb{R}^k$,

$$A_0 \le \left| \sum_n \bar{g}1_Q(x-na)g1_{Q_0}(x-na) \right| \le B_0.$$

Such a cube exists by Lemma 2.3.5.

Let $c_{n,m}(f) = \langle S_1^{-1}f, E_{mb}T_{na}(g1_Q)\rangle$ where

$$S_1 f = \sum_n \sum_m \langle f, E_{mb}T_{na}g1_Q\rangle \, E_{mb}T_{na}g1_{Q_0},$$

Q and Q_0 are cubes, $Q \subset Q_0$, and Q_0 has side length at most $1/b$. Since $f \in L^2(\mathbb{R}^k) \cap L^2_w(\mathbb{R}^k)$ and since $g1_Q$ and $g1_{Q_0}$ are bounded and compactly supported, the sum defining S_1 converges strongly in $L^2(\mathbb{R}^k)$ and in $L^2_w(\mathbb{R}^k)$ and converges to

$$f(x)\, b^{-k} \sum_n \bar{g}1_Q(x-na)g1_{Q_0}(x-na).$$

Thus S_1 is bounded on $L^2(\mathbb{R}^k)$ and $L^2_w(\mathbb{R}^k)$ with a bounded inverse.

Applying Corollary 2.3.3 twice, once with the weight identically 1 and again with weight w, we have that

$$b^k A_0 B_0^{-2} \|f\|_2^2 \le \sum_n \sum_m |c_{n,m}(f)|^2 \le b^k B_0 A_0^{-2} \|f\|_2^2$$

and that

$$b^k A_0 B_0^{-2} B(|Q|)^{-1} \|f\|_{2,w}^2 \le \sum_n \sum_m |c_{n,m}(f)|^2 \; \omega(n;a) \le b^k B_0 A_0^{-2} \|f\|_{2,w}^2.$$

Define

$$Uf = \sum_n \sum_m c_{n,m}(f) \; E_{mb} T_{na} g.$$

We wish to estimate $\|f-Uf\|_2$. To do this, note that the estimates of $\|f-Uf\|_{2,w}$ in Theorem 2.3.9 are valid when $w = 1$. Thus we have

$$\|f-Uf\|_2 \le 3^{k/2} a^{-k/2} d^{k/2} \sum_\nu \left\| (g1_{Q_0} - g) 1_{Q+d\nu} \right\|_\infty [B_0 A_0^{-1}]^{1/2} \|f\|_2.$$

Let Q_0 be so large that

$$\sum_\nu \left\| (g1_{Q_0} - g) 1_{Q+d\nu} \right\|_\infty w_0(d\nu)$$
$$< \lambda \left[3^{k/2} a^{-k/2} d^{k/2} B(|Q|) [B_0 A_0^{-1}]^{1/2} \right]^{-1}.$$

Since $B(|Q|) \ge 1$, we have that

$$\sum_\nu \left\| (g1_{Q_0} - g) 1_{Q+d\nu} \right\|_\infty \le \sum_\nu \left\| (g1_{Q_0} - g) 1_{Q+d\nu} \right\|_\infty w_0(d\nu)$$
$$< \lambda \left[3^{k/2} a^{-k/2} d^{k/2} B(|Q|) [B_0 A_0^{-1}]^{1/2} \right]^{-1}$$
$$\le \lambda \left[3^{k/2} a^{-k/2} d^{k/2} [B_0 A_0^{-1}]^{1/2} \right]^{-1}$$

and also that

$$\sum_\nu \left\| (g1_{Q_0} - g) 1_{Q+d\nu} \right\|_\infty \tilde{w}(d\nu) \le \sum_\nu \left\| (g1_{Q_0} - g) 1_{Q+d\nu} \right\|_\infty w_0(d\nu)$$
$$< \lambda \left[3^{k/2} a^{-k/2} d^{k/2} B(|Q|) [B_0 A_0^{-1}]^{1/2} \right]^{-1}.$$

Therefore $\|f-Uf\|_2 < \lambda \|f\|_2$, where $\lambda < 1$, and consequently, U maps $L^2(\mathbb{R}^k)$ onto $L^2(\mathbb{R}^k)$ and has a continuous inverse. Similarly,

$\|f-Uf\|_{2,w} < \lambda\|f\|_{2,w}$ and U is a continuous bijection on $L^2_w(\mathbb{R}^k)$ with a continuous inverse.

Letting $a_{n,m}(f) = c_{n,m}(U^{-1}f)$ we have that $\sum\sum |a_{n,m}(f)|^2$ is equivalent to $\|f\|_2^2$ and that $f = \sum\sum a_{n,m}(f)\, E_{mb}T_{na}g$ where the sum converges strongly in $L^2(\mathbb{R}^k)$. Thus by Heil's Lemma (Theorem 5.1.7), $a_{n,m}(f) = \langle f, E_{mb}T_{na}S^{-1}g\rangle$ for all $f \in L^2(\mathbb{R}^k) \cap L^2_w(\mathbb{R}^k)$. Since each $a_{n,m}$ is a continuous linear functional on $L^2_w(\mathbb{R}^k)$ and since $L^2(\mathbb{R}^k) \cap L^2_w(\mathbb{R}^k)$ is dense in $L^2_w(\mathbb{R}^k)$, we have that $a_{n,m}(f) = \langle f, E_{mb}T_{na}S^{-1}g\rangle$ for all $f \in L^2_w(\mathbb{R}^k)$.

To complete the proof we must show that in fact $\langle f, E_{mb}T_{na}S^{-1}g\rangle = \langle S^{-1}f, E_{mb}T_{na}g\rangle$ for all $f \in L^2_w(\mathbb{R}^k)$. Certainly, this is true for $f \in L^2(\mathbb{R}^k)$.

By Theorem 4.1.6, S^{-1} is a bounded operator on $L^2_w(\mathbb{R}^k)$ and $L^2_{1/w}(\mathbb{R}^k)$ for all sufficiently small b. Thus let $f_n \in L^2(\mathbb{R}^k) \cap L^2_w(\mathbb{R}^k)$ be such that $f_n \longrightarrow f$ in $L^2_w(\mathbb{R}^k)$. Then

$$
\begin{aligned}
&|\langle f, E_{mb}T_{na}S^{-1}g\rangle - \langle S^{-1}f, E_{mb}T_{na}g\rangle| \\
&\le |\langle f-f_n, E_{mb}T_{na}S^{-1}g\rangle| + |\langle f_n, E_{mb}T_{na}S^{-1}g\rangle - \langle S^{-1}f_n, E_{mb}T_{na}g\rangle| \\
&\qquad + |\langle S^{-1}f_n - S^{-1}f, E_{mb}T_{na}g\rangle| \\
&\le \|f-f_n\|_{2,w}\|E_{mb}T_{na}S^{-1}g\|_{2,1/w} + 0 \\
&\qquad + \|S^{-1}\|_{L^2_w \to L^2_w}\|f-f_n\|_{2,w}\|E_{mb}T_{na}S^{-1}g\|_{2,1/w}.
\end{aligned}
$$

Since $g_{-} \in W(L^\infty, L^1_{\tilde{w}})$, $g \in L^2_{1/w}(\mathbb{R}^k)$ by Proposition 2.2.8. Also, since S^{-1} is a continuous operator on $L^2_{1/w}(\mathbb{R}^k)$ for all sufficiently small b by Theorem 4.1.6, we have that $\|E_{mb}T_{na}S^{-1}g\|_{2,1/w} < \infty$. Thus, we have shown that $\langle f, E_{mb}T_{na}S^{-1}g\rangle = \langle S^{-1}f, E_{mb}T_{na}g\rangle$.

Now, $a_{n,m}(Sf) = \langle f, E_{mb}T_{na}g\rangle$ and $\sum\sum |a_{n,m}(Sf)|^2\ \omega(n;a)$ is equivalent to $\|Sf\|_{2,w}^2$ which is equivalent to $\|f\|_{2,w}^2$. That is, there exist constants c_1 and c_2 independent of f such that

$$0 < c_1\|f\|_{2,w}^2 \le \sum_n \sum_m |\langle f, E_{mb}T_{na}g\rangle|^2\ \omega(n;a) \le c_2\|f\|_{2,w} < \infty. \blacksquare$$

THEOREM 2.6.3. Suppose that g is such that g, $g_- \in W(L^\infty, L_\omega^1)$, and that for some $a > 0$, there exist constants A, B > 0 such that

$$A \le \sum_n |g(x-na)|^2 \le B.$$

Finally suppose that there exists $b_0 > 0$ such that for all $0 < b < b_0$, $\{E_{mb}T_{na}g\}$ is a Banach frame for $L_w^2(\mathbb{R}^k)$, that is, there exist constants c_1, $c_2 > 0$ such that for all $f \in L_w^2(\mathbb{R}^k)$,

$$c_1\|f\|_{2,w}^2 \le \sum_n \sum_m |\langle f, E_{mb}T_{na}g\rangle|^2\ \omega(n;a) \le c_2\|f\|_{2,w}^2.$$

Then there exists a $0 < b_1 \le b_0$ such that for all $0 < b \le b_1$, $\{E_{mb}T_{na}g\}$ is a set of atoms for $L_w^2(\mathbb{R}^k)$. In fact, for each $f \in L_w^2(\mathbb{R}^k)$,

$$\begin{aligned} f &= \sum_n \sum_m \langle S^{-1}f, E_{mb}T_{na}g\rangle\ E_{mb}T_{na}g \\ &= \sum_n \sum_m \langle f, E_{mb}T_{na}S^{-1}g\rangle\ E_{mb}T_{na}g \\ &= \sum_n \sum_m \langle f, E_{mb}T_{na}g\rangle\ E_{mb}T_{na}S^{-1}g \end{aligned}$$

where

$$Sf = \sum_n \sum_m \langle f, E_{mb}T_{na}g\rangle\ E_{mb}T_{na}g.$$

PROOF. Since $g \in W(L^\infty, L_\omega^1)$, since $\{E_{mb}T_{na}g\}$ is a Banach frame, and by Lemma 2.3.6, the sum defining the S operator converges strongly in $L_w^2(\mathbb{R}^k)$. By Theorem 4.1.6, for all sufficiently small b, the S operator is continuously invertible on $L^2_w(\mathbb{R}^k)$. Therefore the formulas for f hold. Also, since $\{E_{mb}T_{na}g\}$ is a Banach frame for $L^2_w(\mathbb{R}^k)$, $\sum \sum |\langle S^{-1}f, E_{mb}T_{na}g\rangle|^2 \, \omega(n;a)$ is equivalent to $\|f\|^2_{2,w}$. ∎

CHAPTER 3

Stability of Atoms in $L^2_w(R^k)$

The stability results in this chapter are in the spirit of similar results proven by Feichtinger and Gröchenig in [FG2]. The results obtained in [FG2] concern W-H decompositions for a wide variety of spaces, namely coorbit spaces for the Heisenberg group. In this chapter, we prove similar but more general results for the spaces $L^2_w(\mathbb{R}^k)$ using the techniques of Chapter 2.

In Section 3.1, we prove that, under suitable hypotheses, a set of W-H atoms for $L^2_w(\mathbb{R}^k)$ continues to be a set of atoms under perturbation of the mother wavelet and of the translation and modualtion lattice.

In Section 3.2, we show that, under suitable hypotheses, a set of W-H atoms for $L^2_w(\mathbb{R}^k)$ continues to be so under a change in the lattice parameters.

Section 3.1. Stability under perturbations

The results of this section fall into two categories: 1) stability with respect to a perturbation of the mother wavelet and 2) stability with respect to a perturbation of the lattice.

The notion of closeness for mother wavelets is simply the metric in $W(L^\infty, L^1_{\tilde{w}})$. The theorem here is valid whether the mother wavelet being perturbed is in $W(L^\infty, L^1_{\tilde{w}})$ or not.

The notion of closeness of two collections of points in this section is simply uniform closeness of the corresponding points, i.e., two collections of points in $\mathbb{R}^k \times \mathbb{R}^k$, (a_n, b_n) and (c_n, d_n) are within a distance ε if $|a_n - c_n| < \varepsilon$ and $|b_n - d_n| < \varepsilon$. This notion is more general than the one in [FG2] which is given in terms of closeness with respect to the topology of the Heisenberg group. The result here is the expected stability result for the W-H wavelet decomposition.

The problem with defining closeness in terms of the Heisenberg group, $\mathbb{H}$ (cf. Definition 2.1.1), is that the toral component of elements in $\mathbb{H}$, while essential to the group structure, is simply superfluous when one is obtaining decompositions of $L^2_w(\mathbb{R}^k)$ in terms of translations and modulations. Requiring that two points in the lattice, x and y, be close in $\mathbb{H}$ requires that the toral component of their difference, i.e., of $y^{-1}x$ be close to 1. In doing so, one may force the two other components to be artificially close together. Specifically, we have the following result.

THEOREM 3.1.1. Let $\varepsilon > 0$ and let $x, y \in \mathbb{H}$ be given by $x = (t_1, a_1, b_1)$ and $y = (t_2, a_2, b_2)$. Then there is a neighborhood U, of the identity, $(1,0,0)$, in $\mathbb{H}$ such that $y^{-1}x \in U$ implies that $|a_1 - a_2| < \varepsilon$ and $|b_1 - b_2| < \varepsilon$. However, the converse is false, that is, there exists a neighborhood U of the identity in $\mathbb{H}$ such that given

$\varepsilon > 0$ and $t_1, t_2 \in \mathbb{T}$, there exist $a_1, a_2 \in \mathbb{R}^k$ and $b_1, b_2 \in \hat{\mathbb{R}}^k$ such that $|a_1 - a_2| < \varepsilon$ and $|b_1 - b_2| < \varepsilon$ but $y^{-1}x$ is not in U.

PROOF. Recall that the Heisenberg group $\mathbb{H}$ is identified with the set $\mathbb{T}\times\mathbb{R}^k\times\hat{\mathbb{R}}^k$, and is equipped with the product topology. To prove the first part, we can take $U = \mathbb{T}\times B(0,\varepsilon)\times B(0,\varepsilon)$, then clearly, if $y^{-1}x \in U$ then $a_1 - a_2 \in B(0,\varepsilon)$ and $b_1 - b_2 \in B(0,\varepsilon)$.

To prove the second part, we may take, for any $0 < \lambda < 1$, $U = \{z \in \mathbb{T}\colon \arg(z) \in [-\pi\lambda, \pi\lambda]\}\times B(0,\varepsilon)\times B(0,\varepsilon)$. Take any a, $b \in \mathbb{R}^k$ such that $t_1/t_2 e^{-2\pi i \langle a,b\rangle} = -1$, and let $a_1 = a_2 = a$ and $b_1 = b_2 = b$. then $y^{-1}x = (-1,0,0)$ which is not in U, but of course $|a_1 - a_2| < \varepsilon$ and $|b_1 - b_2| < \varepsilon$.■

THEOREM 3.1.2. Suppose that $g \in L^2_w(\mathbb{R}^k)$ is such that for some a, b > 0, $\{E_{mb}T_{na}g\}$ is a set of atoms for $L^2_w(\mathbb{R}^k)$ with atomic bounds A, B. Then given $\varepsilon > 0$ there is a $\delta > 0$ such that if $g_1 \in L^2_w(\mathbb{R}^k)$ is such that $\|g - g_1\|_{\tilde{w},1} < \delta$ then $\{E_{mb}T_{na}g_1\}$ is also a set of atoms for $L^2_w(\mathbb{R}^k)$ with atomic bounds A_1, B_1 where $A_1 \geq A(1-\varepsilon)$ and $B_1 \leq B(1+\varepsilon)$.

PROOF. Let $a_{n,m}$ be the collection of coefficient functionals associated to the set of atoms $\{E_{mb}T_{na}g\}$. Then we have

$$f = \sum_n \sum_m a_{n,m}(f)\ E_{mb}T_{na}g$$

for all $f \in L^2_w(\mathbb{R}^k)$. Let $g_1 \in L^2_w(\mathbb{R}^k)$ be such that $g - g_1 \in W(L^\infty, L^1_{\tilde{w}})$ and define the operator S_1 by

$$S_1 f = \sum_n \sum_m a_{n,m}(f)\ E_{mb}T_{na}g_1.$$

We show first that the sum defining S_1 converges strongly in $L^2_w(\mathbb{R}^k)$ for each $f \in L^2_w(\mathbb{R}^k)$. Given a finite set of indices $J = (J_1, J_2) \subset \mathbb{Z}^k \times \mathbb{Z}^k$ define the partial sum operators S^J and S_1^J by

$$S^J f = \sum_{n \in J_1} \sum_{m \in J_2} a_{n,m}(f)\ E_{mb} T_{na} g$$

and

$$S_1^J f = \sum_{n \in J_1} \sum_{m \in J_2} a_{n,m}(f)\ E_{mb} T_{na} g_1.$$

Let $\varepsilon > 0$. Then there is a finite set of indices $F = (F_1, F_2) \subset \mathbb{Z}^k \times \mathbb{Z}^k$ such that for any other finite set of indices $G = (G_1, G_2)$, we have that

$$\sum_{n \in G_1 \setminus F_1} \sum_{m \in G_2 \setminus F_2} |a_{n,m}(f)|^2\ \omega(n;a) < \varepsilon.$$

We know also that, since $\{E_{mb} T_{na} g\}$ is a set of atoms, the sum defining Sf converges strongly. Thus there is a finite set of indices $H = (H_1, H_2)$ such that if $G = (G_1, G_2)$ is any finite set of indices, we have that

$$\|S^{G \setminus H} f\|_{2,w} < \varepsilon.$$

Now let $I = F \cup H = (I_1, I_2)$. Then for any finite set of indices $G = (G_1, G_2)$, we have that

$$\begin{aligned}
\|S_1^{G\setminus I} f\|_{2,w} &\le \|S_1^{G\setminus I} f - S^{G\setminus I} f\|_{2,w} + \|S^{G\setminus I} f\|_{2,w} \\
&\le C\|g - g_1\|_{\tilde{\omega},1} \sum_{n \in G_1 \setminus I_1} \sum_{m \in G_2 \setminus I_2} |a_{n,m}(f)|^2\ \omega(n;a) + \varepsilon \\
&\le \varepsilon(C\|g - g_1\|_{\tilde{\omega},1} + 1)
\end{aligned}$$

where C depends on a, b, and w. Since ε was arbitrary, we are done.

Since by definition, S is the identity, we have that

$$\|f-S_1f\|_{2,w} = \left\| \sum_n \sum_m a_{n,m}(f)\ E_{mb}T_{na}(g-g_1)\right\|_{2,w}$$

$$\leq C\|g-g_1\|_{\tilde{\omega},1}\|f\|_{2,w} \equiv \lambda$$

where C is independent of g, g_1 and f. Assuming that $\lambda < 1$, we have that $\|I-S_1\| \leq \lambda < 1$ which implies that S_1 is a bijective homeomorphism of $L^2_w(\mathbb{R}^k)$. Hence, $\|S_1\| \leq \|I-S_1\|+\|I\| \leq 1+\lambda$, $\|S_1\| \geq \|I\|-\|I-S_1\| \geq 1-\lambda$, and $\|S_1^{-1}\| \leq (1-\lambda)^{-1}$. Letting $c_{n,m}(f) = a_{n,m}(S_1^{-1}f)$, we have that for all $f \in L^2_w(\mathbb{R}^k)$,

$$f = \sum_n \sum_m c_{n,m}(f)\ E_{mb}T_{na}g_1.$$

Now,

$$\sum_n \sum_m |c_{n,m}(f)|^2\ \omega(n;a) = \sum_n \sum_m |a_{n,m}(S_1^{-1}f)|^2\ \omega(n;a)$$

$$\leq B^2\|S_1^{-1}f\|^2_{2,w} \leq B^2(1-\lambda)^{-2}\|f\|^2_{2,w}.$$

Also,

$$\sum_n \sum_m |c_{n,m}(f)|^2\ \omega(n;a) = \sum_n \sum_m |a_{n,m}(S_1^{-1}f)|^2\ \omega(n;a)$$

$$\geq A^2\|S_1^{-1}f\|^2_{2,w} \geq A^2(1+\lambda)^{-2}\|S_1\|^2\|S_1^{-1}f\|^2_{2,w} \geq A^2(1+\lambda)^{-2}\|f\|^2_{2,w}.$$

Now, given $\varepsilon > 0$, there is a $0 < \lambda_0 \leq 1$ such that for all $0 < \lambda < \lambda_0$, $(1-\lambda)^{-2} < 1+\varepsilon$ and $(1+\lambda)^{-2} > 1-\varepsilon$. Let δ be such that $C\delta \leq \lambda_0$. Then if $\|g-g_1\|_{\tilde{\omega},1} < \delta$, $\|I-S_1\| = \lambda < \lambda_0$ and the conclusion follows.∎

LEMMA 3.1.3. Let $c > 0$ be given and let $\{h_n: n \in \mathbb{Z}^k\}$ be a collection of functions such that $\text{supp}(h_n) \subset Q$ for some cube Q with side $d > 0$. Let (c_n) be a collection of points in $\mathbb{R}^k$ such that for some $A < \infty$, $|c_n-cn| < A$. Then there is a constant C depending only on c, d, A, and k such that

$$\left\| \sum_n T_{c_n} h_n \right\|_{2,w} \le C \left[\sum_n \|h_n\|_2^2 \; \omega(n;Q,c) \right]^{1/2}.$$

PROOF. The result will follow as in Lemma 2.3.6 if we can show that for some $R > 0$, we can partition $\mathbb{Z}^k$ into R^k disjoint subsets, I_i, such that if n_1 and $n_2 \in I_i$, then $(Q+c_{n_1}) \cap (Q+c_{n_2}) = \emptyset$. Now, let $R = \lfloor d/c \rfloor + 1 + \lfloor 2A/ck \rfloor$. For each $j = (j^1, j^2, \ldots, j^k)$ with $j^m \in \mathbb{Z}$, $0 \le j^m \le R-1$, $m = 1,2,\ldots,k$, let $I_j = \{j+Rl\colon l \in \mathbb{Z}^k\}$. We have that if n_1, $n_2 \in I_j$, some j, and $n_1 \ne n_2$, then

$$\begin{aligned} |cn_1 - cn_2|_{max} &= c|n_1-n_2| = cR|l_1-l_2| \\ &= c(\lfloor d/c \rfloor + 1 + \lfloor 2A/k \rfloor) \ge c(d/c+1+2A/kc) = d+2A/k. \end{aligned}$$

Hence, we have that

$$\begin{aligned} |c_{n_1} - c_{n_2}|_{max} &= |(c_{n_1}-cn_1)+(cn_1-cn_2)+(cn_2-c_{n_2})|_{max} \\ &\ge |cn_1-cn_2|_{max} - (|c_{n_1}-cn_1|_{max} + |cn_2-c_{n_2}|_{max}) \\ &> |cn_1-cn_2|_{max} - 2A/k \ge d. \end{aligned}$$

Thus, $(Q+c_{n_1}) \cap (Q+c_{n_2}) = \emptyset$. ∎

LEMMA 3.1.4. Let $\omega(n;a) \equiv \omega(n;Q_1,a)$ and denote by $\tilde{B}(v)$ the constant corresponding to the moderate weight $\tilde{w}$ as in Theorem 1.1.6(4). Then given a sequence of numbers $(a_{n,m})$ only finitely many of which are non-zero, and a collection of points (a_n) in $\mathbb{R}^k$ such that there exists a number A such that $\sup_n |a_n - na| \le A$, there is a constant C independent of g and b, for $b \le 1$, such that

$$\left\| \sum_n \sum_m a_{n,m} \, T_{a_n} E_{mb} g \right\|_{2,w} \le C \|g\|_{\tilde{w},1} b^{-k/2} \left[\sum_n \sum_m |a_{n,m}|^2 \; \omega(n;a) \right]^{1/2}.$$

PROOF. Let $d \le 1$. Then by the arguments of Lemma 2.3.7, there is a constant C_1 independent of g and b (as long as $b \le 1$) such that

$$\Big\| \sum_n \sum_m a_{n,m}\, T_{a_n} E_{mb} g \Big\|_{2,w}$$

$$\le C_1 \sum_{\nu} \tilde{w}(d\nu) \Big\| \sum_n T_{a_n}\Big[\sum_m a_{n,m}\, E_{mb} g_{\nu} \Big] \Big\|_{2,w} = (*)$$

where $g_{\nu} = T_{-d\nu}(g1_{Q+d\nu})$. Thus, by Lemma 3.1.3, we have that

$$(*) \le C_1 \|g\|_{\tilde{w},d} [R^k B(1)]^{1/2} \Big[b^{-k} \sum_n \sum_m |a_{n,m}|^2\, \omega(n;Q,a) \Big]^{1/2}$$

$$\le C \|g\|_{\tilde{w},1} \Big[b^{-k} \sum_n \sum_m |a_{n,m}|^2\, \omega(n;a) \Big]^{1/2}$$

where $C = C_1 [R^k B(1)]^{1/2} 2^k \tilde{B}(1)$ and $R = \lfloor 1/c \rfloor + 1 + \lfloor 2A/kc \rfloor$. ∎

THEOREM 3.1.5. Let $g \in W(L^{\infty}, L^1_{\tilde{w}})$, g continuous, be such that for some a, $b > 0$, $\{E_{mb} T_{na} g\}$ is a set of atoms for $L^2_w(\mathbb{R}^k)$ with atomic bounds A, B. Then given $\varepsilon > 0$ there is a $\delta > 0$ such that if (a_n) is any collection of points in $\mathbb{R}^k$ such that $\sup_n |na - a_n| < \delta$ then $\{E_{mb} T_{a_n} g\}$ is also a set of atoms for $L^2_w(\mathbb{R}^k)$ with atomic bounds A_1, B_1 where $A_1 \ge A(1-\varepsilon)$ and $B_1 \le B(1+\varepsilon)$.

PROOF. For any collection of points (a_n) in $\mathbb{R}^k$ we define the operator corresponding to it by

$$S_1 f = \sum_n \sum_m a_{n,m}(f)\, E_{mb} T_{a_n} g.$$

If there is a number A such that $|a_n - an| < A$ for all $n \in \mathbb{Z}^k$, then by Lemma 3.1.4, we have that the sum defining $S_1 f$ converges strongly in $L^2_w(\mathbb{R}^k)$ for each $f \in L^2_w(\mathbb{R}^k)$.

Let $g_0 \in C_c(\mathbb{R}^k)$. Since $\{E_{mb} T_{na} g\}$ is a set of atoms, we have that

$$\|f-S_1f\|_{2,w} = \Big\| \sum_n \sum_m a_{n,m}(f)\ E_{mb}(T_{na}g-T_{a_n}g)\Big\|_{2,w}$$
$$\le \Big\| \sum_n \sum_m a_{n,m}(f)\ E_{mb}T_{na}(g-g_0)\Big\|_{2,w}$$
$$+ \Big\| \sum_n \sum_m a_{n,m}(f)\ E_{mb}T_{na}(g_0-T_{(a_n-na)}g_0)\Big\|_{2,w}$$
$$+ \Big\| \sum_n \sum_m a_{n,m}(f)\ E_{mb}T_{a_n}(g_0-g)\Big\|_{2,w}$$
$$= N_1 + N_2 + N_3.$$

Now, $N_1 \le C_1 b^{-k/2} B\|g-g_0\|_{\tilde{w},1}\|f\|_{2,w}$, where C_1 is given in Corollary 2.3.8, and by Lemma 3.1.3, there is a constant C_2 such that $N_3 \le C_2 b^{-k/2} B\|g_0-g\|_{\tilde{w},1}\|f\|_{2,w}$.

To get the desired estimate on N_2, let $h_n = g_0-T_{(a_n-na)}g_0$. For any $0 < d \le 1/b$, define $h_n^\nu = T_{-d\nu}h_n 1_{Q_d+d\nu}$. Thus, $h_n = \sum_\nu h_n^\nu$ and each h_n^ν is supported in $Q_{1/b}$. Repeating the arguments of Lemma 2.3.7, we have that

$$N_2 \le \sum_\nu \tilde{w}(d\nu)[3^k a^{-k} d^k\ B(d^k)]^{1/2}\ \cdot$$
$$b^{-k/2}\Big[\sum_n \|h_n^\nu\|_\infty \sum_m |a_{n,m}|^2\ \omega(n;Q_d,a)\Big]^{1/2}$$
$$\le C_3 \sum_\nu \tilde{w}(d\nu)\ \sup_n \|h_n^\nu\|_\infty\ b^{-k/2}\Big[\sum_n \sum_m |a_{n,m}|^2\ \omega(n;Q_d,a)\Big]^{1/2}$$

where $C_3 = [3^k a^{-k} d^k B(d^k)]^{1/2}$.

I claim that given $\varepsilon > 0$, there is a $\delta > 0$ such that if $\sup_n |a_n-na| < \delta$, then

$$\sum_\nu \tilde{w}(d\nu)\ \sup_n \|h_n^\nu\|_\infty < \varepsilon.$$

To see this, observe that if $\sup_n |a_n-na| < A$ for some constant A, then the functions $h_n = g_0-T_{(a_n-na)}g_0$ are supported in a single compact set K for all $n\in\mathbb{Z}^k$. Consequently, letting $M =$

$\{m \in \mathbb{Z}^k\colon K \cap (Q_1+m) \neq \emptyset\}$, we see that $\#(M) \equiv C_M$ is finite. Now, since $\tilde{w}$ is locally bounded, there is a constant R such that $\tilde{w}(d\nu) \le R$ for each $\nu \in M$. Also, since g_0 is uniformly continuous, there is a $\delta > 0$ such that if $\sup_n |a_n - na| < \delta$ then $|g_0(x) - T_{(a_n - na)} g_0(x)| < \varepsilon C_M^{-1} R^{-1}$ for all $x \in \mathbb{R}^k$ and $n \in \mathbb{Z}^k$. Consequently, $\sup_n \|h_n^\nu\|_\infty \le \sup_n \|h_n\|_\infty < \varepsilon C_M^{-1} R^{-1}$ for all $\nu \in \mathbb{Z}^k$. Finally, we have that

$$\sum_\nu \tilde{w}(d\nu) \sup_n \|h_n^\nu\|_\infty \le R \sum_{\nu \in M} \sup_n \|h_n^\nu\|_\infty \le RC_M \sup_n \|h_n\|_\infty < \varepsilon$$

whenever $\sup_n |a_n - na| < \delta$.

Now I claim that given $\varepsilon > 0$ there is a $\delta > 0$ such that if $\sup_n |a_n - na| < \delta$ then $\|I - S_1\| \equiv \lambda < \varepsilon$. Given $\varepsilon_1 > 0$ there is a function $g_0 \in C_c(\mathbb{R}^k)$ such that $\|g - g_0\|_{\tilde{w},1} < \varepsilon_1$. Also, by the above paragraph, there is $\delta > 0$ such that if $\sup_n |a_n - na| < \delta$ then

$$\sum_\nu \tilde{w}(d\nu) \sup_n \|h_n^\nu\|_\infty\, b^{-k/2} \left[\sum_n \sum_m |a_{n,m}|^2\, \omega(n; Q_d, a) \right]^{1/2}$$
$$\le B\varepsilon_1 b^{-k/2} \|f\|_{2,w}.$$

Hence,

$$\|f - S_1 f\|_{2,w}$$
$$\le C_1 B b^{-k/2} \|g_0 - g\|_{\tilde{w},1} \|f\|_{2,w} + C_3 B \varepsilon_1 b^{-k/2} \|f\|_{2,w} + C_2 B b^{-k/2} \|g - g_0\|_{2,w}$$
$$< (C_1 + C_2 + C_3) B \varepsilon_1 b^{-k/2} \|f\|_{2,w}.$$

Thus, if ε_1 is small enough, we have that $\|I - S_1\| \equiv \lambda < \varepsilon$.

Now, as in Theorem 3.1.2, we have that $\|S_1\| \le \|I - S_1\| + \|I\| = 1 + \lambda$. If $\lambda < 1$ then we have that S_1 is a bijective homeomorphism of $L_w^2(\mathbb{R}^k)$ and that $\|S_1^{-1}\| \le (1-\lambda)^{-1}$. Letting $c_{n,m}(f) = a_{n,m}(S_1^{-1} f)$, we have that for all $f \in L_w^2(\mathbb{R}^k)$,

$$f = \sum_n \sum_m c_{n,m}(f)\, E_{mb} T_{a_n} g$$

where the sum converges strongly in $L^2_w(\mathbb{R}^k)$. Also, by the same argument as in Theorem 3.1.2, we have that

$$A^2(1+\lambda)^{-2}\|f\|^2_{2,w} \le \sum_n \sum_m |c_{n,m}(f)|^2 \ \omega(n;a) \le B^2(1-\lambda)^{-2}\|f\|^2_{2,w}.$$

Finally, given $\varepsilon > 0$ there is a λ_0 such that if $0 < \lambda < \lambda_0$ then $(1-\lambda)^{-1} \le 1+\varepsilon$ and $(1+\lambda)^{-1} \ge 1-\varepsilon$. Moreover, there is a $\delta > 0$ such that if $\sup_n |a_n - na| < \delta$ then $\lambda < \lambda_0$.∎

The idea for the following lemma has appeared in many places including [PW], [Le], [DE], and [Y].

LEMMA 3.1.6. Let $b > 0$ and $\varepsilon > 0$, Then there exists $\delta > 0$ such that if (b_m) is such that $\sup_m |b_m - mb| < \delta$, then

$$\Big\| \sum_m c_m(e^{2\pi i <mb,t>} - e^{2\pi i <b_m,t>}) \Big\|_{L^2(Q_{1/b})} < \varepsilon \Big(\sum_m |c_m|^2 \Big)^{1/2}$$

for every sequence (c_m) with only finitely many non-zero terms. PROOF. First observe that

$$e^{2\pi i <b_m - mb, t>} - 1 = \sum_{\alpha \ne 0} \frac{(2\pi i(b_m - mb))^\alpha t^\alpha}{\alpha!}$$

where the sum is taken over all non-zero multiindices.

Now,

$$\Big\| \sum_m c_m(e^{2\pi i <mb,t>} - e^{2\pi i <b_m,t>}) \Big\|_{L^2(Q_{1/b})}$$

$$= \Big\| \sum_m c_m e^{2\pi i <mb,t>} (1 - e^{2\pi i <b_m - mb,t>}) \Big\|_{L^2(Q_{1/b})}$$

$$= \Big\| \sum_m c_m e^{2\pi i <mb,t>} \sum_{\alpha \ne 0} \frac{(2\pi i(b_m - mb))^\alpha t^\alpha}{\alpha!} \Big\|_{L^2(Q_{1/b})}$$

$$= \Big\| \sum_{\alpha \ne 0} (2\pi i)^{|\alpha|} \ 1/\alpha! \ t^\alpha \sum_m c_m e^{2\pi i <mb,t>} (b_m - mb)^\alpha \Big\|_{L^2(Q_{1/b})}$$

$$\le \sum_{\alpha\neq 0}(2\pi)^{|\alpha|}\ 1/\alpha!\left\|t^{\alpha}\sum_m c_m(b_m-mb)^{\alpha}e^{2\pi i<mb,t>}\right\|_{L^2(Q_{1/b})}$$

$$\le \sum_{\alpha\neq 0}(2\pi b^{-1})^{|\alpha|}\ 1/\alpha!\left\|\sum_m c_m(b_m-mb)^{\alpha}e^{2\pi i<mb,t>}\right\|_{L^2(Q_{1/b})}$$

(since $t\in Q_{1/b}$ implies $|t|_{max}\le b^{-1}$)

$$= \sum_{\alpha\neq 0}(2\pi b^{-1})^{|\alpha|}\ 1/\alpha!\,b^{-k/2}\left[\sum_m |c_m|^2|(b_m-mb)^{\alpha}|^2\right]^{1/2}$$

$$\le b^{-k/2}\left[\sum_m |c_m|^2\right]^{1/2}\sum_{\alpha\neq 0}(2\pi b^{-1}\delta)^{|\alpha|}\ 1/\alpha! \quad (\text{since } \sup_m|b_m-mb| < \delta)$$

$$= (e^{2\pi b^{-1}k\delta}-1)\ b^{-k/2}\left[\sum_m |c_m|^2\right]^{1/2}.$$

Now, given $\varepsilon > 0$ we can choose $\delta > 0$ such that

$$\left|(e^{2\pi b^{-1}k\delta}-1)\ b^{-k/2}\right| < \varepsilon. \blacksquare$$

COROLLARY 3.1.7. Let $a > 0$ and (b_m) a collection of points in $\mathbb{R}^k$ such that for some $b > 0$, $\sup_m|b_m-mb| \le A < \infty$. Then there is a constant M depending only on A such that for any function g,

$$\left\|\sum_n\sum_m a_{n,m}\ E_{b_m}T_{na}g\right\|_{2,w} \le MC\|g\|_{\tilde{w},1}b^{-k/2}\left[\sum_n\sum_m |a_{n,m}|^2\ \omega(n;a)\right]^{1/2}$$

for each sequence $(a_{n,m})$ with finitely many non-zero terms and where C is the constant from Corollary 2.3.8.

PROOF. First, observe that by Lemma 3.1.6,

$$\left\|\sum_m c_m e^{2\pi i<b_m,t>}\right\|_{L^2(Q_{1/b})}$$

$$\le \left\|\sum_m c_m e^{2\pi i<mb,t>}\right\|_{L^2(Q_{1/b})}+\left\|\sum_m c_m e^{2\pi i<b_m-mb,t>}\right\|_{L^2(Q_{1/b})}$$

$$\le b^{-k/2}\left[\sum_m |c_m|^2\right]^{1/2}+(e^{2\pi b^{-1}kA}-1)b^{-k/2}\left[\sum_m |c_m|^2\right]^{1/2}$$

$$= e^{2\pi b^{-1}kA}\ b^{-k/2}\left[\sum_m |c_m|^2\right]^{1/2}$$

where (c_m) is any sequence with only finitely many non-zero terms. Now, by the arguments of Lemma 2.3.7, we have

$$\Big\| \sum_n \sum_m a_{n,m}\ T_{na}E_{b_m}g \Big\|_{2,w}$$

$$\le C_1 \sum_{\nu} \tilde{w}(d\nu)\|g1_{Q+d\nu}\|_\infty \left[\sum_n \Big\| \sum_m a_{n,m}\ E_{b_m} \Big\|^2_{L^2(Q_{1/b})}\ \omega(n;Q,a) \right]^{1/2}$$

$$\le Ce^{2\pi b^{-1}kA}\|g\|_{\tilde{w},1}\ b^{-k/2}\left[\sum_n \sum_m |a_{n,m}|^2\ \omega(n;a) \right]^{1/2}.$$

Hence we are done with $M = e^{2\pi b^{-1}kA}$. ■

THEOREM 3.1.8. Let $g\in W(L^\infty, L^1_{\tilde{w}})$ be such that for some a, b > 0, $\{E_{mb}T_{na}g\}$ is a set of atoms for $L^2_w(\mathbb{R}^k)$ with atomic bounds A, B. Then given $\varepsilon > 0$ there is a $\delta > 0$ such that if (b_m) is any collection of points in $\hat{\mathbb{R}}^k$ such that $\sup_m |mb-b_m| < \delta$ then $\{E_{b_m}T_{na}g\}$ is also a set of atoms for $L^2_w(\mathbb{R}^k)$ with atomic bounds A_1, B_1 where $A_1 \ge A(1-\varepsilon)$ and $B_1 \le B(1+\varepsilon)$.

PROOF. Observe first that $E_xT_yg = e^{2\pi i<y,x>}T_yE_xg$ for all x, $y\in\mathbb{R}^k$. Let $a^1_{n,m}(f) = e^{2\pi i<na,mb>}a_{n,m}(f)$ and note that $|a^1_{n,m}(f)| = |a_{n,m}(f)|$. Define the operator S_1 by

$$S_1f = \sum_n \sum_m a^1_{n,m}(f)\ T_{na}E_{b_m}g.$$

Then by Corollary 3.1.7 the sum defining S_1 converges strongly in $L^2_w(\mathbb{R}^k)$ for each $f\in L^2_w(\mathbb{R}^k)$.

Now,

$$\|f-S_1f\|_{2,w} = \Big\| \sum_n \sum_m a_{n,m}(f)\ E_{mb}T_{na}g - \sum_n \sum_m a^1_{n,m}(f)\ T_{na}E_{b_m}g \Big\|_{2,w}$$

$$= \Big\| \sum_n \sum_m a^1_{n,m}(f)\ T_{na}(E_{mb}-E_{b_m})g \Big\|_{2,w}.$$

By the arguments of Lemma 2.3.7 and Corollary 2.3.8, there is a constant C such that

$$\|f-S_1f\|_{2,w} \le C\|g\|_{\tilde{\omega},1}\left[\sum_n \left\| \sum_m a^1_{n,m}(f)\ (E_{mb}-E_{b_m})\right\|^2_{L^2(Q_{1/b})} \omega(n;a)\right]^{1/2}.$$

By Lemma 3.1.6, given $\varepsilon > 0$ there is a $\delta > 0$ such that if $\sup_m|b_m-mb| < \delta$ then

$$\left\| \sum_m a^1_{n,m}(f)\ (E_{mb}-E_{b_m})\right\|^2_{L^2(Q_{1/b})} < \varepsilon^2 \sum_m |a^1_{n,m}(f)|^2$$

Consequently, we have that

$$\|f-S_1f\|_{2,w} \le C\|g\|_{\tilde{\omega},1}\varepsilon\left[\sum_n \sum_m |a^1_{n,m}(f)|^2\ \omega(n;a)\right]^{1/2}$$

$$\le CB\|g\|_{\tilde{\omega},1}\varepsilon\|f\|_{2,w} \equiv \lambda\|f\|_{2,w}.$$

Now, as in Theorem 3.1.2, $\|S_1\| \le \|I-S_1\| + \|I\| \le 1+\lambda$. If $\lambda < 1$ then S_1 is a bijective homeomorphism of $L^2_w(\mathbb{R}^k)$ and $\|S_1^{-1}\| \le (1-\lambda)^{-1}$. Let $c_{n,m}(f) = a_{n,m}(S_1^{-1}f)$. Then

$$f = \sum_n \sum_m c_{n,m}(f)\ E_{b_m}T_{na}g$$

and arguing as in Theorem 3.1.2,

$$A^2(1+\lambda)^{-2}\|f\|^2_{2,w} \le \sum_n \sum_m |c_{n,m}(f)|^2\ \omega(n;a) \le B^2(1-\lambda)^{-2}\|f\|^2_{2,w}.$$

Finally, given $\varepsilon > 0$ there is a $\delta > 0$ such that if $\sup_m|b_m-mb| < \delta$ then $(1+\lambda)^{-2} \ge 1-\varepsilon$ and $(1-\lambda)^{-2} \le 1+\varepsilon$.■

The following theorem shows that one can combine the two previous results to obtain the expected theorem on stability of atomic decompositions under simultaneous perturbation of the lattice in both time and frequency.

THEOREM 3.1.9. Let $g \in W(L^{\infty}, L^1_{\tilde{w}})$, g continuous, be such that for some a, $b > 0$, $\{E_{mb}T_{na}g\}$ is a set of atoms for $L^2_w(\mathbb{R}^k)$ with atomic bounds A, B. Then given $\varepsilon > 0$ there is a $\delta > 0$ such that if (a_n, b_m), n, $m \in \mathbb{Z}^k$, is any collection of points in $\mathbb{R}^k \times \hat{\mathbb{R}}^k$ such that $\sup_n |na-a_n| < \delta$ and $\sup_m |mb-b_m| < \delta$ then $\{E_{b_m}T_{a_n}g\}$ is also a set of atoms for $L^2_w(\mathbb{R}^k)$ with atomic bounds A_1, B_1 where $A_1 \geq A(1-\varepsilon)$ and $B_1 \leq B(1+\varepsilon)$.

PROOF. Let us define the operator S_1 by

$$S_1 f = \sum_n \sum_m a_{n,m}(f)\ E_{b_m}T_{a_n}g.$$

Then

$$\begin{aligned}
\|f-S_1f\|_{2,w} &\leq \Big\| \sum_n \sum_m a_{n,m}(f)\ (E_{mb}T_{na}g - E_{b_m}T_{a_n}g) \Big\|_{2,w} \\
&\leq \Big\| \sum_n \sum_m a_{n,m}(f)\ E_{mb}T_{na}(g - T_{(a_n - na)}g) \Big\|_{2,w} \\
&\qquad + \Big\| \sum_n \sum_m a^1_{n,m}(f)\ T_{a_n}(E_{mb} - E_{b_m})g \Big\|_{2,w} \\
&= (1) + (2)
\end{aligned}$$

where $a^1_{n,m}(f) = e^{2\pi i \langle a_n, mb-b_m \rangle} a_{n,m}(f)$ and in particular, $|a^1_{n,m}(f)| = |a_{n,m}(f)|$. Now, by Theorem 3.1.5, given $\varepsilon_1 > 0$, there is a $\delta_1 > 0$ such that if $\sup_n |a_n - na| < \delta_1$ then $(1) < \varepsilon_1 \|f\|_{2,w}$. We wish to obtain a similar estimate on (2). By Lemma 3.1.3 and the arguments of Lemma 2.3.7, we know that there is a constant C such that

$$(2) \leq C\|g\|_{\tilde{w},1} \left[\sum_n \Big\| \sum_m a^1_{n,m}(f)\ (E_{mb} - E_{b_m}) \Big\|^2_{L^2(Q_{1/b})} \omega(n;a) \right]^{1/2}.$$

By Lemma 3.1.6, given $\varepsilon_2 > 0$ there is a $\delta_2 > 0$ such that if $\sup_m |b_m - mb| < \delta_2$ then

$$\left\| \sum_m a^1_{n,m}(f)\ (E_{mb}-E_{b_m}) \right\|^2_{L^2(Q_{1/b})} < \varepsilon_2^2 b^{-k} \sum_m |a_{n,m}|^2.$$

Finally, we have that (2) $\leq \varepsilon_2 CB \|g\|_{\tilde{w},1} \|f\|_{2,w}$.

Thus given $\lambda > 0$ there exist numbers δ_1 and δ_2 such that if $\sup_n |a_n - na|$ and $\sup_m |b_m - mb| < \delta = \min\{\delta_1, \delta_2\}$ then $\|I - S_1\| < \lambda$. Hence the conclusion follows from the same arguments used in Theorems 3.1.2, 3.1.5 and 3.1.8.■

Section 3.2. Stability with respect to lattice parameters

In this section, we show that, under certain assumptions, a set of W-H atoms is stable under a change in the values of the frame parameters. To do this, we require a stronger assumption on the analyzing vector g which is strictly weaker than the assumption that $\sum |g(x-na)|^2$ be bounded above and below (cf. Example 3.2.8).

THEOREM 3.2.1. Let g be such that $g, g_{-} \in W(L^{\infty}, L^1_{\omega})$ and suppose that there exists a cube $R \subset \mathbb{R}^k$ such that

(1) $$0 < \operatorname*{ess\,inf}_{x \in R} |g(x)| \leq \operatorname*{ess\,sup}_{x \in R} |g(x)| < \infty.$$

Then there exist numbers a_0 and b_0 such that for any $0 < a \leq a_0$ and $0 < b \leq b_0$, $\{E_{mb}T_{na}g\}$ is a set of atoms for $L^2_w(\mathbb{R}^k)$.

The proof of this theorem will follow from analogs of certain lemmas in this chapter. First, we show that one may assume without loss of generality that R is centered at the origin.

LEMMA 3.2.2. If $\{E_{mb}T_{na}g\}$ is a set of atoms for $L^2_w(\mathbb{R}^k)$ then for almost every $s \in \mathbb{R}^k$, $\{E_{mb}T_{na}(T_s g)\}$ is also a set of atoms for $L^2_w(\mathbb{R}^k)$.

PROOF. We know that for some collection of linear functionals, $(a_{n,m})$, and for all $f \in L^2_w(\mathbb{R}^k)$,

$$f = \sum_n \sum_m a_{n,m}(f)\ E_{mb}T_{na}g.$$

Thus it follows that for all $f \in L_w^2(\mathbb{R}^k)$,

$$\begin{aligned} f = T_sT_{-s}f &= \sum_n \sum_m a_{n,m}(T_{-s}f)\ T_s(E_{mb}T_{na}g) \\ &= \sum_n \sum_m a_{n,m}(T_{-s}f)e^{-2\pi i\langle mb,s\rangle}\ E_{mb}T_{na}(T_sg) \\ &\equiv \sum_n \sum_m a^1_{n,m}(T_{-s}f)\ E_{mb}T_{na}(T_sg). \end{aligned}$$

Now, there exist constants τ_1, $\tau_2 > 0$ such that for all $f \in L_w^2(\mathbb{R}^k)$,

$$\tau_1\|T_{-s}f\|^2_{2,w} \le \sum_n \sum_m |a^1_{n,m}(T_{-s}f)|^2\ \omega(n;a) \le \tau_2\|T_{-s}f\|^2_{2,w}.$$

But, $\|f\|_{2,w} = \|T_sT_{-s}f\|_{2,w} \le \tilde{w}(s)\|T_{-s}f\|_{2,w}$ and $\|T_{-s}f\|_{2,w} \le \tilde{w}(-s)\|f\|_{2,w}$ and for every $s \in \mathbb{R}^k$, $0 < \tilde{w}(s), \tilde{w}(-s) < \infty$. Thus $\{E_{mb}T_{na}(T_sg)\}$ satisfies the definition of a set of atoms.∎

LEMMA 3.2.3. Let φ be a bounded function supported in a cube Q. Suppose that φ is essentially bounded above and below on some subcube Q_0 of Q. Then there exists $a_0 > 0$ such that for some constants $A, B > 0$ depending only on a_0, φ, and k we have that for all $0 < a \le a_0$ and almost every $x \in \mathbb{R}^k$,

$$a^{-k}A \le \sum_n |\varphi(x-na)|^2 \le a^{-k}B.$$

PROOF. Let s be the side length of Q, s_0 the side length of Q_0, and let $0 < a_0 < s_0$. Then for each $x \in \mathbb{R}^k$ and each $0 < a \le a_0$,

$$\#\{n \in \mathbb{Z}^k:\ x-na \in Q\} \le \left(\lfloor s/a \rfloor + 1\right)^k \le a^{-k}(s+a)^k \le a^{-k}(s+a_0)^k.$$

Therefore,

$$\sum_n |\varphi(x-na)|^2 \le \|\varphi\|^2_\infty \sup_x \#\{n \in \mathbb{Z}^k:\ x-na \in Q\} \le a^{-k}[(s+a_0)^k\|\varphi\|^2_\infty].$$

Observe now that for all $x \in \mathbb{R}^k$ and $0 < a \le a_0$,

$$\#\{n\in\mathbb{Z}^k:\ x-na\in Q_0\} \geq \lfloor s_0/a\rfloor^k \geq a^{-k}(s_0-a)^k \geq a^{-k}(s_0-a_0)^k.$$

Therefore,

$$\sum_n |\varphi(x-na)|^2 \geq \operatorname*{ess\,inf}_{x\in Q_0} |\varphi(x)|^2 \inf_x \#\{n\in\mathbb{Z}^k:\ x-na\in Q_0\}$$
$$\geq a^{-k} \operatorname*{ess\,inf}_{x\in Q_0} |\varphi(x)|^2 (s_0-a_0)^k. \blacksquare$$

We now state results corresponding to Lemmas 2.3.5, 2.3.1, 2.3.2, and Corollary 2.3.3. The proofs of these results are almost exactly the same as the originals, the only difference being in the nature of the constants.

LEMMA 3.2.4. (Lemma 2.3.5) Let g satisfy condition (1) of Theorem 3.2.1 and suppose that g, $g_-\in W(L^\infty, L^1_w)$. Then there is a cube Q and a number a_0 such that there exist constants A', B' > 0 depending on a_0 and g only such that for every cube Q_0 containing Q,

$$a^{-k}A' \leq \left|\sum_n \bar{g}1_Q(x-na)g1_{Q_0}(x-na)\right| \leq a^{-k}B'.$$

In particular,

$$a^{-k}A' \leq \sum_n |g1_Q(x-na)|^2 \leq a^{-k}B'.$$

PROOF. We take as Q the cube R on which |g| is bounded above and below. We may assume without loss of generality that R is centered at the origin because if not we can replace g with an appropriate shift of g such that the correspondingly shifted R is centered at the origin. By Lemma 3.2.2, the shifted g will generate a set of atoms if and only if g does for the same values of a and b.

The result follows immediately from Lemma 3.2.3 because for all $x \in \mathbb{R}^k$, $\bar{g}1_R(x)g1_{Q_0}(x) = |g1_R(x)|^2$ whenever $R \subset Q_0$. ∎

LEMMA 3.2.5. (Lemma 2.3.1). Given g and R as in Theorem 3.2.1, there exist constants a_0 and $b_0 > 0$ such that there exist constants C_1, $C_2 > 0$ depending only on g and k such that for all $0 < a \le a_0$, $0 < b \le b_0$, and $f \in L^2_w(\mathbb{R}^k)$,

$$a^{-k}b^{-k}C_1\|f\|^2_{2,w} \le \sum_n \sum_m |\langle f, E_{mb}T_{na}(g1_R)\rangle|^2 \; \omega(n;a) \le a^{-k}b^{-k}C_2\|f\|^2_{2,w}.$$

PROOF. We choose a_0 as in Lemma 3.2.3, and b_0 so that the side length of R is at most $1/b_0$. Hence we have the estimates required for Lemma 2.3.1 and the result follows identically. ∎

LEMMA 3.2.6 (Lemma 2.3.2). Let g and R be as in the hypotheses of Theorem 3.2.1. Define the operator, S, by

$$Sf = \sum_n \sum_m \langle f, E_{mb}T_{na}(g1_R)\rangle \; E_{mb}T_{na}(g1_Q).$$

Then there is a number $a_0 > 0$ such that given any cube, Q, containing R there is a $b_0 > 0$ such that for all $0 < a \le a_0$, $0 < b \le b_0$, and $f \in L^2_w(\mathbb{R}^k)$, Sf converges strongly in $L^2_w(\mathbb{R}^k)$ to

$$f(x)\; b^{-k} \sum_n \bar{g}1_R(x-na)g1_Q(x-na).$$

Moreover, S is a bijective homeomorphism of $L^2_w(\mathbb{R}^k)$ onto itself and for all $f \in L^2_w(\mathbb{R}^k)$,

$$f = \sum_n \sum_m \langle S^{-1}f, E_{mb}T_{na}(g1_R)\rangle \; E_{mb}T_{na}(g1_Q).$$

PROOF. Choose a_0 as in Lemma 3.2.3, and b_0 so small that the side length of Q is at most $1/b_0$. Thus the hypotheses of Lemma 2.3.2

are satisfied for all $0 < a \le a_0$ and $0 < b \le b_0$ and so the result follows identically.■

COROLLARY 3.2.7 (Corollary 2.3.3). Let g and R be as in the hypotheses of Theorem 3.2.1. There is a number $a_0 > 0$ such that for any cube Q containing R, there is a number $b_0 > 0$ such that for all $0 < a \le a_0$ and $0 < b \le b_0$ there exists a collection of linear functionals, $(c_{n,m})$ such that for all $f \in L^2_w(\mathbb{R}^k)$,

$$f = \sum_n c_{n,m}(f)\ E_{mb}T_{na}(g1_Q)$$

and there exist constants C_1, $C_2 > 0$ depending only on g and k such that

$$C_1\|f\|^2_{2,w} \le a^{-k}b^{-k}\sum_n\sum_m |c_{n,m}(f)|^2\ \omega(n;a) \le C_2\|f\|^2_{2,w}.$$

PROOF. Choose a_0 as in Lemma 3.2.3, and b_0 so small that the side length of Q is at most $1/b_0$. The conclusion follows exactly as in Corollary 2.3.3 with

$$c_{n,m}(f) = \langle S^{-1}f, E_{mb}T_{na}(g1_R)\rangle$$

and

$$Sf = \sum_n\sum_m \langle f, E_{mb}T_{na}(g1_R)\rangle\ E_{mb}T_{na}(g1_Q).■$$

PROOF OF THEOREM 3.2.1. Let a_0 be as in Lemma 3.2.3. Now given $\varepsilon > 0$ there is a cube Q containing R such that $\|g1_Q - g\|_{\tilde{\omega},1} < \varepsilon$. We will choose an appropriate ε and Q later but in the meantime, for any cube Q and numbers a, b define the operator U by

$$Uf = \sum_n c_{n,m}(f)\ E_{mb}T_{na}g$$

where the functionals $c_{n,m}$ are as defined in Corollary 3.2.7. We then have that

$$\|f-Uf\|_{2,w} = \left\| \sum_n \sum_m c_{n,m}(f)\ E_{mb}T_{na}(g1_Q-g) \right\|_{2,w}$$

$$\leq C\|g1_Q-g\|_{\tilde{w},1}\ a^{-k/2}b^{-k/2}\left[\sum_n \sum_m |c_{n,m}(f)|^2\ \omega(n;a)\right]^{1/2}$$

where $C = C_1(\lfloor d \rfloor+2)^k B(1)\tilde{B}(d^k)[B(d^k)3^k d^k]^{1/2}$, $C_1 = B(|Q_\alpha|)$, $\alpha = \max\{1,d\}$, and $0 < d \leq 1/b$. Note that C is independent of a and b.

It follows from Lemma 3.2.7 that for all sufficiently small b there is a constant C_2 depending only on g and k such that,

$$\|f-Uf\|_{2,w} \leq CC_2\|g1_Q-g\|_{\tilde{w},1}\ \|f\|_{2,w}.$$

Now, let $\varepsilon_1 > 0$ be so small that $CC_2\varepsilon_1 = \lambda < 1$, and let Q be a cube containing R such that $\|g1_Q-g\|_{\tilde{w},1} < \varepsilon_1$ and let b_0 be so small that the conclusion of Lemma 3.2.7 holds and the side length of Q is at most $1/b_0$. The result now follows exactly as in Theorem 2.3.9.■

EXAMPLE 3.2.8. There is a function g on $\mathbb{R}$ such that for almost every $x\in\mathbb{R}$,

$$\sum_{n\in Z} |g(x-n)|^2 \equiv 1$$

but for which condition (1) of Theorem 3.2.1 fails. Let $\alpha > 0$ and let $E_0 \subset [0,1]$ be a Cantor set of measure α. Then we can write

$$[0,1]\backslash E_0 = \bigcup_{i=1}^{\infty} I_{i,1}$$

where the $I_{i,1}$ are open, pairwise disjoint intervals. For each i, let $E_{i,1} \subset I_{i,1}$ be a Cantor set of measure $\alpha|I_{i,1}| > 0$, and let $E_1 = \bigcup_{i=1}^{\infty} E_{i,1}$. By the construction of E_1, it is clear that

$[0,1]\setminus(E_0 \cup E_1)$ can be written as a disjoint union of open intervals, i.e. that

$$[0,1]\setminus(E_0 \cup E_1) = \bigcup_{i=1}^{\infty} I_{i,2}.$$

Now for each i, let $E_{i,2} \subset I_{i,2}$ be a Cantor set of measure $\alpha|I_{i,2}| > 0$, let $E_2 = \bigcup_{i=1}^{\infty} E_{i,2}$.

We can continue this process and obtain a countable collection of disjoint sets $\{E_n\}_{n\in Z}$. It is easy to see that

$$\left|[0,1]\setminus \bigcup_{n=0}^{N-1} E_n\right| = \left|\bigcup_{i=1}^{\infty} I_{i,N}\right|$$

and that

$$\left|\bigcup_{i=1}^{\infty} I_{i,N}\right| = (1-\alpha)^N.$$

This implies that $\bigcup_{n=0}^{\infty} E_n = 1$.

Now, let

$$g(x) = \sum_{n\in Z} 1_{E_n+n}(x).$$

Then for every interval $I \subset \mathbb{R}$, $\operatorname{ess\,inf}_{x\in I} |g(x)| = 0$ but since $\bigcup_{n=0}^{\infty} E_n$ has full measure, we see that for almost every $x \in \mathbb{R}$,

$$\sum_{n\in Z} |g(x-n)|^2 \equiv 1. \blacksquare$$

CHAPTER 4

Continuity of the Frame Operator

To every set of atoms in a Banach space B can be associated a Banach frame for B, called the dual frame, which gives one a Fourier series-like expansion of any element of B in terms of the set of atoms. Specifically, if $\{g_i\}$ is a set of atoms for B and $\{e_i\}$ the corresponding dual frame, we have that for each $f \in B$, $f = \sum \langle f, e_i \rangle g_i$.

Whenever it makes sense, we can take $e_i = S^{-1} g_i$ where the operator S is given by $Sf = \sum \langle f, g_i \rangle g_i$. For example, if B is a Hilbert space, the dual frame must be given by the above formula (cf. Theorem 5.1.6). We have also seen that this is the case when $B = L^2_w(\mathbb{R}^k)$ with w a moderate weight and the functions g_i are W-H wavelets generated by an appropriate vector $g \in L^2_w(\mathbb{R}^k)$ for certain parameter values (cf. Theorem 2.6.2).

In this chapter, we investigate the properties of the dual frame when the corresponding set of atoms is a collection of W-H wavelets. In this case, it is well known that the dual frame is also a collection of W-H wavelets generated from a single vector.

In Section 4.1, we show that the vector $S^{-1}g$ reflects many of the decay properties of g by showing that the operators S and S^{-1} preserve many of these same properties. In Section 4.2, we do the same with certain smoothness properties of g, that is, properties defined by the decay of the Fourier transform. Also, in this section, we obtain formulas for derivatives of Sg for a given g, and show that S maps the space $\mathcal{S}(\mathbb{R}^k)$ continuously into itself. In Section 4.3, we give a formula for computing $S^{-1}g$ for any g, and in Section 4.4, we give a generalization of a result of Benedetto ([B]) concerning the invertibility of the continuous frame operator.

Section 4.1. Preservation of decay by the frame operator.

The results in this and the following section can be thought of as stability results because they show that, for the cases we examine, if a function f can be written

$$f = \sum_n \sum_m \langle f, E_{mb}T_{na}\varphi\rangle E_{mb}T_{na}\psi$$

then the function φ is forced to be a slightly perturbed version of ψ. That is, φ has most of the decay and smoothness properties of ψ.

DEFINITION 4.1.1. Given functions φ and ψ on $\mathbb{R}^k$, we define formally *the S-operator corresponding to* φ *and* ψ or simply *the S-operator* by

$$Sf = \sum_n \sum_m \langle f, E_{mb}T_{na}\varphi\rangle \; E_{mb}T_{na}\psi.$$

When we wish to make the auxilliary functions φ and ψ explicit, we write $S(\varphi,\psi)$ for S, and if we wish to make the values of a and b in the definition explicit, we write $S_{a,b}(\varphi,\psi)$ or $S_{a,b}$.

LEMMA 4.1.2. Let $0 < a \le c$, let g and h be any two functions, and m any submultiplicative function. Then

$$\sum_j m(jc)\beta_{g,h}(jc) \le 2^k \|gm\|_{\infty,1,a} \|h_- m\|_{\infty,1,c}$$

where

$$\beta_{g,h}(s) = \operatorname{ess\,sup}_x \left| \sum_n g(x-na)\bar{h}(x-s-na) \right|.$$

PROOF.

$$\sum_j m(jc)\beta_{g,h}(jc)$$
$$= \sum_j m(jc) \operatorname{ess\,sup}_x \left| \sum_n g(x-na)\bar{h}_-(jc-x+na) \right|$$
$$\le \sum_j \operatorname{ess\,sup}_x \left| \sum_n g(x-na)m(x-na)\bar{h}_-(jc-x+na)m(jc-x+na) \right|$$
$$\le \sum_j \sum_n \operatorname{ess\,sup}_{x\in Q_a} |g(x-na)m(x-na)| \ \operatorname{ess\,sup}_{x\in Q_c} |h_-(jc-x+na)m(jc-x+na)|$$

since the sum over n above actually is an a-periodic function so that we need only take the essential supremum over Q_a. Now,

$$\sum_j \sum_n \operatorname{ess\,sup}_{x\in Q_a} |g(x-na)m(x-na)| \ \operatorname{ess\,sup}_{x\in Q_c} |h_-(jc-x+na)m(jc-x+na)|$$
$$= \sum_n \operatorname{ess\,sup}_{x\in Q_a} |g(x-na)m(x-na)| \sum_j \operatorname{ess\,sup}_{x\in Q_c} |h_-(jc-x+na)m(jc-x+na)|$$
$$\le 2^k \|gm\|_{\infty,1,a} \|h_- m\|_{\infty,1,c}$$

by Corollary 2.2.7.■

LEMMA 4.1.3. Let m be a submultiplicative function and let g and h be functions such that gm and $h_- m$ are in $W(L^\infty, L^1)$. Then

$$\lim_{b\to 0} \sum_{j\neq 0} m(j/b)\beta_{g,h}(j/b) = 0$$

where $\beta_{g,h}$ is defined as in Lemma 4.1.2.

PROOF. Let $\varepsilon > 0$. There exists a cube Q such that $Q = -Q$, $\|gm-gm1_Q\|_{\infty,1} < \varepsilon$, $\|(h_-m-h_-m1_Q)_-\|_{\infty,1} < \varepsilon$, $\|gm-gm1_Q\|_{\infty,1,a} < \varepsilon$, and $\|(h_-m-h_-m1_Q)_-\|_{\infty,1,a} < \varepsilon$ with $a > 0$ as in the definition of $\beta_{g,h}$. Now, letting $g_0 = g1_Q$, $g_1 = g-g1_Q$, $h_0 = h1_Q$ and $h_1 = h-h1_Q$, we have that for each j,

$$\beta_{g,h}(j/b) \leq \beta_{g_0,h_0}(j/b)+\beta_{g_0,h_1}(j/b)+\beta_{g_1,h_0}(j/b)+\beta_{g_1,h_1}(j/b).$$

Observe that for all b small enough, $\beta_{g_0,g_0}(j/b) = 0$ for $j\neq 0$, and that if $b \geq 1$ then $\|\cdot\|_{\infty,1,1/b} \leq 2^k\|\cdot\|_{\infty,1}$ (cf. Corollary 2.2.8).

$$\begin{aligned}
&\sum_{j\neq 0} m(j/b)\beta_{g,h}(j/b)\\
&\leq \sum_{j\neq 0} m(j/b)\beta_{g_0,h_1}(j/b)+\sum_{j\neq 0} m(j/b)\beta_{g_1,h_0}(j/b) +\\
&\qquad\qquad + \sum_{j\neq 0} m(j/b)\beta_{g_1,h_1}(j/b)\\
&\leq 2^{2k}(\|g_0m\|_{\infty,1,a}\|h_{1-}m\|_{\infty,1}+\|g_1m\|_{\infty,1,a}\|h_{0-}m\|_{\infty,1}\\
&\qquad\qquad + \|g_1m\|_{\infty,1,a}\|h_{1-}m\|_{\infty,1})\\
&\leq 2^{2k}(\varepsilon\|gm\|_{\infty,1,a}+\varepsilon\|h_-m\|_{\infty,1}+\varepsilon^2).
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we are done.■

THEOREM 4.1.4. Let f, $h\in L^2(\mathbb{R}^k)$ and suppose that φ, $\psi\in W(L^\infty,L^1)$. Then

(1) the sums Sf and Sh converge strongly in $L^2(\mathbb{R}^k)$,

(2) $\langle Sf,h\rangle = b^{-k}\sum_j \int f(x-j/b)\bar{h}(x) \sum_n \psi(x-na)\bar{\varphi}(x-na-j/b)$, and

(3) $Sf = b^{-k}\sum_j f(x-j/b) \sum_n \psi(x-na)\bar{\varphi}(x-na-j/b)$.

PROOF. Let $f\in L^2(\mathbb{R}^k)$. By the argument in Theorem 5.2.1, Lemma 4.1.2, and the fact that $\varphi,\psi\in W(L^\infty,L^1)$, we have that for all

$f \in L^2(\mathbb{R}^k)$,

$$\sum_n \sum_m |<f, E_{mb}T_{na}\varphi>|^2 \le B\|f\|_2^2$$

and

$$\sum_n \sum_m |<f, E_{mb}T_{na}\psi>|^2 \le B\|f\|_2^2$$

for some $B > 0$. Thus, by a very familiar argument, we can show that the sum defining Sf converges in $L^2(\mathbb{R}^k)$. Specifically, we can show that for any sequence $(a_{n,m})$ with $\sum\sum |a_{n,m}|^2 < \infty$,

$$\Big\| \sum_{n\in G} \sum_{m\in F} a_{n,m}E_{mb}T_{na}\psi \Big\|_2 \le B\Big(\sum_{n\in G} \sum_{m\in F} |a_{n,m}|^2\Big)^{1/2}$$

for any finite subsets F, G of $\mathbb{Z}^k$ (cf. [HW]). Thus, (1) holds.

Now suppose that f, $h \in C_c(\mathbb{R}^k)$. Then

$$<Sf, h> = <\sum_n \sum_m <f, E_{mb}T_{na}\varphi>\ E_{mb}T_{na}\psi, h>$$

$$= \sum_n \sum_m <f, E_{mb}T_{na}\varphi>\ \overline{<h, E_{mb}T_{na}\psi>}$$

$$= \sum_n \sum_m \left(\int f(x)\bar{\varphi}(x-na)e^{-2\pi i<mb,x>}\ dx\right)\left(\overline{\int h(t)\bar{\psi}(t-na)e^{-2\pi i<mb,x>}\ dt}\right)$$

$$\overset{(i)}{=} b^{-k}\sum_n \int_{Q_{1/b}} \sum_j f(x-j/b)\bar{\varphi}(x-na-j/b) \sum_l \bar{h}(x-l/b)\psi(x-na-l/b)\ dx$$

$$= b^{-k}\sum_n \int \bar{h}(x)\psi(x-na) \sum_j f(x-j/b)\bar{\varphi}(x-na-j/b)\ dx$$

$$\overset{(ii)}{=} b^{-k}\int \sum_j f(x-j/b)\bar{h}(x) \sum_n \psi(x-na)\bar{\varphi}(x-na-j/b)\ dx.$$

The equality (i) can be justified as follows. First, observe that the numbers

$$b^{k/2}\int f(x)\bar{\varphi}(x-na)e^{-2\pi i<mb,x>}\ dx,$$

and

$$b^{k/2}\int h(t)\bar{\psi}(t-na)e^{-2\pi i\langle mb,t\rangle}\,dt$$

are the Fourier coefficients of the 1/b-periodic functions

$$\sum_j f(x-j/b)\bar{\varphi}(x-na-j/b), \text{ and } \sum_l h(x-l/b)\bar{\psi}(x-na-l/b)$$

respectively. If we can show that each is in $L^2(Q_{1/b})$ for each n, the equality (i) will follow from Parseval's formula. It will be enough to show that each is bounded. Now, since f and h are bounded, we have that,

$$\left(\int\left|\sum_j f(x-j/b)\bar{\varphi}(x-na-j/b)\right|^2 dx\right)^{1/2}$$

$$\leq \sum_j \left(\int_{Q_{1/b}} |f(x-j/b)|^2|\varphi(x-na-j/b)|^2\,dx\right)^{1/2}$$

$$\leq \|f\|_\infty \left(\sum_j \operatorname*{ess\,sup}_{x\in Q_{1/b}} |\varphi(x-na-l/b)|^2\right)^{1/2} b^{-k/2}$$

$$\leq \|f\|_\infty \|T_{na}\varphi\|_{\infty,1} b^{-k/2} < \infty.$$

Similarly,

$$\left(\int\left|\sum_l h(x-l/b)\bar{\psi}(x-na-l/b)\right|^2 dx\right)^{1/2} \leq \|h\|_\infty \|T_{na}\psi\|_{\infty,1} b^{-k/2} < \infty.$$

The equality (ii) is justified since by Lemma 4.1.3 and the fact that $f\in L^2(\mathbb{R}^k)$ and $h\in L^2(\mathbb{R}^k)$, the iterated sums and integral following equality (ii) converge absolutely at each $x\in\mathbb{R}^k$ so that any interchange of summation and integration is justified.

Since $C_c(\mathbb{R}^k)$ is dense in $L^2(\mathbb{R}^k)$, (2) follows. (3) follows immediately from (2).■

THEOREM 4.1.5. Let B be a Banach space of tempered distributions on $\mathbb{R}^k$ satisfying the following conditions.

(1) $L^2(\mathbb{R}^k) \cap B$ is dense in B.

(2) B is a Banach module over $L^\infty(\mathbb{R}^k)$, i.e., for every $h\in L^\infty(\mathbb{R}^k)$ and $f\in L^2(\mathbb{R}^k) \cap B$, we have that $hf\in B$ and $\|hf\|_B \le \|h\|_\infty \|f\|_B$. (We can define hf for arbitrary $f\in B$ by letting $f_n \longrightarrow f$ in B, with $f_n\in L^2(\mathbb{R}^k) \cap B$. Then $\{hf_n\}$ is Cauchy in B. Define hf as the limit of this sequence.)

(3) For each $a\in\mathbb{R}^k$, T_a acts boundedly on B, and $\|T_a\|_{B\to B}$ is denoted by m(a).

If φ and ψ are such that φ and ψ are in $W(L^\infty,L^1) \cap B$, and φ_-m and ψm are in $W(L^\infty,L^1)$, then for any a, b > 0, the S-operator can be extended uniquely to a continuous operator on B.

PROOF. By Theorem 4.1.4, S is defined on $L^2(\mathbb{R}^k) \cap B$ where the sum is taken to mean $L^2(\mathbb{R}^k)$ convergence of the partial sums. Moreover,

$$Sf = b^{-k} \sum_j f(x-j/b) \sum_n \psi(x-na)\overline{\varphi}(x-na-j/b)$$

for such f. Define the operator S_* on B by

$$S_*f = b^{-k} \sum_j f(x-j/b)G_j(x)$$

where

$$G_j(x) = \sum_n \psi(x-na)\overline{\varphi}(x-na-j/b)$$

and where the sum over j converges strongly in B. In fact, the partial sums defining S_* converge in operator norm to S_*.

To see this, observe that by the assumptions on φ and ψ, and by Lemma 4.1.2,

$$\sum_j m(j/b)\|G_j\|_\infty < \infty.$$

Thus, given $\varepsilon > 0$, there is a finite set $F \subset \mathbb{Z}^k$ such that for every finite set $G \subset \mathbb{Z}^k$,

$$\sum_{j \in G \setminus F} m(j/b) \|G_j\|_\infty < \varepsilon.$$

Therefore,

$$\left\| \sum_{j \in G \setminus F} f(x-j/b) G_j(x) \right\|_B \leq \sum_{j \in G \setminus F} \|f(x-j/b) G_j(x)\|_B$$

$$\leq \sum_{j \in G \setminus F} \|T_{j/b} f\|_B \|G_j\|_\infty \leq \|f\|_B \sum_{j \in G \setminus F} m(j/b) \|G_j\|_\infty < \varepsilon \|f\|_B.$$

It follows that the sequence of partial sums defining S_* is Cauchy in operator norm and hence converges to some bounded operator on B. It is also clear that

$$\|S_* f\|_B \leq b^{-k} \|f\|_B \sum_j m(j/b) \|G_j\|_\infty.$$

Hence, S_* is a continuous operator on B which agrees with S on $L^2(\mathbb{R}^k) \cap B$ and so is an extension of S. Since there is only one continuous extension of S to B, S_* is unique.∎

THEOREM 4.1.6. Let B, φ and ψ be as in Theorem 4.1.5 and suppose further that for some constant $A > 0$, and almost every $x \in \mathbb{R}^k$,

$$A \leq \left| \sum_n \psi(x-na) \bar{\varphi}(x-na) \right|.$$

Then there exists $b_0 > 0$ such that for all $0 < b < b_0$, the S-operator is a topological isomorphism from B onto B.

PROOF. Let b_0 be so small that for all $0 < b < b_0$,

$$A^{-1} \sum_{j \neq 0} m(j/b) \|G_j\|_\infty = \lambda < 1.$$

Such a b_0 exists by the assumptions on φ and ψ, and Lemma 4.1.2. Then we have that

$$b^k G_0(x)^{-1} Sf(x) = f(x) + b^k G_0(x)^{-1} \sum_{j\neq 0} f(x-j/b) G_j(x).$$

Thus,

$$\|f - b^k G_0^{-1} Sf\|_B \le A^{-1} \Big\| \sum_{j\neq 0} f(x-j/b) G_j(x) \Big\|_B$$
$$\le \|f\|_B A^{-1} \sum_{j\neq 0} m(j/b) \|G_j\|_\setminus \le \lambda \|f\|_B.$$

Thus the inverse of the operator $b^k G_0^{-1} S$ can be computed by means of an absolutely convergent power series. Thus $b^k G_0^{-1} S$ is a topological isomorphism on B. Since multiplication by $b^k G_0^{-1}$ is obviously a topological isomorphism as well, the theorem is proved.■

The space $L^\infty_w(\mathbb{R}^k)$, w moderate, does not satisfy the hypotheses of Theorems 4.1.5 and 4.1.6 because compactly supported functions are not dense in $L^\infty_w(\mathbb{R}^k)$. The following two theorems give interpretations of S as an operator on the space $L^\infty_w(\mathbb{R}^k)$.

THEOERM 4.1.7. Let $\varphi\_\tilde{w}^2$ and $\psi\tilde{w}^2$ be in $W(L^\infty, L^1)$ and suppose that $f \in L^\infty_w(\mathbb{R}^k)$. Then for each $n \in \mathbb{Z}^k$,

$$\sum_m \langle f, E_{mb}T_{na}\varphi\rangle E_{mb}T_{na}\psi = b^{-k} \sum_j f(x-j/b)\psi(x-na)\bar{\varphi}(x-na-j/b)$$

where the sum on the left converges in $L^2_w(\mathbb{R}^k)$ and that on the right converges absolutely and uniformly. Also,

$$b^{-k} \sum_n \sum_j f(x-j/b)\psi(x-na)\bar{\varphi}(x-na-j/b)$$

converges absolutely and uniformly on compact sets. Thus, Sf converges as an iterated sum, the inner sum converging in $L^2_w(\mathbb{R}^k)$

and the outer sum uniformly on compact sets, and

$$Sf = b^{-k}\sum_j f(x-j/b)\sum_n \psi(x-na)\bar{\varphi}(x-na-j/b).$$

S interpreted in this way is a continuous operator on $L^\infty_w(\mathbb{R}^k)$.

PROOF. We know that the sequence of numbers $\{b^k\langle f, E_{mb}T_{na}\varphi\rangle\}$ are the Fourier coefficients of the 1/b-periodic function

$$\sum_j f(x-j/b)\bar{\varphi}(x-na-j/b).$$

I claim that the above function is bounded and hence is in $L^2(Q_{1/b})$. First, the essential supremum of a 1/b-periodic function is equal to the essential supremum of the function over $Q_{1/b}$. Hence, since $w(0) \le w(x)\tilde{w}^2(-x)$ for all $x\in\mathbb{R}^k$,

$$\Big\| \sum_j f(x-j/b)\bar{\varphi}(x-na-j/b)\Big\|_\infty = \operatorname*{ess\,sup}_{Q_{1/b}}\Big| \sum_j f(x-j/b)\bar{\varphi}(x-na-j/b)\Big|$$

$$\le \operatorname*{ess\,sup}_{Q_{1/b}} \sum_j |f(x-j/b)|w(x-j/b)|T_{na}\bar{\varphi}_-(j/b-x)|\tilde{w}^2(j/b-x)$$

$$\le w(0)^{-1}\|fw\|_\infty \sum_j \operatorname*{ess\,sup}_{Q_{1/b}}|T_{na}\varphi_-\tilde{w}^2(j/b-x)|$$

$$= w(0)^{-1}\|fw\|_\infty\|T_{na}\varphi_-\tilde{w}^2\|_{\infty,1,1/b} < \infty.$$

It follows then that

$$\sum_m \langle f, E_{mb}T_{na}\varphi\rangle E_{mb} = \sum_j f(x-j/b)\bar{\varphi}(x-na-j/b)$$

in $L^2(Q_{1/b})$. It remains to show that

$$T_{na}\psi(x)\sum_m \langle f, E_{mb}T_{na}\varphi\rangle E_{mb}(x)$$

converges in $L^2_w(\mathbb{R}^k)$ to

$$\psi(x-na)\sum_j f(x-j/b)\bar{\varphi}(x-na-j/b).$$

Let $\varepsilon > 0$ and let $F \subset \mathbb{Z}^k$ be a finite set with the property that if $G \subset \mathbb{Z}^k$ is any other finite set, then

$$\Big\| \sum_{m\in G\setminus F} \langle f, E_{mb}T_{na}\varphi\rangle E_{mb} \Big\|_{L^2(Q_{1/b})} < \varepsilon w(0)^{-1/2} \| T_{na}\psi\tilde{w}^2 \|^{-1}_{\infty,1,1/b}.$$

Then we have that, since $w(x) \le w(0)\tilde{w}^2(x)$ for all $x\in\mathbb{R}^k$,

$$\Big\| T_{na}\psi \sum_{m\in G\setminus F} \langle f, E_{mb}T_{na}\varphi\rangle E_{mb} \Big\|^2_{2,w}$$

$$= \int |\psi(x-na)|^2 \Big| \sum_{m\in G\setminus F} \langle f, E_{mb}T_{na}\varphi\rangle E_{mb}(x) \Big|^2 w(x)\, dx$$

$$= \sum_{l} \int_{Q_{1/b}+l/b} |T_{na}\psi(x)|^2 \Big| \sum_{m\in G\setminus F} \langle f, E_{mb}T_{na}\varphi\rangle E_{mb}(x) \Big|^2 w(x)\, dx$$

$$\le w(0) \sum_{l} \operatorname*{ess\,sup}_{Q_{1/b}} |T_{na}\psi\tilde{w}^2(x-l/b)|^2 \int_{Q_{1/b}} \Big| \sum_{m\in G\setminus F} \langle f, E_{mb}T_{na}\varphi\rangle E_{mb}(x) \Big|^2 dx$$

$$\le w(0) \| T_{na}\psi\tilde{w}^2 \|^2_{\infty,1,1/b} \Big\| \sum_{m\in G/F} \langle f, E_{mb}T_{na}\varphi\rangle E_{mb} \Big\|^2_{L^2(Q_{1/b})} < \varepsilon^2.$$

Hence, the series is Cauchy and so converges in $L^2_w(\mathbb{R}^k)$.

We now consider the convergence of the sum over n of the above functions. I claim that this sum converges uniformly on compact sets. Let $\varepsilon > 0$ and let $K \subset \mathbb{R}^k$ be a compact set. There is a cube $Q \subset \mathbb{R}^k$ such that $\|\psi 1_{Q^c}\tilde{w}^2\|_{\infty,1,a} < \varepsilon$. Now, there is a finite set $F \subset \mathbb{Z}^k$ such that if $n\in F^c$ then $\psi 1_Q(x-na) = 0$ for all $x\in K$, that is, pick F so that $n\in F^c$ implies that $(K-na) \cap Q = \emptyset$. Now let $G \subset \mathbb{Z}^k$ be any finite set, let $\psi_1 = \psi 1_Q$ and $\psi_2 = \psi 1_{Q^c}$, and let $m = \inf_{x\in K} w(x) > 0$. Then since

$$\begin{aligned} w(x) &\le w(x-j/b)\tilde{w}^2(j/b) \\ &\le w(x-j/b)\tilde{w}^2(x-na)\tilde{w}^2(j/b+na-x), \end{aligned}$$

we obtain

$$\operatorname*{ess\,sup}_{x\in K} \Big| \sum_{n\in G\setminus F} \psi(x-na) \sum_{j} f(x-j/b)\bar{\varphi}(x-na-j/b) \Big|$$

$$\le \operatorname*{ess\,sup}_{x\in K} \Big| \sum_{n\in G\setminus F} \psi_1(x-na) \sum_{j} f(x-j/b)\bar{\varphi}(x-na-j/b) \Big|$$

$$+ \operatorname*{ess\,sup}_{x\in K}\left| \sum_{n\in G\setminus F} \psi_2(x-na) \sum_j f(x-j/b)\bar{\varphi}(x-na-j/b)\right|$$

$$\le 0 + m^{-1}\operatorname*{ess\,sup}_{x\in K} \sum_n |\psi_2\bar{w}^2(x-na)| \sum_j |fw(x-j/b)|\,|\varphi_-\bar{w}^2(j/b+na-x)|$$

$$\le m^{-1}\left\| \sum_n |\psi_2\bar{w}^2(x-na)| \sum_j |fw(x-j/b)|\,|\varphi_-\bar{w}^2(x-na-j/b)|\right\|_\infty$$

$$\le m^{-1}\|fw\|_\infty \|\psi_2\bar{w}^2\|_{\infty,1,a}2^k\|\varphi_-\bar{w}^2\|_{\infty,1,1/b} < \varepsilon m^{-1}2^k\|fw\|_\infty\|\varphi_-\bar{w}^2\|_{\infty,1,1/b}.$$

Thus, since $\varepsilon > 0$ was arbitrary, the sum in question is uniformly Cauchy on compact sets and so converges uniformly on compact sets. Also,

$$Sf = b^{-k}\sum_j f(x-j/b) \sum_n \psi(x-na)\bar{\varphi}(x-na-j/b).$$

Since $\|(T_a f)w\|_\infty \le \bar{w}^2(a)\|fw\|_\infty$, the argument of Theorem 4.1.5 shows that S is continuous on $L^\infty_w(\mathbb{R}^k)$.■

THEOREM 4.1.8. Let φ and ψ be as in Theorem 4.1.7 and suppose that for some $a > 0$, there exists a constant $A > 0$ such that for almost every $x\in\mathbb{R}^k$,

$$A \le \left|\sum_n \psi(x-na)\bar{\varphi}(x-na)\right|.$$

Then there exists $b_0 > 0$ such that for all $0 < b < b_0$, S interpreted as in Theorem 4.1.7 is a bijective homeomorphism of $L^\infty_w(\mathbb{R}^k)$.

PROOF. The proof is exactly the same as Theorem 4.1.6.■

REMARK 4.1.9. Examples of spaces B satisfying the hypotheses of Theorem 4.1.5 include the following.

(1) $L^p_w(\mathbb{R}^k)$ for w moderate and $1 \le p < \infty$.

(2) $W(L^\infty, L^1_w)$, for w moderate.

(3) $W(L^p(\mathbb{R}^k), L^q_w(\mathbb{R}^k))$ where w is moderate and $1 \le p, q < \infty$ (cf. [F1] for details on these spaces). To see why this is true, observe the following facts.

Certainly, $W(L^p, L^q_w)$ is a Banach space with respect to the norm

$$\|f\| = \left[\int\left[\int |f(x)|^p |k(x-y)|^p \, dx\right]^{q/p} w(y) \, dy\right]^{1/q}$$

where $k \in C_c(\mathbb{R}^k)$ is fixed.

Since w is moderate, $W(L^p, L^q_w)$ is translation invariant.

Clearly, for any $h \in L^\infty(\mathbb{R}^k)$ and $f \in W(L^p, L^q_w)$, $hf \in W(L^p, L^q_w)$ and $\|hf\| \le \|h\|_\infty \|f\|$.

We now show that $C_c(\mathbb{R}^k)$ is dense in $W(L^p, L^q_w)$. Let $f \in W(L^p, L^q_w)$, let $\varepsilon > 0$ and suppose that $\mathrm{supp}(k) \subset Q$, for some compact set Q. Let $K \subset \mathbb{R}^k$ be a compact set so large that

$$\int_{K^c+Q}\left[\int |f(x)|^p |k(x-y)|^p \, dx\right]^{q/p} w(x) \, dx < \varepsilon^q/2$$

and let $g \in C_c(\mathbb{R}^k)$ with $\mathrm{supp}(g) \subset K$ be such that

$$\left[\int_K |f(x)-g(x)|^p \, dx\right]^{q/p} < \varepsilon^q/2\left[|K+Q|\,\|k\|_\infty^q \sup_{x \in K+Q} w(x)\right]^{-1}.$$

Such a g exists because if $f \in W(L^p, L^q_w)$ then f is locally in $L^p(\mathbb{R}^k)$.

Now,

$$\|f-g\|^q = \int\left[\int |f(x)-g(x)|^p |k(x-y)|^p \, dx\right]^{q/p} w(y) \, dy$$

$$= \int_{K+Q}\left[\int_K |f(x)-g(x)|^p |k(x-y)|^p \, dx\right]^{q/p} w(y) \, dy$$

$$+ \int_{K^c+Q}\left[\int_{K^c} |f(x)|^p |k(x-y)|^p\right]^{q/p} w(y) \, dy$$

$$\leq \|k\|_\infty^q |K+Q| \sup_{x\in K+Q} w(x) \left(\int_K |f(x)-g(x)|^p\, dx\right)^{q/p}$$

$$+\int_{K^c+Q} \left(\int |f(x)|^p |k(x-y)|^p\right)^{q/p} w(y)\, dy$$

$$< \varepsilon^q/2+\varepsilon^q/2 = \varepsilon^q.$$

Thus, $\|f-g\| < \varepsilon$.

(4) Note that the space $C_0(\mathbb{R}^k)$ does not satisfy the hypotheses of Theorem 4.1.5 because it is not a Banach module over $L^\infty(\mathbb{R}^k)$. However, the conclusion of the theorem still holds if we assume that φ and ψ are continuous and in $W(L^\infty,L^1)$, i.e., that φ and ψ are in $W(C_0,L^1)$ (cf. [F1]).

To see this, observe that given any bounded, continuous function h and $f\in C_0(\mathbb{R}^k)$, $hf\in C_0(\mathbb{R}^k)$ and $\|hf\|_\infty \leq \|h\|_\infty\|f\|_\infty$. I claim that for each a > 0 and fixed $j\in\mathbb{Z}^k$, the sum

$$\sum_n \psi(x-na)\bar{\varphi}(x-na-j/b)$$

converges uniformly on compact sets. To see this, let $K \subset \mathbb{R}^k$ be a compact set and let $\varepsilon > 0$. There is a cube Q such that $\|\psi 1_{Q^c}\|_{\infty,1,a} < \varepsilon$ and $\|T_{j/b}\varphi 1_{Q^c}\|_{\infty,1,a} < \varepsilon$. Let $F \subset \mathbb{Z}^k$ be a finite set of indices such that if n is not in F then $Q \cap (Q+n) = \emptyset$. Thus, for any other finite set of indices $G \subset \mathbb{Z}^k$, we have that

$$\sup_{x\in K}\left|\sum_{n\in G\setminus F} \psi(x-na)T_{j/b}\bar{\varphi}(x-na)\right|$$

$$\leq \sup_{x\in K}\sum_{n\in G\setminus F} |\psi 1_Q(x-na)|\,|T_{j/b}\varphi 1_Q(x-na)|$$

$$+ \sup_{x\in K}\sum_{n\in G\setminus F} |\psi 1_Q(x-na)|\,|T_{j/b}\varphi 1_{Q^c}(x-na)|$$

$$+ \sup_{x\in K}\sum_{n\in G\setminus F} |\psi 1_{Q^c}(x-na)|\,|T_{j/b}\varphi 1_Q(x-na)|$$

$$+ \sup_{x\in K} \sum_{n\in G\setminus F} |\psi 1_{Q^c}(x-na)|\,|T_{j/b}\varphi 1_{Q^c}(x-na)|$$

$$\le 0 + \|\psi\|_\infty \Big\| \sum_n |T_{1/b}\varphi 1_{Q^c}(x-na)| \Big\|_\infty + 2\|\varphi\|_\infty \Big\| \sum_n |\psi 1_{Q^c}(x-na)| \Big\|_\infty$$

$$< \varepsilon(\|\psi\|_\infty + 2\|\varphi\|_\infty).$$

Since $\varepsilon > 0$ was arbitrary, we are done.

This implies that the functions $G_j(x)$ are continuous and bounded. Note also that since φ and ψ are continuous, the hypothesis of Theorem 4.1.6 is satisfied for sufficiently small a > 0. Therefore, the arguments in Theorem 4.1.5 and 4.1.6 go through unchanged in this case. This says that if f is a continuous function vanishing at infinity, and if φ, ψ, a, and b satisfy appropriate hypotheses, then Sf as well as $S^{-1}f$ are continuous functions.

(5) Suppose that $f\in C_b(\mathbb{R}^k)$, and φ and ψ satisfy the same conditions as in (4) above. If the sum Sf is given the same interpretation as in Theorem 4.1.7, then by the arguments in Theorem 4.1.7 and Theorem 4.1.8, S and S^{-1} are continuous bijections of the space $C_b(\mathbb{R}^k)$ for sufficiently small a and b.

Section 4.2. Preservation of smoothness by the frame operator

DEFINITION 4.2.1. Let B be as in Theorem 4.1.6. We define the space $\mathcal{F}B$ by

$$\mathcal{F}B = \{f \in \mathcal{S}'(\mathbb{R}^k): \quad \hat{f} \in B\}$$

with norm given by $\|f\|_{\mathcal{F}B} = \|\hat{f}\|_B$.

THEOREM 4.2.2. Let f, $h \in L^2(\mathbb{R}^k)$ and suppose that φ and ψ are such that φ, $\psi \in W(L^\infty, L^1)$ and also $\hat{\varphi}$, $\hat{\psi} \in W(L^\infty, L^1)$. Then

(1) the sequence of partial sums defining $(Sf)^\wedge$ converges strongly in $L^2(\hat{\mathbb{R}}^k)$,

(2) $\langle (Sf)^\wedge, \hat{h} \rangle$

$$= a^{-k} \sum_j \int \hat{f}(\gamma - j/a) \bar{\hat{h}}(\gamma) \sum_m \hat{\psi}(\gamma - mb) \bar{\hat{\varphi}}(\gamma - mb - j/a)\, d\gamma, \text{ and}$$

(3) $(Sf)^\wedge = a^{-k} \sum_j \hat{f}(\gamma - j/a) \sum_m \hat{\psi}(\gamma - mb) \bar{\hat{\varphi}}(\gamma - mb - j/a)$.

PROOF. Since φ and ψ are in $W(L^\infty, L^1)$, we know that the sum defining Sf converges strongly in $L^2(\mathbb{R}^k)$ whenever $f \in L^2(\mathbb{R}^k)$. Thus it follows that if $\hat{f} \in L^2(\mathbb{R}^k)$ then Sf will be strongly convergent in $L^2(\hat{\mathbb{R}}^k)$. By Parseval's formula, we have that

$$\begin{aligned}
(Sf)^\wedge &= \sum_n \sum_m \langle \hat{f}, (E_{mb}T_{na}\varphi)^\wedge \rangle (E_{mb}T_{na}\psi)^\wedge \\
&= \sum_n \sum_m \langle \hat{f}, T_{mb}E_{-na}\hat{\varphi} \rangle \; T_{mb}E_{-na}\hat{\psi} \\
&= \sum_m \sum_n \langle \hat{f}, E_{na}T_{mb}\hat{\varphi} \rangle \; E_{na}T_{mb}\hat{\psi}.
\end{aligned}$$

From this formula, all the conclusions follow as in Theorem 4.1.4.■

THEOREM 4.2.3. Let B be as in Theorem 4.1.5 and suppose that $\hat{\varphi}$ and $\hat{\psi}$ are in $W(L^{\infty},L^{1}) \cap B$ and that $\hat{\varphi}_{-m}$ and $\hat{\psi}_{m}$ are in $W(L^{\infty},L^{1})$. Then S can be extended uniquely to a continuous operator on $\mathcal{F}B$.

PROOF. Follows from Theorem 4.1.5, Theorem 4.2.2 and the fact that the collection of functions f such that $\hat{f} \in L^{2}(\mathbb{R}^{k}) \cap B$ is dense in $\mathcal{F}B$.■

THEOREM 4.2.4. Let B, φ and ψ be as in Theorem 4.2.3 and suppose further that for some constant $A > 0$ and almost every $\gamma \in \hat{\mathbb{R}}^{k}$,

$$A \leq \left| \sum_{m} \hat{\psi}(\gamma-mb)\bar{\hat{\varphi}}(\gamma-mb) \right|.$$

Then there exists $a_0 > 0$ such that for all $0 < a < a_0$, the S-operator is a topological isomorphism from $\mathcal{F}B$ onto $\mathcal{F}B$.

PROOF. Follows as in Theorem 4.1.6.■

REMARK 4.2.5. Examples of spaces $\mathcal{F}B$ include the following.

(1) $A(\mathbb{R}^{k})$, the space of absolutely convergent Fourier transforms.

(2) $\mathcal{F}L^{p}_{w}(\mathbb{R}^{k})$, with w moderate and $1 \leq p < \infty$.

The following results show directly that the S-operator preserves smoothness properties of functions. Specifically, we show that S maps $\mathcal{S}(\mathbb{R}^{k})$ continuously into $\mathcal{S}(\mathbb{R}^{k})$ provided that the auxilliary functions φ and ψ are in $\mathcal{S}(\mathbb{R}^{k})$.

LEMMA 4.2.6. Let h be a continuous function on $\mathbb{R}^k$ such that for all $\varphi \in C_c^\infty(\mathbb{R}^k)$ with $\int \varphi(x)dx = 0$, $\int h(x)\varphi(x)dx = 0$. Then h is identically constant.

PROOF. Suppose not. Then we could find a pair of disjoint cubes of the same size, I_1 and I_2 such that $h(x) > 0$ (say) on I_1 and I_2, and for some $\varepsilon > 0$, $h(x_1)-h(x_2) > \varepsilon$ for each $x_1 \in I_1$ and $x_2 \in I_2$. Let $a \in \mathbb{R}^k$ be such that $I_2 = I_1+a$. Let $\varphi_1 \geq 0$ be in $C_c^\infty(\mathbb{R}^k)$ and be supported in I_1. Define $\varphi(x) = \varphi_1(x)$ if $x \in I_1$, $-\varphi_1(x-a)$ if $x \in I_2$, and 0 elsewhere. Then $\varphi \in C_c^\infty(\mathbb{R}^k)$, and $\int \varphi(x)dx = 0$.

Now,

$$\int h(x)\varphi(x)dx = \int_{I_1} h(x)\varphi_1(x)dx - \int_{I_2} h(x)\varphi_1(x-a)dx$$
$$= \int_{I_1} [h(x)-h(x+a)]\varphi_1(x)dx > \varepsilon \int \varphi_1(x)dx > 0.$$

As this is a contradiction, h must be identically constant.■

LEMMA 4.2.7. Let $M \geq 1$ be given and suppose that $\{f_n\}$ is a sequence of continuous functions on $\mathbb{R}^k$ possessing continuous derivatives up to M^{th} order which converges distributionally to a continuous function f. Suppose that for each multi-index α with $|\alpha| \leq M$, we have that $\{D^\alpha f_n\}$ converges distributionally to a continuous function g_α. Then $f \in C^M(\mathbb{R}^k)$ and $D^\alpha f = g_\alpha$ for each $|\alpha| \leq M$.

PROOF. We prove this by induction on M. Suppose first that $M = 1$, and let $|\alpha| = 1$. Specifically, let $\alpha = (0,\ldots,0,1,0,\ldots,0)$ where the "1" is in the i^{th} position. Let φ be a test function on

$\mathbb{R}^k$. Then

$$\int D^\alpha f_n(x)\varphi(x)\ dx = -\int f_n(x)D^\alpha\varphi(x)\ dx \longrightarrow -\int f(x)D^\alpha\varphi(x)\ dx.$$

But

$$\int D^\alpha f_n(x)\varphi(x)\ dx \longrightarrow \int g_\alpha(x)\varphi(x)\ dx = -\int G_\alpha(x)D^\alpha\varphi(x)\ dx$$

where G_α is such that $D^\alpha G_\alpha = g_\alpha$. Specifically, we take

$$G_\alpha(x) = \int_a^{x_i} g_\alpha(x',t)\ dt$$

where we write $x\in\mathbb{R}^k$ as $x = (x',t) = (x_1,\ldots,x_{i-1},t,x_{i+1},\ldots,x_k)$.

Now, since $\varphi\in C_c^\infty(\mathbb{R}^k)$, $D^\alpha\varphi\in C_c^\infty(\mathbb{R}^k)$ and $\int D^\alpha\varphi(x)dx = 0$. Also, every such function can be expressed as the derivative of a test function. Therefore, by Lemma 4.2.6, we have that for all $x\in\mathbb{R}^k$, $f(x) = G_\alpha(x)+c$ for some constant c. Since f and g_α are continuous functions, we have that $D^\alpha f(x) = g_\alpha(x)$ for all $x\in\mathbb{R}^k$.

Now suppose the theorem holds for M-1. Let β be a multiindex with $|\beta| = M$. Then we can write $\beta = \beta'+\alpha$ where $|\beta'| = M-1$ and $|\alpha| = 1$. By hypothesis,

$$D^{\beta'} f_n \longrightarrow D^{\beta'} f$$

and

$$D^\alpha(D^{\beta'} f_n) = D^\beta f_n \longrightarrow g_\beta$$

distributionally on $\mathbb{R}^k$. By the argument in the previous paragraph, we can conclude that $D^\alpha(D^{\beta'} f) = D^\beta f = g_\beta$. Since β was arbitrary, we are done.■

PROPOSITION 4.2.8. A tempered distribution f is in $\mathcal{S}(\mathbb{R}^k)$ if and only if for each pair of multiindices α, β,

$$D^{\alpha}(D^{\beta}f)^{\wedge}\in A(\hat{\mathbb{R}}^k) \cap L^1(\hat{\mathbb{R}}^k).$$

PROOF. ($\Longrightarrow$) Suppose $f\in\mathcal{S}(\mathbb{R}^k)$. Then since $\mathcal{S}$ is invariant under the Fourier transform and differentiation, we have that for multiindices α, β,

$$D^{\alpha}(D^{\beta}f)^{\wedge}\in\mathcal{S}(\mathbb{R}^k) \subset A(\hat{\mathbb{R}}^k) \cap L^1(\hat{\mathbb{R}}^k).$$

($\Longleftarrow$) First observe that if a distribution f satisfies the hypotheses then we can take $\beta=0$, take the inverse Fourier transform and get that $D^{\alpha}f\in A(\mathbb{R}^k) \subset C_0(\mathbb{R}^k)$ for all α. Thus, f is infinitely differentiable and all of its derivatives are bounded. Also, we have that

$$(D^{\beta}[D^{\alpha}f]^{\wedge})^{\vee}(x) = (2\pi i)^{|\beta|}x^{\beta}D^{\alpha}f(x)\in A(\mathbb{R}^k) \cap L^1(\mathbb{R}^k) \subset L^{\infty}(\mathbb{R}^k).$$

Thus, $f\in\mathcal{S}(\mathbb{R}^k)$.■

PROPOSITION 4.2.9. Suppose f, φ and ψ are in $\mathcal{S}(\mathbb{R}^k)$, and let α be a multiindex. Then

$$D^{\alpha}(S(\varphi,\psi)f) = \sum_{|\beta|\le|\alpha|}\ \sum_{|\gamma|\le|\beta|} \binom{\alpha}{\beta}\binom{\beta}{\gamma} S(D^{\beta-\gamma}\varphi, D^{\gamma}\psi)D^{\alpha-\beta}f.$$

PROOF 1. We use the formula from Theorem 4.1.5,

$$S(\varphi,\psi)f(x) = b^{-k}\sum_{j} f(x-j/b) \sum_{n} \psi(x-na)\bar{\varphi}(x-na-j/b)$$

where by the assumptions on f, φ and ψ and by Theorem 4.1.5, the sum over n and j converges uniformly. Now, given finite index sets F, G $\subset \mathbb{Z}^k$, we have

$$b^{-k}D^{\alpha}\left[\sum_{j\in F} f(x-j/b) \sum_{n\in G} \psi(x-na)\bar{\varphi}(x-na-j/b)\right]$$

$$= b^{-k}\sum_{j\in F}\ \sum_{|\beta|\le|\alpha|}\binom{\alpha}{\beta}D^{\alpha-\beta}f(x-j/b)\sum_{n\in G}D^{\beta}[\psi(x-na)\overline{\varphi}(x-na-j/b)]$$

$$= \sum_{|\beta|\le|\alpha|}\ \sum_{|\gamma|\le|\beta|}\binom{\alpha}{\beta}\binom{\beta}{\gamma}$$

$$b^{-k}\sum_{j\in F}D^{\alpha-\beta}f(x-j/b)\sum_{n\in G}D^{\gamma}\psi(x-na)\overline{D^{\beta-\gamma}\varphi}(x-na-j/b).$$

As was noted before, the partial sums defining $S(\varphi,\psi)f$ converge uniformly. Since by assumption, $D^{\alpha-\beta}f$, $D^{\beta-\gamma}\varphi$ and $D^{\gamma}\psi$ are in $\mathcal{S}(\mathbb{R}^k)$ for all α, β and γ, the α-derivative of the partial sums defining $S(\varphi,\psi)f$ converge uniformly to

$$\sum_{|\beta|\le|\alpha|}\ \sum_{|\gamma|\le|\beta|}\binom{\alpha}{\beta}\binom{\beta}{\gamma}S(D^{\beta-\gamma}\varphi,D^{\gamma}\psi)D^{\alpha-\beta}f.$$

From this and Lemma 4.2.7, it follows that $D^{\alpha}S(\varphi,\psi)f$ exists and is a continuous function equal to the above.

PROOF 2. Fix $m,n\in\mathbb{Z}^k$. Then, integrating by parts, we have that

$D^{\alpha}(\langle f,E_{mb}T_{na}\varphi\rangle E_{mb}T_{na}\psi)$

$$= \sum_{|\beta|\le|\alpha|}\binom{\alpha}{\beta}\langle f,E_{mb}T_{na}\varphi\rangle(2\pi imb)^{\beta}E_{mb}T_{na}D^{\alpha-\beta}\psi$$

$$= \sum_{|\beta|\le|\alpha|}\binom{\alpha}{\beta}(-1)^{\beta}\int fT_{na}\overline{\varphi}(x)(-2\pi imb)^{\beta}e^{-2\pi ib\langle m,x\rangle}\,dx\ E_{mb}T_{na}D^{\alpha-\beta}\psi$$

$$= \sum_{|\beta|\le|\alpha|}\binom{\alpha}{\beta}\int D^{\beta}(fT_{na}\overline{\varphi})(x)e^{-2\pi ib\langle m,x\rangle}\,dx\ E_{mb}T_{na}D^{\alpha-\beta}\psi$$

$$= \sum_{|\beta|\le|\alpha|}\binom{\alpha}{\beta}\langle D^{\beta}(fT_{na}\overline{\varphi}),E_{mb}\rangle E_{mb}T_{na}D^{\alpha-\beta}\psi$$

$$= \sum_{|\beta|\le|\alpha|}\ \sum_{|\gamma|\le|\beta|}\binom{\alpha}{\beta}\binom{\beta}{\gamma}\langle D^{\beta-\gamma}f,E_{mb}T_{na}D^{\gamma}\varphi\rangle E_{mb}T_{na}D^{\alpha-\beta}\psi.$$

It follows from this that for finite sets $F, G \in \mathbb{Z}^k$,

$$D^{\alpha}\left[\sum_{n\in F}\ \sum_{m\in G} <f, E_{mb}T_{na}\varphi> E_{mb}T_{na}\psi\right]$$

$$= \sum_{|\beta|\le|\alpha|}\ \sum_{|\gamma|\le|\beta|} \binom{\alpha}{\beta}\binom{\beta}{\gamma} \sum_{n\in F}\ \sum_{m\in G} <D^{\beta-\gamma}f, E_{mb}T_{na}D^{\gamma}\varphi> E_{mb}T_{na}D^{\alpha-\beta}\psi.$$

We know that in this case, the partial sums defining $S(\varphi,\psi)f$ converge in $L^2(\mathbb{R}^k)$ to $S(\varphi,\psi)f$ which is a continuous function since f is continuous, vanishes at infinity, and φ and ψ decay rapidly (cf. Remark 4.1.9(4)). Thus the α-derivative of these partial sums converge in $L^2(\mathbb{R}^k)$ to

$$\sum_{|\beta|\le|\alpha|}\ \sum_{|\gamma|\le|\beta|} \binom{\alpha}{\beta}\binom{\beta}{\gamma} S(D^{\gamma}\varphi, D^{\alpha-\beta}\psi)D^{\beta-\gamma}f$$

which is also a continuous function since $D^{\beta-\gamma}f$ is in $C_0(\mathbb{R}^k)$ and $D^{\gamma}\varphi$ and $D^{\alpha-\beta}\psi$ decay rapidly (cf. Remark 4.1.9(4)). It follows from this and Lemma 4.2.7 that $D^{\alpha}S(\varphi,\psi)f$ is a continuous function and equals the above.

It remains to show that the formula above and that given in proof 1 are equivalent, that is each can be obtained from the other by a change of summation indices. To see that this is so, start with the formula

$$\sum_{|\beta|\le|\alpha|}\ \sum_{|\gamma|\le|\beta|} \binom{\alpha}{\beta}\binom{\beta}{\gamma} S(D^{\beta-\gamma}\varphi, D^{\gamma}\psi)D^{\alpha-\beta}f.$$

For fixed β, let $u=\beta-\gamma$. Then $\gamma=\beta-u$ and $|\gamma|\le|\beta|$ if and only if $|u|\le|\beta|$. Also, $\binom{\beta}{\gamma} = \binom{\beta}{u}$. Thus the above equals

$$\sum_{|\beta|\le|\alpha|}\ \sum_{|u|\le|\beta|} \binom{\alpha}{\beta}\binom{\beta}{u} S(D^{u}\varphi, D^{\beta-u}\psi)D^{\alpha-\beta}f.$$

Now substitute $t=\alpha-\beta+u$. Then $\beta=\alpha-t+u$, $\beta-u=\alpha-t$ and $\alpha-\beta=t-u$. Also, it is easy to see that $|\beta|=|\alpha-t+u|\le|\alpha|$ if and only if $|u|\le|t|$ and that $|u|\le|\alpha-t+u|=|\beta|$ if and only if $|t|\le|\alpha|$. Finally,

$$\binom{\alpha}{\beta}\binom{\beta}{u} = \binom{\alpha}{\alpha-t+u}\binom{\alpha-t+u}{u} = \frac{\alpha!}{(\alpha-t+u)!(t-u)!}\,\frac{(\alpha-t+u)!}{(\alpha-t)!u!}$$

$$= \frac{\alpha!}{(\alpha-t)!t!}\,\frac{t!}{(t-u)!u!} = \binom{\alpha}{t}\binom{t}{u}.$$

Thus, we have

$$\sum_{|\beta|\le|\alpha|}\ \sum_{|u|\le|\beta|}\binom{\alpha}{\beta}\binom{\beta}{u}S(D^{u}\varphi,D^{\beta-u}\psi)D^{\alpha-\beta}f$$

$$= \sum_{|t|\le|\alpha|}\ \sum_{|u|\le|t|}\binom{\alpha}{t}\binom{t}{u}S(D^{u}\varphi,D^{\alpha-t}\psi)D^{t-u}f$$

which is what we wanted to prove.■

PROPOSITION 4.2.10. Let f, φ and ψ be in $\mathscr{S}(\mathbb{R}^k)$, α_1 and α_2 be multiindices. Then

$$D^{\alpha_1}(D^{\alpha_2}S_{a,b}(\varphi,\psi)f)^{\wedge}$$

$$= \sum_{|\beta_2|\le|\alpha_2|}\ \sum_{|\gamma_2|\le|\beta_2|}\ \sum_{|\beta_1|\le|\alpha_1|}\ \sum_{|\gamma_1|\le|\beta_1|}\binom{\alpha_1}{\beta_1}\binom{\beta_1}{\gamma_1}\binom{\alpha_2}{\beta_2}\binom{\beta_2}{\gamma_2}$$

$$S_{b,a}(D^{\beta_1-\gamma_1}[D^{\beta_2-\gamma_2}\varphi]^{\wedge},D^{\gamma_1}[D^{\gamma_2}\psi]^{\wedge})D^{\alpha_1-\beta_1}[D^{\alpha_2-\beta_2}f]^{\wedge}.$$

PROOF. By Proposition 4.2.8, we have that

$$D^{\alpha_2}S_{a,b}(\varphi,\psi)f$$

$$= \sum_{|\beta_2|\le|\alpha_2|}\ \sum_{|\gamma_2|\le|\beta_2|}\binom{\alpha_2}{\beta_2}\binom{\beta_2}{\gamma_2}S_{a,b}(D^{\gamma_2}\varphi,D^{\beta_2-\gamma_2}\psi)D^{\alpha_2-\beta_2}f.$$

By above and Theorem 4.2.2, we have that

$$[D^{\alpha_2}S_{a,b}(\varphi,\psi)f]^{\wedge}$$

$$= \sum_{|\beta_2|\le|\alpha_2|}\ \sum_{|\gamma_2|\le|\beta_2|}\binom{\alpha_2}{\beta_2}\binom{\beta_2}{\gamma_2}S_{b,a}([D^{\gamma_2}\varphi]^{\wedge},[D^{\beta_2-\gamma_2}\psi]^{\wedge})[D^{\alpha_2-\beta_2}f]^{\wedge}.$$

Finally, by Proposition 4.2.9, the conclusion follows.■

THEOREM 4.2.11. Suppose that f, φ and ψ are in $\mathcal{S}(\mathbb{R}^k)$. Then $S_{a,b}(\varphi,\psi)f\in\mathcal{S}(\mathbb{R}^k)$.

PROOF. Let α, β be multiindices. Then by Proposition 4.2.10, $D^\beta[D^\alpha S_{a,b}(\varphi,\psi)f]^\wedge$ can be written as a finite linear combination of "S-operators", with various auxilliary functions, applied to derivatives of f. Since f, φ and ψ are in $\mathcal{S}(\mathbb{R}^k)$, Theorems 4.1.5 and 4.2.3 imply that $D^\beta[D^\alpha S_{a,b}(\varphi,\psi)f]^\wedge\in A(\hat{\mathbb{R}}^k) \cap L^1(\hat{\mathbb{R}}^k)$. Since α, β are arbitrary, this implies that $S_{a,b}(\varphi,\psi)f\in\mathcal{S}(\mathbb{R}^k)$.■

THEOREM 4.2.12. Suppose f, φ and ψ are in $\mathcal{S}(\mathbb{R}^k)$. Then $S(\varphi,\psi)$ is a continuous operator on $\mathcal{S}(\mathbb{R}^k)$.

PROOF. By Theorem 4.2.11, $S(\varphi,\psi)f\in\mathcal{S}(\mathbb{R}^k)$ whenever $f\in\mathcal{S}(\mathbb{R}^k)$. Suppose now that $\{f_n\} \subset \mathcal{S}(\mathbb{R}^k)$ is such that $f_n \longrightarrow 0$ in $\mathcal{S}(\mathbb{R}^k)$, that is,

$$D^\alpha f_n(x)(1+|x|)^m \longrightarrow 0$$

as $n \longrightarrow \infty$ uniformly for each multiindex α and integer $m \geq 0$.

We wish to show that for each multiindex α and integer $m \geq 0$,

$$D^\alpha[S(\varphi,\psi)f_n](1+|x|)^m \longrightarrow 0$$

as $n \longrightarrow \infty$ uniformly. Let α and m be fixed.

By Proposition 4.2.9, we know that $D^\alpha[S(\varphi,\psi)f_n]$ can be written as a finite linear combination of functions of the form $S(D^\beta\varphi,D^\gamma\psi)D^\delta f_n$ where β, γ, and δ are multiindices independent of n. Let $w(x) = (1+|x|)^m$. Then w is a Beurling weight so that $\tilde{w}^2(x) \leq (1+|x|)^m$. Now, since $D^\beta\varphi$, $D^\gamma\psi\in W(L^\infty,L^1)$ and $D^\delta f_n\in L^2(\mathbb{R}^k)$, the sum defining $S(D^{\beta}\varphi, D^{\gamma}\psi)D^{\delta}f_n$ converges in $L^2(\mathbb{R}^k)$ by Theorem 4.1.4 to

$$b^{-k}\sum_j D^{\delta}f_n(x-j/b)G_j(x)$$

where

$$G_j(x) = \sum_l D^{\gamma}\psi(x-na)D^{\beta}\bar{\varphi}(x-na-j/b).$$

Since $D^{\beta}\varphi\_\tilde{w}^2$ and $D^{\gamma}\psi\tilde{w}^2$ are in $W(L^{\infty},L^1)$, Lemma 4.1.3 says that

$$\sum_j \tilde{w}^2(j/b)\|G_j\|_{\infty} < \infty.$$

Then, as in Theorem 4.1.7, we have that

$$\|[S(D^{\beta}\varphi, D^{\gamma}\psi)D^{\delta}f_n]w\|_{\infty} \le \|(D^{\delta}f_n)w\|_{\infty}b^{-k}\sum_j \tilde{w}^2(j/b)\|G_j\|_{\infty} \longrightarrow 0$$

as $n \to \infty$. Thus, it follows that

$$\|D^{\alpha}[S(\varphi,\psi)f_n](x)(1+|x|)^m\|_{\infty} \longrightarrow 0$$

as $n \to \infty$. ∎

Section 4.3. The inverse frame operator

THEOREM 4.3.1. Let B, φ, and ψ be as in Theorem 4.1.6 with the additional assumption that $L_c^\infty(\mathbb{R}^k) \cap B$ is dense in B. (This is true of all the examples in Remark 4.1.9). Define formally the following functions.

(1) $G_j(x) = \sum_n \psi(x-na)\bar{\varphi}(x-na-j/b)$ for all $j \in \mathbb{Z}^k$,

(2) $G_j^{(0)}(x) = \begin{cases} 1 & \text{if } j=0, \\ 0 & \text{if } j \neq 0, \end{cases}$

(3) $G_j^{(1)}(x) = \begin{cases} 0 & \text{if } j=0, \\ -G_0^{-1}(x)G_j(x) & \text{if } j \neq 0, \end{cases}$

(4) $G_j^{(m)}(x) = \sum_n G_{j-n}^{(1)}(x-n/b)G_n^{(m-1)}(x)$, and

(5) $H_m(x) = \sum_{n=0}^{\infty} G_m^{(n)}(x)$.

Suppose that

$$\sum_j m(j/b)\|G_j^{(1)}\|_\infty < 1.$$

Then the sums defining $G_j^{(m)}$ and H_m converge absolutely and uniformly. Moreover, we have that

$$\sum_j m(j/b)\|H_m\|_\infty < \infty$$

and that

$$S^{-1}f(x) = b^k G_0^{-1}(x) \sum_m f(x-m/b)H_m(x)$$

where the sum converges strongly in B.

PROOF. Let $S_0f(x) = b^k G_0^{-1}(x)Sf(x)$. I claim that for any $f \in B$ and for any integer $N \geq 0$, we have that

$$(I-S_0)^N f(x) = \sum_m f(x-m/b)G_m^{(N)}(x).$$

To prove this, first suppose that $f \in L_c^\infty(\mathbb{R}^k) \cap B$ and recall that for such f,

$$Sf(x) = b^{-k}\sum_m f(x-m/b)G_m(x) = b^{-k}G_0(x)f(x)+b^{-k}\sum_{m\neq 0} f(x-m/b)G_m(x).$$

Thus,

$$f(x)-b^k G_0^{-1}(x)Sf(x) = -\sum_m f(x-m/b)G_0^{-1}(x)G_m(x)$$

and so

$$(I-S_0)f = \sum_m f(x-m/b)G_m^{(1)}(x).$$

Thus the conclusion holds for N = 1 and $f \in L_c^\infty(\mathbb{R}^k) \cap B$. Now assume it holds for N. Then

$$\begin{aligned}
(I-S_0)^{N+1} f(x) &= (I-S_0)^N[(I-S_0)f] \\
&= \sum_m (I-S_0)f(x-m/b)G_m^{(N)}(x) \\
&= \sum_m \sum_j f(x-(j+m)/b)G_j^{(1)}(x-m/b)G_m^{(N)}(x) \\
&= \sum_m \sum_i f(x-i/b)G_{i-m}^{(1)}(x-m/b)G_m^{(N)}(x) \qquad \text{(where } i = j+m) \\
&= \sum_i f(x-i/b) \sum_m G_{i-m}^{(1)}(x-m/b)G_m^{(N)}(x) = \sum_i f(x-i/b)G_i^{(N+1)}(x).
\end{aligned}$$

Since f is compactly supported, there are only finitely many non-zero terms in the sum over i and so the last interchange of summations is justified.

In order to pass from $f \in L_c^\infty(\mathbb{R}^k) \cap B$ to arbitrary $f \in B$, it is necessary first to observe that by Theorem 4.1.5, $(I-S_0)^N$ is a bounded operator on B for all integers $N \geq 0$. It also will be necessary to show that the right hand sum above defines a continuous operator on B. For this, the following claim is sufficient.

Claim 1. $\sum_j m(j/b)\|G_j^{(N)}\|_\infty \le \left[\sum_j m(j/b)\|G_j^{(1)}\|_\infty\right]^N$.

Proof: The conclusion is obvious for N = 0 and N = 1. Suppose it holds for N.

$$\sum_j m(j/b)\|G_j^{(N+1)}\|_\infty = \sum_j m(j/b)\left\| \sum_i G_{j-i}^{(1)}(x-i/b)G_i^{(N)}(x)\right\|_\infty$$

$$\le \sum_j \sum_i m((j-i)/b)\|G_{j-i}^{(1)}\|_\infty m(i/b)\|G_i^{(N)}\|_\infty$$

$$= \sum_j m(j/b)\|G_j^{(1)}\|_\infty \sum_i m(i/b)\|G_i^{(N)}\|_\infty$$

$$\le \sum_j m(j/b)\|G_j^{(1)}\|_\infty \left[\sum_j m(j/b)\|G_j^{(1)}\|_\infty\right]^N$$

$$= \left[\sum_j m(j/b)\|G_j^{(1)}\|_\infty\right]^{N+1}.$$

Thus it holds for N+1 and we are done.□

From this it follows that the sum $\sum_i f(x-i/b)G_i^{(N)}(x)$ converges strongly in B for every f∈B and that

$$\left\| \sum_i f(x-i/b)G_i^{(N)}(x)\right\|_B \le \|f\|_B \sum_i m(i/b)\|G_i^{(N)}\|_\infty$$

so that the sum defines a continuous operator on B. Thus, if we choose f∈B and let $f_j \in L_c^\infty(\mathbb{R}^k) \cap B$ be a sequence converging to f in B, the above inequality implies that

$$(I-S_0)^N f = \lim_j (I-S_0)^N f_j$$

$$= \lim_j \sum_i f_j(x-i/b)G_i^{(N)}(x) = \sum_i f(x-i/b)G_i^{(N)}(x).$$

We also see from Claim 1 that the sums defining $G_m^{(N)}(x)$ and $H_m(x)$ converge absolutely and uniformly.

We know that

$$S_0^{-1}f = \sum_{n=0}^{\infty}(I-S_0)^n f$$

provided that the sum on the right converges in B. For $f \in L_c^{\infty}(\mathbb{R}^k) \cap B$, we have

$$\begin{aligned} S_0^{-1}f(x) &= \sum_{n=0}^{\infty}\sum_m f(x-m/b)G_m^{(n)}(x) = \sum_m f(x-m/b)\sum_{n=0}^{\infty}G_m^{(n)}(x) \\ &= \sum_m f(x-m/b)H_m(x) = (*) \end{aligned}$$

where, since $f \in L_c^{\infty}(\mathbb{R}^k)$, the interchange of summation is justified. By Claim 1, we know that

$$\begin{aligned} \sum_m m(m/b)\|H_m\|_\infty &\le \sum_m m(m/b)\sum_{n=0}^{\infty}\|G_m^{(n)}\|_\infty \\ &\le \sum_{n=0}^{\infty}\Big[\sum_m m(m/b)\|G_m^{(1)}\|_\infty\Big]^n < \infty. \end{aligned}$$

Consequently, the sum (*) converges strongly in B for every $f \in B$ and defines a bounded operator on B. Since S_0^{-1} is also a bounded operator on B, it follows that the formula for S_0^{-1} holds for arbitrary $f \in B$. From this in immediately follows that

$$S^{-1}f(x) = b^k G_0^{-1}(x)\sum_m f(x-m/b)H_m(x). \blacksquare$$

Section 4.4. The continuous frame operator

THEOREM 4.4.1. Let $1 \le p < \infty$ be fixed and let $1/p+1/q = 1$. Let g be such that $g \in L_w^p(\mathbb{R}^k) \cap L_{w^{-q/p}}^q(\mathbb{R}^k) \setminus \{0\}$ and that $g_- \in L_m^1(\mathbb{R}^k)$ where for $a \in \mathbb{R}^k$, $m(a) = \|T_a\|_{L_w^p \to L_w^p}$. Let $\rho \in L^1(\mathbb{R}^k)$ be such that $\hat{\rho} \in C_c(\mathbb{R}^k)$, and $\int \hat{\rho}(\gamma)d\gamma = 1$. Define ρ_n by $\rho_n(x) = \rho(x/n)$ for $n \in \mathbb{N}$. Then $(\rho_n)^\wedge(\gamma) = n^k\hat{\rho}(n\gamma)$, and $\{(\rho_n)^\wedge\}_{n=1}^\infty$ is an approximate identity. Then for all $f \in L_w^p(\mathbb{R}^k)$,

$$\lim_{n\to\infty} \|f_n - f\|_{2,w} = 0$$

where

$$f_n(u) = \|g\|_2^{-2} \int_{\hat{\mathbb{R}}^k}\int_{\mathbb{R}^k} \langle f, E_bT_ag\rangle E_bT_ag(u)\rho_n(b)\ da\ db.$$

PROOF.

$$\|g\|_2^2 f_n(u) = \int_{\hat{\mathbb{R}}^k}\int_{\mathbb{R}^k}\int_{\mathbb{R}^k} f(t)\bar{g}(t-a)e^{-2\pi ibt}g(u-a)e^{2\pi ibu}\rho_n(b)\ dt\ da\ db$$

$$= \int_{\mathbb{R}^k} f(t)\left(\int_{\hat{\mathbb{R}}^k}\rho_n(b)e^{-2\pi ib(t-u)}\ db\right)\left(\int_{\mathbb{R}^k}\bar{g}(t-a)g(u-a)\ da\right)\ dt$$

$$= \int_{\mathbb{R}^k} f(t)\hat{\rho}_n(t-u)\left(\int_{\mathbb{R}^k}\bar{g}((t-u)+a)g(a)\ da\right)\ dt$$

$$= (f*\hat{\rho}_nG)(u)$$

where $G(x) = \int_{\mathbb{R}^k}\bar{g}(x+a)g(a)\ da.$

To check that the above interchanges in the order of integration are justified, note that

$$\int_{\hat{\mathbb{R}}^k}\int_{\mathbb{R}^k}\left(\int_{\mathbb{R}^k}|f(t)||g(t-a)|\ dt\right)|g(u-a)||\rho_n(b)|\ da\ db$$

$$\leq \int_{\hat{\mathbb{R}}^k}\int_{\mathbb{R}^k} \|T_a f\|_{p,w}\|g\|_{q,w^{-q/p}}|g(u-a)||\rho_n(b)| \, da \, db$$

$$\leq \|g\|_{q,w^{-q/p}}\|f\|_{p,w}\int_{\mathbb{R}^k}|T_u g_-(a)|m(a) \, da \int_{\hat{\mathbb{R}}^k}|\rho_n(b)| \, db$$

$$\leq \|g\|_{q,w^{-q/p}}\|f\|_{p,w}\|\rho_n\|_1 m(u)\|g_-\|_{1,m} < \infty.$$

Now, let $h\in L^q_{w^{-q/p}}(\mathbb{R}^k)$ with $\|h\|_{q,w^{-q/p}} = 1$. Then

$$|\langle f*\hat{\rho}_n G-f\cdot G(0),h\rangle|$$

$$= \left|\int_{\mathbb{R}^k}((f*\hat{\rho}_n G)(x)-f(x)G(0))\bar{h}(x) \, dx\right|$$

$$= \left|\int_{\mathbb{R}^k}\int_{\mathbb{R}^k}[f(x-y)G(y)-f(x)G(0)]\hat{\rho}_n(y) \, \bar{h}(x) \, dy \, dx\right|$$

$$\leq \int_{\mathbb{R}^k}|\hat{\rho}_n(y)|\left[\int_{\mathbb{R}^k}|f(x-y)G(y)-f(x)G(0)||h(x)| \, dx\right] dy$$

$$\leq \int_{\mathbb{R}^k}|\hat{\rho}_n(y)|\left[\int_{\mathbb{R}^k}|f(x-y)G(y)-f(x)G(0)|^p w(x) \, dx\right]^{1/p} dy$$

$$= (*).$$

Taking the supremum over all such h, we get that

$$\|f*\hat{\rho}_n G-f\cdot G(0)\|_{p,w} \leq (*).$$

Now, since w is locally bounded, $G(x)$ is a continuous function and is also bounded, in fact, $|G(x)| \leq G(0) = \|g\|_2^2$. It follows from this that given $\varepsilon > 0$, there is a $\delta > 0$ such that if $|y| < \delta$ then

$$\left[\int_{\mathbb{R}^k}|f(x-y)G(y)-f(x)G(0)|^p w(x) \, dx\right]^{1/p} < \varepsilon.$$

Therefore,

$$(*) = \int_{|y|<\delta}|\hat{\rho}_n(y)|\left[\int_{\mathbb{R}^k}|f(x-y)G(y)-f(x)G(0)|^p w(x) \, dx\right]^{1/p} dy$$
$$+ \int_{|y|\geq\delta}|\hat{\rho}_n(y)|\left[\int_{\mathbb{R}^k}|f(x-y)G(y)-f(x)G(0)|^p w(x) \, dx\right]^{1/p} dy$$

$$\leq \varepsilon\int_{|y|<\delta}|\hat{\rho}_n(y)| \, dy + G(0)\|f\|_{p,w}\int_{|y|\geq\delta}|\hat{\rho}_n(y)| \, dy$$

$$+ G(0)\|f\|_{p,w}\int_{|y|\geq\delta} |\hat{\rho}_n(y)|m(y)\, dy.$$

$$\leq \varepsilon\|\hat{\rho}\|_1 + G(0)\|f\|_{p,w}\int_{|y|\geq\delta} |\hat{\rho}_n(y)|\, dy$$

$$+ G(0)\|f\|_{p,w}\int_{|y|\geq\delta} |\hat{\rho}_n(y)|m(y)\, dy.$$

Since for n large enough, $\mathrm{supp}(\hat{\rho}_n) \subset \{|y|<\delta\}$, we have,

$$\limsup_{n\to\infty} \|f*\hat{\rho}_n G - f\cdot G(0)\|_{p,w} \leq \varepsilon\|\hat{\rho}\|_1.$$

Since $\varepsilon > 0$ was arbitrary, we have

$$\limsup_{n\to\infty} \|f*\hat{\rho}_n G - f\cdot G(0)\|_{p,w} = 0.$$

Since all of the terms in the sequence are positive, we get finally that

$$\lim_{n\to\infty} \|f*\hat{\rho}_n G - f\cdot G(0)\|_{p,w} = 0.$$

From this, the result follows immediately.■

CHAPTER 5

SPECIAL RESULTS FOR $L^2(R^k)$

In Theorem 2.3.9, we give sufficient conditions on a function g and a number $a > 0$ so that for all sufficiently small b, the set $\{E_{mb}T_{na}g\}$ forms a set of atoms for $L^2_w(\mathbb{R}^k)$.

One important condition on g is that it and g_- be in the Wiener-type space $W(L^\infty, L^1_\omega)$. The reason for this is twofold. First, it guarantees strong convergence of the sum defining the expansion of f (cf. Lemma 2.3.7, Corollary 2.3.8). Second, it guarantees that the S-operator is defined and bounded above and below for all sufficiently small b (cf. Theorems 4.1.5 and 4.1.6). For this second fact, an examination of the proof reveals that it is only necessary that g satisfy a "weighted β-condition", i.e., the conclusion of Lemma 4.1.3 with g = h.

If w = 1, and we are dealing with the Hilbert space $L^2(\mathbb{R}^k)$, then things become simpler. Note that $L^2_w(\mathbb{R}^k)$ is not a Hilbert space with respect to the ordinary inner product. In $L^2(\mathbb{R}^k)$, we have Gröchenig's Lemma (Theorem 5.1.3) which says that frames and sets of atoms are equivalent. This means that as long as we can prove that we have a frame, strong convergence of the S-operator and of the expansion sum are automatic. Thus, the "β-condition" is almost sufficient in this case to guarantee that we have a frame. Additionally, we require only that the sum of the squares of the translates of g be bounded above and away from zero.

In $L^2(\mathbb{R}^k)$, then, we do not necessarily require the rapid decay of g in order to obtain a frame. For example, in $L^2(\mathbb{R}^k)$ we can use the Zak transform to give a sufficient condition that g generates a frame which is not purely a decay condition (cf. [DGM], [D1], [HW]). Thus, even though the weighted $L^2(\mathbb{R}^k)$ theory above contains $L^2(\mathbb{R}^k)$ as a special case, it is still interesting to see how far the theory can be pushed when the weight is 1.

Section 5.1 is a purely expository section in which we present two results, due to Gröchenig and Heil, concerning frames and sets of atoms in Hilbert spaces. Heil's Theorem is used in Section 2.5. In Section 5.2, we examine the β-condition for $L^2(\mathbb{R}^k)$ functions and show how it extends the reservoir $W(L^\infty, L^1)$. We prove an existence theorem for W-H frames in $L^2(\mathbb{R}^k)$ which is a generalization of a theorem of Daubechies (cf. [D1]). We give some examples of functions which generate frames for certain values of the frame parameters but which do not decay very rapidly. Finally, in Séction 5.3, we present a phase-space localization result of Daubechies with a more tractable condition on the mother wavelet g which enables one to obtain explicit estimates on the localization parameters t_ε and ω_ε.

Section 5.1. Frames and atoms in Hilbert space

In this section, we present two known results on frames in Hilbert space, a result of Gröchenig proving the equivalence of frames and atoms in Hilbert space, and a result of Heil on the uniqueness of the dual frame of a Hilbert space frame.

THEOREM 5.1.1. (Gröchenig) Let $\mathbf{H}$ be a Hilbert space and $\{x_n: n \in I\}$ a countable collection of vectors in $\mathbf{H}$ indexed by the set I. Then $\{x_n\}$ is a frame for $\mathbf{H}$ with frame bounds A, B if and only if it is a set of atoms for $\mathbf{H}$ with atomic bounds B^{-1}, A^{-1}.

PROOF. ($\Longrightarrow$) From the basic theory of frames we know that we can write

$$x = \sum_n \langle x, S^{-1}x_n\rangle x_n$$

where

$$Sx = \sum_n \langle x, x_n\rangle x_n,$$

and that

$$B^{-1}\|x\|^2 \le \sum_n |\langle S^{-1}x, x_n\rangle|^2 \le A^{-1}\|x\|^2.$$

($\Longleftarrow$) Suppose that $\{x_n\}$ is a set of atoms for $\mathbf{H}$ with atomic bounds A, B. Observe first that the linear functionals corresponding to $\{x_n\}$ are each continuous by condition (2) of Definition 0.5.4. Thus, there exist vectors $\{e_n\}$ in $\mathbf{H}$ such that

$$a_n(x) = \langle x, e_n\rangle$$

for all $x \in \mathbf{H}$ and $n \in I$. Also, by (2), the collection $\{e_n\}$ is a frame for $\mathbf{H}$ with bounds A, B. Now, we define the operators

$$\begin{aligned} T:\ & \mathbf{H} \longrightarrow \ell^2(I) \\ & x \longmapsto (\langle x, e_n\rangle) \end{aligned}$$

$$\begin{aligned} R:\ & \ell^2(I) \longrightarrow \mathbf{H} \\ & (\lambda_n) \longmapsto \textstyle\sum_n \lambda_n x_n. \end{aligned}$$

We have immediately the following facts.

1. $RT = Id_H$.

2. T is a bounded linear map with a closed range and T is continuously invertible on its range. The norm of T is $B^{1/2}$ and the norm of T^{-1} is $A^{-1/2}$. This follows from the fact that $\{e_n\}$ is a frame for $\mathbf{H}$.

3. For any $x \in \mathbf{H}$, we have that

$$\|x\| = \|(RT)x\| \le \|R\| \|Tx\|_{\ell^2(I)} \le \|R\| \|T\| \|x\|.$$

That is,

$$\|R\|^{-2}\|x\| \le \sum_n |<x, e_n>|^2 \le \|T\|^2\|x\|^2.$$

But this implies that $\|R\|^{-2} \ge A$ hence that $\|R\| \le A^{-1/2}$.

Since $Im(T)$ is a closed subspace of $\mathbf{H}$, it can be thought of as a Hilbert space in its own right. Thus we can compute the adjoint operators

$$T^*: Im(T) \longrightarrow \mathbf{H}$$
$$R^*: \mathbf{H} \longrightarrow Im(T).$$

Now, given $\Lambda \in Im(T)$, $y \in \mathbf{H}$,

$$<T^*\Lambda, y> = <\Lambda, (<y, e_n>)> = \sum_n \lambda_n \overline{<y, e_n>} = <\sum_n \lambda_n e_n, y>$$

where the sum converges strongly since $\{e_n\}$ is a frame. Also, given $y \in \mathbf{H}$ and $\Lambda \in Im(T)$,

$$<R^* y, \Lambda> = <y, \sum_n \lambda_n x_n> = \sum_n <y, x_n> \overline{\lambda_n} = <(<y, x_n>), \Lambda>.$$

Thus

$$T^*(\lambda_n) = \sum_n \lambda_n e_n \text{ and } R^* y = (<y, x_n>).$$

Also, we know that $\|T^*\| = \|T\|$ and that $\|R^*\| = \|R\|$. Finally we have that

$$RT = Id_H = (Id_H)^* = (RT)^* = T^*R^*.$$

Therefore,

$$\|x\| = \|(T^*R^*)x\| \le \|T^*\|\|R^*x\|_{\ell^2(I)} \le \|T^*\|\|R^*\|\|x\|,$$

so that

$$\|T^*\|^{-2}\|x\|^2 \le \sum_n |\langle x, x_n\rangle|^2 \le \|R^*\|^2\|x\|^2.$$

Since $\|T^*\| = \|T\| = B^{1/2}$ and $\|R^*\| = \|R\| \le A^{-1/2}$ we are done.■

THEOREM 5.1.2. Let **H** be a Hilbert space and $\{x_n: n\in I\}$ a countable set of atoms for **H**, and let $\{e_n\}$ be the associated frame, that is, for all $x\in$**H**,

$$x = \sum_n \langle x, e_n\rangle x_n.$$

Then if the operator T is as defined in Theorem 5.1.1, $Im(T) = \ell^2(I)$ if and only if $\{e_n\}$ is a basis for **H**.

PROOF. ($\Longrightarrow$) Suppose that T is surjective. Then T^* is injective(in fact surjective) from $\ell^2(I)$ onto **H**. This is so because if $T^*\Lambda = 0$ then $\langle\Lambda, Ty\rangle = 0$ for all $y\in$**H**. But since T is surjective, this means that $\langle\Lambda,\Gamma\rangle = 0$ for all $\Gamma\in\ell^2(I)$. Thus $\Lambda = 0$ and T^* is injective. In other words, if $(\lambda_n)\in\ell^2(I)$ then $\sum_n\lambda_n e_n = 0$ implies that $\lambda_n = 0$ for all $n\in I$. Thus $\{e_n\}$ is a basis.

($\Longleftarrow$) Suppose $\{e_n\}$ is a basis for **H**. Let $\Lambda\in\ell^2(I)$ be such that $\langle\Lambda,\Gamma\rangle = 0$ for all $\Gamma\in Im(T)$. This means that for all $y\in$**H**, $\langle\Lambda, Ty\rangle = 0$ hence that $\langle T^*\Lambda, y\rangle = 0$ hence that $T^*\Lambda = 0$. In other words, if $\Lambda = (\lambda_n)$ then $\sum_n\lambda_n e_n = 0$ which implies that $\lambda_n = 0$ for all $n\in I$ hence that $\Lambda = 0$. Thus Im(T) is a closed dense subset of **H** and so must be **H**. ∎

Next we present an example which shows that the assumption of strong convergence of the expansion sum in the definition of a set of atoms is necessary in Gröchenig's Lemma. We take **H** to be $L^2(\mathbb{R})$, and find a collection $\{g_i\} \subset$ **H** and a collection of linear functionals $\{\lambda_i\}$ such that $f = \sum \lambda_i(f)g_i$ in some sense (actually pointwise almost everywhere) for each f but for which $\{g_i\}$ is not a frame for **H**.

EXAMPLE 5.1.3. Let φ be a continuous function with support contained in the interval $[0,a_0]$ for some $a_0 > 0$. Suppose that φ does not vanish in the interior of $[0,a_0]$ and fix $0 < a < a_0$. Now let b be such that $a_0 < 1/b$, and let g in $L^2(\mathbb{R})$ be such that g is supported in $[0,1/b]$, $g = 1$ on $[0,a_0]$ and g is unbounded on $[a_0,1/b]$. Therefore we have the following facts.

1. There are constants A, B such that

$$0 < A \leq \sum_n |\varphi(x-na)|^2 \leq B < \infty.$$

2. There are constants A', B' such that

$$0 < A' \leq \sum_n \bar{\varphi}(x-na) \leq B' < \infty.$$

Now, consider the operator defined by

$$Sf(x) = \sum_n \sum_m \langle f, E_{mb}T_{na}\varphi\rangle\, E_{mb}T_{na}g.$$

The sum on the right-hand side converges almost everywhere as an iterated sum by the Carleson-Hunt Theorem to the function

$$f(x)\ b^{-1}\sum_{n} g(x-na)\bar{\varphi}(x-na).$$

Now,

$$b^{-1}\sum_{n} g(x-na)\bar{\varphi}(x-na) = b^{-1}\sum_{n}\bar{\varphi}(x-na)$$

and so is bounded above and below, and hence S defined in this way is continuously invertible on $L^2(\mathbb{R})$. Thus,

$$f(x) = \sum_{n}\sum_{m} \langle S^{-1}f, E_{mb}T_{na}g\rangle\ E_{mb}T_{na}g(x)$$

and

$$AB'\|f\|_2^2 \le \sum_{n}\sum_{m} |\langle S^{-1}f, E_{mb}T_{na}g\rangle|^2 \le B/A'\|f\|_2^2.$$

Thus if the sum defining f converged strongly, then $\{E_{mb}T_{na}g\}$ would be a set of atoms for $L^2(\mathbb{R})$. By Gröchenig's Lemma, this would imply that $\{E_{mb}T_{na}g\}$ was a frame for $L^2(\mathbb{R})$. But this is impossible since g is not bounded. Thus the sum defining f does not converge strongly.

Thus it is seen that while there may be many ways to write

$$f = \sum_{n}\sum_{m} a_{n,m}(f)\ E_{mb}T_{na}g$$

with $\sum_n\sum_m|a_{n,m}(f)|^2$ equivalent to $\|f\|_2^2$, for a particular g, the sum need not converge in the L^2 sense. Such convergence forces some good behavior on g.

Now we present a result due to Chris Heil on the uniqueness of the dual frame.

LEMMA 5.1.4. Let $\{x_n\}$ be a frame for $\mathbf{H}$ with bounds A, B, and let $\{e_n\}$ be a collection of vectors dual to $\{x_n\}$. Further assume that for every $y \in \mathbf{H}$, $\sum_n \langle y, x_n \rangle e_n$ converges strongly in $\mathbf{H}$. Then $\{e_n\}$ is a frame with bounds B^{-1}, A^{-1}.

PROOF. Define the operator I to be

$$Ix = \sum_n \langle x, e_n \rangle x_n.$$

Obviously, I is the identity on $\mathbf{H}$. Now, we wish to compute I^*. Let $y \in \mathbf{H}$. Then

$$\langle I^* y, x \rangle = \langle y, \sum_n \langle x, e_n \rangle x_n \rangle = \sum_n \langle y, x_n \rangle \overline{\langle x, e_n \rangle} = \langle \sum_n \langle y, x_n \rangle e_n, x \rangle$$

where by assumption, the last sum converges strongly. Since $\{x_n\}$ is a frame with bounds A, B, $\{e_n\}$ is a set of atoms with atomic bounds A, B. Therefore, by Gröchenig's Lemma, $\{e_n\}$ is a frame with bounds B^{-1}, A^{-1}.■

LEMMA 5.1.5. Let $\{x_n\}$, $\{e_n\}$ be as in Lemma 5.1.4. Define the operators T, U, T', U' as follows.

$$T: \mathbf{H} \longrightarrow \ell^2(I) \qquad x \longmapsto (\langle x, x_n \rangle) \qquad\qquad T': \ell^2(I) \longrightarrow \mathbf{H} \qquad (\lambda_n) \longmapsto \textstyle\sum_n \lambda_n x_n$$

$$U: \mathbf{H} \longrightarrow \ell^2(I) \qquad x \longmapsto (\langle x, e_n \rangle) \qquad\qquad U': \ell^2(I) \longrightarrow \mathbf{H} \qquad (\lambda_n) \longmapsto \textstyle\sum_n \lambda_n e_n.$$

Then $Im(T) = Im(U)$ and $TU' = UT' = Id_R$, where $R = In(T) = Im(U)$.

PROOF. Observe first that by Lemma 5.1.4, both $\{x_n\}$ and $\{e_n\}$ are frames and consequently the sums defining T' and U' converge strongly for all $(\lambda_n) \in \ell^2(I)$. Also, we know that Im(T) and Im(U) are closed subspaces of $\ell^2(I)$ and that both T and U are continuously invertible on their respective ranges.

Now we show that $UT' = Id_{Im(U)}$. If $\Lambda = (\lambda_n) \in Im(U)$, then $\lambda_n = \langle x, e_n \rangle$ for some $x \in \mathbf{H}$. Now,

$$T'(\Lambda) = \sum_n \langle x, e_n \rangle x_n = x$$

by assumption. Thus

$$UT'(\Lambda) = Ux = (\langle x, e_n \rangle) = (\lambda_n) = \Lambda.$$

Similarly, we can show that $TU' = Id_{Im(T)}$.

Now, I claim that, thought of as a map from Im(U) onto Im(U), the adjoint of UT' is TU' and thought of as a map from Im(T) onto Im(T), the adjoint of TU' is UT'. To see this, let $\Lambda = (\lambda_n)$, $\Gamma = (\gamma_n)$ be in $\ell^2(I)$. Then

$$\begin{aligned}\langle \Gamma, UT'(\Lambda) \rangle &= \sum_m \gamma_m \overline{UT'(\Lambda)_m} \\ &= \sum_m \gamma_m \overline{\langle \sum_n \lambda_n x_n, e_m \rangle} = \langle \sum_m \gamma_m e_m, \sum_n \lambda_n x_n \rangle.\end{aligned}$$

Also,

$$\begin{aligned}\langle TU'(\Gamma), \Lambda \rangle &= \sum_m TU'(\Gamma)_m \overline{\lambda_m} \\ &= \sum_m \langle \sum_n \gamma_n e_n, x_m \rangle \overline{\lambda_m} = \langle \sum_n \gamma_n e_n, \sum_m \lambda_m x_m \rangle.\end{aligned}$$

Thus, $(UT')^* = TU'$ and $(TU')^* = UT'$. Therefore TU' is the identity on Im(U) and UT' is the identity on Im(T). Hence, if $\Lambda \in Im(U)$, then $TU'(\Lambda) = \Lambda$. But $Im(TU') \subset Im(T)$. Thus $\Lambda \in Im(T)$ and so $Im(U) \subseteq Im(T)$. Similarly, $Im(T) \subseteq Im(U)$. Thus $Im(T) = Im(U)$.∎

THEOREM 5.1.6. (Heil) Let $\{x_n\}$ be a frame, and $\{e_n\}$ be as in Lemma 5.1.4. Then $e_n = S^{-1}x_n$ where

$$Sx = \sum_n \langle x, x_n \rangle x_n.$$

PROOF. Let T, T', U, U' be as defined in Lemma 5.1.5. Observe that $S = T'T$. We wish to show that $U'U = S^{-1}$. By definition, $T'U = Id_H$ and by Lemma 5.1.5, $TU' = Id_{Im(U)}$. Thus $T'TU'U = T'U = Id_H$. Thus $U'U = S^{-1}$. Note also that by general results on frames, $S = T'T$ is self-adjoint. Therefore, for all $x \in \mathbf{H}$,

$$\langle x, e_n \rangle = Ux = TU'Ux = TS^{-1}x = \langle S^{-1}x, x_n \rangle = \langle x, S^{-1}x_n \rangle.$$

It therefore follows that $e_n = S^{-1}x_n$ for all $n \in I$. ∎

Section 5.2. Existence of frames in $L^2(R^k)$

In this section, we extend an existence theorem of Daubechies and give some examples which show that the condition in this theorem is not necessary for existence.

THEOREM 5.2.1. Let $g \in L^2(\mathbb{R}^k)$ and let $a > 0$ satisfy

(1) there exist constants A, B > 0 such that for almost every $x \in \mathbb{R}^k$,

$$A \le \sum_n |g(x-na)|^2 \le B,$$

and

(2) $\lim_{b \to 0} \sum_{j \neq 0} \beta_a(j/b) = 0$ where

$$\beta_a(s) = \left\| \sum_n \bar{g}(x-na)g(x-s-na) \right\|_\infty .$$

Then there exists a number $b_0 > 0$ such that if $0 < b < b_0$ then $\{E_{mb}T_{na}g\}$ is a frame for $L^2(\mathbb{R}^k)$.

PROOF. Let $f \in L^2(\mathbb{R}^k)$ be fixed and suppose first that f is compactly supported. Then

$$\sum_n \sum_m |\langle f, E_{mb}T_{na}g\rangle|^2 = \sum_n \sum_m |\langle fT_{na}\bar{g}, E_{mb}\rangle|^2$$

$$= b^{-k} \sum_n \int_{Q_{1/b}} \left| \sum_l f(x-l/b)\bar{g}(x-na-l/b) \right|^2 dx$$

since for each $n \in \mathbb{Z}^k$, $\{b^{-k/2}\langle fT_{na}\bar{g}, E_{mb}\rangle\}$ is the sequence of Fourier coefficients of the 1/b-periodic function

$$\sum_l f(x-l/b)\bar{g}(x-na-l/b).$$

Also, since f is compactly supported, the sum above is finite for each $x\in\mathbb{R}^k$ and each $n\in\mathbb{Z}^k$. Thus, we have

$$\sum_n \sum_m |\langle f, E_{mb}T_{na}g\rangle|^2$$

$$= \sum_n \int_{Q_{1/b}} \sum_l \sum_j f(x-l/b)\bar{f}(x-j/b)\bar{g}(x-na-l/b)g(x-na-j/b)\ dx$$

$$= \sum_n \int f(x)\bar{g}(x-na) \sum_j \bar{f}(x-j/b)g(x-na-j/b)\ dx$$

$$= \sum_j \int f(x)\bar{f}(x-j/b) \sum_n \bar{g}(x-na)g(x-na-j/b)\ dx.$$

The last two equalities involve interchanges of summations and integrals which we will now justify. Certainly, if we can show that the last quantity converges absolutely, all previous interchanges will be justified. First, observe that since f is compactly supported, there are only finitely many non-zero terms in the sum over j, and by assumption (1), we have that

$$\sum_n |g(x-na)||g(x-na-j/b)|$$

$$\le \left(\sum_n |g(x-na)|^2\right)^{1/2}\left(\sum_n |g(x-j/b-na)|^2\right)^{1/2} \le B < \infty.$$

Consequently, we have that

$$\sum_j \int |f(x)||f(x-j/b)| \sum_n |g(x-na)||g(x-na-j/b)|\ dx$$

$$\le \sum_{j\in F} B\int |f(x)||f(x-j/b)|\ dx \le \|f\|_2^2 \sum_{j\in F} B < \infty$$

where $F \subset \mathbb{Z}^k$ is some finite set depending on supp(f).

Now, it follows that if $f\in L^2(\mathbb{R}^k)$ has compact support, then

$$\sum_n \sum_m |\langle f, E_{mb}T_{na}g\rangle|^2$$

$$\le b^{-k} \sum_j \left\| \sum_n \bar{g}(x-na)g(x-na-j/b)\right\|_\infty \|f\|_2^2 \equiv b^{-k}B_0\|f\|_2^2.$$

Now, given an arbitrary $f \in L^2(\mathbb{R}^k)$, we choose a sequence $\{f_i\}$ in $L^2(\mathbb{R}^k)$ where f_i increases to f almost everywhere and each f_i has compact support. Then for each $i \in \mathbb{Z}$,

$$\sum_n \sum_m |\langle f_i, E_{mb}T_{na}g\rangle|^2 \le b^{-k}B_0\|f_i\|_2^2 \le b^{-k}B_0\|f\|_2^2.$$

Also, the Dominated Convergence Theorem implies that for each n, $m \in \mathbb{Z}^k$,

$$\lim_{i\to\infty} \langle f_i, E_{mb}T_{na}g\rangle = \langle f, E_{mb}T_{na}g\rangle.$$

Now, for any finite subsets, F and G, of $\mathbb{Z}^k$, we have that for all $i \in \mathbb{Z}$,

$$\sum_{n\in F} \sum_{m\in G} |\langle f_i, E_{mb}T_{na}g\rangle|^2 \le b^{-k}B_0\|f\|_2^2.$$

Taking the limit as $i\to\infty$, we have that

$$\sum_{n\in F} \sum_{m\in G} |\langle f, E_{mb}T_{na}g\rangle|^2 \le b^{-k}B_0\|f\|_2^2.$$

Taking the supremum over all finite subsets of $\mathbb{Z}^k$, we have finally that

$$\sum_n \sum_m |\langle f, E_{mb}T_{na}g\rangle|^2 \le b^{-k}B_0\|f\|_2^2.$$

If we let $\{f_i\}$ be as in the above paragraph, then we know that

$$\sum_n \sum_m |\langle f_i, E_{mb}T_{na}g\rangle|^2$$

$$= b^{-k}\int |f_i(x)|^2 \sum_n |g(x-na)|^2$$

$$+b^{-k}\sum_{j\neq 0}\int f_i(x)\bar{f}(x-j/b) \sum_n \bar{g}(x-na)g(x-na-j/b)\, dx$$

$$\equiv b^{-k}\int |f_i(x)|^2 \sum_n |g(x-na)|^2 + R(f_i)$$

where

$$|R(f_i)| \le \|f_i\|_2^2 b^{-k} \sum_{j\neq 0} \beta_a(j/b)$$

and hence that

$$\sum_n \sum_m |\langle f_i, E_{mb}T_{na}g\rangle|^2 \ge b^{-k}\left[A - \sum_{j\neq 0} \beta_a(j/b)\right] \|f_i\|_2^2.$$

Moreover, we can use the arguments of the previous paragraph to show that

$$\begin{aligned}
&\left[\sum_n \sum_m |\langle f_i, E_{mb}T_{na}g\rangle|^2\right]^{1/2} \\
&= \left[\sum_n \sum_m |\langle f, E_{mb}T_{na}g\rangle + \langle f_i - f, E_{mb}T_{na}g\rangle|^2\right]^{1/2} \\
&\le \left[\sum_n \sum_m |\langle f, E_{mb}T_{na}g\rangle|^2\right]^{1/2} + \left[\sum_n \sum_m |\langle f_i - f, E_{mb}T_{na}g\rangle|^2\right]^{1/2} \\
&\le \left[\sum_n \sum_m |\langle f, E_{mb}T_{na}g\rangle|^2\right]^{1/2} + b^{-k/2} B_0^{1/2} \|f_i - f\|_2.
\end{aligned}$$

Finally, we have that since both sides above are finite,

$$\begin{aligned}
&\left[\sum_n \sum_m |\langle f, E_{mb}T_{na}g\rangle|^2\right]^{1/2} \\
&\ge \left[\sum_n \sum_m |\langle f_i, E_{mb}T_{na}g\rangle|^2\right]^{1/2} - b^{-k/2} B_0^{1/2} \|f_i - f\|_2 \\
&\ge \left[A - \sum_{j\neq 0} \beta_a(j/b)\right]^{1/2} \|f_i\|_2 - b^{-k/2} B_0^{1/2} \|f_i - f\|_2.
\end{aligned}$$

Letting $i \to \infty$ on the right hand side, we have that for all $f \in L^2(\mathbb{R}^k)$,

$$\sum_n \sum_m |\langle f, E_{mb}T_{na}g\rangle|^2 \ge b^{-k/2}\left[A - \sum_{j\neq 0} \beta_a(j/b)\right] \|f\|_2^2.$$

By hypothesis (2), it follows that for all sufficiently small $b > 0$, $\{E_{mb}T_{na}g\}$ is a frame for $L^2(\mathbb{R}^k)$. ∎

REMARK 5.2.2. If a function g satisfies condition (2) in Theorem 5.2.1, we say that g satisfies the "β-condition."

COROLLARY 5.2.3. If $g \in W(L^\infty, L^1)$, then g satisfies the β-condition.

PROOF. This follows immediately from Theorem 4.1.3, when $w \equiv 1$ and $g = h$. ■

PROPOSITION 5.2.4. There is a function $g \in L^2(\mathbb{R})$ such that

(1) $g \notin W(L^\infty, L^1)$,

(2) g does not satisfy the β-condition, and

(3) (g,a,b) generates a frame for $L^2(\mathbb{R})$ for all $0 < a,\ b \le 1$.

PROOF. Let $g(x) = \dfrac{\sin(\pi x)}{\pi x}$. Then $\hat{g}(\gamma) = 1_{[-1/2,1/2]}(\gamma)$.

Claim: $\operatorname{ess\,sup}_x \left| \sum_{n \in Z} g(x-na)\bar{g}(x-s-na) \right| = \left| \dfrac{\sin(\pi s)}{\pi s} \right|$.

Proof. $\sum_{n \in Z} g(x-na)\bar{g}(x-s-na)$

$$= \sum_{n \in Z} (gT_s g)(x-na)$$

$$= a^{-1} \sum_{j \in Z} (gT_s g)^\wedge(j/a)\ e^{2\pi i x j/a}$$

in $L^2[0,a]$ since $\sum (gT_s g)(x-na)$ is bounded on $[0,a]$ and hence is in $L^2[0,a]$,

$$= \sum_{j \in Z} (\hat{g} * E_s \hat{g})(j/a)\ e^{2\pi i x j/a}$$

$$= \sum_{j \in Z} e^{2\pi i x j/a} \int \hat{g}(\gamma)\hat{g}(j/a-\gamma)E_s(j/a-\gamma)\ d\gamma.$$

Now, since $\hat{g}$ is supported on $[-1/2,1/2]$ and since $a \le 1$, the only non-zero term in the sum is the $j = 0$ term. Thus,

$$\sum_{n \in Z} (gT_s g)(x-na) = \int e^{-2\pi i \gamma s} [\hat{g}(\gamma)]^2\ d\gamma$$

$$= \int_{-1/2}^{1/2} e^{-2\pi i \gamma s}\ d\gamma = \frac{\sin(\pi s)}{\pi s}. \square$$

Now, clearly $g \notin W(L^{\infty}, L^{1})$, and for each $N \in \mathbb{N}$, letting $b = 1/2N$, we have that

$$\sum_{j \in \mathbb{Z} \setminus \{0\}} \left| \frac{\sin(\pi j/b)}{\pi j/b} \right| = \infty$$

since when $j = mN$, where $m \in \mathbb{Z}$, we have that

$$\left| \frac{\sin(\pi j/b)}{\pi j/b} \right| = 2/\pi \ 1/m.$$

Thus, g does not satisfy the β-condition. It is easy to see, however, that $(\hat{g}, b, a)$ generates a frame for $L^{2}(\mathbb{R})$ for every $0 < a, b \leq 1$ which implies that (g, a, b) generates a frame for those same values of a and b.∎

PROPOSITION 5.2.5. There is a function $g \in L^{2}(\mathbb{R})$ such that

(1) $g \notin W(L^{\infty}, L^{1})$,

(2) g is non-negative,

(3) g does not satisfy the β-condition, and

(4) (g, a, b) generates a frame for $L^{2}(\mathbb{R})$ when $a = 1$, $b = 1/N$, $N \in \mathbb{N}$.

PROOF. Let

$$g(x) = \sum_{n=0}^{\infty} 1_{[2^{-n}(2^{n}-1), 2^{-n-1}(2^{n+1}-1))+n}(x).$$

Claim 1:

$$\sum_{n} |g(x-n)|^{2} \equiv 1.$$

Proof. Let $x \in \mathbb{R}$. Then $x = m+r$ for some $m \in \mathbb{Z}$ and $r \in [0,1)$. Now since

$$[0,1) = \bigcup_{n=0}^{\infty} [2^{-n}(2^{n}-1), 2^{-n-1}(2^{n+1}-1))$$

and since those intervals are disjoint, there exists a unique number j such that $r \in [2^{-j}(2^{j}-1), 2^{-j-1}(2^{j+1}-1))$. Now,

$$\sum_n |g(x-n)|^2 = \sum_n |g(r-(n-m))|^2 = |g(r+j)|^2 = 1.\square$$

Claim 2. $\sum_{j\neq 0} \beta_1(jr) = 0$, if $r\in\mathbb{Z}$; $= \infty$, otherwise.

Proof. Suppose that $r\in\mathbb{Z}$. Clearly, we have that $g(x)g(x-jr) = 0$ unless $j = 0$. Thus,

$$\left|\sum_n g(x-n)g(x-jr-n)\right| = 0$$

unless $j = 0$. This gives that $\beta_1(jr) = 0$ if $j \neq 0$.

Suppose that $r\notin\mathbb{Z}$ and that r is rational. Then there is a sequence $\{m_j\} \subset \mathbb{Z}$ and $\{q_j\} \subset (0,1)$ such that $jr = m_j+q_j$ for $j \neq 0$. Let $q_1 = c/d$ in lowest terms. Then $q_{j+d} = q_j$ for all $j \neq 0$ and always, $q_j = p/d$ where $p = 0, 1, 2,\ldots, d-1$. Thus, there are infinitely many j for which $q_j\in[1/2, 2^{-j}(2^j-1)]$. For such j,

$$\operatorname*{ess\,sup}_{[0,1]} g(x)g(x-jr) \geq 1$$

which implies that $\beta_1(jr) \geq 1$ for such j, whence

$$\sum_{j\neq 0} \beta_1(jr) = \infty.$$

Suppose that $r\notin\mathbb{Z}$ and r is irrational. Then the collection of numbers $\{q_j\}$ is dense in $(0,1)$ and also in this case, we have that for infinitely many j, $q_j\in[1/2, 2^{-j}(2^j-1)]$. Thus, we are done by the same argument as above.$\square$

Now, clearly, g is non-negative and $\|g\|_{\infty,1} = \infty$. Claim 2 asserts that

$$\sum_{j\neq 0} \beta_1(j/b) = 0$$

whenever $b = 1/N$, $N\in\mathbb{N}$. Thus, by the argument of Theorem 5.2.1, $(g, 1, 1/N)$ generates a frame for $L^2(\mathbb{R})$ for all $N\in\mathbb{N}$.$\blacksquare$

DEFINITION 5.2.6. For any function g, each $j \in \mathbb{Z}^k$, and $c > 0$, define

$$\lambda_{j,c} = \operatorname*{ess\,sup}_{x \in Q_c} |g(x-jc)|.$$

THEOREM 5.2.7. Let $g \in L^2(\mathbb{R}^k)$ be non-negative and bounded, with $\beta_a(s)$ as defined in Theorem 5.2.1 for some fixed $a > 0$. For any $c \geq a$, there is a sequence of open sets $\{\mathcal{O}_j\}_{j \in \mathbb{Z}^k}$ with $\mathcal{O}_j \subset Q_{2c}+jc$, and a constant $d > 0$ such that for any sequence $\{s_j\}$ with $s_j \in \mathcal{O}_j$,

$$\sum_j \beta_a(s_j) \geq d \sum_j \lambda_{j,c}.$$

PROOF. By the definition of $\lambda_{j,c}$, there are sets $E_j \subset Q_c+jc$ such that $|g(x)| \geq \lambda_{j,c}/2$ for all $x \in E_j$. We may assume without loss of generality that $\lambda_{0,c} > 0$ for if not we could replace g by an appropriate shift of g. This would not alter the result as $\beta_a(s)$ is unaltered when g is shifted. Now, fix $j \in \mathbb{Z}^k$ and let

$$f_j(t) = |(E_0+t) \cap E_j|$$

for $t \in \mathbb{R}^k$. Then f_j is continuous, non-negative and we have that

$$\int f_j(t)\, dt = \iint 1_{E_0}(x-t) 1_{E_j}(x)\, dx\, dt = \int 1_{E_j}(x)\, dx \int 1_{E_0}(x-t)\, dt$$
$$= |E_0|\,|E_j| > 0.$$

Thus, $f_j(t) > 0$ on some open set in $\mathbb{R}^k$, call it $\mathcal{O}_j$. It is clear that $\mathcal{O}_j \subset Q_{2c}+jc$ for if $t \in \mathcal{O}_j$ then certainly we must have $(E_0+t) \cap E_j \neq \emptyset$ and moreover that $(Q_c+t) \cap (Q_c+jc) \neq \emptyset$, or $Q_c \cap (Q_c+jc-t) \neq \emptyset$. This means in particular that $jc-t \in Q_{2c}$ or that $t \in Q_{2c}+jc$.

Now, choose $s_j \in \mathcal{O}_j$ for each j. Then

$$\beta_a(s_j) = \left\| \sum_n |g(x-na)|\,|g(x-s_j-na)| \right\|_\infty$$

$$= \operatorname*{ess\,sup}_{x\in Q_c} \sum_n |g(x-na)||g(x-s_j-na)|$$

$$\geq \operatorname*{ess\,sup}_{x\in Q_c} |g(x)||g(x-s_j)| \geq \lambda_{0,c}/4\ \lambda_{j,c}$$

since $|g(x)||g(x-s_j)| \geq \lambda_{0,c}/4\ \lambda_{j,c}$ for all $x\in E_0$. Thus,

$$\sum_j \beta_a(s_j) \geq \lambda_{0,c}/4 \sum_j \lambda_{j,c}$$

and we are done.∎

REMARK 5.2.8. Clearly, if $g\notin W(L^\infty,L^1)$, then $\sum \lambda_{j,c} = \infty$ for all $c > 0$, so for such a g, Theorem 5.2.7 says that for some sequence of points $\{s_j\}$ in $\mathbb{R}^k$, $\sum \beta_a(s_j) = \infty$. Also, it says that the sequence $\{s_j\}$ can be taken to be arbitrarily "spread out" in $\mathbb{R}^k$, that is, $s_j\in Q_{2c}+jc$ for any sufficiently large c. If for arbitrarily small $b > 0$, we are somehow able to take $s_j = j/b$ then Theorem 5.2.7 says that g fails to satisfy Daubechies' β-condition. In this sense, Theorem 5.2.7 is a partial converse to Corollary 5.2.3.

THEOREM 5.2.9. Let g be as in Theorem 5.2.7. Suppose that for some $\varepsilon > 0$ and $r > 0$, we have that $|g(x)| \geq \varepsilon$ for almost every x in Q_r. Then

$$\sum_j \beta_a(jr) \geq \varepsilon \sum_j \lambda_{j,r}.$$

PROOF. As in the proof of Theorem 5.2.7, we have that

$$\beta_a(jr) \geq \operatorname*{ess\,sup}_{x\in\mathbb{R}^k} |g(x)||g(x-jr)| \geq \operatorname*{ess\,sup}_{x\in Q_r} |g(x)||g(x-jr)|$$

$$\geq \varepsilon \operatorname*{ess\,sup}_{x\in Q_r} |g(x-jr)| = \varepsilon\lambda_{j,r}.$$

The conclusion now follows easily.∎

REMARK 5.2.10. Theorem 5.2.9 is a partial converse to Corollary 5.2.3 for if $g \notin W(L^{\infty}, L^{1})$, and if g satisfies the conditions of Theorem 5.2.9 for each $r > 0$, then we must have that $\sum \beta_a(j/b) = \infty$ for all sufficiently small $b > 0$.

LEMMA 5.2.11. Let A be a countable index set and let $\{x_n\}_{n \in A}$ be a sequence in $\mathbb{R}^k$ such that for some non-negative function h on $\mathbb{R}^k$, we have that $\sum_n h(x_n) = \infty$. Let $\{P_i\}_{i \in I}$ be a countable collection of disjoint subsets of $\mathbb{R}^k$ with the property that $\{x_n\} \subset \bigcup P_i$ and such that there exists a constant $M > 0$ such that for all $i \in I$,

$$\#\{n \in A: \quad x_n \in P_i\} \leq M.$$

Then there is a subsequence $\{x_{n(i)}\}_{i \in I}$ such that for all i, $x_{n(i)} \in P_i$, and

$$\sum_{j \in I} h(x_{n(i)}) = \infty.$$

PROOF. Suppose not, that is, suppose that for each subsequence $\{x_{n(i)}\}$ chosen so that $x_{n(i)} \in P_i$ for each $i \in I$, we have that

$$\sum_{j \in I} h(x_{n(i)}) < \infty.$$

Then by choosing at most M distinct such subsequences, we could exhaust the entire sequence $\{x_n\}$, that is, we could choose for $j = 1, 2, \ldots, M$, $\{x_{n_j(i)}\}$ where for all $i \in I$ and each j, $x_{n_j(i)} \in P_i$, such that $\{x_n\} = \bigcup_{j=1}^{M} \{x_{n_j(i)}\}_{i \in I}$. Thus we would have

$$\sum_{n \in A} h(x_n) = \sum_{j=1}^{M} \sum_{i \in I} h(x_{n_j(i)}) < \infty.$$

As this contradicts our original assumption, the lemma is proved.■

THEOREM 5.2.12. Let g, a, and $\beta_a(s)$ be as in Theorem 5.2.7. Suppose there exists a non-increasing function f on $[0,\infty)$ such that for some constants c_1, $c_2 > 0$,

$$c_1 f(|s|) \le \beta_a(s) \le c_2 f(|s|)$$

for all $s \in \mathbb{R}^k$. Then for every $r \ge a$, there is a constant $c > 0$ such that

$$c \sum_j \lambda_{j,r} \le \sum_j \beta_a(jr).$$

PROOF. For the i^{th} "quadrant" of $\mathbb{R}^k$ (including the adjacent coordinate axes), define the cube $Q_{i,r}$ by $Q_{i,r} = [0, r\varepsilon_i)^k$ where $\varepsilon_i \in \mathbb{R}^k$ is that unique vector in the i^{th} quadrant whose entries are either 1 or -1. Then the collection of cubes $\{Q_{i,r}+nr$: n in the i^{th} quadrant$\}$ forms a partition of the i^{th}quadrant of $\mathbb{R}^k$.

Now, choose a sequence of points $\{s_j\} \subset \mathbb{R}^k$ as in Theorem 5.2.7 so that $s_j \in Q_r + jr$ and each s_j sits in the interior of some cube $Q_{i,r}+nr$ for some i and n. Since $s_j \in Q_r + jr$, it is clear that there is a number $M > 0$ independent of n and i such that

$$\#\{j \in \mathbb{Z}^k: \ s_j \in Q_{i,r}+nr\} \le M.$$

Given n in the i^{th} quadrant, we have that $|nr| \le |x|$ for all $x \in Q_{i,r}+nr$. Thus

$$\sum_{s_j \in Q_{i,r}+nr} \beta_a(s_j) \le c_2 \sum_{s_j \in Q_{i,r}+nr} f(|s_j|)$$

$$\le c_2 M f(|nr|) \le c_2 c_1^{-1} M \beta_a(nr).$$

Then, summing over all n in the i^{th} quadrant gives,

$$\sum_{s_j \in i^{th} \text{ quad.}} \beta_a(s_j) \le c_2 c_1^{-1} M \sum_{n \in i^{th} \text{ quad.}} \beta_a(nr).$$

Finally, we have that

$$\sum_j \beta_a(s_j) = \sum_{i=1}^{2^k} \sum_{s_j \in i^{th} \text{ quad.}} \beta_a(s_j)$$

$$\le c_2 c_1^{-1} M \sum_{i=1}^{2^k} \sum_{n \in i^{th} \text{ quad.}} \beta_a(nr) \le 2^{k-1} c_2 c_1^{-1} M \sum_n \beta_a(nr)$$

where the extra factor of 2^{k-1} comes from the fact that an $n \in \mathbb{Z}^k$ is in 2^{k-1} quadrants when it lies on one of the coordinate axes.

Thus, by Theorem 5.2.7, we have that for some constant c,

$$c \sum_j \lambda_{j,r} \le \sum_j \beta_a(jr). \blacksquare$$

COROLLARY 5.2.13. Let g be a non-negative function in $L^2(\mathbb{R}^k)$ and let f be a non-increasing, function on $[0,\infty)$ such that

(1) $\beta_a(s) \le f(|s|)$ for all $s \in \mathbb{R}^k$, and

(2) for all $r > 0$,

$$\sum_n f(|nr|) < \infty.$$

Then $g \in W(L^\infty, L^1)$.

PROOF. The proof of Theorem 5.2.12 shows that

$$\sum_{s_j \in i^{th} \text{ quad.}} \beta_a(s_j) \le c \sum_{n \in i^{th} \text{ quad.}} f(|nr|)$$

for some constant c which implies that

$$\sum_j \beta_a(s_j) \le c \sum_n f(|nr|) < \infty.$$

By the way the points $\{s_j\}$ were chosen, we have that for some $d > 0$,

$$d\sum_j \lambda_{j,r} \le \sum_j \beta_a(s_j) < \infty$$

whence $g \in W(L^\infty, L^1)$. ■

REMARK 5.2.14. In the existence theorem for W-H frames for $L^2(\mathbb{R})$ found in [D1], Daubechies makes the assumption that for some $\varepsilon > 0$,

$$\beta(s) \le C_\varepsilon (1+|s|^2)^{(1+\varepsilon)/2}$$

for every $s \in \mathbb{R}$ and some $C_\varepsilon < \infty$. Corollary 5.2.13 says that this assumption implies that $g \in W(L^\infty, L^1)$.

Section 5.3. Phase-space localization

The following theorem is similar to a theorem in [D1]. The conclusion in this theorem is the same as that in [D1], but the condition on g required is more transparent and allows the values of important constants to be computed easily. The crucial step is contained in the following lemma. The proof of the theorem is otherwise identical to that in [D1].

LEMMA 5.3.1. Let $g \in W(L^\infty(\mathbb{R}^k), L^1(\mathbb{R}^k))$, let a, b > 0 be given. Then for every $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that for every $T > 0$, and $t \geq t_\varepsilon$, we have

$$\sum_j \operatorname*{ess\,sup}_{\substack{|x| \leq T, \\ |x-j/b| \leq T}} \sum_{|na| \geq T+t} |g(x-na)|\,|g(x-na-j/b)| < \varepsilon.$$

PROOF. Given $\varepsilon_0 > 0$, choose $M > 0$ so large that letting $g_M = g1_{Q_M}$, we have $\|g-g_M\|_{\infty,1,a} < \varepsilon_0$. Also, choose $L > 0$ so large that

$$\sum_{|j|>L} \Big\| \sum_n |g(x-na)|\,|g(x-na-j/b)| \Big\|_\infty < \varepsilon_0.$$

Such an L exists by Lemma 4.1.2. Note that

$$\{x: \ |x| \leq T, \ |x-j/b| \leq T \text{ for all } |j| \leq L\} \subset Q_{T+L/b}.$$

Now,

$$\sum_j \operatorname*{ess\,sup}_{\substack{|x| \leq T, \\ |x-j/b| \leq T}} \sum_{|na| \geq T+t} |g(x-na)|\,|g(x-na-j/b)|$$

$$\leq \sum_{|j|>L} \operatorname*{ess\,sup}_x \sum_n |g(x-na)|\,|g(x-na-j/b)|$$

$$+\sum_{|j|\le L} \operatorname*{ess\,sup}_{\substack{|x|\le T,\\ |x-j/b|\le T}} \sum_{|na|\ge T+t} |g(x-na)||g(x-na-j/b)|$$

$= (*)$.

Now, choose t_0 so large that if $t \ge t_0$ and if $na\in(Q_{T+t})^c$, then $(Q_M+na) \cap Q_{T+L/b} = \emptyset$. For this, it suffices that $T+t_0-M > T+L/b$ or that $t_0 > M+L/b$, *independent of T*. then if $t \ge t_0$, Lemma 4.1.2 says that

$$(*) \le \varepsilon_0+\sum_{|j|\le L} \operatorname*{ess\,sup}_{x\in Q_{T+L/b}} \sum_{|na|\ge T+t} |g_M(x-na)||g(x-na-j/b)|$$

$$+\sum_j \Big\| \sum_n |(g-g_M)(x-na)||g(x-na-j/b)|\Big\|_\infty$$

$$= \varepsilon_0+0+\sum_j \Big\| \sum_n |(g-g_M)(x-na)||g(x-na-j/b)|\Big\|_\infty$$

$\le \varepsilon_0+2^k\|g-g_M\|_{\infty,1,a}\ \|g_-\|_{\infty,1,1/b} < \varepsilon_0(1+2^k\|g_-\|_{\infty,1,1/b})$.

Now, given $\varepsilon > 0$, choose $\varepsilon_0 > 0$ so small that $\varepsilon_0(1+2^k\|g_-\|_{\infty,1,1/b})$ $< \varepsilon$. Let t_ε be the t_0 corresponding to that ε_0.∎

THEOREM 5.3.2. Let $g\in W(L^\infty(\mathbb{R}^k),L^1(\mathbb{R}^k))$, $\hat{g}\in W(L^\infty(\hat{\mathbb{R}}^k),L^1(\hat{\mathbb{R}}^k))$ and suppose that (g,a,b) generates a W-H frame for $L^2(\mathbb{R}^k)$ with bounds A, B. Then given $\varepsilon > 0$, there exists t_ε, $\omega_\varepsilon > 0$ such that for all $f\in L^2(\mathbb{R}^k)$, every T, $\Omega > 0$, and every $t \ge t_\varepsilon$ and $\omega \ge \omega_\varepsilon$,

$$\Big\|f-\sum_{(n,m)\in B} \langle f,E_{mb}T_{na}g\rangle E_{mb}T_{na}S^{-1}g\Big\|_2$$

$$\le (B/A)^{1/2}[\|(I-P_\Omega)f\|_2+\|(I-Q_T)f\|_2]+\varepsilon\|f\|_2$$

where $B = B(\varepsilon,T,\Omega) = \{(n,m)\in\mathbb{Z}^k\times\mathbb{Z}^k:\ |na|\le T+t,\ |mb|\le\Omega+\omega\}$, $Q_Tf = f1_{[-T,T]}$, and $P_\Omega f = (\hat{f}1_{[-\Omega,\Omega]})^\vee$.

PROOF. Fix T, Ω and let

$$B(t,\omega) = \{(n,m)\in\mathbb{Z}^k\times\mathbb{Z}^k: \ |n|\le a^{-1}(T+t), \ |m|\le b^{-1}(\Omega+\omega)\}.$$

Then,

$$\Big\| f - \sum_{(n,m)\in B(t,\omega)} \langle f, E_{mb}T_{na}g\rangle E_{mb}T_{na}S^{-1}g\Big\|_2$$

$$= \Big\| \sum_{(n,m)\in B(t,\omega)^c} \langle f, E_{mb}T_{na}g\rangle E_{mb}T_{na}S^{-1}g\Big\|_2$$

$$= \sup_{\|h\|_2=1} \Big| \sum_{(n,m)\in B(t,\omega)^c} \langle f, E_{mb}T_{na}g\rangle \overline{\langle h, E_{mb}T_{na}S^{-1}g\rangle}\Big|.$$

Now,

$$\Big| \sum_{(n,m)\in B(t,\omega)^c} \langle f, E_{mb}T_{na}g\rangle \overline{\langle h, E_{mb}T_{na}S^{-1}g\rangle}\Big|$$

$$\le \sum_{\substack{|na|\le T+t,\\ m}} |\langle Q_T f, E_{mb}T_{na}g\rangle|\,|\langle h, E_{mb}T_{na}S^{-1}g\rangle|$$

$$+ \sum_{\substack{|na|\le T+t,\\ m}} |\langle (I-Q_T) f, E_{mb}T_{na}g\rangle|\,|\langle h, E_{mb}T_{na}S^{-1}g\rangle|$$

$$+ \sum_{\substack{|mb|\le \Omega+\omega,\\ n}} |\langle P_\Omega f, E_{mb}T_{na}g\rangle|\,|\langle h, E_{mb}T_{na}S^{-1}g\rangle|$$

$$+ \sum_{\substack{|mb|\le \Omega+\omega,\\ n}} |\langle (I-P_\Omega) f, E_{mb}T_{na}g\rangle|\,|\langle h, E_{mb}T_{na}S^{-1}g\rangle|$$

$$= N_1+N_2+N_3+N_4.$$

Now, by Cauchy-Schwarz,

$$N_2 \le \Big(\sum_{\substack{|na|\le T+t,\\ m}} |\langle (I-Q_T) f, E_{mb}T_{na}g\rangle|^2\Big)^{1/2} \Big(\sum_{\substack{|na|\le T+t,\\ m}} |\langle h, E_{mb}T_{na}S^{-1}g\rangle|^2\Big)^{1/2}$$

$$\le (B/A)^{1/2} \|(I-Q_T)f\|_2 \|h\|_2,$$

$$N_4 \le \left[\sum_{\substack{|mb|\le\Omega+\omega,\\ n}} |\langle (I-P_\Omega)f, E_{mb}T_{na}g\rangle|^2 \right]^{1/2}$$

$$\left[\sum_{\substack{|mb|\le\Omega+\omega,\\ n}} |\langle h, E_{mb}T_{na}S^{-1}g\rangle|^2 \right]^{1/2}$$

$$\le (B/A)^{1/2}\|(I-P_\Omega)f\|_2\|h\|_2,$$

since if $\{E_{mb}T_{na}g\}$ is a frame for $L^2(\mathbb{R}^k)$ with bounds A, B, then $\{E_{mb}T_{na}S^{-1}g\}$ is a frame with bounds B^{-1}, A^{-1}.

An application of Cauchy-Schwarz, the argument in the proof of Theorem 5.2.1, the self-adjointness of S, the assumption that (g,a,b) generates a frame for $L^2(\mathbb{R}^k)$, and the fact that $\|Q_T f\|_2 \le \|f\|_2$ gives that

$$N_1 \le \|f\|_2\|S^{-1}h\|_2 b^{-k} \sum_j \operatorname*{ess\,sup}_{\substack{|x|\le T,\\ |x-j/b|\le T}} \left| \sum_{|na|\le T+t} g(x-na)\bar{g}(x-na-j/b) \right|.$$

Now, since

$$|\langle P_\Omega f, E_{mb}T_{na}g\rangle| = |\langle (P_\Omega f)^\wedge, E_{-na}T_{mb}\hat{g}\rangle|$$

and

$$|\langle S^{-1}h, E_{mb}T_{na}g\rangle| = |\langle (S^{-1}h)^\wedge, E_{-na}T_{mb}\hat{g}\rangle|,$$

and since $\|P_\Omega f\|_2 \le \|f\|_2$, we have as above that

$$N_3 \le \|f\|_2\|S^{-1}h\|_2 a^{-k} \sum_j \operatorname*{ess\,sup}_{\substack{|\gamma|\le\Omega,\\ |\gamma-j/a|\le\Omega}} \left| \sum_{|mb|\le\Omega+\omega} \hat{g}(\gamma-mb)\bar{\hat{g}}(\gamma-mb-j/a) \right|.$$

By Lemma 5.3.1, for every $\varepsilon > 0$, there exist numbers t_ε, ω_ε such that for every $t \ge t_\varepsilon$ and $\omega \ge \omega_\varepsilon$, and every T, $\Omega > 0$,

$$\sup_{\|h\|_2=1} (N_1+N_2+N_3+N_4) \le (B/A)^{1/2}[\|(I-Q_T)f\|_2+\|(I-P_\Omega)f\|_2]+\varepsilon\|f\|_2. \blacksquare$$

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