SRC TR 89-2

Topology of the Orbit Space of Generalized Linear Systems

by

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November 30, 1988

* Research partially supported by the National Science Foundation under Grant ECS-8696108 and under the Engineering Research Centers Program Grant NSFD CDR-8803012.

Abstract: We investigate the topology of the orbit space of controllable generalized linear (descriptor) systems modulo restricted system equivalence. We compute the singular homology groups in the complex case, and prove that in both the real and complex cases, this space is a *smooth compactification* of the orbit space of controllable state space systems modulo system similarity.

Keywords: descriptor system, controllability, restricted system equivalence, orbit space, homology

AMS Subject Classification (1980): Primary 93B27, 93B05, 93B10; Secondary 93B30, 57S25.

1. INTRODUCTION

Linear generalized dynamical systems, or descriptor systems, are described in (generalized) state space form by

$$E\dot{x} = Ax + Bu, \quad \det(\lambda E - A) \not\equiv 0, \tag{1.1}$$

where $E, A \in \mathbb{F}^{n\times n}$, $B \in \mathbb{F}^{n\times m}$. These systems provide a natural generalization of the class of regular state space systems

$$\dot{x} = Ax + Bu \tag{1.2}$$

and have attracted widespread interest in recent years. The structure theory of such systems is by now rather well developed, leading to generalizations of pole placement theorems, Rosenbrock's control structure theorem, and state space canonical forms. (See, e.g., the survey paper by Lewis [1].) Of particular relevance to the present paper are the recent papers by Shayman and Zhou [2], Glüsing-Lüerßen and Hinrichsen [3], and Helmke and Shayman [4].

However, geometric properties of spaces of generalized linear systems have not been studied systematically in the literature, in contrast to the rather rich literature dealing with geometric and topological aspects of regular systems (1.2); see e.g., the papers of Brockett [5], Hazewinkel and Kalman [6], Byrnes and Duncan [7], Delchamps [8], Helmke [9,10]. Geometric questions concerning spaces of generalized systems were first raised by Cobb [11,12].

In this paper, we present a detailed topological study of the space of controllable generalized linear systems. Let $\tilde{C}_{n,m}(\mathbb{F})$ denote the open subset of $\mathbb{F}^{n(2n+m)}$ consisting of all controllable generalized systems (1.1). The quotient space

$$C_{n,m}(\mathbb{F}) = \tilde{C}_{n,m}(\mathbb{F})/GL_n(\mathbb{F}) \times GL_n(\mathbb{F})$$

consisting of all equivalence classes $[E, A, B]_\eta$ of state space equivalent systems

$$\{(E', A', B') \in \tilde{C}_{n,m}(\mathbb{F}) \mid (E', A', B') = (MEN^{-1}, MAN^{-1}, MB),\ M, N \in GL_n(\mathbb{F})\}$$

parametrizes all coordinate-free defined controllable generalized systems and thus can be regarded as the proper space of all (abstract) controllable generalized systems.

We prove two main results in this paper. Our first result states that $C_{n,m}(\mathbb{F})$ is a smooth *compact* manifold.

Theorem A: The quotient space $C_{n,m}(\mathbb{F})$ of controllable generalized systems is a smooth compact algebraic manifold of dimension mn.

The result is not obvious in two respects. First, one might believe that by extending the class of controllable regular systems (1.2) (modulo state space similarity) to the quotient space $C_{n,m}(\mathbb{F})$, singularities arise. Our result shows that this is false, and that local neighborhoods of generalized systems (E, A, B) in the quotient space $C_{n,m}(\mathbb{F})$ all look the same, *regardless of whether E is singular or invertible.*

Second, Theorem A shows that $C_{n,m}(\mathbb{F})$ is a (smooth) *compactification* of the orbit space $\Sigma_{n,m}(\mathbb{F})$ of all regular systems (1.2). The construction of a (smooth or singular) compactification of the orbit space $\Sigma_{n,m}(\mathbb{F})$ has been a longstanding open problem (see, e.g., Hazewinkel [13] or Byrnes [14]), which is solved by Theorem A.

In order to prove the compactness of the quotient space, we have to make a detour. We prove the compactness by computing the singular homology groups of $C_{n,m}(\mathbb{C})$, an important and natural class of topological invariants of $C_{n,m}(\mathbb{C})$. Our next main result expresses the homology groups of $C_{n,m}(\mathbb{C})$ in terms of the homology groups of certain products of Grassmann manifolds.

Theorem B: There is an isomorphism of (integral) singular homology groups

$$H_q(C_{n,m}(\mathbb{C})) \cong \bigoplus_{r=0}^{n} H_{q-r}(G_r(\mathbb{C}^{r+m-1}) \times G_{n-r}(\mathbb{C}^{n-r+m-1}))$$

for any $q \geq 0$.

In particular, $C_{n,m}(\mathbb{C})$ is connected. Also, by Theorem B, $C_{n,m}(\mathbb{C})$ has a nontrivial homology group for the maximal dimension $q = 2nm$, and this can be used to conclude the compactness of $C_{n,m}(\mathbb{F})$ for both $\mathbb{F} = \mathbb{R}$ and $\mathbb{F} = \mathbb{C}$; see §8. We are not aware of a more direct proof of the compactness of $C_{n,m}(\mathbb{F})$.

The organization of this paper is as follows: In §2, we review background material on generalized linear systems. In §3, we show that the quotient space $C_{n,m}(\mathbb{F})$ of controllable generalized linear systems (modulo restricted system equivalence) is an analytic manifold. In §4, we construct an analytic stratification $\{C^r_{n,m}(\mathbb{F})\}$ of $C_{n,m}(\mathbb{F})$ indexed by $r :=$ deg $\det(\lambda E - \mu A)$. In §5, we review the Hermite cell decomposition for the quotient space of controllable pairs (modulo similarity). In §6, we construct a generalized Hermite cell decomposition for the stratum $C^r_{n,m}(\mathbb{F})$. This cell decomposition is used in §7 to obtain the singular homology groups and Betti numbers of $C_{n,m}(\mathbb{C})$. Finally, in §8, we apply the homology results in §7 to prove that $C_{n,m}(\mathbb{F})$ is compact.

2. PRELIMINARIES

Let $\mathbb{F}$ denote either the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers, topologized in the usual way. A *generalized linear system*

$$E\dot{x} = Ax + Bu \tag{2.1}$$

with $(E, A, B) \in \mathbb{F}^{n\times n} \times \mathbb{F}^{n\times n} \times \mathbb{F}^{n\times m}$ is called *admissible* if the genericity condition for the homogeneous polynomial

$$\det\,(\lambda E - \mu A) \not\equiv 0 \tag{2.2}$$

in (λ, μ) holds. In the sequel, all generalized systems (2.1) are assumed to satisfy the admissibility condition (2.2). Let

$$\tilde{\sigma}_{n,m}(\mathbb{F}) := \{(E, A, B) \in \mathbb{F}^{n\times(2n+m)} \mid \det\,(\lambda E - \mu A) \not\equiv 0\} \tag{2.3}$$

denote the set of all admissible systems (2.1). $\tilde{\sigma}_{n,m}(\mathbb{F})$ is a Zariski-open subset of the Euclidean space $\mathbb{F}^{n(2n+m)}$ and thus open and dense.

A system $(E, A, B) \in \tilde{\sigma}_{n,m}(\mathbb{F})$ is called *regular* whenever the $n \times n$ matrix E is invertible; otherwise (E, A, B) is called *singular*. In particular, the linear state space systems (A, B)

$$\dot{x} = Ax + Bu \tag{2.4}$$

can be considered as regular systems with $E = I_n$. There is a natural correspondence between the set of all regular systems (E, A, B) and the affine space $\mathbb{F}^{n\times(n+m)}$ of arbitrary linear state space systems (2.4), defined by

$$(E, A, B) \mapsto (E^{-1}A, E^{-1}B) \tag{2.5a}$$

$$(A, B) \mapsto (I_n, A, B) \tag{2.5b}$$

Obviously, the map (2.5b) is a right inverse of (2.5a).

Two linear state space systems $(A, B), (A', B')$ are called *similar*, $(A, B) \sim_\sigma (A', B')$, if

$$(A', B') = (SAS^{-1}, SB) \tag{2.6}$$

for some invertible transformation $S \in GL_n(\mathbb{F})$. $\sim_\sigma$ defines an equivalence relation on $\mathbb{F}^{n\times(n+m)}$ and the equivalence classes

$$[A, B]_\sigma := \{(SAS^{-1}, SB) \mid S \in GL_n(\mathbb{F})\} \tag{2.7}$$

are, by definition, the orbits of the group action

$$\begin{aligned} \sigma : GL_n(\mathbb{F}) \times \mathbb{F}^{n\times(n+m)} &\to \mathbb{F}^{n\times(n+m)} \\ (S, (A, B)) &\mapsto (SAS^{-1}, SB). \end{aligned} \tag{2.8}$$

σ is called the *similarity action* of $GL_n(\mathbb{F})$ on $\mathbb{F}^{n\times(n+m)}$.

Two generalized systems $(E, A, B), (E', A', B') \in \tilde{\sigma}_{n,m}(\mathbb{F})$ are called *equivalent*, in symbols $(E, A, B) \sim_\eta (E', A', B')$, if they belong to the same orbit of the group action of (restricted) system equivalence

$$\begin{aligned} \eta : (GL_n(\mathbb{F}) \times GL_n(\mathbb{F})) \times \tilde{\sigma}_{n,m}(\mathbb{F}) &\to \tilde{\sigma}_{n,m}(\mathbb{F}) \\ ((M, N), (E, A, B)) &\mapsto (MEN^{-1}, MAN^{-1}, MB). \end{aligned} \tag{2.9}$$

The orbits of η are denoted by

$$[E, A, B]_\eta := \{(MEN^{-1}, MAN^{-1}, MB) \mid M, N \in GL_n(\mathbb{F})\} \tag{2.10}$$

and we have

$$(E, A, B) \sim_\eta (E', A', B') \Leftrightarrow [E, A, B]_\eta = [E', A', B']_\eta.$$

Both group actions σ, η are algebraic group actions and the transformation groups $GL_n(\mathbb{F})$, respectively $GL_n(\mathbb{F}) \times GL_n(\mathbb{F})$, are linearly reductive.

There is an important scaling action defined on $\tilde{\sigma}_{n,m}(\mathbb{F})$ which commutes with the action η. For any invertible 2×2 matrix

$$\Omega = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in GL_2(\mathbb{F})$$

let

$$T_\Omega(E, A, B) := (E_\Omega, A_\Omega, B) \tag{2.11a}$$

be defined by

$$(E_\Omega, A_\Omega) := (\alpha E + \beta A, \gamma E + \delta A). \tag{2.11b}$$

Since admissibility of (E, A, B) is invariant under the transformations (2.11), (2.11) defines a $GL_2(\mathbb{F})$-action

$$\begin{aligned} T \; : \; GL_2(\mathbb{F}) \times \tilde{\sigma}_{n,m}(\mathbb{F}) &\to \tilde{\sigma}_{n,m}(\mathbb{F}) \\ (\Omega, (E, A, B)) &\mapsto T_\Omega(E, A, B), \end{aligned} \tag{2.12}$$

called the *scaling action* on $\tilde{\sigma}_{n,m}(\mathbb{F})$. T is an action on $\tilde{\sigma}_{n,m}(\mathbb{F})$ which commutes with η:

$$T_\Omega \circ \eta_{(M,N)} = \eta_{(M,N)} \circ T_\Omega \tag{2.13}$$

for all $\Omega \in GL_2(\mathbb{F})$, $M, N \in GL_n(\mathbb{F})$. Using T, any $(E, A, B) \in \tilde{\sigma}_{n,m}(\mathbb{F})$ is scaling equivalent to a regular system (E_Ω, A_Ω, B)—i.e., $\det E_\Omega \neq 0$ for some $\Omega \in GL_2(\mathbb{F})$. The scaling action (using the subgroup $SO(2)$) was introduced by Shayman and Zhou [2] as an analysis and design tool for generalized linear systems. See also Shayman [15,16].

We briefly recall the well-known Weierstrass decomposition of an admissible generalized system (2.1) into a slow and fast subsystem. (For details, see e.g. Cobb [17].)

Lemma 2.1: Let $(E, A, B) \in \tilde{\sigma}_{n,m}(\mathbb{F})$ and $r := \deg \det(\lambda E - A)$. (E, A, B) is equivalent to a system $(E', A', B') \in \tilde{\sigma}_{n,m}(\mathbb{F})$ with

$$E' = \begin{bmatrix} I_r & 0 \\ 0 & A_2 \end{bmatrix}, \qquad A' = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n-r} \end{bmatrix}. \tag{2.14}$$

Here $A_1 \in \mathbb{F}^{r\times r}$ and the matrix $A_2 \in \mathbb{F}^{(n-r)\times(n-r)}$ is nilpotent. Futhermore,

$$\det(\lambda E - \mu A) = \alpha \mu^{n-r} \det(\lambda I_r - \mu A_1) \tag{2.15}$$

where α is a nonzero constant. ■

Thus every admissible generalized linear system (2.1) is equivalent to a system in *standard form*

$$\dot{x}_1 = A_1 x_1 + B_1 u \tag{2.16a}$$

$$A_2 \dot{x}_2 = x_2 + B_2 u \tag{2.16b}$$

with A_2 nilpotent and $B_1 \in \mathbb{F}^{r\times m}, B_2 \in \mathbb{F}^{(n-r)\times m}$. (2.16a), (2.16b) are called the *slow*, respectively *fast*, *subsystems* of (2.1). They are uniquely defined up to similarity, as shown by the following lemma:

Lemma 2.2 ([4]): For $0 \le r, s \le n$ let

$$E = \begin{bmatrix} I_r & 0 \\ 0 & A_2 \end{bmatrix}, \qquad A = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n-r} \end{bmatrix},$$

$$E' = \begin{bmatrix} I_s & 0 \\ 0 & A_2' \end{bmatrix}, \qquad A' = \begin{bmatrix} A_1' & 0 \\ 0 & I_{n-s} \end{bmatrix}$$

be in standard form (2.16) with A_2, A_2' nilpotent. Assume $(E', A') = (MEN^{-1}, MAN^{-1})$ for $M, N \in GL_n(\mathbb{F})$. Then $r = s$ and

$$M = N = \begin{bmatrix} M_{11} & 0 \\ 0 & M_{22} \end{bmatrix}$$

with $M_{11} \in GL_r(\mathbb{F})$, $M_{22} \in GL_{n-r}(\mathbb{F})$.

■

For $0 \le r \le n$ let

$$\tilde{\sigma}^r_{n,m}(\mathbb{F}) := \{(E, A, B) \in \tilde{\sigma}_{n,m}(\mathbb{F}) \mid \deg\det(\lambda E - A) = r\}.$$

Thus,

$$\tilde{\sigma}_{n,m}(\mathbb{F}) = \bigcup_{r=0}^{n} \tilde{\sigma}^r_{n,m}(\mathbb{F}) \tag{2.17}$$

is a decomposition of $\tilde{\sigma}_{n,m}(\mathbb{F})$ into disjoint quasi-affine subvarieties of $\tilde{\sigma}_{n,m}(\mathbb{F})$. Since $\tilde{\sigma}^r_{n,m}(\mathbb{F})$ is obtained from $\tilde{\sigma}_{n,m}(\mathbb{F})$ by setting $n-r$ independent functions to zero,

$$\dim \tilde{\sigma}^r_{n,m}(\mathbb{F}) = 2n^2 + n(m-1) + r. \tag{2.18}$$

The following topological version of the Weierstrass decomposition (Lemma 2.1) is due to Cobb [18]:

Lemma 2.3: Let $(\overline{E}, \overline{A}, \overline{B}) \in \tilde{\sigma}^r_{n,m}(\mathbb{F})$. There exists an open neighborhood $U \subset \tilde{\sigma}^r_{n,m}(\mathbb{F})$ of $(\overline{E}, \overline{A}, \overline{B})$ and an analytic map

$$\begin{aligned} F : U &\to \tilde{\sigma}^r_{n,m}(\mathbb{F}) \\ (E, A, B) &\mapsto \left(\begin{bmatrix} I_r & 0 \\ 0 & A_2 \end{bmatrix}, \begin{bmatrix} A_1 & 0 \\ 0 & I_{n-r} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \right) \end{aligned} \tag{2.19}$$

which associates to every (E, A, B) an equivalent system in standard form.

■

Following Rosenbrock [19], a system $(E, A, B) \in \tilde{\sigma}_{n,m}(\mathbb{F})$ is called *controllable* if and only if

$$\operatorname{rank}\, [\lambda E - \mu A, B] = n, \quad \forall (\lambda, \mu) \in \mathbb{C}^2 - \{(0,0)\}. \tag{2.20}$$

The following characterization from Yip and Sincovec [20] is an immediate consequence of (2.20).

Lemma 2.4: A system $(E, A, B) \in \tilde{\sigma}_{n,m}(\mathbb{F})$ in standard form (2.16) is controllable if and only if the associated subsystems $(A_1, B_1), (A_2, B_2)$ are controllable.

■

Remark 2.5: Clearly controllability of $(E, A, B) \in \tilde{\sigma}_{n,m}(\mathbb{F})$ is a property which is invariant under the scaling action (2.12).

■

3. THE QUOTIENT SPACE OF CONTROLLABLE GENERALIZED SYSTEMS

In this section the quotient space $C_{n,m}(\mathbb{F})$ of controllable generalized linear systems (E, A, B) modulo restricted system equivalence is introduced. We show that $C_{n,m}(\mathbb{F})$ is an analytic manifold.

Let $\tilde{C}_{n,m}(\mathbb{F})$ denote the set of all controllable systems in $\tilde{\sigma}_{n,m}(\mathbb{F})$. $\tilde{C}_{n,m}(\mathbb{F})$ is a Zariski-open subset of $\mathbb{F}^{n(2n+m)}$ and thus open and dense in $\mathbb{F}^{n(2n+m)}$. We denote by $\tilde{C}^{\text{reg}}_{n,m}(\mathbb{F}) \subset \tilde{C}_{n,m}(\mathbb{F})$ the open and dense subset consisting of all *regular* controllable systems. Similarly, let $\tilde{\Sigma}_{n,m}(\mathbb{F})$ denote the Zariski-open subset of $\mathbb{F}^{n(n+m)}$ consisting of all controllable pairs (A, B).

The group actions σ, η restrict to actions on the spaces $\tilde{C}_{n,m}(\mathbb{F})$, $\tilde{C}^{\text{reg}}_{n,m}(\mathbb{F})$, $\tilde{\Sigma}_{n,m}(\mathbb{F})$ of controllable systems. We refer to

$$\begin{aligned} \sigma :\ GL_n(\mathbb{F}) \times \tilde{\Sigma}_{n,m}(\mathbb{F}) &\to \tilde{\Sigma}_{n,m}(\mathbb{F}) \\ (S,(A,B)) &\mapsto (SAS^{-1}, SB) \end{aligned} \tag{3.1}$$

as the *similarity action* on $\tilde{\Sigma}_{n,m}(\mathbb{F})$. σ is a free group action—i.e., for any $(A, B) \in \tilde{\Sigma}_{n,m}(\mathbb{F})$ the stabilizer subgroup

$$\text{Stab}(A, B) := \{S \in GL_n(\mathbb{F}) \mid (SAS^{-1}, SB) = (A, B)\}$$

is the trivial subgroup consisting of the identity element in $GL_n(\mathbb{F})$.

Similarly,

$$\begin{aligned} \eta :\ (GL_n(\mathbb{F}) \times GL_n(\mathbb{F})) \times \tilde{C}_{n,m}(\mathbb{F}) &\to \tilde{C}_{n,m}(\mathbb{F}) \\ ((M,N),(E,A,B)) &\mapsto (MEN^{-1}, MAN^{-1}, MB), \end{aligned} \tag{3.2}$$

respectively its restriction to $\tilde{C}^{\text{reg}}_{n,m}(\mathbb{F})$

$$\eta :\ (GL_n(\mathbb{F}) \times GL_n(\mathbb{F})) \times \tilde{C}^{\text{reg}}_{n,m} \to \tilde{C}^{\text{reg}}_{n,m}, \tag{3.3}$$

is called the (restricted) system equivalence action on $\tilde{C}_{n,m}(\mathbb{F})$, respectively $\tilde{C}^{\text{reg}}_{n,m}(\mathbb{F})$.

Let

$$C_{n,m}(\mathbb{F}) := \tilde{C}_{n,m}(\mathbb{F})/(GL_n(\mathbb{F}) \times GL_n(\mathbb{F})) \tag{3.4}$$

denote the set of all orbits $[E, A, B]_\eta$ of controllable generalized linear systems. $C_{n,m}(\mathbb{F})$ is endowed with the quotient topology, i.e., with the finest topology for which the quotient map

$$\begin{aligned} \pi : \tilde{C}_{n,m}(\mathbb{F}) &\to C_{n,m}(\mathbb{F}) \\ (E,A,B) &\mapsto [E,A,B]_\eta \end{aligned} \tag{3.5}$$

is continuous. Since equivalent systems $(E, A, B), (E', A', B')$ have the same system theoretic properties, we can regard $C_{n,m}(\mathbb{F})$ as the (abstract) space of all controllable generalized linear systems.

Similarly, let

$$C^{\text{reg}}_{n,m}(\mathbb{F}) := \tilde{C}^{\text{reg}}_{n,m}(\mathbb{F})/(GL_n(\mathbb{F}) \times GL_n(\mathbb{F})) \tag{3.6}$$

$$\Sigma_{n,m}(\mathbb{F}) := \tilde{\Sigma}_{n,m}(\mathbb{F})/GL_n(\mathbb{F}) \tag{3.7}$$

be the orbit spaces of controllable regular systems, respectively of controllable pairs, for the actions (3.3), respectively (3.1). Let

$$\begin{aligned} \pi : \tilde{C}^{\text{reg}}_{n,m}(\mathbb{F}) &\to C^{\text{reg}}_{n,m}(\mathbb{F}) \\ (E, A, B) &\mapsto [E, A, B]_\eta, \end{aligned} \tag{3.8}$$

respectively

$$\begin{aligned} \pi : \tilde{\Sigma}_{n,m}(\mathbb{F}) &\to \Sigma_{n,m}(\mathbb{F}) \\ (A, B) &\mapsto [A, B]_\sigma \end{aligned} \tag{3.9}$$

denote the corresponding quotient maps. We endow $C^{\text{reg}}_{n,m}(\mathbb{F})$, $\Sigma_{n,m}(\mathbb{F})$ with the corresponding quotient topologies. $C^{\text{reg}}_{n,m}(\mathbb{F})$ is an open and dense subset of $C_{n,m}(\mathbb{F})$.

The scaling action (2.12) on $\tilde{\sigma}_{n,m}(\mathbb{F})$ restricts to the *scaling action on* $\tilde{C}_{n,m}(\mathbb{F})$

$$\begin{aligned} T : GL_2(\mathbb{F}) \times \tilde{C}_{n,m}(\mathbb{F}) &\to \tilde{C}_{n,m}(\mathbb{F}) \\ \left(\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}, (E, A, B)\right) &\mapsto (\alpha E + \beta A, \gamma E + \delta A, B). \end{aligned} \tag{3.10}$$

Since the actions T, η commute, T induces a scaling action of $GL_2(\mathbb{F})$ and also of the projective general linear group $PGL_2(\mathbb{F}) := GL_2(\mathbb{F})/\,\mathbb{F}^*$ on the orbit space $C_{n,m}(\mathbb{F})$:

$$\begin{aligned} T : PGL_2(\mathbb{F}) \times C_{n,m}(\mathbb{F}) &\to C_{n,m}(\mathbb{F}) \\ \left(\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}_{\mathbb{F}^*}, [E, A, B]_\eta\right) &\mapsto [\alpha E + \beta A, \gamma E + \delta A, B]_\eta. \end{aligned} \tag{3.11}$$

For $(E, A, B) \in \tilde{\sigma}_{n,m}(\mathbb{F})$, let $\text{Stab}(E, A, B)$ denote the stabilizer subgroup of (E, A, B) for the equivalence action η. Since T, η commute,

$$\text{Stab}(T_\Omega(E, A, B)) = \text{Stab}(E, A, B) \tag{3.12}$$

for all $\Omega \in GL_2(\mathbb{F})$. Suppose $(E, A, B) \in \tilde{\sigma}_{n,m}(\mathbb{F})$ is regular and $(M, N) \in \mathrm{Stab}(E, A, B)$. Then $N \in \mathrm{Stab}(E^{-1}A, E^{-1}B)$ and $M = ENE^{-1}$. Thus

$$\mathrm{Stab}(E, A, B) \cong \mathrm{Stab}(E^{-1}A, E^{-1}B) \tag{3.13}$$

via the group isomorphism which sends each $N \in \mathrm{Stab}(E^{-1}A, E^{-1}B)$ to $(ENE^{-1}, N) \in \mathrm{Stab}(E, A, B)$.

Lemma 3.1: The restricted system equivalence action

$$\eta : (GL_n(\mathbb{F}) \times GL_n(\mathbb{F})) \times \tilde{\sigma}_{n,m}(\mathbb{F}) \to \tilde{\sigma}_{n,m}(\mathbb{F})$$

acts freely on $\tilde{C}_{n,m}(\mathbb{F})$. Moreover, $\tilde{C}_{n,m}(\mathbb{F})$ is the principal orbit type of η—i.e.,

$$\mathrm{Stab}(E, A, B) = \{(I_n, I_n)\} \Leftrightarrow (E, A, B) \in \tilde{C}_{n,m}(\mathbb{F}).$$

Proof: For the similarity action on state space systems $(F, G) \in \mathbb{F}^{n\times n} \times \mathbb{F}^{n\times m}$, Byrnes and Hurt [21] have shown that $\mathrm{Stab}(F, G) = \{I_n\}$ if and only if (F, G) is controllable. Since any $(E, A, B) \in \tilde{\sigma}_{n,m}(\mathbb{F})$ is scaling equivalent to a regular system, the result follows from (3.12), (3.13) and Remark 2.5.

■

Let $\alpha : G \times X \to X$, $(g, x) \mapsto g \cdot x$, denote a group action on a set X. Let $\sim_\alpha$ denote the associated equivalence relation on X. The *graph* of α is defined as the set

$$\Gamma_\alpha := \{(x, y) \in X \times X \mid x \sim_\alpha y\}.$$

Thus Γ_α is the image of the *graph map*

$$\begin{aligned} \Gamma : G \times X &\to X \times X \\ (g, x) &\mapsto (x, g \cdot x). \end{aligned}$$

Proposition 3.2: The graph of the equivalence action

$$\eta : (GL_n(\mathbb{F}) \times GL_n(\mathbb{F})) \times \tilde{C}^{\mathrm{reg}}_{n,m}(\mathbb{F}) \to \tilde{C}^{\mathrm{reg}}_{n,m}(\mathbb{F})$$

is a closed analytic submanifold of $\tilde{C}^{\text{reg}}_{n,m}(\mathbb{F}) \times \tilde{C}^{\text{reg}}_{n,m}(\mathbb{F})$.

Proof: Let $\Gamma^{\text{reg}}_{\eta} \subset \tilde{C}^{\text{reg}}_{n,m}(\mathbb{F}) \times \tilde{C}^{\text{reg}}_{n,m}(\mathbb{F})$ denote the graph of η, and let $\Gamma_{\sigma} \subset \tilde{\Sigma}_{n,m}(\mathbb{F}) \times \tilde{\Sigma}_{n,m}(\mathbb{F})$ denote the graph of the similarity action on the set $\tilde{\Sigma}_{n,m}(\mathbb{F})$ of controllable pairs. By Helmke [10, Lemma 2.1], Γ_{σ} is a closed analytic submanifold of $\tilde{\Sigma}_{n,m}(\mathbb{F}) \times \tilde{\Sigma}_{n,m}(\mathbb{F})$. Consider the analytic mapping

$$\begin{aligned} \Phi : \tilde{C}^{\text{reg}}_{n,m}(\mathbb{F}) \times \tilde{C}^{\text{reg}}_{n,m}(\mathbb{F}) &\to \tilde{\Sigma}_{n,m}(\mathbb{F}) \times \tilde{\Sigma}_{n,m}(\mathbb{F}) \\ ((E,A,B),(\tilde{E},\tilde{A},\tilde{B})) &\mapsto ((E^{-1}A, E^{-1}B),(\tilde{E}^{-1}\tilde{A}, \tilde{E}^{-1}\tilde{B}). \end{aligned} \tag{3.14}$$

For regular $(E,A,B),(\tilde{E},\tilde{A},\tilde{B})$ given, suppose

$$\tilde{E}^{-1}\tilde{A} = N(E^{-1}A)N^{-1}, \; \tilde{E}^{-1}\tilde{B} = N(E^{-1}B)$$

for some $N \in GL_n(\mathbb{F})$. Set $M := \tilde{E}NE^{-1}$. Then

$$(\tilde{E},\tilde{A},\tilde{B}) = (MEN^{-1}, MAN^{-1}, MB).$$

Thus $\Phi^{-1}(\Gamma_{\sigma}) = \Gamma^{\text{reg}}_{\eta}$. Since Φ is a submersion, the inverse image $\Gamma^{\text{reg}}_{\eta}$ is a closed analytic submanifold of $\tilde{C}^{\text{reg}}_{n,m}(\mathbb{F}) \times \tilde{C}^{\text{reg}}_{n,m}(\mathbb{F})$.

■

Using standard results about analytic Lie group actions on manifolds, see Dieudonné [22] or Helmke and Hinrichsen [23], Lemma 3.1 and Proposition 3.2 imply:

Corollary 3.3: The orbit space $C^{\text{reg}}_{n,m}(\mathbb{F}) = \tilde{C}^{\text{reg}}_{n,m}(\mathbb{F})/(GL_n(\mathbb{F}) \times GL_n(\mathbb{F}))$ of regular controllable systems is an analytic manifold of dimension nm. The quotient map $\pi : \tilde{C}^{\text{reg}}_{n,m}(\mathbb{F}) \to C^{\text{reg}}_{n,m}(\mathbb{F})$ is a principal fibre bundle with structure group $GL_n(\mathbb{F}) \times GL_n(\mathbb{F})$.

■

The corresponding result for the orbit space $\Sigma_{n,m}(\mathbb{F})$ of controllable linear systems (A,B) modulo similarity has been well known since the early work of Hazewinkel and Kalman [6] and Byrnes and Hurt [21]. In particular, $\Sigma_{n,m}(\mathbb{F})$ is known to be a connected, nonsingular quasiprojective variety of dimension nm. In addition to the two aforementioned references, see Helmke [9,10] for further results on the topology of $\Sigma_{n,m}(\mathbb{F})$.

Consider the analytic map between manifolds.

$$\begin{aligned} i : \Sigma_{n,m}(\mathbb{F}) &\to C^{\text{reg}}_{n,m}(\mathbb{F}) \\ [A,B]_\sigma &\mapsto [I,A,B]_\eta. \end{aligned} \tag{3.15}$$

i is a bijection with inverse

$$\begin{aligned} i^{-1} = j : C^{\text{reg}}_{n,m}(\mathbb{F}) &\to \Sigma_{n,m}(\mathbb{F}) \\ [E,A,B]_\eta &\mapsto [E^{-1}A, E^{-1}B]_\sigma \end{aligned} \tag{3.16}$$

Since j is analytic, it yields a bianalytical diffeomorphism. This shows

Proposition 3.4: The orbit spaces $\Sigma_{n,m}(\mathbb{F})$ and $C^{\text{reg}}_{n,m}(\mathbb{F})$ are diffeomorphic as analytic manifolds.

■

In order to obtain the full quotient space $C_{n,m}(\mathbb{F})$ as a manifold, we use the scaling action $T : GL_2(\mathbb{F}) \times C_{n,m}(\mathbb{F}) \to C_{n,m}(\mathbb{F})$.

For every $\Omega \in GL_2(\mathbb{F})$, let

$$C_\Omega(n,m) := T(\{\Omega\} \times C^{\text{reg}}_{n,m}(\mathbb{F})). \tag{3.17}$$

Each $C_\Omega(n,m)$ is an open and dense subset of $C_{n,m}(\mathbb{F})$. Furthermore, $C_\Omega(n,m)$ carries a natural structure of an anlytic manifold which makes it analytically diffeomorphic to $C^{\text{reg}}_{n,m}(\mathbb{F})$.

Lemma 3.5: There exist finitely many $\Omega_0, \ldots, \Omega_n \in GL_2(\mathbb{F})$ such that

$$C_{n,m}(\mathbb{F}) = \bigcup_{i=0}^{n} C_{\Omega_i}(n,m). \tag{3.18}$$

Proof: Choose any $\Omega_i = \begin{bmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{bmatrix}^{-1}$ with pairwise linearly independent row vectors (α_i, β_i), $i = 0, \ldots, n$. Suppose there exists $[E,A,B]_\eta \in C_{n,m}(\mathbb{F})$ with $[E,A,B]_\eta \notin C_{\Omega_i}(n,m)$ for all $i = 0, \ldots, n$. Then $\det(\alpha_i E + \beta_i A) = 0$ for $i = 0, \ldots, n$. But the polynomial $\det(\lambda E + \mu A)$ has at most n linearly independent roots (λ, μ), a contradiction.

■

Theorem 3.6: The orbit space $C_{n,m}(\mathbb{F})$ of controllable generalized linear systems is an analytic manifold of dimension nm. Furthermore, the quotient map $\pi : \tilde{C}_{n,m}(\mathbb{F}) \to C_{n,m}(\mathbb{F})$ is a principal fibre bundle with structure group $GL_n(\mathbb{F}) \times GL_n(\mathbb{F})$.

Proof: The proof is by a standard gluing argument. Consider the finite covering (3.18) of $C_{n,m}(\mathbb{F})$ by the open subsets $C_{\Omega_i}(n,m)$, $i = 0, \dots, n$. Consider the homeomorphism

$$\begin{aligned} T_i : C^{\mathrm{reg}}_{n,m}(\mathbb{F}) &\to C_{\Omega_i}(n,m) \\ [E,A,B]_\eta &\mapsto [T_{\Omega_i}(E,A,B)]_\eta \end{aligned} \tag{3.19}$$

$i = 0, \cdots, n$. There exists a unique structure of an $\mathbb{F}$-analytic manifold on $C_{\Omega_i}(n,m)$ for which T_i becomes a bianalytical diffeomorphism. Since

$$\begin{aligned} \varphi_{ij} := T_i^{-1} \circ T_j : \; T_j^{-1}(C_{\Omega_i}(n,m) \cap C_{\Omega_j}(n,m)) &\to T_i^{-1}(C_{\Omega_i}(n,m)) \\ [E,A,B]_\eta &\mapsto [T_{\Omega_i^{-1}\Omega_j}(E,A,B)]_\eta \end{aligned} \tag{3.20}$$

is analytic, there exists a (unique) real analytic manifold structure on $C_{n,m}(\mathbb{F})$ which induces the given analytic manifold structures on $C_{\Omega_i}(n,m)$. Since the maps φ_{ij} are compatible with the quotient maps $\pi : \tilde{C}^{\mathrm{reg}}_{n,m}(\mathbb{F}) \to C^{\mathrm{reg}}_{n,m}(\mathbb{F})$, respectively $\tilde{C}_{n,m}(\mathbb{F}) \to C_{n,m}(\mathbb{F})$, the result follows readily from Corollary 3.3.

■

Remark 3.7: Using the gluing data (3.19), (3.20) it is easy to construct an atlas of local coordinate charts for the manifold $C_{n,m}(\mathbb{F})$. Indeed, let $\{(U_\lambda, \psi_\lambda) \mid \lambda \in \Lambda\}$ be a finite atlas of coordinate charts for the orbit manifold $\Sigma_{n,m}(\mathbb{F})$. For example, use the atlas for $\Sigma_{n,m}(\mathbb{F})$ described in Hazewinkel [13] based on nice selections for the controllability matrix $(B, AB, \dots, A^{n-1}B)$. Using the diffeomorphism $i : \Sigma_{n,m}(\mathbb{F}) \to C^{\mathrm{reg}}_{n,m}(\mathbb{F})$, this defines an atlas $\{(V_\lambda, \varphi_\lambda) \mid \lambda \in \Lambda\}$ for $C^{\mathrm{reg}}_{n,m}(\mathbb{F})$ with $V_\lambda := i(U_\lambda)$, $\varphi_\lambda = \psi_\lambda \circ i^{-1}$, $\lambda \in \Lambda$. Then

$$\{(T_i(V_\lambda), \; \psi_\lambda \circ T_i^{-1}) \mid \lambda \in \Lambda, \; i = 0, \dots, n\}$$

defines a finite atlas of local coordinate charts for $C_{n,m}(\mathbb{F})$.

■

Remark 3.8: It can be shown that $\pi : \tilde{C}_{n,m}(\mathbb{F}) \rightarrow C_{n,m}(\mathbb{F})$ is a *nontrivial* fibre bundle for arbitrary $m, n \in \mathbb{N}$. This implies the nonexistence of continuous canonical forms for restricted system equivalence of controllable generalized systems. See Helmke and Shayman [4] for details.

■

4. A STRATIFICATION OF $C_{n,m}(\mathbb{F})$

Let $c_0(E,A), \ldots, c_n(E,A) \in \mathbb{F}$ be defined by

$$\det(\lambda E - \mu A) = \sum_{i=0}^{n} c_i(E,A)\lambda^i \mu^{n-i}.$$

This defines an analytic map

$$\begin{aligned} \chi : \; & C_{n,m}(\mathbb{F}) \rightarrow \mathbb{P}^n(\mathbb{F}) \\ & [E,A,B]_\eta \mapsto [c_0(E,A) : \ldots : c_n(E,A)] \end{aligned} \tag{4.1}$$

which is called the *characteristic map*. For any $0 \leq r \leq n$, let

$$C_{n,m}^r := \{[E,A,B]_\eta \in C_{n,m}(\mathbb{F}) | \deg \; \det(\lambda E - A) = r\}.$$

Since the functions $c_0(E,A), \ldots, c_n(E,A)$ are algebraically independent, $C_{n,m}^r(\mathbb{F}) = \chi^{-1}$ $(\mathbb{F}^r)$ (with $\mathbb{F}^r \subset \mathbb{P}^n(\mathbb{F})$ the affine subspace defined by $[x_0 : \ldots : x_{r-1} : 1 : 0 \ldots : 0]$) is an analytic subvariety of $C_{n,m}(\mathbb{F})$ with

$$\dim C_{n,m}^r(\mathbb{F}) \;=\; n(m-1) + r.$$

Clearly $C_{n,m}^n(\mathbb{F}) \;=\; C_{n,m}^{\mathrm{reg}}(\mathbb{F})$ is open and dense, and

$$C_{n,m}(\mathbb{F}) \;=\; \bigcup_{r=0}^{n} C_{n,m}^r(\mathbb{F}) \tag{4.2}$$

is a decomposition of $C_{n,m}(\mathbb{F})$ into finitely many disjoint locally closed subsets. Let $\overline{C_{n,m}^r(\mathbb{F})}$ denote the topological closure of $C_{n,m}^r(\mathbb{F})$ in $C_{n,m}(\mathbb{F})$. Since $\overline{C_{n,m}^r(\mathbb{F})} = \chi^{-1}(\mathbb{P}^r)$,

$$\overline{C_{n,m}^r(\mathbb{F})} \;=\; \bigcup_{s=0}^{r} C_{n,m}^s(\mathbb{F}) \tag{4.3}$$

is an analytic subvariety of $C_{n,m}(\mathbb{F})$. This proves

Lemma 4.1: The decomposition (4.2) is a stratification of $C_{n,m}(\mathbb{F})$ by analytic subvarieties $C^r_{n,m}(\mathbb{F})$ with dim $C^r_{n,m}(\mathbb{F}) = n(m-1)+r$.

$\blacksquare$

Recall that a finite decomposition $\{X_i \mid i \in I\}$ of an analytic variety X into nonempty disjoint subsets X_i is called an *analytical stratification* if

(i) X_i is an analytic subvariety of X

(ii) $X_i \cap \overline{X_j} \neq \emptyset \Rightarrow \dim X_i < \dim X_j$ for $i \neq j$.

Let $\mathcal{N}_{k,m}(\mathbb{F}) \subset \Sigma_{k,m}(\mathbb{F})$ denote the subset of $\Sigma_{k,m}(\mathbb{F})$ consisting of all similarity classes $[A,B]_\sigma$ with A nilpotent. $\mathcal{N}_{k,m}(\mathbb{F})$ is a closed algebraic subvariety of the quasi-projective variety $\Sigma_{k,m}(\mathbb{F})$. We can now state the main result of this section which gives an explicit parametrization of $C^r_{n,m}(\mathbb{F})$.

Theorem 4.2: For any $0 \le r \le n$

$$\varphi : \Sigma_{r,m}(\mathbb{F}) \times \mathcal{N}_{n-r,m}(\mathbb{F}) \to C^r_{n,m}(\mathbb{F})$$
$$([A_1,B_1]_\sigma,[A_2,B_2]_\sigma) \mapsto \left[\begin{pmatrix} I_r & 0 \\ 0 & A_2 \end{pmatrix}, \begin{pmatrix} A_1 & 0 \\ 0 & I_{n-r} \end{pmatrix}, \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \right]_\eta$$

is an isomorphism between analytic varieties. In particular, $C^r_{n,m}(\mathbb{F})$ is homeomorphic to the product space $\Sigma_{r,m}(\mathbb{F}) \times \mathcal{N}_{n-r,m}(\mathbb{F})$.

Proof: φ is clearly analytic. Surjectivity of φ follows from Lemma 2.1, while Lemma 2.2 proves the injectivity. Thus φ is bijective. Analyticity of φ^{-1} follows immediately from Lemma 2.3.

$\blacksquare$

5. HERMITE INDICES

In this section we recall some basic facts about Hermite indices of controllable systems (A,B) and an associated cell decomposition of the orbit manifold $\Sigma_{n,m}(\mathbb{F})$. This material is due to Helmke [10] to which we refer for further details.

A *combination* of $n \in \mathbb{N}$ into m parts is any m-tuple of nonnegative integers $K = (K_1, \ldots, K_m)$ with $K_1 + \cdots + K_m = n$. m is called the length and $|K| := \sum_{i=1}^{m} K_i$ the weight of K. Combinations K of fixed length m and weight n form a finite set

$$K_{n,m} := \left\{ K \in \mathbb{Z}^m \mid K_i \geq 0, \ \sum_{i=1}^{m} K_i = n \right\} \tag{5.1}$$

of cardinality

$$\binom{n+m-1}{n} = \frac{(n+m-1)!}{n!(m-1)!}. \tag{5.2}$$

Let $(A, B) \in \tilde{\Sigma}_{n,m}(\mathbb{F})$, and let $b_1, \ldots b_m$ denote the columns of the matrix B. We consider the following elimination procedure on the columns of the controllability matrix of (A, B): Delete in the list

$$(b_1, Ab_1, \ldots, A^{n-1}b_1, \ldots, b_m, Ab_m, \ldots, A^{n-1}b_m),$$

while going from left to right, all vectors $A^k b_\ell$ which are linearly dependent on the set of all previous vectors. By controllability, the remaining vectors

$$T_{AB} := (b_1, \ldots, A^{k_1-1}b_1, \ldots, b_m, \ldots, A^{k_m-1}b_m) \tag{5.3}$$

form a basis of the state space $\mathbb{F}^n$ with nonnegative integers $K_1, \ldots, K_m$ satisfying $K_1 + \cdots + K_m = n$. (5.3) is called the Hermite basis and the combination $K = (K_1, \ldots K_m) \in K_{n,m}$ is called the list of *Hermite indices* of (A, B). Obviously the Hermite indices $K(A, B) = (K_1, \ldots, K_m)$ of (A, B) are similarity invariants:

$$K(SAS^{-1}, SB) = K(A, B), \quad \forall S \in GL_n(\mathbb{F}). \tag{5.4}$$

Expressing (A, B) with respect to the Hermite basis T_{AB} leads to the Hermite canonical form for the similarity action on $\tilde{\Sigma}_{n,m}(\mathbb{F})$. See Mayne [24], Hinrichsen and Prätzel-Wolters [25], Helmke and Shayman [4].

For any combination $K \in K_{n,m}$, a *Hermite stratum* of $\tilde{\Sigma}_{n,m}(\mathbb{F})$ (respectively of $\tilde{\mathcal{N}}_{n,m}(\mathbb{F})$) is defined by

$$\widetilde{Her}_{\mathbb{F}}(K) := \left\{ (A, B) \in \tilde{\Sigma}_{n,m}(\mathbb{F}) \mid K(A, B) = K \right\},$$

respectively

$$\widetilde{Her}^0_{\mathbf{F}}(K) := \left\{ (A,B) \in \tilde{\mathcal{N}}_{n,m}(\mathbb{F}) \mid K(A,B) = K \right\}.$$

The Hermite strata are quasi-affine subvarieties of $\tilde{\Sigma}_{n,m}(\mathbb{F})$, respectively $\tilde{\mathcal{N}}_{n,m}(\mathbb{F})$ [10], and

$$\tilde{\Sigma}_{n,m}(\mathbb{F}) = \bigcup_{K \in K_{n,m}} \widetilde{Her}_{\mathbf{F}}(K), \tag{5.5}$$

respectively

$$\tilde{\mathcal{N}}_{n,m}(\mathbb{F}) = \bigcup_{K \in K_{n,m}} \widetilde{Her}^0_{\mathbf{F}}(K), \tag{5.6}$$

are finite decompositions into disjoint σ-invariant subsets.

Let

$$Her_{\mathbf{F}}(K) := \pi(\widetilde{Her}_{\mathbf{F}}(K)) = \widetilde{Her}_{\mathbf{F}}(K)/GL_n(\mathbb{F}),$$

respectively

$$Her^0_{\mathbf{F}}(K) := \pi(\widetilde{Her}^0_{\mathbf{F}}(K)) = \widetilde{Her}^0_{\mathbf{F}}(K)/GL_n(\mathbb{F})$$

denote the orbit spaces of the similarity action restricted to $\widetilde{Her}_{\mathbf{F}}(K)$, respectively $\widetilde{Her}^0_{\mathbf{F}}(K)$. Thus, $Her_{\mathbf{F}}(K) \subset \Sigma_{n,m}(\mathbb{F})$ and $Her^0_{\mathbf{F}}(K) \subset N_{n,m}(\mathbb{F})$.

Lemma 5.1: Let $K \in K_{n,m}$.

a) $Her_{\mathbf{F}}(K)$ is a cell, i.e., an analytic subvariety of $\Sigma_{n,m}(\mathbb{F})$ which is bianalytically homeomorphic to some $\mathbb{F}^{n(K)}$. The dimension of $Her_{\mathbf{F}}(K)$ is equal to

$$n(K) = \sum_{i=1}^{m} (m-i+1)K_i. \tag{5.7}$$

b) $Her^0_{\mathbf{F}}(K) \cong \mathbb{F}^{n_o(K)}$ is a cell in $N_{n,m}(\mathbb{F})$ of dimension

$$n_0(K) = \sum_{i=1}^{m} (m-i)K_i = n(K) - n. \tag{5.8}$$

Proof: For (a) we refer to [10]. We now prove (b). For $(A,B) \in \widetilde{Her}_{\mathbb{F}}(K)$ and $1 \le j \le m$, there exist uniquely determined numbers $\alpha_{ij\ell}(A,B) \in \mathbb{F}$ with

$$A^{K_j} b_j \;=\; \sum_{\substack{\ell=1\\ i\le j}}^{K_i} \alpha_{ij\ell}(A,B) A^{\ell-1} b_i. \tag{5.9}$$

Here A is nilpotent if and only if

$$\alpha_{jj\ell}(A,B) = 0 \qquad \forall \ell = 1,\ldots,K_j,\;\; j \in \underline{m}. \tag{5.10}$$

(Consider the characteristic polynomial of the block triangular matrix $T_{AB}^{-1}\ A\ T_{AB}$.) Thus $(A,B) \in \widetilde{Her}^0_{\mathbb{F}}(K)$ if and only if (5.9) and (5.10) are satisfied for all $j = 1,\ldots,m$.

By uniqueness, $\alpha_{ij\ell}(SAS^{-1}, SB) = \alpha_{ij\ell}(A,B)$ for all $S \in GL_n(\mathbb{F})$. Let $\alpha_j(A,B) \in \mathbb{F}^{n_j(K)}$ denote the vector with components $\alpha_{ij\ell}(A,B)$, $i < j$ and $\ell = 1,\ldots,K_i$. Thus $n_j(K) = K_1 + \ldots + K_{j-1}$ and

$$n_0(K) \;:= \sum_{j=1}^{m} n_j(K) \;=\; \sum_{j=1}^{m} (m-j)K_j.$$

Define a mapping

$$\begin{aligned} \varphi \;:\; Her^0_{\mathbb{F}}(K) &\;\to\; \mathbb{F}^{n_0(K)} \\ [A,B]_\sigma &\;\mapsto\; (\alpha_1(A,B),\ldots,\alpha_m(A,B)). \end{aligned}$$

φ is clearly a bijection. The functions

$$\alpha_{ij\ell} \;:\; Her^0_{\mathbb{F}}(K) \to \mathbb{F}, \qquad [A,B]_\sigma \mapsto \alpha_{ij\ell}(A,B)$$

are $\mathbb{F}$-analytic [10]. Hence φ is analytic. As in [10], one shows that φ^{-1} is analytic. This completes the proof.

∎

We call $Her_{\mathbb{F}}(K)$, respectively $Her^0_{\mathbb{F}}(K)$, a *Hermite cell* of $\Sigma_{n,m}(\mathbb{F})$, respectively of $N_{n,m}(\mathbb{F})$.

In order to describe how the Hermite cells of $\Sigma_{n,m}(\mathbb{F})$, respectively $N_{n,m}(\mathbb{F})$, are pasted together, we recall the dominance order $\leq$ on the set $K_{n,m}$ of combinations. For any $K, L \in K_{n,m}$, define

$$K \leq L \Leftrightarrow K_1 + \ldots + K_j \leq L_1 + \ldots + L_j \quad \forall j \in \underline{m}. \tag{5.11}$$

This defines a partial ordering on $K_{n,m}$ which is called the *dominance order*.

The following results on the closures of Hermite cells of $\Sigma_{n,m}(\mathbb{F})$ are shown in Helmke [10].

Proposition 5.2: Let $K, L \in K_{n,m}$ and let $\leq$ denote the dominance order on $K_{n,m}$. For $\mathbb{F} = \mathbb{C}$,

$$Her_{\mathbb{C}}(K) \subset \overline{Her_{\mathbb{C}}(L)} \Leftrightarrow Her_{\mathbb{C}}(K) \cap \overline{Her_{\mathbb{C}}(L)} \neq \emptyset \Leftrightarrow K \leq L.$$

■

Proposition 5.2 yields the following explicit characterization of the boundary of a Hermite cell in $\Sigma_{n,m}(\mathbb{C})$:

$$\overline{Her_{\mathbb{C}}(L)} \quad = \quad \bigcup_{K \leq L} Her_{\mathbb{C}}(K). \tag{5.12}$$

For $\mathbb{F} = \mathbb{R}$, the geometric situation is considerably more complicated and Proposition 5.2, respectively the characterization (5.12), becomes in general false. However, we still have the following result:

Proposition 5.3: Let $K, L \in K_{n,m}$ and let $\leq$ denote the dominance order on $K_{n,m}$. Then

$$Her_{\mathbb{F}}(K) \cap \overline{Her_{\mathbb{F}}(L)} \neq \emptyset \Leftrightarrow \; K \leq L.$$

■

Since $Her^0_{\mathbb{C}}(K) \subset Her_{\mathbb{C}}(K)$, we obtain:

Corollary 5.4:

$$Her^0_{\mathbb{F}}(K) \cap \overline{Her^0_{\mathbb{F}}(L)} \neq \emptyset \quad \Rightarrow \; K \leq L.$$

■

In particular we have the following partial description of the boundary of a Hermite cell in $\Sigma_{n,m}(\mathbb{F})$, respectively $\mathcal{N}_{n,m}(\mathbb{F})$.

$$\overline{Her_{\mathbf{F}}(L)} \subset \bigcup_{K \leq L} Her_{\mathbf{F}}(K) \tag{5.13}$$

$$\overline{Her^0_{\mathbf{F}}(L)} \subset \bigcup_{K \leq L} Her^0_{\mathbf{F}}(K). \tag{5.14}$$

In the sequel we use the following terminology from topology:

Definition 5.5: Let X be a locally compact topological space. A finite decomposition $\{X_i \mid i \in I\}$ of X into nonempty disjoint subsets is called a *cell decomposition* provided

(a) Each X_i is homeomorphic to some $\mathbb{R}^{n_i}$, $n_i \in \mathbb{N}$.

(b) The boundary $\partial X_i = \overline{X_i} - X_i$ of X_i is contained in the union of the cells X_j with $\dim X_j < \dim X_i$.

A cell decomposition $\{X_i \mid i \in I\}$ is said to satisfy the *frontier condition*, if

$$X_i \cap \overline{X_j} \neq \emptyset \;\Leftrightarrow\; X_i \subset \overline{X_j} \quad \forall i,j \in I. \tag{5.15}$$

For any $i, j \in I$, we consider the relation

$$i \leq j \quad :\Leftrightarrow \quad X_i \cap \overline{X_j} \neq \emptyset. \tag{5.16}$$

We refer to (5.16) as the *adherence relation* on the cell decomposition $\{X_i \mid i \in I\}$.

■

The decomposition of spaces into cells is a well established technique in topology. For a survey about cell decompositions in linear systems theory, their role in the parametrization of system spaces and their relationship to canonical forms, see Helmke and Hinrichsen [23], Hinrichsen [26].

Theorem 5.6 ([10]): The decomposition $\{Her_{\mathbf{F}}(K) \mid K \in K_{n,m}\}$ of the orbit space $\Sigma_{n,m}(\mathbb{F})$ into Hermite cells is a finite cellular decomposition. The adherence relation on the set of Hermite cells is given by the dominance order. For $\mathbb{F} = \mathbb{C}$, it satisfies the frontier condition.

■

For the subspace $\mathcal{N}_{n,m}(\mathbb{F})$, there is a similar result:

Theorem 5.7: The decomposition $\{Her^0_{\mathbf{F}}(K) \mid K \in K_{n,m}\}$ of $\mathcal{N}_{n,m}(\mathbb{F})$ is a finite cellular decomposition.

Proof: By Lemma (5.1), $Her^0_{\mathbf{F}}(K)$ is a cell of dimension $n_0(K) = \sum_{i=1}^{m}(m-i)K_i = K_1 + (K_1+K_2)+\cdots+(K_1+\cdots+K_{m-1})$. Thus $K < L$ implies $n_0(K) < n_0(L)$. By Corollary 5.4, the boundary of a Hermite cell $Her^0_{\mathbf{F}}(L)$ is contained in $\bigcup_{K<L} Her^0_{\mathbf{F}}(K)$, with dim $Her^0_{\mathbf{F}}(K) = n_0(K) < n_0(L) = \dim Her^0_{\mathbf{F}}(L)$. The result follows.

■

We call $\{Her_{\mathbf{F}}(K) \mid K \in K_{n,m}\}$, respectively $\{Her^0_{\mathbf{F}}(K) \mid K \in K_{n,m}\}$, the *Hermite cell decomposition* of $\Sigma_{n,m}(\mathbb{F})$, respectively $\mathcal{N}_{n,m}(\mathbb{F})$.

6. A CELL DECOMPOSITION OF $C^r_{n,m}(\mathbb{F})$

Let $(E,A,B) \in \tilde{C}_{n,m}(\mathbb{F})$ with deg det $(\lambda E - A) = r$. Using the Weierstrass decomposition, let $(A_1,B_1) \in \tilde{\Sigma}_{r,m}(\mathbb{F})$, respectively $(A_2,B_2) \in \tilde{\mathcal{N}}_{n-r,m}(\mathbb{F})$, denote the slow, respectively fast, subsystems of (E,A,B). Let $K = K(A_1,B_1) \in K_{r,m}$, respectively $L = K(A_2,B_2) \in K_{n-r,m}$, denote the Hermite indices of $(A_1,B_1),(A_2,B_2)$. We call

$$\begin{aligned} K(E,A,B) &:= (r;K(A_1,B_1);K(A_2,B_2)) \in \{r\} \times K_{r,m} \times K_{n-r,m} \\ &= (r;K_1(E,A,B);K_2(E,A,B)) \end{aligned} \tag{6.1}$$

the *generalized Hermite indices* of (E,A,B). By Lemma 2.2 and (5.4), $K(E,A,B)$ is well-defined and satisfies

$$K(MEN^{-1},MAN^{-1},MB) = K(E,A,B) \tag{6.2}$$

for all $M,N \in GL_n(\mathbb{F})$. Let

$$J(n,m) := \{(r,K_1,K_2) \mid 0 \le r \le n,\ K_1 \in K_{r,m},\ K_2 \in K_{n-r,m}\}. \tag{6.3}$$

$J(n,m) \simeq K_{n,2m}$ is a finite set of cardinality

$$\text{card } J(n,m) = \sum_{r=0}^{n} \binom{m+r-1}{r} \binom{m+n-r-1}{n-r} = \binom{n+2m-1}{n}. \tag{6.4}$$

For any $(r, K_1, K_2) \in J(n,m)$, there exists $(E,A,B) \in \tilde{C}_{n,m}(\mathbb{F})$ with deg det $(\lambda E - A) = r$, $K_1(E,A,B) = K_1$, $K_2(E,A,B) = K_2$. Thus $J(n,m) \simeq K_{n,2m}$ is the set of all generalized Hermite indices for $\tilde{C}_{n,m}(\mathbb{F})$.

For any $(r, K_1, K_2) \in J(n,m)$, a generalized *Hermite stratum* of $\tilde{C}^r_{n,m}(\mathbb{F})$ is defined by

$$\widetilde{\mathrm{Her}}_{\mathbf{F}}(K_1, K_2) := \{(E,A,B) \in \tilde{C}^r_{n,m}(\mathbb{F}) \mid (K_1(E,A,B), K_2(E,A,B)) = (K_1, K_2)\}. \tag{6.5}$$

Thus

$$\tilde{C}_{n,m}(\mathbb{F}) = \bigcup_{(r,K_1,K_2) \in J(n,m)} \widetilde{\mathrm{Her}}_{\mathbf{F}}(K_1, K_2) \tag{6.6}$$

is a decomposition into finitely many disjoint η-invariant subsets. (6.6) induces the decomposition

$$\tilde{C}^r_{n,m}(\mathbb{F}) = \bigcup_{\substack{K_1 \in K_{r,n} \\ K_2 \in K_{n-r,m}}} \widetilde{\mathrm{Her}}_{\mathbf{F}}(K_1, K_2) \tag{6.7}$$

for each $0 \le r \le n$.

Let

$$\begin{aligned} Her_{\mathbf{F}}(K_1, K_2) &:= \pi(\widetilde{\mathrm{Her}}_{\mathbf{F}}(K_1, K_2)) \\ &= \widetilde{\mathrm{Her}}_{\mathbf{F}}(K_1, K_2)/(GL_n(\mathbb{F}) \times GL_n(\mathbb{F})) \end{aligned} \tag{6.8}$$

denote the orbit space of the system equivalence action η (3.2) restricted to the generalized Hermite stratum $\widetilde{\mathrm{Her}}_{\mathbf{F}}(K_1, K_2)$. We call $Her_{\mathbf{F}}(K_1, K_2)$ a *generalized Hermite cell* of both $C^r_{n,m}(\mathbb{F})$ and $C_{n,m}(\mathbb{F})$.

Lemma 6.1: Let $K = (K_1, \ldots, K_m) \in K_{r,m}$, $L = (L_1, \ldots, L_m) \in K_{n-r,m}$. $Her_{\mathbf{F}}(K, L)$ is an analytic manifold bianalytically diffeomorphic to $Her_{\mathbf{F}}(K) \times Her^0_{\mathbf{F}}(L)$ and is a cell of dimension

$$n(K,L) = \sum_{i=1}^{m} (m-i)(K_i + L_i) + r.$$

Proof: By Theorem 4.2

$$\varphi : \Sigma_{r,m}(\mathbb{F}) \times \mathcal{N}_{n-r,m}(\mathbb{F}) \to C^r_{n,m}(\mathbb{F})$$

$$([A_1, B_1]_\sigma, [A_2, B_2]_\sigma) \mapsto \left[\begin{pmatrix} I_r & 0 \\ 0 & A_2 \end{pmatrix}, \begin{pmatrix} A_1 & 0 \\ 0 & I_{n-r} \end{pmatrix}, \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}\right]_\eta \tag{6.9}$$

is an analytic isomorphism of varieties. Since $\varphi(Her_{\mathbf{F}}(K) \times Her^0_{\mathbf{F}}(L)) = Her_{\mathbf{F}}(K, L)$, the result follows immediately from Lemma 5.1.

∎

Let $\leq$ denote the ordering by dominance on $K_{r,m}$, $K_{n-r,m}$.

Proposition 6.2: Let $K_1, K_2 \in K_{r,m}$, $L_1, L_2 \in K_{n-r,m}$. If

$$Her_{\mathbf{F}}(K_1, L_1) \cap \overline{Her_{\mathbf{F}}(K_2, L_2)} \neq \emptyset,$$

then $K_1 \leq K_2$ and $L_1 \leq L_2$.

Proof: By Proposition 5.2,

$$\begin{aligned} \overline{Her_{\mathbf{F}}(K_2, L_2)} &= \varphi\left(\overline{Her_{\mathbf{F}}(K_2)} \times \overline{Her^0_{\mathbf{F}}(L_2)}\right) \\ &\subset \bigcup_{\substack{K_1 \leq K_2 \\ L_1 \leq L_2}} \varphi\left(Her_{\mathbf{F}}(K_1) \times Her^0_{\mathbf{F}}(L_1)\right) \\ &= \bigcup_{\substack{K_1 \leq K_2 \\ L_1 \leq L_2}} Her_{\mathbf{F}}(K_1, L_1). \end{aligned}$$

∎

Let $K_i = (K^i_1, \ldots, K^i_m) \in K_{r,m}$, $L_i = (L^i_1, \ldots, L^i_m) \in K_{n-r,m}$, $i = 1, 2$, be given. For $K_1 \leq K_2$, $L_1 \leq L_2$, we have $\dim Her_{\mathbf{F}}(K_1, L_1) = \dim Her_{\mathbf{F}}(K_1) + \dim Her^0_{\mathbf{F}}(L_1) \leq \dim Her_{\mathbf{F}}(K_2) + \dim Her^0_{\mathbf{F}}(L_2) = \dim Her_{\mathbf{F}}(K_2, L_2)$. Here, we have used the "dimension drop" property of the Hermite cell decompositions of $\Sigma_{r,m}(\mathbb{F})$ and $\mathcal{N}_{n-r,m}(\mathbb{F})$. This proves the following main result of this section:

Theorem 6.3: The decomposition $\{Her_{\mathbf{F}}(K, L) \mid K \in K_{r,m},\ L \in K_{n-r,m}\}$ of $C^r_{n,m}(\mathbb{F})$ into generalized Hermite cells is a cell decomposition.

■

We refer to this decomposition as the *(generalized) Hermite cell decomposition of* $C^r_{n,m}(\mathbb{F})$. It contains $\binom{m+r-1}{r}\binom{m+n-r-1}{n-r}$ cells.

Remark 6.4: For $\mathbb{F} = \mathbb{R}$ and $2 \le r \le n$, the Hermite cell decomposition of $C^r_{n,m}(\mathbb{R})$ does *not* satisfy the frontier condition (5.15). Moreover, the relative topological closure $\overline{Her_{\mathbb{R}}(K,L)}$ of a generalized Hermite cell in $C^r_{n,m}(\mathbb{R})$ is in general not an analytic subvariety of $C^r_{n,m}(\mathbb{R})$. For $\mathbb{F} = \mathbb{C}$ and $0 \le r \le n$, we conjecture that the Hermite cell decomposition $\{Her_{\mathbb{C}}(K,L) \mid K \in K_{r,m},\ L \in K_{n-r,m}\}$ of $C^r_{n,m}(\mathbb{C})$ satisfies the frontier condition.

■

We do not know whether the decomposition $\{Her_{\mathbb{F}}(K,L) \mid (r,K,L) \in J(n,m)\}$ of $C_{n,m}(\mathbb{F})$ is a cell decomposition, i.e., if the dimension drop property (b) in Definition 5.5 is satisfied. The problem is that the adherence relation on the whole set $J(n,m)$ of generalized Hermite indices is unknown, while Proposition 6.2 applies only for fixed r. To overcome this difficulty we use the following terminology, which weakens the definition of a cell decomposition:

Definition 6.5: Let X be a locally compact topological space. A finite decomposition $\{X_i \mid i \in I\}$ of X into nonempty disjoint subsets is called a *cellular patch complex* provided

(a′) Each X_i is homeomorphic to some $\mathbb{F}^{n_i}$, $n_i \in N$.

(b′) There exists a *partial ordering* $\le$ on I with

$$X_i \cap \overline{X}_j \neq \emptyset \Rightarrow i \le j.$$

Using this terminology we have

Theorem 6.6: The decomposition $\{Her_{\mathbb{F}}(K,L) | (r,K,L) \in J(n,m)\}$ is a finite cellular patch complex.

Proof: We define a partial ordering on $J(n,m)$ by

$$(r,K_1,L_1) \le (s,K_2,L_2) :\Leftrightarrow \begin{cases} r < s \\ or \\ r = s & \text{and } K_1 \le K_2,\ L_1 \le L_2. \end{cases}$$

By (4.3) and Theorem 6.3, $\{Her_{\mathbb{F}}(K,L)|(r,K,L) \in J(n,m)\}$ is a cellular patch complex for the ordering $\leq$.

■

7. HOMOLOGY GROUPS AND BETTI NUMBERS

An important class of topological invariants of a space X are the *singular homology groups* $H_q(X;G)$, defined for any integer $q \in \mathbb{N}_0$ and an abelian group G. The *Betti numbers* of X are defined by

$$\beta_q(X) := \dim H_q(X;Q), \quad q \geq 0.$$

In this section, we determine the integral homology groups

$$H_*(C_{n,m}(\mathbb{C})) := H_*(C_{n,m}(\mathbb{C});\mathbb{Z})$$

of $C_{n,m}(\mathbb{C})$ leading to an explicit combinatorial formula for the Betti numbers of $C_{n,m}(\mathbb{C})$. Our method of computing $H_*(C_{n,m}(\mathbb{C}))$ is classical and uses the Hermite cell decomposition of the strata $C^r_{n,m}(\mathbb{F})$. In the real case, our method does not allow us to determine $H_*(C_{n,m}(\mathbb{R});\mathbb{Z}/2)$, and we leave the problem of computing the mod 2 homology groups of $C_{n,m}(\mathbb{R})$ as an open question.

First, let us determine the homology groups of the various strata $C^r_{n,m}(\mathbb{C})$ for $r = 0,\ldots,n$. Let $H^{BM}_q(X)$ denote the q-th Borel-Moore homology group with closed support and coefficients in $\mathbb{Z}$. For $p \geq 0$, let $\mathbb{Z}^p = \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$ denote the p-fold direct sum of $\mathbb{Z}$, $\mathbb{Z}^0 := \{0\}$. For a given cell decomposition $\{X_i|i \in I\}$ of a topological space X, the q-th *type number* of $\{X_i|i \in I\}$ is defined by

$$c_q(X) := \operatorname{card}\{i \in I| \dim_{\mathbb{R}} X_i = q\}, \tag{7.1}$$

i.e., the number of cells of real dimension q.

We use the following facts about the Borel-Moore homology groups.

Lemma 7.1: Let X be a topological space with $\emptyset = X_{-1} \subset X_0 \subset X_1 \subset X_2 \subset \ldots \subset X_n = X$ a finite filtration of X by closed subspaces X_i, $i = 0,\ldots,n$. Suppose

$$H^{BM}_{2q+1}(X_i - X_{i-1}) = \{0\} \quad \text{for all } q \geq 0, \quad 0 \leq i \leq n.$$

Then for all $q \geq 0$,

$$H_q^{BM}(X) \cong \bigoplus_{i=0}^{n} H_q^{BM}(X_i - X_{i-1}).$$

Proof: We work by induction on n. For $n = 0$, we have nothing to show. Suppose the result is shown for $n-1$. Thus, for $A := X_{n-1}$,

$$H_q^{BM}(A) \cong \bigoplus_{i=0}^{n-1} H_q^{BM}(X_i - X_{i-1}). \tag{7.2}$$

Consider the homology exact sequence

$$\cdots \to H_q^{BM}(A) \to H_q^{BM}(X) \to H_q^{BM}(X - A) \to H_{q-1}^{BM}(A) \to \cdots.$$

For q odd, we have $H_q^{BM}(X - A) = \{0\}$ and, by (7.2), $H_q^{BM}(A) = \{0\}$. Thus, for q odd, $H_q^{BM}(X) = \{0\}$ and

$$H_q^{BM}(X) \cong H_q^{BM}(X - X_{n-1}) \oplus H_q^{BM}(X_{n-1}).$$

For q even, the homology exact sequence splits into short exact sequences

$$0 \to H_q^{BM}(A) \to H_q^{BM}(X) \to H_q^{BM}(X - A) \to 0,$$

since $H_{q-1}^{BM}(A) = H_{q+1}^{BM}(X - A) = \{0\}$. Thus,

$$\begin{aligned} H_q^{BM}(X) &\cong H_q^{BM}(A) \oplus H_q^{BM}(X - A) \\ &\cong \bigoplus_{i=0}^{n} H_q^{BM}(X_i - X_{i-1}). \end{aligned}$$

■

Corollary 7.2: Let X be a locally compact Hausdorff space and let $\{X_i | i \in I\}$ be a finite cell decomposition of X. If all cells X_i have even dimensions, then for all $q \geq 0$,

$$H_q^{BM}(X) \cong \mathbb{Z}^{c_q(X)}.$$

In particular, $H_*^{BM}(X)$ is torsion free.

■

We now apply Corollary 7.2 to compute the Borel-Moore homology groups of $C_{n,m}(\mathbb{C})$. For any integer $q \geq 0$ and $m, n \geq 1$, let

$$\beta_q(n,m) := \text{card}\{K \in K_{n,m} |\ 2\sum_{i=1}^{m}(m-i)K_i = q\} \tag{7.3}$$

denote the number of Hermite cells $Her^0_{\mathbb{C}}(K)$ of $\mathcal{N}_{n,m}(\mathbb{C})$ of real dimension q. We have the following simple combinatorial formula for $\beta_q(n,m)$.

Lemma 7.3 ([10]): $\beta_q(n,m)$ is a partition number and is equal to the number of sequences $(a_1, \ldots, a_{m-1})$ of nonnegative integers with $0 \leq a_1 \leq \ldots \leq a_{m-1} \leq n$ and $2(a_1 + \cdots + a_{m-1}) = q$.

■

Let $b_q(r;n,m)$ denote the number of generalized Hermite cells $Her_{\mathbb{C}}(K,L)$ of $C^r_{n,m}(\mathbb{C})$ of real dimension q. By Lemma 6.1, we have

Lemma 7.4:

$$b_q(r;n,m) = \sum_{i=0}^{q} \beta_{i-2r}(r,m)\beta_{q-i}(n-r,m).$$

■

The following result is an immediate consequence of Theorem 6.3 and Corollary 7.2:

Theorem 7.5:

(a) For any integer $q \geq 0$,

$$H_q^{BM}(C^r_{n,m}(\mathbb{C})) \cong \mathbb{Z}^{b_q(r;n,m)}$$

(b) The odd-dimensional homology groups vanish:

$$H_{2q+1}^{BM}(C^r_{n,m}(\mathbb{C})) = \{0\}, \quad \forall q \geq 0$$

and $H_*^{BM}(C^r_{n,m}(\mathbb{C}))$ is torsion free.

$\blacksquare$

In order to state the main result of this section, let us introduce the number

$$b_q(n,m) := \sum_{r=0}^{n} b_q(r;n,m) \tag{7.4}$$

$$= \sum_{i=0}^{q}\sum_{r=0}^{n} \beta_{i-2r}(r,m)\beta_{q-i}(n-r,m). \tag{7.5}$$

Let $H_q(X)$ denote the classical q-th *singular homology group* of X with coefficients in $\mathbb{Z}$. These groups $H_q(X)$ are classical topological invariants attached to X, but they have to be clearly distinguished from the Borel-Moore homology groups $H_q^{BM}(X)$. In general, the groups $H_q^{BM}(X)$ and $H_q(X)$ are not isomorphic. Our main interest is in the singular homology groups. However, for technical reasons, we also need to work with the Borel-Moore homology.

Theorem 7.6: The singular (integral) homology groups of the orbit space $C_{n,m}(\mathbb{C})$ are given by

$$H_q(C_{n,m}(\mathbb{C})) \cong \mathbb{Z}^{b_{2nm-q}(n,m)}, \quad q \in \mathbb{N}_0,$$

with $b_q(n,m)$ defined by (7.4),(7.5). In particular, $b_{2nm-q}(n,m)$ is the q-th Betti number of $C_{n,m}(\mathbb{C})$, and $H_*(C_{n,m}(\mathbb{C}))$ is torsion free.

Proof: Consider the filtration

$$\overline{C^0_{n,m}(\mathbb{F})} \subset \overline{C^1_{n,m}(\mathbb{F})} \subset \cdots \subset \overline{C^n_{n,m}(\mathbb{F})} = C_{n,m}(\mathbb{F})$$

by closed subsets $\overline{C^i_{n,m}(\mathbb{F})} = \bigcup_{j=0}^{i} C^j_{n,m}(\mathbb{F})$, $i = 0,\ldots,n$. By Theorem 7.5 and Lemma 7.1, we have for all $q \geq 0$

$$H_q^{BM}(C_{n,m}(\mathbb{C})) \cong \bigoplus_{r=0}^{n} H_q^{BM}(C^r_{n,m}(\mathbb{C})). \tag{7.6}$$

Thus, by Theorem 7.5, (7.4):

$$H_q^{BM}(C_{n,m}(\mathbb{C})) \cong \mathbb{Z}^{b_q(n,m)}, \quad \forall q \geq 0.$$

By Theorem 3.6, $C_{n,m}(\mathbb{C})$ is a complex manifold of real dimension $2nm$. As a complex manifold, $C_{n,m}(\mathbb{C})$ is orientable. Thus, by Poincaré duality

$$\begin{aligned} H_q^{BM}(C_{n,m}(\mathbb{C})) &\cong H_c^q(C_{n,m}(\mathbb{C})) \\ &\cong H_{2nm-q}(C_{n,m}(\mathbb{C})). \end{aligned}$$

Thus,

$$H_q(C_{n,m}(\mathbb{C})) \cong \mathbb{Z}^{b_{2nm-q}(n,m)}, \quad \forall\, q \geq 0.$$

■

In order to reformulate Theorem 7.6 in a more geometric way, let $G_k(\mathbb{F}^n)$ denote the Grassmann manifold of k-dimensional $\mathbb{F}$-linear subspaces of $\mathbb{F}^n$. $G_k(\mathbb{F}^n)$ is a smooth compact manifold. The singular homology groups of $G_k(\mathbb{F}^n)$ are well-known; see Ehresmann [27]. In particular, for $\mathbb{F} = \mathbb{C}$, one has:

$$H_q(G_k(\mathbb{C}^{n+k}) \cong \mathbb{Z}^{\beta_q(n,k+1)} \tag{7.7}$$

where $\beta_q(n, k+1)$ is defined by Lemma 7.3.

Theorem 7.7: There are isomorphisms of singular (integral) homology groups

$$H_q(C_{n,m}(\mathbb{C})) \cong \bigoplus_{r=0}^{n} H_{q-2r}(G_{m-1}(\mathbb{C}^{r+m-1}) \times G_{m-1}(\mathbb{C}^{n-r+m-1}))$$

for all $q \geq 0$.

Proof: By the Künneth formula and (7.7),

$$\begin{aligned} &H_k(G_{m-1}(\mathbb{C}^{r+m-1}) \times G_{m-1}(\mathbb{C}^{n-r+m-1})) \cong \\ &\qquad \cong \bigoplus_{i=0}^{k} H_i(G_{m-1}(\mathbb{C}^{r+m-1})) \otimes H_{k-i}(G_{m-1}(\mathbb{C}^{n-r+m-1})) \\ &\qquad \cong \mathbb{Z}^{\Sigma_{i=0}^{k} \beta_i(r,m)\beta_{k-i}(n-r,m)}. \end{aligned}$$

Since

$$b_{2nm-q}(n,m) = \sum_{r=0}^{n} \sum_{i=0}^{2nm-q} \beta_{i-2r}(r,m)\beta_{2nm-(q+i)}(n-r,m)$$

$$= \sum_{r=0}^{n} \sum_{\ell=0}^{2nm-(q+2r)} \beta_{\ell}(r,m)\beta_{2nm-(q+\ell+2r)}(n-r,m),$$

Theorem 7.6 implies

$$H_q(C_{n,m}(\mathbb{C})) \cong \mathbb{Z}^{b_{2nm-q}(n,m)}$$

$$\cong \bigoplus_{r=0}^{n} H_{2nm-(q+2r)}(G_{m-1}(\mathbb{C}^{r+m-1}) \times G_{m-1}(\mathbb{C}^{n-r+m-1})).$$

Since $G_{m-1}(\mathbb{C}^{r+m-1}) \times G_{m-1}(\mathbb{C}^{n-r+m-1})$ is a compact complex manifold of real dimension $2r(m-1) + 2(n-r)(m-1) = 2n(m-1)$, Poincaré duality gives

$$H_{2nm-(q+2r)}(G_{m-1}(\mathbb{C}^{r+m-1}) \times G_{m-1}(\mathbb{C}^{n-r+m-1}))$$

$$\cong H_{q+2(r-n)}(G_{m-1}(\mathbb{C}^{r+m-1}) \times G_{m-1}(\mathbb{C}^{n-r+m-1})).$$

The result follows.

■

Example 1: Let $m = 1$. By Theorem 7.6, we obtain

$$H_q(C_{n,1}(\mathbb{C})) = \begin{cases} \mathbb{Z} & \text{for } 0 \leq q \leq 2n \text{ even} \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $C_{n,1}(\mathbb{C})$ has the homology groups of the complex projective n-space $\mathbb{P}^n(\mathbb{C})$:

$$H_*(C_{n,1}(\mathbb{C})) \cong H_*(\mathbb{P}^n(\mathbb{C})). \tag{7.8}$$

■

Even more is true:

Proposition 7.8: The spaces $C_{n,1}(\mathbb{F})$ and $\mathbb{P}^n(\mathbb{F})$ are diffeomorphic.

Proof: Consider the analytic mapping $\chi : C_{n,m}(\mathbb{F}) \to \mathbb{P}^n(\mathbb{F})$ defined in (4.1). Using the Weierstrass decomposition (Lemma 2.1), it is easily seen that χ is surjective. Let $m = 1$ and $0 \leq r \leq n$. Consider the analytic isomorphism

$$\varphi_r : \Sigma_{r,1}(\mathbb{F}) \times \mathcal{N}_{n-r,1}(\mathbb{F}) \to C^r_{n,1}(\mathbb{F})$$
$$([A_1, B_1]_\sigma, [A_2, B_2]_\sigma) \mapsto \left[\begin{pmatrix} I_r & 0 \\ 0 & A_2 \end{pmatrix}, \begin{pmatrix} A_1 & 0 \\ 0 & I_{n-r} \end{pmatrix}, \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}\right]_\eta.$$

(See Theorem 4.2.) Thus,

$$\det(\lambda E - \mu A) = \alpha \mu^{n-r} \det(\lambda I_r - A_1),$$

where α is a nonzero scalar. The similarity orbit $[A_1, B_1]_\sigma$ is uniquely defined by the coefficients of the characteristic polynomial $\det(\lambda I_r - A_1)$. Similarly, $[A_2, B_2]_\sigma \in \mathcal{N}_{n-r,1}$ is uniquely determined by $n - r$. Thus, the composed map $\chi \circ \varphi_r$ is injective for every r. The injectivity of χ follows. From Weierstrass form, it is clear that the inverse of the restriction $\chi | C^r_{n,1}(F)$ is analytic for any r. In particular, for $r = n$, $\chi | C^{\mathrm{reg}}_{n,1}(F)$ is a diffeomorphism onto its image $\mathbb{F}^n \subset \mathbb{P}^n$. Let $\tau : GL_2(\mathbb{F}) \to GL_{n+1}(\mathbb{F}) = GL(B(n))$, $\Omega \mapsto \tau_\Omega$ denote the standard irreducible representation of $GL_2(\mathbb{F})$. Here $B(n)$ denotes the $\mathbb{F}$-vector space of all homogeneous polynomials

$$p(\lambda, \mu) = \sum_{j=0}^{n} p_j \lambda^j \mu^{n-j}, \qquad p_j \in \mathbb{F}$$

of degree n in (λ, μ), and $\tau_\Omega p(\lambda, \mu) = p((\lambda, \mu)\Omega)$. We also denote by τ_Ω the corresponding induced map on the projective space of $B(n)$.

Let $C_{n,1}(\mathbb{F}) = \cup_{i=0}^{n} C_{\Omega_i}(n, 1)$ be as in (3.18) with

$$\Omega_i = \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix}, \qquad \Omega'_i = \begin{pmatrix} \alpha_i & -\beta_i \\ -\gamma_i & \delta_i \end{pmatrix}.$$

It is trivial to verify that

$$\chi = \tau_{\Omega'_i} \circ \chi \circ T_{\Omega_i^{-1}}.$$

Since $C_{\Omega_i}(n,1) = T_{\Omega_i}(C^{\mathrm{reg}}_{n,1}(\mathbb{F}))$, and $\chi|C^{\mathrm{reg}}_{n,1}(\mathbb{F})$ is a diffeomorphism onto its image, this implies that $\chi|C_{\Omega_i}(n,1)$ is a diffeomorphism onto its image. It follows that χ is a diffeomorphism.

■

Example 2: Let $n = 1$. By Theorem 7.7,

$$\begin{aligned} H_q(C_{1,m}(\mathbb{C})) &\cong H_q(\mathbb{P}^{m-1}(\mathbb{C})) \oplus H_{q-2}(\mathbb{P}^{m-1}(\mathbb{C})) \\ &\cong H_q(\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^{m-1}(\mathbb{C})) \end{aligned} \tag{7.9}$$

for all $q \geq 0$. ■

Proposition 7.9: The spaces $C_{1,m}(\mathbb{F})$ and $\mathbb{P}^1(\mathbb{F}) \times \mathbb{P}^{m-1}(\mathbb{F})$ are diffeomorphic.

Proof: Consider

$$\begin{aligned} \rho:\ C_{1,m}(\mathbb{F}) &\to \mathbb{P}^1(\mathbb{F}) \times \mathbb{P}^{m-1}(\mathbb{F}) \\ [E, A, (b_1, \ldots, b_m)]_\eta &\mapsto ([c_0(E,A) : c_1(E,A)],\ [b_1 : \ldots : b_m]). \end{aligned}$$

It is trivial to verify that ρ yields a bianalytical diffeomorphism.

■

By Propositions 7.8, 7.9, the orbit spaces $C_{n,m}(\mathbb{F})$ are *compact* for $\min(m,n) = 1$. In the next section, we will prove that $C_{n,m}(\mathbb{F})$ is in fact compact for all m, n.

Example 3: Let $m = n = 2$. Theorem 7.7 implies

$$H_q(C_{2,2}(\mathbb{C})) \cong H_q(\mathbb{P}^2(\mathbb{C})) \oplus H_{q-2}(\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})) \oplus H_{q-4}(\mathbb{P}^2(\mathbb{C})).$$

for all $q \geq 0$. The Betti numbers of $C_{2,2}(\mathbb{C})$ are

$$b_q(C_{2,2}(\mathbb{C})) = \begin{cases} 1 & q = 0 \\ 2 & q = 2 \\ 4 & q = 4 \\ 2 & q = 6 \\ 1 & q = 8 \\ 0 & \text{otherwise} \end{cases}$$

■

8. COMPACTNESS OF $C_{n,m}(\mathbb{F})$

As a topological application of the formula in Theorem 7.7 for the Betti numbers of $C_{n,m}(\mathbb{C})$, we prove the compactness of the orbit space $C_{n,m}(\mathbb{F})$ for $\mathbb{F} = \mathbb{R}, \mathbb{C}$.

For any complex system $(E, A, B) \in \tilde{C}_{n,m}(\mathbb{C})$, let $(\bar{E}, \bar{A}, \bar{B}) \in \tilde{C}_{n,m}(\mathbb{C})$ be defined by complex conjugation of E, A, B. This induces an involution

$$\begin{aligned} \tau : C_{n,m}(\mathbb{C}) &\to C_{n,m}(\mathbb{C}) \\ [E, A, B]_\eta &\mapsto [\bar{E}, \bar{A}, \bar{B}]_\eta \end{aligned} \tag{8.1}$$

on the orbit space $C_{n,m}(\mathbb{C})$. Let

$$\mathrm{Fix}(\tau) := \{[E, A, B]_\eta \in C_{n,m}(\mathbb{C}) \mid [\bar{E}, \bar{A}, \bar{B}]_\eta = [E, A, B]_\eta\}$$

denote the fixed point set of τ. Obviously, $C_{n,m}(\mathbb{R})$ is imbedded in $C_{n,m}(\mathbb{C})$ by associating to the $GL_n(\mathbb{R}) \times GL_n(\mathbb{R})$ orbit of a real system $(E, A, B) \in \tilde{C}_{n,m}(\mathbb{R})$ the complex $GL_n(\mathbb{C}) \times GL_n(\mathbb{C})$ orbit $[E, A, B]_\eta$ in $C_{n,m}(\mathbb{C})$.

Lemma 8.1: $\mathrm{Fix}(\tau) = C_{n,m}(\mathbb{R})$.

Proof: Certainly $C_{n,m}(\mathbb{R})$ is contained in the fixed point set of τ. For any $\Omega \in PGL_2(\mathbb{R})$, the scaling transformation T_Ω commutes with τ:

$$\tau(T_\Omega(C^{\mathrm{reg}}_{n,m}(\mathbb{C}))) = T_\Omega(\tau(C^{\mathrm{reg}}_{n,m}(\mathbb{C}))).$$

If $[A, B]_\sigma$ is a fixed point for complex conjugation on $\Sigma_{n,m}(\mathbb{C})$, then by uniqueness of Hermite form, the Hermite form of (A, B) must be real. Hence, $\Sigma_{n,m}(\mathbb{R})$ is precisely the fixed point set for complex conjugation on $\Sigma_{n,m}(\mathbb{C})$. Since $C^{\mathrm{reg}}_{n,m}(\mathbb{C})$ is τ-equivariantly isomorphic to $\Sigma_{n,m}(\mathbb{C})$, this implies that

$$\mathrm{Fix}(\tau)|_{T_\Omega(C^{\mathrm{reg}}_{n,m}(\mathbb{C}))} = T_\Omega(C^{\mathrm{reg}}_{n,m}(\mathbb{R}))$$

for all $\Omega \in PGL_2(\mathbb{R})$. The result follows from Lemma 3.5.

■

By Lemma 8.1, $C_{n,m}(\mathbb{R})$ is the set of real points of $C_{n,m}(\mathbb{C})$ and thus a closed subspace of $C_{n,m}(\mathbb{C})$.

We can now state and prove one of the main results of this paper.

Theorem 8.2: $C_{n,m}(\mathbb{F})$ is a compact space.

Proof: From the above, it sufficies to prove the compactness of $C_{n,m}(\mathbb{C})$; i.e., in the complex case. By Theorem 3.6, $C_{n,m}(\mathbb{C})$ is a complex manifold of real dimension $2nm$. By Theorem 7.7, the top dimensional singular homology group of $C_{n,m}(\mathbb{C})$ is

$$
\begin{aligned}
&H_{2nm}(C_{n,m}(\mathbb{C})) \cong \\
&\quad \cong \bigoplus_{r=0}^{n} H_{2nm-2r}(G_{m-1}(\mathbb{C}^{r+m-1}) \times G_{m-1}(\mathbb{C}^{n-r+m-1})).
\end{aligned}
$$

The product space of Grassmannians $G_{m-1}(\mathbb{C}^{r+m-1}) \times G_{m-1}(\mathbb{C}^{n-r+m-1})$ is a complex manifold of real dimension $2r(m-1)+2(n-r)(m-1) = 2n(m-1)$. For any n-dimensional manifold X, $H_q(X) = \{0\}$ for $q > n$. Thus,

$$
\begin{aligned}
&H_{2nm}(C_{n,m}(\mathbb{C})) \\
&\quad \cong H_{2n(m-1)}(G_{m-1}(\mathbb{C}^{n+m-1}) \times G_{m-1}(\mathbb{C}^{m-1})) \\
&\quad \cong \mathbb{Z},
\end{aligned}
\tag{8.2}
$$

the last isomorphism being because $G_{m-1}(\mathbb{C}^{n+m-1}) \times G_{m-1}(\mathbb{C}^{m-1})$ is orientable and connected. For any noncompact manifold X of dimension n, the top dimensional homology group must be trivial, i.e., $H_n(X) = \{0\}$. Thus, by (8.2), $C_{n,m}(\mathbb{C})$, and hence $C_{n,m}(\mathbb{R})$, must be compact.

∎

It is easy to show via the methods of Byrnes and Hurt [21] that $C_{n,m}(\mathbb{C})$ is quasiprojective. Combining Theorem 3.6 with Theorem 8.2 thus gives the following result.

Theorem 8.3: The orbit space $C_{n,m}(\mathbb{F})$ of controllable generalized linear systems is a smooth projective variety of $\mathbb{F}$-dimension mn.

∎

Corollary 8.4: For any $q \geq 0$, there are isomorphisms of singular homology groups

(a) $H_q(C_{n,m}(\mathbb{C})) \cong H_{2nm-q}(\mathbb{C}_{n,m}(\mathbb{C}))$.

(b) $H_q(C_{n,m}(\mathbb{R}); \mathbb{Z}/2) \cong H_{nm-q}(C_{n,m}(\mathbb{R}); \mathbb{Z}/2)$.

Proof: Follows immediately from Theorem 8.3 and Poincaré duality.

■

Remark 8.5: The space of generalized controllable systems $C_{n,m}(\mathbb{F})$ contains the set of regular controllable systems $C^{\text{reg}}_{n,m}(\mathbb{F}) \cong \Sigma_{n,m}(\mathbb{F})$ as an open dense subset. The orbit space $\Sigma_{n,m}(\mathbb{F}) \cong C^{\text{reg}}_{n,m}(\mathbb{F})$ of controllable systems is *not* compact. Thus, Theorem 8.3 says that $C_{n,m}(\mathbb{F})$ is a *smooth compactification* of $\Sigma_{n,m}(\mathbb{F})$. The boundary points of $\Sigma_{n,m}(\mathbb{F})$ necessary in order to compactify $\Sigma_{n,m}(\mathbb{F})$ are given by

$$C_{n,m}(\mathbb{F}) - C^{\text{reg}}_{n,m}(\mathbb{F}) = \bigcup_{r=0}^{n-1} C^{r}_{n,m}(\mathbb{F}), \tag{8.3}$$

i.e., consists of singular systems. The important new point is that the compactification $C_{n,m}(\mathbb{F})$ is *smooth* and is *explicitly described* as the quotient space of generalized linear systems. This is quite different from previously described compactifications; see, in particular, Hazewinkel [13] and Byrnes [14].

■

The compactness of $C_{n,m}(\mathbb{F})$ has the following consequence for high gain limits of state feedback orbits. Let $(I, A, B) \in \tilde{C}^{\text{reg}}_{n,m}(\mathbb{F})$ and let $K_\nu \in \mathbb{F}^{m\times n}$, $\nu \in \mathbb{N}$, denote an arbitrary sequence of state feedback gain matrices. In general it is not true that the sequence of systems $[I, A + BK_\nu, B]_\eta \in C^{\text{reg}}_{n,m}(\mathbb{F})$ has a limit point in $C^{\text{reg}}_{n,m}(\mathbb{F})$. However, by Theorem 8.2, there always exists a limit point in $C_{n,m}(\mathbb{F})$. We reformulate this as follows:

Corollary 8.6: Let $(A, B) \in \mathbb{F}^{n\times n} \times \mathbb{F}^{n\times m}$ be controllable, and let $K_\nu \in \mathbb{F}^{m\times n}$, $\nu \in \mathbb{N}$, denote an arbitrary sequence of state feedback gain matrices. There exist invertible matrices $M_\nu, N_\nu \in GL_n(\mathbb{F})$, $\nu \in \mathbb{N}$, such that the sequence of regular systems $(M_\nu N_\nu^{-1}, M_\nu(A + BK_\nu)N_\nu^{-1}, M_\nu B_\nu) \in \tilde{C}^{\text{reg}}_{n,m}(\mathbb{F})$ has a convergent subsequence with limit $(E_\infty, A_\infty, B_\infty) \in \tilde{C}_{n,m}(\mathbb{F})$.

Proof: By Theorem 8.2, the sequence of orbits $[I, A + BK_\nu, B]_\eta \in C^{\text{reg}}_{n,m}(\mathbb{F})$ has a convergent subsequence $[I, A+BK_{\nu'}, B]_\eta$ with

$$\lim_{\nu'\to\infty}[I, A+BK_{\nu'}, B]_\eta := [E_\infty, A_\infty, B_\infty]_\eta \in C_{n,m}(\mathbb{F}).$$

By Theorem 3.6, the quotient map $\pi : \tilde{C}_{n,m}(\mathbb{F}) \to C_{n,m}(\mathbb{F})$ is a locally trivial fibre bundle with structure group $GL_n(\mathbb{F}) \times GL_n(\mathbb{F})$. Thus, there exist invertible matrices $M_\nu, N_\nu \in GL_n(\mathbb{F})$ with

$$\lim_{\nu'\to\infty}(M_{\nu'}N_{\nu'}^{-1}, M_{\nu'}(A+BK_{\nu'})N_{\nu'}^{-1}, M_{\nu'}B) = (E_\infty, A_\infty, B_\infty)$$

with $(E_\infty, A_\infty, B_\infty)$ controllable and admissible.

■

REFERENCES

1. F.L. Lewis, "A survey of linear singular systems," *Circuits, Systems, and Signal Processing*, 5 (1986), 3-36.
2. M.A. Shayman and Z. Zhou, "Feedback control and classification of generalized linear systems," *IEEE Transactions on Automatic Control*, AC-32 (1987), 483-494.
3. H. Glüsing-Lüerßen and D. Hinrichsen, "A Jordan control canonical form for singular systems," *International Journal of Control*, to appear.
4. U. Helmke and M.A. Shayman, "A canonical form for controllable singular systems," *Systems and Control Letters*, to appear.
5. R.W. Brockett, "Some geometric questions in the theory of linear systems," *IEEE Transactions on Automatic Control*, AC-21 (1976), 449-455.
6. M. Hazewinkel and R. Kalman, "On invariants, canonical forms and moduli for linear constant finite-dimensional dynamical systems," *Lecture Notes in Econ.-Math. System Theory*, 131, Springer-Verlag, New York, 1976, 48-60.
7. C.I. Byrnes and T. Duncan, "On certain topological invariants arising in system theory," in *New Directions in Applied Mathematics* (P. Hilton and G. Young, eds.), Springer-Verlag, New York, 1981, 29-71.

8. D.F. Delchamps, *The Geometry of Spaces of Linear Systems with an Application to the Identification Problem*, Ph.D. Thesis, Harvard University, Cambridge, Massachusetts, 1982.

9. U. Helmke, "The topology of a moduli space for linear dynamical systems," *Comment. Math. Helv.*, 60 (1985), 630-655.

10. U. Helmke, "Topology of the moduli space for reachable linear dynamical systems: the complex case," *Math. Sys. Theory*, 19 (1986), 155-187.

11. J.D. Cobb, "Fundamental properties of the manifold of singular and regular linear systems," *Journal of Math. Analysis and Applic.*, 120 (1986), 328-353.

12. J.D. Cobb, "Topological aspects of controllability and observability on the manifold of singular and regular systems," *Journal of Math. Analysis and Applic.*, to appear.

13. M. Hazewinkel, "On families of linear systems: degeneration phenomena," in *Algebraic and Geometric Methods in Linear Systems Theory* (C.I. Byrnes and C.F. Martin, eds.), American Mathematical Society, Providence, Rhode Island, 1980, 157-189.

14. C.I. Byrnes, "On compactifications of spaces of systems and dynamic compensation," *Proceedings of the IEEE Conference on Decision and Control*, San Antonio, Texas, December 1983, 889-894.

15. M.A. Shayman, "Homogeneous indices, feedback invariants and control structure theorem for generalized linear systems," *SIAM Journal on Control and Optimization*, 26 (1988), 387-400.

16. M.A. Shayman, "Pole placement by dynamic compensation for descriptor systems, *Automatica*, 24 (1988), 279-282.

17. J.D. Cobb, "Feedback and pole placement in descriptor variable systems," *International Journal of Control*, 33 (1981), 1135-1146.

18. J.D. Cobb, "Global analyticity of a geometric decomposition for linear singularly perturbed systems," *Circuits, Systems and Signal Processing*, 5 (1986), 139-152.

19. H.H. Rosenbrock, "Structural properties of linear dynamical systems," *International Journal of Control*, 20 (1974), 191-202.

20. E.L Yip and R.F. Sincovec, "Solvability, controllability, and observability of continuous descriptor systems," *IEEE Transactions on Automatic Control*, AC-26 (1981), 702-707.

21. C.I. Byrnes and N.E. Hurt, "On the moduli of linear dynamical systems," *Advances in Math. Studies in Analysis*, 4 (1979), 83-122.

22. J. Dieudonné, *Foundations of Modern Analysis*, Vol. 3, Academic Press, New York, 1982.

23. U. Helmke and D. Hinrichsen, "Canonical forms and orbit spaces of linear systems," *IMA J. Math. Control and Inform.*, 3 (1986), 167-184.

24. D.Q. Mayne, "A canonical model for identification of multivariable systems," *IEEE Transactions on Automatic Control*, AC-17 (1972), 728-729.

25. D. Hinrichsen and D. Prätzel-Wolters, "Generalized Hermite matrices and complete invariants of strict system equivalence," *SIAM Journal on Control and Optimization*, 21 (1983), 289-305.

26. D. Hinrichsen, "Canonical forms and parametrization problems in linear systems theory," in *Fourth IMA International Conference on Control Theory* (P.A. Cook, ed.), Academic Press, New York, 1985.

27. C. Ehresmann, "Sur la topologie de certains espace homogenes," *Ann. of Math.*, 35 (1934), 396-443.