Thesis Report

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Motion Control and Planning for Nonholonomic Kinematic Chains

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MOTION CONTROL AND PLANNING FOR NONHOLONOMIC KINEMATIC CHAINS

by

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ABSTRACT

Title of Dissertation: MOTION CONTROL AND PLANNING FOR NONHOLONOMIC KINEMATIC CHAINS

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In this dissertation we examine a class of systems where nonholonomic kinematic constraints are combined with periodic shape variations, giving rise to a snake–like undulating motion of the system. Within this class, we distinguish two subclasses, one where the system possesses enough kinematic constraints to allow the control of its motion to be based entirely on kinematics and another which does not; in the latter case, the dynamics plays a crucial role in complementing the kinematics and in making motion control possible. An instance of these systems are the Nonholonomic Variable Geometry Truss (NVGT) assemblies, where shape changes are implemented by parallel manipulator modules, while the nonholonomic constraints are imposed by idler wheels attached to the assembly. We assume that the wheels roll without slipping on the ground, thus constraining the instantaneous motion of the assembly. These assemblies can be considered as land locomotion alternatives to systems based on legs or actuated wheels. Their propulsion combines features of both biological systems like skating humans and snakes, and of man–made systems like orbiting satellites with manipulator arms. The NVGT assemblies can be modeled in terms of the Special Euclidean group of rigid motions on the plane. Generalization to nonholonomic kinematic chains on other Lie groups (G) gives rise to the notion of G–Snakes.

Moreover, we examine systems with parallel manipulator subsystems which can be used as sensor–carrying platforms, with potential applications in exploratory and active visual or haptic robotic tasks. We concentrate on specifying a class of configuration space path segments that are optimal in the sense of a curvature–squared cost functional, which

can be specified analytically in terms of elliptic functions and can be used to synthesize a trajectory of the system.

In both cases, a setup of the problem which involves tools from differential geometry and the theory of Lie groups appears to be natural. In the case of G–Snakes, when the number of nonholonomic constraints equals the dimension of the group G, the constraints determine a principal fiber bundle connection. The geometric phase associated to this connection allows us to derive (kinematic) motion control strategies based on periodic shape variations of the system. When the G–Snake assembly has one constraint less than the dimension of the group G, we are still able to synthesize a principal fiber bundle connection by taking into account the Lagrangian dynamics of the system through the so–called nonholonomic momentum. The symmetries of the system are captured by actions of non–abelian Lie groups that leave invariant both the constraints and the Lagrangian and play a significant role in the definition of the momentum and the specification of its evolution. The (dynamic) motion control is now based on periodic shape variations that build up momentum and allow propulsion and steering, as described by the geometric and dynamic phases of the system.

DEDICATION

To Sofia

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TABLE OF CONTENTS

Section **Page**

List of Figures viii

Glossary of Notation x

Chapter 1: Prolegomena 1

Chapter 2: Preliminaries 9

2.1 Introduction 9

2.2 Elliptic Functions and Integrals 9

2.3 Differential Geometry and Lie Groups 17

2.4 Geometric Mechanics 26

2.5 Parallel Manipulators 29

2.5.1 Spatial Parallel Manipulators 29

2.5.2 Planar Parallel Manipulators 36

Chapter 3: Motion Planning based on Path Curvature 42

3.1 Introduction 42

3.2 Curvature–based Planning for 2–dimensional Manifolds 43

3.2.1 Curvature–based Planning Criteria 45

3.2.2 Controllability 47

3.2.3 Hamiltonian functions, Maximum Principle and Poisson Reduction ... 49

3.2.4 The Bryant–Griffiths Optimality Constraint 54

3.2.5 Solutions of the Bryant–Griffiths Optimality Constraint 55

3.2.6 Integral Curves 63

3.3 Curvature–based Planning for 3–dimensional Manifolds 70

3.3.1 Curvature–based Planning 70

3.3.2 The Optimality Constraint 72

3.3.3 Solutions of the Optimality Constraint 72

3.4 Singular Configuration Avoidance for Parallel Manipulators 74

Chapter 4: G–Snakes: Nonholonomic Kinematic Chains on Lie Groups 78
4.1 Introduction 78
4.2 Nonholonomic Kinematic Chains on Lie Groups 79
4.2.1 The ℓ–node Kinematic Chain 79
4.2.2 Nonholonomic Constraints and Connections on Principal Fiber Bundles 81
4.3 Three–node G–Snakes on Three–dimensional Lie Groups 85
4.3.1 G–Snakes on the Special Euclidean Group $SE(2)$ 88
4.3.2 G–Snakes on the Heisenberg Group $H(3)$ 93
4.3.3 G–Snakes on the Special Orthogonal Group $SO(3)$ 97
4.3.4 G–Snakes on the Special Linear Group $SL(2)$ 100

Chapter 5: Nonholonomic Variable Geometry Truss Assemblies 106
5.1 Introduction 106
5.2 Kinematics of the Nonholonomic Variable Geometry Truss (NVGT) Assembly 109
5.2.1 The Nonholonomic Constraint of Rolling–without–Slipping 109
5.2.2 The ℓ–node, $(\ell - 1)$–module Nonholonomic Variable Geometry Truss (NVGT) Assembly 112
5.2.3 Implementation of a NVGT module as a Parallel Manipulator 120
5.3 The 3–node, 2–module NVGT Assembly 122
5.3.1 Kinematics of the 2–module NVGT 123
5.3.2 Motion Control and Planning for the 2–module NVGT 129
5.4 SE(2)–Spiders, SE(2)–Rings and other SE(2)–Snake Kinematic Topologies 145
5.4.1 The 3–module SE(2)–Spider 145
5.4.2 The 3–module SE(2)–Ring 149
5.4.3 The 1–module SE(2)–Snake with more than one constraint per node . 150

Chapter 6: Dynamics of an SE(2)–Snake Assembly 154
6.1 Introduction 154
6.2 Kinematics of the Roller Racer 156
6.3 Symmetry of the Roller Racer 161

6.4 Dynamics of the Roller Racer 165

Chapter 7: Conclusions and Future Research 178

Bibliography 180

LIST OF FIGURES

Number	Page
2.2.1: Jacobi Elliptic Functions	12
2.3.1: Connection on a Principal Fiber Bundle	26
2.5.1: Stewart Platform	29
2.5.2: Planar Parallel Manipulator	36
2.5.3: Forward Kinematics Solutions	38
2.5.4: Kinematic Singularities	39
2.5.5: Singular Surfaces and Singularity Avoidance	40
2.5.6: Effect of Singularities	40
3.2.1: Configuration–Space Paths	44
3.2.2: Curve in the Plane	46
3.2.3: Integral Curves of the Hamiltonian v.f. X_h on the Lie–Poisson Manifold $\mathcal{G}^*_-$	53
3.2.4: Solutions of the Bryant–Griffiths Optimality Constraint when $H > \mathcal{C}$	56
3.2.5: Solutions of the Bryant–Griffiths Optimality Constraint when $H = \mathcal{C}$	56
3.2.6: Solutions of the Bryant–Griffiths Optimality Constraint when $-\mathcal{C} < H < \mathcal{C}$	57
3.2.7: Integral Curves for $H > \mathcal{C}$	67
3.2.8: Integral Curves for $H = \mathcal{C}$	67
3.2.9: Integral Curves for $-\mathcal{C} < H < \mathcal{C}$	69
3.4.1: Singular Surfaces	75
3.4.2: Trajectory III	75
3.4.3: Trajectory I	76
3.4.4: Trajectory II	76
3.4.5: Curvature–Minimizing Trajectory	77
5.1.1: One module of the NVGT assembly	107
5.2.1: Wheel pair rolling–without–slipping	110
5.2.2: The $(i, i+1)$–th module of the NVGT assembly	112
5.3.1: Two–module NVGT assembly	123

5.3.2: Instantaneous Center of Rotation of (i, j)–th Module 126

5.3.3: Nonholonomic Singularities for 2–module NVGT assembly 127

5.3.4: Nonholonomic Singularity Surfaces 127

5.3.5: The 2–module NVGT Model for Motion Control 129

5.3.6: Piecewise Constant Controls and Geometric Phase for Translating 2–module NVGT 135

5.3.7: Geometric Phase for Translating 2–module NVGT (Sinusoidal Controls) 140

5.3.8: Geometric Phase for Rotating 2–module NVGT (Sinusoidal Controls) .. 142

5.3.9: Motion Planning for 2–module NVGT moving on the plane 143

5.3.10: Primitive motions of the 2–module NVGT 144

5.3.11: Obstacle Avoidance with the 2–module NVGT 144

5.4.1: The 3–module $SE(2)$–Spider 146

5.4.2: The 3–module $SE(2)$–Ring 149

6.1.1: The Roller Racer 154

6.1.2: The Roller Racer Model 155

6.4.1: Roller Racer Prototype 174

6.4.2: "Straight–line" motion of the Roller Racer 175

6.4.3: "Turning" motion of the Roller Racer 177

GLOSSARY OF NOTATION

G	Lie Group
n	Dimension of Lie Group G
$g \in G$	Element of Lie Group G
$\mathcal{G}$	Lie Algebra of Lie Group G
$\xi \in \mathcal{G}$	Element of Lie Algebra $\mathcal{G}$
$\{\mathcal{A}_1, \ldots, \mathcal{A}_n\}$	Basis of Lie Algebra $\mathcal{G}$
$\mathcal{G}^*$	Dual Space of the Lie Algebra $\mathcal{G}$
$\{\mathcal{A}_1^\flat, \ldots, \mathcal{A}_n^\flat\}$	Basis of Dual Space $\mathcal{G}^*$
$\xi_i = \mathcal{A}_i^\flat(\xi) \in I\!R$	i–th coordinate of $\xi \in \mathcal{G}$ in basis $\{\mathcal{A}_1, \ldots, \mathcal{A}_n\}$
$\{\gamma_1, \ldots, \gamma_n\}$	Wei–Norman parameters of $g \in G$
$SE(n)$	Special Euclidean group
$SO(n)$	Special Orthogonal group
$H(n)$	Heisenberg group
$SL(n)$	Special Linear group
S^n	n–Sphere
T^n	n–Torus
$se(n)$	Lie Algebra of Special Euclidean group $SE(n)$
$so(n)$	Lie Algebra of Special Orthogonal group $SO(n)$
$h(n)$	Lie Algebra of Heisenberg group $H(n)$
$sl(n)$	Lie Algebra of Special Linear group $SL(n)$
(x, y, θ) or (x, y, ϕ)	Local coordinates for $g \in SE(2)$
(Ξ_1, Ξ_2, ω)	Local coordinates for $\xi \in se(2)$
Q	Configuration Space
$q \in Q$	Element of Configuration Space
$\mathcal{S}$	Shape Space
$s \in \mathcal{S}$	Element of Shape Space
π	Projection $Q \longrightarrow \mathcal{S}$
Φ	Action of Group
ξ_Q	Infinitesimal generator of G acting on Q, corresponding to $\xi \in \mathcal{G}$
$L(q, v)$	Lagrangian with $(q, v) \in TQ$

p	Nonholonomic Momentum
κ	Curvature of Curve
$s \in \mathbb{R}$	Arc–length parametrization parameter

G–Snakes

ℓ	Number of Nodes in Kinematic Chain
$g_{i} \in G$	Configuration of i-th Node
$g_{i,j} \in G$	Shape of (i,j)–th Module
$\gamma_k^i \in \mathbb{R}$	The k–th Wei–Norman parameter of $g_{i} \in G$
$\gamma_k^{i,j} \in \mathbb{R}$	The k–th Wei–Norman parameter of $g_{i,j} \in G$
$\xi_{i} \in \mathcal{G}$	Body Velocity of i-th Node
$\xi_{i,j} \in \mathcal{G}$	Body Velocity of (i,j)–th Module
$\xi_k^i = \mathcal{A}_k^\flat(\xi_{i}) \in \mathbb{R}$	The k–th component of $\xi_{i} \in \mathcal{G}$
$\xi_k^{i,j} = \mathcal{A}_k^\flat(\xi_{i,j}) \in \mathbb{R}$	The k–th component of $\xi_{i,j} \in \mathcal{G}$
$\xi^i \in \mathbb{R}^n$	Vector of coordinates of $\xi_{i} \in \mathcal{G}$ in basis $\{\mathcal{A}_1, \ldots, \mathcal{A}_n\}$
$\xi^{i,j} \in \mathbb{R}^n$	Vector of coordinates of $\xi_{i,j} \in \mathcal{G}$ in basis $\{\mathcal{A}_1, \ldots, \mathcal{A}_n\}$
$\zeta \in \mathbb{R}^{n\ell}$	Composite Velocity Vector of G–Snake. Partitioned as $\binom{\zeta_1}{\zeta_2}$
$\zeta_1 \in \mathbb{R}^{(n-1)\ell}$	Shape Controls Vector of G–Snake
$\zeta_2 \in \mathbb{R}^{\ell}$	Vector of G–Snake Velocities Dependent on Shape Controls ζ_1

CHAPTER ONE

PROLEGOMENA

One of the main topics of this dissertation is the study of a novel class of robotic devices that combine features of such biological systems as snakes and skating humans and of such man–made systems as orbiting satellites equipped with manipulator arms. Those multimodule devices, which we call Nonholonomic Variable Geometry Truss (NVGT) assemblies and which can be used for locomotion as an alternative to cars or bicycles, are equipped with wheels, which we assume that they roll without slipping on the plane that supports the device. Like the front wheel of a bicycle, those wheels are not directly actuated; they merely constrain the instantaneous motion of the assembly so that it does not slip "sideways". This type of constraints belong to the class of so–called nonholonomic constraints, instances of which also appear in the systems mentioned earlier. Their features are presented in more detail later. Each module of the assembly can alter its shape in a periodic way by a suitable mechanism of the Variable Geometry Truss (VGT) variety, implemented as a planar parallel manipulator.

Usual land locomotion systems employ either legs (humans, insects) or an endless rotating element, like actuated wheels or tracks, for propulsion. An alternative, that is currently emerging in robotic studies, employs articulated bodies and suitable motion constraints in a paradigm inspired by snake and worm locomotion. The locomotion of the NVGT is achieved by the interaction of the nonholonomic constraints from the wheels, with the periodic shape variations of the modules. This gives rise to a snake–like undulating motion of the assembly. The kinematic and dynamic analysis of such systems unveils an interesting geometric structure and makes explicit their fundamental property of specifying the global motion of the system as a function of just its shape and of the shape variations. This property is related to the phenomenon of geometric phases that appear in several physical systems and it can be exploited for planning and controlling their motion; however it's not obvious how to solve those problems, since the NVGT systems are very redundant and the proper selection of the shape actuation strategy that will achieve a desired motion of the system is made even more complicated by the presence of the nonholonomic constraints.

This dissertation also examines robotic systems with parallel manipulator subsystems which are used as sensor–carrying platforms to perform exploratory visual or haptic tasks. Exploratory and active tasks occur when a robotic system relocates its mechanical and/or sensory subsystems or alters their characteristics in order to collect information about its environment, about the tasks it is required to perform and, possibly, about the robotic system itself. Actions of this type can also facilitate the subsequent processing and understanding by the system of the sensory information that was obtained, since both spatially and temporally novel information is added in a controlled way. This is a relatively novel paradigm in robotics that is having a profound impact in several related research areas (Negahdaripour & Jain [1991]). Vision is one of the sensory modalities where the active approach has been well documented. This was inspired by the observation that humans tend to move their eyes and heads in a variety of ways, in order to obtain a better view of a scene or in order to focus on a moving object of interest, all of those activities demonstrating that human perception of a dynamically evolving scene is an equally dynamical act, whether conscious or unconscious (Bajcsy [1988]). This has found robotic applications, as in the case where a camera is being moved around the workspace of the robot for obtaining multiple views of it (Zheng, Chen & Tsuji [1991]), for monocular stereo–vision (Sandini & Tistarelli [1990]) or for facilitating dynamic segmentation (Aloimonos [1990]). Furthermore, the implementation of saccadic, smooth pursuit or vergence camera motions, as well as the repositioning of cameras are well known methods for moving target stabilization (Papanikolopoulos, Khosla & Kanade [1991]), for camera fixation and gaze control (Brown [1990]; Raviv [1991]), for easy egomotion parameter computation, for conversion of ill–posed early vision problems into well–posed ones (Aloimonos, Weiss & Bandopadhay [1988]) and for changing the focus of attention of a robotic system (Abbott [1992]; Clark & Ferrier [1989]). In previous work, we examined visual target tracking based on the use of sequences of images that specify the camera reorientation needed so that a target moving in a cluttered 3–dimensional visual environment is kept foveated (Aloimonos & Tsakiris [1991]). In order to bypass the image segmentation and feature correspondence problems inherent in other tracking methods, as well as the restriction to 2–dimensional domains, we employed the optical flow formalism. This however necessitates the use of a dense image sequence as an input to the tracking algorithm, which in turn implies that small and accurate motions of the camera are necessary. Haptic exploratory tasks are another example of the active approach. They can use tactile sensors mounted on fingertips of dextrous hands with the goal to derive information on the shape, the surface texture and the mechanical properties of objects of interest. This may involve suitable accurate motion of the sensors, while contact with those objects is maintained (Bajcsy, Lederman & Klatzky [1989]; Dario [1989]; Ellis [1990]; Loncaric et al. [1989]).

Robotic systems therefore, may include a mechanism that carries the sensors, has the ability to translate or reorient them and is able to perform fine, accurate and, at the same time, fast motions. Previous designs use either conventional robots or pan–and–tilt platforms as a mechanism for carrying the sensors (Ballard & Ozcandarli [1988]; Clark & Ferrier [1988]; Krotkov [1989]). But conventional serial manipulators generally lack the dynamic response characteristics to satisfy the above requirements on the speed and accuracy of motion, while pan–and–tilt platforms have limited degrees–of–freedom. An alternative is to implement this sensor–carrying platform as a *parallel* manipulator (Tsakiris & Aloimonos [1989]; Tsakiris & Krishnaprasad [1993]). An example of such a system is the so–called "Stewart platform", where six legs with linear motors support a platform carrying the sensors. By varying the length of the legs, the platform translates and/or reorients itself. This platform can be attached at the end–effector of a serial robot and provides a light–weight, yet strong and accurate system with full 6 degrees–of–freedom motion. There exists a vast literature on the mechanical design, the kinematics, the dynamics and the control of parallel manipulators (Cleary & Arai [1991]; Do & Yang [1988]; Fichter [1986]; Gosselin & Angeles [1988]; Hudgens & Tesar [1988]; Hunt [1983]; Merlet [1987]; Nanua, Waldron & Murthy [1990]; Pfreundschuh, Kumar & Sugar [1991]; Stewart [1966]; Sugimoto [1989]; Tsai & Tahmasebi [1991]).

In Chapter 2, we collect some tools from Differential Geometry, Geometric Mechanics and the theory of Elliptic functions, which will be needed in the sequel. We also review the kinematics of parallel manipulators.

In Chapter 3 we focus on the *motion planning problem* for parallel manipulators, which is a problem that has not received as much attention as the problems mentioned earlier. We will attempt to describe the geometry of the manifold of singular configurations on the configuration space for a particular design of parallel manipulator, which is generic enough to exhibit interesting kinematic properties, but simple enough to allow our results not to be overshadowed by complex calculations and use the results of motion planning for avoidance of those singular configurations.

Planning sequences of actions is one of the earliest areas of research in Artificial Intelligence, but most of this early work has only considered idealized domains, where a robot can be exactly controlled and where task–level commands are sufficient to describe the behavior of the system (Lozano–Perez [1987]; Nilsson [1980]). In robotic applications there is a need of planning the motions of robots in practical domains, where obstacles, uncertainties, errors and various constraints are present (Brady [1989]). In this spirit, the problem of planning collision–free motions in the workspace of a robot has been considered more recently. Furthermore, the problem of planning motions in cases where certain holonomic or non–holonomic constraints apply, or of motions where certain performance criteria based on kinematic or dynamic characteristics of the motion are optimized, are of significant interest (Latombe [1991]).

In the case of serial manipulators, planning in "joint space" is considered computationally more efficient compared to "Cartesian space" planning, because of the need, in the later case, to solve the inverse kinematics problem at run time (Craig [1986]). This makes a desired Cartesian space trajectory achievable only approximately (Brady et al. [1983]). However, for parallel manipulators this need not be the case, since the inverse kinematics is easily and uniquely solvable, thus planning in Cartesian space becomes feasible. Since the orientation of the platform is also of significant interest, especially when the platform carries vision or tactile sensors, we consider planning in the *configuration space* of the parallel manipulator. Moreover, sensory information interpretation subsystems typically use configuration space–based reasoning and this has to be taken into account in planning the motions of a robotic system.

Our main problem is the specification of a trajectory for the parallel manipulator system that has optimal shape characteristics and meets prespecified boundary conditions.

Such a trajectory is a curve in the configuration space, which is usually a subgroup of the Special Euclidean group $SE(2)$. We examine curves on those spaces, which are optimal in the sense of a curvature–square cost functional. Differential Geometric tools are used to transform this problem into a nonlinear optimal control problem on a left–invariant dynamical system and this is solved using a suitable generalization of the Maximum Principle. In the case of motions of planar parallel manipulators where the configuration space is a 2–dimensional manifold, we characterize the curvature of an optimal curve as the solution of a differential equation of the Bryant–Griffiths type, which, in the generic case, is given by elliptic functions. Parallel manipulators demonstrate an indeterminacy in the specification of their future behavior while they are at singular configurations, as well as an inability to resist forces and torques in specific directions, since, contrary to serial manipulators, they gain degrees–of–freedom in those configurations. Those are undesirable characteristics for our robotic system and we try, either to avoid kinematic singularities as much as possible by proper mechanical design and careful planning of the system trajectories, or to design our crossing of the singular surfaces in a specific way that will allow the dynamics of our system to resolve those indeterminacies. To this end, we map explicitly the singular surfaces of the parallel manipulators that we use and we employ the family of optimal curves derived from motion planning to optimize the singular surface crossing characteristics of the system trajectories. Extensions of this motion planning technique to the general planar case, where the configuration space is a 3–dimensional manifold are also discussed.

The problems in chapter 3 focused on systems with holonomic constraints, i.e. algebraic constraints involving the configuration variables of the system. The class of *nonholonomic constraints*, where the constraints involve the velocities of the system, but cannot be integrated to produce holonomic constraints, is currently receiving attention because of its importance in manipulation, mobile robots and locomotion problems (Latombe [1991]; Murray, Li & Sastry [1994]). A typical example is a car with wheels that roll without slipping on the ground. The motion of the car is restricted instantaneously, since it cannot slip sideways, however this does not restrict its position, since by the well–known parallel parking maneuver, the car can effectively move sideways over a finite time period. The main tool for checking whether a set of constraints (usually expressed as one–forms) is integrable, is Frobenius' Theorem, which states that for this to happen, the distribution which annihilates the constraints should be involutive, i.e. closed under Lie bracketing. In the remainder of this work we will consider systems where this is not true.

In Chapters 4, 5 and 6, we focus on mechanical systems subject to nonholonomic constraints, where variations of shape induce, under the influence of the constraints, a global motion of the system. A well–known example of such systems is a free–floating multibody system in space (e.g. robotic manipulators mounted on orbiting satellites), where periodic movements of the joints induce a reorientation of the system, under the nonholonomic constraint of conservation of angular momentum (Krishnaprasad [1990]; Marsden, Montgomery & Ratiu [1990]).

In Chapter 5, we describe in some detail a novel system, introduced in (Krishnaprasad & Tsakiris [1993]; Krishnaprasad & Tsakiris [1994b]), which uses the above principle for land locomotion. This was inspired by the experimental work of Joel Burdick and his students at Caltech (Chirikjian & Burdick [1991]; Chirikjian & Burdick [1993]). This system, called the Nonholonomic Variable Geometry Truss (NVGT) assembly, consists of longitudinal repetition of truss modules, each one of which is equipped with idler wheels and linear actuators in a planar parallel manipulator configuration, and uses periodic changes of the shape of each module to propel itself. The locomotion principle is not based on direct actuation of wheels, but rather on the nonholonomic constraints imposed on the motion of the system by the rolling without slipping of the idler wheels of each module on the supporting plane. This results in a snake–like motion of the NVGT assembly, which is not too far, at least in principle, from certain modes of actual snake locomotion (Hirose [1993]). Both the shape and the configuration of the NVGT assembly can be described by elements of the Special Euclidean group $SE(2)$, the group of rigid motions on the plane. A system like the NVGT assembly constitutes a kinematic chain evolving on this matrix Lie group G, with the corresponding velocities given by elements of the Lie algebra $\mathcal{G}$ of G. Of these velocities, the shape variations can be considered as the controls of the system and they are referred to as *shape controls.* The nonholonomic constraints allow us to express the global motion of the NVGT assembly as a function of the shape and of the shape controls and to formulate motion control strategies under periodic shape controls.

In Chapter 4, we describe *G–Snakes*, the generalization of NVGT assemblies to nonholonomic kinematics chains (or other kinematic topologies) on arbitrary Lie groups (c.f. (Krishnaprasad & Tsakiris [1994a]; Krishnaprasad & Tsakiris [1994c])). G–Snakes are ℓ–node kinematic chains, where each node evolves on a Lie group G, but its evolution is subject to nonholonomic constraints. The configuration space of such a system is $Q = G^\ell = \underbrace{G \times \cdots \times G}_{\ell \text{ times}}$. A pair of nodes constitutes a module of the G–Snake, whose shape (i.e. the relative configuration of the two nodes) also evolves on the same Lie group G. We consider the variations of these shapes as the controls of the system, assuming that a suitable mechanism is available for this purpose. The shape space $\mathcal{S}$ is $G^{\ell-1}$ and is such that $Q = \mathcal{S} \times G$.

Systems of this type have some interesting geometric properties. First, the manifolds Q and $\mathcal{S}$, the group G and the projection $\pi : Q \longrightarrow \mathcal{S}$ form a principal fiber bundle (Bleecker [1981]; Nomizu [1956]). Second, in the case that the number of nonholonomic constraints is equal to the dimension of G, the constraints specify a *connection* on the above principal fiber bundle. In particular, the intersection of the constraint distribution $\mathcal{D}_q$ at a point $q \in Q$ and of the tangent space $T_q Orb(q)$ to the orbits of the action of G on Q at q is trivial, while the direct sum of the two spaces is the tangent space $T_q Q$. Physically, this provides a splitting of the velocities of the G–Snake in two groups: one is related to shape variations and the other is related to the global motion of the system, as it is characterized by a group trajectory. The connection that the kinematic constraints specify, expresses the second group of velocities as a function of the first and of the shape itself. Notice that this result and the motion control schemes to which it gives rise, are entirely based on kinematics.

In Chapter 6, finally, we consider the *dynamics* of G–Snakes. A particular instance of the case when the number of constraints is one less than the dimension of the group, is presented, namely a 2–node system evolving on the 3–dimensional group $SE(2)$ with the shape space being S^1. The previous kinematic motion control is not feasible here, since there are not enough constraints to specify the principal fiber bundle connection. Even though the sum of the spaces $\mathcal{D}_q$ and $T_q Orb(q)$ is the tangent space $T_q Q$, the intersection of the two spaces is now non–trivial. It is possible however to use the system's Lagrangian dynamics to complement the kinematics. The notion of the nonholonomic momentum, introduced in (Bloch, Krishnaprasad, Marsden & Murray [1994]), can be used, together with the kinematic nonholonomic constraints, to synthesize a principal fiber bundle connection. The evolution of this momentum depends only on the shape variables of the system, therefore the connection again provides a way of specifying the global motion of the system (as it is characterized by a group trajectory) using only the shape and its variation. Successful dynamic motion control strategies depend now on properly varying the shape, so that the system builds up momentum (Krishnaprasad & Tsakiris [1995]).

Numerical simulation results are shown for both the kinematic and the dynamic motion control cases.

CHAPTER TWO

PRELIMINARIES

2.1 Introduction

In this chapter we collect some definitions and facts from the theory of Elliptic functions, from Differential Geometry, from the theory of Lie groups and Lie algebras and from Geometric Mechanics. We also summarize the kinematics of parallel manipulators, which are robotic devices based on closed kinematic chains and we study their kinematic singularities.

2.2 Elliptic Functions and Integrals

In Complex Analysis, a doubly periodic function, the ratio of whose periods is complex and which is regular except for poles, is called *elliptic.* The Jacobi elliptic functions defined below, have a real period, like the trigonometric functions, and an imaginary one, like the hyperbolic functions. Those functions are defined as the inverses, in a sense to be made precise below, of the elliptic integrals. The material in this section is drawn from (Byrd & Friedman [1954]; Davis [1962]; Lawden [1989]) and (Gradstein & Ryzhik [1980]).

Definition 2.2.1 (Elliptic Integrals of the First Kind)

The *elliptic integral of the first kind* is defined as:

$$F(\phi, k) = \int_0^{\phi} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} = \int_0^{\sin\phi} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}\,, \tag{2.2.1}$$

where k is the *modulus* of F and $k' = \sqrt{1-k^2}$ is the *complementary modulus* of F. The integral $K(k) \stackrel{\text{def}}{=} F(\frac{\pi}{2}, k) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$ is called the *complete* elliptic integral of the first kind. Consider also the *complementary* complete elliptic integral of the first kind $K'(k) \stackrel{\text{def}}{=} F(\frac{\pi}{2}, k') = \int\limits_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k'^2x^2)}}$. Observe that $K'(k) = K(k')$.

If the *nome* $q = \exp\left(-\pi\frac{K'}{K}\right)$ is real and $0 \le q < 1$, then $0 \le k < 1$ and $0 < k' \le 1$.

∎

Definition 2.2.2 (Elliptic Integrals of the Second Kind)

The *elliptic integral of the second kind* is defined as:

$$E(\phi, k) = \int\limits_0^\phi \sqrt{1 - k^2 \sin^2\theta}\, d\theta = \int\limits_0^{\sin\phi} \frac{\sqrt{1-k^2x^2}}{\sqrt{1-x^2}} dx \ . \qquad (2.2.2)$$

The integral $E(k) \stackrel{\text{def}}{=} E(\frac{\pi}{2}, k)$ is called the *complete* elliptic integral of the second kind. Consider also the *complementary* complete elliptic integral of the second kind $E'(k) \stackrel{\text{def}}{=} E(\frac{\pi}{2}, k')$. Observe that $E'(k) = E(k')$.

The complete and complementary complete elliptic integrals of the first and second kind are related by Legendre's relation $EK' + E'K - KK' = \frac{\pi}{2}$.

∎

Definition 2.2.3 (Elliptic Integrals of the Third Kind)

The *elliptic integral of the third kind* is defined as:

$$\Pi(\phi, \eta, k) = \int\limits_0^\phi \frac{d\theta}{(1-\eta\sin^2\theta)\sqrt{1-k^2\sin^2\theta}} = \int\limits_0^{\sin\phi} \frac{dx}{(1-\eta x^2)\sqrt{(1-x^2)(1-k^2x^2)}} \ . \qquad (2.2.3)$$

∎

The first form of F, E and Π is the Legendre normal form and the second is the Jacobi normal form.

Definition 2.2.4 (Jacobi Elliptic Functions)

The *Jacobi amplitude function* $\mathrm{am}(u, k)$ is defined as the solution for ϕ of

$$u = F(\phi, k) \qquad (2.2.4) \ .$$

The *Jacobi elliptic functions* sn, cn and dn are then defined as follows (c.f. fig. 2.2.1):

$$\begin{aligned} \mathrm{sn}(u,k) &= \sin\phi = \sin(\mathrm{am}(u,k))\ , \\ \mathrm{cn}(u,k) &= \cos\phi = \cos(\mathrm{am}(u,k))\ , \\ \mathrm{dn}(u,k) &= \frac{d\phi}{du} = \frac{d\,\mathrm{am}(u,k)}{du} = \sqrt{1-k^2\sin^2\phi}\ . \end{aligned} \tag{2.2.5}$$

Nine more elliptic functions are defined by taking reciprocals and quotients:

$$\begin{aligned} \mathrm{ns}\,u &= \frac{1}{\mathrm{sn}\,u}\ ,\ \mathrm{nc}\,u = \frac{1}{\mathrm{cn}\,u}\ ,\ \mathrm{nd}\,u = \frac{1}{\mathrm{dn}\,u}\ , \\ \mathrm{sc}\,u &= \frac{\mathrm{sn}\,u}{\mathrm{cn}\,u}\ ,\ \mathrm{cd}\,u = \frac{\mathrm{cn}\,u}{\mathrm{dn}\,u}\ ,\ \mathrm{ds}\,u = \frac{\mathrm{dn}\,u}{\mathrm{sn}\,u}\ , \\ \mathrm{cs}\,u &= \frac{\mathrm{cn}\,u}{\mathrm{sn}\,u}\ ,\ \mathrm{dc}\,u = \frac{\mathrm{dn}\,u}{\mathrm{cn}\,u}\ ,\ \mathrm{sd}\,u = \frac{\mathrm{sn}\,u}{\mathrm{dn}\,u}\ . \end{aligned} \tag{2.2.6}$$

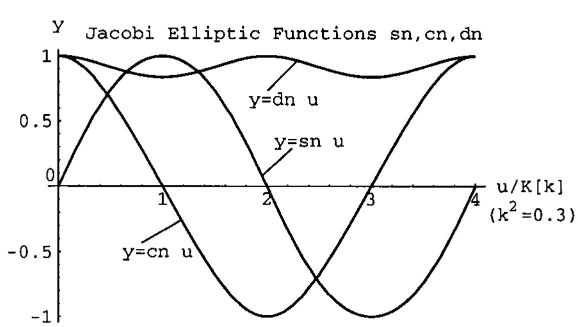
Then

$$\begin{aligned} u = F(\phi,k) &= \mathrm{sn}^{-1}(\sin\phi,k) = \mathrm{cn}^{-1}(\cos\phi,k) = \mathrm{dn}^{-1}(\sqrt{1-k^2\sin^2\phi},k) \\ &= \mathrm{am}^{-1}(\phi,k) = \mathrm{sc}^{-1}(\tan\phi,k)\ . \end{aligned} \tag{2.2.7}$$

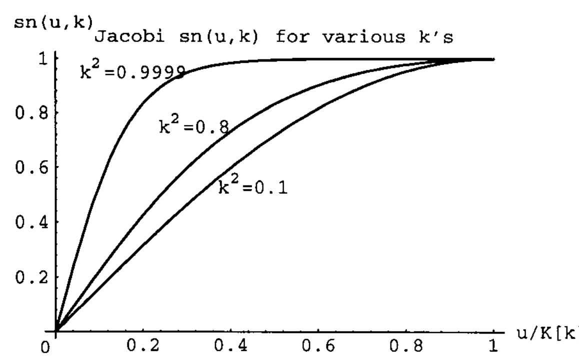
If the nome q is real, the elliptic functions are all real, for all real values of their parameter u.

∎

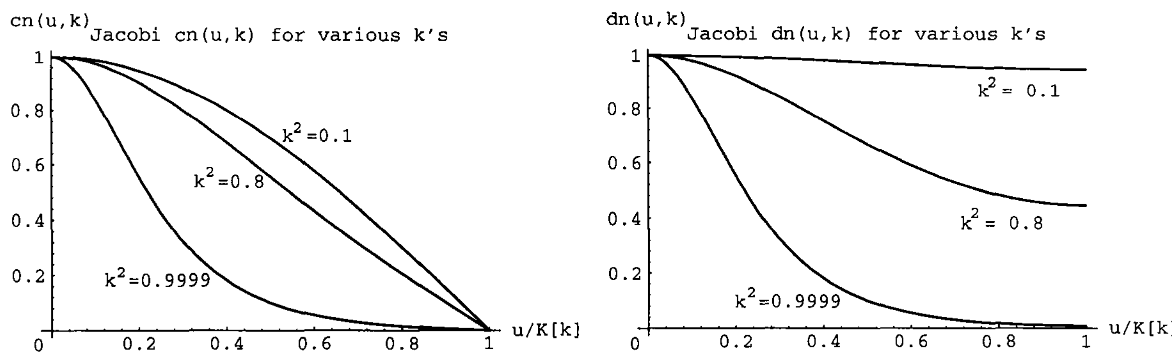
The dependence of the Jacobi elliptic functions on the modulus k is frequently suppressed.



(a)



(b)



(c) (d)

Fig. 2.2.1: Jacobi Elliptic Functions

Proposition 2.2.5 (Special Values)

If u is real, then $-1 \leq \operatorname{sn} u \leq 1$, $-1 \leq \operatorname{cn} u \leq 1$ and $k' \leq \operatorname{dn} u \leq 1$. Moreover,

$$\begin{aligned}
&\operatorname{am}(-u) = -\operatorname{am}(u)\ ,\ \operatorname{am}(0) = 0\ ,\ \operatorname{am}(K) = \frac{\pi}{2}\ ,\ \operatorname{am}(u+K) = \tan^{-1}(k' \operatorname{sc} u) + \frac{\pi}{2}\ , \\
&\operatorname{sn}(-u) = -\operatorname{sn}(u)\ ,\ \operatorname{sn}(0) = 0\ ,\ \operatorname{sn}(K) = 1\ ,\ \operatorname{sn}(u+K) = \operatorname{cd}(u)\ , \\
&\operatorname{cn}(-u) = \operatorname{cn}(u)\ ,\quad \operatorname{cn}(0) = 1\ ,\ \operatorname{cn}(K) = 0\ ,\ \operatorname{cn}(u+K) = -k' \operatorname{sd}(u)\ , \\
&\operatorname{dn}(-u) = \operatorname{dn}(u)\ ,\ \operatorname{dn}(0) = 1\ ,\ \operatorname{dn}(K) = k'\ ,\ \operatorname{dn}(u+K) = k' \operatorname{nd}(u)\ .
\end{aligned} \tag{2.2.8}$$

Some special values of the modulus are as follows:

$$\begin{aligned}
&\operatorname{sn}(u,0) = \sin u\ ,\ \operatorname{cn}(u,0) = \cos u\ ,\ \operatorname{dn}(u,0) = 1\ , \\
&\operatorname{sn}(u,1) = \tanh u\ ,\ \operatorname{cn}(u,1) = \operatorname{dn}(u,1) = \operatorname{sech} u\ .
\end{aligned} \tag{2.2.9}$$

∎

Proposition 2.2.6

$$\begin{aligned}
\operatorname{sn}^2 u + \operatorname{cn}^2 u &= 1\ , \\
\operatorname{dn}^2 u + k^2 \operatorname{sn}^2 u &= 1\ , \\
\operatorname{dn}^2 u - k^2 \operatorname{cn}^2 u &= k'^2\ .
\end{aligned} \tag{2.2.10}$$

Also

$$\operatorname{sn}^{-1} x + \operatorname{cd}^{-1} x = K(k) \,,$$
$$\operatorname{cn}^{-1} x + \operatorname{sd}^{-1}\left(\frac{x}{k'}\right) = K(k) \,, \tag{2.2.11}$$
$$\operatorname{nd}^{-1} x + \operatorname{dn}^{-1}(k'x) = K(k) \,.$$

∎

Proposition 2.2.7 (Addition)

$$\begin{aligned}
\operatorname{sn}(u \pm v) &= \frac{\operatorname{sn} u \operatorname{cn} v \operatorname{dn} v \pm \operatorname{sn} v \operatorname{cn} u \operatorname{dn} u}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v} \,, \\
\operatorname{cn}(u \pm v) &= \frac{\operatorname{cn} u \operatorname{cn} v \mp \operatorname{sn} u \operatorname{sn} v \operatorname{dn} u \operatorname{dn} v}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v} \,, \\
\operatorname{dn}(u \pm v) &= \frac{\operatorname{dn} u \operatorname{dn} v \mp k^2 \operatorname{sn} u \operatorname{sn} v \operatorname{cn} u \operatorname{cn} v}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v} \,.
\end{aligned} \tag{2.2.12}$$

∎

Proposition 2.2.8 (Periodicity)

The elliptic functions have two distinct periods and, if the nome is real, one period is real and the other is purely imaginary. The periods of $\operatorname{sn} u$ are $4K$ and $2iK'$, those of $\operatorname{cn} u$ are $4K$ and $2K + 2iK'$ and those of $\operatorname{dn} u$ are $2K$ and $4iK'$. Thus:

$$\begin{aligned}
&\operatorname{sn}(u + 4K) = \operatorname{sn}(u + 2iK') = \operatorname{sn} u \,, && \operatorname{sn}(u + 2K) = -\operatorname{sn} u \,, \\
&\operatorname{cn}(u + 4K) = \operatorname{cn}(u + 2K + 2iK') = \operatorname{cn} u \,, && \operatorname{cn}(u + 2K) = -\operatorname{cn} u \,, \\
&\operatorname{dn}(u + 2K) = \operatorname{dn}(u + 4iK') = \operatorname{dn} u \,, && \operatorname{dn}(u + K) = \frac{k'}{\operatorname{dn} u} \,.
\end{aligned} \tag{2.2.13}$$

∎

Proposition 2.2.9 (Derivatives, Integrals and Differential Equations)

$$\begin{aligned}
&\frac{d}{du} \operatorname{sn} u = \operatorname{cn} u \operatorname{dn} u \,, && \frac{d}{du} \operatorname{dn} u = -k^2 \operatorname{sn} u \operatorname{cn} u \,, \\
&\frac{d}{du} \operatorname{cn} u = -\operatorname{sn} u \operatorname{dn} u \,, && \frac{d}{du} \operatorname{am} u = \operatorname{dn} u \,,
\end{aligned}$$

$$\int \operatorname{sn} u\, du = \frac{1}{k}\ln(\operatorname{dn} u - k\operatorname{cn} u)\ , \quad \int \operatorname{sn}^2 u\, du = \frac{1}{k^2}\left[u - E(\operatorname{am} u, k)\right]\ ,$$
$$\int \operatorname{cn} u du = \frac{1}{k}\sin^{-1}(k\operatorname{sn} u)\ , \quad \int \operatorname{cn}^2 u\, du = \frac{1}{k^2}\left[E(\operatorname{am} u, k) - k'^2 u\right]\ , \tag{2.2.14}$$
$$\int \operatorname{dn} u du = \sin^{-1}(\operatorname{sn} u)\ , \quad \int \operatorname{dn}^2 u\, du = E(\operatorname{am} u, k)\ ,$$

$$\int \operatorname{sn} u \operatorname{cn} u\, du = -\frac{1}{k^2}\operatorname{dn} u\ , \quad \int \operatorname{sn} u \operatorname{dn} u\, du = -\operatorname{cn} u\ ,$$
$$\int \operatorname{cn} u \operatorname{dn} u\, du = \operatorname{sn} u\ .$$

The Jacobi elliptic functions are solutions of the following first–order differential equations:

$$\frac{d}{du}\operatorname{sn} u = \sqrt{(1-\operatorname{sn}^2 u)(1-k^2\operatorname{sn}^2 u)}\ ,$$
$$\frac{d}{du}\operatorname{cn} u = -\sqrt{(1-\operatorname{cn}^2 u)(k'^2+k^2\operatorname{cn}^2 u)}\ , \tag{2.2.15}$$
$$\frac{d}{du}\operatorname{dn} u = -\sqrt{(1-\operatorname{dn}^2 u)(\operatorname{dn}^2 u - k'^2)}\ ,$$

and of the following second–order differential equations:

$$\frac{d^2}{du^2}\operatorname{sn} u = -(1+k^2)\operatorname{sn} u + 2k^2\operatorname{sn}^3 u\ ,$$
$$\frac{d^2}{du^2}\operatorname{cn} u = (2k^2-1)\operatorname{cn} u - 2k^2\operatorname{cn}^3 u\ , \tag{2.2.16}$$
$$\frac{d^2}{du^2}\operatorname{dn} u = (2-k^2)\operatorname{dn} u - 2\operatorname{dn}^3 u\ ,$$

∎

Proposition 2.2.10 (Approximations)

For $k \ll 1$, we can approximate the Jacobi elliptic functions as follows:

$$\operatorname{sn}(u,k) \approx \sin u - k^2\cos u(u - \sin u\cos u)/4\ ,$$
$$\operatorname{cn}(u,k) \approx \cos u + k^2\sin u(u - \sin u\cos u)/4\ , \tag{2.2.17}$$
$$\operatorname{dn}(u,k) \approx 1 - (k^2\sin^2 u)/2\ .$$

For $k \approx 1$, we can approximate the Jacobi elliptic functions as follows:

$$\begin{aligned}
\operatorname{sn}(u,k) &\approx \tanh u + k'^2 \operatorname{sech}^2 u(\sinh u \cosh u - u)/4\,, \\
\operatorname{cn}(u,k) &\approx \operatorname{sech} u - k'^2 \tanh u \operatorname{sech} u(\sinh u \cosh u - u)/4\,, \\
\operatorname{dn}(u,k) &\approx \operatorname{sech} u + k'^2 \tanh u \operatorname{sech} u(\sinh u \cosh u + u)/4\,.
\end{aligned} \tag{2.2.18}$$

∎

Proposition 2.2.11 (Elliptic Integrals of the First Kind)

Assume u and k are real with $0 < k < 1$. Let $u = \operatorname{sn}^{-1}(x,k)$, then $x = \operatorname{sn} u$. Observe that $\operatorname{sn}^{-1}(x,k)$ is a multivalued function (since $\operatorname{sn}(u+4K) = \operatorname{sn} u$ and $\operatorname{sn}(2K-u) = \operatorname{sn} u$). Restricting to $0 \le x \le 1$ and the corresponding range $0 \le u \le K$, from the Legendre canonical form of the elliptic integral of the first kind, we get:

$$\operatorname{sn}^{-1}(x,k) = u = \int_0^x \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}\,. \tag{2.2.19}$$

The following elliptic integrals, which can be evaluated in terms of the Jacobi elliptic functions, are also reducible, by a suitable change of variables, to equation (2.2.19):

$$\int_0^x \frac{dt}{\sqrt{(a^2-t^2)(b^2-t^2)}} = \frac{1}{a}\operatorname{sn}^{-1}(\frac{x}{b},\frac{b}{a})\,, \tag{2.2.20}$$

where $0 \le x \le b < a$,

$$\int_x^b \frac{dt}{\sqrt{(a^2+t^2)(b^2-t^2)}} = \frac{1}{\sqrt{a^2+b^2}}\operatorname{cn}^{-1}(\frac{x}{b},\frac{b}{\sqrt{a^2+b^2}})\,, \tag{2.2.21}$$

where $0 \le x \le b$,

$$\int_x^b \frac{dt}{\sqrt{(a^2-t^2)(b^2-t^2)}} = \frac{1}{a}\operatorname{cd}^{-1}(\frac{x}{b},\frac{b}{a})\,, \tag{2.2.22}$$

where $0 \le x \le b < a$,

$$\int_0^x \frac{dt}{\sqrt{(a^2+t^2)(b^2-t^2)}} = \frac{1}{\sqrt{a^2+b^2}}\operatorname{sd}^{-1}(\frac{\sqrt{a^2+b^2}\,x}{ab},\frac{b}{\sqrt{a^2+b^2}})\,, \tag{2.2.23}$$

where $0 \le x \le b$,

$$\int_a^x \frac{dt}{\sqrt{(t^2-a^2)(t^2-b^2)}} = \frac{1}{a}\,\mathrm{dc}^{-1}(\frac{x}{a},\frac{b}{a}) \; , \tag{2.2.24}$$

where $b < a \le x$,

$$\int_x^\infty \frac{dt}{\sqrt{(t^2-a^2)(t^2-b^2)}} = \frac{1}{a}\,\mathrm{ns}^{-1}(\frac{x}{a},\frac{b}{a}) \; , \tag{2.2.25}$$

where $b < a \le x$,

$$\int_b^x \frac{dt}{\sqrt{(a^2-t^2)(t^2-b^2)}} = \frac{1}{a}\,\mathrm{nd}^{-1}(\frac{x}{b},\frac{\sqrt{a^2-b^2}}{a}) \; , \tag{2.2.26}$$

where $b \le x \le a$,

$$\int_x^a \frac{dt}{\sqrt{(a^2-t^2)(t^2-b^2)}} = \frac{1}{a}\,\mathrm{dn}^{-1}(\frac{x}{a},\frac{\sqrt{a^2-b^2}}{a}) \; , \tag{2.2.27}$$

where $b \le x \le a$,

$$\int_a^x \frac{dt}{\sqrt{(t^2-a^2)(t^2+b^2)}} = \frac{1}{\sqrt{a^2+b^2}}\,\mathrm{nc}^{-1}(\frac{x}{a},\frac{b}{\sqrt{a^2+b^2}}) \; , \tag{2.2.28}$$

where $a \le x$,

$$\int_x^\infty \frac{dt}{\sqrt{(t^2-a^2)(t^2+b^2)}} = \frac{1}{\sqrt{a^2+b^2}}\,\mathrm{ds}^{-1}(\frac{x}{\sqrt{a^2+b^2}},\frac{b}{\sqrt{a^2+b^2}}) \; , \tag{2.2.29}$$

where $a \le x$,

$$\int_0^x \frac{dt}{\sqrt{(t^2+a^2)(t^2+b^2)}} = \frac{1}{a}\,\mathrm{sc}^{-1}(\frac{x}{b},\frac{\sqrt{a^2-b^2}}{a}) \; , \tag{2.2.30}$$

where $0 < b < a$, $0 \le x$,

$$\int_x^\infty \frac{dt}{\sqrt{(t^2+a^2)(t^2+b^2)}} = \frac{1}{a}\,\mathrm{cs}^{-1}(\frac{x}{a},\frac{\sqrt{a^2-b^2}}{a}) \; , \tag{2.2.31}$$

where $0 < b < a$, $0 \le x$.

∎

2.3 Differential Geometry and Lie Groups

In this section we define Lie groups and Lie algebras and discuss some of their properties that make them relevant to the study of mechanical systems with constraints and with symmetries. The material in this section is drawn from (Abraham & Marsden [1985]; Boothby [1975]; Curtis [1984]; Jacobson [1979]; Lang [1985]; Marsden & Ratiu [1994]; Olver [1986]; Spivak [1970]; Varadarajan [1984]; Warner [1971]).

Definition 2.3.1 (Group)

A *group* $(G, \cdot)$ is a set G, together with an operation $\cdot$ such that for any $g, h \in G$, the product $g \cdot h$ is also an element of G and such that the following properties hold:

i) Associativity: For $f, g, h \in G: \ f \cdot (g \cdot h) = (f \cdot g) \cdot h$

ii) Identity: There is an element $e \in G$ such that for all $g \in G: \ g \cdot e = e \cdot g = g$

iii) Inverse: For each $g \in G$ there is an element of G, denoted g^{-1} such that $g \cdot g^{-1} = g^{-1} \cdot g = e$

∎

Let G and G' be groups. A map $\phi : G \longrightarrow G'$ is a *homomorphism* if $\phi(g \cdot g') = \phi(g) \cdot \phi(g')$. A homomorphism is *surjective* (or *onto*), when $\phi(G) = G'$. It is *injective* (or *one-to-one*), when, if $\phi(g) = \phi(g')$, then $g = g'$. A necessary and sufficient condition for this to happen is that $\phi^{-1}(e') = e$. A homomorphism is *bijective*, if it is onto and one-to-one. A map $\phi : G \longrightarrow G'$ is an *isomorphism* if it is a bijective homomorphism. A map $\phi : G \longrightarrow G$ is an *automorphism* if it is an isomorphism.

Let X and Y be smooth vector fields on a manifold M. Let $X_m, Y_m \in T_m M$, for $m \in M$. For a smooth function $f : M \longrightarrow I\!R$, the operator $f \longrightarrow X_m(Y(f))$ does not define, in general, a tangent vector at m. However, $XY - YX$ does (Boothby [1975]).

Definition 2.3.2 (Jacobi–Lie bracket of vector fields)

Let X and Y be smooth vector fields on a differentiable manifold M. We define a new vector field $[\![X, Y]\!]$, called the *Jacobi–Lie bracket* of X and Y by

$$[\![X, Y]\!]_m(f) = X_m(Y(f)) - Y_m(X(f)) \ . \tag{2.3.1}$$

for $m \in M$ and for all smooth functions $f : M \longrightarrow I\!R$.

∎

For a smooth vector field $\mathcal{X}$ on G, let $\phi_g : \mathbb{R} \to \mathrm{G}$, $t \mapsto \phi_g(t)$ be a *tangent curve* of $\mathcal{X}$ at a fixed point $g \in \mathrm{G}$, i.e. a smooth curve on G such that $\phi_g(0) = g$ and $\frac{d}{dt}\Big|_{t=0} \phi_g(t) = \mathcal{X}(g)$.

Proposition 2.3.3

Consider two curves ϕ_g^1 and ϕ_g^2 tangent to the smooth vector fields $\mathcal{X}_1$ and $\mathcal{X}_2$ respectively, at the point $g \in \mathrm{G}$. Then: $[\![\mathcal{X}_1, \mathcal{X}_2]\!](f)(g) = \frac{d}{dt}\Big|_{t=0} \left[\mathcal{X}_2\left(\phi_g^1(t)\right) - \mathcal{X}_1\left(\phi_g^2(t)\right)\right](f)(g)$.

∎

Definition 2.3.4 (Lie Group)

A *Lie group* is a finite–dimensional smooth manifold G that is also a group and for which the group operations of multiplication $\cdot : G \times G \longrightarrow G : (g, h) \mapsto g \cdot h$ and inversion ${}^{-1} : G \longrightarrow G : g \mapsto g^{-1}$ are smooth. Let e be the group identity.

∎

The definitions of homomorphisms, isomorphisms and automorphisms of Lie groups are similar to the ones for general groups, except that the maps ϕ should be smooth. Then, if ϕ is an isomorphism, it is a diffeomorphism. If $G' = \mathrm{Aut}(V)$, where $\mathrm{Aut}(V)$ is the set of linear nonsingular operators (automorphisms) on a vector space V or if $G' = Gl(n, \mathbb{R})$, then a homomorphism $\phi : G \longrightarrow G'$ is called a *representation* of the Lie group G.

Definition 2.3.5 (Direct and Semidirect Products of Lie Groups)

The *direct product* $G \times H$ of two Lie groups G and H is a new Lie group with the product manifold structure and the direct product structure, which is given for $g_1, g_2 \in G$ and $h_1, h_2 \in H$ by

$$(g_1, h_1) \cdot (g_2, h_2) = (g_1 \cdot g_2, h_1 \cdot h_2) \ . \tag{2.3.2}$$

For Lie groups G and H, suppose G acts on H as a group of transformations via $h \mapsto g \cdot h$, for $h \in H$ with $g \cdot (h_1 \cdot h_2) = (g \cdot h_1) \cdot (g \cdot h_2)$. The *semidirect product* $G \times_S H$ of G and H is the Lie group whose manifold structure is the Cartesian product $G \times H$, but whose group multiplication is

$$(g_1, h_1) \cdot (g_2, h_2) = \big(g_1 \cdot g_2, h_1 \cdot (g_1 \cdot h_2)\big) \ . \tag{2.3.3}$$

∎

For example, the Special Euclidean group $SE(n)$, i.e. the group of rigid motions in $\mathbb{R}^n$, is the semidirect product of the Special Orthogonal group $SO(n)$ with the vector space $\mathbb{R}^n$, where $SO(n)$ acts on $\mathbb{R}^n$ as a group of rotations, with group multiplication $(g_1, h_1) \cdot (g_2, h_2) = (g_1 \cdot g_2, h_1 + g_1 \cdot h_2)$, for $g_1, g_2 \in SO(n)$ and $h_1, h_2 \in \mathbb{R}^n$.

Definition 2.3.6 (Lie Algebra)

A *Lie algebra* is a real vector space $\mathcal{G}$ equipped with a product $[.,.] : \mathcal{G} \times \mathcal{G} \longrightarrow \mathcal{G}$ with the following properties:

i) Bilinearity: For $a, b \in \mathbb{R}$ and $x, y, z \in \mathcal{G}$: $[ax + by, z] = a[x, z] + b[y, z]$, $[z, ax + by] = a[z, x] + b[z, y]$.

ii) Skew–symmetry: For $x, y \in \mathcal{G}$: $[x, y] = -[y, x]$.

iii) Jacobi identity: For $x, y, z \in \mathcal{G}$: $[x, [y, z]] + [y, [z, x] + [z, [x, y]] = 0$.

∎

If $\mathcal{G}$ and $\mathcal{G}'$ are Lie algebras, a map $\psi : \mathcal{G} \longrightarrow \mathcal{G}'$ is a (Lie algebra) homomorphism, if it is linear and it preserves the bracket, i.e. $\psi([X, Y]) = [\psi(X), \psi(Y)]$, for every $X, Y \in \mathcal{G}$. If, in addition, ψ is a bijection, then it is an isomorphism. An isomorphism of $\mathcal{G}$ with itself is an automorphism. If $\mathcal{G}' = \mathrm{End}(V)$, the set of all linear operators (endomorphisms) on a vector space V or if $\mathcal{G}' = gl(n, \mathbb{R})$, then a homomorphism $\psi : \mathcal{G} \longrightarrow \mathcal{G}'$ is called a representation of the Lie algebra $\mathcal{G}$.

A subspace $\mathcal{B}$ of a Lie algebra $\mathcal{G}$ is a *subalgebra* if $[x, y] \in \mathcal{B}$ for all $x, y \in \mathcal{B}$. It is an *ideal* if $[x, y] \in \mathcal{B}$ for all $x \in \mathcal{G}$ and $y \in \mathcal{B}$. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be subspaces of a Lie algebra $\mathcal{G}$ and denote by $[\mathcal{B}_1, \mathcal{B}_2]$ the span of all elements $[b_1, b_2]$ such that $b_1 \in \mathcal{B}_1$ and $b_2 \in \mathcal{B}_2$. Define the *derived series* of $\mathcal{G}$ by

$$\mathcal{G} \supseteq \mathcal{G}' = [\mathcal{G}, \mathcal{G}] \supseteq \mathcal{G}'' = [\mathcal{G}', \mathcal{G}'] \supseteq \cdots \supseteq \mathcal{G}^{(k)} = [\mathcal{G}^{(k-1)}, \mathcal{G}^{(k-1)}] \supseteq \cdots$$

Its terms are ideals. So are the terms of the *lower central series* of $\mathcal{G}$ defined by

$$\mathcal{G} \supseteq \mathcal{G}^2 = [\mathcal{G}, \mathcal{G}] \supseteq \mathcal{G}^3 = [\mathcal{G}^2, \mathcal{G}] \supseteq \cdots \supseteq \mathcal{G}^k = [\mathcal{G}^{k-1}, \mathcal{G}] \supseteq \cdots$$

Definition 2.3.7 (Solvable Lie Algebras)

A Lie algebra $\mathcal{G}$ is *solvable* if $\mathcal{G}^{(n)} = 0$, for some positive integer n.

∎

Abelian algebras are solvable. All Lie algebras $\mathcal{G}$ with dim$\mathcal{G} \leq 3$ and dim$\mathcal{G}' < 3$ are solvable (e.g. $se(2)$). Also, the algebra of triangular matrices is solvable.

Definition 2.3.8 (Nilpotent Lie Algebras)

A Lie algebra $\mathcal{G}$ is *nilpotent* if $\mathcal{G}^n = 0$, for some positive integer n.

∎

The algebra $h(n)$ of nil triangular $n \times n$ matrices (triangular matrices where also the diagonal is zero) is nilpotent. If $\mathcal{G}$ is nilpotent, then it is also solvable (since $\mathcal{G}^{(k)} \subseteq \mathcal{G}^{2^k}$). The converse does not hold.

Definition 2.3.9 (Abelian and Non–Abelian Lie Algebras)

A Lie algebra $\mathcal{G}$ is *abelian* if $\mathcal{G}' = 0$. Otherwise, it is called *non–abelian.*

∎

Definition 2.3.10 (Simple Lie Algebras)

A Lie algebra $\mathcal{G}$ is *simple* if it is non–abelian and its only ideals are 0 and $\mathcal{G}$.

∎

Definition 2.3.11 (Semi–Simple Lie Algebras)

A Lie algebra $\mathcal{G}$ is *semi–simple* if its only abelian ideal is 0.

∎

Simplicity implies semi–simplicity. All 3–dimensional Lie algebras such that dim$\mathcal{G}' = 3$ (i.e. $\mathcal{G}' = \mathcal{G}$) are simple (e.g. $so(3), sl(2)$).

Definition 2.3.12

i) The *left* and *right translation* by $g \in G$ are the maps $L_g : G \to G : h \mapsto gh$ and $R_g : G \to G : h \mapsto hg$. Both are diffeomorphisms.

ii) A vector field X on G is called *left–invariant* if for every $g \in G$, $(L_g)_* X = X$, where $(L_g)_*$ is the differential of L_g. Equivalently, $T_h L_g X(h) = X(gh)$, for every $h \in G$, where $T_h L_g$ is the linearization of the map L_g at h.

∎

The set of smooth vector fields on a manifold M, with the $[\![.,.]\!]$ bracket defined in equation (2.3.1), is a Lie algebra (possibly infinite–dimensional). When M is a Lie group, then this Lie algebra has an important finite–dimensional subalgebra, which is called *the* Lie algebra of M and is defined below.

Consider a Lie group G. For $\xi \in T_eG$, define on G the left–invariant vector field $X_\xi(g) = T_eL_g\xi$, for $g \in G$. Then, T_eG, together with the Lie bracket $[.,.]$ defined by

$$[\xi, \eta] = [\![X_\xi, X_\eta]\!](e) \; , \tag{2.3.4}$$

for $\xi, \eta \in T_eG$, becomes a Lie algebra. Moreover, $[\![X_\xi, X_\eta]\!] = X_{[\xi,\eta]}$.

Definition 2.3.13 (Lie Algebra of Lie Group)

The vector space T_eG with the Lie bracket $[.,.]$ defined in equation (2.3.4) is called the *Lie algebra of* G and denoted $\mathcal{G}$ (or $\mathcal{L}(G)$).

∎

For $\xi \in \mathcal{G}$, let $\phi_\xi : I\!R \longrightarrow G : t \mapsto \exp t\xi$ denote the *one–parameter subgroup* of G, i.e. the integral curve of the left–invariant vector field X_ξ passing through the identity at $t = 0$.

For matrix Lie groups G we have for $g \in G$ and $\xi \in \mathcal{G}$:

$$T_eL_g \cdot \xi = L_g\xi = g\xi \; . \tag{2.3.5}$$

Moreover, for $h \in G$ and for $X_h \in T_hG$ we have:

$$\begin{aligned} T_hL_g : T_hG &\longrightarrow T_{gh}G \\ X_h &\mapsto T_hL_g \cdot X_h = L_gX_h = gX_h \end{aligned} \; . \tag{2.3.6}$$

Consider a left–invariant dynamical system on a matrix Lie group G with n–dimensional Lie algebra $\mathcal{G}$. Consider a curve $g(.) \subset G$. Then, there exists a curve $\xi(.) \in \mathcal{G}$ such that:

$$\dot{g} = T_eL_g \cdot \xi = L_g\xi = g\xi \; . \tag{2.3.7}$$

Let $\{\mathcal{A}_i, \; i = 1, \ldots, n\}$ be a basis of $\mathcal{G}$ and let $[.,.]$ be the usual Lie bracket on $\mathcal{G}$ defined by: $[\mathcal{A}_i, \mathcal{A}_j] = \mathcal{A}_i\mathcal{A}_j - \mathcal{A}_j\mathcal{A}_i$. Then, there exist constants $\Gamma^k_{i,j}$, called *structure constants,* such that:

$$[\mathcal{A}_i, \mathcal{A}_j] = \sum_{k=1}^{n} \Gamma^k_{i,j}\mathcal{A}_k \; , \;\; i, j = 1, \ldots, n. \tag{2.3.8}$$

Let $\mathcal{G}^*$ be the dual space of $\mathcal{G}$, i.e. the space of linear functions from $\mathcal{G}$ to $I\!R$. Let $\{\mathcal{A}_i^\flat,\ i = 1, \ldots, n\}$ be the basis of $\mathcal{G}^*$ such that

$$\mathcal{A}_i^\flat(\mathcal{A}_j) = \delta_i^j, \quad i,j = 1, \ldots, n, \tag{2.3.9}$$

where δ_i^j is the Kronecker symbol. Then the curve $\xi(.) \subset \mathcal{G}$ can be represented as:

$$\xi = \sum_{i=1}^{n} \xi_i \mathcal{A}_i = \sum_{i=1}^{n} \mathcal{A}_i^\flat(\xi) \mathcal{A}_i \ , \tag{2.3.10}$$

for $\xi_i \overset{\text{def}}{=} \mathcal{A}_i^\flat(\xi) \in I\!R,\ i = 1, \ldots, n.$

To obtain a solution of the dynamical system (2.3.7) we use the following product–of–exponentials representation.

Proposition 2.3.14 (Wei & Norman [1964])

Let $g(0) = I$, the identity of G and let $g(t)$ be the solution of (2.3.7). Then, locally around $t = 0$, g is of the form:

$$g(t) = e^{\gamma_1(t)\mathcal{A}_1} e^{\gamma_2(t)\mathcal{A}_2} \ldots e^{\gamma_n(t)\mathcal{A}_n} \ , \tag{2.3.11}$$

where the coefficients γ_i are determined by differentiating (2.3.11) and using (2.3.7). For the coordinates ξ_i of $\xi \in \mathcal{G}$ defined in (2.3.10), we get:

$$\begin{pmatrix} \dot{\gamma}_1 \\ \vdots \\ \dot{\gamma}_n \end{pmatrix} = M(\gamma_1, \ldots, \gamma_n) \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix} . \tag{2.3.12}$$

The matrix M is analytic in γ and depends only on the Lie algebra $\mathcal{G}$ and its structure constants in the given basis. If $\mathcal{G}$ is solvable, then there exists a basis of $\mathcal{G}$ and an ordering of this basis, for which (2.3.11)is global. Then the γ_i's can be found by quadratures.

∎

Definition 2.3.15 (Action of Lie Group on Manifold)

Let Q be a smooth manifold. A (left) *action* of a Lie group G on Q is a smooth mapping $\Phi : G \times Q \longrightarrow Q$ such that:

i) For all $q \in Q$, $\Phi(e, x) = x$.

ii) For every $g, h \in G$, $\Phi(g, \Phi(h, q)) = \Phi(gh, q)$, for all $q \in Q$.

∎

For every $g \in G$, define $\Phi_g : Q \longrightarrow Q : q \longrightarrow \Phi(g, q)$. Then from (i), $\Phi_e = id_Q$ and from (ii), $\Phi_{gh} = \Phi_g \circ \Phi_h$. Moreover, $(\Phi_g)^{-1} = \Phi_{g^{-1}}$. Thus, Φ_g is a diffeomorphism (i.e. one–to–one, onto and both Φ_g and $(\Phi_g)^{-1}$ are smooth).

Definitions 2.3.16

Let Φ be an action of G on Q.

i) For $q \in Q$, the *orbit* (or Φ–orbit) of q is $\mathrm{Orb}(q) = \{\Phi_g(q) | g \in G\}$.

ii) An action is *transitive* if there is only one orbit.

iii) An action Φ is *effective* (or *faithful*) if $g \mapsto \Phi_g$ is one–to–one.

iv) An action Φ is *free* if, for each $q \in Q$, $g \mapsto \Phi_g(q)$ is one–to–one, i.e. the identity e is the only element of G with a fixed point.

v) An action Φ is *proper* if and only if the map $\Phi : G \times Q \longrightarrow Q \times Q : (g, q) \mapsto (q, \Phi(g, q)) = \tilde{\Phi}(g, q)$ is proper, i.e. if $K \subset Q \times Q$ is compact, then $\tilde{\Phi}^{-1}(K)$ is compact.

∎

Definition 2.3.17 (Infinitesimal Generator)

Let $\Phi : G \times Q \longrightarrow Q$ be a smooth action. If $\xi \in \mathcal{G}$, then $\Phi^{\xi} : \mathbb{R} \times Q \longrightarrow Q : (t, q) \mapsto \Phi(\exp t\xi, q)$ is an $\mathbb{R}$–action on Q, i.e. is a flow on Q. The corresponding vector field on Q is called the *infinitesimal generator* of Φ corresponding to ξ and is given by

$$\xi_Q(q) = \frac{d}{dt}\Phi(\exp t\xi, q)\Big|_{t=0} . \tag{2.3.13}$$

Then, the tangent space to the orbit $\mathrm{Orb}(q)$ of q is

$$T_q\mathrm{Orb}(q) = \{\xi_Q(q) | \xi \in \mathcal{G}\} . \tag{2.3.14}$$

∎

Examples 2.3.18

i) Let Φ be the left translation of a matrix Lie group G, considered as an action of G on G, i.e. $\Phi : G \times G \longrightarrow G : (g, h) \mapsto gh = L_g h$. Then the infinitesimal generator ξ_G corresponding to $\xi \in \mathcal{G}$ is $\xi_G(q) = T_e R_g \cdot \xi = \xi g$, which is a right invariant vector field.

ii) Consider the *adjoint action* of G on its Lie algebra $\mathcal{G}$, defined as $\Phi : G \times \mathcal{G} \longrightarrow \mathcal{G} : (g, \eta) \mapsto Ad_g \eta = T_e(R_{g^{-1}} L_g)\eta$. If G is a matrix group, then

$$Ad_g \eta = g \eta g^{-1} , \tag{2.3.15}$$

for $\eta \in \mathcal{G}$. If $\xi \in \mathcal{G}$, then

$$\xi_{\mathcal{G}} \eta = ad_\xi \eta \stackrel{\text{def}}{=} [\xi, \eta] . \tag{2.3.16}$$

Further define $ad_\xi^k \eta \stackrel{\text{def}}{=} ad_\xi(ad_\xi^{k-1}\eta) = [\xi, ad_\xi^{k-1}\eta]$.

The following properties of the adjoint action of a matrix Lie group are easily checked:

i) $Ad_g(a_1\eta_1 + a_2\eta_2) = a_1 Ad_g \eta_1 + a_2 Ad_g \eta_2$, for $a_1, a_2 \in I\!R$ and $\eta_1, \eta_2 \in \mathcal{G}$.

ii) $Ad_{g_1 g_2}(\eta) = Ad_{g_1}(Ad_{g_2}\eta)$, for $g_1, g_2 \in G$.

iii) $Ad_g(\eta_1\eta_2) = Ad_g\eta_1 Ad_g\eta_2$, for $\eta_1, \eta_2 \in \mathcal{G}$.

∎

Definition 2.3.19 (Equivariance)

Let M and N be manifolds and G a Lie group. Let Φ and Ψ be actions of G on M and N respectively and $f : M \longrightarrow N$ be a smooth map. Then f is *equivariant* with respect to those actions, if for all $g \in G :\ f \circ \Phi_g = \Psi_g \circ f$.

∎

Proposition 2.3.20

Let $f : M \longrightarrow N$ be equivariant with respect to actions Φ and Ψ as above. Then, for any $\xi \in \mathcal{G}$, we have $Tf \cdot \xi_M = \xi_N \cdot f$, where ξ_M and ξ_N are infinitesimal generators of Φ and Ψ respectively on M and N corresponding to ξ.

∎

From (2.3.10) we have:

$$Ad_g \xi = \sum_{i=1}^{n} \xi_i Ad_g \mathcal{A}_i = \sum_{i=1}^{n} \mathcal{A}_i^\flat(\xi) Ad_g \mathcal{A}_i . \tag{2.3.17}$$

From (2.3.11) we have:

$$Ad_g \mathcal{A}_i = g \mathcal{A}_i g^{-1} = e^{\gamma_1 \mathcal{A}_1} \cdots e^{\gamma_n \mathcal{A}_n} \mathcal{A}_i e^{-\gamma_n \mathcal{A}_n} \cdots e^{-\gamma_1 \mathcal{A}_1} \tag{2.3.18}$$

and

$$Ad_{g^{-1}}\mathcal{A}_i = g^{-1}\mathcal{A}_i g = e^{-\gamma_n \mathcal{A}_n} \cdots e^{-\gamma_1 \mathcal{A}_1} \mathcal{A}_i e^{\gamma_1 \mathcal{A}_1} \cdots e^{\gamma_n \mathcal{A}_n} \tag{2.3.19}$$

Equation (2.3.18) can be made more explicit by the Baker–Campbell–Hausdorff formula (Wei & Norman [1964]), which for $\xi, \eta \in \mathcal{G}$ states that:

$$Ad_{\exp \xi}\eta = e^{\xi}\eta e^{-\xi} = \left(e^{ad_\xi}\right)\eta = \eta + [\xi, \eta] + \frac{1}{2!}[\xi, [\xi, \eta]] + \cdots \ . \tag{2.3.20}$$

Thus

$$Ad_g \mathcal{A}_i = e^{ad(\gamma_1 \mathcal{A}_1)} \cdots e^{ad(\gamma_n \mathcal{A}_n)} \mathcal{A}_i \tag{2.3.21}$$

and

$$Ad_{g^{-1}}\mathcal{A}_i = e^{ad(-\gamma_n \mathcal{A}_n)} \cdots e^{ad(-\gamma_1 \mathcal{A}_1)} \mathcal{A}_i \ , \tag{2.3.22}$$

for $i = 1, \ldots, n$.

The following material on principal fiber bundles and connections is based on (Bleecker [1981]; Nomizu [1956]). These references consider principal fiber bundles where the group action is a right action. Here we consider left actions and modify appropriately the definition of a principal fiber bundle following (Yang [1992]).

Definition 2.3.21 (Principal Fiber Bundle)

Let $\mathcal{S}$ be a differentiable manifold and G a Lie group. A differentiable manifold Q is called a (differentiable) *principal fiber bundle* if the following conditions are satisfied:

1) G acts on Q to the left, freely and differentiably:

$$\Phi : \ G \times Q \to Q : \ (g, q) \mapsto g \cdot q \stackrel{\text{def}}{=} \Phi_g \cdot q \ .$$

2) $\mathcal{S}$ is the quotient space of Q by the equivalence relation induced by G, i.e. $\mathcal{S} = Q/G$ and the canonical projection $\pi : Q \to \mathcal{S}$ is differentiable.

3) Q is locally trivial, i.e. every point $s \in \mathcal{S}$ has a neighborhood U such that $\pi^{-1}(U) \subset Q$ is isomorphic with $U \times G$, in the sense that $q \in \pi^{-1}(U) \mapsto \big(\pi(q), \phi(q)\big) \in U \times G$ is a diffeomorphism such that $\phi : \pi^{-1}(U) \to G$ satisfies $\phi(g \cdot q) = g\phi(q), \forall g \in G$.

For $s \in \mathcal{S}$, the *fiber over* s is a closed submanifold of Q which is differentiably isomorphic with G. For any $q \in Q$, the *fiber through* q is the fiber over $s = \pi(q)$. When $Q = \mathcal{S} \times G$, then Q is said to be a *trivial* principal fiber bundle (fig. 2.3.1).

∎

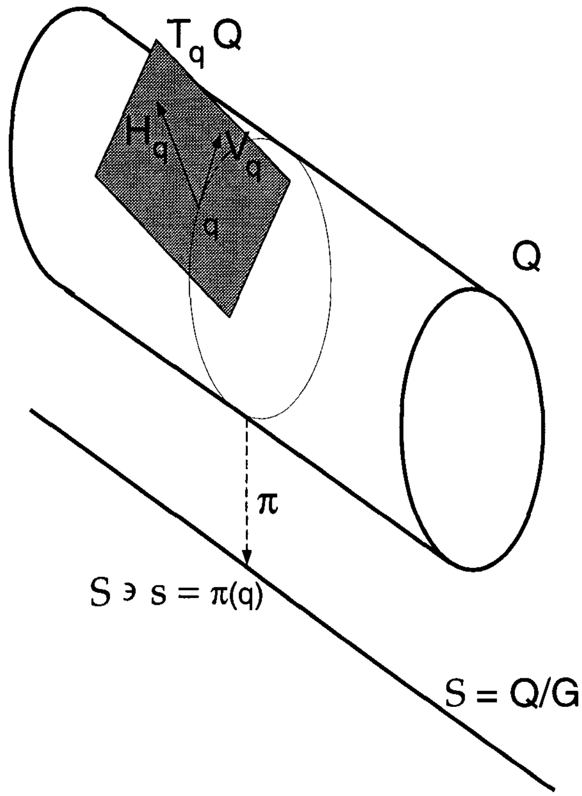


Fig. 2.3.1: Connection on a Principal Fiber Bundle

Definition 2.3.22 (Connection on a Principal Fiber Bundle)

Let $(Q, \mathcal{S}, \pi, G)$ be a principal fiber bundle. A *connection* on the principal fiber bundle is a choice of a tangent subspace $H_q \subset T_qQ$ at each point $q \in Q$ (horizontal subspace) such that, if $V_q \stackrel{\text{def}}{=} \{v \in T_qQ | \pi_{*_q}(v) = 0\}$ is the subspace of T_qQ tangent to the fiber through q (vertical subspace), we have:

1) $T_qQ = H_q \oplus V_q$.

2) For every $g \in G$ and $q \in Q$, $T_q\Phi_g \cdot H_q = H_{g \cdot q}$.

3) H_q depends differentiably on q.

∎

2.4 Geometric Mechanics

The material in this section is drawn from (Abraham & Marsden [1985]; Marsden, Montgomery & Ratiu [1990]; Marsden & Ratiu [1994]; Marsden [1991]) and (Arnold [1978]).

Definition 2.4.1 (Poisson manifold)

Let P be a manifold and let $C^\infty(P)$ be the set of smooth real–valued functions on P. Consider a bracket operation

$$\{.,.\}: C^\infty(P) \times C^\infty(P) \longrightarrow C^\infty(P) .$$

The pair $(P, \{,\})$ is called a *Poisson manifold*, if, for $f, g, h \in C^\infty(P)$, the bracket $\{,\}$ satisfies:

1) Anticommutativity $\{f, g\} = -\{g, f\}$.

2) Bilinearity $\{f, \lambda g + \mu h\} = \lambda\{f, g\} + \mu\{f, h\}$, $\lambda, \mu \in I\!R$.

3) Jacobi's identity $\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$.

4) Leibnitz's rule $\{fg, h\} = f\{g, h\} + g\{f, h\}$.

Properties (1), (2) and (3) above make $C^\infty(P)$ a Lie algebra under the Poisson bracket.

∎

A Poisson bracket is uniquely associated to a contravariant, skew–symmetric two–tensor Λ on P such that:

$$\{f, g\} = \Lambda(df, dg) .$$

If P is a finite dimensional manifold of dimension n, then Λ is given by an $n \times n$ skew–symmetric matrix and the Poisson bracket can be expressed in local coordinates as

$$\{f, g\} = \nabla f^\top \Lambda \nabla g .$$

Definition 2.4.2 (Hamiltonian Vector Field)

Let $(P, \{,\})$ be a Poisson manifold and let $f \in C^\infty(P)$. The *Hamiltonian vector field* X_f is the unique vector field on P such that

$$X_f(g) = \{g, f\} \ , \ \forall g \in C^\infty(P) . \tag{2.4.1}$$

∎

The set of Hamiltonian vector fields on P is a Lie subalgebra of the set of vector fields on P and, in fact, for $f, g \in C^\infty(P)$:

$$[X_f, X_g] = -X_{\{f,g\}} . \tag{2.4.2}$$

An integral curve $\mu(.) \subset P$ of X_f for $f \in C^\infty(P)$ satisfies in local coordinates:

$$\dot{\mu}_i(t) = \{\mu_i(t), f(t)\} \ , \ \ i = 1, \ldots, n \ . \tag{2.4.3}$$

A function $\phi \in C^\infty(P)$ is called a *Casimir function* if

$$\{\phi, \psi\} = 0 \ , \ \forall \psi \in C^\infty(P) \ . \tag{2.4.4}$$

Such a function is constant along the flow of all Hamiltonian vector fields.

Every symplectic manifold is Poisson. However, not every Poisson manifold is symplectic. An important class of non–symplectic Poisson manifolds is the class of *Lie–Poisson manifolds*, an example of which is the dual space $\mathcal{G}^*$ of a Lie algebra $\mathcal{G}$.

Let $<,>$ be the natural pairing between elements of the Lie algebra $\mathcal{G}$ and those of its dual space $\mathcal{G}^*$. Let $\{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n\}$ be a basis of $\mathcal{G}$ and let $\{\mathcal{A}_1^\flat, \mathcal{A}_2^\flat, \ldots, \mathcal{A}_n^\flat\}$ be the dual basis of $\mathcal{G}^*$, which is such that $< \mathcal{A}_i^\flat, \mathcal{A}_j >= \delta_i^j$, where δ_i^j is the Kronecker symbol. Any $\mu \in \mathcal{G}^*$ can be expressed as $\mu = \sum_{i=1}^n \mu_i \mathcal{A}_i^\flat$. The space $\mathcal{G}^*$ becomes a Poisson manifold under either one of the following Lie–Poisson brackets:

$$\{\phi, \psi\}_\pm(\mu) = \pm \Big\langle \mu, [\frac{\delta\phi}{\delta\mu}, \frac{\delta\psi}{\delta\mu}] \Big\rangle \ , \tag{2.4.5}$$

where $\phi, \psi \in C^\infty(\mathcal{G}^*)$ and $\mu \in \mathcal{G}^*$. This Poisson manifold will be denoted $\mathcal{G}^*_\pm$. The variational derivative $\frac{\delta\phi}{\delta\mu} \in \mathcal{G}$ is defined via the Frechet derivative by

$$\langle \nu, \frac{\delta\phi}{\delta\mu} \rangle = D\phi(\mu) \cdot \nu = \frac{d}{dt}\Big|_{t=0} \phi(\mu + t\nu) = \sum_{i=1}^n \frac{\partial\phi}{\partial\mu_i}\nu_i \ , \ \text{for } \nu \in \mathcal{G}^* \ .$$

Let $\Gamma_{i,j}^k$ be the structure constants of $\mathcal{G}$ in the given basis. Then, the Lie–Poisson bracket is expressed in the local coordinates for $\mathcal{G}$ and $\mathcal{G}^*$ as:

$$\{\phi, \psi\}_\pm(\mu) = \pm \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \mu_k \Gamma_{i,j}^k \frac{\partial\phi}{\partial\mu_i} \frac{\partial\psi}{\partial\mu_j} = \sum_{i=1}^n \sum_{j=1}^n \Lambda_{i,j}^\pm \frac{\partial\phi}{\partial\mu_i} \frac{\partial\psi}{\partial\mu_j} = \nabla\phi^\top \Lambda^\pm \nabla\psi \ , \tag{2.4.6}$$

where $\Lambda_{i,j}^\pm \stackrel{\text{def}}{=} \pm \sum_{k=1}^n \Gamma_{i,j}^k \mu_k$ are the elements of the skew–symmetric matrix Λ corresponding to the bracket $\{,\}_\pm$.

2.5 Parallel Manipulators

In section 2.5.1 we examine 3–dimensional (spatial) parallel manipulators. We develop the inverse kinematics and the velocity kinematics with respect both to the spatial and to the body velocities. We define the notions of forward and inverse velocity maps and singular configurations and show that both formulations of the Jacobian (both with respect to the spatial and to body velocities) specify the same set of singular configurations.

In section 2.5.2 we examine 2–dimensional (planar) parallel manipulators. In addition to the above, we define the notion of singular surfaces on the configuration space and we discuss the effect of singularities on an active robotic system.

2.5.1 Spatial Parallel Manipulators

Consider a parallel manipulator of the Stewart platform type (Fichter [1986]) moving in 3-dimensional space (fig. 2.5.1).

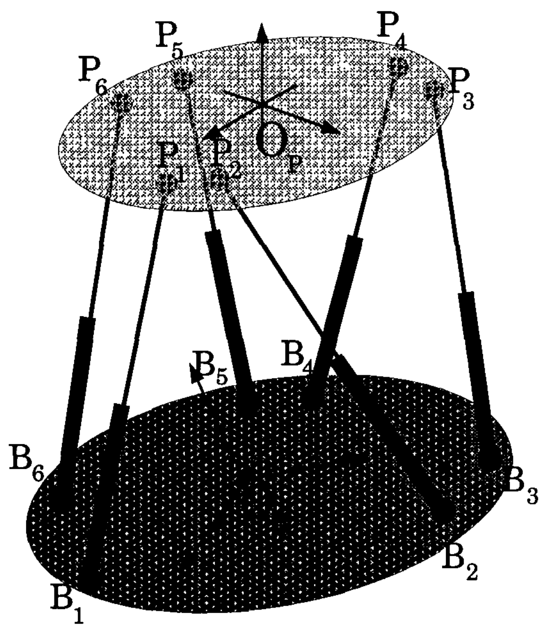


Fig. 2.5.1: Stewart Platform

Let O_B and O_P be, respectively, the basis and platform coordinate systems. Also let $g = \begin{pmatrix} R & T \\ 0 & 1 \end{pmatrix} \in G = SE(3)$, with $R \in SO(3)$, be the configuration matrix of the platform with respect to O_B, B_i be the position of the $i-th$ basis joint with respect to O_B, ${}^pP_i$ be the position of the $i-th$ platform joint with respect to O_P, $P_i = T + R\,{}^pP_i$ be the position of the $i-th$ platform joint with respect to O_B, $S_i = P_i - B_i$ the vector of the $i-th$ leg with respect to O_B, $\sigma_i = \|S_i\|_3$ be the length of the $i-th$ leg, $\sigma \in I\!R^6$ be the vector $(\sigma_1, \ldots, \sigma_6)^\top$ and $M_i = B_i \times P_i = P_i \times S_i = B_i \times S_i$ be the moment of S_i with respect to O_B, where $\|x\|_n = \sqrt{< x, x >_n}$ is the usual Euclidean norm for $x \in I\!R^n$ (for the proper n) and $<,>_n$ is the inner product for $I\!R^n$.

Given a configuration $g \in SE(3)$ of the platform, the **inverse kinematics** problem is to specify the corresponding platform leg lengths σ_i, $i = 1, \ldots, 6$ and define the *inverse kinematic map* $\mathcal{F}^{-1} : SE(3) \to I\!R^6 : g \mapsto \sigma$.

The homogeneous coordinate representation of the vector S_i is $\begin{pmatrix} S_i \\ 0 \end{pmatrix}$ (c.f. (Murray, Li & Sastry [1994])). From fig. 2.5.1 we have:

$$S_i = P_i - B_i = T + R\,{}^pP_i - B_i \implies \begin{pmatrix} S_i \\ 0 \end{pmatrix} = g \begin{pmatrix} {}^pP_i \\ 1 \end{pmatrix} - \begin{pmatrix} B_i \\ 1 \end{pmatrix} . \tag{2.5.1}$$

Then, by definition, the platform leg length σ_i is given by:

$$\sigma_i \stackrel{\text{def}}{=} \|S_i\|_3 = \left\| g \begin{pmatrix} {}^pP_i \\ 1 \end{pmatrix} - \begin{pmatrix} B_i \\ 1 \end{pmatrix} \right\|_4 . \tag{2.5.2}$$

Thus the inverse kinematic map $\mathcal{F}^{-1}$ is:

$$\mathcal{F}^{-1}(g) \stackrel{\text{def}}{=} \sigma(g) = \begin{pmatrix} \sigma_1(g) \\ \vdots \\ \sigma_6(g) \end{pmatrix} = \begin{pmatrix} \|S_1\|_3 \\ \vdots \\ \|S_6\|_3 \end{pmatrix} \tag{2.5.3}$$

Given a set of platform leg lengths σ_i, $i = 1, \ldots, 6$, the **forward kinematics** problem is to specify the corresponding platform configuration $g \in SE(3)$ and define the *forward kinematic map* $\mathcal{F} : I\!R^6 \to SE(3) : \sigma \mapsto g$. This is usually much harder than inverse kinematics for parallel manipulators. Issues related to the forward kinematics problem are addressed in (Fichter [1986]; Merlet [1992]; Nanua & Waldron [1989]; Nanua, Waldron & Murthy [1990]; Tahmasebi & Tsai [1991]).

Let $\hat{\omega}$ be the skew–symmetric matrix corresponding to the vector $\omega = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \in I\!R^3$,

i.e. $\hat{\omega} = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$. The mapping $\hat{\omega} \mapsto \omega$ identifies the Lie algebra $T_eSO(3)$ with $I\!R^3$. Under this identification, the Lie bracket corresponds to the cross product in $I\!R^3$ (Abraham & Marsden [1985]) .

Definition 2.5.1 (Spatial and Body Velocities)

The *spatial angular velocity* $\hat{\omega}$ is defined as the vector corresponding to the skew–symmetric matrix $\hat{\omega} = \dot{R}R^\top$; the *spatial translational velocity* ξ as $\xi = \dot{T} - \dot{R}R^\top T$; the *body angular velocity* $\hat{\Omega}$ as the vector corresponding to the skew–symmetric matrix $\hat{\Omega} = R^\top \dot{R}$ and the *body translational velocity* Ξ as $\Xi = R^\top \dot{T}$.

∎

Lemma 2.5.2 (Left and Right Translation in $SE(3)$)

For $g \in SE(3)$:

$$\dot{g} = \begin{pmatrix} \hat{\omega} & \xi \\ 0 & 0 \end{pmatrix} g = g \begin{pmatrix} \hat{\Omega} & \Xi \\ 0 & 0 \end{pmatrix} . \tag{2.5.4}$$

Also, $\omega = R\Omega$, $\hat{\omega} = Ad_R\hat{\Omega} = R\hat{\Omega}R^\top$, $\xi = R\Xi - R\hat{\Omega}R^\top T$.

Proof

By differentiating $g = \begin{pmatrix} R & T \\ 0 & 1 \end{pmatrix}$, we have:

$$\dot{g} = \begin{pmatrix} \dot{R} & \dot{T} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \dot{R}R^\top & \dot{T} - \dot{R}R^\top T \\ 0 & 0 \end{pmatrix} \begin{pmatrix} R & T \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} R & T \\ 0 & 1 \end{pmatrix} \begin{pmatrix} R^\top \dot{R} & R^\top \dot{T} \\ 0 & 0 \end{pmatrix} .$$

The result follows from Definition 2.5.1.

∎

Let $[a_i]_i$ denote the matrix whose i-th row is a_i, let Σ denote the diagonal matrix with elements $\{\sigma_i,\ i = 1, \ldots, 6\}$ and let $\dot{\sigma}$ denote the vector of leg velocities $\{\dot{\sigma}_i,\ i = 1, \ldots, 6\}$.

Theorem 2.5.3 (Velocity Kinematics)

The spatial velocities (ξ, ω) of the platform and the leg velocities $\dot{\sigma}_i$ of its legs are related as follows:

$$\Sigma(\sigma)\dot{\sigma} = [S_i^\top | M_i^\top]_i(g) \begin{pmatrix} \xi \\ \omega \end{pmatrix} , \tag{2.5.5}$$

The body velocities (Ξ, Ω) of the platform and the leg velocities $\dot{\sigma}_i$ of its legs are related as follows:

$$\Sigma(\sigma)\dot{\sigma} = [S_i^\top R | S_i^\top R \widehat{{}^pP_i}^\top]_i(g) \begin{pmatrix} \Xi \\ \Omega \end{pmatrix} . \tag{2.5.6}$$

Let $J_{SP}(g) = [S_i^\top | M_i^\top]_i$ and $J_B(g) = [S_i^\top R | S_i^\top R \widehat{{}^pP_i}^\top]_i$. Also, let $\mathcal{J}_{SP} = \Sigma^{-1}[S_i^\top | M_i^\top]_i$ and $\mathcal{J}_B = \Sigma^{-1}[S_i^\top R | S_i^\top R \widehat{{}^pP_i}^\top]_i$.

Proof

From (2.5.1) by differentiating:

$$\begin{pmatrix} \dot{S}_i \\ 0 \end{pmatrix} = \dot{g} \begin{pmatrix} {}^pP_i \\ 1 \end{pmatrix} = \begin{pmatrix} \hat{\omega} & \xi \\ 0 & 0 \end{pmatrix} g \begin{pmatrix} {}^pP_i \\ 1 \end{pmatrix} = \begin{pmatrix} \xi + \hat{\omega}(T + R\,{}^pP_i) \\ 0 \end{pmatrix} = \begin{pmatrix} \xi + \hat{\omega}P_i \\ 0 \end{pmatrix}$$

$$= g \begin{pmatrix} \hat{\Omega} & \Xi \\ 0 & 0 \end{pmatrix} \begin{pmatrix} {}^pP_i \\ 1 \end{pmatrix} = \begin{pmatrix} R\Xi + R\hat{\Omega}\,{}^pP_i \\ 0 \end{pmatrix} .$$

Then

$$\dot{S}_i = \xi + \hat{\omega}P_i = R(\Xi + \hat{\Omega}\,{}^pP_i) . \tag{2.5.7}$$

From (2.5.3) we have that $\sigma_i^2 = < S_i, S_i >_3$. By differentiating this we get:

$$\Sigma\dot{\sigma} = [\sigma_i\dot{\sigma}_i]_i = [< S_i, \dot{S}_i >_3]_i = [S_i^\top \dot{S}_i]_i , \tag{2.5.8}$$

Then from (2.5.7) and (2.5.8) we get:

$$\Sigma\dot{\sigma} = [S_i^\top \dot{S}_i]_i = [S_i^\top(\xi + \hat{\omega}P_i)]_i = [S_i^\top \xi + S_i^\top \widehat{P}_i^\top \omega]_i$$

$$= [S_i^\top | M_i^\top]_i \begin{pmatrix} \xi \\ \omega \end{pmatrix} .$$

For the second part of the theorem, again from (2.5.7) and (2.5.8) we get:

$$\Sigma\dot{\sigma} = [S_i^\top \dot{S}_i]_i = [S_i^\top R(\Xi + \hat{\Omega}\,{}^pP_i)]_i = [S_i^\top R\Xi + S_i^\top R\hat{\Omega}\,{}^pP_i]_i = [S_i^\top R\Xi - S_i^\top R \widehat{{}^pP_i}\Omega]_i$$

$$= [S_i^\top R | S_i^\top R \widehat{{}^pP_i}^\top]_i \begin{pmatrix} \Xi \\ \Omega \end{pmatrix} .$$

∎

Corollary 2.5.4 (Forward Velocity Kinematics)

At a platform configuration g with corresponding leg lengths σ, the spatial forward velocity kinematic map $\left(\frac{\partial \mathcal{F}}{\partial \sigma}\right)_{\mathcal{SP}} : \dot{\sigma} \mapsto \begin{pmatrix} \xi \\ \omega \end{pmatrix}$ is given by: $\begin{pmatrix} \xi \\ \omega \end{pmatrix} = \mathcal{J}_{\mathcal{SP}}^{-1}\dot{\sigma}$.

The body forward velocity kinematic map $\left(\frac{\partial \mathcal{F}}{\partial \sigma}\right)_{\mathcal{B}} : \dot{\sigma} \mapsto \begin{pmatrix} \Xi \\ \Omega \end{pmatrix}$ is given by: $\begin{pmatrix} \Xi \\ \Omega \end{pmatrix} = \mathcal{J}_{\mathcal{B}}^{-1}\dot{\sigma}$.

∎

Corollary 2.5.5 (Inverse Velocity Kinematics)

At a platform configuration g with corresponding leg lengths σ, the spatial inverse velocity kinematic map $\left(\frac{\partial \mathcal{F}^{-1}}{\partial \sigma}\right)_{\mathcal{SP}} : \begin{pmatrix} \xi \\ \omega \end{pmatrix} \mapsto \dot{\sigma}$ is given by: $\dot{\sigma} = \mathcal{J}_{\mathcal{SP}} \begin{pmatrix} \xi \\ \omega \end{pmatrix}$.

The body inverse velocity kinematic map $\left(\frac{\partial \mathcal{F}^{-1}}{\partial \sigma}\right)_{\mathcal{B}} : \begin{pmatrix} \Xi \\ \Omega \end{pmatrix} \mapsto \dot{\sigma}$ is given by: $\dot{\sigma} = \mathcal{J}_{\mathcal{B}} \begin{pmatrix} \Xi \\ \Omega \end{pmatrix}$.

∎

Definition 2.5.6 (Kinematic Singularities)

Platform configurations g, where J_{SP} is singular, are called *forward kinematic singularities.* Platform configurations g, where Σ is singular, are called *inverse kinematic singularities* .

∎

Issues related to singular configurations of parallel manipulators are explored in (Gosselin & Angeles [1990]; Hunt [1983]; Ma & Angeles [1991]; Merlet [1988]; Merlet [1989]; Merlet [1988]; Sugimoto, Duffy & Hunt [1982]).

The forward kinematic singularities were defined using the spatial velocity formulation. It can be shown that it would be equivalent to define them using the body velocity formulation.

Proposition 2.5.7

The following conditions are equivalent for a configuration $g \in SE(3)$ to be a forward kinematic singularity

(1) $detJ_{SP}(g) = 0$.

(2) there exists $(\xi, \omega) \neq 0$ such that $< S_i, \xi + \hat{\omega} P_i >_3 = 0$, for all $i = 1, \ldots, 6$.

(3) $detJ_B(g) = 0$.

(4) there exists $(\Xi, \Omega) \neq 0$ such that $< S_i, R(\Xi + \hat{\Omega}^p P_i) >_3 = 0$, for all $i = 1, \ldots, 6$.

Proof

(1) $\iff$ (2) : Observe that (using $< a \times b, c >_3 = < b, c \times a >_3$):

$$J_{SP} \begin{pmatrix} \xi \\ \omega \end{pmatrix} = [S_i^\top | M_i^\top]_i \begin{pmatrix} \xi \\ \omega \end{pmatrix} = [S_i^\top \xi + (P_i \times S_i)^\top \omega]_i$$

$$= [S_i^\top \xi + S_i^\top (\omega \times P_i)]_i = [S_i^\top (\xi + \hat{\omega} P_i)]_i = [< S_i, \xi + \hat{\omega} P_i >_3]_i$$

Thus if there exists $(\xi, \omega) \neq 0$ such that $J_{SP} \begin{pmatrix} \xi \\ \omega \end{pmatrix} = 0$, then $< S_i, \xi + \hat{\omega} P_i >_3 = 0$, $\forall i = 1, \ldots, 6$ and vice–versa.

(2) $\iff$ (4) : Easy to see from Lemma 2.5.2 :

$$< S_i, \xi + \hat{\omega} P_i >_3 = S_i^\top (\xi + \hat{\omega} P_i) = S_i^\top (R\Xi - R\hat{\Omega} R^\top T + R\hat{\Omega} R^\top P_i)$$

$$= S_i^\top R[\Xi + \hat{\Omega} R^\top (P_i - T)] = S_i^\top R[\Xi + \hat{\Omega} R^\top R\, {}^p P_i)] \ .$$

$$= S_i^\top R(\Xi + \hat{\Omega}\, {}^p P_i) = < S_i, R(\Xi + \hat{\Omega}^p P_i) >_3$$

(3) $\iff$ (4) : Observe that:

$$J_B \begin{pmatrix} \Xi \\ \Omega \end{pmatrix} = [S_i^\top R\Xi + (R\, {}^p P_i)^\top R\Omega]_i = [S_i^\top R\Xi + S_i^\top (R\Omega) \times (R\, {}^p P_i)]_i$$

$$= [S_i^\top R\Xi + S_i^\top \hat{R\Omega} R\, {}^p P_i]_i = [S_i^\top R(\Xi + \hat{\Omega}\, {}^p P_i)]_i$$

∎

Corollary 2.5.8

If we start at a singular configuration $g_* \in SE(3)$, the input $\dot{\sigma} \in I\!R^6$ determines the platform motion only up to an element of the null space $\mathcal{N}_{g_*}(J_{SP})$ of $J_{SP}(g_*)$. This element corresponds to such motions that make the velocity of all joints P_i either perpendicular to the corresponding leg vector S_i or zero.

Proof

If we lock the legs of the platform ($\dot{\sigma} = 0$) and apply Corollary 2.5.5 and Proposition 2.5.7, we get $J_{SP}(g)\begin{pmatrix}\xi\\ \omega\end{pmatrix} = 0$. If g is a singular configuration, then there exists a non–zero vector $\begin{pmatrix}\xi_0\\ \omega_0\end{pmatrix}$ such that $J_{SP}(g)\begin{pmatrix}\xi_0\\ \omega_0\end{pmatrix} = 0$. Therefore, even though the legs of the platform are locked, it can still move with velocity $\begin{pmatrix}\xi_0\\ \omega_0\end{pmatrix}$ in directions specified by the eigenvectors corresponding to the null eigenvalue of $J_{SP}(g)$. From condition (2) of Proposition 2.5.7, we see that, geometrically, those directions can be specified by the property that the velocity of the joint P_i is either perpendicular to the i-th leg vector S_i or is zero, for all $i = 1, \ldots 6$.

Now, for some $\dot{\sigma} \neq 0$, suppose $\begin{pmatrix}\xi\\ \omega\end{pmatrix}$ is such that $J_{SP}(g)\begin{pmatrix}\xi\\ \omega\end{pmatrix} = \Sigma\dot{\sigma}$. But if g is a singular configuration and $\begin{pmatrix}\xi_0\\ \omega_0\end{pmatrix} \in \mathcal{N}_g(J_{SP})$, we also have $J_{SP}(g)[\begin{pmatrix}\xi\\ \omega\end{pmatrix}+\begin{pmatrix}\xi_0\\ \omega_0\end{pmatrix}] = \Sigma\dot{\sigma}$. Therefore, $\dot{\sigma}$ determines the platform motion only up to an element of $\mathcal{N}_g(J_{SP})$.

∎

Definition 2.5.9 (Singular Surfaces)

The loci of singular configurations in configuration space are called *singular surfaces.*

∎

From Corollary 2.5.8, there exists an indeterminacy in the motion of our system, whenever it is in a singular configuration. This indeterminacy is undesirable when we use the platform for e.g. repositioning vision sensors and perform exploratory visual tasks. An example of this is shown in section 2.5.2. Moreover, by the duality of kinematics and dynamics, there exist wrenches that the platform cannot resist while at singular configurations. This may be equally undesirable if the system is used with tactile sensors to perform haptic exploration. On the other hand, limiting the motion of the platform only to areas of the configuration space which are free of singularities, as is frequently done in practice, is not possible without excessively restricting the workspace of the manipulator. Therefore, we should either try to avoid singular configurations as much as possible or just drive through them, in such a way that the dynamics of the system resolves the indeterminacy in the platform motion.

2.5.2 Planar Parallel Manipulators

In order to present concrete examples of our ideas about planning for parallel manipulators, we chose a particular planar platform architecture (fig. 2.5.2), which is simple enough to allow analytic derivation of the forward kinematic map and easy visualization of the singular surfaces, but also generic enough to demonstrate the applicability of our results.

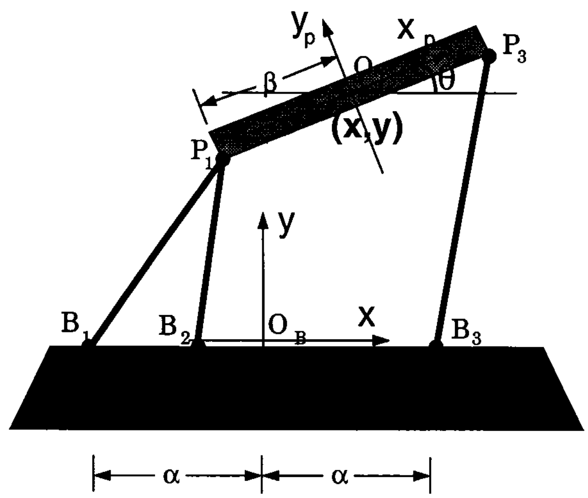


Fig. 2.5.2: Planar Parallel Manipulator

We maintain the notational conventions of section 2.5.1, with the necessary adaptation to the planar case. In particular $g = \begin{pmatrix} R & T \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{pmatrix} \in G = SE(2)$ is the configuration matrix of the platform with respect to the basis coordinate system centered at O_B, where θ is the angle of the platform with the horizontal and $T = (x, y)^{\top}$ is the position of the center O_P of the platform coordinate system. Again $P_i = T + R\ {}^{p}P_i$, where $S_i \stackrel{\text{def}}{=} \begin{pmatrix} S_{i1} \\ S_{i2} \end{pmatrix} = P_i - B_i = T + R\ {}^{p}P_i - B_i$ or $\begin{pmatrix} S_i \\ 1 \end{pmatrix} = g \begin{pmatrix} {}^{p}P_i \\ 1 \end{pmatrix} - \begin{pmatrix} B_i \\ 1 \end{pmatrix}$, for $g \in SE(2)$.

In the case of fig. 2.5.2, ${}^{p}P_1 = {}^{p}P_2 = (-\beta,\ 0)^{\top}$, ${}^{p}P_3 = (\beta,\ 0)^{\top}$, $B_1 =$

$(-\alpha, \ \ 0)^\top$, $B_2 = (-\frac{\alpha}{2}, \ \ 0)^\top$, $B_3 = (\alpha, \ \ 0)^\top$. Then,

$$
\begin{aligned}
S_1 &\stackrel{\text{def}}{=} (S_{11}, \ \ S_{12})^\top = (\alpha + x - \beta\cos\theta, y - \beta\sin\theta)^\top , \\
S_2 &\stackrel{\text{def}}{=} (S_{21}, \ \ S_{22})^\top = (\frac{\alpha}{2} + x - \beta\cos\theta, y - \beta\sin\theta)^\top , \\
S_3 &\stackrel{\text{def}}{=} (S_{31}, \ \ S_{32})^\top = (-\alpha + x + \beta\cos\theta, y + \beta\sin\theta)^\top .
\end{aligned}
\tag{2.5.9}
$$

Moreover,

$$
M_1 = -\alpha S_{12} , \ \ M_2 = -\frac{\alpha}{2} S_{22} , \ \ M_3 = \alpha S_{32} . \tag{2.5.10}
$$

The *inverse kinematic map* is derived directly from (2.5.3):

$$
\sigma(g) = \begin{pmatrix} \sigma_1(g) \\ \sigma_2(g) \\ \sigma_3(g) \end{pmatrix} = \mathcal{F}^{-1}(g) = \begin{pmatrix} \sqrt{S_{11}^2 + S_{12}^2} \\ \sqrt{S_{21}^2 + S_{22}^2} \\ \sqrt{S_{31}^2 + S_{32}^2} \end{pmatrix} (g) \tag{2.5.11}
$$

The *forward kinematic map*, i.e. the platform configuration $g \in SE(2)$ as a function of the link lengths $\sigma_1, \sigma_2, \sigma_3$ is obtained analytically if we observe that (x, y, θ) can be easily determined provided the position of the end–points of the platform P_1 and P_3 is known. But point P_1 lies on the intersection of cycles (B_1, σ_1) and (B_2, σ_2) and point P_3 lies on the intersection of cycles $(P_1, 2\beta)$ and (B_3, σ_3) (fig. 2.5.3). Thus, this problem has in the generic case four solutions, while in special cases (which correspond to singular configurations of the platform) it may have one, two or infinite solutions.

Let $\xi \stackrel{\text{def}}{=} (\xi_1, \xi_2)^\top$ and $\Xi \stackrel{\text{def}}{=} (\Xi_1, \Xi_2)^\top$. From Definition 2.5.1, adapted to the case of $SE(2)$, we get the spatial and body velocities:

$$
\begin{aligned}
\begin{pmatrix} \hat{\omega} & \xi \\ 0 & 0 \end{pmatrix} &\stackrel{\text{def}}{=} \dot{g} g^{-1} = \begin{pmatrix} 0 & -\dot{\theta} & \dot{x} + \dot{\theta} y \\ \dot{\theta} & 0 & \dot{y} - \dot{\theta} x \\ 0 & 0 & 0 \end{pmatrix} \\
\begin{pmatrix} \hat{\Omega} & \Xi \\ 0 & 0 \end{pmatrix} &\stackrel{\text{def}}{=} g^{-1} \dot{g} = \begin{pmatrix} 0 & -\dot{\theta} & \dot{x}\cos\theta + \dot{y}\sin\theta \\ \dot{\theta} & 0 & -\dot{x}\sin\theta + \dot{y}\cos\theta \\ 0 & 0 & 0 \end{pmatrix}
\end{aligned}
\tag{2.5.12}
$$

Conversely,

$$
\begin{aligned}
\dot{\theta} &= \omega = \Omega \\
\dot{x} &= \xi_1 - \omega y = \cos\theta \Xi_1 - \sin\theta \Xi_2 \\
\dot{y} &= \xi_2 + \omega x = \sin\theta \Xi_1 + \cos\theta \Xi_2 .
\end{aligned}
$$

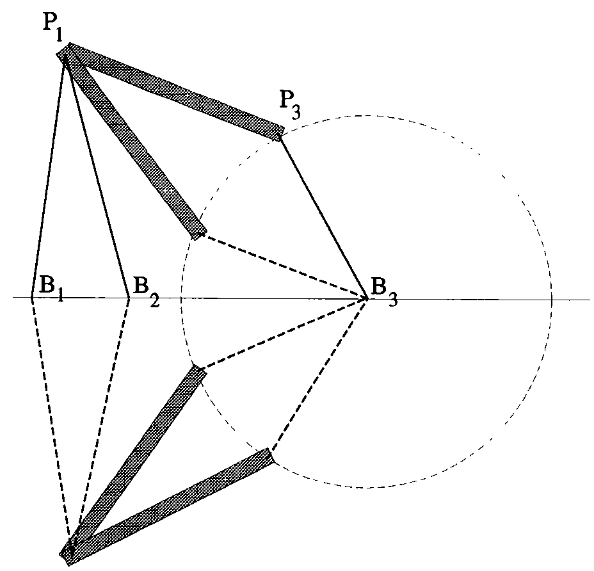


Fig. 2.5.3: Forward Kinematics Solutions

From Theorem 2.5.3 we have:

$$\Sigma(\sigma)\begin{pmatrix}\dot{\sigma}_1\\ \dot{\sigma}_2\\ \dot{\sigma}_3\end{pmatrix} = J_{SP}(g)\begin{pmatrix}\xi_1\\ \xi_2\\ \omega\end{pmatrix} = J_B(g)\begin{pmatrix}\Xi_1\\ \Xi_2\\ \Omega\end{pmatrix}, \tag{2.5.13}$$

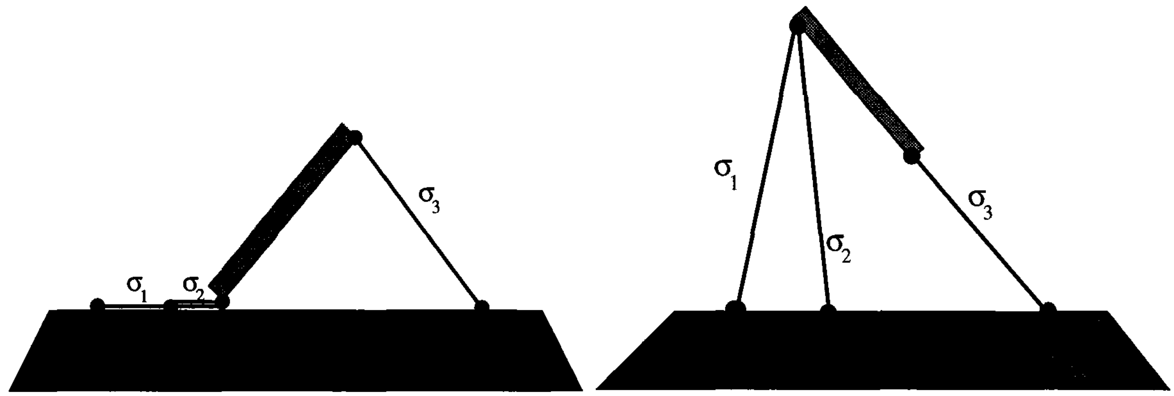
where $\Sigma(\sigma) \stackrel{\text{def}}{=} \begin{pmatrix}\sigma_1 & 0 & 0\\ 0 & \sigma_2 & 0\\ 0 & 0 & \sigma_3\end{pmatrix}$, $J_{SP}(g) \stackrel{\text{def}}{=} \begin{pmatrix}S_{11} & S_{12} & -\alpha S_{12}\\ S_{21} & S_{22} & -\frac{\alpha}{2}S_{22}\\ S_{31} & S_{32} & \alpha S_{32}\end{pmatrix}$ and

$$J_B(g) \stackrel{\text{def}}{=} \begin{pmatrix}(S_{11}\ \ S_{12})R & \beta(\sin\theta S_{11} - \cos\theta S_{12})\\ (S_{21}\ \ S_{22})R & \beta(\sin\theta S_{21} - \cos\theta S_{22})\\ (S_{31}\ \ S_{32})R & \beta(-\sin\theta S_{31} + \cos\theta S_{32})\end{pmatrix}.$$

From Definition 2.5.6, *forward kinematics singularities* are platform configurations $g \in SE(2)$ where J_{SP} is singular, i.e. where $det J_{SP}(g) = 0$. From (2.5.13) we have $det J_{SP}(g) = \alpha\beta(y - \beta\sin\theta)[y\cos\theta - (x-\alpha)\sin\theta]$. This leads to two kinds of singular configurations:

1) Those where $\mathcal{X}_* = \{(x,y,\theta) | y = \beta\sin\theta\}$ shown in the generic case in fig. 2.5.4a. This condition corresponds to configurations where the leg axes intersect at point B_3. The locus of those configurations in (x,y,θ)–space is a ruled surface shown in fig. 2.5.5.

2) Those where $\mathcal{X}_* = \{(x,y,\theta) | y\cos\theta = (x-\alpha)\sin\theta\}$ shown in the generic case in fig. 2.5.4.b. This condition corresponds to configurations where the leg axes intersect at point P_1. The locus of those configurations in (x,y,θ)–space is a helicoid shown in fig.



(a) (b)

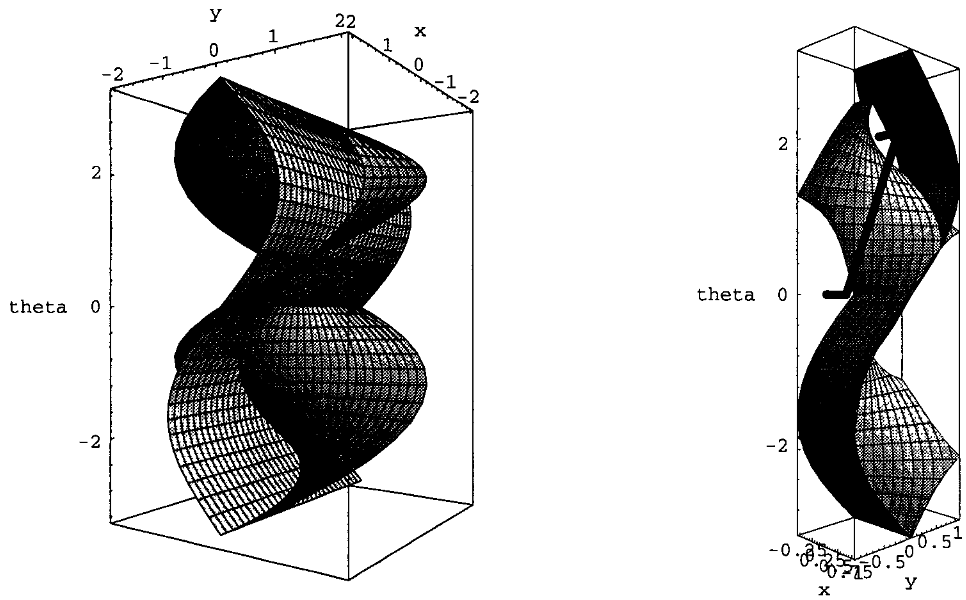
Fig. 2.5.4: Kinematic Singularities

2.5.5.

Thus, the singular surfaces for the manipulator of fig. 2.5.2 are two 2–dimensional manifolds. Their intersection is composed of the lines $\{y = 0, \theta = \kappa\pi\}$ and a circular helix with radius β and axis the line $\{x = \alpha, y = 0\}$ (see fig. 2.5.5). The singular surfaces divide the configurations space $SE(2)$ of the planar parallel manipulator in four disjoint parts corresponding to $y > \beta\sin\theta$, $y < \beta\sin\theta$, $y\cos\theta > (x-\alpha)\sin\theta$ and $y\cos\theta < (x-\alpha)\sin\theta$.

Inverse kinematic singularities occur when Σ is singular, i.e. when $det(\Sigma) = 0$. Then at least one of the platform legs has zero length. Observe that they also correspond to forward kinematic singularities, therefore they will not be considered separately.

As observed in Corollary 2.4.8, at singular configurations the input $\dot{\sigma}$ determines the platform motion only up to an element of the null space of J_{SP} or of J_B. In the case of singular configurations of the first kind, where $\mathcal{X}_* = \{(x, y, \theta) | y = \beta\sin\theta, \sin\theta \neq 0\}$, we can find that the null space $\mathcal{N}_{\mathcal{X}_*}(J)$ is a one–parameter family of rotations around point B_3. In particular, $\mathcal{N}_{\mathcal{X}_*}(J_{SP}) = \left\{ \begin{pmatrix} \xi_1 \\ \xi_2 \\ \omega \end{pmatrix} \middle| \begin{pmatrix} \xi_1 \\ \xi_2 \\ \omega \end{pmatrix} = \begin{pmatrix} 0 \\ -\alpha \\ 1 \end{pmatrix} \omega \right\}$. In the case of singular configurations of the second kind, where $\mathcal{X}_* = \{(x, y, \theta) | y\cos\theta = (x-\alpha)\sin\theta, y \neq \beta\sin\theta, x \neq \alpha, \sin\theta \neq 0, \cos\theta \neq 0\}$ we can find that the null space $\mathcal{N}_{\mathcal{X}_*}(J)$ is a one–parameter family of rotations around point P_1. In particular, $\mathcal{N}_{\mathcal{X}_*}(J_{SP}) = \left\{ \begin{pmatrix} \xi_1 \\ \xi_2 \\ \omega \end{pmatrix} \middle| \begin{pmatrix} \xi_1 \\ \xi_2 \\ \omega \end{pmatrix} = \right.$



(a) (b)

Fig. 2.5.5: Singular Surfaces and Singularity Avoidance

$$\begin{pmatrix} (x-\alpha-\beta\cos\theta)\tan\theta \\ -(x-\beta\cos\theta) \\ 1 \end{pmatrix} \omega \Bigg\} \text{ or } \mathcal{N}_{\mathcal{X}_*}(J_B) = \left\{ \begin{pmatrix} \Xi_1 \\ \Xi_2 \\ \Omega \end{pmatrix} \middle| \begin{pmatrix} \Xi_1 \\ \Xi_2 \\ \Omega \end{pmatrix} = \begin{pmatrix} 0 \\ \beta \\ 1 \end{pmatrix} \Omega \right\}.$$

Physically, this means that those rotations cannot be controlled by our system input $\dot{\sigma}$. If we start at configuration I of fig. 2.5.6 and apply input $\{\dot{\sigma}_1 = \dot{\sigma}_2 = 0, \dot{\sigma}_3 \neq 0\}$, the platform is expected to rotate around point P_1. From the input alone though, the final configuration cannot be determined. We may end up in configuration II or in configuration III. This is the source of the problems discussed at the end of the previous section.

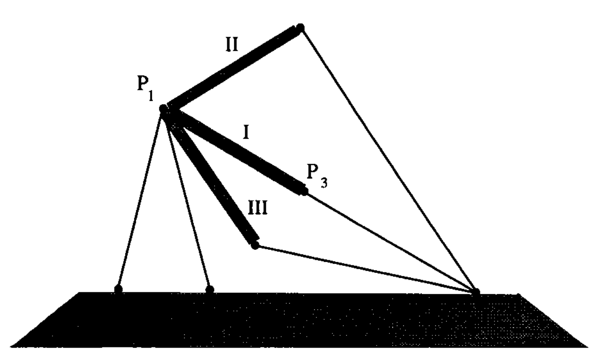


Fig. 2.5.6: Effect of Singularities

However, by properly planning the system's trajectory, it is often possible to avoid singular configurations. An example is shown in fig. 2.5.5.b. This will be further discussed in the next chapter.

CHAPTER THREE

MOTION PLANNING BASED ON PATH CURVATURE

3.1 Introduction

In this chapter we consider solving *analytically* some instances of the motion planning problem for robotic systems with holonomic constraints. The motion planning problem that we consider is to specify a trajectory of the system that has certain desired kinematic and dynamic properties and that connects a given initial and final configuration. Because of the holonomic constraints, the configuration space of the robotic systems can be considered as an n–dimensional Lie group G, thus its trajectory is a curve $g(.) \subset G$, with given end–points $g(0)$ and $g(T)$. The curve $g(.)$ specifies a corresponding curve $\xi(.)$ on the Lie algebra $\mathcal{G}$ of G, which, through identifying $\mathcal{G}$ with $I\!R^n$ by a choice of coordinates, corresponds to a curve $\sigma(.)$ on $I\!R^n$. There, we can employ differential geometric tools such as its curvature and the associated Frenet–Serret moving frame to describe σ. We will consider the problem of finding the curve that optimizes a curvature–squared cost functional among all the curves on G that connect the given end–points. This can naturally be viewed as an *optimal control* problem involving a left–invariant dynamical system that evolves on $SE(n)$, the group of rigid motions on $I\!R^n$. The nonholonomic nature of this system allows us to achieve controllability with fewer controls than the dimension of $SE(n)$ ($(n-1)$ controls, while the dimension of $SE(n)$ is $\frac{1}{2}n(n+1)$).

The *motion planning problem* is related to the specification of a path in configuration space, so that certain objectives related to a specific task are optimally met. Planning in the configuration space of the manipulator allows us to solve the problem without worrying about the specifics of the manipulator architecture. The specification of the corresponding joint trajectories is done through the inverse kinematics, which, in the case of parallel manipulators, is easily solvable.

Suppose we are interested in driving our system from an initial configuration to a final one is such a way that the path is as little "curly" as possible. We do not want to specify further the shape of the path a-priori (e.g. by choosing a straight line or circular arc), in order to maintain some necessary flexibility. Such a situation occurs when we try to move from one side of a *singular surface* of a parallel manipulator to the other (see section 3.4). A "curly" path will force us to move close to this surface for a long time, or, worse, cross it multiple times. On the other hand, a straight path between an initial and a final configuration may not allow us the necessary flexibility to cross at some "favorable" point.

This situation of a desired path which is as little "curly" as possible is very similar to the classical *elastica* problem, where a flexible thin rod is allowed to deform while its ends are fixed (Jurdjevic [a]; Jurdjevic [b]; Jurdjevic [1990]; Jurdjevic [1992]). Its equilibrium configuration is known to correspond to the extremals of a variational problem with constraints, where a cost functional involving the square of the geodesic curvature of the rod is minimized.

We consider the optimal control problem of minimizing a curvature–squared cost functional. This is similar to the classical problem of the elastica. We obtain analytical solutions of the optimal trajectories and apply them to the solution of the singularity avoidance problem for the planar parallel manipulators that we considered in chapter 2. The solutions to the problem of the elastica provide us with trajectory segments which are (locally) optimal. Those can be patched together to form a trajectory that links the desired end–points. Because we have a family of optimal segments which is richer than the usual straight–line and circular–arc segments most motion planners consider, we can choose those segments so that the path curvature is continuous, even at the joints of the segments. This is important in various applications (e.g. autonomous vehicles on a manufacturing floor (Fleury et al. [1993]; Kanayama & Hartman [989]; Nelson [1989a]; Nelson [1989b])). In the case of a general motion of the manipulator, the configuration space is $SE(2)$, while, if the motion is restricted to a 2–dimensional manifold, it can be $\mathbb{R}^2$ or $\mathbb{R}\times S^1$. This last case is considered in section 3.2, while the case of a 3–dimensional manifold is considered in section 3.3.

3.2 Curvature–based Planning for 2–dimensional Manifolds

First we consider the case of a *translating* planar parallel manipulator. The configuration space of the system is then $I\!R^2$ with coordinates $\sigma = (x, y)$. A path in configuration space is a regular curve $\sigma(t) = (x(t), y(t))$ in $I\!R^2$. Alternatively, consider the case where the planar platform *translates* along a specific line and also *rotates*. Then the configuration space is $I\!R \times S^1$. If we consider motion on a chart of this group, the problem setup is exactly the same as in the translational case.

Intuitively, given a fixed length L and end–points $\sigma(0)$ and $\sigma(L)$ in $I\!R^2$, the result of a planning scheme based on minimizing the curvature–squared cost functional $\frac{1}{2}\int \kappa^2(s)\, ds$, will be a curve $\sigma(s)$ similar to curve (I) in fig. 3.2.1, rather than similar to the "curlier" curves (II) and (III). The flexibility of this approach stems from our ability to choose the length L at will, thus manipulating the shape of the path.

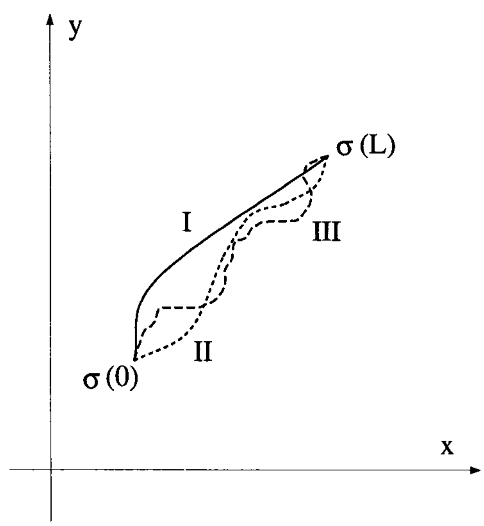


Fig. 3.2.1: Configuration–Space Paths

Such a solution to the planning problem only affects the geometric invariants of the path, i.e. it specifies its shape, but not the velocity with which this is traversed. This extra degree of freedom can be used to optimize *dynamical* properties of the system trajectory, such as minimizing acceleration or optimizing the power requirements of the system actuators. Of course, in the case when the magnitude of the velocity is constant (a non–trivial case since the direction of the velocity may vary, which is often of great interest, as in the case of autonomously guided vehicles), we know that the curvature is proportional to the magnitude of the acceleration, therefore a minimal–curvature–squared path minimizes acceleration and, thus, the current to the motors driving the platform (Kuo [1987]). The curvature of the optimal paths is bounded and its bounds can be selected by the designer. In addition, the curvature varies continuously, thus the acceleration does not have sudden jumps. Moreover, the jerk, which is defined as the derivative of the acceleration, is proportional to the derivative of the curvature, which can be shown to be bounded on minimal curvature–squared paths. Therefore, the proposed planning scheme can optimize the system trajectory with respect to both kinematic and dynamic criteria.

3.2.1 Curvature-based Planning Criteria

Let $\sigma(s) = (x(s), y(s))$ be an arc–length parametrization of a curve σ in $I\!R^2$, where the arclength is $s(t) = \int_0^t \| \frac{d\sigma}{dt} \|_2 d\tau = \int_0^t \sqrt{\dot{x}^2(\tau) + \dot{y}^2(\tau)}\, d\tau$, for $0 \leq t \leq T$. Then $\sigma(s)$ is a unit speed curve with total length $L = s(T)$. After properly defining its tangent vector $v_1(s)$ and its normal vector $v_2(s)$ (Millman & Parker [1977]), we define the (plane) curvature of the curve as $\kappa(s) =< \frac{dv_1}{ds}(s), v_2(s) >_2$, where $<,>_n$ is the inner product in $I\!R^n$. We are interested in minimizing

$$\eta = \frac{1}{2} \int_0^L \kappa^2(s) ds \ .$$

If θ is the angle that the tangent v_1 makes with the horizontal (fig. 3.2.2), we have that $v_1(s) = \begin{pmatrix} \cos\theta(s) \\ \sin\theta(s) \end{pmatrix}$ and $v_2(s) = \begin{pmatrix} -\sin\theta(s) \\ \cos\theta(s) \end{pmatrix}$. Then $\kappa(s) = \dot{\theta}(s)$ and from the *Frenet–Serret* equations we get:

$$\frac{d\sigma}{ds}(s) = v_1(s) \ , \quad \frac{dv_1}{ds}(s) = \kappa(s) v_2(s) \ , \quad \frac{dv_2}{ds}(s) = -\kappa(s) v_1(s) \ . \tag{3.2.1}$$

Defining $g(s) = \begin{pmatrix} v_1 & v_2 & \sigma \\ 0 & 0 & 1 \end{pmatrix}(s) = \begin{pmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{pmatrix}(s) \in G = SE(2)$, the system (3.2.1) can be viewed as a *left–invariant* system on the Lie group G $= SE(2)$ of the form:

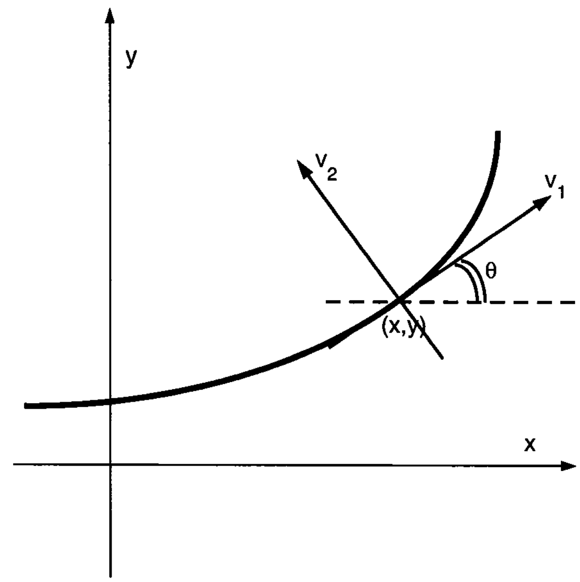


Fig. 3.2.2: Curve in the Plane

$$\frac{dg}{ds}(s) = g(s)\begin{pmatrix} 0 & -\kappa(s) & 1 \\ \kappa(s) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$= g(s)\Big(\mathcal{A}_2 + \kappa(s)\mathcal{A}_1\Big) = \mathcal{X}_2(g(s)) + \kappa(s)\mathcal{X}_1(g(s)) \;, \tag{3.2.2}$$

where the left–invariant vector fields $\mathcal{X}_i$ are defined as $\mathcal{X}_i(g(s)) \stackrel{\text{def}}{=} g(s)\mathcal{A}_i$ and where the matrices $\mathcal{A}_i$ belong to the following basis of the Lie algebra $\mathcal{G}$ of G :

$$\{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3\} = \left\{ \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \right\}. \tag{3.2.3}$$

For $\mathcal{X}_i$ a left–invariant vector field of the form $\mathcal{X}_i(g) = g\mathcal{A}_i \;, i = 1,2,3$, with $g = \begin{pmatrix} R & T \\ 0 & 1 \end{pmatrix}$ and $\mathcal{A}_i = \begin{pmatrix} A_i & a_i \\ 0 & 0 \end{pmatrix}$, we have $\phi^i_g(t) = ge^{t\mathcal{A}_i} = \begin{pmatrix} Re^{tA_i} & T + RA_i^{-1}(e^{tA_i} - I)a_i \\ 0 & 1 \end{pmatrix}$. Then from (2.3.4): $[\![\mathcal{X}_i, \mathcal{X}_j]\!] = g(\mathcal{A}_i\mathcal{A}_j - \mathcal{A}_j\mathcal{A}_i) = g[\mathcal{A}_i, \mathcal{A}_j]$, where $[\mathcal{A}_i, \mathcal{A}_j] = \mathcal{A}_i\mathcal{A}_j - \mathcal{A}_j\mathcal{A}_i$ is the commutator of the matrices $\mathcal{A}_i$ and $\mathcal{A}_j$. It is easy to verify the following results:

$$[\mathcal{A}_1, \mathcal{A}_2] = \mathcal{A}_3 \;, \;\; [\mathcal{A}_1, \mathcal{A}_3] = -\mathcal{A}_2 \;.$$

Then, for the Jacobi–Lie brackets of the vector fields $\mathcal{X}_i$, we get corresponding results.

Consider now the following variational problem:

Problem P_2 :

Minimize $\eta = \frac{1}{2}\int_0^L \kappa^2(s)ds$

on the dynamical system $\frac{dg}{ds} = g(\mathcal{A}_2 + \kappa\mathcal{A}_1)$, with $g \in G = SE(2)$,

with $\kappa \in U = \mathbb{R}$ and with given boundary conditions $\sigma(0)$ and $\sigma(L)$.

This problem falls in the framework of problems studied by (Bryant & Griffiths [1986]) and (Jurdjevic [1990]), where it was shown that the curvature of the optimal paths satisfies an equation of the form: $(\frac{d\kappa}{ds})^2 + (\mathcal{C}_1 - \frac{1}{2}\kappa^2)^2 = \mathcal{C}_2^2$. Subsequently we will derive this (Bryant–Griffiths) equation for our system using the Maximum Principle formalism.

3.2.2 Controllability

Since we are dealing with an optimal control problem with boundary conditions, we should examine the controllability of our dynamical system.

The group $G = SE(2)$ is non–compact (therefore the controllability results of e.g. (Jurdjevic & Sussmann [1972]) do not apply directly), but can be viewed as the semi–direct product of the vector space $V = \mathbb{R}^2$ and of the compact connected Lie group $K = SO(2)$, denoted $G = V \times_S K$, with Lie algebra $\mathcal{G} = \mathbb{R}^2 \times so(2)$.

Consider a subset Γ of $\mathcal{G}$ and the associated family $\mathcal{X}_\Gamma$ of left–invariant vector fields on G. Then $\mathcal{X} \in \mathcal{X}_\Gamma$ if and only if $\mathcal{X}_e \in \Gamma$, where $\mathcal{X}_e$ is the value of $\mathcal{X}$ at the identity e of G. Moreover, consider the semi–group $S(\Gamma)$ generated by $\bigcup_{\mathcal{A}\in\Gamma} \{\exp \mathcal{A}t \,:\, t \geq 0\}$.

Definition 3.2.1 The family $\mathcal{X}_\Gamma$ of left–invariant vector fields is *transitive* on G if $S(\Gamma) = G$.

∎

Notice that if Γ is the set of elements of $\mathcal{G}$ associated with a left–invariant control system (Γ, U), where U is a class of admissible controls, then we know that for every $g_0 \in$ G, there exists a unique solution g of (Γ, U) corresponding to $u \in U$ and defined for $0 \leq t \leq \infty$ such that $g(0) = g_0$. Denote this solution by $g(g_0, u, t)$. If for some

$T \geq 0$, $g(g_0, u, T) = g_1$, we say that the control u *steers* g_0 to g_1 in T units of time. If there exists $u \in U$ which steers g_0 to g_1 in T units of time, we say that g_1 is *reachable* from g_0 at time T. We denote the set of all elements $g_1 \in G$ which are reachable from g_0 at time T by $\mathcal{R}(g_0, T)$ and the set of all elements of G reachable from g_0 by $\mathcal{R}(g_0)$. Then $\mathcal{R}(g_0) = \bigcup_{0 \leq T \leq \infty} \mathcal{R}(g_0, T)$.

Definition 3.2.2 The system (Γ, U) is *controllable* from g_0, if $\mathcal{R}(g_0) = G$.

∎

From the left–invariance of (Γ, U), it follows that $\mathcal{R}(g_0) = g_0 \mathcal{R}(e)$. Therefore, in order to study controllability of (Γ, U), it suffices to study its controllability from the identity.

Definition 3.2.3 A vector $v \in V$ is a *fixed point* under K, denoted $Kv = v$, if $Kv \stackrel{\text{def}}{=} \{kv | k \in K\} = \{v\}$.

∎

Proposition 3.2.4 (Bonnard et al. [1982])

Consider a group $G = V \times_S K$, which is the semi–direct product of a vector space V and of a compact connected Lie group K. Suppose that V and K are such that V admits no fixed non–zero points with respect to K. Consider a subset Γ of $\mathcal{G}$ and the associated family $\mathcal{X}_\Gamma$ of left–invariant vector fields on G. Then $\mathcal{X}_\Gamma$ is transitive on G if and only if the Lie algebra $\mathcal{L}(\Gamma)$ generated by the elements of Γ is equal to $\mathcal{G}$.

∎

Lemma 3.2.5

The system of equations (3.2.2) is controllable.

Proof

For the system (3.2.2) we have G $= SE(2)$, $V = I\!R^2$ and $K = SO(2)$. The only fixed point of V under K is $\{0\}$, since it is the only point (x, y) in $I\!R^2$ for which $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$, $\forall \theta$.

The system (3.2.2) can be written in the form $\frac{dg}{ds} = g(\mathcal{A}_2 + \kappa\mathcal{A}_1)$, for $\kappa \in I\!R$. Consider $\Gamma = \{\mathcal{A}_2 + \kappa\mathcal{A}_1 \,|\, \kappa \in I\!R\}$. It is easy to see that $\mathcal{L}(\Gamma) = sp\{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3\}$. Therefore, $\mathcal{L}(\Gamma) = \mathcal{G}$ and from Proposition 3.2.4, we conclude that the family of vector fields $\mathcal{X}_\Gamma$ is transitive on G.

Observe that the set $S(\Gamma)$ defined above is (by definition) the set of all elements of G reachable from the identity under the action of the vector fields in $\mathcal{X}_\Gamma$. Therefore, the notion of transitivity of $\mathcal{X}_\Gamma$ on G is exactly that of controllability of (Γ, U) from the identity. In view of the left–invariance of $\mathcal{X}_\Gamma$, controllability from the identity is equivalent to controllability from any $g_0 \in$ G . Thus, the system (3.2.2) is controllable from any $g_0 \in$ G .

∎

3.2.3 Hamiltonian functions, Maximum Principle and Poisson Reduction

Consider a left–invariant control system, on an n–dimensional matrix Lie group G with Lie algebra $\mathcal{G}$, of the form

$$\dot{g} = T_e L_g \cdot \xi_u = g\Big(\xi_0 + \sum_{i=1}^{m} u_i(t)\xi_i\Big) \, , \tag{3.2.4}$$

where $g \in G$, $\xi_i \in \mathcal{G}$ and $u_i(t) \in I\!R$, such that $\{\xi_0, \xi_1, \ldots, \xi_m\}$ span an $(m+1)$–dimensional subspace h of $\mathcal{G}$, where $\dim h \leq \dim \mathcal{G}$. Let $U = I\!R^m$ be the set of admissible controls.

Consider the optimal control problem of minimizing

$$\eta = \int_0^T L(u(t))dt = \frac{1}{2}\int_0^T \sum_{i=1}^{m} I_i u_i^2 dt$$

on the system (3.2.4), subject to the boundary conditions $g(0) = g_0$ and $g(T) = g_1$. Assume $I_i > 0$ for $i = 1, \ldots, m$. Notice that L does not depend on $g \in G$.

Let $<, >$ be the natural pairing between elements of a vector space V and those of its dual space V^*. Let $\{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n\}$ be a basis of the Lie algebra $\mathcal{G}$ and let

$\{\mathcal{A}_1^\flat, \mathcal{A}_2^\flat, \ldots, \mathcal{A}_n^\flat\}$ be the basis of its dual space $\mathcal{G}^*$, which is such that $< \mathcal{A}_i^\flat, \mathcal{A}_j >= \delta_i^j$, where δ_i^j is the Kronecker symbol. Any $\mu \in \mathcal{G}^*$ can be expressed as $\mu = \sum_{i=1}^n \mu_i \mathcal{A}_i^\flat$.

The tangent and cotangent bundle TG and T^*G of a Lie group G are trivial vector bundles, therefore can be represented as G $\times\, \mathcal{G}$ and G $\times\, \mathcal{G}^*$ respectively, where $\mathcal{G}^*$ is the dual space of $\mathcal{G}$.

A generic element of $\mathcal{G} = se(2)$ is represented in the basis $\{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3\}$ as $\mathcal{A} = \begin{pmatrix} A & a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -a_1 & a_2 \\ a_1 & 0 & a_3 \\ 0 & 0 & 0 \end{pmatrix}$, $a_i \in I\!R$. Consider two elements $\mathcal{A}_1 = \begin{pmatrix} A_1 & a_1 \\ 0 & 0 \end{pmatrix}$ and $\mathcal{A}_2 = \begin{pmatrix} A_2 & a_2 \\ 0 & 0 \end{pmatrix}$ of $\mathcal{G} = se(2)$ and consider the following inner product $< . , . >_{\mathcal{G}}$ on $\mathcal{G}$:

$$< \mathcal{A}_1, \mathcal{A}_2 >_{\mathcal{G}} = < a_1, a_2 >_n + \frac{1}{2} tr(A_1^\top A_2) ,$$

where $<, >_n$ is the inner product in $I\!R^n$ and tr is the trace operator in $I\!R^{n \times n}$. A generic element μ of $\mathcal{G}^*$ can be identified, using the Riesz Representation Theorem, with an element $\mu^\sharp \stackrel{\text{def}}{=} \begin{pmatrix} A_\mu & a_\mu \\ 0 & 0 \end{pmatrix}$ of $\mathcal{G}$ via the pairing $\mu(\mathcal{A})$ (also denoted as $< \mu, \mathcal{A} >$), with any $\mathcal{A} \in \mathcal{G}$, defined as:

$$\begin{aligned} \mu(\mathcal{A}) &= < \mu, \mathcal{A} > \stackrel{\text{def}}{=} < \mu^\sharp, \mathcal{A} >_{\mathcal{G}} \\ &= < \begin{pmatrix} A_\mu & a_\mu \\ 0 & 0 \end{pmatrix} , \begin{pmatrix} A & a \\ 0 & 0 \end{pmatrix} >_{\mathcal{G}} = < a_\mu, a >_n + \frac{1}{2} tr(A_\mu^\top A) . \end{aligned}$$

Consider an element $\alpha_g \stackrel{\text{def}}{=} (g, \mu)$ of T^*G $\approx G \times \mathcal{G}^*$ with $g \in G$ and $\mu \in \mathcal{G}^*$ and define the pre–Hamiltonian:

$$H_\lambda(\alpha_g, u) = -\lambda L(u) + < \alpha_g, T_e L_g \cdot \xi_u > = -\lambda \frac{1}{2} \sum_{i=1}^m I_i u_i^2 + \left\langle \alpha_g, T_e L_g \cdot \left(\xi_0 + \sum_{i=1}^m u_i(t) \xi_i\right) \right\rangle , \tag{3.2.5}$$

where $\lambda = 0$ or $\lambda = 1$ and where $<, >$ is the natural pairing between the elements of TG and those of T^*G.

Proposition 3.2.6 (Maximum Principle)

Suppose $g^*(.)$ is a trajectory of (3.2.4) that corresponds to the control $u^*(.)$ that minimizes η. This trajectory is a projection of an integral curve $\alpha_g^*(.) = (g^*(.), \mu^*(.))$

of the Hamiltonian vector field $X_{H_\lambda(\alpha_g^*,u^*)}$ defined on the entire interval $[0,L]$, which satisfies:

i) If $\lambda = 0$, then $\mu^*(s) \neq 0$, for some $s \in [0,L]$ (α_g^* is not the zero section of T^*G).

ii) $H_\lambda(\alpha_g^*(s), u^*(s)) \geq H_\lambda(\alpha_g^*(s), u(s))$, for any $u(.) \in U$ and for almost all $s \in [0,L]$.

iii) When L is fixed, $H_\lambda(\alpha_g^*(s), u^*(s))$ is constant for almost all $s \in [0,L]$. When L is free to vary, $H_\lambda(\alpha_g^*(s), u^*(s)) = 0$ for almost all $s \in [0,L]$.

∎

For $\lambda = 0$, the integral curve $\alpha_g^*(.)$ is called an *exceptional* extremal and is independent of the cost functional. For $\lambda = 1$, $\alpha_g^*(.)$ is a *regular* extremal. In either case, since the set of admissible controls U is unbounded, condition (ii) above implies $\frac{\partial H_\lambda}{\partial u}(\alpha_g^*(s), u^*(s)) = 0$ and $\frac{\partial^2 H_\lambda}{\partial u^2}(\alpha_g^*(s), u^*(s)) \leq 0$, for almost all $s \in [0,L]$.

Proposition 3.2.7 (Krishnaprasad [1993])

Consider the left–invariant control system of (3.2.4) and the optimal control problem of minimizing η subject to the boundary conditions $g(0) = g_0$ and $g(T) = g_1$.

Consider the hamiltonian

$$h = <\mu, \xi_0> + \frac{1}{2}\sum_{i=1}^{m} \frac{<\mu, \xi_i>^2}{I_i}, \tag{3.2.6}$$

where $\mu \subset \mathcal{G}^*$ is an integral curve of the Hamiltonian vector field X_h on the Lie–Poisson manifold $\mathcal{G}^*_- = (\mathcal{G}^*, \{\,,\,\}_-)$, given in local coordinates by (2.4.3) with the bracket $\{\,,\,\}_-$ given by (2.4.5) and (2.4.6). Then, the *regular extremals* of the problem are given by

$$u_i = \frac{<\mu, \xi_i>}{I_i}. \tag{3.2.7}$$

∎

In order to apply this result to problem (P_2), let $u_1 \stackrel{\text{def}}{=} \kappa$ and consider the pre-hamiltonian for $\lambda = 1$:

$$H = -\frac{1}{2}u_1^2 + \left\langle \alpha_g, T_e L_g \cdot \left(\mathcal{A}_2 + u_1 \mathcal{A}_1\right)\right\rangle, \tag{3.2.8}$$

where $\alpha_g \in T^*G \cong G \times \mathcal{G}^*$. However,

$$\begin{aligned}\left\langle \alpha_g, T_e L_g \cdot \left(\mathcal{A}_2 + u_1 \mathcal{A}_1\right)\right\rangle &= \left\langle (T_e L_g)^* \cdot \alpha_g, \mathcal{A}_2 + u_1 \mathcal{A}_1\right\rangle \\ &= \left\langle \mu, \mathcal{A}_2 + u_1 \mathcal{A}_1\right\rangle \ ,\end{aligned} \tag{3.2.9}$$

where $\mu = (T_e L_g)^* \cdot \alpha_g \in \mathcal{G}^*$. Then,

$$H = -\frac{1}{2}u_1^2 + < \mu, \mathcal{A}_2 > + u_1 < \mu, \mathcal{A}_1 > \ .$$

From the Maximum Principle, we have:

$$\frac{\partial H}{\partial u_1} = -u_1 + < \mu, \mathcal{A}_1 >= 0 \implies u_1 =< \mu, \mathcal{A}_1 >= \mu_1 \ , \tag{3.2.10}$$

then we can specify the problem's hamiltonian, which is a function $H \in C^\infty(T^*G)$:

$$H =< \mu, \mathcal{A}_2 > + \frac{1}{2} < \mu, \mathcal{A}_1 >^2 = \mu_2 + \frac{1}{2}\mu_1^2 \ . \tag{3.2.11}$$

Thus, the trajectory $g \in G$ corresponding to the optimal control u is a projection of an integral curve $\alpha_g = (g, \mu)$ of the hamiltonian vector field X_H on T^*G, that corresponds to the function $H \in C^\infty(T^*G)$. However, from (3.2.11) we observe that H is G–invariant, thus the hamiltonian vector field X_H can be reduced to a hamiltonian vector field X_h on $\mathcal{G}^*$, that corresponds to the function $h \in C^\infty(\mathcal{G}^*)$:

$$h =< \mu, \mathcal{A}_2 > + \frac{1}{2} < \mu, \mathcal{A}_1 >^2 = \mu_2 + \frac{1}{2}\mu_1^2 \ . \tag{3.2.12}$$

For the given basis of $\mathcal{G} = se(2)$ we have from (2.4.6):

$$\Lambda^- = \begin{pmatrix} 0 & -\mu_3 & \mu_2 \\ \mu_3 & 0 & 0 \\ -\mu_2 & 0 & 0 \end{pmatrix} \ . \tag{3.2.13}$$

Corollary 3.2.8

The regular extrema of problem (P_2) are given by

$$\kappa =< \mu, \mathcal{A}_1 >= \mu_1 \ , \tag{3.2.14}$$

where $\mu \subset \mathcal{G}^*$ is an integral curve of the Hamiltonian vector field X_h on $\mathcal{G}^*_-$ corresponding to the hamiltonian

$$h = < \mu, \mathcal{A}_2 > + \frac{1}{2} < \mu, \mathcal{A}_1 >^2 = \mu_2 + \frac{1}{2}\mu_1^2 \, . \tag{3.2.15}$$

This integral curve is given in local coordinates of $\mathcal{G}^*$ by

$$\begin{aligned} \dot{\mu}_1 &= -\mu_3 \, , \\ \dot{\mu}_2 &= \mu_1 \mu_3 \, , \\ \dot{\mu}_3 &= -\mu_1 \mu_2 \, . \end{aligned} \tag{3.2.16}$$

The integral curves $\mu \subset \mathcal{G}^*$ of X_h are the intersections of the level sets of the hamiltonian h with the symplectic leaves of $\mathcal{G}^*_-$. The level sets of the hamiltonian h are the parabolic cylinders $\{\mu : \frac{1}{2}\mu_1^2 + \mu_2 = h\}$. The Casimir functions of the Lie–Poisson manifold $\mathcal{G}^*_-$ are of the form $\Phi = \Phi(\mu_2^2 + \mu_3^2)$. Their level sets (symplectic leaves of $\mathcal{G}^*_-$) are the circular cylinders $\{\mu : \mu_2^2 + \mu_3^2 = C^2\}$ (fig. 3.2.3).

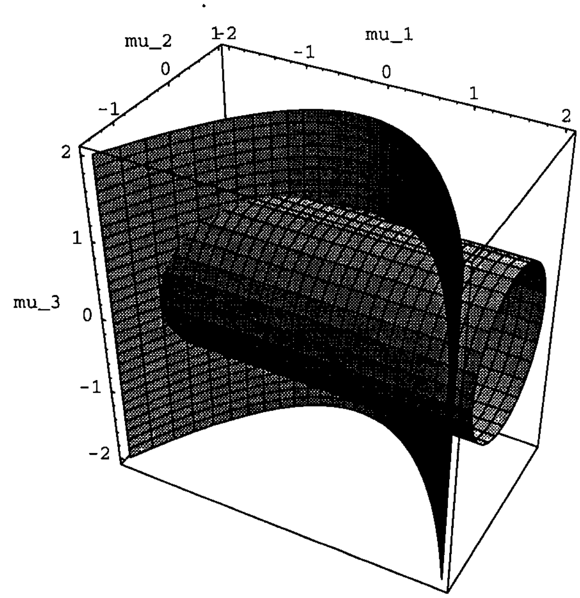


Fig. 3.2.3: Integral Curves of the Hamiltonian v.f. X_h on the Lie–Poisson Manifold $\mathcal{G}^*_-$

Proof

The result follows from Proposition 3.2.7, using (2.4.3) and (3.2.7).

By differentiation, it is easy to verify that $\frac{1}{2}\mu_1^2 + \mu_2$ and $\mu_2^2 + \mu_3^2$ are constant on the integral curves (3.2.16):

$$\frac{d}{dt}\Big(\frac{1}{2}\mu_1^2 + \mu_2\Big) = \mu_1\dot{\mu}_1 + \dot{\mu}_2 \overset{(3.2.16)}{=} -\mu_1\mu_3 + \mu_1\mu_3 = 0$$

$$\frac{d}{dt}(\mu_2^2 + \mu_3^2) = 2(\mu_2\dot{\mu}_2 + \mu_3\dot{\mu}_3) \overset{(3.2.16)}{=} 2(\mu_1\mu_2\mu_3 - \mu_1\mu_2\mu_3) = 0 \,.$$

■

3.2.4 The Bryant–Griffiths Optimality Constraint

Theorem 3.2.9 (Bryant–Griffiths Optimality Constraint)

The curvature $\kappa(.)$ that corresponds to regular extremals of the variational problem (P_2) satisfies the following optimality constraint when L is fixed:

$$\Big(\frac{d\kappa}{ds}(s)\Big)^2 + \Big(H - \frac{1}{2}\kappa^2(s)\Big)^2 = \mathcal{C}^2 \,, \qquad (3.2.17)$$

where $\mathcal{C}^2 \overset{\text{def}}{=} \big(\frac{d\kappa}{ds}(0)\big)^2 + \big(H - \frac{1}{2}\kappa^2(0)\big)^2$ with $\mathcal{C} \geq 0$.

When L is free, the curvature satisfies:

$$\Big(\frac{d\kappa}{ds}(s)\Big)^2 + \frac{1}{4}\kappa^4(s) = \mathcal{C}^2 \,, \qquad (3.2.18)$$

where $\mathcal{C}^2 \overset{\text{def}}{=} \big(\frac{d\kappa}{ds}(0)\big)^2 + \frac{1}{4}\kappa^4(0)$ with $\mathcal{C} \geq 0$.

Proof

From (3.2.16), by differentiating

$$\ddot{\mu}_1 = -\dot{\mu}_2 = \mu_1\mu_2 = \mu_1(H - \frac{1}{2}\mu_1^2) \,. \qquad (3.2.19)$$

Observe that

$$\frac{d}{dt}\Big[\frac{1}{2}\dot{\mu}_1^2\Big] = \dot{\mu}_1\ddot{\mu}_1 \overset{(3.2.19)}{=} \mu_1(H - \frac{1}{2}\mu_1^2)\dot{\mu}_1 \,.$$

By integrating we get

$$(\dot{\mu}_1)^2 + (H - \frac{1}{2}\mu_1^2)^2 = \mathcal{C}^2 \,,$$

where $C \stackrel{\text{def}}{=} \sqrt{\dot{\mu}_1^2(0) + (H - \frac{1}{2}\mu_1^2(0))^2}$. Then (3.2.17) follows from (3.2.14).

When L is free, then $H = 0$ and (3.2.18) follows from (3.2.17).

∎

3.2.5 Solutions of the Bryant–Griffiths Optimality Constraint

The solutions of the Bryant–Griffiths optimality constraint for free L (equation (3.2.18)) are the special case for $H = 0$ of the solutions of the corresponding constraint for fixed L (equation (3.2.17)) (see case 3 below). Thus we examine only the solutions of (3.2.17) in detail.

Theorem 3.2.10

The generic solution of the Bryant–Griffiths optimality constraint (equation (3.2.17)) involves Jacobi elliptic functions. Some special cases involve only elementary functions. The solutions of equation (3.2.17) can be classified as follows, when $C \neq 0$:

1. $H > C$: The curvature κ is given by $\kappa = \pm\kappa_1 \operatorname{dn}(\frac{\kappa_1}{2}s + \delta, k)$, where $k = \frac{\sqrt{\kappa_1{}^2 - \kappa_3{}^2}}{\kappa_1}$, $\kappa_1 = \sqrt{2(H + C)}$ and $\kappa_3 = \sqrt{2(H - C)}$. The constant δ depends on the initial condition $\kappa(0)$. The solutions and the corresponding $(\kappa, \dot{\kappa})$–plot are shown in fig. 3.2.4.

2. $H = C$: The curvature κ is given by $\kappa = \pm\kappa_1 \operatorname{sech}(\frac{\kappa_1}{2}s + \delta)$, where $\kappa_1 = 2\sqrt{C}$ and the constant δ depends on the initial condition $\kappa(0)$. The solutions and the corresponding $(\kappa, \dot{\kappa})$–plot are shown in fig. 3.2.5.

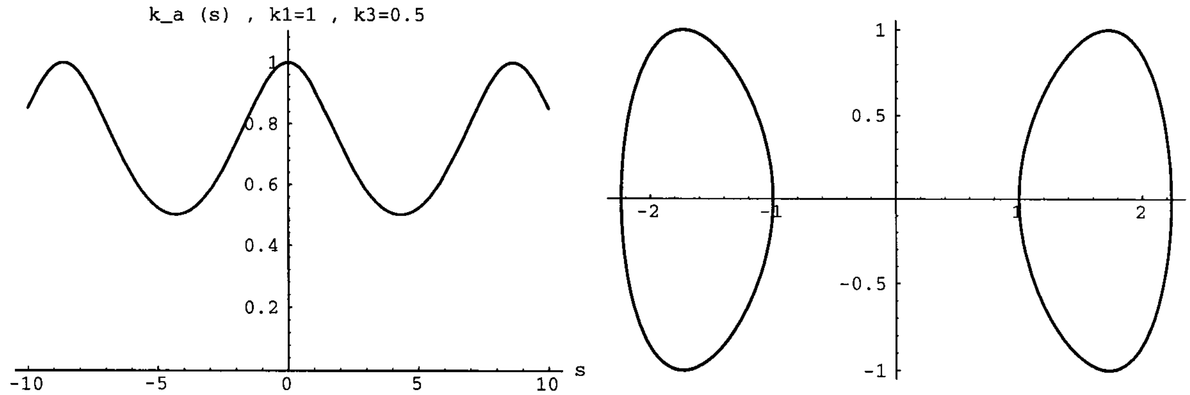
3. $-C < H < C$: The curvature κ is given by $\kappa = \kappa_1 \operatorname{cn}(\frac{\sqrt{\kappa_1{}^2 + \kappa_3{}^2}}{2}s + \delta, k)$, where $k = \frac{\kappa_1}{\sqrt{\kappa_1{}^2 + \kappa_3{}^2}}$, $\kappa_1 = \sqrt{2(H + C)}$ and $\kappa_3 = \sqrt{2(C - H)}$. The constant δ depends on the initial condition $\kappa(0)$. There are three subcases, namely $H < 0$, $H = 0$ and $H > 0$. The solutions and the corresponding $(\kappa, \dot{\kappa})$–plots are shown in fig. 3.2.6.

4. $H = -C$: The curvature κ is given by $\kappa = 0$.

5. $H < -C$: There are no real solutions.

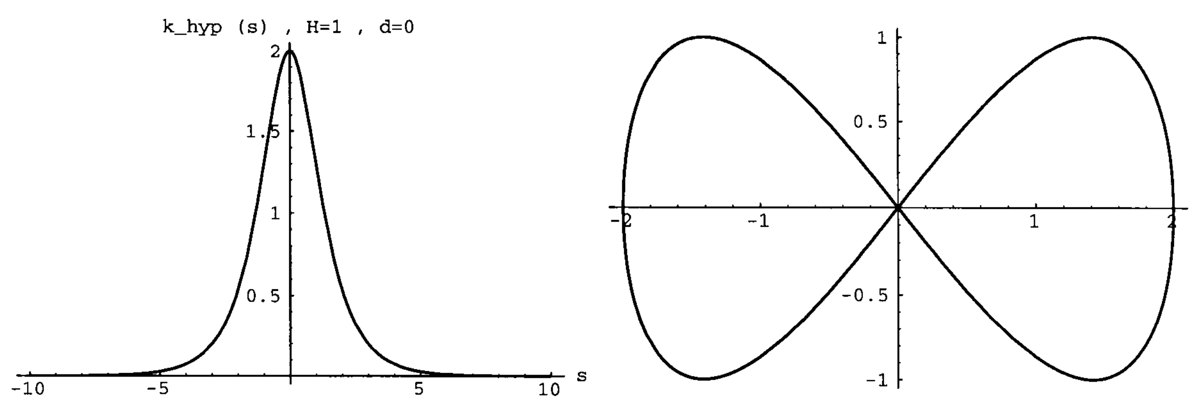
When $C = 0$, equation (3.2.17) admits only the constant real solutions $\kappa = \pm\sqrt{2H}$.

The $(\kappa, \dot{\kappa})$–plots in figs. 3.2.4.b-3.2.6.b are the projections on the (μ_1, μ_3)–plane of the integral curves of the Hamiltonian vector field X_h. As mentioned already, those are the intersections of the level sets of the hamiltonian h (the parabolic cylinder in fig. 3.2.3) with the symplectic leaves of $\mathcal{G}^*_-$ (the circular cylinder in fig. 3.2.3). The projection of their intersection, for various relative positions of the two ruled surfaces, produces those $(\kappa, \dot{\kappa})$–plots and gives rise to the above classification of the solutions.



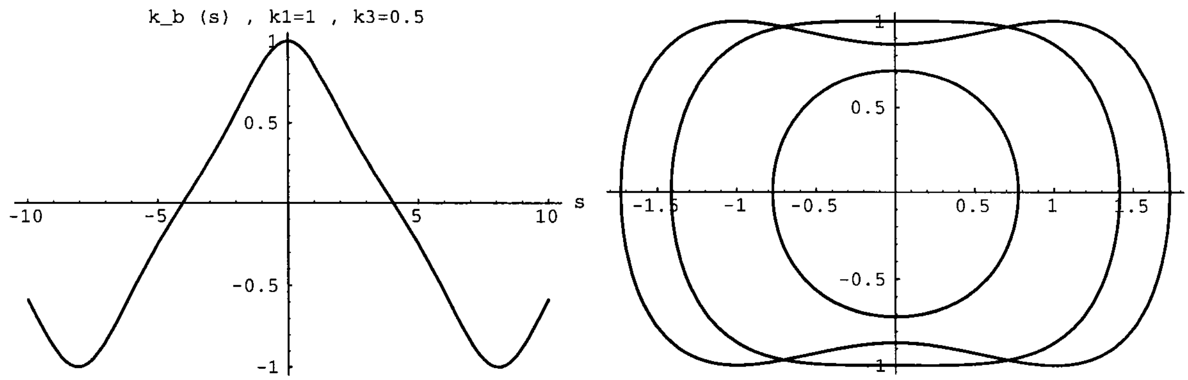
(a) (b)

Fig. 3.2.4: Solutions of the Bryant–Griffiths Optimality Constraint when $H > \mathcal{C}$



(a) (b)

Fig. 3.2.5: Solutions of the Bryant–Griffiths Optimality Constraint when $H = \mathcal{C}$



(a) (b)

Fig. 3.2.6: Solutions of the Bryant–Griffiths Optimality Constraint when $-\mathcal{C} < H < \mathcal{C}$

Proof

Let $\mathcal{C} \neq 0$. From (3.2.17) we have:

$$(\frac{d\kappa}{ds})^2 = \mathcal{C}^2 - (H - \frac{1}{2}\kappa^2)^2 = -\frac{1}{4}\left[\kappa^2 - 2(H+\mathcal{C})\right]\left[\kappa^2 - 2(H-\mathcal{C})\right] . \qquad (3.2.20)$$

Define $g_1(\kappa) \stackrel{\text{def}}{=} \left[\kappa^2 - 2(H+\mathcal{C})\right]\left[\kappa^2 - 2(H-\mathcal{C})\right]$. To specify its roots, let $\mathcal{K} \stackrel{\text{def}}{=} \kappa^2$ and define $g_2(\mathcal{K}) \stackrel{\text{def}}{=} [\mathcal{K} - 2(H+\mathcal{C})][\mathcal{K} - 2(H-\mathcal{C})]$, whose roots are $\mathcal{K}_1 = 2(H+\mathcal{C})$ and $\mathcal{K}_3 = 2(H-\mathcal{C})$. Observe that $\mathcal{K}_1, \mathcal{K}_3 \in I\!R$ and $\mathcal{K}_3 < \mathcal{K}_1$. In order to solve (3.2.20), it remains to specify the roots of $g_1(\kappa)$. The real roots of $g_1(\kappa)$ specify the constant solutions of (3.2.17). Consider the following cases:

1. $H > \mathcal{C}$

In this case $0 < \mathcal{K}_3 < \mathcal{K}_1$. Define $\kappa_1 \stackrel{\text{def}}{=} \sqrt{\mathcal{K}_1}$ and $\kappa_3 \stackrel{\text{def}}{=} \sqrt{\mathcal{K}_3}$ and observe that $0 < \kappa_3 < \kappa_1$. Then, the roots of $g_1(\kappa)$ are $\{\kappa_1, -\kappa_1, \kappa_3, -\kappa_3\}$ and from (3.2.20)

$$(\frac{d\kappa}{ds})^2 = -\frac{1}{4}(\kappa^2 - \mathcal{K}_1)(\kappa^2 - \mathcal{K}_3) = \frac{1}{4}(\kappa_1^2 - \kappa^2)(\kappa^2 - \kappa_3^2) . \qquad (3.2.21)$$

From this we see that for $\frac{d\kappa}{ds}$ to be real, we should have $(-\kappa_1 \leq \kappa \leq -\kappa_3)$ or $(\kappa_3 \leq \kappa \leq \kappa_1)$.

Let $\kappa_3 < \kappa < \kappa_1$ and assume $\kappa(0) = \kappa_3$. Then $\frac{d\kappa}{ds} \geq 0$ and from (3.2.21) and (2.2.26) we have:

$$
\begin{aligned}
\frac{d\kappa}{ds}(s) &= \frac{1}{2}\sqrt{(\kappa_1^2 - \kappa^2)(\kappa^2 - \kappa_3^2)} \\
&\Rightarrow \int_{\kappa_3}^{\kappa(s)} \frac{d\kappa}{\sqrt{(\kappa_1^2 - \kappa^2)(\kappa^2 - \kappa_3^2)}} = \frac{1}{2}\int_0^s dt \\
&\Rightarrow \frac{1}{\kappa_1}\,\mathrm{nd}^{-1}\Big(\frac{\kappa(s)}{\kappa_3}, \frac{\sqrt{\kappa_1{}^2 - \kappa_3{}^2}}{\kappa_1}\Big) = \frac{s}{2} \\
&\Rightarrow \kappa(s) = \kappa_1 k'\,\mathrm{nd}\big(\frac{\kappa_1}{2}s, k\big) = \kappa_1\,\mathrm{dn}\big(\frac{\kappa_1}{2}s + K(k), k\big)\ ,
\end{aligned}
\tag{3.2.22}
$$

where nd and dn are the Jacobi elliptic functions with *modulus* $k = \frac{\sqrt{\kappa_1{}^2 - \kappa_3{}^2}}{\kappa_1}$ and *complementary modulus* $k' = \frac{\kappa_3}{\kappa_1}$. Also, let $m \stackrel{\text{def}}{=} k^2$. The solution $\kappa(.)$ has period $T = \frac{2}{\kappa_1}2K(k) = \frac{4K(k)}{\kappa_1}$, where $K(k)$ is the complete elliptic integral of the first kind. This solution is shown in fig. 3.2.4 (for $\kappa_1 = 1$ and $\kappa_3 = 0.5$).

Now assume $\kappa(0) = \kappa_1$. Then $\frac{d\kappa}{ds} \leq 0$ and from (3.2.21) and (2.2.27) we have:

$$
\begin{aligned}
\frac{d\kappa}{ds}(s) &= -\frac{1}{2}\sqrt{(\kappa_1^2 - \kappa^2)(\kappa^2 - \kappa_3^2)} \\
&\Rightarrow \int_{\kappa(s)}^{\kappa_1} \frac{d\kappa}{\sqrt{(\kappa_1^2 - \kappa^2)(\kappa^2 - \kappa_3^2)}} = \frac{1}{2}\int_0^s dt \\
&\Rightarrow \frac{1}{\kappa_1}\,\mathrm{dn}^{-1}\Big(\frac{\kappa(s)}{\kappa_1}, \frac{\sqrt{\kappa_1{}^2 - \kappa_3{}^2}}{\kappa_1}\Big) = \frac{s}{2} \\
&\Rightarrow \kappa(s) = \kappa_1\,\mathrm{dn}\big(\frac{\kappa_1 s}{2}, k\big)\ .
\end{aligned}
\tag{3.2.23}
$$

Let $-\kappa_1 < \kappa < -\kappa_3$ and assume $\kappa(0) = -\kappa_1$. Then $\frac{d\kappa}{ds} \geq 0$ and from (3.2.21) and

(2.2.27) we have:

$$
\begin{aligned}
\frac{d\kappa}{ds}(s) &= \frac{1}{2}\sqrt{(\kappa_1^2-\kappa^2)(\kappa^2-\kappa_3^2)} \\
&\Rightarrow \int_{-\kappa_1}^{\kappa(s)} \frac{d\kappa}{\sqrt{(\kappa_1^2-\kappa^2)(\kappa^2-\kappa_3^2)}} = \frac{1}{2}\int_0^s dt \\
&\Rightarrow \frac{1}{\kappa_1}\,\mathrm{dn}^{-1}\Big(\frac{-\kappa(s)}{\kappa_1}, \frac{\sqrt{{\kappa_1}^2-{\kappa_3}^2}}{\kappa_1}\Big) = \frac{s}{2} \\
&\Rightarrow \kappa(s) = -\kappa_1\,\mathrm{dn}\Big(\frac{\kappa_1 s}{2}, k\Big)\ .
\end{aligned}
\tag{3.2.24}
$$

Now assume $\kappa(0) = -\kappa_3$. Then $\frac{d\kappa}{ds} \leq 0$ and from (3.2.21) and (2.2.26) we have:

$$
\begin{aligned}
\frac{d\kappa}{ds}(s) &= -\frac{1}{2}\sqrt{(\kappa_1^2-\kappa^2)(\kappa^2-\kappa_3^2)} \\
&\Rightarrow \int_{-\kappa_3}^{\kappa(s)} \frac{d\kappa}{\sqrt{(\kappa_1^2-\kappa^2)(\kappa^2-\kappa_3^2)}} = -\frac{1}{2}\int_0^s dt \\
&\Rightarrow \frac{1}{\kappa_1}\,\mathrm{nd}^{-1}\Big(\frac{-\kappa(s)}{\kappa_3}, \frac{\sqrt{{\kappa_1}^2-{\kappa_3}^2}}{\kappa_1}\Big) = \frac{s}{2} \\
&\Rightarrow \kappa(s) = -\kappa_1 k'\,\mathrm{nd}\Big(\frac{\kappa_1}{2}s, k\Big) = -\kappa_1\,\mathrm{dn}\Big(\frac{\kappa_1}{2}s + K(k), k\Big)\ .
\end{aligned}
\tag{3.2.25}
$$

2. $H = \mathcal{C}$

In this case $0 = \mathcal{K}_3 < \mathcal{K}_1$. Define $\kappa_1 \stackrel{\text{def}}{=} \sqrt{\mathcal{K}_1}$ and $\kappa_3 \stackrel{\text{def}}{=} \sqrt{\mathcal{K}_3} = 0$ and observe that $0 = \kappa_3 < \kappa_1$. Then, the roots of $g_1(\kappa)$ are $\{\kappa_1, -\kappa_1, 0 \text{ (double root)}\}$ and from (3.2.20) we get

$$
\Big(\frac{d\kappa}{ds}\Big)^2 = \frac{1}{4}\kappa^2({\kappa_1}^2 - \kappa^2)\ . \tag{3.2.26}
$$

Observe that $\frac{d\kappa}{ds}$ is real only when $-\kappa_1 \leq \kappa \leq \kappa_1$. From (3.2.26) we get:

$$
\frac{d\kappa}{ds} = \pm\frac{1}{2}\kappa\sqrt{\kappa_1^2 - \kappa^2}\ . \tag{3.2.27}
$$

By integrating and after defining $\delta \stackrel{\text{def}}{=} \ln\big(\frac{\kappa_0}{\kappa_1 + \sqrt{\kappa_1^2 - \kappa_0^2}}\big)$, we get for $0 \leq \kappa \leq \kappa_1$:

$$
\kappa(s) = \kappa_1\,\mathrm{sech}\Big(\frac{\kappa_1}{2}s \pm \delta\Big) \tag{3.2.28}
$$

and for $-\kappa_1 \leq \kappa \leq 0$:

$$\kappa(s) = -\kappa_1 \operatorname{sech}(\frac{\kappa_1}{2}s \pm \delta) . \tag{3.2.29}$$

The positive solution is shown in fig. 3.2.5 (for $\delta = 0$ and $H = 1$).

3. $-\mathcal{C} < H < \mathcal{C}$

In this case $\mathcal{K}_3 < 0 < \mathcal{K}_1$. Define $\kappa_1 \stackrel{\text{def}}{=} \sqrt{\mathcal{K}_1}$ and $\kappa_3 \stackrel{\text{def}}{=} \sqrt{-\mathcal{K}_3}$. Observe that when $H < 0$ we have $0 < \kappa_1 < \kappa_3$, when $H = 0$ we have $0 < \kappa_1 = \kappa_3$, and when $H > 0$ we have $0 < \kappa_3 < \kappa_1$. The roots of $g_1(\kappa)$ are $\{\kappa_1, -\kappa_1, i\kappa_3, -i\kappa_3\}$ and from (3.2.20):

$$(\frac{d\kappa}{ds})^2 = \frac{1}{4}(\mathcal{K}_1 - \kappa^2)(\kappa^2 + \mathcal{K}_3) = \frac{1}{4}(\kappa_1^2 - \kappa^2)(\kappa^2 + \kappa_3^2) . \tag{3.2.30}$$

From this we see that for $\frac{d\kappa}{ds}$ to be real, we should have $-\kappa_1 \leq \kappa \leq \kappa_1$. We distinguish the cases $0 \leq \kappa \leq \kappa_1$ and $-\kappa_1 \leq \kappa \leq 0$.

Let $0 \leq \kappa \leq \kappa_1$ and assume $\kappa(0) = 0$. Then $\frac{d\kappa}{ds} \geq 0$ and from (3.2.30) and (2.2.23) we have:

$$\begin{aligned}
&\frac{d\kappa}{ds} = \frac{1}{2}\sqrt{(\kappa_1^2 - \kappa^2)(\kappa^2 + \kappa_3^2)} \\
&\qquad \Rightarrow \int_0^{\kappa(s)} \frac{d\kappa}{\sqrt{(\kappa_1^2 - \kappa^2)(\kappa^2 + \kappa_3^2)}} = \frac{1}{2}\int_0^s dt \\
&\qquad \Rightarrow \frac{1}{\sqrt{\kappa_1^2 + \kappa_3^2}} \operatorname{sd}^{-1}(\frac{\sqrt{\kappa_1^2 + \kappa_3^2}}{\kappa_1\kappa_3}\kappa(s) , k) = \frac{s}{2} \\
&\qquad \Rightarrow \kappa(s) = \kappa_1 k' \operatorname{sd}(\frac{\sqrt{\kappa_1^2 + \kappa_3^2}}{2}s , k) = \kappa_1 \operatorname{cn}(\frac{\sqrt{\kappa_1^2 + \kappa_3^2}}{2}s + 3K(k) , k) ,
\end{aligned} \tag{3.2.31}$$

where the modulus is $k = \frac{\kappa_1}{\sqrt{\kappa_1^2+\kappa_3^2}}$ and the complementary modulus is $k' = \frac{\kappa_3}{\sqrt{\kappa_1^2+\kappa_3^2}}$.
The solution κ has period $T = \frac{2}{\sqrt{\kappa_1^2+\kappa_3^2}}4K(k) = \frac{8K(k)}{\sqrt{\kappa_1^2+\kappa_3^2}}$.

Now assume that $\kappa(0) = \kappa_1$. Then $\frac{d\kappa}{ds} \leq 0$ and from (3.2.30) and (2.2.21) we have:

$$
\begin{aligned}
\frac{d\kappa}{ds}(s) &= -\frac{1}{2}\sqrt{(\kappa_1^2 - \kappa^2)(\kappa^2 + \kappa_3^2)} \\
&\Rightarrow -\int_{\kappa_1}^{\kappa(s)} \frac{d\kappa}{\sqrt{(\kappa_1^2 - \kappa^2)(\kappa^2 + \kappa_3^2)}} = \frac{1}{2}\int_0^s dt \\
&\Rightarrow \frac{1}{\sqrt{\kappa_1^2 + \kappa_3^2}} \operatorname{cn}^{-1}\left(\frac{\kappa}{\kappa_1}, k\right) = \frac{s}{2} \\
&\Rightarrow \kappa(s) = \kappa_1 \operatorname{cn}\left(\frac{\sqrt{\kappa_1^2 + \kappa_3^2}}{2} s, k\right),
\end{aligned}
\tag{3.2.32}
$$

where the modulus is the same as before.

Let $-\kappa_1 \leq \kappa \leq 0$ and assume $\kappa(0) = 0$. Then $\frac{d\kappa}{ds} \leq 0$ and from (3.2.30) and (2.2.23) we have:

$$
\begin{aligned}
\frac{d\kappa}{ds}(s) &= -\frac{1}{2}\sqrt{(\kappa_1^2 - \kappa^2)(\kappa^2 + \kappa_3^2)} \\
&\Rightarrow -\int_0^{\kappa(s)} \frac{d\kappa}{\sqrt{(\kappa_1^2 - \kappa^2)(\kappa^2 + \kappa_3^2)}} = \frac{1}{2}\int_0^s dt \\
&\Rightarrow \frac{1}{\sqrt{\kappa_1^2 + \kappa_3^2}} \operatorname{sd}^{-1}\left(\frac{\sqrt{\kappa_1^2 + \kappa_3^2}}{\kappa_1\kappa_3} - \kappa(s), k\right) = \frac{s}{2} \\
&\Rightarrow \kappa(s) = -\kappa_1 k' \operatorname{sd}\left(\frac{\sqrt{\kappa_1^2 + \kappa_3^2}}{2} s, k\right) = \kappa_1 \operatorname{cn}\left(\frac{\sqrt{\kappa_1^2 + \kappa_3^2}}{2} s + K(k), k\right),
\end{aligned}
\tag{3.2.33}
$$

where the modulus and complementary modulus are the same as before.

Now assume that $\kappa(0) = -\kappa_1$. Then $\frac{d\kappa}{ds} \geq 0$ and from (3.2.30) and (2.2.21) we have:

$$\frac{d\kappa}{ds}(s) = \frac{1}{2}\sqrt{(\kappa_1^2 - \kappa^2)(\kappa^2 + \kappa_3^2)}$$

$$\Rightarrow \int_{-\kappa_1}^{\kappa(s)} \frac{d\kappa}{\sqrt{(\kappa_1^2 - \kappa^2)(\kappa^2 + \kappa_3^2)}} = \frac{1}{2}\int_0^s dt$$

$$\Rightarrow \frac{1}{\sqrt{\kappa_1^2 + \kappa_3^2}} \operatorname{cn}^{-1}(\frac{-\kappa}{\kappa_1}\,, k) = \frac{s}{2}$$

$$\Rightarrow \kappa(s) = -\kappa_1 \operatorname{cn}(\frac{\sqrt{\kappa_1^2 + \kappa_3^2}}{2}s\,, k) = \kappa_1 \operatorname{cn}(\frac{\sqrt{\kappa_1^2 + \kappa_3^2}}{2}s + 2K(k)\,, k)\,, \tag{3.2.34}$$

where the modulus is the same as before.

4. $H = -\mathcal{C}$

In this case $\mathcal{K}_3 < \mathcal{K}_1 = 0$. Define $\kappa_1 \overset{\text{def}}{=} \sqrt{-\mathcal{K}_1}$ and $\kappa_3 \overset{\text{def}}{=} \sqrt{-\mathcal{K}_3}$ and observe that $0 = \kappa_1 < \kappa_3$. Then, the roots of $g_1(\kappa)$ are $\{0 \text{ (double root)}, i\kappa_3, -i\kappa_3\}$ and from (3.2.20)

$$(\frac{d\kappa}{ds})^2 = -\frac{1}{4}\kappa^2(\kappa^2 - \mathcal{K}_3) = -\frac{1}{4}\kappa^2(\kappa^2 + \kappa_3^2) \leq 0\,. \tag{3.2.35}$$

From this we see that the only real solution in this case is $\kappa(s) = 0$.

5. $H < -\mathcal{C}$

In this case $\mathcal{K}_3 < \mathcal{K}_1 < 0$. Define $\kappa_1 \overset{\text{def}}{=} \sqrt{-\mathcal{K}_1}$ and $\kappa_3 \overset{\text{def}}{=} \sqrt{-\mathcal{K}_3}$ and observe that $0 < \kappa_1 < \kappa_3$. Then, the roots of $g_1(\kappa)$ are $\{i\kappa_1, -i\kappa_1, i\kappa_3, -i\kappa_3\}$ and from (3.2.20)

$$(\frac{d\kappa}{ds})^2 = -\frac{1}{4}(\kappa^2 - \mathcal{K}_1)(\kappa^2 - \mathcal{K}_3) = -\frac{1}{4}(\kappa^2 + \kappa_1^2)(\kappa^2 + \kappa_3^2) < 0\,. \tag{3.2.36}$$

From this we see that there are no real solutions in this case.

6. (Constant solutions)

Consider the case $\frac{d\kappa}{ds}(s) = 0$. The real roots of $g_1(\kappa) \overset{\text{def}}{=} \kappa^4 - 4H\kappa^2 + 4(H^2 - \mathcal{C}^2)$ are then:

$$\kappa(s) = \begin{cases} \pm\sqrt{2(H \pm \mathcal{C})}, & \text{for } \mathcal{C} < H; \\ 0 \text{ (double root)}, \pm 2\sqrt{\mathcal{C}}, & \text{for } H = \mathcal{C}; \\ \pm\sqrt{2(H + \mathcal{C})}, & \text{for } -\mathcal{C} < H < \mathcal{C}; \\ 0 \text{ (double root)}, & \text{for } H = -\mathcal{C}; \\ \text{No real solution}, & \text{for } H < -\mathcal{C}; \end{cases}$$

From the real roots of $g_1(\kappa)$ we get the constant solutions of (3.2.17), which correspond to linear segments $(\kappa(s) = 0)$ or circular arcs $(\kappa(s) = \kappa_0 \neq 0)$.

Observe that the solutions for $H = \mathcal{C}$ and $H = -\mathcal{C}$ are limiting cases of the elliptic solutions. In the first case, the modulus k tends to 1 as $H \longrightarrow \mathcal{C}$ and in the second k tends to 0 as $H \longrightarrow -\mathcal{C}$ (c.f. equations (2.2.9)).

Let $\mathcal{C} = 0$. From (3.2.17) we have:

$$(\frac{d\kappa}{ds})^2 = -(H - \frac{1}{2}\kappa^2)^2 \leq 0 \,. \tag{3.2.37}$$

This leads to imaginary $\frac{d\kappa}{ds}$, except when $\kappa(s) = \pm\sqrt{2H}$.

∎

3.2.6 Integral Curves

Assume that the optimal curvature $\kappa(s)$ has been specified from the Bryant–Griffiths optimality constraint of the previous section for the system (3.2.1) parametrized by arclength for $0 \leq s \leq L$ and with known boundary conditions $\sigma(0)$ and $\sigma(L)$, i.e. $x(0) = x_0$, $y(0) = y_0$, $x(L) = x_1$ and $y(L) = y_1$.

By rearranging (3.2.1) we get:

$$\begin{aligned} \begin{pmatrix} \dot{v}_{11} \\ \dot{v}_{21} \end{pmatrix}(s) &= \begin{pmatrix} 0 & \kappa(s) \\ -\kappa(s) & 0 \end{pmatrix} \begin{pmatrix} v_{11} \\ v_{21} \end{pmatrix}(s) \,, \\ \begin{pmatrix} \dot{v}_{12} \\ \dot{v}_{22} \end{pmatrix}(s) &= \begin{pmatrix} 0 & \kappa(s) \\ -\kappa(s) & 0 \end{pmatrix} \begin{pmatrix} v_{12} \\ v_{22} \end{pmatrix}(s) \,. \end{aligned} \tag{3.2.38}$$

Both systems are of the form $\dot{x}(s) = A(s)x(s)$, where $x(.) \in \mathbb{R}^2$, with $A(s)$ and $\int A(\tau)d\tau$ commuting. Then, defining $\Lambda(s) = \int_0^s \kappa(\tau)d\tau$, their solution is:

$$\begin{aligned} \begin{pmatrix} v_{11}(s) \\ v_{21}(s) \end{pmatrix} &= \exp\left[\int_0^s \begin{pmatrix} 0 & \kappa(\tau) \\ -\kappa(\tau) & 0 \end{pmatrix} d\tau\right] \begin{pmatrix} v_{11}(0) \\ v_{21}(0) \end{pmatrix} \\ &= \begin{pmatrix} \cos\Lambda(s) & \sin\Lambda(s) \\ -\sin\Lambda(s) & \cos\Lambda(s) \end{pmatrix} \begin{pmatrix} v_{11}(0) \\ v_{21}(0) \end{pmatrix} \,, \end{aligned} \tag{3.2.39}$$

$$\begin{pmatrix} v_{12}(s) \\ v_{22}(s) \end{pmatrix} = \exp\left[\int_0^s \begin{pmatrix} 0 & \kappa(\tau) \\ -\kappa(\tau) & 0 \end{pmatrix} d\tau\right] \begin{pmatrix} v_{12}(0) \\ v_{22}(0) \end{pmatrix}$$
$$= \begin{pmatrix} \cos\Lambda(s) & \sin\Lambda(s) \\ -\sin\Lambda(s) & \cos\Lambda(s) \end{pmatrix} \begin{pmatrix} v_{12}(0) \\ v_{22}(0) \end{pmatrix} . \tag{3.2.40}$$

Moreover, since $\kappa = \frac{d\theta}{ds}$, we have:

$$\theta(s) = \theta_0 + \Lambda(s) = \theta_0 + \int_0^s \kappa(\tau) d\tau . \tag{3.2.41}$$

If $\theta(0) = \theta_0$ and $\theta(L) = \theta_1$ are known, then

$$\Lambda(L) = \theta_1 - \theta_0 . \tag{3.2.42}$$

By integrating (3.2.1) we get:

$$x(s) = x(0) + \int_0^s \cos\theta(\tau) d\tau$$
$$= x(0) + \cos\theta_0 \int_0^s \cos\Lambda(\tau)\, d\tau - \sin\theta_0 \int_0^s \sin\Lambda(\tau)\, d\tau , \tag{3.2.43}$$

$$y(s) = y(0) + \int_0^s \sin\theta(\tau) d\tau$$
$$= y(0) + \sin\theta_0 \int_0^s \cos\Lambda(\tau)\, d\tau + \cos\theta_0 \int_0^s \sin\Lambda(\tau)\, d\tau . \tag{3.2.44}$$

Define $A(s) \stackrel{\text{def}}{=} \int_0^s \cos\Lambda(\tau)\, d\tau$ and $B(s) \stackrel{\text{def}}{=} \int_0^s \sin\Lambda(\tau)\, d\tau$. Then from (3.2.43) and (3.2.44) and after taking the boundary conditions $x(0) = x_0, y(0) = y_0, x(L) = x_1$ and $y(L) = y_1$ into account, we get:

$$A(L)\cos\theta_0 - B(L)\sin\theta_0 = x_1 - x_0 ,$$
$$A(L)\sin\theta_0 + B(L)\cos\theta_0 = y_1 - y_0 \tag{3.2.45}$$

and from these:

$$A^2(L) + B^2(L) = [x_1 - x_0]^2 + [y_1 - y_0]^2 \,. \tag{3.2.46}$$

Observe that for a periodic solution with $x_1 = x_0$ and $y_1 = y_0$, we have $A^2(L)+B^2(L) = 0$. Provided $A^2(L) + B^2(L) > 0$, we can solve (3.2.45) and (3.2.46) for $\cos\theta_0$ and $\sin\theta_0$:

$$\begin{aligned} \cos\theta_0 &= \frac{1}{A^2(L)+B^2(L)}\left[A(L)(x_1 - x_0) + B(L)(y_1 - y_0)\right] \,, \\ \sin\theta_0 &= \frac{1}{A^2(L)+B^2(L)}\left[A(L)(y_1 - y_0) - B(L)(x_1 - x_0)\right] \,. \end{aligned} \tag{3.2.47}$$

If L is given, together with (x_0, y_0) and (x_1, y_1) (**elastica problem**), then, after we specify κ from the Bryant–Griffiths equation (3.2.17), we can use (3.2.47) to find the necessary θ_0 and (3.2.41), (3.2.43) and (3.2.44) to specify the corresponding trajectory g of our system.

If L is free, but $\theta(0) = \theta_0$ and $\theta(L) = \theta_1$ are given, in addition to (x_0, y_0) and (x_1, y_1) (**free elastica problem**), then, after we specify κ from the Bryant–Griffiths equation (3.2.18), we can use (3.2.45) or (3.2.42) and (3.2.46) to specify L and the constant C of κ. Then, from (3.2.43) and (3.2.44) we specify the corresponding trajectory g of our system.

We now apply the above integration procedure to the various solutions of the Bryant–Griffiths equation.

1. $H > C$

Consider $\kappa(s) = \kappa_1 \operatorname{dn}(\frac{\kappa_1 s}{2}, k)$, with $k = \frac{\sqrt{\kappa_1{}^2 - \kappa_3{}^2}}{\kappa_1}$. Then from (2.2.14) $\Lambda(s) = 2\sin^{-1}\big(\operatorname{sn}(\frac{\kappa_1 s}{2}, k)\big)$, for $s \in [-\frac{T}{2}, \frac{T}{2}]$. Moreover, from (2.2.21), (2.2.8), (2.2.10) and

(2.2.14) we have:

$$A(s) = \int_0^s \cos\Lambda(\tau)d\tau = \int_0^s \Big[1 - 2\sin^2\Big(\frac{\Lambda(\tau)}{2}\Big)\Big]d\tau$$

$$= \int_0^s \Big[1 - 2\,\mathrm{sn}^2(\frac{\kappa_1\tau}{2}, k)\Big]d\tau$$

$$= s + \frac{4}{\kappa_1 k^2}\Big[E\big(\mathrm{am}(\frac{\kappa_1 s}{2}), k\big) - \frac{\kappa_1 s}{2}\Big]$$

$$= (1 - \frac{2}{k^2})s + \frac{4}{\kappa_1 k^2}E\big(\mathrm{am}(\frac{\kappa_1 s}{2}), k\big)\ ,$$

$$B(s) = \int_0^s \sin\Lambda(\tau)d\tau = \int_0^s 2\sin\Big(\frac{\Lambda(\tau)}{2}\Big)\sqrt{1 - \sin^2\Big(\frac{\Lambda(\tau)}{2}\Big)}d\tau$$

$$= \int_0^s 2\,\mathrm{sn}(\frac{\kappa_1\tau}{2}, k)\,\mathrm{cn}(\frac{\kappa_1\tau}{2}, k)d\tau$$

$$= \frac{4}{\kappa_1 k^2}\big[1 - \mathrm{dn}(\frac{\kappa_1 s}{2}, k)\big]\ ,$$

for $s \in [-\frac{T}{2}, \frac{T}{2}]$. From this and (3.2.41), (3.2.43), (3.2.44) and (3.2.47), we get the system trajectory and θ_0. In fig. 3.2.7, the curve $(x(s), y(s))$ is shown (for $\kappa_1 = 1$ and $\kappa_3 = 0.5$).

2. $H = \mathcal{C}$

Consider $\kappa(s) = \kappa_1\,\mathrm{sech}(\frac{\kappa_1}{2}s + \delta)$. Then $\Lambda(s) = 4\big[\tan^{-1}(e^{\frac{\kappa_1}{2}s+\delta}) - \tan^{-1}(e^{\delta})\big]$ and

$$A(s) = \int_0^s \cos\Lambda(\tau)d\tau = \frac{\mathcal{E}}{(1+\gamma^2)^2}s + \frac{8}{\kappa_1(1+\gamma^2)^2}\left[\frac{\mathcal{D}e^{\frac{\kappa_1}{2}s} + \mathcal{E}}{1+\gamma^2 e^{\kappa_1 s}} - \frac{\mathcal{D}+\mathcal{E}}{1+\gamma^2}\right],$$

$$B(s) = \int_0^s \sin\Lambda(\tau)d\tau = -\frac{\mathcal{D}}{\gamma(1+\gamma^2)^2}s + \frac{8}{\kappa_1\gamma(1+\gamma^2)^2}\left[\frac{\mathcal{E}\gamma^2 e^{\frac{\kappa_1}{2}s} - \mathcal{D}}{1+\gamma^2 e^{\kappa_1 s}} - \frac{\mathcal{E}\gamma^2 - \mathcal{D}}{1+\gamma^2}\right],$$

where $\gamma = e^{\delta}$, $\mathcal{D} = 4\gamma^2(1-\gamma^2)$ and $\mathcal{E} = 1 - 6\gamma^2 + \gamma^4$. When $\delta = 0$:

$$A(s) = -s + \frac{4}{\kappa_1}\tanh(\frac{\kappa_1}{2}s)\ \text{ and }\ B(s) = \frac{4}{\kappa_1}\Big[1 - \mathrm{sech}(\frac{\kappa_1}{2}s)\Big]\ .$$

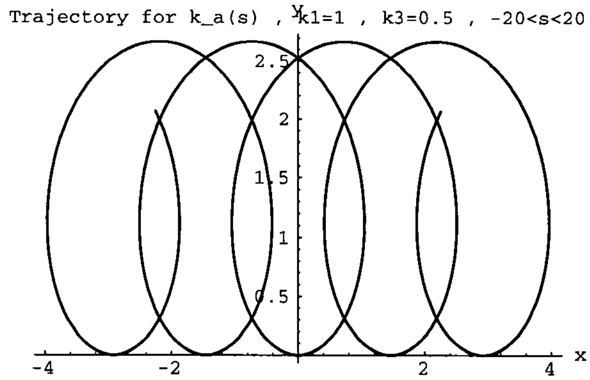


Fig. 3.2.7: Integral Curves for $H > \mathcal{C}$

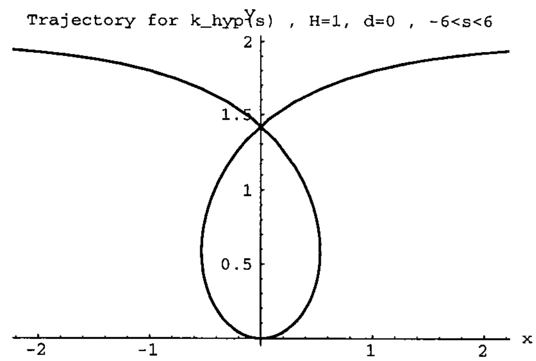


Fig. 3.2.8: Integral Curves for $H = \mathcal{C}$

From those and equations (3.2.41), (3.2.43), (3.2.44) and (3.2.47), we get the system trajectory and θ_0. The curve $(x(s), y(s))$ is shown in fig. 3.2.8 (for $h = 1$ and $\delta = 0$).

3. $-\mathcal{C} < H < \mathcal{C}$

Consider $\kappa(s) = \kappa_1 \operatorname{cn}(\frac{\sqrt{\kappa_1^2+\kappa_3^2}}{2} s\,,\ k)$, with $k = \frac{\kappa_1}{\sqrt{\kappa_1^2+\kappa_3^2}}$. Then from (2.2.14) we have: $\Lambda(s) = 2\sin^{-1}\big(k \operatorname{sn}(\frac{\sqrt{\kappa_1^2+\kappa_3^2}}{2} s, k)\big)$, for $s \in [-\frac{T}{2}, \frac{T}{2}]$. Moreover, from (2.2.10) and (2.2.14) we have:

$$A(s) = \int_0^s \cos\Lambda(\tau) d\tau = \int_0^s \Big[1 - 2\sin^2\Big(\frac{\Lambda(\tau)}{2}\Big)\Big] d\tau$$

$$= \int_0^s \Big[1 - 2k^2 \operatorname{sn}^2(\frac{\sqrt{\kappa_1^2+\kappa_3^2}}{2}\tau, k)\Big] d\tau$$

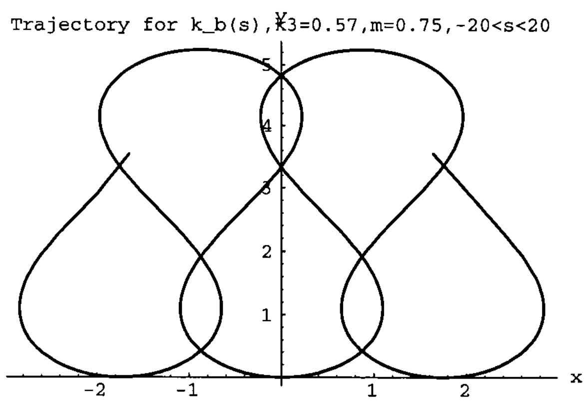
$$= -s + \frac{4}{\sqrt{\kappa_1^2+\kappa_3^2}} E\big(\operatorname{am}(\frac{\sqrt{\kappa_1^2+\kappa_3^2}}{2} s), k\big)\ ,$$

$$B(s) = \int_0^s \sin\Lambda(\tau)d\tau = \int_0^s 2\sin\Big(\frac{\Lambda(\tau)}{2}\Big)\sqrt{1-\sin^2\Big(\frac{\Lambda(\tau)}{2}\Big)}d\tau$$

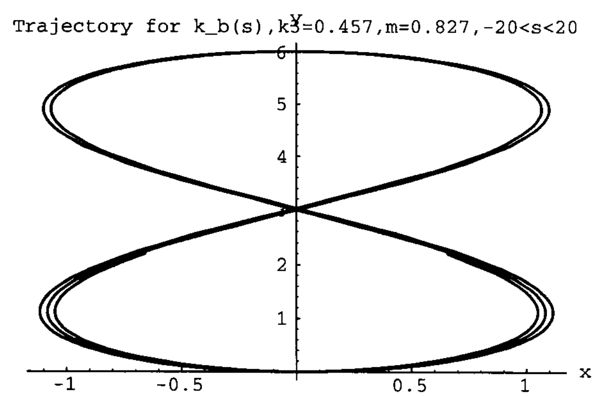
$$= \int_0^s 2k\,\mathrm{sn}(\frac{\sqrt{\kappa_1^2+\kappa_3^2}}{2}\tau, k)\,\mathrm{dn}(\frac{\sqrt{\kappa_1^2+\kappa_3^2}}{2}\tau, k)d\tau$$

$$= \frac{4k}{\sqrt{\kappa_1^2+\kappa_3^2}}[1-\mathrm{cn}(\frac{\sqrt{\kappa_1^2+\kappa_3^2}}{2}s, k)]\ ,$$

for $s \in [-\frac{T}{2}, \frac{T}{2}]$. From those and equations (3.2.41), (3.2.43), (3.2.44) and (3.2.47), we get the system trajectory and θ_0. In fig. 3.2.9, the curve $(x(s), y(s))$ is shown for various values of κ_1 and κ_3 (notice that the symbol m in those figures is defined as $m \stackrel{\text{def}}{=} k^2 = \frac{\kappa_1^2}{\kappa_1^2+\kappa_3^2}$).



(a)



(b)

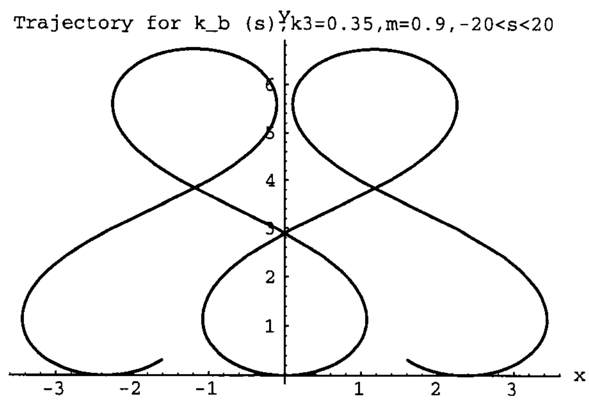
6. (Constant solutions)

Let $\kappa(s) = 0$. Then $\Lambda(s) = 0$. From (3.2.41), (3.2.43) and (3.2.44):

$$\theta(s) = \theta_0\ ,$$

$$x(s) = x_0 + \cos\theta_0 s\ ,$$

$$y(s) = y_0 + \sin\theta_0 s\ ,$$



(c)

Fig. 3.2.9: Integral Curves for $-\mathcal{C} < H < \mathcal{C}$

for all $s \in [0, L]$. Moreover, from (3.2.47) for $A(s) = s$ and $B(s) = 0$, we get:

$$\cos\theta_0 = \frac{x_1 - x_0}{L} \;, \quad \sin\theta_0 = \frac{y_1 - y_0}{L} \;.$$

This is a straight line in $I\!R^2$.

Let $\kappa(s) = \kappa_0 \neq 0$. Then $\Lambda(s) = \kappa_0 s$. From (3.2.41), (3.2.43) and (3.2.44):

$$\theta(s) = \theta_0 + \kappa_0 s \;,$$

$$x(s) = x_0 + \frac{1}{\kappa_0}\cos\theta_0 \sin(\kappa_0 s) - \frac{1}{\kappa_0}\sin\theta_0[1 - \cos(\kappa_0 s)] \;,$$

$$y(s) = y_0 + \frac{1}{\kappa_0}\sin\theta_0 \sin(\kappa_0 s) + \frac{1}{\kappa_0}\cos\theta_0[1 - \cos(\kappa_0 s)] \;,$$

for all $s \in [0, L]$. Moreover, from (3.2.47) for $A(s) = \frac{1}{\kappa_0}\sin(\kappa_0 s)$ and $B(s) = \frac{1}{\kappa_0}[1 - \cos(\kappa_0 s)]$, we get::

$$\cos\theta_0 = \frac{\kappa_0}{2\big(1 - \cos(\kappa_0 L)\big)}\big[(x_1 - x_0)\sin(\kappa_0 L) + (y_1 - y_0)(1 - \cos(\kappa_0 L))\big] \;,$$

$$\sin\theta_0 = \frac{\kappa_0}{2\big(1 - \cos(\kappa_0 L)\big)}\big[(x_1 - x_0)(\cos(\kappa_0 L) - 1) + (y_1 - y_0)\sin(\kappa_0 L)\big] \;.$$

This is a circular arc in $I\!R^2$ with center $(x_0 - \frac{1}{\kappa_0}\sin\theta_0, y_0 + \frac{1}{\kappa_0}\cos\theta_0)$ and radius $\frac{1}{\kappa_0}$ that starts at the point (x_0, y_0) and ends at the point (x_1, y_1).

3.3 Curvature–based Planning for 3–dimensional Manifolds

3.3.1 Curvature–based Planning

Consider now the case when the planar platform translates and rotates or when the spatial platform translates in $I\!R^3$. The configuration space is then a 3–dimensional manifold, in particular $SE(2)$ in the first case and $I\!R^3$ in the second. If we consider motion on a chart of $SE(2)$, the problem reduces to $I\!R^3$ in both cases.

Let $\sigma(s) = (x(s), y(s), z(s))$ be a curve in $I\!R^3$ parametrized by arc–length and with total length L. The *tangent vector field* to σ is defined as $v_1(s) = \frac{d\sigma}{ds}(s)$. The *curvature* of σ is defined as $\kappa(s) = \left\|\dot{v}_1(s)\right\|_3$, where $\|.\|_n$ is the Euclidean norm in $I\!R^n$. The *principal normal vector field* to σ is defined as $v_2(s) = \frac{\frac{dv_1}{ds}}{\|\frac{dv_1}{ds}\|_3} = \frac{\dot{v}_1(s)}{\kappa(s)}$. The *binormal vector field* to σ is defined as $v_3(s) = v_1(s) \times v_2(s)$. The *torsion* of σ is defined as $\tau(s) = -<\dot{v}_3(s), v_2(s)>_3$, where $<,>_n$ is the inner product in $I\!R^n$. The *Frenet–Serret apparatus* for the unit speed curve σ is $\{\kappa(s), \tau(s), v_1(s), v_2(s), v_3(s)\}$. From the above definitions and the *Frenet–Serret equations* we get:

$$
\begin{aligned}
\frac{d\sigma}{ds}(s) &= v_1(s) \ , \\
\frac{dv_1}{ds}(s) &= \kappa(s)v_2(s) \ , \\
\frac{dv_2}{ds}(s) &= -\kappa(s)v_1(s) + \tau(s)v_3(s) \ , \\
\frac{dv_3}{ds}(s) &= -\tau(s)v_2(s) \ .
\end{aligned}
\tag{3.3.1}
$$

Defining $g = \begin{pmatrix} R & T \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} v_1 & v_2 & v_3 & \sigma \\ 0 & 0 & 0 & 1 \end{pmatrix} \in G = SE(3)$, the system (3.3.1) can be viewed as a *left–invariant* system on the Lie group $G = SE(3)$ of the form:

$$
\begin{aligned}
\frac{dg}{ds}(s) &= g(s)\begin{pmatrix} 0 & -\kappa(s) & 0 & 1 \\ \kappa(s) & 0 & -\tau(s) & 0 \\ 0 & \tau(s) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
&= g(s)\big(\mathcal{A}_4 + \kappa(s)\mathcal{A}_3 + \tau(s)\mathcal{A}_1\big) \\
&= \mathcal{X}_4(g(s)) + \kappa(s)\mathcal{X}_3(g(s)) + \tau(s)\mathcal{X}_1(g(s)) \ ,
\end{aligned}
\tag{3.3.2}
$$

where the left–invariant vector field $\mathcal{X}_i(g(s)) \stackrel{\text{def}}{=} g(s)\mathcal{A}_i$ and where the matrix $\mathcal{A}_i$ belongs to the set

$$
\{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5, \mathcal{A}_6\} =
$$

$$
\left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \right.
$$

$$
\left. \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\} .
\tag{3.3.3}
$$

As we see from the following Lie bracket relationships, this set forms a basis for the Lie Algebra $\mathcal{G} = se(3)$.

$$
\begin{aligned}
&[\mathcal{A}_1, \mathcal{A}_2] = \mathcal{A}_3,\ [\mathcal{A}_1, \mathcal{A}_3] = -\mathcal{A}_2,\ [\mathcal{A}_1, \mathcal{A}_4] = 0,\ [\mathcal{A}_1, \mathcal{A}_5] = \mathcal{A}_6,\ [\mathcal{A}_1, \mathcal{A}_6] = -\mathcal{A}_5, \\
&[\mathcal{A}_2, \mathcal{A}_3] = \mathcal{A}_1,\ [\mathcal{A}_2, \mathcal{A}_4] = -\mathcal{A}_6,\ [\mathcal{A}_2, \mathcal{A}_5] = 0,\ [\mathcal{A}_2, \mathcal{A}_6] = \mathcal{A}_4, \\
&[\mathcal{A}_3, \mathcal{A}_4] = \mathcal{A}_5,\ [\mathcal{A}_3, \mathcal{A}_5] = -\mathcal{A}_4,\ [\mathcal{A}_3, \mathcal{A}_6] = 0, \\
&[\mathcal{A}_4, \mathcal{A}_5] = [\mathcal{A}_4, \mathcal{A}_6] = [\mathcal{A}_5, \mathcal{A}_6] = 0.
\end{aligned}
$$

Similar results hold for the left–invariant vector fields $\mathcal{X}_i$ under the Jacobi–Lie bracket operation.

Consider now the following variational problem:

Problem P_3 :

Minimize

$$
\eta = \frac{1}{2}\int_0^L \kappa^2(s)ds \ . \tag{3.3.4}
$$

on the dynamical system $\frac{dg}{ds} = g(s)\big(\mathcal{A}_4 + \kappa(s)\mathcal{A}_3 + \tau(s)\mathcal{A}_1\big)$ with $g \in G = SE(3)$, and with $(\kappa,\ \tau) \in U = I\!R_+ \times I\!R$ and with given boundary conditions $\sigma(0)$ and $\sigma(L)$.

This problem falls in the framework of problems studied by (Griffiths [1983]; Langer & Singer [1984]) and (Jurdjevic [1990]), where it was shown that the curvature and torsion of the optimal paths satisfy: $\kappa^2\tau = \mathcal{C}_1$ and $2\frac{d^2\kappa}{ds^2} + \kappa^3 - 2\mathcal{C}_2\kappa - 2\tau^2\kappa = 0$, where $\mathcal{C}_1$ and $\mathcal{C}_2$ are constants.

3.3.2 The Optimality Constraint

Theorem 3.3.1 (Griffiths [1983]; Jurdjevic [1990])

The curvature κ and torsion τ corresponding to the regular extremals of the variational problem of minimizing $\frac{1}{2}\int_0^L \kappa^2(s)ds$ on the system (3.3.1) satisfy the optimality conditions:

$$\kappa^2\tau = \mathcal{C}_1 \quad \text{and} \quad 2\frac{d^2\kappa}{ds^2} + \kappa^3 - 2\mathcal{C}_2\kappa - 2\tau^2\kappa = 0 \ , \tag{3.3.5}$$

where $\mathcal{C}_1$ and $\mathcal{C}_2$ are constants.

∎

3.3.3 Solutions of the Optimality Constraint

Combining equations (3.3.5), we get:

$$2\frac{d^2\kappa}{ds^2} + \kappa^3 - 2\mathcal{C}_2\kappa - 2\frac{\mathcal{C}_1^2}{\kappa^3} = 0 \ .$$

Multiplying by $\frac{d\kappa}{ds}$ and integrating, we get:

$$(\frac{d\kappa}{ds})^2 + \frac{1}{4}\kappa^4 - \mathcal{C}_2\kappa^2 + \frac{\mathcal{C}_1^2}{\kappa^2} = \mathcal{C}_3 \ , \tag{3.3.6}$$

where $\mathcal{C}_3 \stackrel{\text{def}}{=} \dot{\kappa}_0^2 + \frac{1}{4}{\kappa_0}^4 - \mathcal{C}_2{\kappa_0}^2 + \frac{\mathcal{C}_1^2}{\kappa_0^2}$.

Let $u \stackrel{\text{def}}{=} \kappa^2$. Then:

$$(\frac{du}{ds})^2 + u^3 - 4\mathcal{C}_2u^2 - 4\mathcal{C}_3u + 4\mathcal{C}_1^2 = 0 \ , \tag{3.3.7}$$

which is solvable by elliptic functions.

Let $P_1(u) \stackrel{\text{def}}{=} u^3 - 4\mathcal{C}_2 u^2 - 4\mathcal{C}_3 u + 4\mathcal{C}_1^2$. This cubic will have in general three roots, which will be denoted u_1, u_2, u_3.

By the substitution $y \stackrel{\text{def}}{=} u - \frac{4}{3}\mathcal{C}_2$, we bring $P_1(u)$ to the reduced form

$$P_2(y) \stackrel{\text{def}}{=} y^3 + py + q \ ,$$

where $p \stackrel{\text{def}}{=} -\frac{4}{3}(3\mathcal{C}_3 + 4\mathcal{C}_2^2)$ and $q \stackrel{\text{def}}{=} \frac{4}{27}[-32\mathcal{C}_2^3 - 36\mathcal{C}_2\mathcal{C}_3 + 27\mathcal{C}_1^2]$. Let y_1, y_2, y_3 be the roots of $P_2(y)$.

Consider the discriminant $D \stackrel{\text{def}}{=} (\frac{p}{3})^3 + (\frac{q}{2})^2$ of $P_2(y)$. If $D > 0$, then $P_2(y)$ has one real root y_1 and two complex conjugate ones y_2 and y_3, while if $D \leq 0$, $P_2(y)$ has three real roots y_1, y_2 and y_3 (may not be distinct).

Let $w \stackrel{\text{def}}{=} (-\frac{q}{2} + \sqrt{D})^{\frac{1}{3}}$ and $v \stackrel{\text{def}}{=} (-\frac{q}{2} - \sqrt{D})^{\frac{1}{3}}$. The roots of $P_2(y)$ are then :

$$\begin{aligned} y_1 &= w + v \\ y_2 &= -\frac{1}{2}(w+v) + i\frac{\sqrt{3}}{2}(w-v) \\ y_3 &= -\frac{1}{2}(w+v) - i\frac{\sqrt{3}}{2}(w-v) \end{aligned}$$

and the roots of $P_1(u)$ are

$$u_j = y_j + \frac{4}{3}\mathcal{C}_2 \quad , \quad \text{for } j = 1,2,3 \ .$$

Observe that u_1 is always real.

The generic solution of (3.3.7) involves *elliptic* functions. Consider the case $D < 0$:

In this case all roots of $P_1(u)$ are real. Observe that if $\mathcal{C}_2 < 0$, then $\mathcal{C}_3 \geq 0$. Then, by Descartes' rule of signs, $P_1(u)$ should have two positive and one negative root. Let $-u_1 \leq 0 \leq u_2 \leq u_3$ be those roots (this is a permutation, with possible sign change of the roots we defined earlier). Then: $P_1(u) = (u+u_1)(u-u_2)(u-u_3)$ and from (3.3.7): $(\frac{du}{ds})^2 = -P_1(u) = (u+u_1)(u-u_2)(u_3-u)$. In order for $\frac{du}{ds}$ to be real, we should have $P_1(u) < 0$ whenever u is not constant. Thus, we either have $u \leq -u_1 \leq 0$ or $0 \leq u_2 \leq u \leq u_3$. In the first case, since $\kappa^2 = u$, we will not get real κ. We only consider then the case $u_2 \leq u \leq u_3$, when:

$$\frac{du}{ds} = \pm\sqrt{(u+u_1)(u-u_2)(u_3-u)}\ , \tag{3.3.8}$$

Let $u(0) = u_3$ and $\frac{du}{ds} \leq 0$. From (3.3.8) we have:

$$\frac{du}{ds} = -\sqrt{(u+u_1)(u-u_2)(u_3-u)}$$

$$\Rightarrow -\int_{u_3}^{u(s)} \frac{du}{\sqrt{(u+u_1)(u-u_2)(u_3-u)}} = \int_0^s ds$$

$$\overset{\text{(Gradstein \& Ryzhik [1980])}}{\Rightarrow} \frac{2}{\sqrt{u_3+u_1}}\,\mathrm{sn}^{-1}\Big(\sqrt{\frac{u_3-u}{u_3-u_2}}, k\Big) = s$$

$$\begin{aligned} \Rightarrow u(s) &= u_3 - (u_3-u_2)\,\mathrm{sn}^2\Big(\frac{\sqrt{u_3+u_1}}{2}s, k\Big) \\ &= u_3\Big[1 - \frac{u_3-u_2}{u_3}\,\mathrm{sn}^2\Big(\frac{\sqrt{u_3+u_1}}{2}s, k\Big)\Big] \\ &= u_3\Big[1 - \frac{k^2}{w^2}\,\mathrm{sn}^2(rs, k)\Big]\ , \end{aligned} \tag{3.3.9}$$

where $k = \sqrt{\frac{u_3-u_2}{u_3+u_1}}$, $w^2 = \frac{u_3}{u_3+u_1}$ and $r = \frac{\sqrt{u_3+u_1}}{2}$.

The solutions of (3.3.5) are then $\kappa(s) = \pm\sqrt{u(s)}$ and $\tau(s) = \frac{C_1}{u(s)}$.

3.4 Singular Configuration Avoidance for Parallel Manipulators

Consider the planar parallel manipulator examined in section 2.5. The kinematic singularities of the system divide the configuration space into four areas (fig. 2.5.5). We saw that for several grasping and sensor reorientation tasks it is beneficial to optimize the crossing of the singular surfaces. From fig. 2.5.5 it can be seen that we may have to cross them at most twice for every choice of initial and final configuration. In case the crossing of the singular surfaces cannot be avoided, the most favorable point for that has to be specified, since, if we only drive through the singular surfaces, the dynamics of the system will determine the platform configuration after crossing the singular surface. Therefore, we should either try to avoid singular configurations as much as possible or just drive through them, in such a way that the dynamics of the system resolves the indeterminacy in the platform motion. A planning scheme not restricted to the traditional linear configuration–space trajectories can be very important in this respect. If we consider only linear configuration–space trajectories, our system may have to cross unnecessarily the singular surfaces or follow paths that are longer than needed.

Example 3.4.1

An example of this is the case when the platform translates and rotates, keeping the x–coordinate constant (say $x = x_0$). The corresponding singular surfaces are: $\mathcal{X}_*^{(1)} = \{(x, y, \theta) | y = \beta \sin\theta \,, \, x = x_0\}$ and $\mathcal{X}_*^{(2)} = \{(x, y, \theta) | y \cos\theta = (x_0 - \alpha) \sin\theta_0 \,, \, x = x_0 \}$. Those singular surfaces are shown in fig. 3.4.1 for the case $x_0 - \alpha > \beta$. If we consider moving from a configuration g_0 to g_1 along the linear trajectories I or II, we'll have to cross unnecessarily the singular surface $\mathcal{X}_*^{(1)}$ twice (fig. 3.4.3 and 3.4.4). Trajectory III is better in this respect, but is also undesirable, since it is not smooth and it is much longer than necessary (fig. 3.4.2). It is then beneficial to have a family of configuration–space paths, which on one hand is richer than the set of straight line segments, but on the other retains some of the fundamental properties of the straight lines.

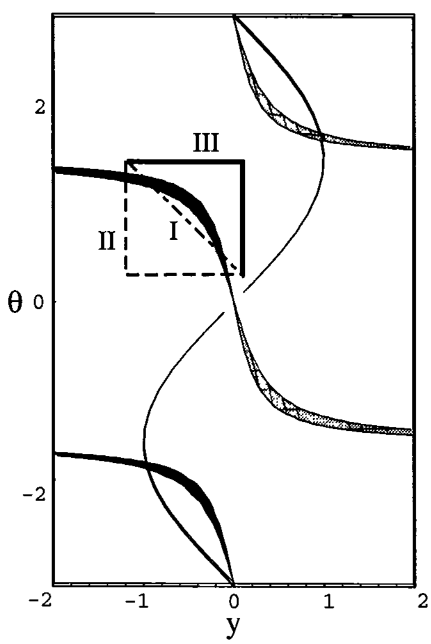


Fig. 3.4.1: Singular Surfaces

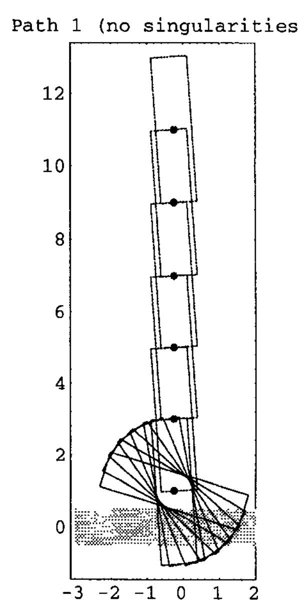


Fig. 3.4.2: Trajectory III

We can use the results of section 3.2, in order to specify analytically a system

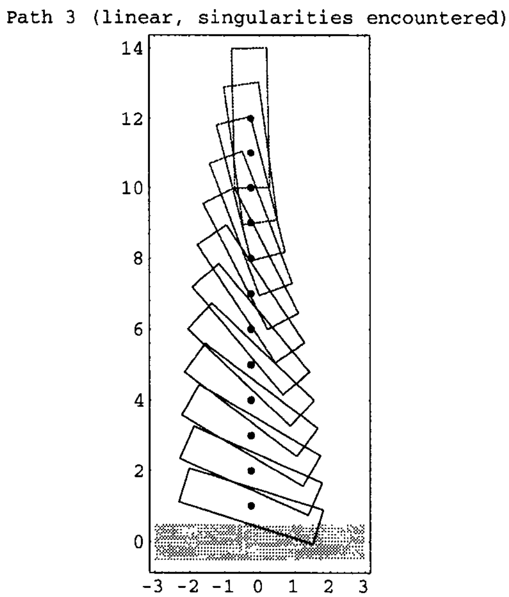


Fig. 3.4.3: Trajectory I

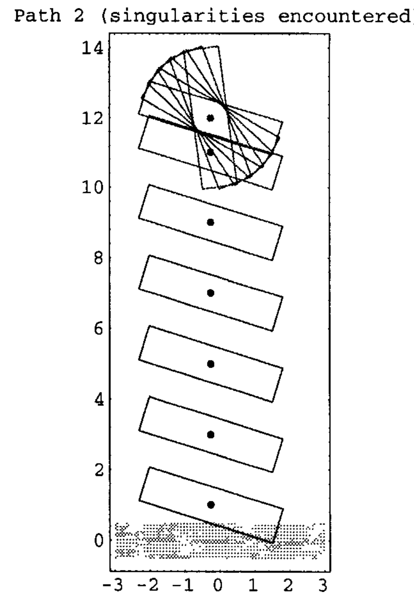


Fig. 3.4.4: Trajectory II

trajectory $g(.)$ that avoids singular configurations. Assume that we are given $g_0 \in SE(2)$ and $g_1 \in SE(2)$, i.e. the positions and tangent vectors at the end–points of the path. In this case, the length L of the path is free, thus $H = 0$, $\kappa_1 = \kappa_3 = \sqrt{2C}$ and $\kappa(s, C) = \kappa_1 \operatorname{cn}\big(\frac{\sqrt{\kappa_1^2+\kappa_3^2}}{2} s, k\big)$, with $k = \frac{\kappa_1}{\sqrt{\kappa_1^2+\kappa_3^2}} = \frac{\sqrt{2}}{2}$. Then $\Lambda(s, C) = 2\sin^{-1}\big(\frac{1}{\sqrt{2}}\operatorname{sn}(\sqrt{C}s, k)\big)$. Moreover, $A(s, C) = -s + \frac{2}{\sqrt{C}} E\big(\operatorname{am}(\sqrt{C}s), k\big)$ and $B(s, C) = \frac{2k}{\sqrt{C}}\left[1 - \operatorname{cn}(\sqrt{C}s, k)\right]$.

We need to specify C and L from the end–point data. From the boundary conditions (3.2.42) and (3.2.46), i.e. from $\Lambda(L, C) = \theta_1 - \theta_0$ and from $A^2(L) + B^2(L) = (x_1 - x_0)^2 + (y_1 - y_0)^2$, we get:

$$C = \frac{\alpha^2(\Delta\theta) + \beta^2(\Delta\theta)}{(x_1 - x_0)^2 + (y_1 - y_0)^2}$$

and

$$L = \frac{1}{\sqrt{C}} \operatorname{sn}^{-1}\left(\frac{1}{k}\sin(\frac{\theta_1 - \theta_0}{2}), k\right) = \frac{1}{\sqrt{C}} F\left(\sin^{-1}\big(\frac{1}{k}\sin(\frac{\theta_1 - \theta_0}{2})\big), k\right),$$

where α and β are known functions of $\Delta\theta = \theta_1 - \theta_0$. The last expression is valid for $0 < \frac{1}{k}\sin(\frac{\theta_1 - \theta_0}{2}) < 1$, i.e. for $0 < \Delta\theta < \frac{\pi}{2}$ or $\frac{3\pi}{2} < \Delta\theta < 2\pi$. Outside those segments, we need to split the path in several segments and repeat this process for each segment.

The curvature at the joints of those segments should be continuous. If we define

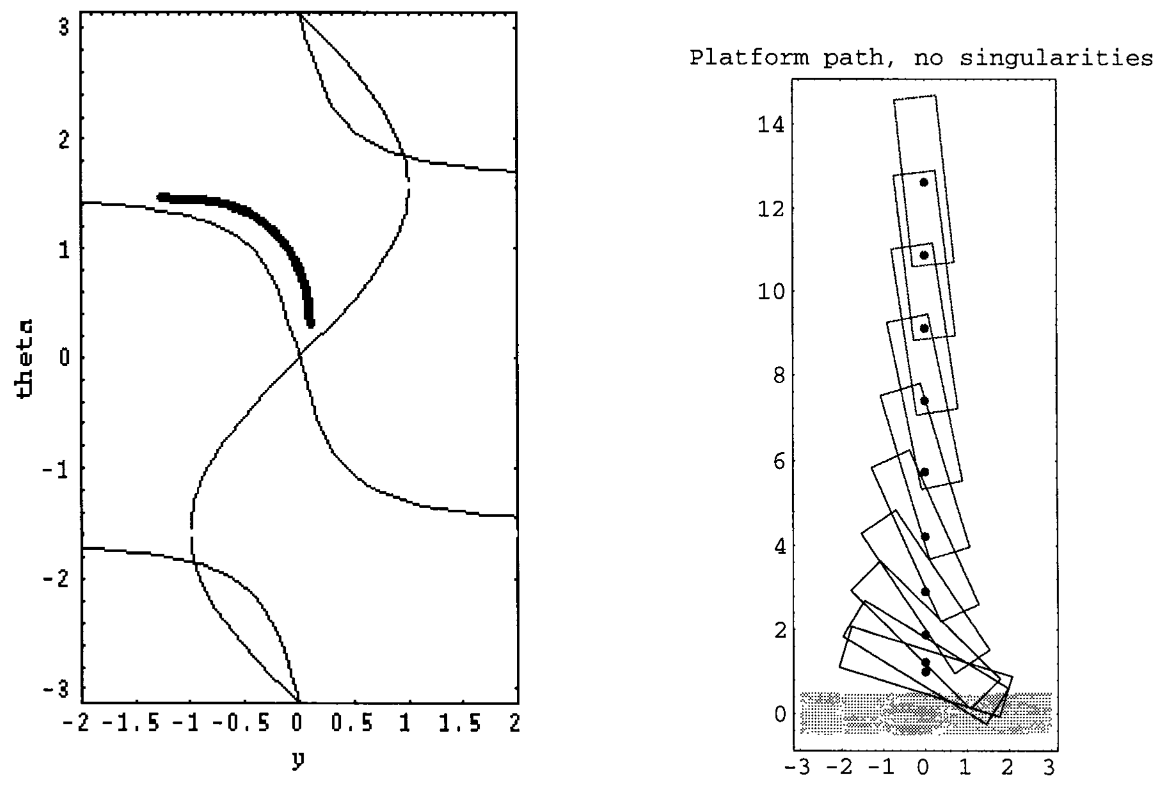
$$\gamma(\theta) \stackrel{\text{def}}{=} \operatorname{sn}^{-1}\left(\frac{1}{k}\sin(\frac{\theta}{2}), k\right) = F\left(\sin^{-1}\left(\frac{1}{k}\sin(\frac{\theta_1-\theta_0}{2})\right), k\right) ,$$

we have:

$$\alpha(\theta) = -\gamma(\theta) + 2E\big(\operatorname{am}(\gamma(\theta)), k\big) \quad \text{and} \quad \beta(\theta) = 1 - \operatorname{cn}(\gamma(\theta), k) \ .$$

Then we know $\Lambda(s)$, $A(s)$ and $B(s)$, thus the system trajectory $g(s)$ can be specified by (3.2.41), (3.2.43) and (3.2.44) (fig. 3.4.5).

∎



(a) (b)

Fig. 3.4.5: Curvature–Minimizing Trajectory

CHAPTER FOUR

G–SNAKES: NONHOLONOMIC KINEMATIC CHAINS ON LIE GROUPS

4.1 Introduction

In this chapter we are interested in groups with a real finite–dimensional non–abelian Lie algebra $\mathcal{G}$ (of dimension n) and ℓ–node kinematic chains evolving on them under a certain class of nonholonomic constraints, where the constraints force the velocities of the system to lie in a subspace of $\mathcal{G}$, which is not a subalgebra of $\mathcal{G}$ but which generates the whole algebra $\mathcal{G}$ under Lie bracketing. We refer to systems of this type as *G–Snakes* and observe that they possess an interesting geometric structure: When $\ell = n$ and the codimension of the constraints is one, the configuration and shape spaces of the system specify a principal fiber bundle and the nonholonomic constraints determine a (partial) connection on it, at least away from certain configurations which we call *nonholonomic singularities* (higher codimension cases will be treated briefly in section 5.4).

In section 4.2, we consider the kinematics of an ℓ–node kinematic chain evolving on an n–dimensional Lie group. The Wei–Norman representation of G, which expresses an element of the group as a product of the one–parameter subgroups of G, and the adjoint action of G on $\mathcal{G}$ allow us to express in a compact form how the motion of each node of the kinematic chain relates to that of the other nodes and to the global motion of the system and how this latter becomes a function of just the shape and the shape controls because of the nonholonomic constraints. We show that the configuration and shape spaces of the G–snake specify a principal fiber bundle and that the nonholonomic constraints determine a connection on it.

In section 4.3, we focus on 3–node G–Snakes evolving on 3–dimensional Lie groups ($\ell = n = 3$). In particular, we examine, apart from $SE(2)$, the Heisenberg group $H(3)$, the Special Orthogonal group $SO(3)$ and the Special Linear group $SL(2)$. We derive the corresponding Wei–Norman representation, the system kinematics, the connection and the nonholonomic singularities in each case.

For reasons having to do with ease of exposition, we limit ourselves to matrix Lie groups. Extensions to arbitrary Lie groups are easy.

4.2 Nonholonomic Kinematic Chains on Lie Groups

In section 4.2.1 we derive the kinematics of the ℓ–node kinematic chain. In section 4.2.2 we examine the geometric structure of the chain kinematics when nonholonomic constraints are present, using the theory of connections on principal fiber bundles.

4.2.1 The ℓ–node Kinematic Chain

We consider an ℓ–node dynamical system that evolves on the Cartesian product $Q = \underbrace{G \times \cdots \times G}_{\ell \text{ times}}$. Its trajectory is a curve $g(.) = (g_1(.), \ldots, g_\ell(.)) \subset Q$. On each copy of G, the system traces a curve $g_i(.) \subset G$, which represents the trajectory of the i–th node in time and is such that

$$\dot{g}_i = T_e L_{g_i} \cdot \xi_i = g_i \xi_i \ , \ i = 1, \ldots, \ell, \tag{4.2.1}$$

where $\xi_i(.) \in \mathcal{G}, \ i = 1, \ldots, \ell$. We think of the g_i's as the nodes of a kinematic chain, in which case Q is its configuration space.

Let the instantaneous *shape* of the kinematic chain be given by the $(\ell - 1)$–tuple $(g_{1,2}, g_{2,3}, \ldots, g_{\ell-1,\ell}) \in S = \underbrace{G \times \cdots \times G}_{(\ell-1) \text{ times}}$, where

$$g_{i,i+1} = g_i^{-1} g_{i+1}, \ i = 1, \ldots, \ell - 1 \ . \tag{4.2.2}$$

We call S the shape space of the kinematic chain. A pair of adjacent nodes of the chain constitutes a *module.* The $g_{i,i+1}$'s can be regarded as the shapes of the modules of an $(\ell-1)$–module kinematic chain. We refer to the corresponding curves $\xi_{i,i+1} \subset \mathcal{G}$ specified by (2.3.7) as the *shape variations:*

$$\dot{g}_{i,i+1} = T_e L_{g_{i,i+1}} \cdot \xi_{i,i+1} = g_{i,i+1} \xi_{i,i+1} \ , \ i = 1, \ldots, \ell - 1 \ . \tag{4.2.3}$$

We can generalize the notion of a module to include pairs of nodes which may not be adjacent. Consider the module that consists of nodes i and j The corresponding shape $g_{i,j}$ and $\xi_{i,j}$ are

$$g_{i,j} = g_i^{-1} g_j = g_{i,i+1} \cdots g_{j-1,j}, \quad i \leq j \tag{4.2.4}$$

and

$$\dot{g}_{i,j} = T_e L_{g_{i,j}} \cdot \xi_{i,j} = g_{i,j} \xi_{i,j}, \quad i \leq j \, . \tag{4.2.5}$$

We can think of the ξ_i's as characterizing the global motion of the G–snake system with respect to some global coordinate system, while the $\xi_{i,j}$'s capture the relative motion (or shape variation) of nodes i and j.

From (4.2.1), (4.2.2) and (4.2.3) we get:

$$\xi_i = \xi_{i-1,i} + Ad_{g^{-1}_{i-1,i}} \xi_{i-1}, \; i = 2, \ldots, \ell. \tag{4.2.6}$$

Applying (4.2.6) iteratively we can express any ξ_i as a function of ξ_1 and of the shape controls $\xi_{1,2}, \ldots, \xi_{i-1,i}$ as follows:

$$\xi_i = \xi_{i-1,i} + Ad_{g^{-1}_{i-1,i}} \xi_{i-2,i-1} + \cdots + Ad_{g^{-1}_{2,i}} \xi_{1,2} + Ad_{g^{-1}_{1,i}} \xi_1 \, . \tag{4.2.7}$$

The variations in Q can then be parametrized either by $(\xi_1, \xi_2, \ldots, \xi_\ell)$ or by $(\xi_1, \xi_{1,2}, \ldots, \xi_{\ell-1,\ell})$. In the latter case, the first element (ξ_1) captures the global motion of the system, while the remaining ones $(\xi_{1,2}, \ldots, \xi_{\ell-1,\ell})$ describe its shape.

Using (2.3.10) we get:

$$\begin{aligned} \xi_i = & \sum_{j=1}^{n} \mathcal{A}_j^\flat(\xi_{i-1,i}) \mathcal{A}_j + \sum_{j=1}^{n} \mathcal{A}_j^\flat(\xi_{i-2,i-1}) Ad_{g^{-1}_{i-1,i}} \mathcal{A}_j \\ & + \cdots + \sum_{j=1}^{n} \mathcal{A}_j^\flat(\xi_{1,2}) Ad_{g^{-1}_{2,i}} \mathcal{A}_j + \sum_{j=1}^{n} \mathcal{A}_j^\flat(\xi_1) Ad_{g^{-1}_{1,i}} \mathcal{A}_j \, . \end{aligned} \tag{4.2.8}$$

4.2.2 Nonholonomic Constraints and Connections on Principal Fiber Bundles

In this section we consider nonholonomic constraints acting on the G–Snake and we show that they determine a connection on the principal fiber bundle associated to our problem.

Codimension 1 Constraint Hypothesis: Assume that the evolution of system (4.2.1) on each copy of G is constrained to lie on an $(n-1)$–dimensional subspace h of the Lie algebra $\mathcal{G}$, where h is not a subalgebra of $\mathcal{G}$, i.e.

$$\xi_i \in h,\ i = 1, \ldots, \ell\,. \tag{4.2.9}$$

Then, for some $\mathcal{A}^\flat_\kappa \in \mathcal{G}^*$ (not necessarily an element of the basis $\{\mathcal{A}^\flat_i,\ i = 1, \ldots, n\}$) we have:

$$h = \mathrm{Ker}(\mathcal{A}^\flat_\kappa). \tag{4.2.10}$$

The constraints (4.2.9) can be expressed as:

$$\mathcal{A}^\flat_\kappa(\xi_i) = 0,\ i = 1, \ldots, \ell. \tag{4.2.11}$$

The constraints (4.2.11) are linear in the components of ξ_1 and those of the shape controls $\xi_{i,j}$. This can be made explicit by defining the composite $n\ell$–dimensional *velocity vector* ζ of the kinematic chain:

$$\begin{aligned}\zeta &\stackrel{\text{def}}{=} (\,\xi^1_1\ \cdots\ \xi^1_n\ \xi^{1,2}_1\ \cdots\ \xi^{\ell-1,\ell}_n\,)^\top \\ &= \left(\,\mathcal{A}^\flat_1(\xi_1)\ \cdots\ \mathcal{A}^\flat_n(\xi_1)\ \mathcal{A}^\flat_1(\xi_{1,2})\ \cdots\ \mathcal{A}^\flat_n(\xi_{\ell-1,\ell})\,\right)^\top.\end{aligned}$$

Theorem 4.2.1

The ℓ nonholonomic constraints (4.2.11) can be written in matrix form as:

$$A(g_{1,2}, \ldots, g_{\ell-1,\ell})\,\zeta = 0\,, \tag{4.2.12}$$

where A is a function of only the shape of the system and is a block lower triangular $\ell \times n\ell$ matrix of maximal rank of the form

$$A = \begin{pmatrix} *_{1,1} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ *_{1,2} & *_{2,2} & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \ddots & 0 & \cdots & 0 & 0 \\ *_{1,i} & *_{2,i} & \cdots & *_{i-1,i} & *_{i,i} & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \cdots & \ddots & 0 \\ *_{1,\ell} & *_{2,\ell} & \cdots & *_{i-1,\ell} & *_{i,\ell} & \cdots & *_{\ell-1,\ell} & *_{\ell,\ell} \end{pmatrix}, \tag{4.2.13}$$

with the $1 \times n$ block $*_{p,q}$, defined for $p \leq q$ as:

$$*_{p,q} = \begin{pmatrix} \mathcal{A}^\flat_\kappa(Ad_{g^{-1}_{p,q}} \mathcal{A}_1) & \cdots & \mathcal{A}^\flat_\kappa(Ad_{g^{-1}_{p,q}} \mathcal{A}_n) \end{pmatrix}.$$

Proof

From (4.2.8) and (4.2.11):

$$\begin{aligned} \mathcal{A}^\flat_\kappa(\xi_1) &= \sum_{j=1}^n \mathcal{A}^\flat_j(\xi_1)\mathcal{A}^\flat_\kappa(\mathcal{A}_j) = 0, \\ \mathcal{A}^\flat_\kappa(\xi_i) &= \sum_{j=1}^n \mathcal{A}^\flat_j(\xi_{i-1,i})\mathcal{A}^\flat_\kappa(\mathcal{A}_j) + \sum_{j=1}^n \mathcal{A}^\flat_j(\xi_{i-2,i-1})\mathcal{A}^\flat_\kappa(Ad_{g^{-1}_{i-1,i}} \mathcal{A}_j) \\ &\quad + \cdots + \sum_{j=1}^n \mathcal{A}^\flat_j(\xi_{1,2})\mathcal{A}^\flat_\kappa(Ad_{g^{-1}_{2,i}} \mathcal{A}_j) + \sum_{j=1}^n \mathcal{A}^\flat_j(\xi_1)\mathcal{A}^\flat_\kappa(Ad_{g^{-1}_{1,i}} \mathcal{A}_j), \\ &\qquad i = 2, \ldots, \ell. \end{aligned} \tag{4.2.14}$$

The diagonal blocks $*_{p,p}$ of A have the form $\begin{pmatrix} \mathcal{A}^\flat_\kappa(\mathcal{A}_1) & \cdots & \mathcal{A}^\flat_\kappa(\mathcal{A}_n) \end{pmatrix}$, therefore they contain at least one non-zero constant term. Thus A has always maximal rank.

∎

Corollary 4.2.2

Assume $\ell \geq n$. After possibly reordering its elements, we partition ζ as $\binom{\zeta_1}{\zeta_2}$, with ζ_2 an ℓ–dimensional vector containing the components of ξ_1 (and possibly some components of shape variations), while ζ_1 is an $(n-1)\ell$–dimensional vector containing *only* components of shape variations. Let the corresponding partition of A be $(A_1\ A_2)$, with A_1 a $\ell \times (n-1)\ell$ matrix and A_2 a locally invertible $\ell \times \ell$ matrix. Then from (4.2.12):

$$\zeta_2 = -A_2^{-1}(g_{1,2}, \ldots, g_{\ell-1,\ell})A_1(g_{1,2}, \ldots, g_{\ell-1,\ell})\zeta_1 \,. \tag{4.2.15}$$

The elements of ζ_1 will be referred to as *shape controls.*

Proof

Follows from the smooth dependence of A on the shape variables and the maximal rank property of Theorem 4.2.1.

∎

The physical significance of this result is that, if the global motion of the ℓ–node system is characterized by the global motion of its first module (i.e. by ξ_1), then variations of the shape (at least those which are elements of ζ_1) induce a global motion of the system.

Definition 4.2.3 (Nonholonomic Singularities)

G–Snake configurations $q \in Q$ where the matrix A_2 becomes singular for *all* possible partitions $(\zeta_1,\ \zeta_2)$ of the composite velocity vector ζ, which are such that ζ_2 contains all the components of ξ_1 (and possibly some of the shape variations), will be called *nonholonomic singularities.*

∎

There may be configurations where the matrix A_2 that corresponds to a particular partition of ζ is singular, but by considering a different partition of ζ, the corresponding A_2 ceases to be singular. These configurations are *not* nonholonomic singularities.

Consider now the manifolds Q and S defined in section 4.2.1 and the canonical projection $\pi : Q \to S$ defined by equation (4.2.2), i.e.

$$\pi(g_1, \ldots, g_\ell) \stackrel{\text{def}}{=} (g_1^{-1} g_2, \ldots, g_{\ell-1}^{-1} g_\ell) = (g_{1,2}, \ldots, g_{\ell-1,\ell}) \,. \tag{4.2.16}$$

Lemma 4.2.4

The quadruple (Q, S, π, G), together with the (left) action Φ of G on Q defined by

$$\Phi :\ G \times Q \to Q :\ (g, q) = (g, (g_1, \ldots, g_\ell)) \mapsto g \cdot q = (gg_1, \ldots, gg_\ell) \,, \tag{4.2.17}$$

is a (trivial) *principal fiber bundle.*

Proof

The canonical projection π defined in (4.2.16) is differentiable and its differential is

$$\pi_{*_q} :\ T_q Q \to T_{\pi(q)} S :\ (g_1 \xi_1, \ldots, g_\ell \xi_\ell) \mapsto (g_{1,2} \xi_{1,2}, \ldots, g_{\ell-1,\ell} \xi_{\ell-1,\ell}) \,, \tag{4.2.18}$$

where the $\xi_{i-1,i}$ are given by (4.2.6):

$$\xi_{i-1,i} = \xi_i - Ad_{g_{i-1,i}^{-1}} \xi_{i-1} ,\ i = 2, \ldots, \ell \,. \tag{4.2.19}$$

Thus, (Q, S, π, G) meets the requirements of Definition 2.3.21.

$\blacksquare$

Theorem 4.2.5 (Connection for case $\ell = n$)

Away from nonholonomic singularities and when $\ell = n$, the nonholonomic constraints (4.2.11) determine a *connection* on the principal fiber bundle (Q, S, π, G), with the horizontal subspace defined as follows:

$$\begin{aligned} H_q &= \{v \in T_qQ \mid v = (g_1\xi_1, \dots, g_\ell\xi_\ell) \text{ and } \xi_i \in h\} \\ &= \{v \in T_qQ \mid v = (g_1\xi_1, \dots, g_\ell\xi_\ell) \text{ and } \zeta_2 = -A_2^{-1}(\pi(q))A_1(\pi(q))\zeta_1 \}\,, \end{aligned} \tag{4.2.20}$$

where $\zeta_1 = \begin{pmatrix}\xi_1^{1,2} & \cdots & \xi_n^{1,2} & \xi_1^{2,3} & \cdots & \xi_n^{\ell-1,\ell}\end{pmatrix}^\top$ and $\zeta_2 = \begin{pmatrix}\xi_1^{1} & \cdots & \xi_n^{1}\end{pmatrix}^\top$.

Proof

Due to the left–invariance of our system, $T_qQ = \{(g_1\xi_1, \dots, g_\ell\xi_\ell) \mid \xi_i \in \mathcal{G}\}$. The vertical subspace is (from (4.2.7), (4.2.17)–(4.2.19))

$$\begin{aligned} V_q &= \{v \in T_qQ \mid \pi_{*_q}(v) = 0\} \\ &= \{(g_1\xi_1, \dots, g_\ell\xi_\ell) \mid (g_{1,2}\xi_{1,2},\ \dots,\ g_{\ell-1,\ell}\xi_{\ell-1,\ell}) = 0\} \\ &= \{(g_1\xi_1, \dots, g_\ell\xi_\ell) \mid \xi_{1,2} = \dots = \xi_{\ell-1,\ell} = 0\} \\ &= \{(g_1\xi_1, \dots, g_\ell\xi_\ell) \mid \xi_i = Ad_{g_{1,i}^{-1}}\xi_1,\ i = 2, \dots, \ell\}\,. \end{aligned} \tag{4.2.21}$$

Physically, the vertical subspace contains all infinitesimal motions of the kinematic chain that do not alter its shape (those are not necessarily motions that satisfy the nonholonomic constraints).

To show property (1) of Definition 2.3.22, we first prove that $H_q \cap V_q = \{0\}$ and then that $\dim(T_qQ) = \dim(H_q) + \dim(V_q)$.

To show $H_q \cap V_q = \{0\}$, assume that there exists a non–trivial $v \in H_q \cap V_q$. By the definition of V_q, the corresponding shape controls are zero. Thus $\zeta_1 = 0$ and, by the definition of H_q, also $\zeta_2 = 0$. But then $\xi_1 = 0$ and from (4.2.21) also $\xi_i = 0$, $i = 2, \dots, \ell$. Thus $\zeta = 0$. Thus $H_q \cap V_q = \{0\}$.

Now observe that, away from the nonholonomic singularities $\dim(H_q) = n\ell - \ell$. Further, $\dim(V_q) = n$. So, when $\ell = n$, $\dim(H_q \oplus V_q) = (n\ell - \ell) + n = (n^2 - n) + n = n^2 = \dim(T_qQ)$. It follows that $H_q \oplus V_q = T_qQ$.

To show property (2) of Definition 2.3.22, consider $T_q\Phi_g \cdot H_q = g \cdot H_q = g \cdot \{(g_1\xi_1, \ldots, g_\ell\xi_\ell) \mid \xi_i \in h\} \stackrel{\text{def}}{=} \{(gg_1\xi_1, \ldots, gg_\ell\xi_\ell) \mid \xi_i \in h\}$ and $H_{g\cdot q} = \{v \in T_{g\cdot q}Q \mid v = (g \cdot q) \cdot (\xi_1, \ldots, \xi_\ell)$ and $\xi_i \in h\} = \{(gg_1\xi_1, \ldots, gg_\ell\xi_\ell) \mid \xi_i \in h\}$. Then, obviously, $T_q\Phi_g \cdot H_q = H_{g\cdot q}$.

Property (3) of Definition 2.3.22 is immediate from the smooth dependence of A on the shape and from the left–invariance of our system.

∎

4.3 Three–node G–Snakes on Three–dimensional Lie Groups

Here we specialize the results of the previous section to kinematic chains on Lie groups with 3-dimensional real non-abelian Lie algebras ($n = 3$). In section 4.3.1 we consider the Special Euclidean group $SE(2)$, in section 4.3.2 the Heisenberg group $H(3)$, in section 4.3.3 the Special Orthogonal group $SO(3)$ and in section 4.3.4 the Special Linear group $SL(2)$.

We study 3–node, 2–module kinematic chains on each of these groups ($\ell = n = 3$) by deriving their Wei–Norman representation and by defining the partial connection on the corresponding principal fiber bundle.

Let G be one of the above four matrix Lie groups and $\mathcal{G}$ be the corresponding Lie algebra. Consider the system (2.3.7) on G :

$$\dot{g} = T_eL_g \cdot \xi = g\xi \ , \tag{4.3.1}$$

with $g(.) \subset G$ and $\xi(.) \subset \mathcal{G}$. The curve $\xi(.) \subset \mathcal{G}$ can be represented as in (2.3.10):

$$\xi = \sum_{i=1}^{3} \xi_i \mathcal{A}_i = \sum_{i=1}^{3} \mathcal{A}_i^\flat(\xi)\mathcal{A}_i \ , \tag{4.3.2}$$

with $\xi_i = \mathcal{A}_i^\flat(\xi) \in I\!R$. From Proposition 2.3.14, any $g(.) \subset G$ with $g(0) = I$, has a *local* Wei–Norman representation of the form (2.3.11):

$$g(t) = e^{\gamma_1(t)\mathcal{A}_1} e^{\gamma_2(t)\mathcal{A}_2} e^{\gamma_3(t)\mathcal{A}_3} \ . \tag{4.3.3}$$

The coefficients $\gamma_i \in I\!R$ are related to the components of ξ by (2.3.12):

$$\begin{pmatrix} \dot\gamma_1 \\ \dot\gamma_2 \\ \dot\gamma_3 \end{pmatrix} = M(\gamma_1, \gamma_2, \gamma_3) \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} . \tag{4.3.4}$$

By differentiating (4.3.3) and using (2.3.20) to get:

$$\begin{aligned} \frac{dg}{dt} &= \dot\gamma_1 e^{\gamma_1 \mathcal{A}_1} \mathcal{A}_1 e^{\gamma_2 \mathcal{A}_2} e^{\gamma_3 \mathcal{A}_3} + \dot\gamma_2 e^{\gamma_1 \mathcal{A}_1} e^{\gamma_2 \mathcal{A}_2} \mathcal{A}_2 e^{\gamma_3 \mathcal{A}_3} + \dot\gamma_3 e^{\gamma_1 \mathcal{A}_1} e^{\gamma_2 \mathcal{A}_2} e^{\gamma_3 \mathcal{A}_3} \mathcal{A}_3 \\ &= e^{\gamma_1 \mathcal{A}_1} e^{\gamma_2 \mathcal{A}_2} e^{\gamma_3 \mathcal{A}_3} [\dot\gamma_1 e^{-\gamma_3 \mathcal{A}_3} e^{-\gamma_2 \mathcal{A}_2} \mathcal{A}_1 e^{\gamma_2 \mathcal{A}_2} e^{\gamma_3 \mathcal{A}_3} + \dot\gamma_2 e^{-\gamma_3 \mathcal{A}_3} \mathcal{A}_2 e^{\gamma_3 \mathcal{A}_3} + \dot\gamma_3 \mathcal{A}_3] \\ &= g\, [\dot\gamma_1 e^{ad(-\gamma_3 \mathcal{A}_3)} e^{ad(-\gamma_2 \mathcal{A}_2)} \mathcal{A}_1 + \dot\gamma_2 e^{ad(-\gamma_3 \mathcal{A}_3)} \mathcal{A}_2 + \dot\gamma_3 \mathcal{A}_3] . \end{aligned}$$

Thus

$$\xi = \dot\gamma_1 e^{ad(-\gamma_3 \mathcal{A}_3)} e^{ad(-\gamma_2 \mathcal{A}_2)} \mathcal{A}_1 + \dot\gamma_2 e^{ad(-\gamma_3 \mathcal{A}_3)} \mathcal{A}_2 + \dot\gamma_3 \mathcal{A}_3 . \tag{4.3.5}$$

From (2.3.21) and (2.3.22), we have for $i = 1, 2, 3$:

$$Ad_g \mathcal{A}_i = e^{ad(\gamma_1 \mathcal{A}_1)} e^{ad(\gamma_2 \mathcal{A}_2)} e^{ad(\gamma_3 \mathcal{A}_3)} \mathcal{A}_i . \tag{4.3.6}$$

and

$$Ad_{g^{-1}} \mathcal{A}_i = e^{ad(-\gamma_3 \mathcal{A}_3)} e^{ad(-\gamma_2 \mathcal{A}_2)} e^{ad(-\gamma_1 \mathcal{A}_1)} \mathcal{A}_i . \tag{4.3.7}$$

Consider now the 2–module kinematic chain on G. From the system kinematics (equations (4.2.1)–(4.2.8)) we have:

$$\begin{aligned} g_2 &= g_1 g_{1,2} , \\ g_3 &= g_2 g_{2,3} = g_1 g_{1,2} g_{2,3} , \\ g_{1,3} &= g_{1,2} g_{2,3} . \end{aligned} \tag{4.3.8}$$

From (4.2.6) we get for the corresponding velocities:

$$\begin{aligned} \xi_2 &= \xi_{1,2} + Ad_{g_{1,2}^{-1}} \xi_1 , \\ \xi_3 &= \xi_{2,3} + Ad_{g_{2,3}^{-1}} \xi_2 = \xi_{2,3} + Ad_{g_{2,3}^{-1}} \xi_{1,2} + Ad_{g_{1,3}^{-1}} \xi_1 , \\ \xi_{1,3} &= \xi_{2,3} + Ad_{g_{2,3}^{-1}} \xi_{1,2} , \end{aligned} \tag{4.3.9}$$

where $\xi_1, \xi_2, \xi_3, \xi_{1,2}, \xi_{2,3}, \xi_{1,3} \in \mathcal{G}$.

Assume that the evolution of system (4.3.1) on each copy of G is constrained to lie on a 2–dimensional subspace h of the Lie algebra $\mathcal{G}$, where h is not a subalgebra of $\mathcal{G}$. Define

$$\zeta_1 = \left(\xi_1^{1,2}\, \xi_2^{1,2}\, \xi_3^{1,2}\, \xi_1^{2,3}\, \xi_2^{2,3}\, \xi_3^{2,3}\right)^{\mathsf{T}} \text{ and } \zeta_2 = \left(\xi_1^1\, \xi_2^1\, \xi_3^1\right)^{\mathsf{T}} .$$

Theorem 4.2.1 holds with $\zeta = \binom{\zeta_2}{\zeta_1}$. From Corollary 4.2.2 we conclude that the global velocity of the 2–module kinematic chain, as it is characterized by ξ_1, can be expressed as a function of only the shape variables $g_{1,2}, g_{2,3}$ and shape controls $\xi_{1,2}, \xi_{2,3}$ of the assembly:

$$\zeta_2 = -A_2^{-1}(g_{1,2}, g_{2,3}) A_1(g_{1,2}, g_{2,3}) \zeta_1 \,. \tag{4.3.10}$$

From Theorem 4.2.5, equation (4.3.10) defines (away from the singularities of A_2) a connection on the trivial principal bundle $(S \times G, S, \pi, G)$ with $S = G \times G$ and with horizontal subspace:

$$\begin{aligned} H_q &= \{v \in T_qQ \mid v = (g_1\xi_1, g_2\xi_2, g_3\xi_3) \text{ and } \xi_i \in h\} \\ &= \{v \in T_qQ \mid v = (g_1\xi_1, g_2\xi_2, g_3\xi_3) \text{ and } \zeta_2 = -A_2^{-1}(g_{1,2}, g_{2,3})\, A_1(g_{1,2}, g_{2,3})\, \zeta_1 \,\} \,. \end{aligned} \tag{4.3.11}$$

Subsequently, we will derive explicitly the Wei–Norman representation for each of the Lie groups mentioned earlier and we will define the connection (4.3.11) for specific 2–dimensional subspaces h of $\mathcal{G}$.

Proposition 4.3.1 (Vershik & Gershkovich [1994])

For each of $H(3), SO(3)$ and $SE(2)$, all 2–dimensional subspaces h of $\mathcal{G}$, which are not subalgebras, are equivalent under $Aut(\mathcal{G})$, the group of automorphisms of $\mathcal{G}$, and can be represented by $h = \mathrm{sp}\{\mathcal{A}_1, \mathcal{A}_2\}$ in the basis of $\mathcal{G}$ (specified in the following sections). For $SL(2)$, there are 2 classes of such equivalent subspaces that can be represented, respectively, by $h = \mathrm{sp}\{\mathcal{A}_1, \mathcal{A}_2\}$ and by $h = \mathrm{sp}\{\mathcal{A}_3, \mathcal{A}_1 + \mathcal{A}_2\}$.

∎

Due to this result, only nonholonomic constraints corresponding to these subspaces of $\mathcal{G}$ need to be considered here.

Our main purpose in this section is to set the stage for a deeper understanding of this novel class of kinematic chains, by cataloguing the low–dimensional possibilities. One case, corresponding to $SE(2)$ has already found a concrete mechanical realization (c.f. chapter 5). Others might follow, for instance, there are possible connections between $SO(3)$–Snakes and the kinematics of long chain molecules (Karplus & McCammon [1986]).

4.3.1 G–Snakes on the Special Euclidean Group $SE(2)$

Let $G = SE(2)$ be the Special Euclidean group of rigid motions on the plane and $\mathcal{G} = se(2)$ be the corresponding algebra with the following basis:

$$\mathcal{A}_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \mathcal{A}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \mathcal{A}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} . \tag{4.3.12}$$

Then:

$$[\mathcal{A}_1 , \mathcal{A}_2] = \mathcal{A}_3 , [\mathcal{A}_1 , \mathcal{A}_3] = -\mathcal{A}_2 , [\mathcal{A}_2 , \mathcal{A}_3] = 0 . \tag{4.3.13}$$

Proposition 4.3.2

Let $g(0) = I$, the identity of G. There exists a *global* representation of the curve $g(.) \subset G$ of the form (4.3.3) with

$$\begin{pmatrix} \dot{\gamma}_1 \\ \dot{\gamma}_2 \\ \dot{\gamma}_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \gamma_3 & 1 & 0 \\ -\gamma_2 & 0 & 1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} . \tag{4.3.14}$$

Equation (4.3.14) is solvable by quadratures:

$$
\begin{aligned}
\gamma_1(t) &= \gamma_1(0) + \int_0^t \xi_1(\tau)d\tau \, , \\
\gamma_2(t) &= \gamma_2(0)\cos\left(\int_0^t \xi_1(\sigma)d\sigma\right) + \gamma_3(0)\sin\left(\int_0^t \xi_1(\sigma)d\sigma\right) \\
&\quad + \int_0^t \xi_2(\tau)\cos\left(\int_\tau^t \xi_1(\sigma)d\sigma\right)d\tau + \int_0^t \xi_3(\tau)\sin\left(\int_\tau^t \xi_1(\sigma)d\sigma\right)d\tau \, , \\
\gamma_3(t) &= -\gamma_2(0)\sin\left(\int_0^t \xi_1(\sigma)d\sigma\right) + \gamma_3(0)\cos\left(\int_0^t \xi_1(\sigma)d\sigma\right) \\
&\quad - \int_0^t \xi_2(\tau)\sin\left(\int_\tau^t \xi_1(\sigma)d\sigma\right)d\tau + \int_0^t \xi_3(\tau)\cos\left(\int_\tau^t \xi_1(\sigma)d\sigma\right)d\tau \, .
\end{aligned}
\tag{4.3.15}
$$

For the initial condition $g(0) = I$, we have $\gamma_i(0) = 0, \; i = 1, 2, 3$.

Proof

Since $\mathcal{G} = se(2)$ is solvable, the existence of a global representation is immediate by (Wei & Norman [1964]). To see that it has the form (4.3.3) with coefficients given by (4.3.14) and (4.3.15), we compute the RHS of (4.3.5), using (4.3.13) and (2.3.20):

$$
\begin{aligned}
& e^{ad(-\gamma_3\mathcal{A}_3)}e^{ad(-\gamma_2\mathcal{A}_2)}\mathcal{A}_1 = e^{ad(-\gamma_3\mathcal{A}_3)}(\mathcal{A}_1 + \gamma_2\mathcal{A}_3) = \mathcal{A}_1 - \gamma_3\mathcal{A}_2 + \gamma_2\mathcal{A}_3 \, , \\
& e^{ad(-\gamma_3\mathcal{A}_3)}\mathcal{A}_2 = \mathcal{A}_2 \, .
\end{aligned}
\tag{4.3.16}
$$

From (4.3.1), (4.3.2), (4.3.5) and (4.3.16), we have:

$$
\begin{aligned}
\xi &= \xi_1\mathcal{A}_1 + \xi_2\mathcal{A}_2 + \xi_3\mathcal{A}_3 \\
&= \dot\gamma_1(\mathcal{A}_1 - \gamma_3\mathcal{A}_2 + \gamma_2\mathcal{A}_3) + \dot\gamma_2\mathcal{A}_2 + \dot\gamma_3\mathcal{A}_3 = \dot\gamma_1\mathcal{A}_1 + (\dot\gamma_2 - \gamma_3\dot\gamma_1)\mathcal{A}_2 + (\dot\gamma_3 + \gamma_2\dot\gamma_1)\mathcal{A}_3 \, .
\end{aligned}
$$

Since $\{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3\}$ is a basis, we have:

$$\xi_1 = \dot\gamma_1 \, , \quad \xi_2 = \dot\gamma_2 - \gamma_3\dot\gamma_1 \, , \quad \xi_3 = \dot\gamma_3 + \gamma_2\dot\gamma_1 \, .$$

Solving for the $\dot{\gamma}_i$'s we get (4.3.14), which can be rewritten as:

$$\dot{\gamma}_1 = \xi_1 \ ,$$
$$\begin{pmatrix} \dot{\gamma}_2 \\ \dot{\gamma}_3 \end{pmatrix} = \begin{pmatrix} 0 & \xi_1 \\ -\xi_1 & 0 \end{pmatrix} \begin{pmatrix} \gamma_2 \\ \gamma_3 \end{pmatrix} + \begin{pmatrix} \xi_2 \\ \xi_3 \end{pmatrix} .$$

This system can be solved by quadratures, giving (4.3.15).

∎

Lemma 4.3.3

Consider the Wei–Norman representation (4.3.3) of $g \in SE(2)$ determined by (4.3.14) and (4.3.15). Then:

$$\begin{aligned} Ad_g\mathcal{A}_1 &= \mathcal{A}_1 + (\gamma_2 \sin\gamma_1 + \gamma_3 \cos\gamma_1)\mathcal{A}_2 + (-\gamma_2 \cos\gamma_1 + \gamma_3 \sin\gamma_1)\mathcal{A}_3 \ , \\ Ad_g\mathcal{A}_2 &= \cos\gamma_1 \mathcal{A}_2 + \sin\gamma_1 \mathcal{A}_3 \ , \\ Ad_g\mathcal{A}_3 &= -\sin\gamma_1 \mathcal{A}_2 + \cos\gamma_1 \mathcal{A}_3 \ . \end{aligned} \tag{4.3.17}$$

Moreover,

$$\begin{aligned} Ad_{g^{-1}}\mathcal{A}_1 &= \mathcal{A}_1 - \gamma_3\mathcal{A}_2 + \gamma_2\mathcal{A}_3 \ , \\ Ad_{g^{-1}}\mathcal{A}_2 &= \cos\gamma_1 \mathcal{A}_2 - \sin\gamma_1 \mathcal{A}_3 \ , \\ Ad_{g^{-1}}\mathcal{A}_3 &= \sin\gamma_1 \mathcal{A}_2 + \cos\gamma_1 \mathcal{A}_3 \ . \end{aligned} \tag{4.3.18}$$

Proof

We need to compute (4.3.6). For $\mathcal{A}_1$ we have:

$$\begin{aligned} e^{ad(\gamma_3\mathcal{A}_3)}\mathcal{A}_1 &= \mathcal{A}_1 + \gamma_3\mathcal{A}_3 \\ e^{ad(\gamma_2\mathcal{A}_2)}e^{ad(\gamma_3\mathcal{A}_3)}\mathcal{A}_1 &= \mathcal{A}_1 - \gamma_3\mathcal{A}_2 + \gamma_2\mathcal{A}_3 \\ e^{ad(\gamma_1\mathcal{A}_1)}e^{ad(\gamma_2\mathcal{A}_2)}e^{ad(\gamma_3\mathcal{A}_3)}\mathcal{A}_1 &= \mathcal{A}_1 + (\gamma_2 \sin\gamma_1 + \gamma_3 \cos\gamma_1)\mathcal{A}_2 \\ &\quad + (-\gamma_2 \cos\gamma_1 + \gamma_3 \sin\gamma_1)\mathcal{A}_3 \ . \end{aligned}$$

For $\mathcal{A}_2$:

$$\begin{aligned} e^{ad(\gamma_3\mathcal{A}_3)}\mathcal{A}_2 &= \mathcal{A}_2 \\ e^{ad(\gamma_2\mathcal{A}_2)}e^{ad(\gamma_3\mathcal{A}_3)}\mathcal{A}_2 &= \mathcal{A}_2 \\ e^{ad(\gamma_1\mathcal{A}_1)}e^{ad(\gamma_2\mathcal{A}_2)}e^{ad(\gamma_3\mathcal{A}_3)}\mathcal{A}_2 &= \cos\gamma_1\mathcal{A}_2 + \sin\gamma_1\mathcal{A}_3 \ . \end{aligned}$$

For $\mathcal{A}_3$:

$$e^{ad(\gamma_3\mathcal{A}_3)}\mathcal{A}_3 = \mathcal{A}_3$$

$$e^{ad(\gamma_2\mathcal{A}_2)}e^{ad(\gamma_3\mathcal{A}_3)}\mathcal{A}_3 = \mathcal{A}_3$$

$$e^{ad(\gamma_1\mathcal{A}_1)}e^{ad(\gamma_2\mathcal{A}_2)}e^{ad(\gamma_3\mathcal{A}_3)}\mathcal{A}_3 = -\sin\gamma_1\mathcal{A}_2 + \cos\gamma_1\mathcal{A}_3 \ .$$

From the above, (4.3.17) is immediate.

We also need to compute (4.3.7). For $\mathcal{A}_1$ we have:

$$e^{ad(-\gamma_1\mathcal{A}_1)}\mathcal{A}_1 = \mathcal{A}_1$$

$$e^{ad(-\gamma_2\mathcal{A}_2)}e^{ad(-\gamma_1\mathcal{A}_1)}\mathcal{A}_1 = e^{ad(-\gamma_2\mathcal{A}_2)}\mathcal{A}_1 = \mathcal{A}_1 + \gamma_2\mathcal{A}_3$$

$$e^{ad(-\gamma_3\mathcal{A}_3)}e^{ad(-\gamma_2\mathcal{A}_2)}e^{ad(-\gamma_1\mathcal{A}_1)}\mathcal{A}_1 = e^{ad(-\gamma_3\mathcal{A}_3)}(\mathcal{A}_1 + \gamma_2\mathcal{A}_3) = \mathcal{A}_1 - \gamma_3\mathcal{A}_2 + \gamma_2\mathcal{A}_3 \ .$$

For $\mathcal{A}_2$:

$$e^{ad(-\gamma_1\mathcal{A}_1)}\mathcal{A}_2 = \cos\gamma_1\mathcal{A}_2 - \sin\gamma_1\mathcal{A}_3$$

$$e^{ad(-\gamma_2\mathcal{A}_2)}e^{ad(-\gamma_1\mathcal{A}_2)}\mathcal{A}_2 = e^{ad(-\gamma_2\mathcal{A}_2)}(\cos\gamma_1\mathcal{A}_2 - \sin\gamma_1\mathcal{A}_3) = \cos\gamma_1\mathcal{A}_2 - \sin\gamma_1\mathcal{A}_3$$

$$\begin{aligned} e^{ad(-\gamma_3\mathcal{A}_3)}e^{ad(-\gamma_2\mathcal{A}_2)}e^{ad(-\gamma_1\mathcal{A}_1)}\mathcal{A}_2 &= e^{ad(-\gamma_3\mathcal{A}_3)}(\cos\gamma_1\mathcal{A}_2 - \sin\gamma_1\mathcal{A}_3) \\ &= \cos\gamma_1\mathcal{A}_2 - \sin\gamma_1\mathcal{A}_3 \ . \end{aligned}$$

For $\mathcal{A}_3$:

$$e^{ad(-\gamma_1\mathcal{A}_1)}\mathcal{A}_3 = \sin\gamma_1\mathcal{A}_2 + \cos\gamma_1\mathcal{A}_3$$

$$e^{ad(-\gamma_2\mathcal{A}_2)}e^{ad(-\gamma_1\mathcal{A}_3)}\mathcal{A}_3 = e^{ad(-\gamma_2\mathcal{A}_2)}(\sin\gamma_1\mathcal{A}_2 + \cos\gamma_1\mathcal{A}_3) = \sin\gamma_1\mathcal{A}_2 + \cos\gamma_1\mathcal{A}_3$$

$$\begin{aligned} e^{ad(-\gamma_3\mathcal{A}_3)}e^{ad(-\gamma_2\mathcal{A}_2)}e^{ad(-\gamma_1\mathcal{A}_1)}\mathcal{A}_3 &= e^{ad(-\gamma_3\mathcal{A}_3)}(\sin\gamma_1\mathcal{A}_2 + \cos\gamma_1\mathcal{A}_3) \\ &= \sin\gamma_1\mathcal{A}_2 + \cos\gamma_1\mathcal{A}_3 \ . \end{aligned}$$

From the above, (4.3.18) is immediate.

∎

Using the definition of the basis $\{\mathcal{A}_i\}$ from (4.3.12), we have:

$$e^{\gamma_1\mathcal{A}_1} = \begin{pmatrix} \cos\gamma_1 & -\sin\gamma_1 & 0 \\ \sin\gamma_1 & \cos\gamma_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} , \quad e^{\gamma_2\mathcal{A}_2} = \begin{pmatrix} 1 & 0 & \gamma_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} , \quad e^{\gamma_3\mathcal{A}_3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \gamma_3 \\ 0 & 0 & 1 \end{pmatrix} . \tag{4.3.19}$$

Then, from (4.3.3):

$$g = e^{\gamma_1 \mathcal{A}_1} e^{\gamma_2 \mathcal{A}_2} e^{\gamma_3 \mathcal{A}_3} = \begin{pmatrix} \cos\gamma_1 & -\sin\gamma_1 & \gamma_2\cos\gamma_1 - \gamma_3\sin\gamma_1 \\ \sin\gamma_1 & \cos\gamma_1 & \gamma_2\sin\gamma_1 + \gamma_3\cos\gamma_1 \\ 0 & 0 & 1 \end{pmatrix} . \tag{4.3.20}$$

Define:

$$\begin{aligned} \phi &\stackrel{\text{def}}{=} \gamma_1 \\ x &\stackrel{\text{def}}{=} \gamma_2\cos\gamma_1 - \gamma_3\sin\gamma_1 \\ &= x_0 + \int_0^t \xi_2(\tau)\cos\phi(\tau)\, d\tau - \int_0^t \xi_3(\tau)\sin\phi(\tau)\, d\tau \ , \\ y &\stackrel{\text{def}}{=} \gamma_2\sin\gamma_1 + \gamma_3\cos\gamma_1 \\ &= y_0 + \int_0^t \xi_2(\tau)\sin\phi(\tau)\, d\tau + \int_0^t \xi_3(\tau)\cos\phi(\tau)\, d\tau \ , \end{aligned} \tag{4.3.21}$$

where $x_0 \stackrel{\text{def}}{=} \gamma_2(0)\cos\gamma_1(0) - \gamma_3(0)\sin\gamma_1(0)$ and $y_0 \stackrel{\text{def}}{=} \gamma_2(0)\sin\gamma_1(0) + \gamma_3(0)\cos\gamma_1(0)$. Then we take from (4.3.20) the usual form of elements of $SE(2)$

$$g = \begin{pmatrix} \cos\phi & -\sin\phi & x \\ \sin\phi & \cos\phi & y \\ 0 & 0 & 1 \end{pmatrix} . \tag{4.3.22}$$

Differentiating and using (4.3.14), we have:

$$\begin{aligned} \dot{x} &= \xi_2\cos\gamma_1 - \xi_3\sin\gamma_1 \ , \\ \dot{y} &= \xi_2\sin\gamma_1 + \xi_3\cos\gamma_1 \ , \\ \dot{\phi} &= \xi_1 \ . \end{aligned} \tag{4.3.23}$$

Observe that

$$\begin{aligned} \gamma_1 &= \phi \ , \\ \gamma_2 &= x\cos\phi + y\sin\phi \ , \\ \gamma_3 &= -x\sin\phi + y\cos\phi \end{aligned} \tag{4.3.24}$$

and

$$\xi_1 = \dot{\phi} \,,$$
$$\xi_2 = \dot{x}\cos\phi + \dot{y}\sin\phi \,, \tag{4.3.25}$$
$$\xi_3 = -\dot{x}\sin\phi + \dot{y}\cos\phi \,.$$

Consider now the 3–node kinematic chain on $SE(2)$. A concrete mechanical realization of such a system is the Variable Geometry Truss assembly detailed in the next chapter. The system kinematics of equations (4.3.8),(4.3.9) apply.

From (4.3.13) we can see that there are at least two possible 2–dimensional subspaces h of $\mathcal{G}$ that can generate the whole algebra under Lie bracketing:

$$h_3 = \mathrm{sp}\{\mathcal{A}_1, \mathcal{A}_2\} = \mathrm{Ker}(\mathcal{A}_3^\flat) \;\text{ and }\; h_2 = \mathrm{sp}\{\mathcal{A}_1, \mathcal{A}_3\} = \mathrm{Ker}(\mathcal{A}_2^\flat) \,. \tag{4.3.26}$$

From Proposition 4.3.1 we know that those subspaces are equivalent under automorphisms of $\mathcal{G}$ (e.g. for $\xi \in se(2)$ with coordinates (ξ_1, ξ_2, ξ_3) with respect to the basis defined in (4.3.12), consider the automorphism $\psi : se(2) \longrightarrow se(2) : (\xi_1, \xi_2, \xi_3) \mapsto (\xi_1, -\xi_3, \xi_2)$, noting that $\psi(h_3) = h_2$). Thus, subsequently, we will consider only the subspace h_2. The nonholonomic constraints $\xi_i \in h_2$ can, then, be expressed as:

$$\mathcal{A}_2^\flat(\xi_i) = 0, \;\; i = 1, 2, 3 \,. \tag{4.3.27}$$

The exact form of A_1, A_2, ζ_1 and ζ_2, as well as a description of the system's nonholonomic singularities, is presented in section 5.3, as well as possible motion control schemes based on periodic shape variations.

4.3.2 G–Snakes on the Heisenberg Group $H(3)$

Let $G = H(3)$ be the Heisenberg group of real 3×3 upper triangular matrices of the form $\begin{pmatrix} 1 & \alpha & \beta \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{pmatrix}$ and let $\mathcal{G} = h(3)$ be the algebra of 3×3 nil–triangular matrices with the following basis:

$$\mathcal{A}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; \mathcal{A}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \; \mathcal{A}_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \,. \tag{4.3.28}$$

Then:

$$[\mathcal{A}_1 , \mathcal{A}_2] = \mathcal{A}_3 , [\mathcal{A}_1 , \mathcal{A}_3] = 0 , [\mathcal{A}_2 , \mathcal{A}_3] = 0 . \tag{4.3.29}$$

Proposition 4.3.4

The algebra $\mathcal{G} = h(3)$ is nilpotent (thus solvable) and, from Proposition 2.3.14, any $g \in G = H(3)$ has a *global* Wei–Norman representation of the form (4.3.3) with

$$\begin{pmatrix} \dot\gamma_1 \\ \dot\gamma_2 \\ \dot\gamma_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\gamma_2 & 0 & 1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} . \tag{4.3.30}$$

Equation (4.3.30) is solvable by quadratures:

$$\begin{aligned} \gamma_1(t) &= \gamma_1(0) + \int_0^t \xi_1(\tau) d\tau , \\ \gamma_2(t) &= \gamma_2(0) + \int_0^t \xi_2(\tau) d\tau , \\ \gamma_3(t) &= \gamma_3(0) - \int_0^t \gamma_2(\tau)\xi_1(\tau) d\tau + \int_0^t \xi_3(\tau) d\tau \\ &= \gamma_3(0) - \gamma_2(0) \int_0^t \xi_1(\sigma) d\sigma - \int_0^t \xi_1(\tau) \Big(\int_0^\tau \xi_2(\sigma) d\sigma \Big) d\tau + \int_0^t \xi_3(\tau) d\tau . \end{aligned} \tag{4.3.31}$$

Proof

Since $\mathcal{G} = h(3)$ is nilpotent, the existence of a global representation is immediate by (Wei & Norman [1964]). To see that it has the form (4.3.3) with coefficients given by (4.3.30) and (4.3.31), we compute the RHS of (4.3.5), using (4.3.29) and (2.3.20):

$$\begin{aligned} & e^{ad(-\gamma_2 \mathcal{A}_2)} \mathcal{A}_1 = \mathcal{A}_1 + \gamma_2 \mathcal{A}_3 , \\ & e^{ad(-\gamma_3 \mathcal{A}_3)} e^{ad(-\gamma_2 \mathcal{A}_2)} \mathcal{A}_1 = \mathcal{A}_1 + \gamma_2 \mathcal{A}_3 , \\ & e^{ad(-\gamma_3 \mathcal{A}_3)} \mathcal{A}_2 = \mathcal{A}_2 . \end{aligned} \tag{4.3.32}$$

From (4.3.1), (4.3.2), (4.3.5) and (4.3.32), we have:

$$\begin{aligned} \xi &= \xi_1 \mathcal{A}_1 + \xi_2 \mathcal{A}_2 + \xi_3 \mathcal{A}_3 \\ &= \dot\gamma_1 (\mathcal{A}_1 + \gamma_2 \mathcal{A}_3) + \dot\gamma_2 \mathcal{A}_2 + \dot\gamma_3 \mathcal{A}_3 . \end{aligned}$$

Since $\{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3\}$ is a basis, we have:

$$\xi_1 = \dot{\gamma}_1 \ , \quad \xi_2 = \dot{\gamma}_2 \ , \quad \xi_3 = \gamma_2\dot{\gamma}_1 + \dot{\gamma}_3 \ .$$

Solving for the $\dot{\gamma}_i$'s we get (4.3.30). This system can be solved by quadratures, giving (4.3.31).

∎

Lemma 4.3.5

Consider the Wei–Norman representation (4.3.3) of $g \in H(3)$ determined by (4.3.30) and (4.3.31). Then:

$$\begin{aligned} Ad_{g^{-1}}\mathcal{A}_1 &= \mathcal{A}_1 + \gamma_2\mathcal{A}_3 \ , \\ Ad_{g^{-1}}\mathcal{A}_2 &= \mathcal{A}_2 - \gamma_1\mathcal{A}_3 \ , \\ Ad_{g^{-1}}\mathcal{A}_3 &= \mathcal{A}_3 \ . \end{aligned} \tag{4.3.33}$$

Proof

We need to compute (4.3.7). For $\mathcal{A}_1$ we have:

$$\begin{aligned} &e^{ad(-\gamma_1\mathcal{A}_1)}\mathcal{A}_1 = \mathcal{A}_1 \\ &e^{ad(-\gamma_2\mathcal{A}_2)}e^{ad(-\gamma_1\mathcal{A}_1)}\mathcal{A}_1 = \mathcal{A}_1 + \gamma_2\mathcal{A}_3 \\ &e^{ad(-\gamma_3\mathcal{A}_3)}e^{ad(-\gamma_2\mathcal{A}_2)}e^{ad(-\gamma_1\mathcal{A}_1)}\mathcal{A}_1 = \mathcal{A}_1 + \gamma_2\mathcal{A}_3 \ . \end{aligned}$$

For $\mathcal{A}_2$:

$$\begin{aligned} &e^{ad(-\gamma_1\mathcal{A}_1)}\mathcal{A}_2 = \mathcal{A}_2 - \gamma_1\mathcal{A}_3 \\ &e^{ad(-\gamma_2\mathcal{A}_2)}e^{ad(-\gamma_1\mathcal{A}_2)}\mathcal{A}_2 = \mathcal{A}_2 - \gamma_1\mathcal{A}_3 \\ &e^{ad(-\gamma_3\mathcal{A}_3)}e^{ad(-\gamma_2\mathcal{A}_2)}e^{ad(-\gamma_1\mathcal{A}_1)}\mathcal{A}_2 = \mathcal{A}_2 - \gamma_1\mathcal{A}_3 \ . \end{aligned}$$

For $\mathcal{A}_3$:

$$\begin{aligned} &e^{ad(-\gamma_1\mathcal{A}_1)}\mathcal{A}_3 = \mathcal{A}_3 \\ &e^{ad(-\gamma_2\mathcal{A}_2)}e^{ad(-\gamma_1\mathcal{A}_3)}\mathcal{A}_3 = \mathcal{A}_3 \\ &e^{ad(-\gamma_3\mathcal{A}_3)}e^{ad(-\gamma_2\mathcal{A}_2)}e^{ad(-\gamma_1\mathcal{A}_1)}\mathcal{A}_3 = \mathcal{A}_3 \ . \end{aligned}$$

From the above, (4.3.33) is immediate.

∎

Using the definition of the basis $\{\mathcal{A}_i\}$ from (4.3.28), we have:

$$e^{\gamma_1 \mathcal{A}_1} = \begin{pmatrix} 1 & \gamma_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} , \quad e^{\gamma_2 \mathcal{A}_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \gamma_2 \\ 0 & 0 & 1 \end{pmatrix} , \quad e^{\gamma_3 \mathcal{A}_3} = \begin{pmatrix} 1 & 0 & \gamma_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

Then, from (4.3.3):

$$g = e^{\gamma_1 \mathcal{A}_1} e^{\gamma_2 \mathcal{A}_2} e^{\gamma_3 \mathcal{A}_3} = \begin{pmatrix} 1 & \gamma_1 & \gamma_1\gamma_2 + \gamma_3 \\ 0 & 1 & \gamma_2 \\ 0 & 0 & 1 \end{pmatrix} . \tag{4.3.34}$$

Consider now the 3–node kinematic chain on $G = H(3)$. The system kinematics of equations (4.3.8),(4.3.9) apply.

From (4.3.29) we can see that there is at least one possible 2–dimensional subspace h of $\mathcal{G}$ that can generate the whole algebra under Lie bracketing:

$$h = \mathrm{sp}\{\mathcal{A}_1, \mathcal{A}_2\} = \mathrm{Ker}(\mathcal{A}_3^\flat) . \tag{4.3.35}$$

The nonholonomic constraints can, then, be expressed as:

$$\mathcal{A}_3^\flat(\xi_{i}) = 0, \quad i = 1, 2, 3 . \tag{4.3.36}$$

Equation (4.3.10) holds with ζ_1 and ζ_2 defined as above and with:

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \mathcal{A}_3^\flat(Ad_{g^{-1}_{2,3}}\mathcal{A}_1) & \mathcal{A}_3^\flat(Ad_{g^{-1}_{2,3}}\mathcal{A}_2) & \mathcal{A}_3^\flat(Ad_{g^{-1}_{2,3}}\mathcal{A}_3) & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \gamma_2^{2,3} & -\gamma_1^{2,3} & 1 & 0 & 0 & 1 \end{pmatrix}$$

and

$$A_2 = \begin{pmatrix} 0 & 0 & 1 \\ \mathcal{A}_3^\flat(Ad_{g^{-1}_{1,2}}\mathcal{A}_1) & \mathcal{A}_3^\flat(Ad_{g^{-1}_{1,2}}\mathcal{A}_2) & \mathcal{A}_3^\flat(Ad_{g^{-1}_{1,2}}\mathcal{A}_3) \\ \mathcal{A}_3^\flat(Ad_{g^{-1}_{1,3}}\mathcal{A}_1) & \mathcal{A}_3^\flat(Ad_{g^{-1}_{1,3}}\mathcal{A}_2) & \mathcal{A}_3^\flat(Ad_{g^{-1}_{1,3}}\mathcal{A}_3) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 \\ \gamma_2^{1,2} & -\gamma_1^{1,2} & 1 \\ \gamma_2^{1,3} & -\gamma_1^{1,3} & 1 \end{pmatrix} .$$

The nonholonomic singularities of the system are the configurations where:

$$\begin{aligned}\det(A_2) &= \mathcal{A}_3^\flat(Ad_{g^{-1}_{1,2}}\mathcal{A}_1)\mathcal{A}_3^\flat(Ad_{g^{-1}_{1,3}}\mathcal{A}_2) - \mathcal{A}_3^\flat(Ad_{g^{-1}_{1,3}}\mathcal{A}_1)\mathcal{A}_3^\flat(Ad_{g^{-1}_{1,2}}\mathcal{A}_2) \\ &= -\gamma_2^{1,2}\gamma_1^{1,3} + \gamma_1^{1,2}\gamma_2^{1,3} = -\gamma_2^{1,2}\gamma_1^{2,3} + \gamma_1^{1,2}\gamma_2^{2,3} = 0 \,.\end{aligned}$$

4.3.3 *G*–Snakes on the Special Orthogonal Group $SO(3)$

Let $G = SO(3)$ be the Special Orthogonal group of real orthogonal 3×3 matrices with determinant equal to one and let $\mathcal{G} = so(3)$ be the algebra of 3×3 real skew–symmetric matrices. Consider the following basis for $\mathcal{G}$:

$$\mathcal{A}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \mathcal{A}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \mathcal{A}_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.3.37}$$

Then:

$$[\mathcal{A}_1\,,\,\mathcal{A}_2] = \mathcal{A}_3\,,\ [\mathcal{A}_1\,,\,\mathcal{A}_3] = -\mathcal{A}_2\,,\ [\mathcal{A}_2\,,\,\mathcal{A}_3] = \mathcal{A}_1\,. \tag{4.3.38}$$

Proposition 4.3.6

Let $g(0) = I$, the identity of $G = SO(3)$. The algebra $\mathcal{G} = so(3)$ is simple, thus, the Wei–Norman representation (4.3.3) is only local (defined when $\cos\gamma_2 \neq 0$) with coefficients:

$$\begin{pmatrix} \dot\gamma_1 \\ \dot\gamma_2 \\ \dot\gamma_3 \end{pmatrix} = \begin{pmatrix} \sec\gamma_2\cos\gamma_3 & -\sec\gamma_2\sin\gamma_3 & 0 \\ \sin\gamma_3 & \cos\gamma_3 & 0 \\ -\tan\gamma_2\cos\gamma_3 & \tan\gamma_2\sin\gamma_3 & 1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}. \tag{4.3.39}$$

Proof

To see that the Wei–Norman representation has the form (4.3.3) with coefficients given by (4.3.39), we compute the RHS of (4.3.5), using (4.3.38) and (2.3.20):

$$\begin{aligned} &e^{ad(-\gamma_2\mathcal{A}_2)}\mathcal{A}_1 = \cos\gamma_2\mathcal{A}_1 + \sin\gamma_2\mathcal{A}_3\,, \\ &e^{ad(-\gamma_3\mathcal{A}_3)}e^{ad(-\gamma_2\mathcal{A}_2)}\mathcal{A}_1 = \cos\gamma_2\cos\gamma_3\mathcal{A}_1 - \cos\gamma_2\sin\gamma_3\mathcal{A}_2 + \sin\gamma_2\mathcal{A}_3\,, \\ &e^{ad(-\gamma_3\mathcal{A}_3)}\mathcal{A}_2 = \sin\gamma_3\mathcal{A}_1 + \cos\gamma_3\mathcal{A}_2\,. \end{aligned} \tag{4.3.40}$$

From (4.3.1), (4.3.2), (4.3.5) and (4.3.40), we have:

$$\begin{aligned}\xi &= \xi_1\mathcal{A}_1 + \xi_2\mathcal{A}_2 + \xi_3\mathcal{A}_3\\ &= \dot{\gamma}_1(\cos\gamma_2\cos\gamma_3\mathcal{A}_1 - \cos\gamma_2\sin\gamma_3\mathcal{A}_2 + \sin\gamma_2\mathcal{A}_3)\\ &\qquad\qquad + \dot{\gamma}_2(\sin\gamma_3\mathcal{A}_1 + \cos\gamma_3)\mathcal{A}_2) + \dot{\gamma}_3\mathcal{A}_3\ .\end{aligned}$$

Since $\{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3\}$ is a basis, we have:

$$\xi_1 = \dot{\gamma}_1\cos\gamma_2\cos\gamma_3 + \dot{\gamma}_2\sin\gamma_3\ ,\ \ \xi_2 = -\dot{\gamma}_1\cos\gamma_2\sin\gamma_3 + \dot{\gamma}_2\cos\gamma_3\ ,\ \ \xi_3 = \dot{\gamma}_1\sin\gamma_2 + \gamma_3\ .$$

Solving for the $\dot{\gamma}_i$'s we get (4.3.39).

∎

Lemma 4.3.7

Consider the Wei–Norman representation (4.3.3) of $g \in SO(3)$ determined by (4.3.39). Then:

$$\begin{aligned}Ad_{g^{-1}}\mathcal{A}_1 &= \cos\gamma_2\cos\gamma_3\mathcal{A}_1 - \cos\gamma_2\sin\gamma_3\mathcal{A}_2 + \sin\gamma_2\mathcal{A}_3\ ,\\ Ad_{g^{-1}}\mathcal{A}_2 &= (\sin\gamma_1\sin\gamma_2\cos\gamma_3 + \cos\gamma_1\sin\gamma_3)\mathcal{A}_1\\ &\qquad + (-\sin\gamma_1\sin\gamma_2\sin\gamma_3 + \cos\gamma_1\cos\gamma_3)\mathcal{A}_2 - \sin\gamma_1\cos\gamma_2\mathcal{A}_3\ ,\\ Ad_{g^{-1}}\mathcal{A}_3 &= (-\cos\gamma_1\sin\gamma_2\cos\gamma_3 + \sin\gamma_1\sin\gamma_3)\mathcal{A}_1\\ &\qquad + (\cos\gamma_1\sin\gamma_2\sin\gamma_3 + \sin\gamma_1\cos\gamma_3)\mathcal{A}_2 + \cos\gamma_1\cos\gamma_2\mathcal{A}_3\ .\end{aligned} \tag{4.3.41}$$

Proof

We need to compute (4.3.7). For $\mathcal{A}_1$ we have:

$$\begin{aligned}&e^{ad(-\gamma_1\mathcal{A}_1)}\mathcal{A}_1 = \mathcal{A}_1\\ &e^{ad(-\gamma_2\mathcal{A}_2)}e^{ad(-\gamma_1\mathcal{A}_1)}\mathcal{A}_1 = \cos\gamma_2\mathcal{A}_1 + \sin\gamma_2\mathcal{A}_3\\ &e^{ad(-\gamma_3\mathcal{A}_3)}e^{ad(-\gamma_2\mathcal{A}_2)}e^{ad(-\gamma_1\mathcal{A}_1)}\mathcal{A}_1 = \cos\gamma_2\cos\gamma_3\mathcal{A}_1 - \cos\gamma_2\sin\gamma_3\mathcal{A}_2 + \sin\gamma_2\mathcal{A}_3\ .\end{aligned}$$

For $\mathcal{A}_2$:

$$\begin{aligned}&e^{ad(-\gamma_1\mathcal{A}_1)}\mathcal{A}_2 = \cos\gamma_1\mathcal{A}_2 - \sin\gamma_1\mathcal{A}_3\\ &e^{ad(-\gamma_2\mathcal{A}_2)}e^{ad(-\gamma_1\mathcal{A}_2)}\mathcal{A}_2 = \sin\gamma_1\sin\gamma_2\mathcal{A}_1 + \cos\gamma_1\mathcal{A}_2 - \sin\gamma_1\cos\gamma_2\mathcal{A}_3\\ &e^{ad(-\gamma_3\mathcal{A}_3)}e^{ad(-\gamma_2\mathcal{A}_2)}e^{ad(-\gamma_1\mathcal{A}_1)}\mathcal{A}_2 = (\sin\gamma_1\sin\gamma_2\cos\gamma_3 + \cos\gamma_1\sin\gamma_3)\mathcal{A}_1\\ &\qquad\qquad + (-\sin\gamma_1\sin\gamma_2\sin\gamma_3 + \cos\gamma_1\cos\gamma_3)\mathcal{A}_2 - \sin\gamma_1\cos\gamma_2\mathcal{A}_3\ ,\end{aligned}$$

For $\mathcal{A}_3$:

$$e^{ad(-\gamma_1\mathcal{A}_1)}\mathcal{A}_3 = \sin\gamma_1\mathcal{A}_2 + \cos\gamma_1\mathcal{A}_3$$

$$e^{ad(-\gamma_2\mathcal{A}_2)}e^{ad(-\gamma_1\mathcal{A}_3)}\mathcal{A}_3 = -\cos\gamma_1\sin\gamma_2\mathcal{A}_1 + \sin\gamma_1\mathcal{A}_2 + \cos\gamma_1\cos\gamma_2\mathcal{A}_3$$

$$\begin{aligned} e^{ad(-\gamma_3\mathcal{A}_3)}e^{ad(-\gamma_2\mathcal{A}_2)}e^{ad(-\gamma_1\mathcal{A}_1)}\mathcal{A}_3 &= (-\cos\gamma_1\sin\gamma_2\cos\gamma_3 + \sin\gamma_1\sin\gamma_3)\mathcal{A}_1 \\ &+ (\cos\gamma_1\sin\gamma_2\sin\gamma_3 + \sin\gamma_1\cos\gamma_3)\mathcal{A}_2 + \cos\gamma_1\cos\gamma_2\mathcal{A}_3 \ . \end{aligned}$$

From the above, (4.3.41) is immediate.

∎

Using the definition of the basis $\{\mathcal{A}_i\}$ from (4.3.38), we have:

$$e^{\gamma_1\mathcal{A}_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma_1 & -\sin\gamma_1 \\ 0 & \sin\gamma_1 & \cos\gamma_1 \end{pmatrix} , \quad e^{\gamma_2\mathcal{A}_2} = \begin{pmatrix} \cos\gamma_2 & 0 & \sin\gamma_2 \\ 0 & 1 & 0 \\ -\sin\gamma_2 & 0 & \cos\gamma_2 \end{pmatrix} ,$$

$$e^{\gamma_3\mathcal{A}_3} = \begin{pmatrix} \cos\gamma_3 & -\sin\gamma_3 & 0 \\ \sin\gamma_3 & \cos\gamma_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

Then, from (4.3.3):

$$g = e^{\gamma_1\mathcal{A}_1}e^{\gamma_2\mathcal{A}_2}e^{\gamma_3\mathcal{A}_3}$$

$$= \begin{pmatrix} \cos\gamma_2\cos\gamma_3 & -\cos\gamma_2\sin\gamma_3 & \sin\gamma_2 \\ \cos\gamma_1\sin\gamma_3 + \sin\gamma_1\sin\gamma_2\cos\gamma_3 & \cos\gamma_1\cos\gamma_3 - \sin\gamma_1\sin\gamma_2\sin\gamma_3 & -\sin\gamma_1\cos\gamma_2 \\ \sin\gamma_1\sin\gamma_3 - \cos\gamma_1\sin\gamma_2\cos\gamma_3 & \sin\gamma_1\cos\gamma_3 + \cos\gamma_1\sin\gamma_2\sin\gamma_3 & \cos\gamma_1\cos\gamma_2 \end{pmatrix} . \tag{4.3.42}$$

Consider now the 3–node kinematic chain on $G = SO(3)$. The system kinematics of equations (4.3.8),(4.3.9) apply.

From (4.3.38) we can see that there are at least three possible 2–dimensional subspaces h of $\mathcal{G}$ that can generate the whole algebra under Lie bracketing:

$$\begin{aligned} h_3 = \mathrm{sp}\{\mathcal{A}_1, \mathcal{A}_2\} = \mathrm{Ker}(\mathcal{A}_3^\flat) \ , \quad h_2 = \mathrm{sp}\{\mathcal{A}_1, \mathcal{A}_3\} = \mathrm{Ker}(\mathcal{A}_2^\flat), \\ \text{and } \ h_1 = \mathrm{sp}\{\mathcal{A}_2, \mathcal{A}_3\} = \mathrm{Ker}(\mathcal{A}_1^\flat). \end{aligned} \tag{4.3.43}$$

From Proposition 4.3.1 we know that those are equivalent, so we consider only $h_3 \subset \mathcal{G}$. The nonholonomic constraints $\xi_i \in h_3$ can, then, be expressed as:

$$\mathcal{A}_3^\flat(\xi_i) = 0, \ \ i = 1, 2, 3 \ . \tag{4.3.44}$$

Equation (4.3.10) holds with ζ_1 and ζ_2 defined as above and with:

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \mathcal{A}_3^\flat\big(Ad_{g^{-1}_{2,3}}\mathcal{A}_1\big) & \mathcal{A}_3^\flat\big(Ad_{g^{-1}_{2,3}}\mathcal{A}_2\big) & \mathcal{A}_3^\flat\big(Ad_{g^{-1}_{2,3}}\mathcal{A}_3\big) & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \sin\gamma_2^{2,3} & -\sin\gamma_1^{2,3}\cos\gamma_2^{2,3} & \cos\gamma_1^{2,3}\cos\gamma_2^{2,3} & 0 & 0 & 1 \end{pmatrix}$$

and

$$A_2 = \begin{pmatrix} 0 & 0 & 1 \\ \mathcal{A}_3^\flat\big(Ad_{g^{-1}_{1,2}}\mathcal{A}_1\big) & \mathcal{A}_3^\flat\big(Ad_{g^{-1}_{1,2}}\mathcal{A}_2\big) & \mathcal{A}_3^\flat\big(Ad_{g^{-1}_{1,2}}\mathcal{A}_3\big) \\ \mathcal{A}_3^\flat\big(Ad_{g^{-1}_{1,3}}\mathcal{A}_1\big) & \mathcal{A}_3^\flat\big(Ad_{g^{-1}_{1,3}}\mathcal{A}_2\big) & \mathcal{A}_3^\flat\big(Ad_{g^{-1}_{1,3}}\mathcal{A}_3\big) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 \\ \sin\gamma_2^{1,2} & -\sin\gamma_1^{1,2}\cos\gamma_2^{1,2} & \cos\gamma_1^{1,2}\cos\gamma_2^{1,2} \\ \sin\gamma_2^{1,3} & -\sin\gamma_1^{1,3}\cos\gamma_2^{1,3} & \cos\gamma_1^{1,3}\cos\gamma_2^{1,3} \end{pmatrix} .$$

The nonholonomic singularities of the system are the configurations where:

$$\det(A_2) = \mathcal{A}_3^\flat\big(Ad_{g^{-1}_{1,2}}\mathcal{A}_1\big)\mathcal{A}_3^\flat\big(Ad_{g^{-1}_{1,3}}\mathcal{A}_2\big) - \mathcal{A}_3^\flat\big(Ad_{g^{-1}_{1,3}}\mathcal{A}_1\big)\mathcal{A}_3^\flat\big(Ad_{g^{-1}_{1,2}}\mathcal{A}_2\big)$$

$$= -\sin\gamma_2^{1,2}\sin\gamma_1^{1,3}\cos\gamma_2^{1,3} + \sin\gamma_2^{1,3}\sin\gamma_1^{1,2}\cos\gamma_2^{1,2} = 0 .$$

4.3.4 G–Snakes on the Special Linear Group $SL(2)$

Let $G = SL(2)$ be the Special Linear group of real 2×2 matrices with determinant one and let $\mathcal{G} = sl(2)$ be the algebra of real 2×2 matrices of trace zero. Consider the following basis for $\mathcal{G}$:

$$\mathcal{A}_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} , \; \mathcal{A}_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} , \; \mathcal{A}_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \qquad (4.3.45)$$

Then:

$$[\mathcal{A}_1 \, , \, \mathcal{A}_2] = 2\mathcal{A}_3 \, , \; [\mathcal{A}_1 \, , \, \mathcal{A}_3] = -\mathcal{A}_1 \, , \; [\mathcal{A}_2 \, , \, \mathcal{A}_3] = \mathcal{A}_2 \, . \qquad (4.3.46)$$

Proposition 4.3.8

Let $g(0) = I$, the identity of $G = SL(2)$. The Wei–Norman representation (4.3.3) is only local (defined when $e^{\gamma_3} \neq 0$) with coefficients:

$$\begin{pmatrix} \dot{\gamma}_1 \\ \dot{\gamma}_2 \\ \dot{\gamma}_3 \end{pmatrix} = \begin{pmatrix} e^{\gamma_3} & 0 & 0 \\ \gamma_2^2 e^{\gamma_3} & e^{-\gamma_3} & 0 \\ -2\gamma_2 e^{\gamma_3} & 0 & 1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} . \tag{4.3.47}$$

(See however comments in (Wei & Norman [1964]) and their Theorem 3. A global representation of $SL(2)$ can be obtained using $\{\mathcal{A}_1, \mathcal{A}_1 - \mathcal{A}_2, \mathcal{A}_3\}$ as a basis).

Proof

To see that the Wei–Norman representation has the form (4.3.3) with coefficients given by (4.3.47), we compute the RHS of (4.3.5), using (4.3.46) and (2.3.20):

$$\begin{aligned} &e^{ad(-\gamma_2 \mathcal{A}_2)}\mathcal{A}_1 = \mathcal{A}_1 - \gamma_2^2 \mathcal{A}_2 + 2\gamma_2 \mathcal{A}_3 , \\ &e^{ad(-\gamma_3 \mathcal{A}_3)} e^{ad(-\gamma_2 \mathcal{A}_2)}\mathcal{A}_1 = e^{-\gamma_3}\mathcal{A}_1 - \gamma_2^2 e^{\gamma_3}\mathcal{A}_2 + 2\gamma_2 \mathcal{A}_3 , \\ &e^{ad(-\gamma_3 \mathcal{A}_3)}\mathcal{A}_2 = e^{\gamma_3}\mathcal{A}_2 . \end{aligned} \tag{4.3.48}$$

From (4.3.1), (4.3.2), (4.3.5) and (4.3.48), we have:

$$\begin{aligned} \xi &= \xi_1 \mathcal{A}_1 + \xi_2 \mathcal{A}_2 + \xi_3 \mathcal{A}_3 \\ &= \dot{\gamma}_1 (e^{-\gamma_3}\mathcal{A}_1 - \gamma_2^2 e^{\gamma_3}\mathcal{A}_2 + 2\gamma_2 \mathcal{A}_3) + \dot{\gamma}_2 e^{\gamma_3}\mathcal{A}_2 + \dot{\gamma}_3 \mathcal{A}_3 . \end{aligned}$$

Since $\{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3\}$ is a basis, we have:

$$\xi_1 = \dot{\gamma}_1 e^{-\gamma_3} , \quad \xi_2 = -\dot{\gamma}_1 \gamma_2^2 e^{\gamma_3} + \dot{\gamma}_2 e^{\gamma_3} , \quad \xi_3 = \dot{\gamma}_1 2\gamma_2 + \dot{\gamma}_3 .$$

Solving for the $\dot{\gamma}_i$'s we get (4.3.47).

∎

Lemma 4.3.9

Consider the Wei–Norman representation (4.3.3) of $g = SL(2)$ determined by (4.3.47). Then:

$$\begin{aligned} Ad_{g^{-1}}\mathcal{A}_1 &= e^{-\gamma_3}\mathcal{A}_1 - \gamma_2^2 e^{\gamma_3}\mathcal{A}_2 + 2\gamma_2 \mathcal{A}_3 , \\ Ad_{g^{-1}}\mathcal{A}_2 &= -\gamma_1^2 e^{-\gamma_3}\mathcal{A}_1 + (\gamma_1\gamma_2 + 1)^2 e^{\gamma_3}\mathcal{A}_2 - 2\gamma_1(\gamma_1\gamma_2 + 1)\mathcal{A}_3 , \\ Ad_{g^{-1}}\mathcal{A}_3 &= \gamma_1 e^{-\gamma_3}\mathcal{A}_1 - \gamma_2(\gamma_1\gamma_2 + 1)e^{\gamma_3}\mathcal{A}_2 + (2\gamma_1\gamma_2 + 1)\mathcal{A}_3 . \end{aligned} \tag{4.3.49}$$

Proof

We need to compute (4.3.7). For $\mathcal{A}_1$ we have:

$$e^{ad(-\gamma_1\mathcal{A}_1)}\mathcal{A}_1 = \mathcal{A}_1$$

$$e^{ad(-\gamma_2\mathcal{A}_2)}e^{ad(-\gamma_1\mathcal{A}_1)}\mathcal{A}_1 = \mathcal{A}_1 - \gamma_2^2\mathcal{A}_2 + 2\gamma_2\mathcal{A}_3$$

$$e^{ad(-\gamma_3\mathcal{A}_3)}e^{ad(-\gamma_2\mathcal{A}_2)}e^{ad(-\gamma_1\mathcal{A}_1)}\mathcal{A}_1 = e^{-\gamma_3}\mathcal{A}_1 - \gamma_2^2e^{\gamma_3}\mathcal{A}_2 + 2\gamma_2\mathcal{A}_3 \ .$$

For $\mathcal{A}_2$:

$$e^{ad(-\gamma_1\mathcal{A}_1)}\mathcal{A}_2 = -\gamma_1^2\mathcal{A}_1 + \mathcal{A}_2 - 2\gamma_1\mathcal{A}_3$$

$$e^{ad(-\gamma_2\mathcal{A}_2)}e^{ad(-\gamma_1\mathcal{A}_2)}\mathcal{A}_2 = -\gamma_1^2\mathcal{A}_1 + (1+\gamma_1\gamma_2)^2\mathcal{A}_2 - 2\gamma_1(1+\gamma_1\gamma_2)\mathcal{A}_3$$

$$\begin{aligned}e^{ad(-\gamma_3\mathcal{A}_3)}e^{ad(-\gamma_2\mathcal{A}_2)}e^{ad(-\gamma_1\mathcal{A}_1)}\mathcal{A}_2 = {} & -\gamma_1^2e^{-\gamma_3}\mathcal{A}_1 + (1+\gamma_1\gamma_2)^2e^{\gamma_3}\mathcal{A}_2 \\ & - 2\gamma_1(1+\gamma_1\gamma_2)\mathcal{A}_3 \ .\end{aligned}$$

For $\mathcal{A}_3$:

$$e^{ad(-\gamma_1\mathcal{A}_1)}\mathcal{A}_3 = \gamma_1\mathcal{A}_1 + \mathcal{A}_3$$

$$e^{ad(-\gamma_2\mathcal{A}_2)}e^{ad(-\gamma_1\mathcal{A}_3)}\mathcal{A}_3 = \gamma_1\mathcal{A}_1 - \gamma_2(1+\gamma_1\gamma_2)\mathcal{A}_2 + (1+2\gamma_1\gamma_2)\mathcal{A}_3$$

$$e^{ad(-\gamma_3\mathcal{A}_3)}e^{ad(-\gamma_2\mathcal{A}_2)}e^{ad(-\gamma_1\mathcal{A}_1)}\mathcal{A}_3 = \gamma_1e^{-\gamma_3}\mathcal{A}_1 - \gamma_2(1+\gamma_1\gamma_2)e^{\gamma_3}\mathcal{A}_2 + (1+2\gamma_1\gamma_2)\mathcal{A}_3 \ .$$

From the above, (4.3.49) is immediate.

∎

Using the definition of the basis $\{\mathcal{A}_i\}$ from (4.3.45), we have:

$$e^{\gamma_1\mathcal{A}_1} = \begin{pmatrix} 1 & \gamma_1 \\ 0 & 1 \end{pmatrix} \ , \ e^{\gamma_2\mathcal{A}_2} = \begin{pmatrix} 1 & 0 \\ \gamma_2 & 1 \end{pmatrix} \ ,$$

$$e^{\gamma_3\mathcal{A}_3} = \begin{pmatrix} \cosh\gamma_3 + \frac{1}{2}\sinh\gamma_3 & 0 \\ 0 & \cosh\gamma_3 - \frac{1}{2}\sinh\gamma_3 \end{pmatrix}$$

Then, from (4.3.3):

$$g = e^{\gamma_1\mathcal{A}_1}e^{\gamma_2\mathcal{A}_2}e^{\gamma_3\mathcal{A}_3} = \begin{pmatrix} (1+\gamma_1\gamma_2)(\cosh\gamma_3 + \frac{1}{2}\sinh\gamma_3) & \gamma_1(\cosh\gamma_3 - \frac{1}{2}\sinh\gamma_3) \\ \gamma_2(\cosh\gamma_3 + \frac{1}{2}\sinh\gamma_3) & \cosh\gamma_3 - \frac{1}{2}\sinh\gamma_3 \end{pmatrix} \ . \tag{4.3.50}$$

Consider now the 3–node kinematic chain on $G = SL(2)$. The system kinematics of equations (4.3.8),(4.3.9) apply.

From (4.3.49) we can see that there are at least two possible 2–dimensional subspaces h of $\mathcal{G}$ that can generate the whole algebra under Lie bracketing:

$$h_3 = \mathrm{sp}\{\mathcal{A}_1, \mathcal{A}_2\} = \mathrm{Ker}(\mathcal{A}_3^\flat) \ \text{ and } \ h_{1,2} = \mathrm{sp}\{\mathcal{A}_3, \mathcal{A}_1 + \mathcal{A}_2\} = \mathrm{Ker}(\mathcal{A}_1^\flat - \mathcal{A}_2^\flat) \ . \quad (4.3.51)$$

From Proposition 4.3.1 we know that those are *not* equivalent. We first consider $h_3 \subset \mathcal{G}$. The nonholonomic constraints $\xi_i \in h_3$ can, then, be expressed as:

$$\mathcal{A}_3^\flat(\xi_i) = 0, \ \ i = 1, 2, 3 \ . \quad (4.3.52)$$

Equation (4.3.10) holds with ζ_1 and ζ_2 defined as above and with:

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \mathcal{A}_3^\flat(Ad_{g^{-1}_{2,3}}\mathcal{A}_1) & \mathcal{A}_3^\flat(Ad_{g^{-1}_{2,3}}\mathcal{A}_2) & \mathcal{A}_3^\flat(Ad_{g^{-1}_{2,3}}\mathcal{A}_3) & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 2\gamma_2^{2,3} & -2\gamma_1^{2,3}(\gamma_1^{2,3}\gamma_2^{2,3} + 1) & 2\gamma_1^{2,3}\gamma_2^{2,3} + 1 & 0 & 0 & 1 \end{pmatrix}$$

and

$$A_2 = \begin{pmatrix} 0 & 0 & 1 \\ \mathcal{A}_3^\flat(Ad_{g^{-1}_{1,2}}\mathcal{A}_1) & \mathcal{A}_3^\flat(Ad_{g^{-1}_{1,2}}\mathcal{A}_2) & \mathcal{A}_3^\flat(Ad_{g^{-1}_{1,2}}\mathcal{A}_3) \\ \mathcal{A}_3^\flat(Ad_{g^{-1}_{1,3}}\mathcal{A}_1) & \mathcal{A}_3^\flat(Ad_{g^{-1}_{1,3}}\mathcal{A}_2) & \mathcal{A}_3^\flat(Ad_{g^{-1}_{1,3}}\mathcal{A}_3) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 \\ 2\gamma_2^{1,2} & -2\gamma_1^{1,2}(\gamma_1^{1,2}\gamma_2^{1,2} + 1) & 2\gamma_1^{1,2}\gamma_2^{1,2} + 1 \\ 2\gamma_2^{1,3} & -2\gamma_1^{1,3}(\gamma_1^{1,3}\gamma_2^{1,3} + 1) & 2\gamma_1^{1,3}\gamma_2^{1,3} + 1 \end{pmatrix} .$$

The nonholonomic singularities of the system are the configurations where:

$$\det(A_2) = \mathcal{A}_3^\flat(Ad_{g^{-1}_{1,2}}\mathcal{A}_1)\mathcal{A}_3^\flat(Ad_{g^{-1}_{1,3}}\mathcal{A}_2) - \mathcal{A}_3^\flat(Ad_{g^{-1}_{1,3}}\mathcal{A}_1)\mathcal{A}_3^\flat(Ad_{g^{-1}_{1,2}}\mathcal{A}_2)$$

$$= -4\gamma_2^{1,2}\gamma_1^{1,3}(\gamma_1^{1,3}\gamma_2^{1,3} + 1) + 4\gamma_2^{1,3}\gamma_1^{1,2}(\gamma_1^{1,2}\gamma_2^{1,2} + 1) = 0 \ .$$

Now we consider the subspace $h_{1,2} \subset \mathcal{G}$ in (4.3.51). The nonholonomic constraints can, then, be expressed as:

$$(\mathcal{A}_1^\flat - \mathcal{A}_2^\flat)(\xi_i) = 0, \ \ i = 1, 2, 3 \ . \quad (4.3.53)$$

Equation (4.3.10) holds with ζ_1 and ζ_2 defined as above and with:

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ (\mathcal{A}_1^\flat - \mathcal{A}_2^\flat)(Ad_{g^{-1}_{2,3}}\mathcal{A}_1) & (\mathcal{A}_1^\flat - \mathcal{A}_2^\flat)(Ad_{g^{-1}_{2,3}}\mathcal{A}_2) & (\mathcal{A}_1^\flat - \mathcal{A}_2^\flat)(Ad_{g^{-1}_{2,3}}\mathcal{A}_3) & 1 & -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ e^{-\gamma_3^{2,3}} + (\gamma_2^{2,3})^2 e^{\gamma_3^{2,3}} & -(\gamma_1^{2,3})^2 e^{-\gamma_3^{2,3}} - (\gamma_1^{2,3}\gamma_2^{2,3}+1)^2 e^{\gamma_3^{2,3}} & \gamma_1^{2,3} e^{-\gamma_3^{2,3}} + \gamma_2^{2,3}(\gamma_1^{2,3}\gamma_2^{2,3}+1)e^{\gamma_3^{2,3}} & 1 & -1 & 0 \end{pmatrix}$$

and

$$A_2 = \begin{pmatrix} 1 & -1 & 0 \\ (\mathcal{A}_1^\flat - \mathcal{A}_2^\flat)(Ad_{g^{-1}_{1,2}}\mathcal{A}_1) & (\mathcal{A}_1^\flat - \mathcal{A}_2^\flat)(Ad_{g^{-1}_{1,2}}\mathcal{A}_2) & (\mathcal{A}_1^\flat - \mathcal{A}_2^\flat)(Ad_{g^{-1}_{1,2}}\mathcal{A}_3) \\ (\mathcal{A}_1^\flat - \mathcal{A}_2^\flat)(Ad_{g^{-1}_{1,3}}\mathcal{A}_1) & (\mathcal{A}_1^\flat - \mathcal{A}_2^\flat)(Ad_{g^{-1}_{1,3}}\mathcal{A}_2) & (\mathcal{A}_1^\flat - \mathcal{A}_2^\flat)(Ad_{g^{-1}_{1,3}}\mathcal{A}_3) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -1 & 0 \\ e^{-\gamma_3^{1,2}} + (\gamma_2^{1,2})^2 e^{\gamma_3^{1,2}} & -(\gamma_1^{1,2})^2 e^{-\gamma_3^{1,2}} - (\gamma_1^{1,2}\gamma_2^{1,2}+1)^2 e^{\gamma_3^{1,2}} & \gamma_1^{1,2} e^{-\gamma_3^{1,2}} + \gamma_2^{1,2}(\gamma_1^{1,2}\gamma_2^{1,2}+1)e^{\gamma_3^{1,2}} \\ e^{-\gamma_3^{1,3}} + (\gamma_2^{1,3})^2 e^{\gamma_3^{1,3}} & -(\gamma_1^{1,3})^2 e^{-\gamma_3^{1,3}} - (\gamma_1^{1,3}\gamma_2^{1,3}+1)^2 e^{\gamma_3^{1,3}} & \gamma_1^{1,3} e^{-\gamma_3^{1,3}} + \gamma_2^{1,3}(\gamma_1^{1,3}\gamma_2^{1,3}+1)e^{\gamma_3^{1,3}} \end{pmatrix}.$$

The nonholonomic singularities of the system are the configurations where:

$$\begin{aligned} \det(A_2) = {} & (\mathcal{A}_1^\flat - \mathcal{A}_2^\flat)(Ad_{g^{-1}_{1,2}}(\mathcal{A}_1 + \mathcal{A}_2))(\mathcal{A}_1^\flat - \mathcal{A}_2^\flat)(Ad_{g^{-1}_{1,3}}\mathcal{A}_3) \\ & - (\mathcal{A}_1^\flat - \mathcal{A}_2^\flat)(Ad_{g^{-1}_{1,3}}(\mathcal{A}_1 + \mathcal{A}_2))(\mathcal{A}_1^\flat - \mathcal{A}_2^\flat)(Ad_{g^{-1}_{1,2}}\mathcal{A}_3) \end{aligned}$$

$$= \left(e^{-\gamma_3^{1,2}} + (\gamma_2^{1,2})^2 e^{\gamma_3^{1,2}} - (\gamma_1^{1,2})^2 e^{-\gamma_3^{1,2}} - (\gamma_1^{1,2}\gamma_2^{1,2}+1)^2 e^{\gamma_3^{1,2}}\right)$$

$$\cdot \left(\gamma_1^{1,3} e^{-\gamma_3^{1,3}} + \gamma_2^{1,3}(\gamma_1^{1,3}\gamma_2^{1,3}+1) e^{\gamma_3^{1,3}}\right)$$

$$- \left(e^{-\gamma_3^{1,3}} + (\gamma_2^{1,3})^2 e^{\gamma_3^{1,3}} - (\gamma_1^{1,3})^2 e^{-\gamma_3^{1,3}} - (\gamma_1^{1,3}\gamma_2^{1,3}+1)^2 e^{\gamma_3^{1,3}}\right)$$

$$\cdot \left(\gamma_1^{1,2} e^{-\gamma_3^{1,2}} + \gamma_2^{1,2}(\gamma_1^{1,2}\gamma_2^{1,2}+1) e^{\gamma_3^{1,2}}\right) = 0 \, .$$

CHAPTER FIVE

NONHOLONOMIC VARIABLE GEOMETRY TRUSS ASSEMBLIES

5.1 Introduction

In this chapter we consider an instance of $SE(2)$–Snakes, based on a particular type of Variable Geometry Truss (VGT) assemblies (Miura, Furuya & Suzuki [1985]; Wada [1990]). These are structures consisting of longitudinal repetition of similar truss modules. In the present instance, each module is implemented as a planar parallel manipulator consisting of two platforms connected by legs whose lengths can vary under the control of linear actuators. Each platform is equipped with a pair of wheels, so that it can move on the plane that supports the structure (fig. 5.1.1). The wheels of each platform are free and not actuated and their motion is independent of each other, while we assume that the wheels roll without slipping on the plane. This imposes a nonholonomic constraint on the motion of each platform, namely the requirement that its velocity is perpendicular to the axis connecting the wheels. When the legs of the individual modules are expanded or contracted, the shape of the whole structure changes. As a consequence of the nonholonomic constraints imposed by the rolling–without–slipping assumption on the wheels, this shape change induces a global motion of the structure, as we shall show. These structures will be called *Nonholonomic Variable Geometry Truss (NVGT) assemblies.*

The motion planning problem for such an assembly is of the nonholonomic variety. There is a significant body of research related to such problems (see e.g. (Latombe [1991]; Li & Canny [1993]; Murray, Li & Sastry [1994])), which in general assumes cart–type mobile robots moving under direct actuation of a set of wheels. The main difference in our case is the prominence of shape changes as the means which, together with the action of the nonholonomic constraints, induces global motion of the system. This is analogous to

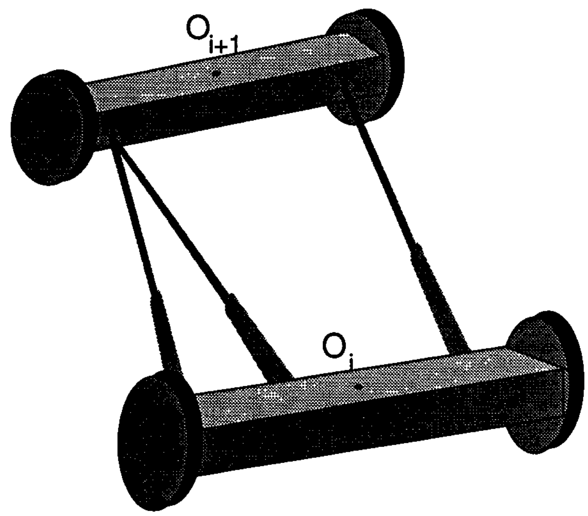


Fig. 5.1.1: One module of the NVGT assembly

the idea of reorientation in free–floating multibody systems, induced by closed joint space trajectories under the nonholonomic constraint of conservation of angular momentum (Krishnaprasad [1990]; Krishnaprasad & Yang [1991]; Marsden, Montgomery & Ratiu [1990]).

VGT assemblies have been examined in the past (see (Chirikjian & Burdick [1991]; Wada [1990]) and references there), but the emphasis was on its capabilities as a redundant manipulator and on locomotion using snake–like motions, not on the special problems introduced by nonholonomic constraints. A system similar to the one described here was built by (Chirikjian & Burdick [1993]) using castors instead of wheels in the platforms of the modules and therefore the nonholonomic constraints that we consider here were *not* present.

In section 5.2, we examine the kinematics of an NVGT assembly with ℓ platforms and $\ell - 1$ modules. Each platform can be considered as a node of a kinematic chain. The *configuration* of the ℓ–node NVGT assembly can be described by its shape and by the position and orientation of the assembly with respect to some fixed (world) coordinate system, thus by a total of ℓ elements of $SE(2)$. Consider the i–th module that includes the i–th and $(i+1)$–th platforms (fig. 5.1.1). Its *shape* can be described by the relative position and orientation of the coordinate frame centered at the point O_{i+1} with respect to the coordinate frame centered at the point O_i. Then, the shape of each module corresponds to an element of the Special Euclidean group $SE(2)$ that describes rigid motions on the plane and, as a result, the shape of the $(\ell - 1)$–module NVGT can be described by $(\ell - 1)$ elements of $SE(2)$. In (Brockett, Stokes & Park [1993]) a systematic way of deriving the kinematics of serial linkages is presented based on the "product of exponentials" formula, where the configuration of the system is described by an element of the appropriate $SE(n)$ group and is expressed as a product of its one–parameter subgroups, with one element of the product corresponding to each of the one–degree–of–freedom joints of the linkage. The NVGT assembly that we consider here is a structure similar to the ones described there, but the joints are more complicated parallel manipulator modules with more than one degree–of–freedom each. Moreover, the whole assembly is not anchored to a base, but is free to move on a plane and, finally, nonholonomic constraints are present, in addition to the holonomic ones. However, an extension of the above method, allows us to systematically derive the kinematics of the NVGT assembly as follows: Using the Wei–Norman representation of $SE(2)$, we express the shape of each module as a product of the one–parameter subgroups of $SE(2)$. Then, the configuration of the whole assembly can be expressed as a product of such one–parameter subgroups. Using the notion of the adjoint action of $SE(2)$ on its Lie algebra, we determine, in section 5.2.2, how the motion of a module relates to the motion of the other modules of the assembly. We also express the nonholonomic constraints in a compact form that can be used to make explicit the dependence of the assembly configuration on the shape of its modules. This allows us to characterize the dependence of the global motion of the assembly on the *shape controls*, namely the changes in the shape of each module, which are expressed as elements of the Lie algebra of $SE(2)$. In section 5.2.3, we consider the implementation of each module as a planar parallel manipulator. The shape of each module is determined by the lengths of the legs of the parallel manipulator. From the velocity kinematics of the parallel manipulator we conclude that motion planning schemes for the NVGT assembly can disregard the particular details of the implementation of the modules and only consider the shape of each module. Thus, instead of considering the changes in leg lengths as controls for the NVGT assembly, we can use the corresponding shape controls of each module.

In section 5.3, we specialize the previous discussion to the 3–node, 2–module NVGT. Unlike the generic ℓ–node case, here we have exactly the number of nonholonomic constraints that we need in order to determine the position and orientation of the NVGT assembly with respect to the world coordinate frame based on a sequence of shape changes from a reference shape. As a result, we can demonstrate how shape changes of the NVGT assembly induce a global snake–like motion due to the nonholonomic constraints. We consider the motion planning problem under a specific shape actuation scheme, where one of the two modules is responsible for the motion of the assembly through periodic changes of its shape and the other module is responsible for steering. We demonstrate how to generate primitive "straight line motion" and "turning" behaviors and we show by computer simulations how to synthesize these into more complex ones, like avoidance of obstacles.

In section 5.4, we consider non–sequential arrangements of NVGT modules, namely tree–like arrangements which we refer to as $SE(2)$–Spiders and ring–like arrangements which we refer to as $SE(2)$–Rings. We show how motion control for such systems can be reduced to the corresponding problem for the 2–module NVGT. In section 5.4.3, we consider $SE(2)$–Snake assemblies with more than one constraint per node. Thus, we relax the Codimension 1 Constraint Hypothesis of the previous chapter.

5.2 Kinematics of the Nonholonomic Variable Geometry Truss (NVGT) Assembly

In section 5.2.1 we discuss the nonholonomic constraint of rolling–without–slipping and in section 5.2.2 we apply the Wei–Norman representation of curves in $SE(2)$ to the derivation of the kinematics of the ℓ–node NVGT. In section 5.2.3 we consider the implementation of a module of the NVGT assembly as a planar parallel manipulator.

5.2.1 The Nonholonomic Constraint of Rolling–without–Slipping

Consider the wheel pair assembly of fig. 5.2.1 consisting of two idler wheels mounted on an axis, so that they can rotate freely and independently of each other.

Let ϕ_r (resp. ϕ_l) be the angle of the right (left) wheel of the pair with respect to the vertical. Let r be the diameter of each wheel and $2\,L$ the distance between the wheels. Consider a coordinate system centered midway between the wheels at point $O_{1,}$, with the x–axis along the axis joining them and pointing towards the right one and with the z–axis vertical to the plane supporting the wheel assembly. Let the wheel assembly move

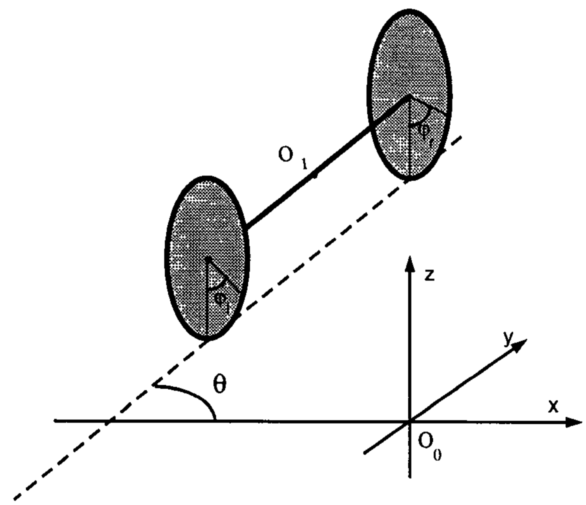


Fig. 5.2.1: Wheel pair rolling–without–slipping

with respect to an inertial coordinate system centered at O_0. Let $g \in G = SE(2)$ be the current configuration of the assembly and $\xi = \begin{pmatrix} 0 & -\omega & \Xi_1 \\ \omega & 0 & \Xi_2 \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{G}$ be the body velocity of the assembly. Then $\dot{g} = g\xi$. It is a simple exercise (c.f. (Goldstein [1980])) to derive the nonholonomic constraints imposed by this assembly and relate the motion of each wheel to the body velocity ξ.

Proposition 5.2.1

The angular velocity of each wheel is related to the body velocity of the wheel assembly by:

$$\begin{aligned} \dot{\phi}_r &= \frac{1}{r}(\Xi_2 + L\omega) , \\ \dot{\phi}_l &= \frac{1}{r}(\Xi_2 - L\omega) . \end{aligned} \tag{5.2.1}$$

Proof

Let P_r (P_l) be the position of the center of the right (left) wheel with respect to the coordinate system centered at O_1 and p_r (p_l) be the position of the right (left) wheel with respect to the inertial coordinate system at O_0. Then $P_r = (L \quad 0 \quad 1)^{\mathsf{T}}$, $P_l = (-L \quad 0 \quad 1)^{\mathsf{T}}$, while $p_r = gP_r = (x + L\cos\theta \quad y + L\sin\theta \quad 1)^{\mathsf{T}}$ and $p_l = gP_l =$

$(\, x - L\cos\theta \quad y + L\sin\theta \quad 1\,)^{\top}$. By differentiating,

$$\dot{p}_r = \dot{g}P_r = g\xi P_r = \begin{pmatrix} -\omega L\sin\theta + (\Xi_1\cos\theta - \Xi_2\sin\theta) \\ \omega L\cos\theta + (\Xi_1\sin\theta + \Xi_2\cos\theta) \\ 0 \end{pmatrix} \tag{5.2.2}$$

and

$$\dot{p}_l = \dot{g}P_l = g\xi P_l = \begin{pmatrix} \omega L\sin\theta + (\Xi_1\cos\theta - \Xi_2\sin\theta) \\ -\omega L\cos\theta + (\Xi_1\sin\theta + \Xi_2\cos\theta) \\ 0 \end{pmatrix} . \tag{5.2.3}$$

Assuming that both wheels roll without slipping, we must have:

$$\dot{p}_r = r\dot{\phi}_r \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix} \tag{5.2.4}$$

and

$$\dot{p}_l = r\dot{\phi}_l \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix} . \tag{5.2.5}$$

From (5.2.2) and (5.2.4):

$$\Xi_1 = 0\,,$$

$$\Xi_2 = r\dot{\phi}_r - L\omega$$

and from (5.2.3) and (5.2.5):

$$\Xi_1 = 0\,,$$

$$\Xi_2 = r\dot{\phi}_l + L\omega\,.$$

Thus:

$$\Xi_1 = \dot{x}\cos\theta + \dot{y}\sin\theta = 0\,, \tag{5.2.6}$$

$$\dot{\phi}_r + \dot{\phi}_l = \frac{2}{r}\Xi_2\,, \tag{5.2.7}$$

$$\dot{\phi}_r - \dot{\phi}_l = \frac{2}{r}L\omega = \frac{2}{r}L\dot{\theta}\,. \tag{5.2.8}$$

Observe that (5.2.8) is a holonomic constraint, while (5.2.6) and (5.2.7) are nonholonomic ones. The proof of this appears in (Latombe [1991]). Since $\omega = \frac{d\theta}{dt}$, from (5.2.8) we get:

$$\theta(t) = \theta(0) - \frac{r}{2L}\big(\phi_r(0) - \phi_l(0)\big) + \frac{r}{2L}\big(\phi_r(t) - \phi_l(t)\big) \tag{5.2.9}$$

Equations (5.2.1) follow from (5.2.7) and (5.2.8).

∎

5.2.2 The ℓ–node, $(\ell-1)$–module Nonholonomic Variable Geometry Truss (NVGT) Assembly

We consider a chain of $(\ell-1)$ modules of the type shown in fig. 5.1.1 and 5.2.2. Each module consists of two platforms in a planar parallel manipulator configuration with one pair of wheels per platform and with each wheel rotating independently from the other around its axis, both forward and backwards. Neither wheel pair is actuated and we assume that the wheels roll without slipping. This system is composed by ℓ planar platforms, thus has 3ℓ degrees–of–freedom, its configuration space is $Q = \underbrace{SE(2) \times \cdots \times SE(2)}_{\ell \text{ times}}$ and it is subject to $3(\ell-1)$ holonomic constraints from the parallel manipulator legs and to ℓ nonholonomic constraints from the rolling–without–slipping wheel motion. The configuration of the assembly can be determined by its shape (which is an element of the shape space $S = \underbrace{SE(2) \times \cdots \times SE(2)}_{(\ell-1) \text{ times}}$) *and* by the position and orientation of the assembly with respect to the world coordinate system (which is an element of $G = SE(2)$) . Then $Q = G \times S$.

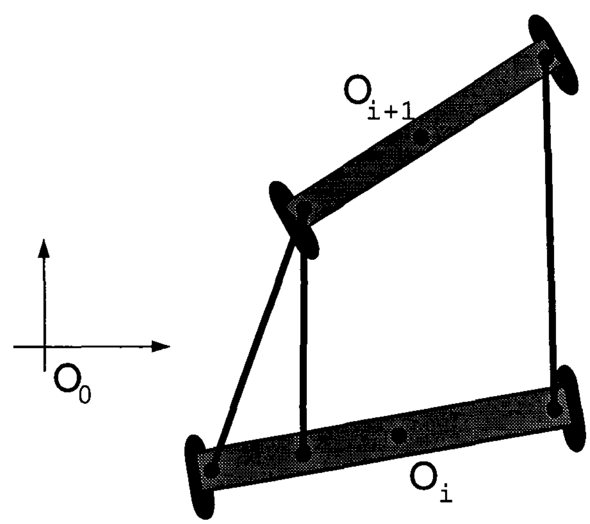


Fig. 5.2.2: The $(i, i+1)$–th module of the NVGT assembly

Consider a world coordinate system centered at O_0 and platform coordinate systems centered at O_i, $i = 1, \ldots, \ell$. Let $g_i \in G = SE(2)$ be the configuration matrix of the i–th platform with respect to the world coordinate system. Define $\xi_i \in \mathcal{G} = se(2)$ by:

$$\dot{g}_i = g_i \xi_i \,, \; i = 1, \ldots, \ell \,. \tag{5.2.10}$$

From (4.3.2) we have (with a slight abuse of notation we denote by $\xi^i_j \in I\!R$ the j–th component of ξ_i, i.e. $\mathcal{A}^\flat_j(\xi_i)$):

$$\xi_i = \sum_{j=1}^{3} \xi^i_j \mathcal{A}_j = \sum_{j=1}^{3} \mathcal{A}^\flat_j(\xi_i) \mathcal{A}_j \,. \tag{5.2.11}$$

Also define the vector ξ^i of the components of $\xi_i \in \mathcal{G}$ as $\xi^i = (\xi^i_1 \, \xi^i_2 \, \xi^i_3)^\top = (\, \mathcal{A}^\flat_1(\xi_i) \; \mathcal{A}^\flat_2(\xi_i) \; \mathcal{A}^\flat_3(\xi_i) \,)^\top \in I\!R^3$.

Let $g_{j,i+1} \in G$ be the configuration matrix of the $(i+1)$–th platform with respect to the coordinate system of the j–th platform. Define $\xi_{j,i+1} \in \mathcal{G}$ by:

$$\dot{g}_{j,i+1} = g_{j,i+1} \xi_{j,i+1} \,, \quad \text{for } i = 1, \ldots, \ell - 1 \; \text{ and } \; 1 \leq j < i+1 \,. \tag{5.2.12}$$

From (4.3.2) we have:

$$\xi_{j,i+1} = \sum_{k=1}^{3} \xi^{j,i+1}_k \mathcal{A}_k = \sum_{k=1}^{3} \mathcal{A}^\flat_k(\xi_{j,i+1}) \mathcal{A}_k \,. \tag{5.2.13}$$

Also define the vector $\xi^{j,i+1}$ of the components of $\xi_{j,i+1} \in \mathcal{G}$ as $\xi^{j,i+1} = (\xi^{j,i+1}_1 \, \xi^{j,i+1}_2 \, \xi^{j,i+1}_3)^\top = (\, \mathcal{A}^\flat_1(\xi_{j,i+1}) \; \mathcal{A}^\flat_2(\xi_{j,i+1}) \; \mathcal{A}^\flat_3(\xi_{j,i+1}) \,)^\top$. Observe that the $g_{j,i+1}$'s and the $\xi_{j,i+1}$'s are the *shape variables* of the chain.

The *shape* of the NVGT assembly is determined by $\{g_{i,i+1}, \; i = 1, \ldots, \ell - 1\}$. The velocities $\{V_{i,i+1}, \; i = 1, \ldots, \ell - 1\}$ are called *shape variations.* A subset of them are the *shape controls.*

By (4.3.3), (4.3.21) and (4.3.23) we have (note the definitions of R_i and T_i):

$$g_i(t) = e^{\gamma^i_1(t)\mathcal{A}_1} e^{\gamma^i_2(t)\mathcal{A}_2} e^{\gamma^i_3(t)\mathcal{A}_3} = \begin{pmatrix} R_i & T_i \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\phi_i & -\sin\phi_i & x_i \\ \sin\phi_i & \cos\phi_i & y_i \\ 0 & 0 & 1 \end{pmatrix} \tag{5.2.14}$$

and

$$g_{j,i+1}(t) = e^{\gamma_1^{j,i+1}(t)\mathcal{A}_1} e^{\gamma_2^{j,i+1}(t)\mathcal{A}_2} e^{\gamma_3^{j,i+1}(t)\mathcal{A}_3}$$

$$= \begin{pmatrix} R_{j,i+1} & T_{j,i+1} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\theta_{j,i+1} & -\sin\theta_{j,i+1} & x_{j,i+1} \\ \sin\theta_{j,i+1} & \cos\theta_{j,i+1} & y_{j,i+1} \\ 0 & 0 & 1 \end{pmatrix}, \tag{5.2.15}$$

where the γ's and the corresponding ξ's are related by (4.3.14) and (4.3.15). By the system kinematics we have:

$$g_{i+1} = g_i g_{i,i+1} = g_1 g_{1,2} \cdots g_{i,i+1}\,,\ i = 1, \ldots, \ell - 1 \tag{5.2.16}$$

and

$$g_{j,i+1} = g_{j,i} g_{i,i+1} = g_{j,j+1} \cdots g_{i,i+1}\,,\ \text{for } i = 1, \ldots, \ell - 1 \text{ and } 1 \leq j < i+1\,. \tag{5.2.17}$$

Define $g_{j,j} = I$, where I is the identity in G.

Equations (5.2.16) with (5.2.14) and (5.2.15) can be seen as a generalization of the "product–of–exponentials" formula of (Brockett, Stokes & Park [1993]) for kinematic chains with more than one degree–of–freedom per joint.

Lemma 5.2.2

The velocities of the $(i+1)$–th node depend on the velocities of the previous nodes as follows:

$$\begin{aligned} \xi_{i+1} &= Ad_{g^{-1}_{i,i+1}} \xi_i + \xi_{i,i+1} \\ &= Ad_{g^{-1}_{i,i+1}} \cdots Ad_{g^{-1}_{1,2}} \xi_1 + Ad_{g^{-1}_{i,i+1}} \cdots Ad_{g^{-1}_{2,3}} \xi_{1,2} \\ &\qquad + \cdots + Ad_{g^{-1}_{i,i+1}} \xi_{i-1,i} + \xi_{i,i+1}\,. \end{aligned}$$

The velocities of the $(j, i+1)$–th module depend on those of modules $(j, j+1), (j+1, j+2), \ldots, (i, i+1)$ as follows:

$$\begin{aligned} \xi_{j,i+1} &= Ad_{g^{-1}_{i,i+1}} \xi_{j,i} + \xi_{i,i+1} \\ &= Ad_{g^{-1}_{i,i+1}} \cdots Ad_{g^{-1}_{j+1,j+2}} \xi_{j,j+1} + Ad_{g^{-1}_{i,i+1}} \cdots Ad_{g^{-1}_{j+2,j+3}} \xi_{j+1,j+2} \\ &\qquad + \cdots + Ad_{g^{-1}_{i,i+1}} \xi_{i-1,i} + \xi_{i,i+1}\,. \end{aligned} \tag{5.2.18}$$

Proof

From (5.2.10) and (5.2.16):

$$\begin{aligned}\dot{g}_{i+1} &= \dot{g}_i g_{i,i+1} + g_i \dot{g}_{i,i+1} = g_i \xi_i g_{i,i+1} + g_i g_{i,i+1} \xi_{i,i+1} \\ &= g_i g_{i,i+1} [g_{i,i+1}^{-1} \xi_i g_{i,i+1} + \xi_{i,i+1}] \\ &= g_{i+1} [Ad_{g_{i,i+1}^{-1}} \xi_i + \xi_{i,i+1}] ,\end{aligned} \tag{5.2.19}$$

$$\begin{aligned}\dot{g}_{i+1} &= \dot{g}_1 g_{1,2} \cdots g_{i,i+1} + g_1 \dot{g}_{1,2} \cdots g_{i,i+1} + \cdots + g_1 g_{1,2} \cdots \dot{g}_{i,i+1} \\ &= g_{i+1} [Ad_{g_{i,i+1}^{-1}} \cdots Ad_{g_{1,2}^{-1}} \xi_1 + Ad_{g_{i,i+1}^{-1}} \cdots Ad_{g_{2,3}^{-1}} \xi_{1,2} \\ &\qquad + \cdots + Ad_{g_{i,i+1}^{-1}} \xi_{i-1,i} + \xi_{i,i+1}] .\end{aligned} \tag{5.2.20}$$

Then (5.2.18) follows from those and (5.2.10). The equation for $\xi_{j,i+1}$ is derived similarly. ∎

Corollary 5.2.3

Equations (5.2.18) induce the following relationships between the positions and velocities of the $(i+1)$–th node and those of the previous nodes:

$$\begin{aligned}\xi_1^{i+1} &= \xi_1^i + \xi_1^{i,i+1} , \\ \xi_2^{i+1} &= -\xi_1^i \gamma_3^{i,i+1} + \xi_2^i \cos\gamma_1^{i,i+1} + \xi_3^i \sin\gamma_1^{i,i+1} + \xi_2^{i,i+1} , \\ \xi_3^{i+1} &= \xi_1^i \gamma_2^{i,i+1} - \xi_2^i \sin\gamma_1^{i,i+1} + \xi_3^i \cos\gamma_1^{i,i+1} + \xi_3^{i,i+1} .\end{aligned} \tag{5.2.21}$$

Also:

$$\begin{aligned}\xi_1^{j,i+1} &= \xi_1^{j,i} + \xi_1^{i,i+1} , \\ \xi_2^{j,i+1} &= -\xi_1^{j,i} \gamma_3^{i,i+1} + \xi_2^{j,i} \cos\gamma_1^{i,i+1} + \xi_3^{j,i} \sin\gamma_1^{i,i+1} + \xi_2^{i,i+1} , \\ \xi_3^{j,i+1} &= \xi_1^{j,i} \gamma_2^{i,i+1} - \xi_2^{j,i} \sin\gamma_1^{i,i+1} + \xi_3^{j,i} \cos\gamma_1^{i,i+1} + \xi_3^{i,i+1} .\end{aligned} \tag{5.2.22}$$

Moreover, for the corresponding Wei–Norman parameters we have:

$$\begin{aligned}\gamma_1^{i+1} &= \gamma_1^i + \gamma_1^{i,i+1} , \\ \gamma_2^{i+1} &= \gamma_2^i \cos\gamma_1^{i,i+1} + \gamma_3^i \sin\gamma_1^{i,i+1} + \gamma_2^{i,i+1} , \\ \gamma_3^{i+1} &= -\gamma_2^i \sin\gamma_1^{i,i+1} + \gamma_3^i \cos\gamma_1^{i,i+1} + \gamma_3^{i,i+1}\end{aligned} \tag{5.2.23}$$

and

$$\begin{aligned}
\gamma_1^{j,i+1} &= \gamma_1^{j,i} + \gamma_1^{i,i+1} , \\
\gamma_2^{j,i+1} &= \gamma_2^{j,i} \cos\gamma_1^{i,i+1} + \gamma_3^{j,i} \sin\gamma_1^{i,i+1} + \gamma_2^{i,i+1} , \\
\gamma_3^{j,i+1} &= -\gamma_2^{j,i} \sin\gamma_1^{i,i+1} + \gamma_3^{j,i} \cos\gamma_1^{i,i+1} + \gamma_3^{i,i+1} .
\end{aligned} \tag{5.2.24}$$

Proof

From (4.3.2), (5.2.18), (2.3.10) and (4.3.18), observing that the $\{\mathcal{A}_i\}$ form a basis, we get equations (5.2.21) and (5.2.22). From the Wei–Norman representation of elements of $SE(2)$ (equation (4.3.20)) and from equation (5.2.16), we get equations (5.2.23) and (5.2.24).

∎

The nonholonomic constraint of rolling–without–slipping on the wheels of each platform can be expressed, using (4.3.23), as:

$$\xi_2^i = \mathcal{A}_2^\flat(\xi_i) = \dot{x}_i \cos\phi_i + \dot{y}_i \sin\phi_i = 0 , \quad i = 1, \ldots, \ell . \tag{5.2.25}$$

Define the composite *velocity vector* ζ of the NVGT assembly as the components of the velocity ξ_1 of the first platform and those of the shape variations of all the modules of the assembly:

$$\begin{aligned}
\zeta &\stackrel{\text{def}}{=} (\xi^1 \,|\, \xi^{1,2} \,|\, \cdots \,|\, \xi^{\ell-1,\ell})^\top \\
&= (\xi_1^1 \xi_2^1 \xi_3^1 \,|\, \xi_1^{1,2} \cdots \,|\, \xi_1^{\ell-1,\ell} \xi_2^{\ell-1,\ell} \xi_1^{\ell-1,\ell})^\top \\
&= \left(\mathcal{A}_1^\flat(\xi_1)\, \mathcal{A}_2^\flat(\xi_1)\, \mathcal{A}_3^\flat(\xi_1) \,|\, \mathcal{A}_1^\flat(\xi_{1,2}) \cdots \,|\, \mathcal{A}_1^\flat(\xi_{\ell-1,\ell})\, \mathcal{A}_2^\flat(\xi_{\ell-1,\ell})\, \mathcal{A}_3^\flat(\xi_{\ell-1,\ell})\right)^\top .
\end{aligned}$$

Theorem 5.2.4

The ℓ nonholonomic constraints (5.2.25) can be written in matrix form as:

$$A(g_{1,2}, \cdots, g_{\ell-1,\ell})\, \zeta = 0 , \tag{5.2.26}$$

where A is a function of only the shape of the NVGT assembly $\{g_{i,i+1}, i = 1, \cdots, \ell - 1\}$ and is an $\ell \times 3\ell$ block lower triangular matrix of maximal rank of the form:

$$A(g_{1,2}, \cdots, g_{\ell-1,\ell}) = \begin{pmatrix} *_{1,1} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ *_{1,2} & *_{2,2} & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \ddots & 0 & \cdots & 0 & 0 \\ *_{1,i+1} & *_{2,i+1} & \cdots & *_{i,i+1} & *_{i+1,i+1} & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \cdots & \ddots & 0 \\ *_{1,\ell} & *_{2,\ell} & \cdots & *_{i,\ell} & *_{i+1,\ell} & \cdots & *_{\ell-1,\ell} & *_{\ell,\ell} \end{pmatrix} \tag{5.2.27}$$

with the k–th block of the $(i+1)$–th line, for $i = 1, \ldots, \ell - 1$ defined as:

$$\begin{aligned} *_{k,i+1} &= \Big(\mathcal{A}_2^\flat(Ad_{g^{-1}_{k,i+1}} \mathcal{A}_1) \;\; \mathcal{A}_2^\flat(Ad_{g^{-1}_{k,i+1}} \mathcal{A}_2) \;\; \mathcal{A}_2^\flat(Ad_{g^{-1}_{k,i+1}} \mathcal{A}_3) \Big) \\ &= \Big(\mathcal{A}_2^\flat(Ad_{g^{-1}_{i,i+1}} \cdots Ad_{g^{-1}_{k,k+1}} \mathcal{A}_1) \;\; \mathcal{A}_2^\flat(Ad_{g^{-1}_{i,i+1}} \cdots Ad_{g^{-1}_{k,k+1}} \mathcal{A}_2) \\ &\qquad\qquad \mathcal{A}_2^\flat(Ad_{g^{-1}_{i,i+1}} \cdots Ad_{g^{-1}_{k,k+1}} \mathcal{A}_3) \Big) \\ &= \big(-\gamma_3^{k,i+1} \;\; \cos\gamma_1^{k,i+1} \;\; \sin\gamma_1^{k,i+1} \big) \,, \qquad \text{for } 1 \le k < i+1 \,, \\ *_{k,k} &= \Big(\mathcal{A}_2^\flat(Ad_{g^{-1}_{k,k}} \mathcal{A}_1) \;\; \mathcal{A}_2^\flat(Ad_{g^{-1}_{k,k}} \mathcal{A}_2) \;\; \mathcal{A}_2^\flat(Ad_{g^{-1}_{k,k}} \mathcal{A}_3) \Big) \\ &= (0\,1\,0) \,, \qquad\qquad \text{for } k = i+1 \,, \\ *_{k,i+1} &= (0\,0\,0) \,, \qquad\qquad \text{for } i+1 < k \le \ell \,. \end{aligned} \tag{5.2.28}$$

Proof

This result follows from Theorem 4.2.1.

∎

Since A has always maximal rank ℓ, its null space $\mathcal{N}(A)$ has dimension $m \stackrel{\text{def}}{=} 3\ell - \ell = 2\ell$.

Corollary 5.2.5

Assume that the velocities ζ can be reordered, so that the matrix A is partitioned as $A = (\, A_1 \quad A_2 \,)$, with A_1 an $\ell \times m$ matrix and A_2 a locally invertible $\ell \times \ell$ matrix.

Let the corresponding partition of the velocity vector be $\zeta = \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix}$, with $\zeta_1 \in I\!R^m$ and $\zeta_2 \in I\!R^\ell$. Then, there exists an $3\ell \times m$ matrix B such that:

$$\zeta = \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} = B\, \zeta_1 \,. \tag{5.2.29}$$

Proof

Follows from Corollary 4.2.2. The matrix $B = \begin{pmatrix} I_{m\times m} \\ -A_2^{-1} A_1 \end{pmatrix}$ works.

∎

The physical meaning of this result is that the global motion of the NVGT assembly depends only on the current shape of the assembly and on the variations of this shape. Direct actuation of the wheels (as in a car) is not necessary for propulsion of this system.

Notice that, since A depends only on the shape, so does A_2. In addition, the choice of the locally invertible matrix A_2 is dictated by the choice of the splitting of ζ into ζ_1 and ζ_2. For a particular choice, as the shape is altered, A_2 will become singular at certain shapes. However, a different splitting of ζ may provide a nonsingular A_2. If this is not the case, the corresponding configurations of the NVGT assembly shall be referred to as *nonholonomic singularities.* Those singularities are not removable by merely choosing alternative splittings of ζ or alternative parametrizations of the configuration space (e.g. how we assign orderings to the kinematic chain at hand). In these configurations, the system kinematics, together with the shape control, may not provide sufficient information to determine the motion of NVGT assembly and the dynamics of the system may have to be used in the form of a ballistic motion through the nonholonomic singularity.

Remarks 5.2.6

1) Unlike previous work on nonholonomic motion planning, in our case the ζ_1's of equation (5.2.29) do not correspond directly to the controls of the system and, thus, are not at our disposal to alter at will. Our real controls are the leg velocities $\dot{\sigma}$ of the parallel manipulator modules (see next section). However, as we will see in the next section, *off* the *kinematic* singularities of the parallel manipulators, the shape controls can easily determine the corresponding leg velocities. Therefore, in order to simplify the

discussion of motion planning, once the partitioning of the velocity vector ζ as $\begin{pmatrix}\zeta_1\\ \zeta_2\end{pmatrix}$ is done in such a way that all the ζ_1's are controllable from the $\dot{\sigma}$'s, we will disregard the particulars of the implementation of the modules and only consider the shape controls. Note that when we have only one module, i.e. $\ell = 2$, such a partitioning of the velocity vector cannot be done.

2) The $3(\ell - 1)$ holonomic constraints imposed by the legs of the modules of the $(\ell-1)$–module NVGT assembly determine the shape $g_{1,2}, g_{2,3}, \ldots, g_{\ell-1,\ell}$ of the assembly. In order to determine completely its configuration, we also need to specify the position and orientation of the assembly with respect to the world coordinate frame, given by an element (g_1 in this case) of $G = SE(2)$. Thus, we need 3 more constraints. These come from the ℓ nonholonomic constraints provided by the platform wheels. Consider now some special cases: *i*) If we have a one–module NVGT, i.e. if $\ell = 2$, we have 3 holonomic and 2 nonholonomic constraints, but we need to determine 6 degrees–of–freedom, thus we do not have enough kinematic constraints to determine the motion of the assembly. We either have to consider its dynamics or we need to impose additional constraints (e.g. unidirectional wheel motion). The first alternative is particularly interesting and gives rise to the "Roller Racer"system that we consider in chapter 6. *ii*) If we have a two–module NVGT, i.e. if $\ell = 3$, we have 6 holonomic and 3 nonholonomic constraints and we need to determine 9 degrees–of–freedom, thus we have exactly the required number of constraints. *iii*) If we have an NVGT assembly with more than two–modules, i.e. if $\ell > 3$, we have $3(\ell - 1)$ holonomic and ℓ nonholonomic constraints and we need to determine 3ℓ degrees–of–freedom, thus we have $3(\ell-1)+\ell-3\ell = \ell-3$ extra constraints that have to be satisfied. Therefore, from the $3(\ell - 1)$ shape velocity components in ζ, *only* $3(\ell - 1) - (\ell - 3) = 2\ell$ can be determined independently and will be elements of the vector ζ_1 in equation (5.2.29). The remaining $\ell - 3$ shape velocity components will be determined by the $\ell - 3$ extra constraints, i.e. they will be elements of the vector ζ_2, together with the velocities that characterize the global motion of the assembly (here they are the ξ^1's). Observe that if there exists non–negative integer k such that $\ell - 3 = 3k$, we can choose $\ell - k$ of the ℓ modules and alter their shape at will, while the shape of the remaining k modules will be determined by the extra constraints. In brief, for $\ell > 3$, the

problem is over–constrained.

3) The 2–module NVGT is "canonical" in the following sense: Suppose that we change the $(\ell-1)$–module NVGT architecture (for $\ell > 3$) so that we have wheels on only 3 platforms and castors on the remaining $\ell - 3$ platforms. Then we have only 3 nonholonomic constraints (instead of ℓ). Such a system has exactly the number of constraints needed to determine its motion, which, in our original NVGT assembly is the case only for $\ell = 2$. The motion planning strategies that will be determined for the 2–module NVGT will work also for this $(\ell-1)$–module assembly. Moreover, this $(\ell-1)$–module assembly possesses $3\ell - 3$ degrees–of–freedom more than the 2–module NVGT and those can be exploited for other purposes (e.g. for motion in a constrained environment, or for generating a richer *repertoire* of motions of the type described in (Chirikjian & Burdick [1991])).

5.2.3 Implementation of an NVGT module as a Parallel Manipulator

Consider the $(i, i+1)$–th module of the NVGT assembly, implemented as a planar parallel manipulator (c.f. section 2.5). This consists of the i–th and $(i+1)$–th platforms, which are connected by three legs of variable length. One possible architecture is shown in fig. 5.1.1.

In previous sections we saw that the shape of the $(i, i+1)$–th module is determined by $g_{i,i+1} \in SE(2)$. Let ${}^{p}P_j^k$, for $k = i, i+1$, $j = 1, 2, 3$ be the position of the j–th joint of the k–th platform with respect to the coordinate system of this platform. Consider the following quantities, defined with respect to the coordinate system of the i–th platform: the position P_j^{i+1} of the $(i+1)$–th platform, the vector S_j^i and the length σ_j^i of the j–th leg of the module. Let σ^i be the vector of the leg lengths of the $(i, i+1)$–th module, i.e. $\sigma^i \stackrel{\text{def}}{=} (\sigma_1^i \; \sigma_2^i \; \sigma_3^i)^\top$.

The *inverse kinematic map* $\mathcal{F}^{-1} : SE(2) \to I\!R_+^3$ specifies the leg lengths of the module as functions of its shape $g_{i,i+1}$. We can easily see that for the j–th leg:

$$\begin{pmatrix} S_j^i \\ 0 \end{pmatrix} = g_{i,i+1} \begin{pmatrix} {}^{p}P_j^{i+1} \\ 1 \end{pmatrix} - \begin{pmatrix} {}^{p}P_j^i \\ 1 \end{pmatrix} , \tag{5.2.30}$$

where $g_{i,i+1} = \begin{pmatrix} R_{i,i+1} & T_{i,i+1} \\ 0 & 1 \end{pmatrix}$. Then the leg lengths of the module are given by

$$\sigma_j^i = ||S_j^i||_3 = \sqrt{< S_j^i, S_j^i >_3}\ , \quad \text{for } j = 1, 2, 3\ , \tag{5.2.31}$$

where $<\,,\,>_n$ is the inner product in $\mathbb{R}^n$ and $||\,.\,||_n$ is the corresponding norm. Then the inverse kinematic map is:

$$\mathcal{F}^{-1}(g_{i,i+1}) = \sigma^i\ . \tag{5.2.32}$$

Applying Theorem 2.5.3 in this case, we get the following results.

Proposition 5.2.7 (Velocity Kinematics)

The body velocities of the i–th module $\xi^{i,i+1}$ and the changes of length of its legs $\dot\sigma^i \stackrel{\text{def}}{=} (\dot\sigma_1^i\ \dot\sigma_2^i\ \dot\sigma_3^i)^\top$ are related by:

$$\Sigma(\sigma^i)\dot\sigma^i = J(g_{i,i+1})\xi^{i,i+1}\ , \tag{5.2.33}$$

where $\Sigma(\sigma^i) = diag\{\sigma_1^i\ , \sigma_2^i\ , \sigma_3^i\}$ and $J(g_{i,i+1}) = \left[{S_j^i}^\top R_{i,i+1}\, \widehat{{}^pP_j^{i+1}}^\top \ \mid\ {S_j^i}^\top R_{i,i+1} \right]_j$.

Proof

From (5.2.31): $(\sigma_j^i)^2 =< S_j^i, S_j^i >_3$. Differentiating both sides and defining $\Omega \stackrel{\text{def}}{=} \xi_1^{i,i+1}$ and $\Xi \stackrel{\text{def}}{=} \begin{pmatrix} \xi_2^{i,i+1} \\ \xi_3^{i,i+1} \end{pmatrix}$, we get:

$$\begin{aligned}\sigma_j^i\dot\sigma_j^i &=< S_j^i, \dot S_j^i >_3 =< S_j^i, R_{i,i+1}(\Xi + \hat\Omega\, {}^pP_j^{i+1}) >_3 \\ &=< {R_{i,i+1}}^\top S_j^i, \Xi + \widehat{{}^pP_j^{i+1}}^\top \Omega >_3 =< {}^pP_j^{i+1} {R_{i,i+1}}^\top S_j^i, \Omega >_3 + < {R_{i,i+1}}^\top S_j^i, \Xi >_3\ ,\end{aligned}$$

where $R_{i,i+1}$ was defined in (5.2.15). Then (5.2.33) follows.

∎

Configurations where $J(g)$ or $\Sigma(\sigma(g))$ is singular are called *kinematic singularities.* Those have nothing to do with the nonholonomic singularities of the NVGT assembly defined in the previous section.

Corollary 5.2.8

Suppose the configuration $g_{i,i+1} \in SE(2)$ of the $(i, i+1)$–th module is not a kinematic singularity and that the corresponding leg lengths are σ^i. Then:

$$\xi^{i,i+1} = J^{-1}(g_{i,i+1})\Sigma(\sigma^i)\dot{\sigma}^i \tag{5.2.34}$$

and

$$\dot{\sigma}^i = \Sigma^{-1}(\sigma^i)J(g_{i,i+1})\xi^{i,i+1}\ . \tag{5.2.35}$$

∎

Observe that after we specify the shape variations $\xi^{i,i+1}$, the corresponding leg length changes can be easily determined from (5.2.35). Therefore, our discussion of the motion planning problem can disregard, without loss of generality, the particular implementation details of the modules and consider only the shape variations of the modules.

5.3 The 3–node, 2–module NVGT Assembly

In section 5.3.1 we consider the kinematics of the 3–node, 2-module NVGT (fig. 5.3.1) as a special case of the kinematics of the ℓ-node NVGT. We show that the velocity vector ζ can be partitioned in two parts, one of which (ζ_1) constitutes the independent shape controls and the other (ζ_2) being velocities dependent on the shape variables, which characterize the motion of the assembly with respect to the world coordinate system. In section 5.3.2 we examine the motion planning problem for the 2-module NVGT assembly. We show that shape actuation strategies, where the shape of one module is kept fixed and the shape of other is varied periodically, induce a rotation of the NVGT assembly around the instantaneous center of rotation of the first module. If the platforms of the *fixed shape* module are parallel, the induced motion is a translation along a direction perpendicular to the platforms. We allow the shape of the second module to describe a closed path in shape space and show that a net displacement of the NVGT assembly with respect to the world coordinate system is induced after each traversal of the shape space path.

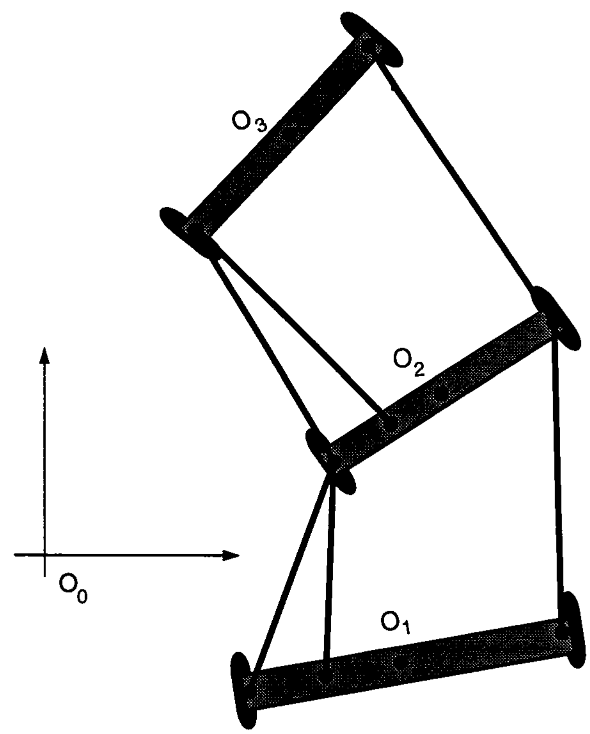


Fig. 5.3.1: Two–module NVGT assembly

5.3.1 Kinematics of the 2–module NVGT

In the assembly of fig. 5.3.1, we consider a chain of two NVGT modules. This system has $3\ell = 9$ degrees–of–freedom, its configuration space is $Q = G \times S$, where $G = SE(2)$ and the shape space is $S = SE(2) \times SE(2)$ and it is subject to $3(\ell - 1) = 6$ holonomic constraints from the parallel manipulator legs and to $\ell = 3$ nonholonomic constraints from the rolling–without–slipping assumption on the wheels. From the system kinematics we have (specializing the results of section 5.2):

$$\begin{aligned} g_2 &= g_1 g_{1,2} \,, \\ g_3 &= g_2 g_{2,3} = g_1 g_{1,2} g_{2,3} \,, \\ g_{13} &= g_{1,2} g_{2,3} \,. \end{aligned} \tag{5.3.1}$$

From (5.2.18) we get for the corresponding velocities:

$$\xi_2 = Ad_{g^{-1}_{1,2}} \xi_1 + \xi_{1,2} \,,$$

$$\xi_3 = Ad_{g^{-1}_{2,3}} \xi_2 + \xi_{2,3} = Ad_{g^{-1}_{2,3}} Ad_{g^{-1}_{1,2}} \xi_1 + Ad_{g^{-1}_{2,3}} \xi_{1,2} + \xi_{2,3} \,, \tag{5.3.2}$$

$$\xi_{1,3} = Ad_{g^{-1}_{2,3}} \xi_{1,2} + \xi_{2,3} \,.$$

The nonholonomic constraint of rolling–without–slipping on the wheels of each platform can be expressed using (5.2.25) as:

$$\xi^i_2 = \mathcal{A}^\flat_2(\xi_i) = \dot{x}_i \cos\phi_i + \dot{y}_i \sin\phi_i = 0 \,, \quad i = 1,2,3 \,. \tag{5.3.3}$$

The 3 nonholonomic constraints can be put in the matrix form of (5.2.26):

$$A(g_{1,2}, g_{2,3})\, \zeta = 0 \,, \tag{5.3.4}$$

where $\zeta = \begin{pmatrix} \xi^1 \\ \xi^{1,2} \\ \xi^{2,3} \end{pmatrix}$. The matrix A is a function of only the shape variables $g_{1,2}$, $g_{2,3}$ of the chain and is a block lower triangular matrix of the form:

$$A(g_{1,2}, g_{2,3}) = \begin{pmatrix} *_{1,1} & 0 & 0 \\ *_{1,2} & *_{2,2} & 0 \\ *_{1,3} & *_{2,3} & *_{3,3} \end{pmatrix}$$

$$= \left(\begin{matrix} 0 & 1 & 0 \\ \mathcal{A}^\flat_2(Ad_{g^{-1}_{1,2}} \mathcal{A}_1) & \mathcal{A}^\flat_2(Ad_{g^{-1}_{1,2}} \mathcal{A}_2) & \mathcal{A}^\flat_2(Ad_{g^{-1}_{1,2}} \mathcal{A}_3) \\ \mathcal{A}^\flat_2(Ad_{g^{-1}_{1,3}} \mathcal{A}_1) & \mathcal{A}^\flat_2(Ad_{g^{-1}_{1,3}} \mathcal{A}_2) & \mathcal{A}^\flat_2(Ad_{g^{-1}_{1,3}} \mathcal{A}_3) \end{matrix} \right.$$

$$\left. \begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \mathcal{A}^\flat_2(Ad_{g^{-1}_{2,3}} \mathcal{A}_1) & \mathcal{A}^\flat_2(Ad_{g^{-1}_{2,3}} \mathcal{A}_2) & \mathcal{A}^\flat_2(Ad_{g^{-1}_{2,3}} \mathcal{A}_3) & 0 & 1 & 0 \end{matrix} \right)$$

$$= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\gamma^{1,2}_3 & \cos\gamma^{1,2}_1 & \sin\gamma^{1,2}_1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -\gamma^{1,3}_3 & \cos\gamma^{1,3}_1 & \sin\gamma^{1,3}_1 & -\gamma^{2,3}_3 & \cos\gamma^{2,3}_1 & \sin\gamma^{2,3}_1 & 0 & 1 & 0 \end{pmatrix} \tag{5.3.5}$$

Equation (5.3.4) can be put in the form (5.2.29) by partitioning ζ as $\zeta = \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix}$ with

$$\zeta_1 = \begin{pmatrix} \xi^{1,2} \\ \xi^{2,3} \end{pmatrix} \quad \text{and} \quad \zeta_2 = \xi^1 \tag{5.3.6}$$

and by partitioning A as $(A_1\ A_2)$ with

$$\begin{aligned}
A_1(g_{2,3}) &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \mathcal{A}_2^\flat(Ad_{g^{-1}_{2,3}}\mathcal{A}_1) & \mathcal{A}_2^\flat(Ad_{g^{-1}_{2,3}}\mathcal{A}_2) & \mathcal{A}_2^\flat(Ad_{g^{-1}_{2,3}}\mathcal{A}_3) & 0 & 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -\gamma_3^{2,3} & \cos\gamma_1^{2,3} & \sin\gamma_1^{2,3} & 0 & 1 & 0 \end{pmatrix}
\end{aligned} \tag{5.3.7}$$

and

$$\begin{aligned}
A_2(g_{1,2}, g_{2,3}) &= \begin{pmatrix} 0 & 1 & 0 \\ \mathcal{A}_2^\flat(Ad_{g^{-1}_{1,2}}\mathcal{A}_1) & \mathcal{A}_2^\flat(Ad_{g^{-1}_{1,2}}\mathcal{A}_2) & \mathcal{A}_2^\flat(Ad_{g^{-1}_{1,2}}\mathcal{A}_3) \\ \mathcal{A}_2^\flat(Ad_{g^{-1}_{1,3}}\mathcal{A}_1) & \mathcal{A}_2^\flat(Ad_{g^{-1}_{1,3}}\mathcal{A}_2) & \mathcal{A}_2^\flat(Ad_{g^{-1}_{1,3}}\mathcal{A}_3) \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 & 0 \\ -\gamma_3^{1,2} & \cos\gamma_1^{1,2} & \sin\gamma_1^{1,2} \\ -\gamma_3^{1,3} & \cos\gamma_1^{1,3} & \sin\gamma_1^{1,3} \end{pmatrix} .
\end{aligned} \tag{5.3.8}$$

Then the velocity of the 2–module NVGT assembly with respect to the world coordinate system, as it is characterized by ξ^1, can be expressed as a function of only the shape variables of the assembly:

$$\xi^1 = -A_2^{-1}(g_{1,2}, g_{2,3})A_1(g_{2,3}) \begin{pmatrix} \xi^{1,2} \\ \xi^{2,3} \end{pmatrix} . \tag{5.3.9}$$

Proposition 5.3.1

The nonholonomic singularities of the 2–module NVGT assembly are configurations where *all 3 axes* of the platforms of the assembly either intersect at one point or are parallel.

Proof

The nonholonomic singularities of the system are specified by considering the determinant of the matrix A_2 :

$$\begin{aligned}
det(A_2) &= \mathcal{A}_2^\flat(Ad_{g^{-1}_{1,3}}\mathcal{A}_1)\mathcal{A}_2^\flat(Ad_{g^{-1}_{1,2}}\mathcal{A}_3) - \mathcal{A}_2^\flat(Ad_{g^{-1}_{1,2}}\mathcal{A}_1)\mathcal{A}_2^\flat(Ad_{g^{-1}_{1,3}}\mathcal{A}_3) \\
&= -\gamma_3^{1,3}\sin\gamma_1^{1,2} + \gamma_3^{1,2}\sin\gamma_1^{1,3} \\
&= y_{1,2}\sin\gamma_1^{2,3} - \gamma_3^{2,3}\sin\gamma_1^{1,2} .
\end{aligned} \tag{5.3.10}$$

Observe that when $\sin\gamma_1^{1,2} \neq 0$ and $\sin\gamma_1^{1,3} \neq 0$:

$$
\begin{aligned}
det(A_2) &= \sin\gamma_1^{1,2}\sin\gamma_1^{1,3}\left[\left(-\frac{\gamma_3^{1,3}}{\sin\gamma_1^{1,3}}\right) - \left(-\frac{\gamma_3^{1,2}}{\sin\gamma_1^{1,2}}\right)\right] \\
&= \sin\gamma_1^{1,2}\sin\gamma_1^{1,3}\left[\Delta x_{O_{1,3}} - \Delta x_{O_{1,2}}\right] ,
\end{aligned}
\tag{5.3.11}
$$

where $\Delta x_{O_{1,j}} \stackrel{\text{def}}{=} -\frac{\gamma_3^{1,j}}{\sin\gamma_1^{1,j}}$, $j = 2, 3$, is the distance of the intersection $O_{1,j}$ of the axis of platform 1 with the axis of platform j from the point O_1, as shown in fig. 5.3.2. It will be shown in the next section that the point $O_{1,j}$ coincides with the Instantaneous Center of Rotation (ICR) of the module composed of the platforms 1 and j. Then, the matrix A_2 is singular whenever the points $O_{1,2}$ and $O_{1,3}$ coincide (the point $O_{1,2,3}$ may be at infinity as in fig. 5.3.3.a).

∎

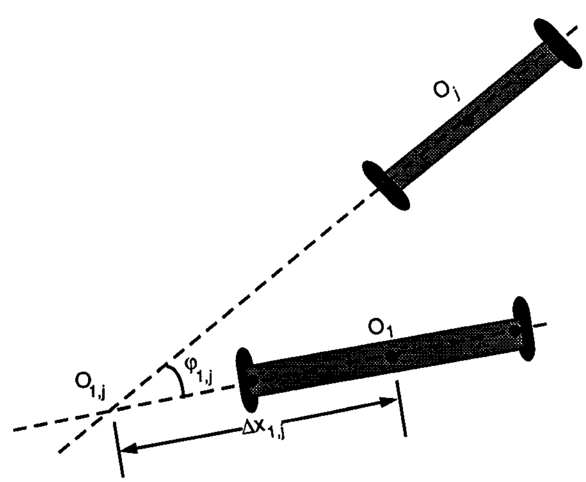


Fig. 5.3.2: Instantaneous Center of Rotation of the (i, j)–th Module

It can be shown that nonholonomic singularities are not removable by reparametrizing T_qQ, i.e. if the global motion of the 2–module NVGT is characterized by ξ_2 or ξ_3, as opposed to ξ_1, the equations corresponding to (5.3.4) and (5.3.9) will provide us with the same set of nonholonomic singularities that appear in the above Proposition.

Even in the case the system is at a nonholonomic singularity, the 3 nonholonomic constraints *remain independent* (c.f. equation (5.3.5), where $rank(A) = 3$), but, since

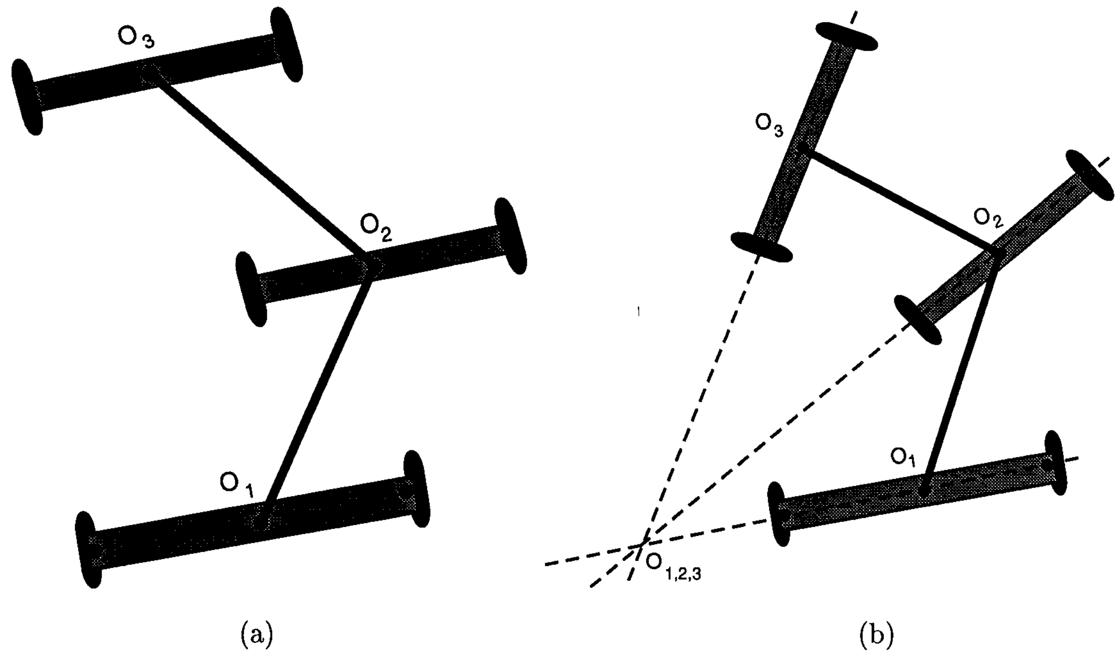


Fig. 5.3.3: Nonholonomic Singularities for 2–module NVGT assembly

the platforms have a common instantaneous center of rotation, equation (5.3.4) cannot be recast in the form of (5.3.9). Therefore, motion with respect to point $O_{1,2,3}$ cannot be controlled by the system shape variables alone and the dynamics of the system ought to be considered. This is analogous to what practicing engineers refer to as loss of control authority.

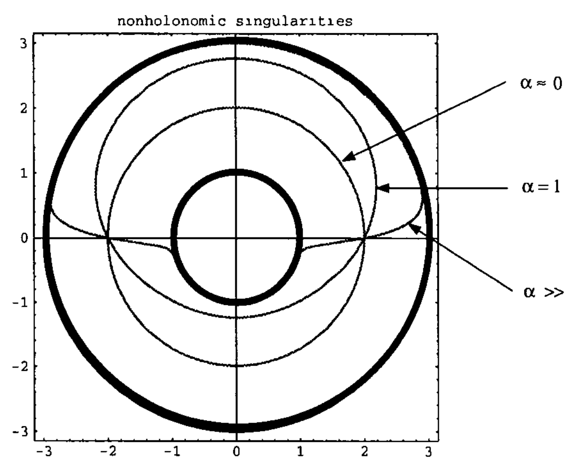


Fig. 5.3.4: Nonholonomic Singularity Surfaces

In the motion control strategies that we will consider in the next section, the (1, 2)–

module will be used for steering, therefore its shape will remain constant for relatively large periods of time. We are interested in finding out what the locus of nonholonomic singularities (as a function of the shape of the (2, 3)–module) is under this condition. From (5.3.10), the condition $det A_2 = 0$ becomes $\gamma_3^{2,3} = \alpha(\gamma_1^{1,2}) \sin\gamma_1^{2,3}$, with $\alpha(\gamma_1^{1,2}) \stackrel{\text{def}}{=} \frac{y_{1,2}}{\sin\gamma_1^{1,2}}$ when $\sin\gamma_1^{1,2} \neq 0$ or $\sin\gamma_1^{2,3} = 0$, when $\sin\gamma_1^{1,2} = 0$. The first relationship is shown in fig. 5.3.4, where the usual geometric picture of $SE(2)$ as a "thick" cylinder is used to visualize the loci of nonholonomic singularities for various values of α. Assuming that the γ_1–axis runs around the cylinder, the γ_2–axis runs along its length and the γ_3–axis runs along its radial direction, we consider a cut of this cylinder vertical to the $\gamma_2^{2,3}$–axis. Observe that in either one of the above cases, the locus splits $SE(2)$ in two disconnected parts. If we want to go from one part to the other we have to go through this manifold of nonholonomic singularities. Crossing this locus in the most appropriate way may require path planning of the type presented in chapter 3, in order to move transversally to this manifold and allow the dynamics of the system to carry the system from one part of $SE(2)$ to the other.

From (5.3.8):

$$A_2^{-1} = \frac{1}{det(A_2)} \begin{pmatrix} \sin\gamma_1^{2,3} & -\sin\gamma_1^{1,3} & \sin\gamma_1^{1,2} \\ \gamma_3^{1,2}\sin\gamma_1^{1,3} - \gamma_3^{1,3}\sin\gamma_1^{1,2} & 0 & 0 \\ -\gamma_3^{1,2}\cos\gamma_1^{1,3} + \gamma_3^{1,3}\cos\gamma_1^{1,2} & -\gamma_3^{1,3} & \gamma_3^{1,2} \end{pmatrix}. \tag{5.3.12}$$

Then, from (5.3.9) we get:

$$\xi^1 = \frac{-1}{det(A_2)} \begin{pmatrix} -\sin\gamma_1^{1,2}\gamma_3^{2,3} & -\sin\gamma_1^{1,3} + \sin\gamma_1^{1,2}\cos\gamma_1^{2,3} & \sin\gamma_1^{1,2}\sin\gamma_1^{2,3} \\ 0 & 0 & 0 \\ -\gamma_3^{1,2}\gamma_3^{2,3} & -\gamma_3^{1,3} + \gamma_3^{1,2}\cos\gamma_1^{2,3} & \gamma_3^{1,2}\sin\gamma_1^{2,3} \end{pmatrix}$$
$$\begin{pmatrix} 0 & \sin\gamma_1^{1,2} & 0 \\ 0 & 0 & 0 \\ 0 & \gamma_3^{1,2} & 0 \end{pmatrix} \begin{pmatrix} \xi^{1,2} \\ \xi^{2,3} \end{pmatrix}. \tag{5.3.13}$$

Observe that the partitioning of ζ in equation (5.3.6) is the only one that assigns to ζ_1 shape controls which can be affected by leg length changes in the parallel manipulator modules. Moreover, it assigns to ζ_2 the velocities that characterize the global motion of the NVGT assembly with respect to the world coordinate system.

Proposition 5.3.2 (Controllability)

The 2–module NVGT assembly is controllable from any initial position $g_1 \in G = SE(2)$ to any final one $\bar{g}_1 \in G$.

Proof

The left–invariant system $\dot{g}_1 = g_1 \xi_1$ with ξ_1 given by (5.3.13) as $\xi_1 = u_1\mathcal{A}_1 + u_3\mathcal{A}_3$ is such that $[\mathcal{A}_1, \mathcal{A}_3] = -\mathcal{A}_2$, thus $\mathrm{sp}\{\mathcal{A}_1, \mathcal{A}_3, [\mathcal{A}_1, \mathcal{A}_3]\} = \mathcal{G}$. Thus, it is controllable from any initial position $g_1 \in G = SE(2)$ to any final one $\bar{g}_1 \in G$, whenever u_1 and u_3 are independent controls. For a generic set of shape controls $\begin{pmatrix} \xi^{1,2} \\ \xi^{2,3} \end{pmatrix}$ this will happen whenever the 1st and 3rd rows of the matrix $A_2^{-1}A_1$ in (5.3.13) are linearly independent. By checking all the 2×2 determinants generated by those rows, we easily see that this is always true away from the nonholonomic singularities.

∎

5.3.2 Motion Control and Planning for the 2–module NVGT

There are several possible *actuation strategies* for the 2–module NVGT. We will consider a simple one where the first module "steers" the system, while the second provides the translation mechanism through periodic variations of its shape parameters.

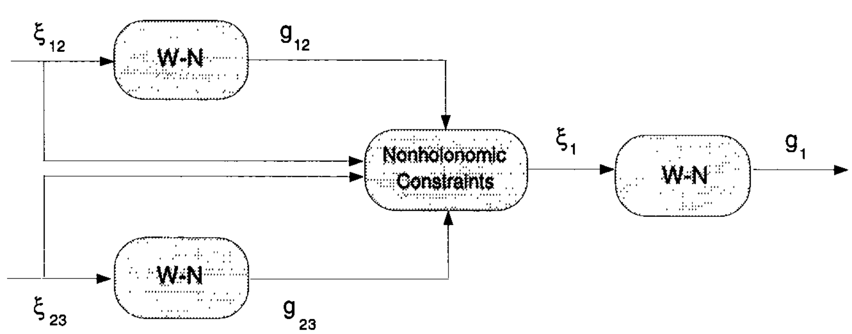


Fig. 5.3.5: The 2–module NVGT Model for Motion Control

The Motion Control scheme that we consider here is open–loop and is based on periodic oscillations of the shape of the assembly. The model for the 2–module NVGT assembly is summarized in fig. 5.3.5. When we specify the shape controls $\xi_{1,2}$ and $\xi_{2,3}$, the corresponding shape trajectories $g_{1,2}(t)$ and $g_{2,3}(t)$ are given by the Wei–Norman procedure (the "W-N" block in the above figure, which refers to equations (4.3.3), (4.3.14) and (4.3.15)). The nonholonomic constraints are then used to calculate the instantaneous motion of the whole assembly with respect to the world coordinate system, as it is described by ξ^1 (equation (5.3.13)). Finally the Wei–Norman procedure allows us to determine the corresponding system trajectory g_1. This whole process is detailed below.

From (5.3.13) we observe that the motion of the system is determined completely, at least away from the nonholonomic singularities, by the *shape controls* vectors $\xi^{1,2}$ and $\xi^{2,3}$. Here we will consider the special case of motions that are generated by keeping the shape of the first module fixed, i.e. $\xi^{1,2} = 0$, and vary the shape controls $\xi^{2,3}$ of the second module periodically. Then from (4.3.14):

$$\dot{\gamma}^{1,2} = \begin{pmatrix} \dot{\gamma}_1^{1,2} \\ \dot{\gamma}_2^{1,2} \\ \dot{\gamma}_3^{1,2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \gamma_3^{1,2} & 1 & 0 \\ -\gamma_2^{1,2} & 0 & 1 \end{pmatrix} \begin{pmatrix} \xi_1^{1,2} \\ \xi_2^{1,2} \\ \xi_3^{1,2} \end{pmatrix} = 0 \,. \tag{5.3.14}$$

Thus from (4.3.15):

$$\gamma_1^{1,2} = \gamma_1^{1,2}(0) \;,\;\; \gamma_2^{1,2} = \gamma_2^{1,2}(0) \;,\;\; \gamma_3^{1,2} = \gamma_3^{1,2}(0) \,. \tag{5.3.15}$$

Also :

$$\dot{\gamma}^{2,3} = \begin{pmatrix} \dot{\gamma}_1^{2,3} \\ \dot{\gamma}_2^{2,3} \\ \dot{\gamma}_3^{2,3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \gamma_3^{2,3} & 1 & 0 \\ -\gamma_2^{2,3} & 0 & 1 \end{pmatrix} \begin{pmatrix} \xi_1^{2,3} \\ \xi_2^{2,3} \\ \xi_3^{2,3} \end{pmatrix} \,. \tag{5.3.16}$$

Then from (4.3.15):

$$\gamma_1^{2,3}(t) = \gamma_1^{2,3}(0) + \int_0^t \xi_1^{2,3}(\tau)d\tau \ ,$$

$$\gamma_2^{2,3}(t) = \gamma_2^{2,3}(0)\cos\left(\int_0^t \xi_1^{2,3}(\sigma)d\sigma\right) + \gamma_3^{2,3}(0)\sin\left(\int_0^t \xi_1^{2,3}(\sigma)d\sigma\right)$$

$$+ \int_0^t \xi_2^{2,3}(\tau)\cos\left(\int_\tau^t \xi_1^{2,3}(\sigma)d\sigma\right)d\tau + \int_0^t \xi_3^{2,3}(\tau)\sin\left(\int_\tau^t \xi_1^{2,3}(\sigma)d\sigma\right)d\tau \ ,$$

$$\gamma_3^{2,3}(t) = -\gamma_2^{2,3}(0)\sin\left(\int_0^t \xi_1^{2,3}(\sigma)d\sigma\right) + \gamma_3^{2,3}(0)\cos\left(\int_0^t \xi_1^{2,3}(\sigma)d\sigma\right)$$

$$- \int_0^t \xi_2^{2,3}(\tau)\sin\left(\int_\tau^t \xi_1^{2,3}(\sigma)d\sigma\right)d\tau + \int_0^t \xi_3^{2,3}(\tau)\cos\left(\int_\tau^t \xi_1^{2,3}(\sigma)d\sigma\right)d\tau \ . \tag{5.3.17}$$

From (5.2.24), (5.3.10) and (5.3.14):

$$\begin{aligned} det(A_2) &= -\gamma_3^{1,3}\sin\gamma_1^{1,2}(0) + \gamma_3^{1,2}(0)\sin\gamma_1^{1,3} \\ &= -\left[\gamma_3^{2,3} - \gamma_2^{1,2}(0)\sin\gamma_1^{2,3} + \gamma_3^{1,2}(0)\cos\gamma_1^{2,3}\right]\sin\gamma_1^{1,2}(0) \\ &\qquad\qquad + \gamma_3^{1,2}(0)\sin(\gamma_1^{1,2}(0) + \gamma_1^{2,3}) \\ &= y_{1,2}(0)\sin\gamma_1^{2,3} - \gamma_3^{2,3}\sin\gamma_1^{1,2}(0) \ . \end{aligned} \tag{5.3.18}$$

From (5.3.13):

$$\xi^1 = \begin{pmatrix} \xi_1^1 \\ \xi_2^1 \\ \xi_3^1 \end{pmatrix} = \frac{-1}{det(A_2)} \begin{pmatrix} 0 & \sin\gamma_1^{1,2}(0) & 0 \\ 0 & 0 & 0 \\ 0 & \gamma_3^{1,2}(0) & 0 \end{pmatrix} \begin{pmatrix} \xi_1^{2,3} \\ \xi_2^{2,3} \\ \xi_3^{2,3} \end{pmatrix} = \frac{-\xi_2^{2,3}}{det(A_2)} \begin{pmatrix} \sin\gamma_1^{1,2}(0) \\ 0 \\ \gamma_3^{1,2}(0) \end{pmatrix} \ . \tag{5.3.19}$$

Observe that $\xi_2^1 = 0$, as is expected from the nonholonomic constraints. Then from (4.3.14) we have

$$\dot\gamma^1 = \begin{pmatrix} \dot\gamma_1^1 \\ \dot\gamma_2^1 \\ \dot\gamma_3^1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \gamma_3^1 & 1 & 0 \\ -\gamma_2^1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \xi_1^1 \\ 0 \\ \xi_3^1 \end{pmatrix} \tag{5.3.20}$$

and from (4.3.15):

$$\gamma_1^1(t) = \gamma_1^1(0) + \int_0^t \xi_1^1(\tau) d\tau \ ,$$

$$\gamma_2^1(t) = \gamma_2^1(0) \cos\left(\int_0^t \xi_1^1(\sigma) d\sigma\right) + \gamma_3^1(0) \sin\left(\int_0^t \xi_1^1(\sigma) d\sigma\right)$$
$$+ \int_0^t \xi_3^1(\tau) \sin\left(\int_\tau^t \xi_1^1(\sigma) d\sigma\right) d\tau \ ,$$

$$\gamma_3^1(t) = -\gamma_2^1(0) \sin\left(\int_0^t \xi_1^1(\sigma) d\sigma\right) + \gamma_3^1(0) \cos\left(\int_0^t \xi_1^1(\sigma) d\sigma\right)$$
$$+ \int_0^t \xi_3^1(\tau) \cos\left(\int_\tau^t \xi_1^1(\sigma) d\sigma\right) d\tau \ . \tag{5.3.21}$$

Theorem 5.3.3

i) If $\xi^{1,2} = 0$ and $\gamma_1^{1,2} = 0$, the 2–module NVGT instantaneously *translates* along an axis perpendicular to platforms 1 and 2.

ii) If $\xi^{1,2} = 0$ and $\gamma_1^{1,2} \neq 0$, the 2–module NVGT instantaneously *rotates* around the intersection of the axes of platforms 1 and 2.

Proof

i) Let $\gamma_1^{1,2} = 0$. From (5.3.18) and (5.3.19):

$$\begin{aligned} \xi_1^1 &= 0 \ , \\ \xi_2^1 &= 0 \ , \\ \xi_3^1 &= -\frac{\xi_2^{2,3}}{\sin \gamma_1^{2,3}} \ . \end{aligned} \tag{5.3.22}$$

From (5.3.16):

$$\left.\begin{array}{l}\dot{\gamma}_1^1 = 0 \\ \dot{\gamma}_2^1 = 0 \\ \dot{\gamma}_3^1 = \xi_3^1\end{array}\right\} \Rightarrow \begin{array}{l}\gamma_1^1(t) = \gamma_1^1(0)\ , \\ \gamma_2^1(t) = \gamma_2^1(0)\ , \\ \gamma_3^1(t) = \gamma_3^1(0) + \int\limits_0^t \xi_3^1(\tau)d\tau\ .\end{array} \tag{5.3.23}$$

From (4.3.23):

$$\begin{pmatrix}\dot{x}_1 \\ \dot{y}_1 \\ \dot{\phi}_1\end{pmatrix} = \xi_3^1 \begin{pmatrix}-\sin\gamma_1^1(0) \\ \cos\gamma_1^1(0) \\ 0\end{pmatrix}\ . \tag{5.3.24}$$

Thus, platform 1 translates along the axis that passes through the point O_1 and is parallel to the vector $(-\sin\gamma_1^1(0)\ , \cos\gamma_1^1(0))$, i.e. perpendicular to the platform. This is a constant vector, thus the point O_1 traces a straight line. Moreover, since ϕ_1 is constant, the whole platform translates along this line.

ii) Let $\gamma_1^{1,2}(t) = \gamma_1^{1,2}(0) \neq 0$. The *instantaneous center of rotation (ICR)* of the velocity distribution of equation (5.3.19) (Bottema & Roth [1979]) can be proven to be the point $O_{1,2}$, where the axes of platform 1 and 2 intersect. To see this, assume that the ICR has coordinates (x_P^1, y_P^1) with respect to the fixed world coordinate system and $(x_P^{1,2}, y_P^{1,2})$ with respect to the moving coordinate system of platform 1. Define $x_P = (x_P^1\ y_P^1\ 1)^\top$ and $X_P = (x_P^{1,2}\ y_P^{1,2}\ 1)^\top$. By its definition, $\dot{X}_P = 0$. Then:

$$x_P = g_1 X_P \ \Rightarrow\ \dot{x}_P = \dot{g}_1 X_P + g_1 \dot{X}_P = g_1 \xi_1 X_P\ . \tag{5.3.25}$$

The ICR is defined as the point where $\dot{x}_P = 0$. From this and (5.3.25) we get:

$$\xi_1 X_P = 0 \ \Rightarrow\ x_P^{1,2} = -\frac{\gamma_3^{1,2}(0)}{\sin\gamma_1^{1,2}(0)}\ ,\ y_P^{1,2} = 0\ . \tag{5.3.26}$$

From fig. 5.3.2, it is easy to see that this is point $O_{1,2}$, the intersection of the axes of platform 1 and 2.

Furthermore, it is possible to show that the ICR is constant, not only with respect to the coordinate system of platform 1, as is immediately evident from the expressions for $(x_P^{1,2}, y_P^{1,2})$ in (5.3.26), but also with respect to the world coordinate system. Therefore, the motion of the system, is indeed a rotation around this point. To see this, consider the vector x_P as a function of time t and expand in Taylor series around a fixed time instant t_0. Then, defining $\Delta t = t - t_0$, we get:

$$x_P(t) = x_P(t_0) + \frac{dx_P(t_0)}{dt}\Delta t + \frac{1}{2!}\frac{d^2x_P(t_0)}{dt^2}\Delta t^2 + \dots \ .$$

By definition: $\frac{dx_P}{dt} = \dot{g}_1 X_P + g_1 \dot{X}_P = g_1 \xi_1 X_P = 0$. Moreover, from (5.3.19) and (5.3.26):

$$\frac{d^2x_P}{dt^2} = \ddot{g}_1 X_P + \dot{g}_1 \dot{X}_P + \dot{g}_1 \dot{X}_P + g_1 \ddot{X}_P = \ddot{g}_1 X_P = (\dot{g}_1 \xi_1 + g_1 \dot{\xi}_1) X_P = 0 \ .$$

Similarly, all the higher derivatives of x_P are zero under the given velocity distribution. Thus the ICR is a constant point.

∎

We attempt to specify the global motion of the NVGT assembly induced by the shape controls, as characterized by the position and orientation γ^1 of platform 1. We saw that instantaneously this motion is a translation whenever $\gamma_1^{1,2} = 0$ or a rotation whenever $\gamma_1^{1,2} \neq 0$. We want to find out if, after a period T of the shape controls, there is a net motion $\Delta\gamma^1 \stackrel{\text{def}}{=} \gamma^1(T) - \gamma^1(0)$ of the NVGT assembly. This is equivalent to the idea of using *geometric phases* (Krishnaprasad [1990]; Marsden, Montgomery & Ratiu [1990]).

Proposition 5.3.4 (Geometric Phase for Piecewise Constant Shape Controls)

Consider the 2–module NVGT actuation scheme where the shape of the $(1,2)$–module is kept constant ($\xi_{1,2} = 0$) and the shape of the $(2,3)$–module traces the rectangular path of fig. 5.3.6.a in $(x_{2,3}, \phi_{2,3})$–space.

In the case of "straight–line" motion ($\gamma_1^{1,2} = 0$), if the shape–space path is traced clockwise, after a period of the shape controls, the assembly translates forward by

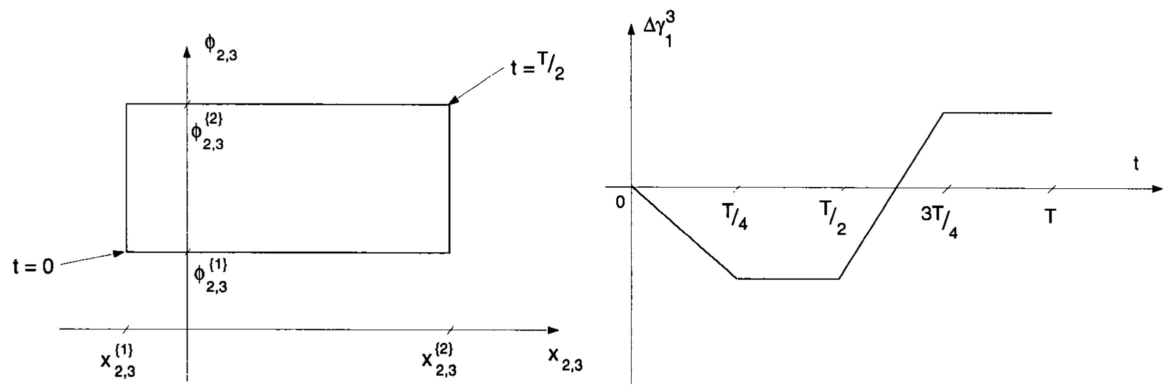
$$\Delta\gamma_3^1(T) = (x_{2,3}^{\{2\}} - x_{2,3}^{\{1\}})\left(\frac{1}{\tan\phi_{2,3}^{\{1\}}} - \frac{1}{\tan\phi_{2,3}^{\{2\}}}\right) ,$$

where $x_{2,3}^{\{1\}}, x_{2,3}^{\{2\}}$ and $\phi_{2,3}^{\{1\}}, \phi_{2,3}^{\{2\}}$ are the extreme values of $x_{2,3}$ and $\phi_{2,3}$ (we assume here that $x_{2,3}^{\{1\}} < x_{2,3}^{\{2\}}$) and $\phi_{2,3}^{\{1\}} < \phi_{2,3}^{\{2\}}$. If the shape–space path is traced counterclockwise, the assembly translates backwards by the same amount.

In the case of "turning" motion ($\gamma_1^{1,2} \neq 0$), after a period of the shape controls, the assembly rotates by

$$\Delta\gamma_1^1(T) = -\frac{\cos\phi_{2,3}^{\{1\}}}{\sin\phi_{2,3}^{\{1\}}}\ln\left|\frac{A_1 + B_1\frac{T}{4}}{A_1}\right| - \frac{\cos\phi_{2,3}^{\{2\}}}{\sin\phi_{2,3}^{\{2\}}}\ln\left|\frac{A_2 - B_2\frac{3T}{4}}{A_2 - B_2\frac{T}{2}}\right|,$$

where A_1, B_1, A_2, B_2 are defined in the proof below.



(a) (b)

Fig. 5.3.6: Piecewise Constant Controls and Geometric Phase for Translating 2–module NVGT

Proof

Consider the following *piecewise constant* periodic shape variations for the second module (let T be the corresponding period):

$$\dot\phi_{2,3}(t) = \begin{cases} 0, & \text{for } t \in [0, \frac{T}{4}]; \\ \beta, & \text{for } t \in [\frac{T}{4}, \frac{T}{2}]; \\ 0, & \text{for } t \in [\frac{T}{2}, \frac{3T}{4}]; \\ -\beta, & \text{for } t \in [\frac{3T}{4}, T]; \end{cases}, \quad \dot x_{2,3}(t) = \begin{cases} \alpha, & \text{for } t \in [0, \frac{T}{4}]; \\ 0, & \text{for } t \in [\frac{T}{4}, \frac{T}{2}]; \\ -\alpha, & \text{for } t \in [\frac{T}{2}, \frac{3T}{4}]; \\ 0, & \text{for } t \in [\frac{3T}{4}, T]; \end{cases}.$$

This makes the shape trace (counterclockwise) a rectangular path in $(x_{2,3}, \phi_{2,3})$–space. Let $x_{2,3}^{\{1\}}, x_{2,3}^{\{2\}}$ and $\phi_{2,3}^{\{1\}}, \phi_{2,3}^{\{2\}}$ be the extreme values of $x_{2,3}$ and $\phi_{2,3}$. Then $\alpha = \frac{4}{T}(x_{2,3}^{\{2\}} - x_{2,3}^{\{1\}})$ and $\beta = \frac{4}{T}(\phi_{2,3}^{\{2\}} - \phi_{2,3}^{\{1\}})$. Those shape variations correspond to the following shape controls:

$$\xi_1^{2,3}(t) = \dot{\phi}_{2,3} \ ,$$

$$\xi_2^{2,3}(t) = \dot{x}_{2,3} \cos\phi_{2,3} = \begin{cases} \alpha\cos\phi_{2,3}^{\{1\}}, & \text{for } t \in [0, \frac{T}{4}]; \\ 0, & \text{for } t \in [\frac{T}{4}, \frac{T}{2}]; \\ -\alpha\cos\phi_{2,3}^{\{2\}}, & \text{for } t \in [\frac{T}{2}, \frac{3T}{4}]; \\ 0, & \text{for } t \in [\frac{3T}{4}, T]; \end{cases} ,$$

$$\xi_3^{2,3}(t) = -\dot{x}_{2,3} \sin\phi_{2,3} = \begin{cases} -\alpha\sin\phi_{2,3}^{\{1\}}, & \text{for } t \in [0, \frac{T}{4}]; \\ 0, & \text{for } t \in [\frac{T}{4}, \frac{T}{2}]; \\ \alpha\sin\phi_{2,3}^{\{2\}}, & \text{for } t \in [\frac{T}{2}, \frac{3T}{4}]; \\ 0, & \text{for } t \in [\frac{3T}{4}, T]; \end{cases} .$$

The corresponding Wei–Norman parameters for the $(2,3)$–module are given by (5.3.17) as follows:

$$\gamma_1^{2,3}(t) = \gamma_1^{2,3}(0) + \int_0^t \xi_1^{2,3}(\tau)d\tau = \begin{cases} \gamma_1^{2,3}(0) = \phi_{2,3}^{\{1\}}, & \text{for } t \in [0, \frac{T}{4}]; \\ \gamma_1^{2,3}(0) + \beta(t - \frac{T}{4}), & \text{for } t \in [\frac{T}{4}, \frac{T}{2}]; \\ \gamma_1^{2,3}(0) + \beta\frac{T}{4} = \phi_{2,3}^{\{2\}}, & \text{for } t \in [\frac{T}{2}, \frac{3T}{4}]; \\ \gamma_1^{2,3}(0) + \beta(T - t), & \text{for } t \in [\frac{3T}{4}, T]; \end{cases} .$$

Let $\Delta\gamma_1^{2,3}(t) \overset{\text{def}}{=} \int_0^t \xi_1^{2,3}(\tau)d\tau$. Then from (5.3.17) we get:

$$\begin{aligned} \gamma_3^{2,3}(t) = & -\gamma_2^{2,3}(0)\sin\left(\Delta\gamma_1^{2,3}(t)\right) + \gamma_3^{2,3}(0)\cos\left(\Delta\gamma_1^{2,3}(t)\right) \\ & - \int_0^t \xi_2^{2,3}(\tau)\sin\left(\Delta\gamma_1^{2,3}(t) - \Delta\gamma_1^{2,3}(\tau)\right)d\tau \\ & + \int_0^t \xi_3^{2,3}(\tau)\cos\left(\Delta\gamma_1^{2,3}(t) - \Delta\gamma_1^{2,3}(\tau)\right)d\tau \end{aligned}$$

$$= \begin{cases} \gamma_3^{2,3}(0) - \alpha\sin\phi_{2,3}^{\{1\}} t, \\ \qquad \text{for } t \in [0, \frac{T}{4}]; \\ -\gamma_2^{2,3}(0)\sin\left(\beta(t - \frac{T}{4})\right) + \gamma_3^{2,3}(0)\cos\left(\beta(t - \frac{T}{4})\right) - \alpha\sin\phi_{2,3}^{\{1\}}\frac{T}{4}, \\ \qquad \text{for } t \in [\frac{T}{4}, \frac{T}{2}]; \\ -\gamma_2^{2,3}(0)\sin(\beta\frac{T}{4}) + \gamma_3^{2,3}(0)\cos(\beta\frac{T}{4}) - \alpha\sin\phi_{2,3}^{\{1\}}\frac{T}{4} + \alpha\sin\phi_{2,3}^{\{2\}}(t - \frac{T}{2}), \\ \qquad \text{for } t \in [\frac{T}{2}, \frac{3T}{4}]; \\ -\gamma_2^{2,3}(0)\sin(\beta(T - t)) + \gamma_3^{2,3}(0)\cos(\beta(T - t)) + \alpha\left(\sin\phi_{2,3}^{\{2\}} - \sin\phi_{2,3}^{\{1\}}\right)\frac{T}{4}, \\ \qquad \text{for } t \in [\frac{3T}{4}, T]; \end{cases}$$

In the case that the system translates (i.e. $\gamma_1^{1,2} = 0$), we get from (5.3.22) and (5.3.23):

$$\Delta\gamma_3^1(t) = \gamma_3^1(t) - \gamma_3^1(0) = \int\limits_0^t \xi_3^1(\tau)d\tau = -\int\limits_0^t \frac{\xi_2^{2,3}(\tau)}{\sin\gamma_1^{2,3}(\tau)}d\tau$$

$$= \begin{cases} -\alpha\frac{\cos\phi_{2,3}^{\{1\}}}{\sin\phi_{2,3}^{\{1\}}}t, & \text{for } t \in [0, \frac{T}{4}]; \\ -\alpha\frac{\cos\phi_{2,3}^{\{1\}}}{\sin\phi_{2,3}^{\{1\}}}\frac{T}{4}, & \text{for } t \in [\frac{T}{4}, \frac{T}{2}]; \\ -\alpha\frac{\cos\phi_{2,3}^{\{1\}}}{\sin\phi_{2,3}^{\{1\}}}\frac{T}{4} + \alpha\frac{\cos\phi_{2,3}^{\{2\}}}{\sin\phi_{2,3}^{\{2\}}}(t - \frac{T}{2}), & \text{for } t \in [\frac{T}{2}, \frac{3T}{4}]; \\ \alpha\left(\frac{\cos\phi_{2,3}^{\{2\}}}{\sin\phi_{2,3}^{\{2\}}} - \frac{\cos\phi_{2,3}^{\{1\}}}{\sin\phi_{2,3}^{\{1\}}}\right)\frac{T}{4}, & \text{for } t \in [\frac{3T}{4}, T]; \end{cases}$$

Then, after a period of the shape controls, the assembly has translated (backwards, if the relationship between the extreme values of $x_{2,3}$ and $\phi_{2,3}$ is as shown in fig. 5.3.6.b) by

$$\Delta\gamma_3^1(T) = \alpha\left(\frac{\cos\phi_{2,3}^{\{2\}}}{\sin\phi_{2,3}^{\{2\}}} - \frac{\cos\phi_{2,3}^{\{1\}}}{\sin\phi_{2,3}^{\{1\}}}\right)\frac{T}{4} = (x_{2,3}^{\{2\}} - x_{2,3}^{\{1\}})\left(\frac{1}{\tan\phi_{2,3}^{\{2\}}} - \frac{1}{\tan\phi_{2,3}^{\{1\}}}\right).$$

If we trace the shape–space loop in reverse (clockwise), we get the following global motion:

$$\Delta\gamma_3^1(t) = \begin{cases} 0, & \text{for } t \in [0, \frac{T}{4}]; \\ -\alpha\frac{\cos\phi_{2,3}^{\{2\}}}{\sin\phi_{2,3}^{\{2\}}}(t - \frac{T}{4}), & \text{for } t \in [\frac{T}{4}, \frac{T}{2}]; \\ -\alpha\frac{\cos\phi_{2,3}^{\{2\}}}{\sin\phi_{2,3}^{\{2\}}}\frac{T}{4}, & \text{for } t \in [\frac{T}{2}, \frac{3T}{4}]; \\ -\alpha\frac{\cos\phi_{2,3}^{\{2\}}}{\sin\phi_{2,3}^{\{2\}}}\frac{T}{4} + \alpha\frac{\cos\phi_{2,3}^{\{1\}}}{\sin\phi_{2,3}^{\{1\}}}(t - \frac{3T}{4}), & \text{for } t \in [\frac{3T}{4}, T]; \end{cases}$$

Then, after a period of the shape controls, the assembly has translated by

$$\Delta\gamma_3^1(T) = -\alpha\left(\frac{\cos\phi_{2,3}^{\{2\}}}{\sin\phi_{2,3}^{\{2\}}} - \frac{\cos\phi_{2,3}^{\{1\}}}{\sin\phi_{2,3}^{\{1\}}}\right)\frac{T}{4} = -(x_{2,3}^{\{2\}} - x_{2,3}^{\{1\}})\left(\frac{1}{\tan\phi_{2,3}^{\{2\}}} - \frac{1}{\tan\phi_{2,3}^{\{1\}}}\right),$$

which is the opposite from before. Thus, if we trace the shape–space loop in the reverse direction (clockwise), the system translates in the opposite direction from before (forward).

Consider now the case $g_1^{1,2} \neq 0$, when the 2–module NVGT assembly will rotate around the intersection of the axes of the two platforms of the $(1,2)$–module.

$$\Delta\gamma_1^1(t) = \gamma_1^1(t) - \gamma_1^1(0) = \int_0^t \xi_1^1(\tau)d\tau = -\sin\gamma_1^{1,2}(0)\int_0^t \frac{\xi_2^{2,3}(\tau)}{det(A_2(\tau))}d\tau$$

$$= \begin{cases} -\frac{\cos\phi_{2,3}^{\{1\}}}{\sin\phi_{2,3}^{\{1\}}}\ln\left|\frac{A_1+B_1 t}{A_1}\right|, & \text{for } t\in[0,\frac{T}{4}]; \\ -\frac{\cos\phi_{2,3}^{\{1\}}}{\sin\phi_{2,3}^{\{1\}}}\ln\left|\frac{A_1+B_1\frac{T}{4}}{A_1}\right|, & \text{for } t\in[\frac{T}{4},\frac{T}{2}]; \\ -\frac{\cos\phi_{2,3}^{\{1\}}}{\sin\phi_{2,3}^{\{1\}}}\ln\left|\frac{A_1+B_1\frac{T}{4}}{A_1}\right| - \frac{\cos\phi_{2,3}^{\{2\}}}{\sin\phi_{2,3}^{\{2\}}}\ln\left|\frac{A_2-B_2 t}{A_2-B_2\frac{T}{2}}\right|, & \text{for } t\in[\frac{T}{2},\frac{3T}{4}]; \\ -\frac{\cos\phi_{2,3}^{\{1\}}}{\sin\phi_{2,3}^{\{1\}}}\ln\left|\frac{A_1+B_1\frac{T}{4}}{A_1}\right| - \frac{\cos\phi_{2,3}^{\{2\}}}{\sin\phi_{2,3}^{\{2\}}}\ln\left|\frac{A_2-B_2\frac{3T}{4}}{A_2-B_2\frac{T}{2}}\right|, & \text{for } t\in[\frac{3T}{4},T]; \end{cases} \quad . \tag{5.3.27}$$

where $A_1 = y_{1,2}(0)\sin\gamma_1^{2,3}(0) - \sin\gamma_1^{1,2}(0)\gamma_3^{2,3}(0)$, $B_1 = \alpha\sin\gamma_1^{1,2}(0)\sin\phi_{2,3}^{\{1\}}$, $A_2 = y_{1,2}(0)\sin\left(\gamma_1^{2,3}(0)+\beta\frac{T}{4}\right) - \sin\gamma_1^{1,2}(0)\left[-\gamma_2^{2,3}(0)\sin(\beta\frac{T}{4})\gamma_3^{2,3}(0)\cos(\beta\frac{T}{4}) - \alpha\sin\phi_{2,3}^{\{1\}}\frac{T}{4} - \alpha\sin\phi_{2,3}^{\{2\}}\frac{T}{2}\right]$ and $B_2 = \alpha\sin\gamma_1^{1,2}(0)\sin\phi_{2,3}^{\{2\}}$. Then, after a period of the shape controls, the assembly has rotated by

$$\Delta\gamma_1^1(T) = -\frac{\cos\phi_{2,3}^{\{1\}}}{\sin\phi_{2,3}^{\{1\}}}\ln\left|\frac{A_1+B_1\frac{T}{4}}{A_1}\right| - \frac{\cos\phi_{2,3}^{\{2\}}}{\sin\phi_{2,3}^{\{2\}}}\ln\left|\frac{A_2-B_2\frac{3T}{4}}{A_2-B_2\frac{T}{2}}\right| .$$

∎

Proposition 5.3.5 (Geometric Phase for Sinusoidal Shape Controls)

Consider the 2–module NVGT actuation scheme where the shape of the $(1,2)$–module is kept constant ($\xi_{1,2} = 0$) and the shape of the $(2,3)$–module traces an elliptical path in $(x_{2,3}, \phi_{2,3})$–space, given by the following sinusoidal controls:

$$\dot{\phi}_{2,3}(t) = \alpha_1\omega\cos\omega t\ ,\ \ \dot{x}_{2,3} = \alpha_2\omega\sin\omega t\ ,\ \ \dot{y}_{2,3} = 0\ . \tag{5.3.28}$$

In the case of "straight-line" motion ($\gamma_1^{1,2} = 0$), after a period T of the shape controls, the assembly translates forward by

$$\Delta\gamma_3^1(T) = -\alpha_2 \int_0^t \frac{\omega \sin\omega\tau}{\tan(\alpha_3 + \alpha_1 \sin\omega\tau)} d\tau \ ,$$

where $\omega = \frac{2\pi}{T}$ (c.f. fig. 5.3.7). If the shape–space path is traced in reverse, the assembly translates backwards by the same amount.

In the case of "turning" motion ($\gamma_1^{1,2} \neq 0$), after a period T of the shape controls, the assembly rotates by

$$\Delta\gamma_1^1(T) = -\sin\gamma_1^{1,2}(0) \int_0^t \frac{\xi_2^{2,3}(\tau)}{det(A_2(\tau))} d\tau$$

(c.f. fig. 5.3.8). If the shape–space path is traced in reverse, the assembly rotates backwards by the same amount.

Proof

Consider the following *sinusoidal* periodic shape controls $\xi_{2,3}$ that correspond to those of equation (5.3.28):

$$\begin{aligned}
\xi_1^{2,3}(t) &= \alpha_1\omega\cos\omega t \ , \\
\xi_2^{2,3}(t) &= \alpha_2\omega\sin\omega t\cos\gamma_1^{2,3} \ , \\
\xi_3^{2,3}(t) &= -\alpha_2\omega\sin\omega t\sin\gamma_1^{2,3} \ .
\end{aligned} \tag{5.3.29}$$

From (5.3.17):

$$\begin{aligned}
\gamma_1^{2,3}(t) &= \gamma_1^{2,3}(0) + \alpha_1\sin\omega t \ , \\
\gamma_2^{2,3}(t) &= \gamma_2^{2,3}(0)\cos(\alpha_1\sin\omega t) + \gamma_3^{2,3}(0)\sin(\alpha_1\sin\omega t) + \alpha_2(1-\cos\omega t)\cos\gamma_1^{2,3} \ , \\
\gamma_3^{2,3}(t) &= -\gamma_2^{2,3}(0)\sin(\alpha_1\sin\omega t) + \gamma_3^{2,3}(0)\cos(\alpha_1\sin\omega t) - \alpha_2(1-\cos\omega t)\sin\gamma_1^{2,3} \ .
\end{aligned} \tag{5.3.30}$$

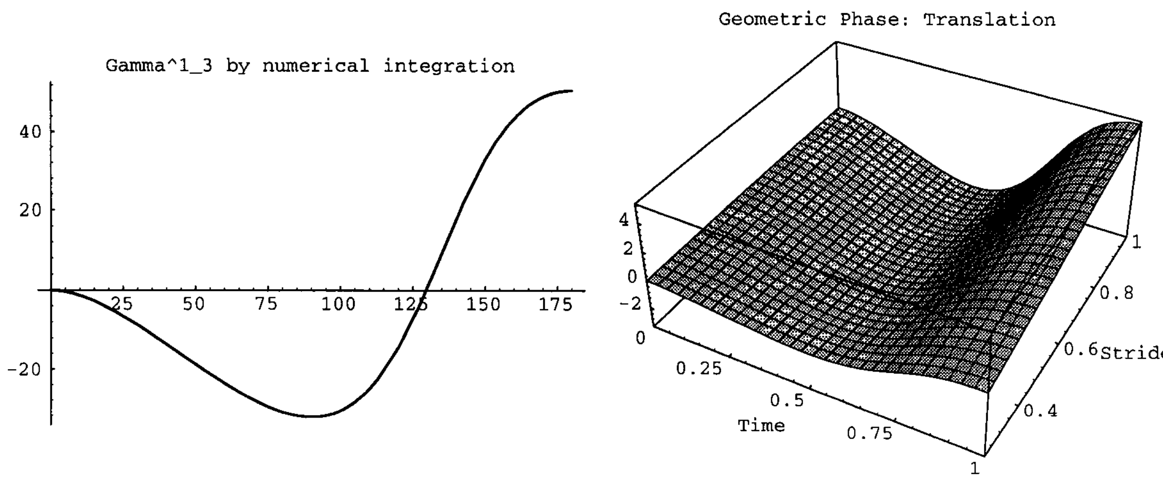
From (4.3.21) and (5.3.30) we see that those shape controls correspond to a closed elliptical path in $(x_{2,3}, \phi_{2,3})$–space.

Let $\gamma_1^{1,2} = \gamma_1^{1,2}(0) = 0$ so that the 2–module NVGT assembly translates.

From (5.3.22) and (5.3.23):

$$
\begin{aligned}
&\Delta\gamma_1^1(t) = \gamma_1^1(t) - \gamma_1^1(0) = 0 \,, \\
&\Delta\gamma_2^1(t) = \gamma_2^1(t) - \gamma_2^1(0) = 0 \,, \\
&\Delta\gamma_3^1(t) = \gamma_3^1(t) - \gamma_3^1(0) = \int_0^t \xi_3^1(\tau)d\tau = -\alpha_2 \int_0^t \frac{\omega \sin \omega\tau}{\tan(\alpha_3 + \alpha_1 \sin \omega\tau)} d\tau \,,
\end{aligned}
\tag{5.3.31}
$$

where $\alpha_3 \stackrel{\text{def}}{=} \gamma_1^{2,3}(0)$.



(a) (b)

Fig. 5.3.7: Geometric Phase for Translating 2–module NVGT (Sinusoidal Controls)

From (4.3.21) and (5.3.31) we have:

$$
\begin{aligned}
\phi_1 &= \gamma_1^1(0) \,, \\
x_1 &= \gamma_2^1(0) \cos \gamma_1^1(0) - \gamma_3^1 \sin \gamma_1^1(0) \,, \\
y_1 &= \gamma_2^1(0) \sin \gamma_1^1(0) + \gamma_3^1 \cos \gamma_1^1(0) \,.
\end{aligned}
\tag{5.3.32}
$$

From this we get:

$$
y_1 = \frac{\gamma_2^1(0)}{\sin \gamma_1^1(0)} - \frac{1}{\tan \gamma_1^1(0)} x_1 \,. \tag{5.3.33}
$$

Thus the locus of the point O_1 is the straight line given by equation (5.3.33), which is perpendicular to the axis of platform 1. Using Mathematica, we can integrate (5.3.31) numerically and verify that after a period of the shape controls, the 2–module NVGT assembly has moved forward by a distance specified by $\Delta\gamma_3^1(\frac{2\pi}{\omega}) = \gamma_3^1(\frac{2\pi}{\omega}) - \gamma_3^1(0)$ (fig. 5.3.7). If we trace the closed shape–space path of (5.3.29) in the reverse direction, the assembly will move backwards by the same distance.

Let $\gamma_1^{1,2} = \gamma_1^{1,2}(0) \neq 0$, so that the 2–module NVGT assembly rotates instantaneously around the point $O_{1,2}$. The position of the assembly with respect to this point can be characterized by the angle γ_1^1.

From (5.2.24) and (5.3.30):

$$\begin{aligned}
\gamma_1^{1,3}(t) &= \gamma_1^{1,2} + \gamma_1^{2,3} = \gamma_1^{1,2}(0) + \gamma_1^{2,3}(0) + \alpha_1 \sin\omega t \,, \\
\gamma_3^{1,3}(t) &= \gamma_3^{2,3} - \gamma_2^{1,2}\sin\gamma_1^{2,3} + \gamma_3^{1,2}\cos\gamma_1^{2,3} \\
&= -\gamma_2^{2,3}(0)\sin(\alpha_1\sin\omega t) + \gamma_3^{2,3}(0)\cos(\alpha_1\sin\omega t) \\
&\qquad - \left[\gamma_2^{1,2}(0) + \alpha_2(1-\cos\omega t)\right]\sin\left(\gamma_1^{2,3}(0) + \alpha_1\sin\omega t\right) \\
&\qquad + \gamma_3^{1,2}(0)\cos\left(\gamma_1^{2,3}(0) + \alpha_1\sin\omega t\right) \,.
\end{aligned} \tag{5.3.34}$$

Then, from (5.3.18):

$$det(A_2(t)) = -\gamma_3^{1,3}(t)\sin\gamma_1^{1,2}(0) + \gamma_3^{1,2}(0)\sin\gamma_1^{1,3}(t) \,. \tag{5.3.35}$$

From(5.3.19):

$$\xi_1^1(t) = -\sin\gamma_1^{1,2}(0)\frac{\xi_2^{2,3}(t)}{det(A_2(t))} \tag{5.3.36}$$

and from (5.3.21):

$$\Delta\gamma_1^1(t) = \gamma_1^1(t) - \gamma_1^1(0) = \int_0^t \xi_1^1(\tau)d\tau = -\sin\gamma_1^{1,2}(0)\int_0^t \frac{\xi_2^{2,3}(\tau)}{det(A_2(\tau))}d\tau \,. \tag{5.3.37}$$

Using Mathematica, we can integrate (5.3.37) numerically and verify that for e.g. $\gamma_1^{1,2} = -\frac{\pi}{4}$, after a period of the shape controls, the 2–module NVGT assembly rotates clockwise around the point $O_{1,2}$ by an angle specified by $\Delta\gamma_1^1(\frac{2\pi}{\omega}) = \gamma_1^1(\frac{2\pi}{\omega}) - \gamma_1^1(0)$ (fig. 5.3.8). If we trace the closed shape–space path of (26) in the reverse direction, the assembly will rotate counter–clockwise by the same angle.

∎

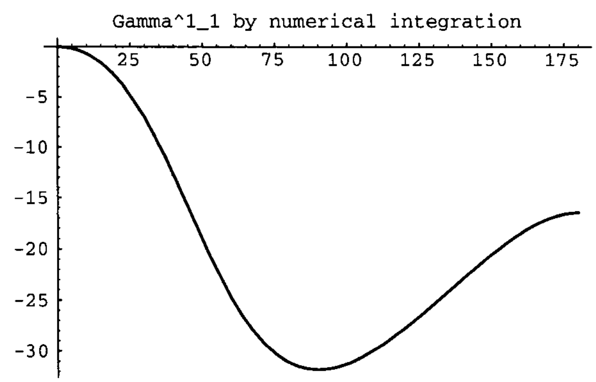


Fig. 5.3.8: Geometric Phase for Rotating 2–module NVGT (Sinusoidal Controls)

The kinematics of the 2–module NVGT assembly were simulated on Silicon Graphics IRIS 4D/120 and Indigo 2 graphics workstations. The primitive "straight–line" and "turning" motions described above are shown in fig. 5.3.10. They describe only the average motion of the system, however, they can be very useful in *motion planning*, since these primitive motions can be synthesized to display more complex behaviors of the system, like obstacle avoidance (fig. 5.3.11).

Standard motion planning strategies (Voronoi diagrams, cell decomposition, potential fields) can be used for this task (Latombe [1991]), but they will result in paths that may require sharp changes of direction for our system. Those paths could be made smoother, as was done in fig. 5.3.11, however this is not necessary. Notice that the 2–module NVGT can execute very sharp turns, even rotate (on the average) in place. Thus no constraints on the curvature of the average path of the system need to be placed. Moreover, by properly adjusting the size of the shape–space loop, the "stride length", as it is quantified by the geometric phase, can be varied. If greater accuracy of motion is needed, then smaller "steps" can be taken (c.f. related results in (Leonard [1994])).

Suppose then that we want to move from $g_1 = g_1(x_1, y_1, \phi_1) \in SE(2)$ to $\bar{g}_1 = \bar{g}_1(\bar{x}_1, \bar{y}_1, \bar{\phi}_1) \in SE(2)$, corresponding to points O_1 and $\bar{O}_1$ respectively. This can be done by synthesizing the primitive "straight–line" and "turning" motions described above in three steps, as follows:

i) First, "turn" in place (i.e. move with $\gamma_1^{1,2} = \frac{\pi}{2}$) from the initial orientation ϕ_1 to an orientation along the line $O_1\bar{O}_1$ (fig. 5.3.9).

ii) Then, "move straight" (i.e. with $\gamma_1^{1,2} = 0$) along the line $O_1\bar{O}_1$ from point O_1 to the desired point $\bar{O}_1$.

iii) Finally, "turn" in place until the final orientation $\bar{\phi}_1$ is reached.

Since controllability has been established for $SE(2)$, we know that any such motions are possible, as long as we stay away from nonholonomic singularities. Thus, the shape variations used to implement the above primitive motions, should be designed so that this happens.

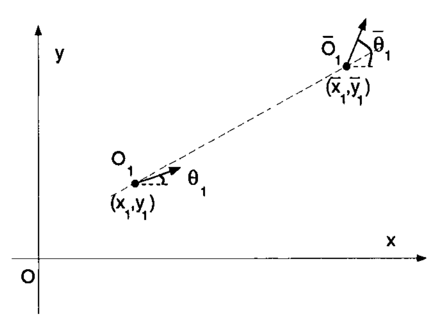
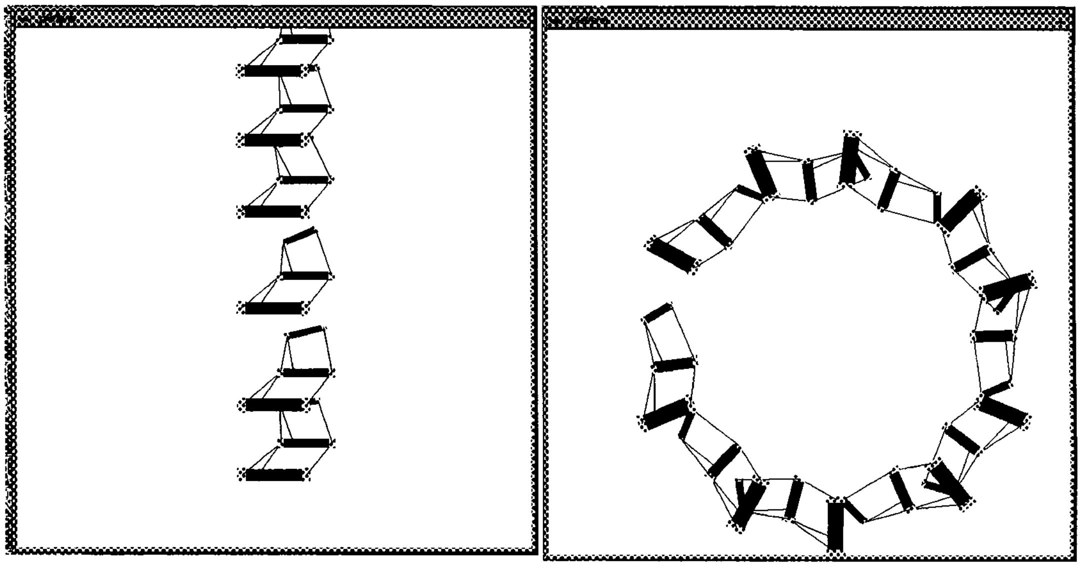


Fig. 5.3.9: Motion Planning for 2–module NVGT moving on the plane



(a) "Straight–line" (b) "Turning"

Fig. 5.3.10: Primitive motions of the 2–module NVGT

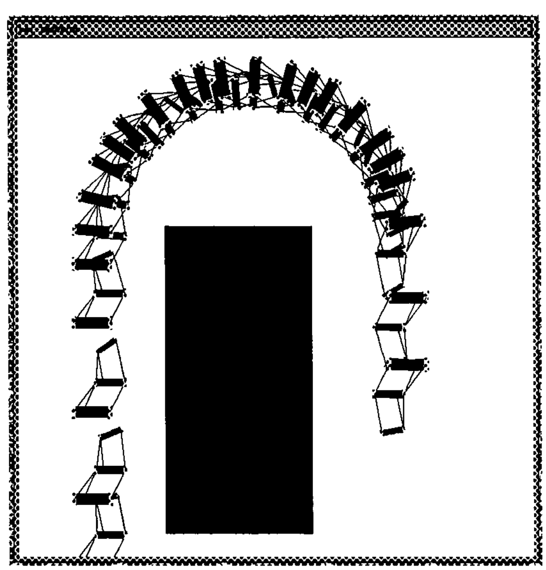


Fig. 5.3.11: Obstacle Avoidance with the 2–module NVGT

5.4 SE(2)–Spiders, SE(2)–Rings and other SE(2)–Snake Kinematic Topologies

The framework presented in section 4.2 is not restricted to nodes in serial (chain) arrangement. It can be applied to non–serial tree–like and ring–like arrangements of nodes.

In section 5.4.1 we analyze the case of a *tree*–structured assembly in $SE(2)$. We refer to those assemblies as $SE(2)$*–Spiders.* In section 5.4.2 we analyze the case of a *ring*–structured assembly in $SE(2)$. We refer to those assemblies as $SE(2)$*–Rings.* After developing their kinematics, we show how motion control for such assemblies can be reduced to the corresponding problem for the $SE(2)$–Snake (NVGT) that was discussed in section 5.3.2. In section 5.4.3 we consider the case of a 1–module $SE(2)$–Snake assembly, where the Codimension 1 Constraint Hypothesis introduced in section 4.2.2 is relaxed so that one of the nodes is constrained by one nonholonomic constraint as usual, but the other one is constrained by two such constraints. This provides us enough constraints to allow the expression of the global velocity of the assembly as a function of the shape and the shape variations, which was not possible previously, as discussed in section 5.2.2.

5.4.1 The 3–module SE(2)–Spider

Consider the 4–node assembly of fig. 5.4.1 consisting of one central node connected by linear actuators in a planar parallel manipulator configuration with each of the 3 other nodes. These nodes are subject to one nonholonomic constraint per node, due to idler wheels.

The trajectory of each node is a curve $g_i \subset G = SE(2)$, while its velocity is a curve $\xi_i \subset \mathcal{G} = se(2)$, such that:

$$\dot{g}_i = T_e L_{g_i} \cdot \xi_i = g_i \xi_i \ , \ i = 1, \ldots, 4. \tag{5.4.1}$$

A pair of nodes i and j of the chain constitutes the (i,j)–th module and its shape $g_{i,j} \subset G$ and shape variation $\xi_{i,j} \subset \mathcal{G}$ are defined by:

$$g_{i,j} = g_i^{-1} g_j \ , \tag{5.4.2}$$

and

$$\dot{g}_{i,j} = T_e L_{g_{i,j}} \cdot \xi_{i,j} = g_{i,j} \xi_{i,j} \ . \tag{5.4.3}$$

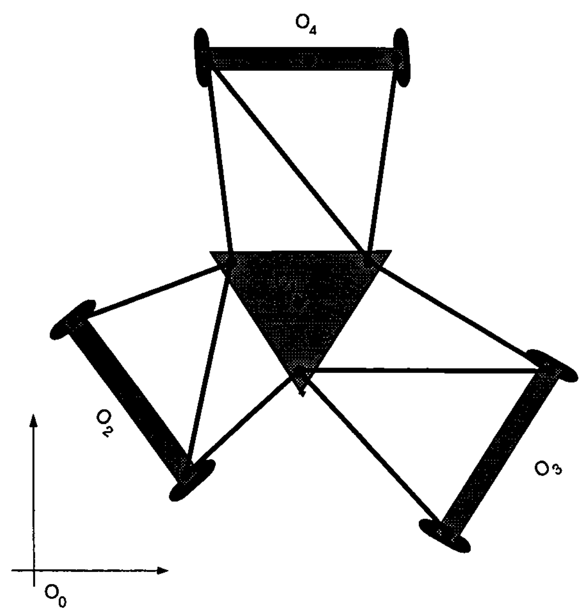


Fig. 5.4.1: The 3–module $SE(2)$–Spider

From this, we can easily see that

$$g_i = g_1 g_{1,j} \tag{5.4.4}$$

and

$$\xi_i = \xi_{1,i} + Ad_{g_{1,i}^{-1}} \xi_1 \,, \quad \text{for } i = 2, 3, 4 \,. \tag{5.4.5}$$

The system is subject to the following 3 nonholonomic constraints:

$$\begin{aligned} \mathcal{A}_2^\flat(\xi_2) &= 0 \,, \\ \mathcal{A}_2^\flat(\xi_3) &= 0 \,, \\ \mathcal{A}_2^\flat(\xi_4) &= 0 \,. \end{aligned} \tag{5.4.6}$$

Consider the composite velocity vector of the system $\zeta = (\xi^{1^\top} \xi^{1,2^\top} \xi^{1,3^\top} \xi^{1,4^\top})^\top$ and let the global velocity of the assembly be characterized by ξ_1, while the shape variations $\xi_{1,2}, \xi_{1,3}, \xi_{1,4}$ of the 3 modules are the system's controls.

The constraints (5.4.6) can be put in matrix form as:

$$A(g_{1,2}, g_{1,3}, g_{1,4})\, \zeta = 0 \,, \tag{5.4.7}$$

where $A = (A_2\,,\, A_1)$ with

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

and

$$A_2 = \begin{pmatrix} -\gamma_3^{1,2} & \cos\gamma_1^{1,2} & \sin\gamma_1^{1,2} \\ -\gamma_3^{1,3} & \cos\gamma_1^{1,3} & \sin\gamma_1^{1,3} \\ -\gamma_3^{1,4} & \cos\gamma_1^{1,4} & \sin\gamma_1^{1,4} \end{pmatrix} .$$

Then, the composite velocity vector can be partitioned as $\zeta = \binom{\zeta_1}{\zeta_2}$, with $\zeta_1 = (\,\xi^{1,2\top}\;\xi^{1,3\top}\;\xi^{1,4\top}\,)^\top$ and $\zeta_2 = \xi^1$. Then, when A_2 is nonsingular, the constraints (5.4.6) take the form:

$$\zeta_2 = \begin{pmatrix} \xi_1^1 \\ \xi_2^1 \\ \xi_3^1 \end{pmatrix} = -A_2^{-1} A_1 \zeta_1 = -A_2^{-1} \begin{pmatrix} \xi_2^{1,2} \\ \xi_2^{1,3} \\ \xi_2^{1,4} \end{pmatrix} . \qquad (5.4.8)$$

It is easy to see that A_2 is singular whenever the axes of platforms $2, 3$ and 4 intersect at one point or are parallel. Motion control for this system will use periodic $\xi_2^{1,2}, \xi_2^{1,3}$ and $\xi_2^{1,4}$ in generating appropriate motions of the system.

However, a simpler strategy would be to reduce the motion control for the 3–module SE(2)–spider, to that of motion control for the 2–module NVGT that was examined in detail in the previous section. To do this, we will characterize the global motion of the system by ξ_2, not ξ_1, while regarding $\xi_{2,3}$ and $\xi_{3,4}$ as the shape controls. The real shape controls $\xi_2^{1,2}, \xi_2^{1,3}$ will be used to implement the desired variation of $\xi_{2,3}$ and $\xi_{3,4}$.

From the system's kinematics we have

$$\begin{aligned} g_3 &= g_2 g_{2,3}\,, \\ g_4 &= g_3 g_{3,4} = g_2 g_{2,3} g_{3,4}\,, \\ g_{2,3} &= g_2^{-1} g_1 g_1^{-1} g_3 = g_{1,2}^{-1} g_{1,3}\,, \\ g_{3,4} &= g_3^{-1} g_1 g_1^{-1} g_4 = g_{1,3}^{-1} g_{1,4}\,, \\ g_{2,4} &= g_{2,3} g_{3,4}\,. \end{aligned} \qquad (5.4.9)$$

Then:

$$[\mathcal{A}_1 \, , \, \mathcal{A}_2] = \mathcal{A}_3 \, , \, [\mathcal{A}_1 \, , \, \mathcal{A}_3] = 0 \, , \, [\mathcal{A}_2 \, , \, \mathcal{A}_3] = 0 \, . \tag{4.3.29}$$

Proposition 4.3.4

The algebra $\mathcal{G} = h(3)$ is nilpotent (thus solvable) and, from Proposition 2.3.14, any $g \in G = H(3)$ has a *global* Wei–Norman representation of the form (4.3.3) with

$$\begin{pmatrix} \dot{\gamma}_1 \\ \dot{\gamma}_2 \\ \dot{\gamma}_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\gamma_2 & 0 & 1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} \, . \tag{4.3.30}$$

Equation (4.3.30) is solvable by quadratures:

$$\begin{aligned}
\gamma_1(t) &= \gamma_1(0) + \int_0^t \xi_1(\tau) d\tau \, , \\
\gamma_2(t) &= \gamma_2(0) + \int_0^t \xi_2(\tau) d\tau \, , \\
\gamma_3(t) &= \gamma_3(0) - \int_0^t \gamma_2(\tau)\xi_1(\tau) d\tau + \int_0^t \xi_3(\tau) d\tau \\
&= \gamma_3(0) - \gamma_2(0) \int_0^t \xi_1(\sigma) d\sigma - \int_0^t \xi_1(\tau) \left(\int_0^\tau \xi_2(\sigma) d\sigma \right) d\tau + \int_0^t \xi_3(\tau) d\tau \, .
\end{aligned} \tag{4.3.31}$$

Proof

Since $\mathcal{G} = h(3)$ is nilpotent, the existence of a global representation is immediate by (Wei & Norman [1964]). To see that it has the form (4.3.3) with coefficients given by (4.3.30) and (4.3.31), we compute the RHS of (4.3.5), using (4.3.29) and (2.3.20):

$$\begin{aligned}
& e^{ad(-\gamma_2 \mathcal{A}_2)} \mathcal{A}_1 = \mathcal{A}_1 + \gamma_2 \mathcal{A}_3 \, , \\
& e^{ad(-\gamma_3 \mathcal{A}_3)} e^{ad(-\gamma_2 \mathcal{A}_2)} \mathcal{A}_1 = \mathcal{A}_1 + \gamma_2 \mathcal{A}_3 \, , \\
& e^{ad(-\gamma_3 \mathcal{A}_3)} \mathcal{A}_2 = \mathcal{A}_2 \, .
\end{aligned} \tag{4.3.32}$$

From (4.3.1), (4.3.2), (4.3.5) and (4.3.32), we have:

$$\begin{aligned}
\xi &= \xi_1 \mathcal{A}_1 + \xi_2 \mathcal{A}_2 + \xi_3 \mathcal{A}_3 \\
&= \dot{\gamma}_1 (\mathcal{A}_1 + \gamma_2 \mathcal{A}_3) + \dot{\gamma}_2 \mathcal{A}_2 + \dot{\gamma}_3 \mathcal{A}_3 \, .
\end{aligned}$$

The corresponding shape variations are

$$\xi_3 = \xi_{2,3} + Ad_{g^{-1}_{3,4}} \xi_3 \,, \tag{5.4.10}$$

$$\xi_4 = \xi_{3,4} + Ad_{g^{-1}_{3,4}} \xi_3 = \xi_{3,4} + Ad_{g^{-1}_{3,4}} \xi_{2,3} + Ad_{g^{-1}_{2,4}} \xi_2 \,, \tag{5.4.11}$$

$$\xi_{2,3} = \xi_{1,3} - Ad_{g^{-1}_{1,2} g_{1,3}{}^{-1}} \xi_{1,2} \,, \tag{5.4.12}$$

$$\xi_{3,4} = \xi_{1,4} - Ad_{g^{-1}_{1,3}} \xi_{1,3} \,. \tag{5.4.13}$$

The nonholonomic constraints (5.4.6) can be expressed, using (5.4.10) and (5.4.11) as

$$A(g_{2,3}, g_{3,4})\,\zeta = 0 \,, \tag{5.4.14}$$

where the composite velocity vector is now $\zeta = (\xi^{2\top}\, \xi^{2,3\top}\, \xi^{3,4\top})^\top$ and $A = (A_2\,,\, A_1)$ with

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -\gamma_3^{3,4} & \cos\gamma_1^{3,4} & \sin\gamma_1^{3,4} & 0 & 1 & 0 \end{pmatrix}$$

and

$$A_2 = \begin{pmatrix} 0 & 1 & 0 \\ -\gamma_3^{2,3} & \cos\gamma_1^{2,3} & \sin\gamma_1^{2,3} \\ -\gamma_3^{2,4} & \cos\gamma_1^{2,4} & \sin\gamma_1^{2,4} \end{pmatrix} .$$

Then, the composite velocity vector can be partitioned as $\zeta = \binom{\zeta_1}{\zeta_2}$, with $\zeta_1 = (\xi^{2,3\top}\, \xi^{3,4\top})^\top$ and $\zeta_2 = \xi^2$. Then, when A_2 is nonsingular, the constraints (5.4.14) take the form:

$$\zeta_2 = -A_2^{-1} A_1 \zeta_1 \,. \tag{5.4.15}$$

The motion control strategies that were devised for the 2–module NVGT can be applied to this system in terms of periodic controls for $\xi_{2,3}$ and $\xi_{3,4}$. They can be implemented by solving (5.4.12) and (5.4.13) for $\xi_{1,2}, \xi_{1,3}$ and $\xi_{1,4}$. This can be done by e.g. setting $\xi_{1,3} = 0$, when

$$\xi_{1,2} = -Ad_{g_{2,3}} \xi_{2,3} \,,$$

$$\xi_{1,4} = \xi_{3,4} \,.$$

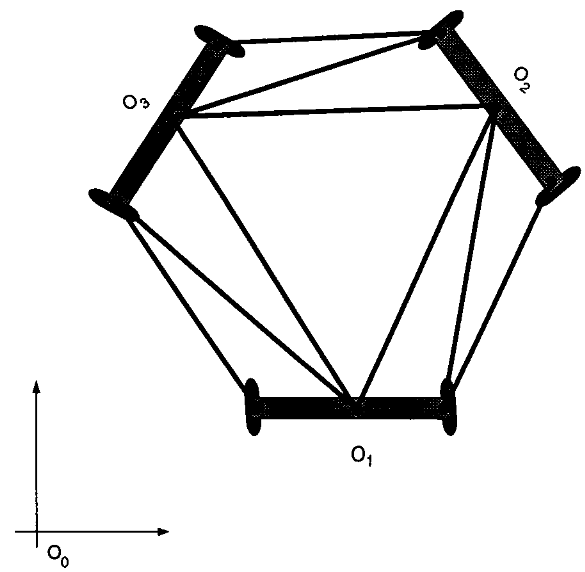


Fig. 5.4.2: The 3–module $SE(2)$–Ring

5.4.2 The 3–module SE(2)–Ring

Consider the 3–node ring assembly of fig. 5.4.2. Each node is connected with both of the others in a planar parallel manipulator configuration and is subject to a nonholonomic constraint coming from idler wheels rolling–without–slipping on the plane supporting the assembly.

From the system kinematics we get:

$$\begin{aligned} g_2 &= g_1 g_{1,2} , \\ g_3 &= g_2 g_{2,3} . \end{aligned} \tag{5.4.16}$$

From the loop equation we have:

$$g_{1,3} = g_{1,2} g_{2,3} . \tag{5.4.17}$$

The corresponding velocities are

$$\begin{aligned} \xi_2 &= \xi_{1,2} + Ad_{g_{1,2}^{-1}} \xi_1 , \\ \xi_3 &= \xi_{2,3} + Ad_{g_{2,3}^{-1}} \xi_2 = \xi_{2,3} + Ad_{g_{2,3}^{-1}} \xi_{1,2} + Ad_{(g_{1,2} g_{2,3})^{-1}} \xi_1 . \end{aligned} \tag{5.4.18}$$

Furthermore, the shape variations should satisfy the holonomic constraint coming from the loop equations:

$$\xi_{1,3} = \xi_{2,3} + Ad_{g_{2,3}^{-1}}\xi_{1,2} \,. \tag{5.4.19}$$

The nonholonomic constraints of rolling–without–slipping can be expressed as:

$$\mathcal{A}_2^\flat(\xi_i) = 0 \,, \quad \text{for } i = 1, 2, 3. \tag{5.4.20}$$

The nonholonomic constraints (5.4.20) can be expressed, using (5.4.18) as

$$A(g_{1,2}, g_{2,3})\,\zeta = 0 \,, \tag{5.4.21}$$

where the composite velocity vector is now $\zeta = (\,{\xi^1}^\top\, {\xi^{1,2}}^\top\, {\xi^{2,3}}^\top\,)^\top$ and $A = (A_2\,,\, A_1)$ with

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -\gamma_3^{2,3} & \cos\gamma_1^{2,3} & \sin\gamma_1^{2,3} & 0 & 1 & 0 \end{pmatrix}$$

and

$$A_2 = \begin{pmatrix} 0 & 1 & 0 \\ -\gamma_3^{1,2} & \cos\gamma_1^{1,2} & \sin\gamma_1^{1,2} \\ -\gamma_3^{1,3} & \cos\gamma_1^{1,3} & \sin\gamma_1^{1,3} \end{pmatrix} .$$

Then, the composite velocity vector can be partitioned as $\zeta = \binom{\zeta_1}{\zeta_2}$, with $\zeta_1 = (\,{\xi^{1,2}}^\top\, {\xi^{2,3}}^\top\,)^\top$ and $\zeta_2 = \xi^1$. Then, when A_2 is nonsingular, the constraints (5.4.21) take the form:

$$\zeta_2 = -A_2^{-1}A_1\zeta_1 \,. \tag{5.4.22}$$

It is easy to see that A_2 is singular whenever the axes of all 3 platforms are parallel or intersect at one point. Motion control strategies for the 2–module NVGT in terms of periodic controls for $\xi_{1,2}$ and $\xi_{2,3}$ can be applied to this system. However, now $\xi_{1,3}$ will have to be adjusted using (5.4.19) to account for the loop constraint.

5.4.3 The 1–module SE(2)–Snake with more than one constraint per node

In section 4.2.2 we introduced the Codimension 1 Constraint Hypothesis, according to which, every node of a G–Snake was constrained to evolve on an $(n-1)$–dimensional subspace h of the Lie algebra $\mathcal{G}$, which, in addition, was not a subalgebra of $\mathcal{G}$. The constraints $\xi \in h$ could then be expressed as $\mathcal{A}_k^\flat(\xi) = 0$, for some $\mathcal{A}_k^\flat \in \mathcal{G}^*$, such that $h = \mathrm{Ker}(\mathcal{A}_k^\flat)$.

In this section we examine how this hypothesis can be relaxed, by allowing some nodes of the assembly to evolve on a subspace h of $\mathcal{G}$ of dimension $n - r$, for $r \geq 1$. Suppose $h = \mathrm{Ker}\big(\mathrm{sp}\{\mathcal{A}_{k_1}^\flat, \ldots, \mathcal{A}_{k_r}^\flat\}\big)$. Then the requirement that $\xi \in h$ can be expressed as a set of constraints

$$\mathcal{A}_{k_1}^\flat(\xi) = 0, \; \ldots \; , \mathcal{A}_{k_r}^\flat(\xi) = 0 \; ,$$

for a set of r linearly independent elements of $\mathcal{G}^*$. However, in this case, the system may have to evolve, not on the whole of G, but on a subspace of G.

As an example of this case, we consider a 1–module NVGT assembly where two platforms with idler wheels are connected with 3 legs in a planar parallel manipulator configuration (fig. 5.1.1). One platform (say platform 2) has two wheels that move independently of each other, so they provide a nonholonomic constraint from rolling–without–slipping exactly similar to the previous assemblies of this chapter. The other platform (platform 1) has two wheels which are joined by a shaft, which forces them to rotate simultaneously. Thus they are not independent of each other any more, but they are otherwise free to rotate. Such an arrangement prevents both sideways motion and rotation of the platform, which is only able to translate along the perpendicular to the axis of the wheels.

The system kinematics are now

$$g_2 = g_1 g_{1,2} \tag{5.4.23}$$

and

$$\xi_2 = \xi_{1,2} + Ad_{g_{1,2}^{-1}} \xi_1 \; . \tag{5.4.24}$$

The two nonholonomic constraints imposed by the motion of the wheels of platform 1 can be expressed as

$$\begin{aligned} \mathcal{A}_1^\flat(\xi_1) &= 0 \; , \\ \mathcal{A}_2^\flat(\xi_1) &= 0 \; , \end{aligned} \tag{5.4.25}$$

while the constraint imposed by platform 2 can be expressed as

$$\mathcal{A}_2^\flat(\xi_2) = 0\,. \tag{5.4.26}$$

Consider the composite velocity vector of the system $\zeta = (\xi^{1^\top}\ \xi^{1,2^\top})^\top$ and let the global velocity of the assembly be characterized by ξ_1, while the shape variations $\xi_{1,2}$ of the module are the system's controls.

The constraints (5.4.25) and (5.4.26) can be put in matrix form as:

$$A(g_{1,2})\,\zeta = 0\,, \tag{5.4.27}$$

where $A = (A_2\,,\, A_1)$ with

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

and

$$A_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\gamma_3^{1,2} & \cos\gamma_1^{1,2} & \sin\gamma_1^{1,2} \end{pmatrix}.$$

Then, the composite velocity vector can be partitioned as $\zeta = \binom{\zeta_1}{\zeta_2}$, with $\zeta_1 = \xi^{1,2}$ and $\zeta_2 = \xi^1$. The matrix A_2 is singular whenever $detA_2 = \sin\gamma_1^{1,2} = 0$. When A_2 is nonsingular, the constraints (5.4.27) take the form:

$$\zeta_2 = \begin{pmatrix} \xi_1^1 \\ \xi_2^1 \\ \xi_3^1 \end{pmatrix} = -A_2^{-1}A_1\zeta_1 = -\frac{\xi_2^{1,2}}{\sin\gamma_1^{1,2}}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\,. \tag{5.4.28}$$

Only $\xi_3^{1,2}$ is nonzero, as expected by the constraints, thus the assembly moves perpendicular to the axis of the first platform. Configurations outside this line cannot be reached.

Assume that the shape of the 1–module NVGT assembly traces a rectangular path in shape–space similar to the one described in Proposition 5.3.4, i.e.

$$\xi_1^{1,2} = \begin{cases} 0, & \text{for } t\in[0,\frac{T}{4}]; \\ \beta, & \text{for } t\in[\frac{T}{4},\frac{T}{2}]; \\ 0, & \text{for } t\in[\frac{T}{2},\frac{3T}{4}]; \\ -\beta, & \text{for } t\in[\frac{3T}{4},T]; \end{cases}\quad,\quad \xi_2^{1,2} = \begin{cases} \alpha\cos\phi_{1,2}^{\{1\}}, & \text{for } t\in[0,\frac{T}{4}]; \\ 0, & \text{for } t\in[\frac{T}{4},\frac{T}{2}]; \\ -\alpha\cos\phi_{1,2}^{\{2\}}, & \text{for } t\in[\frac{T}{2},\frac{3T}{4}]; \\ 0, & \text{for } t\in[\frac{3T}{4},T]; \end{cases}\,.$$

This makes the shape trace (counterclockwise) a rectangular path in $(x_{1,2}, \phi_{1,2})$–space. Let $x_{1,2}^{\{1\}}, x_{1,2}^{\{2\}}$ and $\phi_{1,2}^{\{1\}}, \phi_{1,2}^{\{2\}}$ be the extreme values of $x_{1,2}$ and $\phi_{1,2}$. Then $\alpha = \frac{4}{T}(x_{1,2}^{\{2\}} - x_{1,2}^{\{1\}})$ and $\beta = \frac{4}{T}(\phi_{1,2}^{\{2\}} - \phi_{1,2}^{\{1\}})$.

The corresponding Wei–Norman parameters for the $(1,2)$–module are given by (5.3.17) as follows:

$$\gamma_1^{1,2}(t) = \gamma_1^{1,2}(0) + \int_0^t \xi_1^{1,2}(\tau)d\tau = \begin{cases} \gamma_1^{1,2}(0) = \phi_{1,2}^{\{1\}}, & \text{for } t \in [0, \frac{T}{4}]; \\ \gamma_1^{1,2}(0) + \beta(t - \frac{T}{4}), & \text{for } t \in [\frac{T}{4}, \frac{T}{2}]; \\ \gamma_1^{1,2}(0) + \beta\frac{T}{4} = \phi_{1,2}^{\{2\}}, & \text{for } t \in [\frac{T}{2}, \frac{3T}{4}]; \\ \gamma_1^{1,2}(0) + \beta(T - t), & \text{for } t \in [\frac{3T}{4}, T]; \end{cases}.$$

Then we get from (5.4.28):

$$\Delta\gamma_3^1(t) = \gamma_3^1(t) - \gamma_3^1(0) = \int_0^t \xi_3^1(\tau)d\tau = -\int_0^t \frac{\xi_2^{1,2}(\tau)}{\sin\gamma_1^{1,2}(\tau)}d\tau$$

$$= \begin{cases} -\alpha\frac{\cos\phi_{1,2}^{\{1\}}}{\sin\phi_{1,2}^{\{1\}}}t, & \text{for } t \in [0, \frac{T}{4}]; \\ -\alpha\frac{\cos\phi_{1,2}^{\{1\}}}{\sin\phi_{1,2}^{\{1\}}}\frac{T}{4}, & \text{for } t \in [\frac{T}{4}, \frac{T}{2}]; \\ -\alpha\frac{\cos\phi_{1,2}^{\{1\}}}{\sin\phi_{1,2}^{\{1\}}}\frac{T}{4} + \alpha\frac{\cos\phi_{1,2}^{\{2\}}}{\sin\phi_{1,2}^{\{2\}}}(t - \frac{T}{2}), & \text{for } t \in [\frac{T}{2}, \frac{3T}{4}]; \\ \alpha\left(\frac{\cos\phi_{1,2}^{\{2\}}}{\sin\phi_{1,2}^{\{2\}}} - \frac{\cos\phi_{1,2}^{\{1\}}}{\sin\phi_{1,2}^{\{1\}}}\right)\frac{T}{4}, & \text{for } t \in [\frac{3T}{4}, T]; \end{cases}.$$

Then, after a period of the shape controls, the assembly has translated by

$$\Delta\gamma_3^1(T) = \alpha\left(\frac{\cos\phi_{1,2}^{\{2\}}}{\sin\phi_{1,2}^{\{2\}}} - \frac{\cos\phi_{1,2}^{\{1\}}}{\sin\phi_{1,2}^{\{1\}}}\right)\frac{T}{4} = (x_{1,2}^{\{2\}} - x_{1,2}^{\{1\}})\left(\frac{1}{\tan\phi_{1,2}^{\{2\}}} - \frac{1}{\tan\phi_{1,2}^{\{1\}}}\right).$$

If we trace the shape–space loop in reverse, the assembly moves in the opposite direction.

CHAPTER SIX

DYNAMICS OF AN SE(2)–SNAKE ASSEMBLY

6.1 Introduction

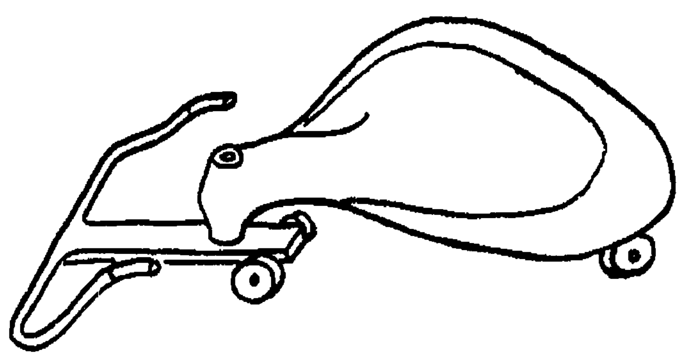


Fig. 6.1.1: The Roller Racer

The effect of *dynamics* on systems of the G–snake type that we examined *kinematically* in chapter 5, can be seen by considering the 1–module $SE(2)$–snake system of fig. 6.1.1. This system is sold commercially under the name "Roller Racer" and its design is based on a U.S. patent held by W. E. Hendricks [Patent # 3663038 of May 16, 1972]. The system consists of a seat, with a pair of idler wheels mounted in its rear and with an elongated steering arm pivoted to the front of the seat, with a pair of idler wheels also mounted on it, on the rear of the pivot axis. A rider provides the propulsion and steering mechanism of this vehicle, by swinging, with his arms and feet, the steering arm around the pivot axis and by leaning his body to the left and to the right, following the

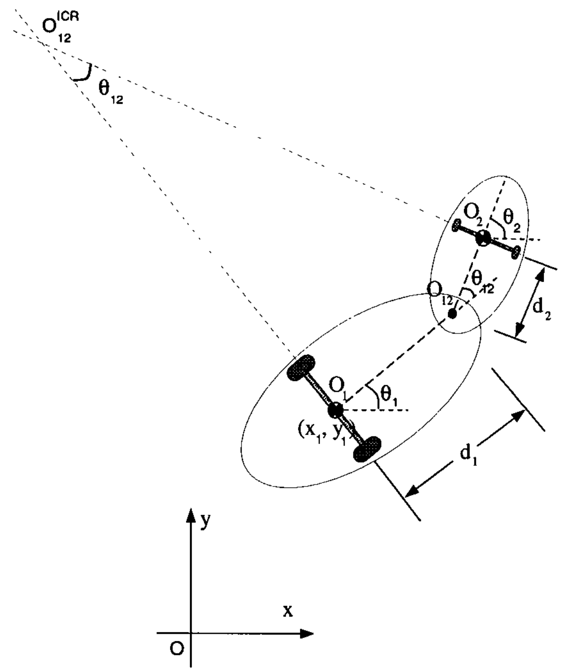


Fig. 6.1.2: The Roller Racer Model

swinging of the arm.

The model of fig. 6.1.2 will be used in our analysis. Two planar platforms with centers of mass (c.o.m.) located at points O_1 and O_2 are connected with a rotary joint at $O_{1,2}$. A pair of idler wheels is attached on each of the platforms, with the axis of the wheels perpendicular to the line connecting the c.o.m. with the joint. A coordinate frame centered at the c.o.m. and with its x–axis along the line $O_iO_{1,2}$ for $i = 1, 2$ connecting the c.o.m. with the joint, will be used to describe the configuration of each platform with respect to a global coordinate system at some reference point O. For simplicity, it will be assumed that the axis of the wheels passes through the c.o.m. of each platform.

The effects of the rider's body motion will be ignored at first approximation. Experiments with the Roller Racer show that, even though those body motions may amplify the resulting motion of the system, the fundamental means of its propulsion is the pivoting of the steering arm around the joint axis and the nonholonomic constraints coming from the wheels' rolling–without–slipping on the plane supporting the vehicle. In this respect, this system is very different from the Snakeboard, a variation of the skateboard, where the motion of the rider is essential for the propulsion of the system (Lewis et al.

[1994]). Riderless prototypes of the Roller Racer built at the Intelligent Servosystems Laboratory verified this. The propulsion and steering mechanism in these vehicles comes from a rotary (stepper) motor at the joint $O_{1,2}$, whose torque can be considered as the control of our system. As discussed in section 5.2, the purely kinematic analysis of such a system does not allow us to determine the global motion of the system by just the shape variations (the joint velocity in this case), since (unlike the 2-module case) it does not possess a sufficient number of nonholonomic constraints for this to happen (c.f. Remarks 5.2.6). Our goal here is to complement this kinematic analysis with the dynamics of the system, which will provide the necessary information. Thus, certain fundamental behaviors of the system ("straight–line" motion, "turning" motion) can be achieved by proper oscillatory relative motions of the two platforms. In both numerical simulations and experiments with prototypes, we observed such behaviors.

An alternative to the usual approach of solving the full Lagrange–d' Alembert equations of motion of the system is considered here. In (Bloch, Krishnaprasad, Marsden & Murray [1994]), the notion of the momentum map is examined for systems with nonholonomic constraints and symmetries and its evolution law, the momentum equation, is derived from the Lagrange–d' Alembert equations. By applying this method to the problem at hand, a useful decomposition of the equations of motion is obtained: Given a shape–space trajectory (which corresponds to the controls of our system), first we compute the nonholonomic momentum from the momentum equation. This only involves the solution of a linear ordinary differential equation. Subsequently, we use the momentum to reconstruct the group trajectory, which corresponds to the global motion of the system. The corresponding velocities depend linearly on the momentum. This process is very useful for the derivation of motion control laws for this system and can be extended to 1–module $SE(2)$–snakes with more general shape–changing mechanisms.

6.2 Kinematics of the Roller Racer

Let $g_i = \begin{pmatrix} \cos\theta_i & -\sin\theta_i & x_i \\ \sin\theta_i & \cos\theta_i & y_i \\ 0 & 0 & 1 \end{pmatrix} \in SE(2)$, for $i = 1, 2$, be the configuration of platform i with respect to the global coordinate frame at O, where x_i, y_i and θ_i are indicated in fig. 6.1.2. Let $g_{1,2} = \begin{pmatrix} \cos\theta_{1,2} & -\sin\theta_{1,2} & x_{1,2} \\ \sin\theta_{1,2} & \cos\theta_{1,2} & y_{1,2} \\ 0 & 0 & 1 \end{pmatrix} \in SE(2)$ be the configuration of platform 2 with respect to the coordinate frame of platform 1 at O_1. Because of the special structure of the joint, we have

$$x_{1,2} = d_1 + d_2 \cos\theta_{1,2} \ , \quad y_{1,2} = d_2 \sin\theta_{1,2} \ , \tag{6.2.1}$$

where d_i is the distance of O_i from the joint $O_{1,2}$. We consider non–negative d_1 and d_2. In fact, we assume $d_1 > 0$. However, we allow for the case $d_2 = 0$ and we examine it in detail.

Since the platforms form a kinematic chain, we have

$$g_2 = g_1 g_{1,2} \ , \tag{6.2.2}$$

thus

$$\begin{aligned} \theta_2 &= \theta_1 + \theta_{1,2} \ , \\ x_2 &= x_1 + x_{1,2}\cos\theta_1 - y_{1,2}\sin\theta_1 = x_1 + d_1\cos\theta_1 + d_2\cos\theta_2 \ , \\ y_2 &= y_1 + x_{1,2}\sin\theta_1 + y_{1,2}\cos\theta_1 = y_1 + d_1\sin\theta_1 + d_2\sin\theta_2 \ . \end{aligned} \tag{6.2.3}$$

The system kinematics are a special case of those of the VGT assembly, i.e. for $\xi_i = \begin{pmatrix} 0 & -\dot\theta_i & \Xi_1^i \\ \dot\theta_i & 0 & \Xi_2^i \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{G} = se(2)$ we have:

$$\dot g_i = g_i \xi_i \ , \ i = 1, 2 \tag{6.2.4}$$

and

$$\dot g_{1,2} = g_{1,2}\xi_{1,2} \ , \tag{6.2.5}$$

where $\xi_{1,2} = \dot\theta_{1,2} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & d_2 \\ 0 & 0 & 0 \end{pmatrix}$. Moreover, we can show that

$$\xi_2 = Ad_{g_{1,2}^{-1}}\xi_1 + \xi_{1,2} \ , \tag{6.2.6}$$

where

$$
\begin{aligned}
Ad_{g^{-1}}\mathcal{A}_1 &= \mathcal{A}_1 - \gamma_3\mathcal{A}_2 + \gamma_2\mathcal{A}_3 \ , \\
Ad_{g^{-1}}\mathcal{A}_2 &= \cos\gamma_1\mathcal{A}_2 - \sin\gamma_1\mathcal{A}_3 \ , \\
Ad_{g^{-1}}\mathcal{A}_3 &= \sin\gamma_1\mathcal{A}_2 + \cos\gamma_1\mathcal{A}_3 \ ,
\end{aligned} \tag{6.2.7}
$$

with γ_i the Wei–Norman parameters of g. Alternatively,

$$
\begin{aligned}
\dot\theta_2 &= \dot\theta_1 + \dot\theta_{1,2} \ , \\
\dot x_2 &= \dot x_1 + \dot x_{1,2}\cos\theta_1 - \dot y_{1,2}\sin\theta_1 - (x_{1,2}\sin\theta_1 + y_{1,2}\cos\theta_1)\dot\theta_1 \\
&= \dot x_1 - \dot\theta_1[d_1\sin\theta_1 + d_2\sin(\theta_1+\theta_{1,2})] - \dot\theta_{1,2}d_2\sin(\theta_1+\theta_{1,2}) \ , \\
\dot y_2 &= \dot y_1 + \dot x_{1,2}\sin\theta_1 + \dot y_{1,2}\cos\theta_1 + (x_{1,2}\cos\theta_1 - y_{1,2}\sin\theta_1)\dot\theta_1 \\
&= \dot y_1 + \dot\theta_1[d_1\cos\theta_1 + d_2\cos(\theta_1+\theta_{1,2})] + \dot\theta_{1,2}d_2\cos(\theta_1+\theta_{1,2}) \ .
\end{aligned} \tag{6.2.8}
$$

From (6.2.3) we see that the configuration space for the Roller Racer system is $Q = SE(2)\times S^1$. Consider the space $\mathcal{S} = S^1\times S^1$ and consider the projection $\pi : Q \to \mathcal{S}$. Let Q be parametrized by $(g_1(x_1,y_1,\theta_1),\theta_{1,2})$ with $g_1(x_1,y_1,\theta_1)\in G$ and $\mathcal{S}$ be parametrized by $(\theta_1,\theta_{1,2})$. Then $v_q\in T_qQ$ can be represented as $(\dot x_1,\dot y_1,\dot\theta_1,\dot\theta_{1,2})$. It can be easily seen that $T\pi$ is onto, thus π is a submersion. Thus, $\pi : Q\to\mathcal{S}$ is a bundle with vertical space

$$
V_q = \mathrm{Ker}(T_q\pi) = \left\{v_q\in T_qQ \,\middle|\, \dot\theta_1 = 0, \dot\theta_{1,2} = 0\right\} .
$$

The constraints specify an Ehresman connection (Bloch, Krishnaprasad, Marsden & Murray [1994]), (Marsden, Montgomery & Ratiu [1990]) on this bundle. By also considering the dynamics (or more precisely the momentum equation), we can synthesize a principal fiber bundle connection for this system.

Consider the usual bases $\{\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3\}$ for $\mathcal{G} = se(2)$ and $\{\mathcal{A}_1^\flat,\mathcal{A}_2^\flat,\mathcal{A}_3^\flat\}$ for its dual space $\mathcal{G}^*$. The nonholonomic constraints on the wheels of the two platforms can be expressed as:

$$
\xi_3^1 = \mathcal{A}_3^\flat(\xi_1) = -\dot x_1\sin\theta_1 + \dot y_1\cos\theta_1 = 0 \ , \tag{6.2.9}
$$

$$
\xi_3^2 = \mathcal{A}_3^\flat(\xi_2) = -\dot x_2\sin\theta_2 + \dot y_2\cos\theta_2 = 0 \ , \tag{6.2.10}
$$

From (6.2.8) and (6.2.10), we get

$$\xi_3^2 = \mathcal{A}_3^\flat(\xi_2) = -\dot{x}_1 \sin(\theta_1 + \theta_{1,2}) + \dot{y}_1 \cos(\theta_1 + \theta_{1,2}) + \dot{\theta}_1(d_1 \cos\theta_{1,2} + d_2) + \dot{\theta}_{1,2} d_2 = 0 \ . \tag{6.2.11}$$

Observe that for $d_2 = 0$, neither one of the constraints (6.2.9) and (6.2.11) involves $\dot{\theta}_{1,2}$. From (6.2.9) and (6.2.11), we get:

$$\xi_3^2 = \mathcal{A}_3^\flat(\xi_2) = -(\dot{x}_1 \cos\theta_1 + \dot{y}_1 \sin\theta_1) \sin\theta_{1,2} + (d_1 \cos\theta_{1,2} + d_2)\dot{\theta}_1 + d_2 \dot{\theta}_{1,2} = 0 \ . \tag{6.2.12}$$

Proposition 6.2.1

The nonholonomic constraints (6.2.9) and (6.2.11) are linearly independent for all $q \in Q$.

Proof

We can rewrite the constraints in the form:

$$\begin{pmatrix} -\sin\theta_1 & \cos\theta_1 & 0 & 0 \\ -\sin(\theta_1 + \theta_{1,2}) & \cos(\theta_1 + \theta_{1,2}) & d_1 \cos\theta_{1,2} + d_2 & d_2 \end{pmatrix} \begin{pmatrix} \dot{x}_1 \\ \dot{y}_1 \\ \dot{\theta}_1 \\ \dot{\theta}_{1,2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \ .$$

Consider the determinants of all possible 2×2 minors:

$$\begin{aligned}
\Delta_1 &= -\sin\theta_1 \cos(\theta_1 + \theta_{1,2}) + \cos\theta_1 \sin(\theta_1 + \theta_{1,2}) = \sin\theta_{1,2} \ , \\
\Delta_2 &= -\sin\theta_1 (d_1 \cos\theta_{1,2} + d_2) \ , \\
\Delta_3 &= -\sin\theta_1 \ d_2 \ , \\
\Delta_4 &= \cos\theta_1 (d_1 \cos\theta_{1,2} + d_2) \ , \\
\Delta_5 &= \cos\theta_1 \ d_2 \ , \\
\Delta_6 &= 0 \ .
\end{aligned}$$

When $d_2 \neq 0$, consider $\Delta_1, \ldots, \Delta_5$. If $\sin\theta_{1,2} \neq 0$, then $\Delta_1 \neq 0$ establishes linear independence. If $\sin\theta_{1,2} = 0$, then $\Delta_3 \neq 0$ or $\Delta_5 \neq 0$. When $d_2 = 0$, consider Δ_1, Δ_2 and Δ_4. If $\sin\theta_{1,2} \neq 0$, then $\Delta_1 \neq 0$. If $\sin\theta_{1,2} = 0$, then $\cos\theta_{1,2} \neq 0$ and we have either $\Delta_2 \neq 0$ or $\Delta_4 \neq 0$. Thus, the null space of this matrix is always of dimension 2.

∎

Define the *constraint one–forms:*

$$\omega_q^1 = -\sin\theta_1 dx_1 + \cos\theta_1 dy_1 \ ,$$

$$\omega_q^2 = -\sin(\theta_1+\theta_{1,2})dx_1 + \cos(\theta_1+\theta_{1,2})dy_1 + (d_1\cos\theta_{1,2}+d_2)d\theta_1 + d_2 d\theta_{1,2} \ . \tag{6.2.13}$$

The *constraint distribution* $\mathcal{D}_q$ is the subspace of T_qQ which is the intersection of the kernels of the constraint one–forms, i.e.

$$\mathcal{D}_q = \mathrm{Ker}\ \omega_q^1 \cap \mathrm{Ker}\ \omega_q^2 \ . \tag{6.2.14}$$

Since the constraints are linearly independent, we know that $\mathcal{D}_q$ is always 2–dimensional. Next, we will specify a basis for it.

Proposition 6.2.2

The constraint distribution $\mathcal{D}_q \subset T_qQ$ is

$$\mathcal{D}_q = \mathrm{sp}\{\xi^1_Q, \xi^2_Q\} \ , \tag{6.2.15}$$

where in the case $d_2 \neq 0$:

$$\begin{aligned} \xi_Q^1 &= d_2(\cos\theta_1\frac{\partial}{\partial x_1} + \sin\theta_1\frac{\partial}{\partial y_1}) + \sin\theta_{1,2}\frac{\partial}{\partial\theta_{1,2}} \ , \\ \xi_Q^2 &= d_2\frac{\partial}{\partial\theta_1} - (d_1\cos\theta_{1,2}+d_2)\frac{\partial}{\partial\theta_{1,2}} \ , \end{aligned} \tag{6.2.16}$$

while in the case $d_2 = 0$:

$$\begin{aligned} \xi_Q^1 &= d_1\cos\theta_{1,2}(\cos\theta_1\frac{\partial}{\partial x_1} + \sin\theta_1\frac{\partial}{\partial y_1}) + \sin\theta_{1,2}\frac{\partial}{\partial\theta_1} \ , \\ \xi_Q^2 &= \frac{\partial}{\partial\theta_{1,2}} \ . \end{aligned} \tag{6.2.17}$$

Proof

Let $X_q = v_1\frac{\partial}{\partial x_1} + v_2\frac{\partial}{\partial y_1} + v_3\frac{\partial}{\partial\theta_1} + v_4\frac{\partial}{\partial\theta_{1,2}} \in \mathcal{D}_q$, with $v_i \in I\!R$, be a vector field in the constraint distribution. By definition it should annihilate both constraint one–forms. Consider first the case $d_2 \neq 0$. From the constraints we have:

$$\omega_q^1(X_q) = -v_1\sin\theta_1 + v_2\cos\theta_1 = 0 \ ,$$

$$\omega_q^2(X_q) = -(v_1\cos\theta_1 + v_2\sin\theta_1)\sin\theta_{1,2} + (d_1\cos\theta_{1,2}+d_2)v_3 + d_2v_4 = 0 \ .$$

Let $v_1 = d_2 \cos\theta_1 v_5,\ v_2 = d_2 \sin\theta_1 v_5$ and $v_4 = \sin\theta_{1,2} v_5 - \frac{d_1 \cos\theta_{1,2} + d_2}{d_2} v_3$. Then:

$$X_q = v_5\big[d_2 \cos\theta_1 \frac{\partial}{\partial x_1} + d_2 \sin\theta_1 \frac{\partial}{\partial y_1} + \sin\theta_{1,2} \frac{\partial}{\partial \theta_{1,2}}\big]$$
$$+ v_3\big[d_2 \frac{\partial}{\partial \theta_1} - (d_1 \cos\theta_{1,2} + d_2) \frac{\partial}{\partial \theta_{1,2}}\big] ,$$

for arbitrary v_3, v_5, satisfies both constraints. The two vector fields are obviously linearly independent.

Consider the case $d_2 = 0$. The constraints now become:

$$\omega_q^1(X_q) = -v_1 \sin\theta_1 + v_2 \cos\theta_1 = 0 ,$$
$$\omega_q^2(X_q) = -(v_1 \cos\theta_1 + v_2 \sin\theta_1) \sin\theta_{1,2} + d_1 \cos\theta_{1,2} v_3 = 0 .$$

Let $v_1 = d_1 \cos\theta_1 v_5,\ v_2 = d_1 \sin\theta_1 v_5$ and $v_3 = \sin\theta_{1,2} v_5$. Then:

$$X_q = v_5\big[d_1 \cos\theta_1 \frac{\partial}{\partial x_1} + d_1 \sin\theta_1 \frac{\partial}{\partial y_1} + \sin\theta_{1,2} \frac{\partial}{\partial \theta_1}\big] + v_4 \frac{\partial}{\partial \theta_{1,2}} ,$$

for arbitrary v_4, v_5, satisfies both constraints. The two vector fields are obviously linearly independent.

∎

6.3 Symmetry of the Roller Racer

Consider now the effect of symmetries on this system. In particular, consider the action Φ of the group $G = SE(2)$ on the configuration space Q defined by:

$$\begin{aligned}\Phi :& G \times Q \to Q \\ & (g, (g_1, \theta_{1,2})) \mapsto (g g_1, \theta_{1,2}) \\ & ((b, c, a), (x_1, y_1, \theta_1, \theta_{1,2})) \mapsto (x_1 \cos a - y_1 \sin a + b, x_1 \sin a + y_1 \cos a + c, \theta_1 + a, \theta_{1,2}),\end{aligned} \tag{6.3.1}$$

where $g = g(b, c, a) \in G$. The tangent space at $q \in Q$ to the orbit of Φ is given by

$$T_q \mathrm{Orb}(q) = \mathrm{sp}\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial \theta_1}\} . \tag{6.3.2}$$

Notice that

$$\mathcal{D}_q + T_q\mathrm{Orb}(q) = T_qQ \ .$$

In (Bloch, Krishnaprasad, Marsden & Murray [1994]), this is referred to as the *principal* case. Our goal is to show that the nonholonomic constraints, together with a momentum equation, can specify a connection on the principal fiber bundle $Q \longrightarrow Q/G$.

The intersection $\mathcal{S}_q$ of $\mathcal{D}_q$ and $T_q\mathrm{Orb}(q)$ is non–trivial. Contrast this with the 3–node G–snake where $T_qQ = \mathcal{D}_q \oplus T_q\mathrm{Orb}(q)$, thus the intersection of $\mathcal{D}_q$ and $T_q\mathrm{Orb}(q)$ is trivial. We specify a basis for $\mathcal{S}_q$ as follows:

Proposition 6.3.1

Consider the intersection

$$\mathcal{S}_q = \mathcal{D}_q \cap T_q\mathrm{Orb}(q) \ .$$

In the case $d_1 \neq d_2$, the distribution $\mathcal{S}_q$ is 1–dimensional and is given by:

$$\mathcal{S}_q = \mathrm{sp}\{\xi^q_Q\} \ , \tag{6.3.3}$$

where

$$\xi^q_Q = (d_1\cos\theta_{1,2} + d_2)(\cos\theta_1\frac{\partial}{\partial x_1} + \sin\theta_1\frac{\partial}{\partial y_1}) + \sin\theta_{1,2}\frac{\partial}{\partial\theta_1} \ . \tag{6.3.4}$$

Proof

Consider $X_q \in \mathcal{S}_q = \mathcal{D}_q \cap T_q\mathrm{Orb}(q)$. Because $X_q \in \mathcal{D}_q$, we have $X_q = u_1\xi^1_Q + u_2\xi^2_Q$, for $u_i \in I\!R$. Because $X_q \in T_q\mathrm{Orb}(q)$, we have $X_q = v_1\frac{\partial}{\partial x_1} + v_2\frac{\partial}{\partial y_1} + v_3\frac{\partial}{\partial\theta_1}$, for $v_i \in I\!R$. In order for X_q to lie in the intersection of the two spaces, we should have:

$$u_1\xi^1_Q + u_2\xi^2_Q = v_1\frac{\partial}{\partial x_1} + v_2\frac{\partial}{\partial y_1} + v_3\frac{\partial}{\partial\theta_1} \ . \tag{6.3.5}$$

In the case $d_2 \neq 0$, we have from (6.2.16):

$$\begin{aligned}&[d_2(\cos\theta_1\frac{\partial}{\partial x_1} + \sin\theta_1\frac{\partial}{\partial y_1}) + \sin\theta_{1,2}\frac{\partial}{\partial\theta_{1,2}}]u_1 + [d_2\frac{\partial}{\partial\theta_1} - (d_1\cos\theta_{1,2} + d_2)\frac{\partial}{\partial\theta_{1,2}}]u_2\\ &\qquad = v_1\frac{\partial}{\partial x_1} + v_2\frac{\partial}{\partial y_1} + v_3\frac{\partial}{\partial\theta_1} \ .\end{aligned}$$

This corresponds to a system of four equations:

$$\begin{pmatrix} d_2\cos\theta_1 & 0 & -1 & 0 & 0 \\ d_2\sin\theta_1 & 0 & 0 & -1 & 0 \\ 0 & d_2 & 0 & 0 & -1 \\ \sin\theta_{1,2} & -(d_1\cos\theta_{1,2}+d_2) & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} .$$

When $d_1 \neq d_2$, the 4×5 matrix above is always of maximal rank, thus $\dim \mathcal{S}_q = 5 - 4 = 1$, for all $q \in Q$. Pick $u_1 = (d_1\cos\theta_{1,2} + d_2)u_5$ and $u_2 = \sin\theta_{1,2}u_5$. Then, $v_1 = d_2\cos\theta_1(d_1\cos\theta_{1,2}+d_2)u_5$, $v_2 = d_2\sin\theta_1(d_1\cos\theta_{1,2}+d_2)u_5$ and $v_3 = d_2\sin\theta_{1,2}u_5$. Thus

$$X_q = \big[(d_1\cos\theta_{1,2} + d_2)(\cos\theta_1\frac{\partial}{\partial x_1} + \sin\theta_1\frac{\partial}{\partial y_1}) + \sin\theta_{1,2}\frac{\partial}{\partial\theta_1}\big]d_2u_5 ,$$

for arbitrary u_5. Observe that when $d_1 \neq d_2$, the vector field X_q is nontrivial for all $q \in Q$.

In the case $d_2 = 0$, we have from (6.2.17):

$$\left[d_1\cos\theta_{1,2}(\cos\theta_1\frac{\partial}{\partial x_1}+\sin\theta_1\frac{\partial}{\partial y_1})+\sin\theta_{1,2}\frac{\partial}{\partial\theta_1}\right]u_1+u_2\frac{\partial}{\partial\theta_{1,2}} = v_1\frac{\partial}{\partial x_1}+v_2\frac{\partial}{\partial y_1}+v_3\frac{\partial}{\partial\theta_1} .$$

From this, we get:

$$\begin{aligned} u_2 &= 0 , \\ v_1 &= d_1\cos\theta_{1,2}\cos\theta_1 u_1 , \\ v_2 &= d_1\cos\theta_{1,2}\sin\theta_1 u_1 , \\ v_3 &= \sin\theta_{1,2}u_1 . \end{aligned}$$

Therefore,

$$X_q = \big[d_1\cos\theta_{1,2}(\cos\theta_1\frac{\partial}{\partial x_1} + \sin\theta_1\frac{\partial}{\partial y_1}) + \sin\theta_{1,2}\frac{\partial}{\partial\theta_1}\big]u_1 ,$$

for arbitrary u_1. Thus, $\mathcal{S}_q$ is again a 1–dimensional distribution.

The two cases can be unified in the expression (6.3.4).

∎

Proposition 6.3.2

For the action Φ, the infinitesimal generators corresponding to the basis elements of $\mathcal{G} = se(2)$, defined in (4.3.12), at the point $q \in Q$, are:

$$\begin{aligned} \mathcal{A}_{1\,Q}^{\;q} &= -y_1 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial y_1} + \frac{\partial}{\partial \theta_1} \,, \\ \mathcal{A}_{2\,Q}^{\;q} &= \frac{\partial}{\partial x_1} \,, \\ \mathcal{A}_{3\,Q}^{\;q} &= \frac{\partial}{\partial y_1} \,. \end{aligned} \tag{6.3.6}$$

The infinitesimal generator corresponding to $\xi^q = \xi_1 \mathcal{A}_1 + \xi_2 \mathcal{A}_2 + \xi_3 \mathcal{A}_3 \in \mathcal{G}$ is

$$\xi_Q^q = (\xi_2 - y_1 \xi_1) \frac{\partial}{\partial x_1} + (\xi_2 + x_1 \xi_1) \frac{\partial}{\partial y_1} + \xi_1 \frac{\partial}{\partial \theta_1} \,. \tag{6.3.7}$$

A given vector field $\xi_Q^q = v_1 \frac{\partial}{\partial x_1} + v_2 \frac{\partial}{\partial y_1} + v_3 \frac{\partial}{\partial \theta_1}$ can be considered as the infinitesimal generator of an element $\xi^q \in \mathcal{G} = se(2)$, under the action Φ. Then, ξ^q is:

$$\xi^q = v_3 \mathcal{A}_1 + (v_1 + y_1 v_3) \mathcal{A}_2 + (v_2 - x_1 v_3) \mathcal{A}_3 \,. \tag{6.3.8}$$

Proof

From (4.3.19) and (6.3.1):

$$\begin{aligned} \mathcal{A}_{1\,Q}^{\;q} &= \left.\frac{d}{dt}\right|_{t=0} \Phi(e^{t\mathcal{A}_1}, q) = \left.\frac{d}{dt}\right|_{t=0} (x_1 \cos t - y_1 \sin t, x_1 \sin t + y_1 \cos t, \theta_1 + t, \theta_{1,2}) \\ &= (-y_1, x_1, 1, 0) \longleftrightarrow -y_1 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial y_1} + \frac{\partial}{\partial \theta_1} \,. \end{aligned}$$

The infinitesimal generators for the other elements of the basis are specified similarly.

∎

The vector field ξ_Q^q in (6.3.4) corresponds to the following element ξ^q of $se(2)$:

$$\begin{aligned} \xi^q = \sin\theta_{1,2} \mathcal{A}_1 &+ [(d_1 \cos\theta_{1,2} + d_2) \cos\theta_1 + y_1 \sin\theta_{1,2}] \mathcal{A}_2 \\ &+ [(d_1 \cos\theta_{1,2} + d_2) \sin\theta_1 - x_1 \sin\theta_{1,2}] \mathcal{A}_3 \,. \end{aligned} \tag{6.3.9}$$

By differentiating (6.3.9), we get

$$\begin{aligned}\frac{d\xi^q}{dt} &= \cos\theta_{1,2}\dot\theta_{1,2}\mathcal{A}_1 \\ &+ [-d_1\sin\theta_{1,2}\cos\theta_1\dot\theta_{1,2} - (d_1\cos\theta_{1,2}+d_2)\sin\theta_1\dot\theta_1 + \dot y_1\sin\theta_{1,2} + y_1\cos\theta_{1,2}\dot\theta_{1,2}]\mathcal{A}_2 \\ &+ [-d_1\sin\theta_{1,2}\sin\theta_1\dot\theta_{1,2} + (d_1\cos\theta_{1,2}+d_2)\cos\theta_1\dot\theta_1 - \dot x_1\sin\theta_{1,2} - x_1\cos\theta_{1,2}\dot\theta_{1,2}]\mathcal{A}_3\ .\end{aligned} \tag{6.3.10}$$

The corresponding infinitesimal generator is

$$\begin{aligned}\left[\frac{d\xi^q}{dt}\right]_Q &= \big[-d_1\sin\theta_{1,2}\cos\theta_1\dot\theta_{1,2} - (d_1\cos\theta_{1,2}+d_2)\sin\theta_1\dot\theta_1 + \dot y_1\sin\theta_{1,2}\big]\frac{\partial}{\partial x_1} \\ &+ \big[-d_1\sin\theta_{1,2}\sin\theta_1\dot\theta_{1,2} + (d_1\cos\theta_{1,2}+d_2)\cos\theta_1\dot\theta_1 - \dot x_1\sin\theta_{1,2}\big]\frac{\partial}{\partial y_1} \\ &+ \cos\theta_{1,2}\dot\theta_{1,2}\frac{\partial}{\partial\theta_1}\ .\end{aligned} \tag{6.3.11}$$

6.4 Dynamics of the Roller Racer

Consider the Lagrangian:

$$\begin{aligned}L(\dot q) &= \frac{1}{2}m_1(\dot x_1^2+\dot y_1^2) + \frac{1}{2}I_{z_1}\dot\theta_1^2 + \frac{1}{2}I_{z_2}(\dot\theta_1+\dot\theta_{1,2})^2 \\ &= \frac{1}{2}\begin{pmatrix}\dot x_1 & \dot y_1 & \dot\theta_1 & \dot\theta_{1,2}\end{pmatrix}\begin{pmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_1 & 0 & 0 \\ 0 & 0 & I_{z_1}+I_{z_2} & I_{z_2} \\ 0 & 0 & I_{z_2} & I_{z_2}\end{pmatrix}\begin{pmatrix}\dot x_1 \\ \dot y_1 \\ \dot\theta_1 \\ \dot\theta_{1,2}\end{pmatrix} \\ &= \frac{1}{2}\dot q^\top \mathit{I\!I}_q \dot q\ ,\end{aligned} \tag{6.4.1}$$

for $q=(x_1,y_1,\theta_1,\theta_{1,2})$, where m_i and I_{z_i} is respectively the mass and moment of inertia of platform i. The choice of Lagrangian reflects our assumption that the mass and linear momentum of platform 2 are much smaller than those of platform 1 and can be ignored. However, the inertia of platform 2 is not ignored.

From (6.4.1):

$$\frac{\partial L}{\partial \dot{q}} = I\!I_q \dot{q} = \begin{pmatrix} m_1 \dot{x}_1 \\ m_1 \dot{y}_1 \\ (I_{z_1} + I_{z_2})\dot{\theta}_1 + I_{z_2}\dot{\theta}_{1,2} \\ I_{z_2}\dot{\theta}_1 + I_{z_2}\dot{\theta}_{1,2} \end{pmatrix} . \tag{6.4.2}$$

Proposition 6.4.1

The constraints (6.2.9) and (6.2.11) and the Lagrangian (6.4.1) are invariant under the action Φ given by (6.3.1).

Proof

Fix $g(b, c, a) \in G = SE(2)$. Consider $q = (x_1, y_1, \theta_1, \theta_{1,2}) \in Q$. Under the action Φ of G on Q, $\bar{q} = (\bar{x}_1, \bar{y}_1, \bar{\theta}_1, \bar{\theta}_{1,2}) = \Phi(g, q)$:

$$\bar{x}_1 \stackrel{\text{def}}{=} x_1 \cos a - y_1 \sin a + b \Longrightarrow \dot{\bar{x}}_1 = \dot{x}_1 \cos a - \dot{y}_1 \sin a \ ,$$

$$\bar{y}_1 \stackrel{\text{def}}{=} x_1 \sin a + y_1 \cos a + c \Longrightarrow \dot{\bar{y}}_1 = \dot{x}_1 \sin a + \dot{y}_1 \cos a \ ,$$

$$\bar{\theta}_1 \stackrel{\text{def}}{=} \theta_1 + a \Longrightarrow \dot{\bar{\theta}}_1 = \dot{\theta}_1 \ ,$$

$$\bar{\theta}_{1,2} \stackrel{\text{def}}{=} \theta_{1,2} \Longrightarrow \dot{\bar{\theta}}_{1,2} = \dot{\theta}_{1,2} \ .$$

Consider the effect of Φ on the constraints (6.2.9) and (6.2.11):

$$\begin{aligned}
&- \dot{\bar{x}}_1 \sin \bar{\theta}_1 + \dot{\bar{y}}_1 \cos \bar{\theta}_1 = \\
&\qquad = - \sin(\theta_1 + a)(\dot{x}_1 \cos a - \dot{y}_1 \sin a) + \cos(\theta_1 + a)(\dot{x}_1 \sin a + \dot{y}_1 \cos a) \\
&\qquad = \dot{x}_1 \sin \theta_1 + \dot{y}_1 \cos \theta_1 \ , \\
&- \dot{\bar{x}}_1 \sin(\bar{\theta}_1 + \bar{\theta}_{1,2}) + \dot{\bar{y}}_1 \cos(\bar{\theta}_1 + \bar{\theta}_{1,2}) + \dot{\bar{\theta}}_1 (d_1 \cos \bar{\theta}_{1,2} + d_2) + \dot{\bar{\theta}}_{1,2} d_2 = \\
&\qquad = - \sin(\theta_1 + \theta_{1,2} + a)(\dot{x}_1 \cos a - \dot{y}_1 \sin a) \\
&\qquad\qquad + \cos(\theta_1 + \theta_{1,2} + a)(\dot{x}_1 \sin a + \dot{y}_1 \cos a) \\
&\qquad\qquad\qquad + \dot{\theta}_1 (d_1 \cos \theta_{1,2} + d_2) + \dot{\theta}_{1,2} d_2 \\
&\qquad = -\dot{x}_1 \sin(\theta_1 + \theta_{1,2}) + \dot{y}_1 \cos(\theta_1 + \theta_{1,2}) + \dot{\theta}_1 (d_1 \cos \theta_{1,2} + d_2) + \dot{\theta}_{1,2} d_2 \ .
\end{aligned}$$

Consider the effect of Φ on the Lagrangian (6.4.1):

$$\begin{aligned} L(\dot{\bar{x}}_1, \dot{\bar{y}}_1, \dot{\bar{\theta}}_1, \dot{\bar{\theta}}_{1,2}) &= \\ &= \frac{1}{2}m_1(\dot{\bar{x}}_1^2 + \dot{\bar{y}}_1^2) + \frac{1}{2}I_{z_1}\dot{\bar{\theta}}_1^2 + \frac{1}{2}I_{z_2}(\dot{\bar{\theta}}_1 + \dot{\bar{\theta}}_{1,2})^2 \\ &= \frac{1}{2}m_1\left[(\dot{x}_1 \cos a - \dot{y}_1 \sin a)^2 + (\dot{x}_1 \sin a + \dot{y}_1 \cos a)^2\right] + \frac{1}{2}I_{z_1}\dot{\theta}_1^2 + \frac{1}{2}I_{z_2}(\dot{\theta}_1 + \dot{\theta}_{1,2})^2 \\ &= L(\dot{x}_1, \dot{y}_1, \dot{\theta}_1, \dot{\theta}_{1,2}) \ . \end{aligned}$$

∎

Define the *nonholonomic momentum* as:

$$p = \sum_i \frac{\partial L}{\partial \dot{q}^i}(\xi_Q^q)^i \ . \tag{6.4.3}$$

Proposition 6.4.2 (Nonholonomic Momentum)

The nonholonomic momentum for the Roller Racer system is

$$p = m_1(d_1 \cos\theta_{1,2} + d_2)(\dot{x}_1 \cos\theta_1 + \dot{y}_1 \sin\theta_1) + [(I_{z_1} + I_{z_2})\dot{\theta}_1 + I_{z_2}\dot{\theta}_{1,2}]\sin\theta_{1,2} \ . \tag{6.4.4}$$

Proof

From (6.4.2) and (6.3.4), we get (6.4.4).

∎

Let

$$\Delta(\theta_{1,2}) \stackrel{\text{def}}{=} (I_{z_1} + I_{z_2})\sin^2\theta_{1,2} + m_1(d_1\cos\theta_{1,2} + d_2)^2 \ . \tag{6.4.5}$$

For $d_1 \neq d_2$, we have $\Delta > 0$ for all $q \in Q$.

Proposition 6.4.3

The angular velocity $\dot{\theta}_1$ is a linear function of the nonholonomic momentum

$$\dot{\theta}_1 = \frac{1}{\Delta(\theta_{1,2})}\sin\theta_{1,2}\, p - \frac{1}{\Delta(\theta_{1,2})}[I_{z_2}\sin^2\theta_{1,2} + m_1 d_2(d_1\cos\theta_{1,2} + d_2)]\dot{\theta}_{1,2} \ . \tag{6.4.6}$$

Proof

Multiplying both sides of (6.4.4) by $\sin\theta_{1,2}$ we get

$$\sin\theta_{1,2} p = m_1(d_1\cos\theta_{1,2} + d_2)(\dot{x}_1\cos\theta_1 + \dot{y}_1\sin\theta_1)\sin\theta_{1,2}$$
$$+ [(I_{z_1} + I_{z_2})\dot{\theta}_1 + I_{z_2}\dot{\theta}_{1,2}]\sin^2\theta_{1,2} \, .$$

From (6.2.11) we get

$$\begin{aligned}\sin\theta_{1,2} p &= m_1(d_1\cos\theta_{1,2} + d_2)[(d_1\cos\theta_{1,2} + d_2)\dot{\theta}_1 + d_2\dot{\theta}_{1,2}] \\ &\quad + [(I_{z_1} + I_{z_2})\dot{\theta}_1 + I_{z_2}\dot{\theta}_{1,2}]\sin^2\theta_{1,2} \\ &= \left[(I_{z_1} + I_{z_2})\sin^2\theta_{1,2} + m_1(d_1\cos\theta_{1,2} + d_2)^2\right]\dot{\theta}_1 \\ &\quad + \left[I_{z_2}\sin^2\theta_{1,2} + m_1 d_2(d_1\cos\theta_{1,2} + d_2)\right]\dot{\theta}_{1,2} \, .\end{aligned}$$

Solving for $\dot{\theta}_1$, we get (6.4.6).

∎

Note that for $d_2 = 0$

$$\dot{\theta}_1 = \frac{\sin\theta_{1,2}}{\Delta}(p - I_{z_2}\sin\theta_{1,2}\dot{\theta}_{1,2}) \, . \tag{6.4.7}$$

The momentum equation presented in the next Theorem is derived from the Lagrange–d'Alembert principle by considering only variations that satisfy the constraints and that depend on the symmetry, as it is expressed by a free group action. The equation does not depend on internal torques and depends only on the shape variables and not on the group variables.

Theorem 6.4.4 (Bloch, Krishnaprasad, Marsden & Murray [1994])

Consider a Lagrangian L which is invariant under the action Φ of a group G on a configuration space Q. Let $\mathcal{D}_q$ be a constraint distribution on T_qQ and consider the intersection $\mathcal{S}_q$ of $\mathcal{D}_q$ with the tangent space to the orbit of Φ at q. Let $\xi^q_Q \in \mathcal{S}_q$ and let ξ^q be the corresponding element of the Lie algebra $\mathcal{G}$. The evolution of the nonholonomic momentum p, defined as in equation (6.4.3), satisfies the equation:

$$\frac{dp}{dt} = \sum_i \frac{\partial L}{\partial \dot{q}^i}\left[\frac{d\xi^q}{dt}\right]^i_Q \, . \tag{6.4.8}$$

∎

This result generalizes the classical Noether's Theorem, which specifies conserved quantities for solutions of the Euler–Lagrange equations.

Proposition 6.4.5 (Momentum Equation)

The momentum equation for the Roller Racer system is

$$\frac{dp}{dt} = A_1^4(\theta_{1,2})\dot{\theta}_{1,2}p + A_2^4(\theta_{1,2})\dot{\theta}_{1,2}^2 \ , \tag{6.4.9}$$

where

$$A_1^4(\theta_{1,2}) = \frac{(I_{z_1} + I_{z_2})\cos\theta_{1,2} - m_1 d_1(d_1\cos\theta_{1,2} + d_2)}{(I_{z_1} + I_{z_2})\sin^2\theta_{1,2} + m_1(d_1\cos\theta_{1,2} + d_2)^2}\sin\theta_{1,2} = \frac{1}{\Delta(\theta_{1,2})}\beta(\theta_{1,2})\sin\theta_{1,2}$$

and

$$A_2^4(\theta_{1,2}) = \frac{m_1(d_1 + d_2\cos\theta_{1,2})(-I_{z_1}d_2 + I_{z_2}d_1\cos\theta_{1,2})}{(I_{z_1} + I_{z_2})\sin^2\theta_{1,2} + m_1(d_1\cos\theta_{1,2} + d_2)^2} = \frac{m_1}{\Delta(\theta_{1,2})}\lambda(\theta_{1,2})\gamma(\theta_{1,2}),$$

where

$$\beta(\theta_{1,2}) \stackrel{\text{def}}{=} (I_{z_1} + I_{z_2})\cos\theta_{1,2} - m_1 d_1(d_1\cos\theta_{1,2} + d_2) = I\cos\theta_{1,2} - m_1 d_1 r \ ,$$

$$\gamma(\theta_{1,2}) \stackrel{\text{def}}{=} -I_{z_1}d_2 + I_{z_2}d_1\cos\theta_{1,2} \ ,$$

$$r(\theta_{1,2}) \stackrel{\text{def}}{=} d_1\cos\theta_{1,2} + d_2 \ ,$$

$$\lambda(\theta_{1,2}) \stackrel{\text{def}}{=} d_1 + d_2\cos\theta_{1,2} \ ,$$

$$I \stackrel{\text{def}}{=} I_{z_1} + I_{z_2} \ .$$

Proof

From (6.4.8), (6.4.2) and (6.3.11) we get

$$\begin{aligned}
\frac{dp}{dt} &= m_1\dot{x}_1\big[-d_1\sin\theta_{1,2}\cos\theta_1\dot{\theta}_{1,2} - (d_1\cos\theta_{1,2} + d_2)\sin\theta_1\dot{\theta}_1 + \dot{y}_1\sin\theta_{1,2}\big] \\
&\qquad + m_1\dot{y}_1\big[-d_1\sin\theta_{1,2}\sin\theta_1\dot{\theta}_{1,2} + (d_1\cos\theta_{1,2} + d_2)\cos\theta_1\dot{\theta}_1 - \dot{x}_1\sin\theta_{1,2}\big] \\
&\qquad + \big[(I_{z_1} + I_{z_2})\dot{\theta}_1 + I_{z_2}\dot{\theta}_{1,2}\big]\cos\theta_{1,2}\dot{\theta}_{1,2} \\
&= -m_1 d_1(\dot{x}_1\cos\theta_1 + \dot{y}_1\sin\theta_1)\sin\theta_{1,2}\dot{\theta}_{1,2} \\
&\qquad + m_1(d_1\cos\theta_{1,2} + d_2)(-\dot{x}_1\sin\theta_1 + \dot{y}_1\cos\theta_1)\dot{\theta}_1 \\
&\qquad + \big[(I_{z_1} + I_{z_2})\dot{\theta}_1 + I_{z_2}\dot{\theta}_{1,2}\big]\cos\theta_{1,2}\dot{\theta}_{1,2}
\end{aligned}$$

Using the nonholonomic constraints (6.2.9) and (6.2.11) we get

$$\frac{dp}{dt} = -m_1 d_1 \big[(d_1 \cos\theta_{1,2} + d_2)\dot{\theta}_1 + d_2\dot{\theta}_{1,2}\big] + \big[(I_{z_1} + I_{z_2})\dot{\theta}_1 + I_{z_2}\dot{\theta}_{1,2}\big] \cos\theta_{1,2}\dot{\theta}_{1,2}$$

$$= \big[(I_{z_1} + I_{z_2})\cos\theta_{1,2} - m_1 d_1 (d_1 \cos\theta_{1,2} + d_2)\big]\dot{\theta}_1\dot{\theta}_{1,2} + \big[I_{z_2}\cos\theta_{1,2} - m_1 d_1 d_2\big]\dot{\theta}_{1,2}^2 \,.$$

Using (6.4.6), we substitute $\dot{\theta}_1$ in the above expression and we get (6.4.9).

∎

Proposition 6.4.6

The solution of the momentum equation (6.4.9) is:

$$p(t) = \Phi(t,t_0)p(t_0) + \int_{t_0}^{t} \Phi(t,\tau)A_2^4(\theta_{1,2}(\tau))\dot{\theta}_{1,2}^2(\tau)d\tau \,, \tag{6.4.10}$$

where

$$\Phi(t,t_0) = \exp\Big[\int_{t_0}^{t} A_1^4(\theta_{1,2}(\tau))\dot{\theta}_{1,2}(\tau)d\tau\Big] = \exp\Big[\int_{\theta_{1,2}(t_0)}^{\theta_{1,2}(t)} A_1^4(\theta_{1,2})d\theta_{1,2}\Big] = \sqrt{\frac{\Delta(\theta_{1,2}(t))}{\Delta(\theta_{1,2}(t_0))}} \tag{6.4.11}$$

is the state transition matrix of (6.4.9) and where $\Delta(\theta_{1,2})$ is defined in equation (6.4.5).

Proof

Equation (6.4.9) is a first–order linear time–varying ODE with state transition matrix $\Phi(t,t_0)$. Thus, (6.4.10) is obvious. To compute the state transition matrix $\Phi(t,t_0)$, observe that we get from (6.4.5):

$$\frac{d\Delta}{d\theta_{1,2}} = 2\sin\theta_{1,2}\big[(I_{z_1} + I_{z_2})\cos\theta_{1,2} - m_1 d_1(d_1\cos\theta_{1,2} + d_2)\big] = 2\beta(\theta_{1,2})\sin\theta_{1,2} \,.$$

From this and the definition of A_1^4 in (6.4.9) we get

$$A_1^4(\theta_{1,2}) = \frac{\beta(\theta_{1,2})}{\Delta(\theta_{1,2})}\sin\theta_{1,2} = \frac{1}{2\Delta}\frac{d\Delta}{d\theta_{1,2}} \,.$$

Thus

$$\Phi(t,t_0) = \exp\Big[\int_{t_0}^{t} A_1^4(\theta_{1,2}(\tau))\dot{\theta}_{1,2}(\tau)d\tau\Big] = \exp\Big[\int_{\theta_{1,2}(t_0)}^{\theta_{1,2}(t)} A_1^4(\theta_{1,2})d\theta_{1,2}\Big]$$
$$= \exp\Big[\int_{\Delta(\theta_{1,2}(t_0))}^{\Delta(\theta_{1,2}(t))} \frac{1}{2}\frac{d\Delta}{\Delta}\Big] = \exp\Big[\ln\Big(\sqrt{\frac{\Delta(\theta_{1,2}(t))}{\Delta(\theta_{1,2}(t_0))}}\Big)\Big] = \sqrt{\frac{\Delta(\theta_{1,2}(t))}{\Delta(\theta_{1,2}(t_0))}} \tag{6.4.12}$$

∎

Assume a shape–space trajectory $\theta_{1,2}(.) \subset \mathcal{S}$ has been specified. The corresponding nonholonomic momentum can be determined from the solution of the momentum equation (6.4.10). From the formula for the nonholonomic momentum (equation (6.4.4)) and from the nonholonomic constraints (equations (6.2.9) and (6.2.11)), we can reconstruct the group trajectory $g_1(.) = g_1(x_1(.), y_1(.), \theta_1(.)) \subset SE(2)$. This can be done either by first specifying $(\dot{x}_1, \dot{y}_1, \dot{\theta}_1)$ and then integrating to find (x_1, y_1, θ_1) or by first specifying $\xi_1 \in \mathcal{G}$ and then applying the Wei–Norman procedure to find the corresponding $g_1 \in G$.

Proposition 6.4.7 (Reconstruction of Group Trajectory)

For $g_1 = g_1(x_1, y_1, \theta_1) \in SE(2)$, the corresponding curve in the Lie algebra $\xi_1 = g_1^{-1}\dot{g}_1$ is given by

$$\xi_1 = \xi_1^1(\theta_{1,2}, \dot{\theta}_{1,2})\mathcal{A}_1 + \xi_2^1(\theta_{1,2}, \dot{\theta}_{1,2})\mathcal{A}_2 \,, \tag{6.4.13}$$

where for $d_1 \neq d_2$, the components of ξ_1 are

$$\xi_1^1(\theta_{1,2}, \dot{\theta}_{1,2}) = \dot{\theta}_1 = \frac{\sin\theta_{1,2}}{\Delta(\theta_{1,2})}p - \frac{I_{z_2}\sin^2\theta_{1,2} + m_1 d_2(d_1\cos\theta_{1,2} + d_2)}{\Delta(\theta_{1,2})}\dot{\theta}_{1,2} \tag{6.4.14}$$

and

$$\xi_2^1(\theta_{1,2}, \dot{\theta}_{1,2}) = \dot{x}_1\cos\theta_1 + \dot{y}_1\sin\theta_1 = \frac{d_1\cos\theta_{1,2} + d_2}{\Delta(\theta_{1,2})}p + \frac{I_{z_1}d_2 - I_{z_2}d_1\cos\theta_{1,2}}{\Delta(\theta_{1,2})}\sin\theta_{1,2}\dot{\theta}_{1,2} \tag{6.4.15}$$

and where $\{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3\}$ is the usual basis of $\mathcal{G} = se(2)$. Moreover,

$$
\begin{aligned}
\dot{x}_1 &= \cos\theta_1 \, \xi_2^1 = \frac{\cos\theta_1}{\Delta(\theta_{1,2})}\left[(d_1\cos\theta_{1,2} + d_2)p + (I_{z_1}d_2 - I_{z_2}d_1\cos\theta_{1,2})\sin\theta_{1,2}\dot{\theta}_{1,2}\right] , \\
\dot{y}_1 &= \sin\theta_1 \, \xi_2^1 = \frac{\sin\theta_1}{\Delta(\theta_{1,2})}\left[(d_1\cos\theta_{1,2} + d_2)p + (I_{z_1}d_2 - I_{z_2}d_1\cos\theta_{1,2})\sin\theta_{1,2}\dot{\theta}_{1,2}\right] .
\end{aligned}
\tag{6.4.16}
$$

Proof

Equation (6.4.14) is immediate from (6.4.6). From (6.2.11) and (6.4.6) we get

$$
\begin{aligned}
&(\dot{x}_1\cos\theta_1 + \dot{y}_1\sin\theta_1)\sin\theta_{1,2} \\
&\qquad = \frac{\sin\theta_{1,2}}{\Delta}\left[(d_1\cos\theta_{1,2} + d_2)p + (I_{z_1}d_2 - I_{z_2}d_1\cos\theta_{1,2})\sin\theta_{1,2}\dot{\theta}_{1,2}\right] .
\end{aligned}
$$

When $\sin\theta_{1,2} \neq 0$ we get

$$
\dot{x}_1\cos\theta_1 + \dot{y}_1\sin\theta_1 = \frac{1}{\Delta}\left[(d_1\cos\theta_{1,2} + d_2)p + (I_{z_1}d_2 - I_{z_2}d_1\cos\theta_{1,2})\sin\theta_{1,2}\dot{\theta}_{1,2}\right] . \tag{6.4.17}
$$

From (6.4.4) and (6.4.6) we get

$$
\begin{aligned}
&m_1(d_1\cos\theta_{1,2} + d_2)(\dot{x}_1\cos\theta_1 + \dot{y}_1\sin\theta_1) \\
&\quad = \frac{m_1}{\Delta}(d_1\cos\theta_{1,2} + d_2)\left[(d_1\cos\theta_{1,2} + d_2)p + (I_{z_1}d_2 - I_{z_2}d_1\cos\theta_{1,2})\sin\theta_{1,2}\dot{\theta}_{1,2}\right] .
\end{aligned}
$$

When $d_1\cos\theta_{1,2} + d_2 \neq 0$ we get from this exactly (6.4.17).
When $d_1 \neq d_2$, either $\sin\theta_{1,2} \neq 0$ or $d_1\cos\theta_{1,2} + d_2 \neq 0$. In either case (6.4.17) holds. From this we get (6.4.15).
Finally, from (6.2.9) we get $\xi_3^1 = 0$. Equations (6.4.16) are immediate from (6.4.15) and (6.2.9).

∎

From (6.4.15), note that for $d_2 = 0$, the velocity component ξ_2^1 is

$$
\xi_2^1 = \dot{x}_1\cos\theta_1 + \dot{y}_1\sin\theta_1 = \frac{d_1\cos\theta_{1,2}}{\Delta}(p - I_{z_2}\sin\theta_{1,2}\dot{\theta}_{1,2}) . \tag{6.4.18}
$$

Now observe that from (6.4.14) and (6.4.15) we get

$$\begin{pmatrix} \xi_1^1 \\ \xi_2^2 \end{pmatrix} = \frac{1}{\Delta} \begin{pmatrix} \sin\theta_{1,2} & -[I_{z_2}\sin^2\theta_{1,2} + m_1 d_2(d_1\cos\theta_{1,2} + d_2)] \\ d_1\cos\theta_{1,2} + d_2 & I_{z_1}d_2 - I_{z_2}d_1\cos\theta_{1,2})\sin\theta_{1,2} \end{pmatrix} \begin{pmatrix} p \\ \dot\theta_{1,2} \end{pmatrix}$$
$$= B(\theta_{1,2}) \begin{pmatrix} p \\ \dot\theta_{1,2} \end{pmatrix} \tag{6.4.19}$$

and notice that $\det B(\theta_{1,2}) = \frac{d_2}{\Delta}$. Thus, in the case $d_2 = 0$, given a group trajectory $\xi_1 \subset \mathcal{G}$, we cannot always solve (6.4.19) for p and $\dot\theta_{1,2}$.

Equation (6.4.10) can be used to derive qualitative information about the momentum, which can be useful in motion control.

Proposition 6.4.8 (Sign of the Nonholonomic Momentum)

Assume $d_1 > d_2$.

a) Assume the angle $\theta_{1,2}$ remains in an ϵ–neighborhood of $\theta_{1,2} = \pi$, with $\epsilon < \frac{\pi}{2}$ and that the initial momentum of the system is non–positive. Then, the momentum p is negative at all times.

b) Assume $I_{z_1}d_2 < I_{z_2}d_1$ and that the angle $\theta_{1,2}$ remains in an $\tilde\epsilon$–neighborhood of $\theta_{1,2} = 0$, with $\tilde\epsilon \leq \cos^{-1}(\frac{I_{z_1}d_2}{I_{z_2}d_1})$. Assume further that the initial momentum of the system is non–negative. Then, the momentum p is positive at all times.

Proof

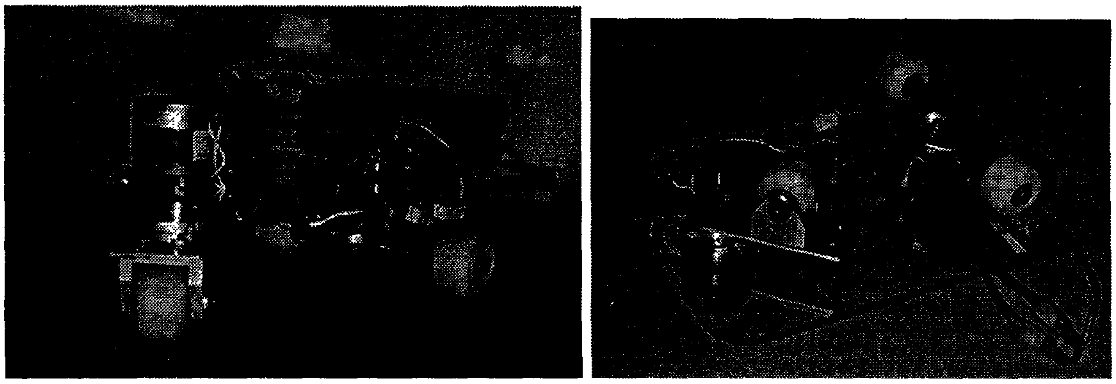
Since $d_1 > d_2$, we know that $\lambda = d_1 + d_2\cos\theta_{1,2} > 0$, for all $\theta_{1,2}$.

a) By our choice of the ϵ–neighborhood, we have $\cos\theta_{1,2} < 0$, thus $\gamma = -I_{z_1}d_2 + I_{z_2}d_1\cos\theta_{1,2} < 0$, for all $\theta_{1,2}$ in this neighborhood. Thus $A_2^4 = \frac{m_1}{\Delta}\lambda\gamma < 0$, for all such $\theta_{1,2}$ and, thus, the second term of (6.4.10) is negative. If $p(t_0) \leq 0$, then $p < 0$.

b) By our choice of the $\tilde\epsilon$–neighborhood, $\gamma > 0$, for all $\theta_{1,2}$ in this neighborhood. Then, $A_2^4 > 0$ and the second term of (6.4.10) is positive. If $p(t_0) \geq 0$, then $p > 0$.

∎

A computer–controlled prototype of the Roller Racer system was built at the Intelligent Servosystems Laboratory by Vikram Manikonda (fig. 6.4.1). The assumption of our model that the only feature of the body motion of a Roller Racer rider which is



(a) (b)

Fig. 6.4.1: Roller Racer Prototype

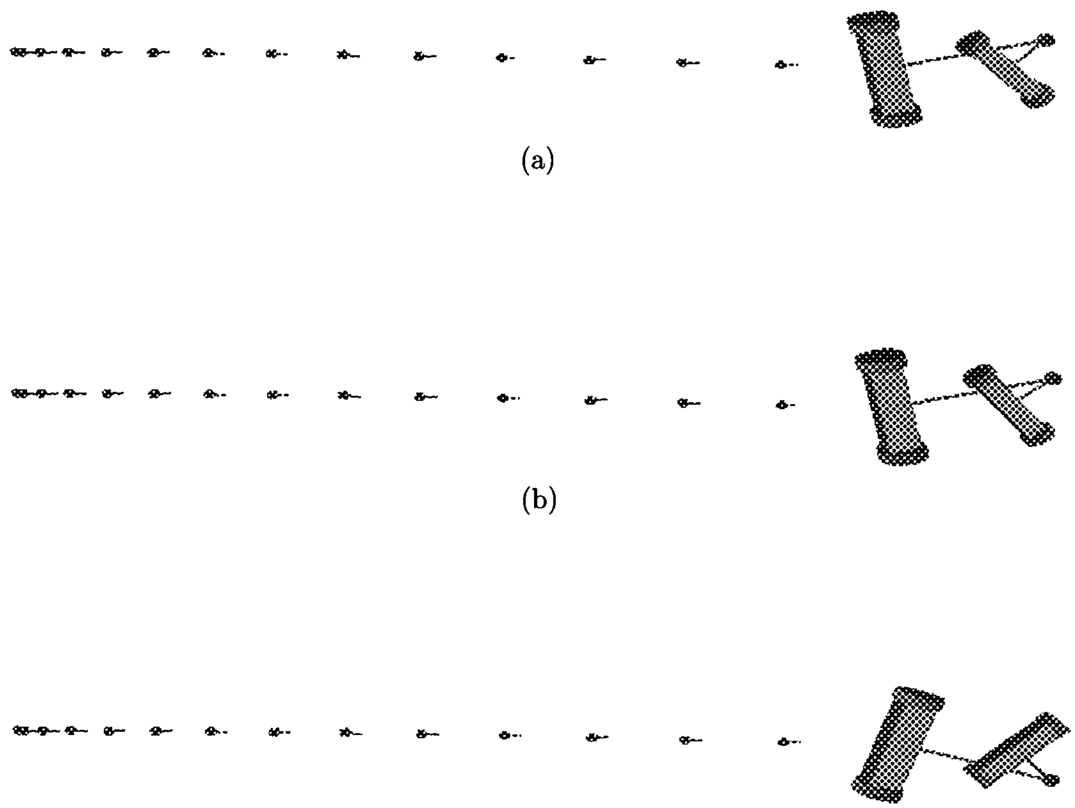
crucial to the propulsion of this mechanism is the swinging of the steering arm around the pivot axis, was verified using this and other similar prototypes.

The model of the dynamics of the Roller Racer system, which was developed in this section, was used in numerical simulations of the system on a Silicon Graphics workstation.

A periodic shape trajectory of period $T_{1,2}$ of the form

$$\theta_{1,2}(t) = \theta_{1,2}(t_0) + \alpha_{1,2} \sin(\omega_{1,2} t + \phi_{1,2}) ,$$

with $\omega_{1,2} = \frac{2\pi}{T_{1,2}}$ is used in the simulations. The average value of $\theta_{1,2}$ is $\theta_{1,2}(t_0)$. Setting this average to π, as in fig. 6.4.2, generates a "straight–line" motion. In fig. 6.4.2 three successive snapshots of the system's motion are shown. The trajectory of the system is shown by needle–like markers, so that the position and orientation of the system at the end of a period of the shape control becomes evident. As the system moves to the right, the spacing between those needles becomes larger, since, as momentum is built up, the system moves faster. Setting $\theta_{1,2}(t_0)$ to a value other than π or zero, as in fig. 6.4.3 (here $\theta_{1,2}(t_0) = 3.31$ rad), generates a rotation around the point where the axes of the platforms intersect, when the system is in the configuration corresponding to this average value. Once momentum is built up through periodic shape variations, we can stop varying the shape periodically and, for some time, use $\theta_{1,2}$ just to steer the system.

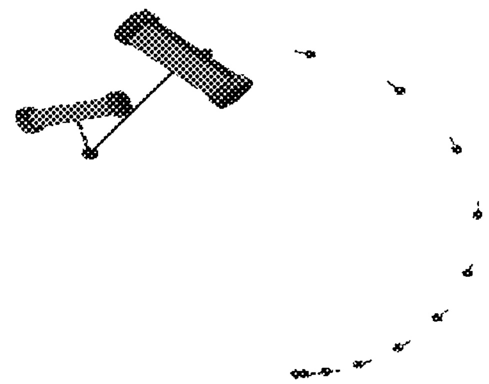


(a)

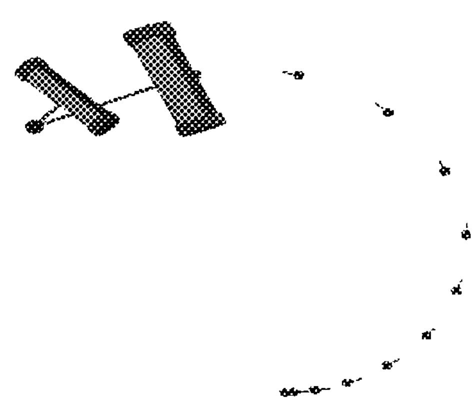
(b)

(c)

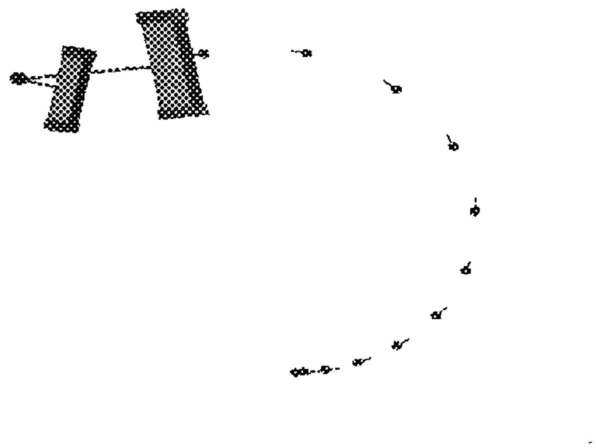
Fig. 6.4.2: “Straight–line” motion of the Roller Racer



(a)



(b)



(c)

Fig. 6.4.3: "Turning" motion of the Roller Racer

CHAPTER SEVEN

CONCLUSIONS AND FUTURE RESEARCH

In this dissertation we examined several robotic systems with holonomic and nonholonomic constraints that include parallel manipulator subsystems.

In Chapter 3, we considered motion planning based on minimizing a curvature-squared cost functional, we showed how the problem is related to the classical problem of the elastica and we derived analytical solutions for the optimal path segments in the case of a 2–dimensional manifold and for the optimal curvature and torsion in the case of a 3–dimensional manifold.

In Chapter 4, we introduced the concept of G–Snakes, a class of kinematic chains with nonholonomic constraints evolving on a Lie group G. Shape variations of the system modules were shown to induce a snake–like global motion of the system. We provided a framework upon which motion planning strategies based on periodic shape variations can be developed.

In Chapter 5, a concrete mechanical realization of G–Snakes, associated with the Special Euclidean group $G = SE(2)$ and related to the concept of Variable Geometry Truss assembly with nonholonomic constraints was presented. We derived the associated kinematics and examined motion planning by showing how periodic shape changes induce global translation or rotation of the assembly under the influence of the nonholonomic constraints.

In Chapter 6, we considered the dynamics of a particular instance of the 2–node $SE(2)$–Snake, namely the Roller Racer, where the shape space is the subgroup S^1 of $SE(2)$. We used a novel approach based on the notion of the nonholonomic momentum to decouple the Lagrange–d'Alembert equations from the group variables and we considered motion control schemes based on periodic shape variations.

Immediate extensions to this work include the generalization of the dynamic motion control scheme to 2–node $SE(2)$ Snakes and to $(n-1)$–node G–Snakes and the detailed study of motion planning in those cases, as well as the extension of the dynamic model to allow for joint flexibility and to include the effect of the rider on the Roller Racer system. Also, we expect expansions to: the systematic study and classification of gaits that both the kinematic and the dynamic motion control systems can produce, the exploration of alternative shape change strategies (integrally related sinusoids, curvature–squared minimizing closed shape–space paths, etc), the study of geometric phase and motion control for the other 3–dimensional groups considered in chapter 4.

Of significant interest would be the use of these systems as analytical and computational tools to explore motion control based on coupled oscillators, since their prominent role in the physiology of the human and animal walk in the form of Central Pattern Generators might signal their importance to the motion control of very redundant robotic systems, like the ones at hand (Bässler [1986]; Krishnaprasad [1995]; Matsuoka [1987]).

Finally, the integration of sensory feedback either to monitor the state of the system or that of a dynamically changing environment, brings up certain control theoretic questions related to stabilization and trajectory tracking and raises several challenging issues related to navigation and obstacle avoidance in the presence of partial or noisy information, to the robust interpretation of the sensory information and the inclusion, in an appropriate way, of this information in a motion control scheme. Recent studies related to formal languages and hybrid models for motion control, able to incorporate both symbolic strings and real–valued functions are certainly relevant here (Brockett [1990]; Brockett [1993]).

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