# ABSTRACT

Title of dissertation: CENTER OF PRO-$p$-IWAHORI-HECKE ALGEBRA

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Let $\mathbf{G}$ be a connected reductive group over a $p$-adic field $F$. The study of representations of $\mathbf{G}(F)$ naturally involves the pro-$p$-Iwahori-Heche algebra of $\mathbf{G}(F)$. The pro-$p$-Iwahori-Hecke algebra is a deformation of the group algebra of the pro-$p$-Iwahori Weyl group of $\mathbf{G}(F)$ with generic parameters. The pro-$p$-Iwahori-Hecke algebra with zero parameters plays an important role in the study of mod-$p$ representations of $\mathbf{G}(F)$.

In a series of paper, Vigneras introduced a generic algebra $\mathcal{H}_R(q_{\tilde{s}}, c_{\tilde{s}})$ which generalizes the pro-$p$-Iwahori-Hecke algebra of a reductive $p$-adic group. Vigneras also gave a basis of the center of $\mathcal{H}_R(q_{\tilde{s}}, c_{\tilde{s}})$ when $\mathcal{H}_R(q_{\tilde{s}}, c_{\tilde{s}})$ is associated with a pro-$p$-Iwahori Weyl group. This basis is defined by using the Bernstein presentation of $\mathcal{H}_R(q_{\tilde{s}}, c_{\tilde{s}})$ and the alcove walk. In this article, we restrict to the case where $q_{\tilde{s}} = 0$ and give an explicit description of the center of $\mathcal{H}_R(0, c_{\tilde{s}})$ using the Iwahori-Matsumoto presentation.

First, we introduce the generic algebra. Let $W$ be the semidirect product of a Coxeter group and a group acting on the Coxeter group and stabilizing the

generating set of the Coxeter group. Let $W(1)$ be an extension of $W$ with a commutative group. Let $R$ be a commutative ring. We give the definition of the $R$-algebra $\mathcal{H}_R(q_{\tilde{s}}, c_{\tilde{s}})$ of $W(1)$ with parameters $(q_{\tilde{s}}, c_{\tilde{s}})$. Then for any pair $(v, w)$ in $W \times W$ with $v \leq w$, we define a linear operator $r_{v,w}$ between $R$-submodules of $\mathcal{H}_R(q_{\tilde{s}}, c_{\tilde{s}})$. It takes some work to show that $r_{v,w}$ is well defined.

Next, we restrict $W$ to be an Iwahori Weyl group. We show that the maximal length terms of a central element in $\mathcal{H}_R(q_{\tilde{s}}, c_{\tilde{s}})$ is given by a union of finite conjugacy classes in $W(1)$. Then we prove some techical results regarding $r_{v,w}$ acting on the maximal length terms of a central element in $\mathcal{H}_R(q_{\tilde{s}}, c_{\tilde{s}})$.

In the last part, we restrict to the case when $q_{\tilde{s}} = 0$ and give a explicit basis of the center of $\mathcal{H}_R(0, c_{\tilde{s}})$ in the Iwahori-Matsumoto presentation by using the operator $r_{v,w}$. Two examples are given to help understand how this basis looks like.

# CENTER OF PRO-$p$-IWAHORI-HECKE ALGEBRA

by

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# Chapter 1: Introduction

Iwahori-Hecke algebras are deformations of the group algebras of Coxeter groups $W_0$. When $W_0$ is finite, they play an important role in the study of representations of finite groups of Lie type. In [2], Geck and Rouquier gave a basis of the center of Iwahori-Hecke algebras associated to finite Coxeter groups. The basis is closely related to minimal length elements in the conjugacy classes of $W_0$.

The 0-Hecke algebra was used by Carter and Lusztig in [1] in the study of $p$-modular representations of finite groups of Lie type. 0-Hecke algebras are deformations of the group algebras of finite Coxeter groups with zero parameter. In [7], He gave a basis of the center of 0-Hecke algebras associated to finite Coxeter groups. The basis is closely related to maximal length elements in the conjugacy classes of $W_0$.

Affine Hecke algebras are deformations of the group algebras of affine Weyl groups $W^{\mathrm{aff}}$. They appear naturally in the representation theory of reductive $p$-adic groups. In [9], Lusztig gave a basis of the center of affine Hecke algebras. In [7], He mentioned that a proof similar to his proof of Theorem 4.4 could be applied to give a basis of the center of affine 0-Hecke algebras. The basis is closely related to finite conjugacy classes in $W^{\mathrm{aff}}$.

Let $\mathbf{G}$ be a connected reductive group over a $p$-adic field $F$. The study of mod-$p$ representations of $\mathbf{G}(F)$ naturally involves the pro-$p$-Iwahori Hecke algebra of $\mathbf{G}(F)$. Let $R$ be a commutative ring. Let $W$ be the semidirect product of a Coxeter group and a group $\Omega$, where the action of $\Omega$ on the Coxeter group stabilizes the generating set of the Coxeter group. Let $W(1)$ be an extension of $W$ with a commutative group. In [14], Vigneras discussed the $R$-algebra $\mathcal{H}_R(q_{\tilde{s}}, c_{\tilde{s}})$ associated to $W(1)$, which generalizes the pro-$p$-Iwahori Hecke algebra of $\mathbf{G}(F)$. In [15], Vigneras gave a basis of the center of $\mathcal{H}_R(q_{\tilde{s}}, c_{\tilde{s}})$ by using the Bernstein relation and alcove walks (the definition of alcove walk can be found in [3]). The basis of the center is closely related to the finite conjugacy classes in $W(1)$.

In general, the expression of the center in [15] is complicated if we want to write it out explicitly by Iwahori-Matsumoto presentation. But for $R$-algebras $\mathcal{H}_R(0, c_{\tilde{s}})$, we can give an explicit description of the center by Iwahori-Matsumoto presentation. This is the main result of this article. In Chapter 2, we review the definition of $\mathcal{H}_R(q_{\tilde{s}}, c_{\tilde{s}})$ and define a new operator $r_{v,w}$. In Chapter 3, we give a brief review of the Iwahori Weyl group and show that the maximal length terms of a central element in $\mathcal{H}_R(q_{\tilde{s}}, c_{\tilde{s}})$ come from finite conjugacy classes in $W(1)$. Then we prove some technical results regarding $r_{v,w}$, where $w$ is in some finite conjugacy class and give a basis of the center of $\mathcal{H}_R(0, c_{\tilde{s}})$. In Chapter 4, we give some examples to show how the main result works.

# Chapter 2: A new operator

## 2.1 Generic algebra

The symbols $\mathbb{N}, \mathbb{Z}, \mathbb{R}$ refers to the natural numbers, the integers and the real numbers.

Let $R$ be a commutative ring. Let

$$W^{\mathrm{aff}}, S^{\mathrm{aff}}, \Omega, W, Z, W(1),$$

satisfying:

- $(W^{\mathrm{aff}}, S^{\mathrm{aff}})$ is a Coxeter system.
- $\Omega$ is a group acting on $W^{\mathrm{aff}}$ and stabilizing $S^{\mathrm{aff}}$.
- $W$ is the semi-direct product $W^{\mathrm{aff}} \rtimes \Omega$.
- $Z$ is a commutative group.
- $1 \to Z \to W(1) \xrightarrow{\pi} W \to 1$ is an extension of $W$ by $Z$.

In the setting of a reductive $p$-adic group $\mathbf{G}$, $W$ is the Iwahori-Weyl group and $Z$ corresponds to a finite torus of $\mathbf{G}$. More details of pro-$p$-Iwahori-Hecke algebra of reductive $p$-adic groups can be found in [14].

We denote by $X(1)$ the inverse image in $W(1)$ of a subset $X \subseteq W$.

In general, $Z$ may not be finite. The length function $\ell : W^{\mathrm{aff}} \to \mathbb{N}$ of $(W^{\mathrm{aff}}, S^{\mathrm{aff}})$ being invariant by conjugation by $\Omega$, extends to a length function $\ell$ of $W$ constant on the double cosets of $\Omega$, and inflates to a length function on $W(1)$, still denoted by $\ell$, such that $\ell(\tilde{w}) = \ell(\pi(\tilde{w}))$ for $\tilde{w} \in W(1)$. The subgroup of length 0 elements in $W$ is $\Omega$, and in $W(1)$ is $\Omega(1)$. The inverse image of $W^{\mathrm{aff}}$ in $W(1)$ is a normal subgroup $W^{\mathrm{aff}}(1)$ such that $Z = W^{\mathrm{aff}}(1) \cap \Omega(1)$ and $W(1) = W^{\mathrm{aff}}(1)\Omega(1)$. The Bruhat order on $W$ can also be defined. Let $v = v'\tau, w = w'\tau'$ be two elements in $W$ where $v', w' \in W^{\mathrm{aff}}$ and $\tau, \tau' \in \Omega$, then $v \leq w$ if and only if $v' \leq w'$ and $\tau = \tau'$. We will use the following result of Bruhat order on $W$.

**Lemma 2.1.** *Let $x, y \in W$ with $x \leq y$. Let $s \in S^{\mathit{aff}}$. Then*

- $\min\{x, sx\} \leq \min\{y, sy\}$ *and* $\max\{x, sx\} \leq \max\{y, sy\}$.
- $\min\{x, xs\} \leq \min\{y, ys\}$ *and* $\max\{x, xs\} \leq \max\{y, ys\}$.

*Proof.* When $\Omega$ is trivial, this is all well-known: see Corollary 2.5 in [10]. The more general statement is immediate by definition of the Bruhat order on $W$ because $W = W^{\mathrm{aff}} \rtimes \Omega$. □

For $\tilde{w} \in W(1)$ and $t \in Z$, $\tilde{w}(t) = \tilde{w}t\tilde{w}^{-1}$ depends only on the image of $\tilde{w}$ in $W$ because $Z$ is commutative. By linearity the conjugation defines an action

$$(\tilde{w}, c) \mapsto \tilde{w}(c) : W(1) \times R[Z] \to R[Z]$$

of $W(1)$ on $R[Z]$ factoring through the map $\pi : W(1) \to W$.

We recall the definition of the generic algebra $\mathcal{H}_R(q_{\tilde{s}}, c_{\tilde{s}})$ introduced in [14].

**Theorem 2.2.** *Let $(q_{\tilde{s}}, c_{\tilde{s}}) \in R \times R[Z]$ for all $\tilde{s} \in S^{aff}(1)$. Suppose*

- $q_{\tilde{s}} = q_{\tilde{s}t} = q_{\tilde{s'}}$,
- $c_{\tilde{s}t} = c_{\tilde{s}}t$ *and* $\tilde{w}(c_{\tilde{s}}) = c_{\tilde{w}\tilde{s}\tilde{w}^{-1}} = c_{\tilde{s'}}$,

*for any $t \in Z, \tilde{w} \in W(1)$, and $\tilde{s}, \tilde{s'} \in S^{aff}(1)$ satisfying $\tilde{s'} = \tilde{w}\tilde{s}\tilde{w}^{-1}$.*

*Then the free $R$-module $\mathcal{H}_R(q_{\tilde{s}}, c_{\tilde{s}})$ of basis $(T_{\tilde{w}})_{\tilde{w}\in W(1)}$ admits a unique $R$-algebra structure satisfying*

- *the braid relations: $T_{\tilde{w}}T_{\tilde{w'}} = T_{\tilde{w}\tilde{w'}}$ for $\tilde{w}, \tilde{w'} \in W(1), \ell(\tilde{w}) + \ell(\tilde{w'}) = \ell(\tilde{w}\tilde{w'})$,*
- *the quadratic relations: $T_{\tilde{s}}^2 = q_{\tilde{s}}T_{\tilde{s}^2} + c_{\tilde{s}}T_{\tilde{s}}$ for $\tilde{s} \in S^{aff}(1)$,*

*where $c_{\tilde{s}} = \sum_{t\in Z} c_{\tilde{s}}(t)t \in R[Z]$ is identified with $\sum_{t\in Z} c_{\tilde{s}}(t)T_t$.*

The algebra $\mathcal{H}_R(q_{\tilde{s}}, c_{\tilde{s}})$ is called the $R$-algebra of $W(1)$ with parameters $(q_{\tilde{s}}, c_{\tilde{s}})$.

For convenience, we define a $W(1)$-action on $\mathcal{H}_R(q_{\tilde{s}}, c_{\tilde{s}})$ given by $\tilde{w} \bullet T_{\tilde{w'}} = T_{\tilde{w}\tilde{w'}\tilde{w}^{-1}}$ for any $\tilde{w}, \tilde{w'} \in W(1)$, extended linearly to all elements in $\mathcal{H}_R(q_{\tilde{s}}, c_{\tilde{s}})$.

The following lemma is useful in later discussion:

**Lemma 2.3.** *Let $\tilde{w}_1, \tilde{w}_2, \tilde{v}_1, \tilde{v}_2 \in W(1), \tilde{s}_1, \tilde{s}_2 \in S^{aff}(1)$, and suppose $\tilde{w}_1\tilde{s}_1\tilde{v}_1 = \tilde{w}_2\tilde{s}_2\tilde{v}_2$ and $\pi(\tilde{w}_1\tilde{v}_1) = \pi(\tilde{w}_2\tilde{v}_2)$. Then $\tilde{w}_1c_{\tilde{s}_1}\tilde{v}_1 = \tilde{w}_2c_{\tilde{s}_2}\tilde{v}_2$.*

*Proof.* Since $\pi(\tilde{w}_1\tilde{v}_1) = \pi(\tilde{w}_2\tilde{v}_2)$, we have $\tilde{w}_2\tilde{v}_2 = \tilde{w}_1t\tilde{v}_1$ for some $t \in Z$, hence $\tilde{w}_1^{-1}\tilde{w}_2 = t\tilde{v}_1\tilde{v}_2^{-1}$. Then $\tilde{s}_1 = \tilde{w}_1^{-1}\tilde{w}_2\tilde{s}_2\tilde{v}_2\tilde{v}_1^{-1} = t(\tilde{v}_1\tilde{v}_2^{-1})\tilde{s}_2(\tilde{v}_1\tilde{v}_2^{-1})^{-1}$, therefore $c_{\tilde{s}_1}$ $= t(\tilde{v}_1\tilde{v}_2^{-1})c_{\tilde{s}_2}(\tilde{v}_1\tilde{v}_2^{-1})^{-1} = \tilde{w}_1^{-1}\tilde{w}_2c_{\tilde{s}_2}\tilde{v}_2\tilde{v}_1^{-1}$, i.e., $\tilde{w}_1c_{\tilde{s}_1}\tilde{v}_1 = \tilde{w}_2c_{\tilde{s}_2}\tilde{v}_2$. □

## 2.2 Operator $r_{v,w}$

In this section, we will define an operator $r_{v,w}$ for any pair $(v,w) \in W \times W$ with $v \leq w$. This operator is the main ingredient of this article.

For every $s \in S^{\text{aff}}$, pick a lifing $\tilde{s}$ in $S^{\text{aff}}(1)$, and for every $\tau \in \Omega$, pick a lifting $\tilde{\tau}$ in $\Omega(1)$. Let $w \in W$ with $\ell(w) = n$ and $\underline{w} = s_{i_1} s_{i_2} \cdots s_{i_n} \tau$ be a reduced expression of $w$. A subexpression of $\underline{w}$ is a word $s_{i_1}^{e_{i_1}} s_{i_2}^{e_{i_2}} \cdots s_{i_n}^{e_{i_n}} \tau$ with $(e_{i_1}, e_{i_2}, \cdots, e_{i_n}) \in \{0,1\}^n$. A subexpression is called non-decreasing if $\ell(s_{i_1}^{e_{i_1}} s_{i_2}^{e_{i_2}} \cdots s_{i_n}^{e_{i_n}} \tau) = \sum_{k=1}^{n} e_{i_k}$. Let $v \leq w$, then there exists $(e_{i_1}, e_{i_2}, \cdots, e_{i_n}) \in \{0,1\}^n$ such that $\underline{v_{w}} = s_{i_1}^{e_1} s_{i_2}^{e_2} \cdots s_{i_n}^{e_n} \tau$ equals $v$ and is also a non-decreasing subexpression of $\underline{w}$. Let $\tilde{w} \in W(1)$ be a lifting of $w$, then $\tilde{w}$ has an expression $\underline{\tilde{w}} = t \tilde{s}_{i_1} \tilde{s}_{i_2} \cdots \tilde{s}_{i_n} \tilde{\tau}$ for some $t \in Z$. Then the operator

$$r_{\underline{v_w}} : \bigoplus_{\tilde{w} \in W(1), \pi(\tilde{w}) = w} RT_{\tilde{w}} \longrightarrow \bigoplus_{\tilde{v} \in W(1), \pi(\tilde{v}) = v} RT_{\tilde{v}}$$

is defined term by term and extended linearly, where

$$r_{\underline{v_w}}(T_{\tilde{w}}) = T_t T_{\tilde{s}_{i_1}}^{e_1} (-c_{\tilde{s}_{i_1}})^{1-e_1} T_{\tilde{s}_{i_2}}^{e_2} (-c_{\tilde{s}_{i_2}})^{1-e_2} \cdots T_{\tilde{s}_{i_n}}^{e_n} (-c_{\tilde{s}_{i_n}})^{1-e_n} T_{\tilde{\tau}}.$$

Here the codomain of $r_{\underline{v_w}}$ is regarded as a submodule of $\mathcal{H}_R(q_{\tilde{s}}, c_{\tilde{s}})$.

In other words, we fix $T_{\tilde{s}_{i_k}}$'s for $e_k = 1$, and replace all the other $T_{\tilde{s}_{i_k}}$'s with $-c_{\tilde{s}_{i_k}}$'s. It is easy to see that $r_{\underline{v_w}}$ is independent of choice of liftings.

**Example 2.4.** *In the $SL_3$ case, $W$ is generated by three elements $s_0, s_1, s_2$ with relations $s_i^2 = 1$ for all $i$ and $s_i s_j s_i = s_j s_i s_j$ if $i \neq j$. Let $\tilde{s}_0, \tilde{s}_1, \tilde{s}_2$ be liftings of $s_0, s_1, s_2$ respectively. Let $\underline{w} = s_0 s_1 s_2 s_0 s_1 s_2$, $\underline{\tilde{w}} = t \tilde{s}_0 \tilde{s}_1 \tilde{s}_2 \tilde{s}_0 \tilde{s}_1 \tilde{s}_2$ for some $t \in Z$. Let $(e_1, e_2, e_3, e_4, e_5, e_6) = (1,1,1,0,1,0)$ so that $\underline{v_w} = s_0 s_1 s_2 1 s_1 1$. Then*

$$r_{\underline{v_w}}(T_{\tilde{w}}) = T_t T_{\tilde{s}_0} T_{\tilde{s}_1} T_{\tilde{s}_2} (-c_{\tilde{s}_0}) T_{\tilde{s}_1} (-c_{\tilde{s}_2}) = T_t T_{\tilde{s}_0} T_{\tilde{s}_1} T_{\tilde{s}_2} c_{\tilde{s}_0} T_{\tilde{s}_1} c_{\tilde{s}_2}.$$

A priori, $r_{\underline{v_w}}$ depends not only on the choice of reduced expression $\underline{w}$ but also on the choice of non-decreasing subexpression $\underline{v_w}$. In the following part, we will show that, in fact, $r_{\underline{v_w}}$ is independent of these choices, so the notation $r_{v,w}$ makes sense.

**Lemma 2.5.** *Let $w \in W$ with $\ell(w) = n$ and let $\underline{w} = s_{i_1} s_{i_2} \cdots s_{i_n} \tau$ be a reduced expression of $w$. Let $\tilde{w} \in W(1)$ be a lifting of $w$ with $\underline{\tilde{w}} = t\tilde{s}_{i_1}\tilde{s}_{i_2} \cdots \tilde{s}_{i_n}\tilde{\tau}$ for some $t \in Z$. Let $v \leq w$, and let $\underline{v_w} = s_{i_1}^{e_1} s_{i_2}^{e_2} \cdots s_{i_n}^{e_n}\tau$ and $\underline{v'_w} = s_{i_1}^{f_1}, s_{i_2}^{f_2} \cdots s_{i_n}^{f_n}\tau$ be two non-decreasing subexpressions of $\underline{w}$ which both equal $v$. Then $r_{\underline{v_w}}(T_{\tilde{w}}) = r_{\underline{v'_w}}(T_{\tilde{w}})$.*

*Proof.* We show this by induction on $l = \ell(w) + \ell(v)$.

If $l = 0$, then $r_{\underline{v_w}}(T_{\tilde{w}}) = r_{\underline{v'_w}}(T_{\tilde{w}}) = T_{t\tilde{\tau}}$.

If $l = 1$, then $\ell(w) = 1$ and $\ell(v) = 0$, so $r_{\underline{v_w}}(T_{\tilde{w}}) = r_{\underline{v'_w}}(T_{\tilde{w}}) = T_t(-c_{\tilde{s}_{i_1}})T_{\tilde{\tau}}$.

Now suppose that the statement is correct for $l < k$, and we consider the case when $l = k$.

- If $e_1 = f_1$, then by induction, the statement is correct.

- If $e_1 \neq f_1$, then without loss of generality, we may assume that $e_1 = 1, f_1 = 0$, then

$$r_{\underline{v_w}}(T_{\tilde{w}}) = T_t T_{\tilde{s}_{i_1}} T_{\tilde{s}_{i_2}}^{e_2}(-c_{\tilde{s}_{i_2}})^{1-e_2} \cdots T_{\tilde{s}_{i_n}}^{e_n}(-c_{\tilde{s}_{i_n}})^{1-e_n} T_{\tilde{\tau}},$$

$$r_{\underline{v'_w}}(T_{\tilde{w}}) = T_t(-c_{\tilde{s}_{i_1}}) T_{\tilde{s}_{i_2}}^{f_2}(-c_{\tilde{s}_{i_2}})^{1-f_2} \cdots T_{\tilde{s}_{i_n}}^{f_n}(-c_{\tilde{s}_{i_n}})^{1-f_n} T_{\tilde{\tau}}.$$

Let $\ell(v) = m$, then we may assume that $f_{i_j} = 1$ for $j \in \{j_1, \cdots, j_m\} \subseteq \{2, \cdots, n\}$ and $f_{i_j} = 0$ otherwise in subexpression $\underline{v'_w}$. But $s_{i_1}v < v$, so by exchange condition, $s_{i_1}v = s_{i_{j_1}} \cdots \widehat{s_{i_{j_d}}} \cdots s_{i_{j_m}}$ for some $j_d$. Then by induction,

$$r_{\underline{v}_{\underline{w}}}(T_{\tilde{w}}) = T_t T_{\tilde{s}_{i_1}} T_{\tilde{s}_{i_2}}^{e'_2}(-c_{\tilde{s}_{i_2}})^{1-e'_2} \cdots T_{\tilde{s}_{i_n}}^{e'_n}(-c_{\tilde{s}_{i_n}})^{1-e'_n} T_{\tilde{\tau}},$$

where $e'_{i_j} = 1$ for $j \in \{j_1, \cdots, \widehat{j_d}, \cdots, j_m\}$ and $e'_{i_j} = 0$ otherwise.

Now the only difference between $r_{\underline{v}_{\underline{w}}}(T_{\tilde{w}})$ and $r_{\underline{v'}_{\underline{w}}}(T_{\tilde{w}})$ is that $r_{\underline{v}_{\underline{w}}}(T_{\tilde{w}})$ has $T_{\tilde{s}_{i_1}}$ and $-c_{\tilde{s}_{i_{j_d}}}$ as factors in the first and $j_d$th position respectively, while $r_{\underline{v'}_{\underline{w}}}(T_{\tilde{w}})$ has $-c_{\tilde{s}_{i_1}}$ and $T_{\tilde{s}_{i_{j_d}}}$ as factors in the first and $j_d$th position respectively. The factors in all other positions are the same for $r_{\underline{v}_{\underline{w}}}(T_{\tilde{w}})$ and $r_{\underline{v'}_{\underline{w}}}(T_{\tilde{w}})$.

Since $c_{\tilde{s}}$ is just an $R$-linear combination of elements in $Z$, it suffices to show that

$$\tilde{s}_{i_1} t_1 \tilde{s}_{i_{j_1}} t_2 \tilde{s}_{i_{j_2}} \cdots t_{j_d} c_{\tilde{s}_{i_{j_d}}} \cdots t_m \tilde{s}_{i_{j_m}} = c_{\tilde{s}_{i_1}} t_1 \tilde{s}_{i_{j_1}} t_2 \tilde{s}_{i_{j_2}} \cdots t_{j_d} \tilde{s}_{i_{j_d}} \cdots t_m \tilde{s}_{i_{j_m}}$$

for any $m$-tuple $(t_1, \cdots, t_m) \in Z^m$, which holds by Lemma 2.3.

This finishes the proof. $\square$

This lemma tells us that $r_{\underline{v}_{\underline{w}}}$ is independent of the choice of the non-decreasing subexpression $\underline{v}_{\underline{w}}$. So we can rewrite the operator as $r_{v,\underline{w}}$.

**Theorem 2.6.** *Let $w \in W$ with $\ell(w) = n$ and let $\underline{w}_1 = s_{11}s_{12} \cdots s_{1n}\tau$ and $\underline{w}_2 = s_{21}s_{22} \cdots s_{2n}\tau$ be two reduced expressions of $w$. Let $\tilde{w} \in W(1)$ be a lifting of $w$. Let $v \leq w$ with $\ell(v) = m$, then $r_{v,\underline{w}_1}(T_{\tilde{w}}) = r_{v,\underline{w}_2}(T_{\tilde{w}})$.*

*Proof.* Since $\underline{w}_1$ and $\underline{w}_2$ are two reduced expressions of $w$, then by Theorem 1.9 in [10] there exists a sequence

$$\underline{w}_1 = (\underline{w})_1, (\underline{w})_2, ..., (\underline{w})_d = \underline{w}_2$$

of reduced expressions of $w$ such that $(\underline{w})_i$ and $(\underline{w})_{i+1}$ differ only by a braid relation. So without loss of generality, we may assume that $\underline{w}_1$ and $\underline{w}_2$ differ only by a braid relation, and even more we may assume $n, m$ are both even and the other cases for $n, m$ follow by similar proofs. Then

$$\underline{\tilde{w}}_1 = \underbrace{\tilde{s}_\alpha \tilde{s}_\beta \cdots \tilde{s}_\alpha \tilde{s}_\beta}_{n},$$

$$\underline{\tilde{w}}_2 = t\, \underbrace{\tilde{s}_\beta \tilde{s}_\alpha \cdots \tilde{s}_\beta \tilde{s}_\alpha}_{n},$$

$$v = \underbrace{s_\alpha s_\beta \cdots s_\alpha s_\beta}_{m}.$$

for some $t \in Z$. Therefore,

$$r_{v,\underline{w}_1}(T_{\tilde{w}}) = \underbrace{T_{\tilde{s}_\alpha} \cdots T_{\tilde{s}_\beta}}_{m} \underbrace{(-c_{\tilde{s}_\alpha})(-c_{\tilde{s}_\beta}) \cdots (-c_{\tilde{s}_\beta})}_{n-m}$$

$$= \underbrace{T_{\tilde{s}_\alpha} \cdots T_{\tilde{s}_\beta}}_{m} \underbrace{c_{\tilde{s}_\alpha} c_{\tilde{s}_\beta} \cdots c_{\tilde{s}_\beta}}_{n-m},$$

$$r_{v,\underline{w}_2}(T_{\tilde{w}}) = T_t(-c_{\tilde{s}_\beta}) \underbrace{T_{\tilde{s}_\alpha} \cdots T_{\tilde{s}_\beta}}_{m} \underbrace{(-c_{\tilde{s}_\alpha})(-c_{\tilde{s}_\beta}) \cdots (-c_{\tilde{s}_\alpha})}_{n-m-1}$$

$$= T_t c_{\tilde{s}_\beta} \underbrace{T_{\tilde{s}_\alpha} \cdots T_{\tilde{s}_\beta}}_{m} \underbrace{c_{\tilde{s}_\alpha} c_{\tilde{s}_\beta} \cdots c_{\tilde{s}_\alpha}}_{n-m-1}.$$

It is enough to show that $\underbrace{\tilde{s}_\alpha \cdots \tilde{s}_\beta}_{m} \underbrace{c_{\tilde{s}_\alpha} c_{\tilde{s}_\beta} \cdots c_{\tilde{s}_\beta}}_{n-m} = t c_{\tilde{s}_\beta} \underbrace{\tilde{s}_\alpha \cdots \tilde{s}_\beta}_{m} \underbrace{c_{\tilde{s}_\alpha} c_{\tilde{s}_\beta} \cdots c_{\tilde{s}_\alpha}}_{n-m-1}$. But $t\tilde{s}_\beta = \underbrace{\tilde{s}_\alpha \tilde{s}_\beta \cdots \tilde{s}_\alpha}_{n-1} \tilde{s}_\beta \underbrace{\tilde{s}_\alpha^{-1} \cdots \tilde{s}_\beta^{-1} \tilde{s}_\alpha^{-1}}_{n-1}$, so $t c_{\tilde{s}_\beta} = \underbrace{\tilde{s}_\alpha \tilde{s}_\beta \cdots \tilde{s}_\alpha}_{n-1} c_{\tilde{s}_\beta} \underbrace{\tilde{s}_\alpha^{-1} \cdots \tilde{s}_\beta^{-1} \tilde{s}_\alpha^{-1}}_{n-1}$. Therefore

$$
\begin{aligned}
tc_{\tilde{s}_\beta}\underbrace{\tilde{s}_\alpha\cdots\tilde{s}_\beta}_{m}\underbrace{c_{\tilde{s}_\alpha}c_{\tilde{s}_\beta}\cdots c_{\tilde{s}_\alpha}}_{n-m-1} &= \underbrace{\tilde{s}_\alpha\tilde{s}_\beta\cdots\tilde{s}_\alpha}_{n-1}c_{\tilde{s}_\beta}\underbrace{\tilde{s}_\alpha^{-1}\cdots\tilde{s}_\beta^{-1}\tilde{s}_\alpha^{-1}}_{n-1}\underbrace{\tilde{s}_\alpha\cdots\tilde{s}_\beta}_{m}\underbrace{c_{\tilde{s}_\alpha}c_{\tilde{s}_\beta}\cdots c_{\tilde{s}_\alpha}}_{n-m-1}\\
&= \underbrace{\tilde{s}_\alpha\cdots\tilde{s}_\beta}_{m}\underbrace{\tilde{s}_\alpha\tilde{s}_\beta\cdots\tilde{s}_\alpha}_{n-m-1}c_{\tilde{s}_\beta}\underbrace{\tilde{s}_\alpha^{-1}\cdots\tilde{s}_\beta^{-1}\tilde{s}_\alpha^{-1}}_{n-m-1}\underbrace{c_{\tilde{s}_\alpha}c_{\tilde{s}_\beta}\cdots c_{\tilde{s}_\alpha}}_{n-m-1}\\
&= \underbrace{\tilde{s}_\alpha\cdots\tilde{s}_\beta}_{m}c_{\tilde{s}_\alpha}\underbrace{\tilde{s}_\beta\cdots\tilde{s}_\alpha}_{n-m-2}c_{\tilde{s}_\beta}\underbrace{\tilde{s}_\alpha^{-1}\cdots\tilde{s}_\beta^{-1}}_{n-m-2}\underbrace{c_{\tilde{s}_\beta}\cdots c_{\tilde{s}_\alpha}}_{n-m-2}\\
&= \underbrace{\tilde{s}_\alpha\cdots\tilde{s}_\beta}_{m}c_{\tilde{s}_\alpha}c_{\tilde{s}_\beta}\underbrace{\tilde{s}_\alpha\tilde{s}_\beta\cdots\tilde{s}_\alpha}_{n-m-3}c_{\tilde{s}_\beta}\underbrace{\tilde{s}_\alpha^{-1}\cdots\tilde{s}_\beta^{-1}\tilde{s}_\alpha^{-1}}_{n-m-3}\\
&\quad\underbrace{c_{\tilde{s}_\alpha}c_{\tilde{s}_\beta}\cdots c_{\tilde{s}_\alpha}}_{n-m-3}\\
&\cdots\\
&= \underbrace{\tilde{s}_\alpha\cdots\tilde{s}_\beta}_{m}\underbrace{c_{\tilde{s}_\alpha}c_{\tilde{s}_\beta}\cdots c_{\tilde{s}_\beta}}_{n-m}.
\end{aligned}
$$

The third equality holds since

$$
\underbrace{\tilde{s}_\alpha\tilde{s}_\beta\cdots\tilde{s}_\alpha}_{n-m-1}c_{\tilde{s}_\beta}\underbrace{\tilde{s}_\alpha^{-1}\cdots\tilde{s}_\beta^{-1}\tilde{s}_\alpha^{-1}}_{n-m-1}c_{\tilde{s}_\alpha} = c_{\tilde{s}_\alpha}\underbrace{\tilde{s}_\beta\cdots\tilde{s}_\alpha}_{n-m-2}c_{\tilde{s}_\beta}\underbrace{\tilde{s}_\alpha^{-1}\cdots\tilde{s}_\beta^{-1}}_{n-m-2}
$$

which is true because $\underbrace{\tilde{s}_\beta\cdots\tilde{s}_\alpha}_{n-m-2}c_{\tilde{s}_\beta}\underbrace{\tilde{s}_\alpha^{-1}\cdots\tilde{s}_\beta^{-1}}_{n-m-2} \in R[Z]$ and $\tilde{s}_\alpha t'\tilde{s}_\alpha^{-1}c_{\tilde{s}_\alpha} = c_{\tilde{s}_\alpha t'\tilde{s}_\alpha^{-1}\tilde{s}_\alpha}$ $= c_{\tilde{s}_\alpha t'} = c_{\tilde{s}_\alpha}t'$ for any $t' \in Z$. And all subsequent equalities hold for a similar reason. $\square$

As the main result of this section, this theorem guarantees that $r_{v,\underline{w}}$ is independent of the choice of reduced expression of $w$. So we can rewrite the operator as $r_{v,w}$, which is what we need and will be used later.

By definition of the operator, we can easily get the following propositions.

**Proposition 2.7.** *Let $u, v, w \in W$ and suppose $u \leq v \leq w$, then*

$$r_{u,v} r_{v,w} = r_{u,w}.$$

**Proposition 2.8.** *Let $u, v, w \in W$ and $\tilde{u}, \tilde{w} \in W(1)$ be liftings of $u, w$ respectively.*

*(1) If $v \leq w$ and $\ell(uv) = \ell(u) + \ell(v), \ell(uw) = \ell(u) + \ell(w)$, then*

$$T_{\tilde{u}} r_{v,w}(T_{\tilde{w}}) = r_{uv,uw}(T_{\tilde{u}\tilde{w}}).$$

*(2) If $v \leq w$ and $\ell(vu) = \ell(v) + \ell(u), \ell(wu) = \ell(w) + \ell(u)$, then*

$$r_{v,w}(T_{\tilde{w}}) T_{\tilde{u}} = r_{vu,wu}(T_{\tilde{w}\tilde{u}}).$$

# Chapter 3: Center of $\mathcal{H}_R(0, c_{\tilde{s}})$

## 3.1 Iwahori Weyl Group

From this section, we will assume that $W$ is an Iwahori Weyl group which is a special case of the Coxeter group. We recall some basic settings of the Iwahori Weyl group.

Let $\Sigma$ be a reduced root system with simple system $\Delta$. Let $W_0$ be the finite Weyl group of $\Sigma$, and $S_0$ be the set of simple reflections corresponding to $\Delta$. Then $S_0$ is a generating set of $W_0$.

Let $\mathcal{V} = \mathbb{Z}\Sigma^\vee \otimes_{\mathbb{Z}} \mathbb{R}$ be the $\mathbb{R}$-vector space spanned by the dual root system $\Sigma^\vee$. Let $\Sigma^{\text{aff}}$ be the affine root system associated to $\Sigma$, i.e. the set $\Sigma + \mathbb{Z}$ of affine functionals on $\mathcal{V}$. The term hyperplane always means the null-set of an element of $\Sigma^{\text{aff}}$.

Choose a special vertex $\mathfrak{v}_0 \in \mathcal{V}$ such that $\mathfrak{v}_0$ is stabilized by the action of $W_0$. Let $\mathfrak{C}_0$ be the Weyl chamber at $\mathfrak{v}_0$ corresponding to $S_0$ and let $\mathfrak{A}_0 \in \mathfrak{C}_0$ be the alcove for which $\mathfrak{v}_0 \in \bar{\mathfrak{A}}_0$ where $\bar{\mathfrak{A}}_0$ is the closure of $\mathfrak{A}_0$.

Let $W^{\text{aff}}$ be the affine Weyl group of $\Sigma^{\text{aff}}$ and $S^{\text{aff}}$ be the set of affine reflections corresponding to walls of $\mathfrak{A}_0$. Then $S^{\text{aff}}$ is a generating set of $W^{\text{aff}}$ extended from $S_0$. $(W^{\text{aff}}, S^{\text{aff}})$ is a Coxeter system, and we can equip $W^{\text{aff}}$ with the length function $\ell$ and the Bruhat order $\leq$.

Let $F$ be a non-archimedean local field and let $\mathbf{G}$ be a connected reductive $F$-group. Let $\mathbf{T} \subseteq \mathbf{G}$ be a maximal $F$-split torus and set $\mathbf{Z}$ and $\mathbf{N}$ be $\mathbf{G}$-centralizer and $\mathbf{G}$-normalizer of $\mathbf{T}$ respectively. Let $\mathbf{G}(F), \mathbf{T}(F), \mathbf{Z}(F), \mathbf{N}(F)$ be the groups of $F$-points of $\mathbf{G}, \mathbf{T}, \mathbf{Z}, \mathbf{N}$. Then the group $\mathbf{Z}(F)$ admits a unique parahoric subgroup $\mathbf{Z}(F)_0$. We may define the Iwahori-Weyl group of $(\mathbf{G}, \mathbf{T})$ to be the quotient $W := \mathbf{N}(F)/\mathbf{Z}(F)_0$.

There are two ways to express the Iwahori-Weyl group as a semidirect product. By the work of Bruhat and Tits, it is known that there exists a reduced root system $\Sigma$ such that the corresponding affine Weyl group is a subgroup of $W$. Denoting by $W_0$ the finite Weyl group of $\Sigma$, it can be shown that $W = \Lambda \rtimes W_0$ and that $W = W^{\text{aff}} \rtimes \Omega$. For more details of these semidirect products, consult [12] and [6]. The action of $W^{\text{aff}}$ on $\mathcal{V}$ extends to an action of $W$. The subgroup $\Lambda$ acts on $\mathcal{V}$ by translations and the subgroup $\Omega$ acts on $\mathcal{V}$ by invertible affine transformations that stabilize the base alcove $\mathfrak{A}_0$ in $\mathcal{V}$.

The group $\Omega$ stabilizes $S^{\text{aff}}$. By the semidirect product $W = W^{\text{aff}} \rtimes \Omega$, we know that $W^{\text{aff}}, S^{\text{aff}}, \Omega, W$ satisfy the assumptions mentioned in the beginning of Section 2.

The group $\Lambda$ is finitely generated and abelian. In general, $\Lambda$ may not be torsion free. The action of $\Lambda$ on $\mathcal{V}$ is given by the homomorphism

$$\nu : \Lambda \to \mathcal{V}$$

such that $\lambda \in \Lambda$ acts as translation by $\nu(\lambda)$ in $\mathcal{V}$. The group $\Lambda$ is normalized by $x \in W_0$: $x\lambda x^{-1}$ acts as translation by $x(\nu(\lambda))$. The length $\ell$ is constant on each

$W_0$-conjugacy class in $\Lambda$. By Lemma 2.1 in [15], a conjugacy class of $W$ is finite if and only if it is contained in $\Lambda$.

In addition, $\Lambda(1)$ is normal in $W(1)$ and $W(1) = \Lambda(1)W_0(1)$, $Z = \Lambda(1) \cap W_0(1)$. Any finite conjugacy class of $W(1)$ is contained in $\Lambda(1)$.

We'll later use the following geometric characterization of length (see Lemma 5.1.1 in [13]):

**Lemma 3.1.** *Let $w \in W$ and $s \in S^{aff}$. If $H_s$ is the hyperplane stabilized by $s$, then*

- *$\ell(sw) > \ell(w)$ if and only if $\mathfrak{A}_0$ and $w(\mathfrak{A}_0)$ are on the same side of $H_s$,*
- *$\ell(ws) > \ell(w)$ if and only if $\mathfrak{A}_0$ and $w(\mathfrak{A}_0)$ are on the same side of $w(H_s)$.*

The following result of Bruhat order on $W$ is also useful.

**Lemma 3.2.** *Let $w \in W$ and $s \in S^{aff}$. Suppose $\ell(w) = \ell(sws)$.*

- *If $w \in \Lambda$ and $sws = w$, then $sw = ws > w$.*
- *If $sws \neq w$, then $sw > w > ws$ or $ws > w > sw$.*

*Proof.* The first statement follows from Lemma 3.1. When $\Omega$ is trivial, the second statement follows from Lemma in 7.2 of [11]. The more general statement is immediate by definition of the Bruhat order and length function on $W$ because $W = W^{\mathrm{aff}} \rtimes \Omega$ and $\Omega$ stabilizes $S^{\mathrm{aff}}$. □

When $W$ is an Iwahori Weyl group. A basis of the center of the $R$-algebra $\mathcal{H}_R(q_{\tilde{s}}, c_{\tilde{s}})$ associated to $W(1)$ is given in [15] by using the Bernstein presentation. This basis can be very complicated when written explicitly by Iwahori-Matsumoto presentation. But when $q_{\tilde{s}} = 0$, we can write out a basis explicitly.

## 3.2 Maximal Length Elements

Let $\mathcal{Z}_R(q_{\tilde{s}}, c_{\tilde{s}})$ be the center of $\mathcal{H}_R(q_{\tilde{s}}, c_{\tilde{s}})$ and $h \in \mathcal{Z}_R(q_{\tilde{s}}, c_{\tilde{s}})$. Then

$$h = \sum_{\tilde{w} \in W(1)} a_{\tilde{w}} T_{\tilde{w}}, \quad \text{for some } \ a_{\tilde{w}} \in R.$$

Set $\operatorname{supp}(h) = \{\tilde{w} \in W(1) | a_{\tilde{w}} \neq 0\}$. Let $\operatorname{supp}(h)_{\max}$ be the set of maximal length elements in $\operatorname{supp}(h)$. The following theorem tells what $\operatorname{supp}(h)_{\max}$ is comprised of.

**Theorem 3.3.** *Suppose $h \in \mathcal{Z}_R(q_{\tilde{s}}, c_{\tilde{s}})$, then $supp(h)_{\max}$ is a union of conjugacy classes in $W(1)$.*

This theorem comes from the following results.

**Lemma 3.4.** *Let $\tilde{s} \in S^{aff}(1)$, $h \in \mathcal{Z}_R(q_{\tilde{s}}, c_{\tilde{s}})$ and $\tilde{w} \in supp(h)_{\max}$. If $\ell(\tilde{s}\tilde{w}) > \ell(\tilde{w})$ or $\ell(\tilde{w}\tilde{s}) > \ell(\tilde{w})$, then $\tilde{s}\tilde{w}\tilde{s}^{-1} \in supp(h)_{\max}$ and $a_{\tilde{s}\tilde{w}\tilde{s}^{-1}} = a_{\tilde{w}}$.*

*Proof.* Without loss of generality, we may assume that $\ell(\tilde{s}\tilde{w}) > \ell(\tilde{w})$. Then $\tilde{s}\tilde{w} \in \operatorname{supp}(T_{\tilde{s}}h) = \operatorname{supp}(hT_{\tilde{s}})$ since $T_{\tilde{s}}h = hT_{\tilde{s}}$, and

$$\operatorname{supp}(T_{\tilde{s}}h)_{\max} = \{\tilde{s}\tilde{x} | \tilde{x} \in \operatorname{supp}(h)_{\max}, \ell(\tilde{s}\tilde{x}) > \ell(\tilde{x})\},$$

$$\operatorname{supp}(hT_{\tilde{s}})_{\max} = \{\tilde{y}\tilde{s} | \tilde{y} \in \operatorname{supp}(h)_{\max}, \ell(\tilde{y}\tilde{s}) > \ell(\tilde{y})\}.$$

Both sets are nonempty because $\tilde{s}\tilde{w} \in \operatorname{supp}(T_{\tilde{s}}h)_{\max}$. Therefore, $\tilde{s}\tilde{w}\tilde{s}^{-1} \in \operatorname{supp}(h)_{\max}$ and $\ell(\tilde{s}\tilde{w}\tilde{s}^{-1}) = \ell(\tilde{w})$. The $R$-coefficient of $T_{\tilde{s}\tilde{w}}$ in $T_{\tilde{s}}h$ is $a_{\tilde{w}}$ and the $R$-coefficient of $T_{\tilde{s}\tilde{w}}$ in $hT_{\tilde{s}}$ is $a_{\tilde{s}\tilde{w}\tilde{s}^{-1}}$. Thus $a_{\tilde{s}\tilde{w}\tilde{s}^{-1}} = a_{\tilde{w}}$. □

We recall the Main Theorem in [13]:

**Theorem 3.5.** *Fix $w \in W$. If $w \notin \Lambda$ then there exists $s \in S^{aff}$ and $s_1, \cdots, s_n \in S^{aff}$ such that, setting $w' \overset{def}{=} s_n \cdots s_1 w s_1 \cdots s_n$,*

- *$\ell(s_i \cdots s_1 w s_1 \cdots s_i) = \ell(w)$ for all $i$,*
- *$\ell(sw's) > \ell(w')$.*

**Lemma 3.6.** *Suppose $h \in \mathcal{Z}_R(q_{\tilde{s}}, c_{\tilde{s}})$ and $\tilde{w} \in supp(h)_{\max}$, then $\tilde{w} \in \Lambda(1)$.*

*Proof.* We prove by contradiction. Assume $\tilde{w} \in \mathrm{supp}(h)_{\max}$ but $\tilde{w} \notin \Lambda(1)$.

By Theorem 3.5, there exist $\tilde{s} \in S^{\mathrm{aff}}(1)$ and $\tilde{s}_1, \tilde{s}_2, \cdots, \tilde{s}_n \in S^{\mathrm{aff}}(1)$ such that

- $\ell(\tilde{s}_i \cdots \tilde{s}_1 \tilde{w} \tilde{s}_1^{-1} \cdots \tilde{s}_i^{-1}) = \ell(\tilde{w})$ for all $i$,
- $\pi(\tilde{s}_i \tilde{s}_{i-1} \cdots \tilde{s}_1 \tilde{w} \tilde{s}_1^{-1} \cdots \tilde{s}_{i-1}^{-1} \tilde{s}_i^{-1}) \neq \pi(\tilde{s}_{i-1} \cdots \tilde{s}_1 \tilde{w} \tilde{s}_1^{-1} \cdots \tilde{s}_{i-1}^{-1})$ for all $i$,
- $\ell(\tilde{s}\tilde{w'}\tilde{s}^{-1}) > \ell(\tilde{w'})$, where $\tilde{w'} = \tilde{s}_n \cdots \tilde{s}_2 \tilde{s}_1 \tilde{w} \tilde{s}_1^{-1} \tilde{s}_2^{-1} \cdots \tilde{s}_n^{-1}$.

By Lemma 3.2 and Lemma 3.4, $\tilde{s}_i \cdots \tilde{s}_1 \tilde{w} \tilde{s}_1^{-1} \cdots \tilde{s}_i^{-1} \in \mathrm{supp}(h)_{\max}$ for all $i$, in particular, $\tilde{w'} = \tilde{s}_n \cdots \tilde{s}_2 \tilde{s}_1 \tilde{w} \tilde{s}_1^{-1} \tilde{s}_2^{-1} \cdots \tilde{s}_n^{-1} \in \mathrm{supp}(h)_{\max}$.

By Lemma 3.4 again, $\tilde{s}\tilde{w'}\tilde{s}^{-1} \in \mathrm{supp}(h)_{\max}$. But $\ell(\tilde{s}\tilde{w'}\tilde{s}^{-1}) > \ell(\tilde{w'})$, which is a contradiction. $\square$

*Proof of Theorem 3.3.* It suffices to show that if $h \in \mathcal{Z}_R(q_{\tilde{s}}, c_{\tilde{s}})$, $\tilde{w} \in \mathrm{supp}(h)_{\max}$ and $Cl(\tilde{w})$ is the $W(1)$-conjugacy class of $\tilde{w}$ in $W(1)$, then $Cl(\tilde{w}) \subseteq \mathrm{supp}(h)_{\max}$ and $a_{\tilde{w'}} = a_{\tilde{w}}$ for any $\tilde{w'} \in Cl(\tilde{w})$.

By Lemma 3.4 and Lemma 3.6, $\tilde{x}\tilde{w}\tilde{x}^{-1} \in \mathrm{supp}(h)_{\max}$ and $a_{\tilde{x}\tilde{w}\tilde{x}^{-1}} = a_{\tilde{w}}$ for any $\tilde{x} \in W^{\mathrm{aff}}(1)$. It remains to show that $\tilde{\tau}\tilde{w}\tilde{\tau}^{-1} \in \mathrm{supp}(h)_{\max}$ and $a_{\tilde{\tau}\tilde{w}\tilde{\tau}^{-1}} = a_{\tilde{w}}$ for any

$\tilde\tau \in \Omega(1)$. But $\tilde\tau\tilde w \in \mathrm{supp}(T_{\tilde\tau}h) = \mathrm{supp}(hT_{\tilde\tau})$, and

$$\mathrm{supp}(T_{\tilde\tau}h)_{\max} = \{\tilde\tau\tilde x | \tilde x \in \mathrm{supp}(h)_{\max}\},$$

$$\mathrm{supp}(hT_{\tilde\tau})_{\max} = \{\tilde y\tilde\tau | \tilde y \in \mathrm{supp}(h)_{\max}\}.$$

Both sets are nonempty because $\tilde\tau\tilde w \in \mathrm{supp}(T_{\tilde\tau}h)_{\max}$. Therefore, $\tilde\tau\tilde w\tilde\tau^{-1} \in \mathrm{supp}(h)_{\max}$. The $R$-coefficient of $T_{\tilde\tau\tilde w}$ in $T_{\tilde\tau}h$ is $a_{\tilde w}$ and the $R$-coefficient of $T_{\tilde\tau\tilde w}$ in $hT_{\tilde\tau}$ is $a_{\tilde\tau\tilde w\tilde\tau^{-1}}$. Thus $a_{\tilde\tau\tilde w\tilde\tau^{-1}} = a_{\tilde w}$. □

By Lemma 1.1 in [15], a conjugacy class $C$ of $W$ is finite if and only if $C$ is contained in $\Lambda$. In $W(1)$, we can only conclude that any finite conjugacy class is contained in $\Lambda(1)$. So $\mathrm{supp}(h)_{\max}$ is a union of some conjugacy classes in $\Lambda(1)$.

## 3.3 Some Technical Results

Let $C$ be a finite conjugacy class in $W(1)$. Set

$$h_{\lambda,C} = \sum_{\tilde\lambda \in \pi^{-1}(\lambda)\cap C} T_{\tilde\lambda},$$

for every $\lambda \in \pi(C)$.

In the rest of this section, we fix a finite conjugacy class $C$ in $W(1)$ and write $h_\lambda$ for $h_{\lambda,C}$ without ambiguity. Now we prove some properties of $r_{x,\lambda}(h_\lambda)$.

**Lemma 3.7.** *Let $\lambda \in \pi(C)$ and $s \in S^{aff}$. Let $x \in W$ with $x < sx$ or $x < xs$. Suppose that $x \leq \lambda$ and $x \leq s\lambda s$. Then*

$$r_{x,\lambda}(h_\lambda) = r_{x,s\lambda s}(h_{s\lambda s}).$$

*Proof.* Without loss of generality, we may assume $x < sx$.

If $s\lambda s = \lambda$, then it is clearly true.

If $s\lambda s \neq \lambda$, then by Lemma 3.2 and without loss of generality, we may assume $s\lambda < \lambda$. In this case, $x \leq s\lambda$ by Lemma 2.1. Thus

$$r_{x,\lambda}(h_\lambda) = r_{x,s\lambda}(r_{s\lambda,\lambda}(h_\lambda)), \quad r_{x,s\lambda s}(h_{s\lambda s}) = r_{x,s\lambda}(r_{s\lambda,s\lambda s}(h_{s\lambda s})).$$

It suffices to show that $r_{s\lambda,\lambda}(h_\lambda) = r_{s\lambda,s\lambda s}(h_{s\lambda s})$.

Since $c_{\tilde{s}^{-1}} \in R[Z]$, we may assume that

$$c_{\tilde{s}^{-1}} = \sum_{t\in Z} b_t t, \quad \text{for some } \ b_t \in R.$$

Then

$$\begin{aligned}
& r_{s\lambda,\lambda}(h_\lambda) = r_{s\lambda,s\lambda s}(h_{s\lambda s}) \\
\Longleftrightarrow & \sum_{\tilde{\lambda}\in\pi^{-1}(\lambda)\cap C} -c_{\tilde{s}^{-1}} T_{\tilde{s}\tilde{\lambda}} = \sum_{\tilde{\lambda}\in\pi^{-1}(\lambda)\cap C} -T_{\tilde{s}\tilde{\lambda}} c_{\tilde{s}^{-1}} \\
\Longleftrightarrow & \sum_{\tilde{\lambda}\in\pi^{-1}(\lambda)\cap C} -c_{\tilde{s}^{-1}} T_{\tilde{s}\tilde{\lambda}} = \sum_{\tilde{\lambda}\in\pi^{-1}(\lambda)\cap C} -T_{\tilde{s}\tilde{\lambda}} (\tilde{s} c_{\tilde{s}^{-1}} \tilde{s}^{-1}) \\
\Longleftrightarrow & \sum_{\tilde{\lambda}\in\pi^{-1}(\lambda)\cap C} -(\sum_{t\in Z} b_t t) T_{\tilde{s}\tilde{\lambda}} = \sum_{\tilde{\lambda}\in\pi^{-1}(\lambda)\cap C} -T_{\tilde{s}\tilde{\lambda}} (\tilde{s}(\sum_{t\in Z} b_t t)\tilde{s}^{-1}) \\
\Longleftrightarrow & \sum_{t\in Z} b_t \sum_{\tilde{\lambda}\in\pi^{-1}(\lambda)\cap C} -T_{t\tilde{s}\tilde{\lambda}} = \sum_{t\in Z} b_t \sum_{\tilde{\lambda}\in\pi^{-1}(\lambda)\cap C} -T_{\tilde{s}\tilde{\lambda}(\tilde{s}t\tilde{s}^{-1})}.
\end{aligned}$$

We want to show the last equation. It suffices to show that

$$\begin{aligned}
& \sum_{\tilde{\lambda}\in\pi^{-1}(\lambda)\cap C} t\tilde{s}\tilde{\lambda} = \sum_{\tilde{\lambda}\in\pi^{-1}(\lambda)\cap C} \tilde{s}\tilde{\lambda}(\tilde{s}t\tilde{s}^{-1}) \\
\Longleftrightarrow & \sum_{\tilde{\lambda}\in\pi^{-1}(\lambda)\cap C} \tilde{s}^{-1}t\tilde{s}\tilde{\lambda} = \sum_{\tilde{\lambda}\in\pi^{-1}(\lambda)\cap C} \tilde{\lambda}(\tilde{s}t\tilde{s}^{-1}) \\
\Longleftrightarrow & (\tilde{s}^{-1}t\tilde{s})(\sum_{\tilde{\lambda}\in\pi^{-1}(\lambda)\cap C} \tilde{\lambda})(\tilde{s}^{-1}t\tilde{s})^{-1} = \sum_{\tilde{\lambda}\in\pi^{-1}(\lambda)\cap C} \tilde{\lambda}
\end{aligned}$$

for any $t \in Z$ in the group algebra $R[W(1)]$. The last equation holds because $\sum_{\tilde{\lambda} \in \pi^{-1}(\lambda) \cap C} \tilde{\lambda}$ is fixed by $Z$. $\square$

For $w, w' \in W$, we write $w \xrightarrow{s} w'$ if $w' = sws$ and $\ell(w') = \ell(w) - 2$.

For $w, w' \in W$, we write $w \overset{s}{\sim} w'$ if $w' = sws$, $\ell(w') = \ell(w)$, and $sw > w$ or $ws > w$. We write $w \sim w'$ if $\exists$ a sequence

$$w = w_0, w_1, ..., w_n = w'$$

such that $w_{i-1} \overset{s_i}{\sim} w_i$ for every $i$ and some $s_i \in S^{\text{aff}}$. If $\lambda, \lambda'$ are in the same finite conjugacy class in $W$, then $\lambda' = w\lambda w^{-1}$ for some $w \in W$. Since $W = \Lambda \rtimes W_0$, we can write $w = w_0\lambda''$ for some $w_0 \in W_0$ and $\lambda'' \in \Lambda$. Thus by commutativity of $\Lambda$, $\lambda' = (w_0\lambda'')\lambda(w_0\lambda'')^{-1} = w_0\lambda w_0^{-1}$. By Lemma 3.2, we have $\lambda \sim \lambda'$.

**Lemma 3.8.** *Let $\lambda \in \pi(C)$. Let $x, x' \in W$ and $x \leq \lambda$. Suppose*

$$x = x_0 \overset{s_1}{\sim} x_1 \overset{s_2}{\sim} \cdots \overset{s_n}{\sim} x_n = x',$$

*for some $s_i \in S^{\text{aff}}$. Let $w = s_n \cdots s_1$. Then there exist $\lambda' \sim \lambda$ and $\tilde{w} \in W(1)$ a lifting of $w$, such that $x' \leq \lambda'$ and*

$$\tilde{w} \bullet (r_{x,\lambda}(h_\lambda)) = r_{x',\lambda'}(h_{\lambda'}).$$

*Proof.* It suffices to consider the case where $x \overset{s}{\sim} x'$ for some $s \in S^{\text{aff}}$, i.e. $x' = sxs$. Without loss of generality, we may assume that $sx > x$.

- If $s\lambda > \lambda$, then by Lemma 2.1 $sxs \leq s\lambda s$. It is enough to show that

$T_{\tilde{s}} r_{x,\lambda}(h_\lambda) = r_{sxs,s\lambda s}(h_{s\lambda s}) T_{\tilde{s}}$ for any $\tilde{s} \in S^{aff}(1)$ with $\pi(\tilde{s}) = s$. But

$$\begin{aligned} T_{\tilde{s}} r_{x,\lambda}(h_\lambda) &= r_{sx,s\lambda}(T_{\tilde{s}} h_\lambda) \\ &= r_{sx,s\lambda}(h_{s\lambda s} T_{\tilde{s}}) \\ &= r_{sxs,s\lambda s}(h_{s\lambda s}) T_{\tilde{s}}. \end{aligned}$$

The second the equality holds because $\tilde{s} \bullet h_\lambda = h_{s\lambda s}$, and the other equalities hold by Proposition 2.8. Therefore,

$$\tilde{s} \bullet (r_{x,\lambda}(h_\lambda)) = r_{x',s\lambda s}(h_{s\lambda s}).$$

- If $s\lambda < \lambda$, then by Lemma 2.1 and Lemma 3.2, $sxs < sx \leq \lambda$ and $x \leq s\lambda < s\lambda s$. Therefore, for any $\tilde{s} \in S^{aff}(1)$ with $\pi(\tilde{s}) = s$, we have

$$\begin{aligned} T_{\tilde{s}}(r_{x,\lambda}(h_\lambda)) &= T_{\tilde{s}}(r_{x,s\lambda s}(h_{s\lambda s})) \\ &= r_{sx,\lambda s}(T_{\tilde{s}} h_{s\lambda s}) \\ &= r_{sx,\lambda s}(h_\lambda T_{\tilde{s}}) \\ &= (r_{sxs,\lambda}(h_\lambda)) T_{\tilde{s}}. \end{aligned}$$

The first equality holds by Lemma 3.7. The third equality holds because $\tilde{s} \bullet h_{s\lambda s} = h_\lambda$ and the other equalities hod by Proposition 2.8. Thus

$$\tilde{s} \bullet (r_{x,\lambda}(h_\lambda)) = r_{x',\lambda}(h_\lambda).$$

This finishes the proof. $\square$

Recall that $\nu$ is the homomorphism which defines the action of $\Lambda$. Set $\Lambda^+ = \{\lambda \in \Lambda | \beta(\nu(\lambda)) \geq 0, \forall \beta \in \Sigma^+\}$ where $\Sigma^+$ is the set of positive roots in $\Sigma$. A element in $\Lambda$ is called dominant if it is contained in $\Lambda^+$. Let $\mu_0 \in \Lambda^+$ and $\lambda \in \Lambda$. Let $\lambda_0$ be a dominant element in $\{\lambda' \in \Lambda | \lambda' \sim \lambda\}$. In fact, $\lambda_0$ is unique. Suppose $\lambda_0, \lambda_0'$ are both dominant and in $\{\lambda' \in \Lambda | \lambda' \sim \lambda\}$, then $\lambda_0' = w\lambda_0 w^{-1}$ for some $w \in W$. We know $w = w_0\lambda''$ for some $w_0 \in W_0$ and $\lambda'' \in \Lambda$. Hence $\lambda_0' = w_0\lambda_0 w_0^{-1}$ since $\Lambda$ is abelian. But $\nu(\lambda_0') = \nu(w_0\lambda_0 w_0^{-1}) = w_0(\nu(\lambda_0))$ and $\nu(\lambda_0)$ are not in the same chamber unless $w_0 = 1$, that is, $\lambda_0' = \lambda_0$. Suppose $\mu_0 \leq \lambda$, then by Corollary 4.4 in [4], $\mu_0 \leq \lambda_0$. We have the following result.

**Lemma 3.9.** *Let $\mu_0 \in \Lambda^+$ and $\lambda \in \Lambda$. Let $\lambda_0$ be the unique dominant element in $\{\lambda' \in \Lambda | \lambda' \sim \lambda\}$. Suppose $\mu_0 \leq \lambda$, then there exists a sequence*

$$\lambda_0, \lambda_1, \cdots, \lambda_n = \lambda$$

*such that $\lambda_{i-1} \overset{s_i}{\sim} \lambda_i$ for every $i$ and some $s_i \in S_0$, and $\mu_0 \leq \lambda_i$ for all $i$.*

*Proof.* Since $\lambda \sim \lambda_0$, there exists $w \in W_0$ such that $\lambda = w\lambda_0 w^{-1}$. We prove the statement by induction on $l = \ell(w)$.

If $l = 0, 1$, then it is obvious.

Now suppose that the statement is correct for $l < k$, and we consider the case when $l = k$. Let $w = s_{i_k} \cdots s_{i_1}$ and it suffices to show that $\mu_0 \leq s_{i_k}\lambda s_{i_k}$.

If $s_{i_k}\lambda s_{i_k} = \lambda$, then it is obvious.

If $s_{i_k}\lambda s_{i_k} \neq \lambda$, then by $s_{i_k}w < w$ and Lemma 3.1, $w(\mathfrak{A}_0)$ and $\mathfrak{A}_0$ are on different sides of $H_{s_{i_k}}$. On the other hand, $s_{i_k}\lambda s_{i_k} \neq \lambda$, then $\nu(\lambda) = \nu(w\lambda_0 w^{-1}) = w(\nu(\lambda_0)) \in w(\bar{\mathfrak{C}}_0) \backslash H_{s_{i_k}}$. Thus $\lambda(\mathfrak{A}_0) = \mathfrak{A}_0 + \nu(\lambda)$ and $\mathfrak{A}_0$ are on different sides of $H_{s_{i_k}}$, i.e. $s_{i_k}\lambda < \lambda$ by Lemma 3.1. We also have $s_{i_k}\mu_0 > \mu_0$, thus by Lemma 2.1, $\mu_0 \leq s_{i_k}\lambda < s_{i_k}\lambda s_{i_k}$, which finishes the proof. $\square$

**Theorem 3.10.** *Let $\lambda_1, \lambda_2 \in \pi(C)$ and $x \in W$. Suppose $x \le \lambda_1, \lambda_2$, then*

$$r_{x,\lambda_1}(h_{\lambda_1}) = r_{x,\lambda_2}(h_{\lambda_2}).$$

*Proof.* We prove it by induction on $d = \ell(\lambda_1) - \ell(x) = \ell(\lambda_2) - \ell(x)$.

If $d = 0$, then it is obvious since $x = \lambda_1 = \lambda_2$.

Now suppose $d > 0$.

- If $x \notin \Lambda$, then by Theorem 3.5 there exist $s_1, s_2, \cdots, s_n, s' \in S^{\mathrm{aff}}$ such that $s_i s_{i-1} \cdots s_1 x s_1 \cdots s_{i-1} s_i \overset{s_{i+1}}{\sim} s_{i+1} s_i s_{i-1} \cdots s_1 x s_1 \cdots s_{i-1} s_i s_{i+1}$ for all $i$ and $s' s_n s_{n-1} \cdots s_1 x s_1 \cdots s_{n-1} s_n s' \xrightarrow{s'} s_n s_{n-1} \cdots s_1 x s_1 \cdots s_{n-1} s_n$. Let $\tilde{w} \in W^{\mathrm{aff}}(1)$ be a lifting of $s_n s_{n-1} \cdots s_1$ and $x' = s_n s_{n-1} \cdots s_1 x s_1 \cdots s_{n-1} s_n$. Then by Lemma 3.8,

  $$\tilde{w} \bullet (r_{x,\lambda_1}(h_{\lambda_1})) = r_{x',\lambda_1'}(h_{\lambda_1'}), \quad \tilde{w} \bullet (r_{x,\lambda_2}(h_{\lambda_2})) = r_{x',\lambda_2'}(h_{\lambda_2'}),$$

  for some $\lambda_1' \sim \lambda_1, \lambda_2' \sim \lambda_2$. We have $\lambda_1' \sim \lambda_2'$ because $\lambda_1 \sim \lambda_2$.

  It suffices to show that $r_{x',\lambda_1'}(h_{\lambda_1'}) = r_{x',\lambda_2'}(h_{\lambda_2'})$. It can be checked using Lemma 2.1 that $s'x' \le \lambda_j'$ or $s'\lambda_j's'$ for $j = 1, 2$. By Lemma 3.7 and without loss of generality, we may assume that $s'x' \le \lambda_1', \lambda_2'$, then

  $$r_{x',\lambda_1'}(h_{\lambda_1'}) = r_{x',s'x'}(r_{s'x',\lambda_1'}(h_{\lambda_1'})) = r_{x',s'x'}(r_{s'x',\lambda_2'}(h_{\lambda_2'})) = r_{x',\lambda_2'}(h_{\lambda_2'}),$$

  where the second equality holds by induction. If $s'x' \le s'\lambda_j's'$, then $x' < s'\lambda_j's'$. By Lemma 3.7,

  $$r_{x',\lambda_j'}(h_{\lambda_j'}) = r_{x',s'\lambda_j's'}(h_{s'\lambda_j's'}) = r_{x',s'x'}(r_{s'x',s'\lambda_j's'}(h_{s'\lambda_j's'})),$$

  and we can apply a similar proof as above.

- If $x \in \Lambda$, then there exists $w = s_n \cdots s_1$ with $s_i \in W_0$ such that $x = x_0 \overset{s_1}{\sim} x_1 \overset{s_2}{\sim} \cdots \overset{s_n}{\sim} x_n = x'$ and $x' \in \Lambda^+$. Let $\tilde{w} \in W(1)$ be a lifting of $w$, then by Lemma 3.8,

$$\tilde{w} \bullet (r_{x,\lambda_1}(h_{\lambda_1})) = r_{x',\lambda_1'}(h_{\lambda_1'}), \quad \tilde{w} \bullet (r_{x,\lambda_2}(h_{\lambda_2})) = r_{x',\lambda_2'}(h_{\lambda_2'}),$$

for some $\lambda_1' \sim \lambda_1, \lambda_2' \sim \lambda_2$. We have $\lambda_1' \sim \lambda_2'$ because $\lambda_1 \sim \lambda_2$.

It suffices to show that $r_{x',\lambda_1'}(h_{\lambda_1'}) = r_{x',\lambda_2'}(h_{\lambda_2'})$. By Lemma 3.7 and 3.9, $r_{x',\lambda_1'}(h_{\lambda_1'}) = r_{x',\lambda_0}(h_{\lambda_0}) = r_{x',\lambda_2'}(h_{\lambda_2'})$ where $\lambda_0 \in \Lambda^+$ and $\lambda_0 \sim \lambda_1', \lambda_0 \sim \lambda_2'$.

This finishes the proof. $\square$

## 3.4 Main Theorem

From this section, all our discussions will be under the condition where $q_{\tilde{s}} = 0$ for all $\tilde{s} \in S^{\mathrm{aff}}(1)$, that is, we will consider the algebra $\mathcal{H}_R(0, c_{\tilde{s}})$ and the center $\mathcal{Z}_R(0, c_{\tilde{s}})$. In this case, the quadratic relations become $T_{\tilde{s}}^2 = c_{\tilde{s}} T_{\tilde{s}}$.

Let $C$ be a finite conjugacy class in $W(1)$. Then $C \subset \Lambda(1), \pi(C) \subset \Lambda$ and there is a unique element $\lambda_0 \in \pi(C) \cap \Lambda^+$. Set

$$\mathrm{Adm}(C) = \mathrm{Adm}(\lambda_0) = \{w \in W | w \leq \lambda \text{ for some } \lambda \in \pi(C)\}.$$

We define

$$h_C = \sum_{w \in \mathrm{Adm}(C)} h_w,$$

where $h_w = r_{w,\lambda}(h_\lambda)$ for any $\lambda \in \pi(C)$ with $\lambda > w$. By Theorem 3.10, $h_C$ is well defined.

**Lemma 3.11.** *Suppose $C$ is a finite conjugacy class in $W(1)$. Then $h_C \in \mathcal{Z}_R(0, c_{\tilde{s}})$.*

*Proof.* For any $\tilde{\tau} \in \Omega(1)$ with $\pi(\tilde{\tau}) = \tau$,

$$\begin{aligned} T_{\tilde{\tau}} h_C &= \sum_{w \in \mathrm{Adm}(C)} T_{\tilde{\tau}} h_w \\ &= \sum_{w \in \mathrm{Adm}(C)} h_{\tau w \tau^{-1}} T_{\tilde{\tau}} \\ &= (\tilde{\tau} \bullet (\sum_{w \in \mathrm{Adm}(C)} h_w)) T_{\tilde{\tau}} \\ &= h_C T_{\tilde{\tau}}. \end{aligned}$$

The second equality holds by definition of $h_C$ and Proposition 2.8, and the third equality holds because $h_C$ is stable under the action of $W(1)$.

It remains to show that for any $\tilde{s} \in S^{\mathrm{aff}}(1)$ with $\pi(\tilde{s}) = s$, $T_{\tilde{s}} h_C = h_C T_{\tilde{s}}$. The left hand side

$$T_{\tilde{s}} h_C = \sum_{w \in \mathrm{Adm}(C)} T_{\tilde{s}} h_w = \sum_{x, sx \in \mathrm{Adm}(C)} T_{\tilde{s}} h_x + \sum_{y \in \mathrm{Adm}(C), sy \notin \mathrm{Adm}(C)} T_{\tilde{s}} h_y.$$

If $x, sx \in \mathrm{Adm}(C)$, then without loss of generality, we may assume $x < sx \leq \lambda \in \pi(C)$. In this case,

$$\begin{aligned} T_{\tilde{s}} h_x + T_{\tilde{s}} h_{sx} &= T_{\tilde{s}} r_{x,\lambda}(h_\lambda) + T_{\tilde{s}} r_{sx,\lambda}(h_\lambda) \\ &= T_{\tilde{s}} r_{x,\lambda}(h_\lambda) + c_{\tilde{s}} r_{sx,\lambda}(h_\lambda) \\ &= T_{\tilde{s}} r_{x,\lambda}(h_\lambda) + T_{\tilde{s}}(-r_{x,sx}(r_{sx,\lambda}(h_\lambda))) \\ &= T_{\tilde{s}} r_{x,\lambda}(h_\lambda) + T_{\tilde{s}}(-r_{x,\lambda}(h_\lambda)) \\ &= 0. \end{aligned}$$

The second equality holds because $T_{\tilde{s}} T_{\tilde{s}\tilde{x}} = c_{\tilde{s}} T_{\tilde{s}\tilde{x}}$ for any $\tilde{x} \in W(1)$ with $\pi(\tilde{x}) = x$. The third equality holds because $c_{\tilde{s}} T_{\tilde{s}} = T_{\tilde{s}} c_{\tilde{s}}$ and $c_{\tilde{s}} T_{\tilde{s}\tilde{x}} = T_{\tilde{s}}(c_{\tilde{s}} T_{\tilde{x}}) = T_{\tilde{s}}(-r_{x,sx}(T_{\tilde{s}\tilde{x}}))$

for any $\tilde{x} \in W(1)$ with $\pi(\tilde{x}) = x$. The fourth equality holds by Proposition 2.8. Therefore,

$$T_{\tilde{s}} h_C = \sum_{x \in \mathrm{Adm}(C), sx \notin \mathrm{Adm}(C)} T_{\tilde{s}} h_x.$$

Similarly,

$$h_C T_{\tilde{s}} = \sum_{x \in \mathrm{Adm}(C), xs \notin \mathrm{Adm}(C)} h_x T_{\tilde{s}}.$$

But it is easy to check by Lemma 2.1 that there is a one-to-one correspondence between the two sets $\{x \in \mathrm{Adm}(C) | sx \notin \mathrm{Adm}(C)\}$ and $\{x \in \mathrm{Adm}(C) | xs \notin \mathrm{Adm}(C)\}$, i.e., $y \in \{x \in \mathrm{Adm}(C) | sx \notin \mathrm{Adm}(C)\}$ if and only if $sys \in \{x \in \mathrm{Adm}(C) | xs \notin \mathrm{Adm}(C)\}$. Therefore, it is enough to show that if $x \in \mathrm{Adm}(C)$ and $sx \notin \mathrm{Adm}(C)$, then

$$T_{\tilde{s}} h_x = h_{sxs} T_{\tilde{s}}.$$

Now $x < sx$, and we suppose $x \leq \lambda \in \pi(C)$. If $s\lambda > \lambda$, then by Lemma 2.1 $sxs \leq s\lambda s$, thus

$$\begin{aligned} T_{\tilde{s}} h_x &= T_{\tilde{s}} r_{x,\lambda}(h_\lambda) \\ &= r_{sx,s\lambda}(T_{\tilde{s}} h_\lambda) \\ &= r_{sx,s\lambda}(h_{sxs} T_{\tilde{s}}) \\ &= r_{sxs,s\lambda s}(h_{s\lambda s}) T_{\tilde{s}} \\ &= h_{sxs} T_{\tilde{s}}. \end{aligned}$$

The second and fourth equalities hold by Proposition 2.8. The third equality holds because $\tilde{s} \bullet h_\lambda = h_{s\lambda s}$.

If $s\lambda < \lambda$, then by Lemma 2.1 $sx \leq \lambda$, but $\lambda < \lambda s$ so by Lemma 2.1 again

$sxs \leq \lambda$ and $sx \leq \lambda s$, therefore $x \leq s\lambda s$. Now let $y = sxs$, then $y \leq \lambda$ and $sys \leq s\lambda s$, therefore applying a similar proof as above, we have $h_y T_{\tilde{s}} = T_{\tilde{s}} h_{sys}$, i.e., $T_{\tilde{s}} h_x = h_{sxs} T_{\tilde{s}}$.

This finishes the proof. $\square$

**Theorem 3.12** (Main Theorem)**.** *The center $\mathcal{Z}_R(0, c_{\tilde{s}})$ of $\mathcal{H}_R(0, c_{\tilde{s}})$ has a basis $\{h_C\}_{C \in \mathcal{F}(W(1))}$, where $\mathcal{F}(W(1))$ is the family of finite conjugacy classes in $W(1)$.*

*Proof.* First, we show that $\{h_C\}_{C \in \mathcal{F}(W(1))}$ is linearly independent.

Let $C_1, C_2, \cdots, C_n$ be distinct conjugacy classes in $\mathcal{F}(W(1))$. Suppose that $h = \sum_{i=1}^{n} a_i h_{C_i} = 0$ for some $a_i \in R$. We show that $a_i = 0$ for all $i$ by induction on $n$.

If $n = 1$, apparently $a_1 = 0$.

Suppose the statement is correct for $n < k$, and we consider the case when $n = k$. We write $\ell(C)$ as the common length of elements in a finite conjugacy class $C$. Choose $C_j$ from $\{C_1, C_2, \cdots, C_k\}$ such that $\ell(C_j)$ is maximal. Let $w \in C_j$, then only $h_{C_j}$ contains the term $T_w$ and the $R$-coefficient of $T_w$ in $h$ is $a_j$, so $a_j = 0$. Then by induction, we also have $a_i = 0$ for all $i \neq j$. Therefore, $\{h_C\}_{C \in \mathcal{F}(W(1))}$ is linearly independent.

By Lemma 3.11, we know $h_C \in \mathcal{Z}_R(0, c_{\tilde{s}})$ for all $C \in \mathcal{F}(W(1))$. Next, we show that $\{h_C\}_{C \in \mathcal{F}(W(1))}$ spans $\mathcal{Z}_R(0, c_{\tilde{s}})$. For any $h \in \mathcal{Z}_R(0, c_{\tilde{s}})$, we show that $h$ is an $R$-linear combination of elements in $\{h_C\}_{C \in \mathcal{F}(W(1))}$. We prove this by induction on $n = \max_{w \in \mathrm{supp}(h)} \ell(w)$.

If $n = 0$, then by Theorem 3.3 and its proof, we know that the statement is correct.

Suppose the statement is correct for $n < k$. We consider the case when $n = k$.

By Theorem 3.3, we know that $\mathrm{supp}(h)_{\max} = \cup_{i=1}^{m} C_i$ for some $C_i \in \mathcal{F}(W(1))$. By the proof of Theorem 3.3, we know that, for any $i$, if we choose two arbitrary elements $w, w'$ from $C_i$, then the $R$-coefficients of $T_w$ and $T_{w'}$ are the same in $h$, so we can write this common coefficient as $a_{C_i}$. Then the element

$$h' = h - \sum_{i=1}^{n} a_{C_i} h_{C_i}$$

is also in $\mathcal{Z}_R(0, c_{\tilde{s}})$, and $\max_{w \in \mathrm{supp}(h')} \ell(w) < k$. By induction, $h'$ is an $R$-linear combination of elements in $\{h_C\}_{C \in \mathcal{F}(W(1))}$. Therefore, $h$ is also an $R$-linear combination of elements in $\{h_C\}_{C \in \mathcal{F}(W(1))}$.

This finishes the proof. $\square$

# Chapter 4: Examples

Given a finite conjugacy class $C$ in $W(1)$, we can write out the corresponding central element $h_C$ as follow.

Since we know what $\pi(C)$ is, we can write out $h_{\lambda,C}$ for each $\lambda \in \pi(C)$. For other $x \in \mathrm{Adm}(C)$, it is easy to find a $\lambda \in \pi(C)$ such that $x < \lambda$. Then we can apply the operator $r_{x,\lambda}$ on $h_{\lambda,C}$ by changing some factors $T_{\tilde{s}}$ to $-c_{\tilde{s}}$. Adding up all these terms, we get $h_C$.

In this section, we give two examples to show how the above process works.

**Example 4.1.** *In the $GL_2$ case, the Iwahori Weyl group $W = W^{aff} \rtimes \Omega$. The affine Weyl group $W^{aff}$ is generated by $S^{aff} = \{s_0, s_1\}$. The group $\Omega$ is generated by $\tau$ and $\tau s_0 = s_1 \tau, \tau s_1 = s_0 \tau$.*

*Suppose $C_1$ is a finite conjugacy class in $W(1)$ with*

$$\pi(C_1) = \{s_0 s_1 s_0 s_1, s_1 s_0 s_1 s_0\}.$$

*Then $Adm(C_1) = \{s_0 s_1 s_0 s_1, s_1 s_0 s_1 s_0, s_0 s_1 s_0, s_1 s_0 s_1, s_0 s_1, s_1 s_0, s_0, s_1, 1\}$.*

*Suppose*

$$h_{s_0 s_1 s_0 s_1, C_1} = \sum_{t \in Z_1} T_{\tilde{s}_0 \tilde{s}_1 \tilde{s}_0 \tilde{s}_1 t},$$

*for some subset $Z_1 \subseteq Z$. Then*

$$h_{s_1 s_0 s_1 s_0, C_1} = \sum_{t \in Z_1} T_{\tilde{s}_1 \tilde{s}_0 \tilde{s}_1 t \tilde{s}_0},$$

*where $\tilde{s}_1 \tilde{s}_0 \tilde{s}_1 t \tilde{s}_0$ is indeed a lifting of $s_1 s_0 s_1 s_0$.*

*Since $s_0 s_1 s_0, s_1 s_0 s_1 < s_0 s_1 s_0 s_1$, we have*

$$h_{s_0 s_1 s_0, C_1} = r_{s_0 s_1 s_0, s_0 s_1 s_0 s_1}(h_{s_0 s_1 s_0 s_1, C_1}) = \sum_{t \in Z_1} -T_{\tilde{s}_0 \tilde{s}_1 \tilde{s}_0} c_{\tilde{s}_1 t},$$

$$h_{s_1 s_0 s_1, C_1} = r_{s_1 s_0 s_1, s_0 s_1 s_0 s_1}(h_{s_0 s_1 s_0 s_1, C_1}) = \sum_{t \in Z_1} -c_{\tilde{s}_0} T_{\tilde{s}_1 \tilde{s}_0 \tilde{s}_1 t}.$$

*Since $s_0 s_1, s_1 s_0 < s_0 s_1 s_0 s_1$, we have*

$$h_{s_0 s_1, C_1} = r_{s_0 s_1, s_0 s_1 s_0 s_1}(h_{s_0 s_1 s_0 s_1, C_1}) = \sum_{t \in Z_1} c_{\tilde{s}_0} c_{\tilde{s}_1} T_{\tilde{s}_0 \tilde{s}_1 t},$$

$$h_{s_1 s_0, C_1} = r_{s_1 s_0, s_0 s_1 s_0 s_1}(h_{s_0 s_1 s_0 s_1, C_1}) = \sum_{t \in Z_1} c_{\tilde{s}_0} T_{\tilde{s}_1 \tilde{s}_0} c_{\tilde{s}_1 t}.$$

*Since $s_0, s_1 < s_0 s_1 s_0 s_1$, we have*

$$h_{s_0, C_1} = r_{s_0, s_0 s_1 s_0 s_1}(h_{s_0 s_1 s_0 s_1, C_1}) = \sum_{t \in Z_1} -T_{\tilde{s}_0} c_{\tilde{s}_1} c_{\tilde{s}_0} c_{\tilde{s}_1 t},$$

$$h_{s_1, C_1} = r_{s_1, s_0 s_1 s_0 s_1}(h_{s_0 s_1 s_0 s_1, C_1}) = \sum_{t \in Z_1} -c_{\tilde{s}_0} c_{\tilde{s}_1} c_{\tilde{s}_0} T_{\tilde{s}_1 t}.$$

*Since $1 < s_0 s_1 s_0 s_1$, we have*

$$h_{1, C_1} = r_{1, s_0 s_1 s_0 s_1}(h_{s_0 s_1 s_0 s_1, C_1}) = \sum_{t \in Z_1} c_{\tilde{s}_0} c_{\tilde{s}_1} c_{\tilde{s}_0} c_{\tilde{s}_1 t}.$$

*We can easily tell that the parity of the sign is determined by the length difference.*

*Therefore the corresponding central element is*

$$h_{C_1} = \sum_{t \in Z_1} T_{\tilde{s}_0\tilde{s}_1\tilde{s}_0\tilde{s}_1 t} + T_{\tilde{s}_1\tilde{s}_0\tilde{s}_1 t\tilde{s}_0} - T_{\tilde{s}_0\tilde{s}_1\tilde{s}_0} c_{\tilde{s}_1 t}$$

$$- c_{\tilde{s}_0} T_{\tilde{s}_1\tilde{s}_0\tilde{s}_1 t} + c_{\tilde{s}_0} c_{\tilde{s}_1} T_{\tilde{s}_0\tilde{s}_1 t} + c_{\tilde{s}_0} T_{\tilde{s}_1\tilde{s}_0} c_{\tilde{s}_1 t}$$

$$- T_{\tilde{s}_0} c_{\tilde{s}_1} c_{\tilde{s}_0} c_{\tilde{s}_1 t} - c_{\tilde{s}_0} c_{\tilde{s}_1} c_{\tilde{s}_0} T_{\tilde{s}_1 t} + c_{\tilde{s}_0} c_{\tilde{s}_1} c_{\tilde{s}_0} c_{\tilde{s}_1 t}.$$

*Suppose $C_2$ is another finite conjugacy class in $W(1)$ with*

$$\pi(C_2) = \{s_0 s_1 s_0 \tau, s_1 s_0 s_1 \tau\}.$$

*Then $Adm(C_2) = \{s_0 s_1 s_0 \tau, s_1 s_0 s_1 \tau, s_0 s_1 \tau, s_1 s_0 \tau, s_0 \tau, s_1 \tau, \tau\}$.*

*Suppose*

$$h_{s_0 s_1 s_0 \tau, C_2} = \sum_{t \in Z_2} T_{\tilde{s}_0\tilde{s}_1\tilde{s}_0\tilde{\tau} t},$$

*for some subset $Z_2 \subseteq Z$. Then*

$$h_{s_1 s_0 s_1 \tau, C_2} = \sum_{t \in Z_2} T_{\tilde{s}_1\tilde{s}_0\tilde{s}_1\tilde{s}_0\tilde{\tau} t \tilde{s}_1^{-1}} = \sum_{t \in Z_2} T_{\tilde{s}_1\tilde{s}_0\tilde{s}_1\tilde{\tau}(\tilde{\tau}^{-1}\tilde{s}_0\tilde{\tau} t \tilde{s}_1^{-1})},$$

*where $\tilde{\tau}^{-1}\tilde{s}_0\tilde{\tau} t \tilde{s}_1^{-1}$ is an element in $Z$. So $\tilde{s}_1\tilde{s}_0\tilde{s}_1\tilde{\tau}(\tilde{\tau}^{-1}\tilde{s}_0\tilde{\tau} t \tilde{s}_1^{-1})$ is indeed a lifting of $s_1 s_0 s_1 \tau$.*

*Since $s_0 s_1 \tau, s_1 s_0 \tau < s_0 s_1 s_0 \tau$, we have*

$$h_{s_0 s_1 \tau, C_2} = r_{s_0 s_1 \tau, s_0 s_1 s_0 \tau}(h_{s_0 s_1 s_0 \tau, C_2}), \quad h_{s_1 s_0 \tau, C_2} = r_{s_1 s_0 \tau, s_0 s_1 s_0 \tau}(h_{s_0 s_1 s_0 \tau, C_2}).$$

*Since $s_0 \tau, s_1 \tau < s_0 s_1 s_0 \tau$, we have*

$$h_{s_0 \tau, C_2} = r_{s_0 \tau, s_0 s_1 s_0 \tau}(h_{s_0 s_1 s_0 \tau, C_2}), \quad h_{s_1 \tau, C_2} = r_{s_1 \tau, s_0 s_1 s_0 \tau}(h_{s_0 s_1 s_0 \tau, C_2}).$$

*Since $\tau < s_0 s_1 s_0 \tau$, we have*

$$h_{\tau, C_2} = r_{\tau, s_0 s_1 s_0 \tau}(h_{s_0 s_1 s_0 \tau, C_2}).$$

*Therefore the corresponding central element is*

$$h_{C_2} = \sum_{t \in Z_2} T_{\tilde{s}_0\tilde{s}_1\tilde{s}_0\tilde{\tau}t} + T_{\tilde{s}_1\tilde{s}_0\tilde{s}_1\tilde{\tau}(\tilde{\tau}^{-1}\tilde{s}_0\tilde{\tau}t\tilde{s}_1^{-1})} - T_{\tilde{s}_0\tilde{s}_1}c_{\tilde{s}_0}T_{\tilde{\tau}t} - c_{\tilde{s}_0}T_{\tilde{s}_1\tilde{s}_0\tilde{\tau}t}$$

$$+ c_{\tilde{s}_0}c_{\tilde{s}_1}T_{\tilde{s}_0\tilde{\tau}t} + c_{\tilde{s}_0}T_{\tilde{s}_1}c_{\tilde{s}_0}T_{\tilde{\tau}t} - c_{\tilde{s}_0}c_{\tilde{s}_1}c_{\tilde{s}_0}T_{\tilde{\tau}t}.$$

**Example 4.2.** *In the $SL_3$ case, the Iwahori Weyl group $W = W^{aff}$. The affine Weyl group $W^{aff}$ is generated by $S^{aff} = \{s_0, s_1, s_2\}$ with braid relations $s_is_js_i = s_js_is_j$ for $i \neq j$.*

*Suppose $C$ is a finite conjugacy class in $W(1)$ with*

$$\pi(C) = \{s_0s_1s_2s_1, s_1s_0s_1s_2, s_2s_0s_2s_1, s_1s_2s_1s_0, s_2s_1s_0s_1, s_1s_2s_0s_2\}.$$

*Then*

$$Adm(C) = \{s_0s_1s_2s_1, s_1s_0s_1s_2, s_2s_0s_2s_1, s_1s_2s_1s_0, s_2s_1s_0s_1, s_1s_2s_0s_2,$$

$$s_1s_2s_1, s_1s_0s_1, s_2s_0s_2, s_0s_1s_2, s_0s_2s_1, s_1s_0s_2, s_1s_2s_0, s_2s_1s_0, s_2s_0s_1,$$

$$s_0s_1, s_0s_2, s_1s_2, s_2s_1, s_1s_0, s_2s_0, s_0, s_1, s_2, 1\}.$$

*Suppose*

$$h_{s_0s_1s_2s_1,C} = \sum_{t \in Z'} T_{\tilde{s}_0\tilde{s}_1\tilde{s}_2\tilde{s}_1t},$$

*for some subset $Z' \subseteq Z$. Then*

$$h_{s_1s_0s_1s_2,C} = \sum_{t\in Z'} T_{\tilde{s}_1 t\tilde{s}_0\tilde{s}_1\tilde{s}_2},$$

$$h_{s_2s_0s_2s_1,C} = \sum_{t\in Z'} T_{\tilde{s}_2\tilde{s}_0\tilde{s}_1\tilde{s}_2\tilde{s}_1 t\tilde{s}_2^{-1}} = \sum_{t\in Z'} T_{\tilde{s}_2\tilde{s}_0\tilde{s}_2\tilde{s}_1(\tilde{s}_1^{-1}\tilde{s}_2^{-1}\tilde{s}_1\tilde{s}_2\tilde{s}_1 t\tilde{s}_2^{-1})},$$

$$h_{s_1s_2s_1s_0,C} = \sum_{t\in Z'} T_{\tilde{s}_1\tilde{s}_2\tilde{s}_1 t\tilde{s}_0},$$

$$h_{s_2s_1s_0s_1,C} = \sum_{t\in Z'} T_{\tilde{s}_2\tilde{s}_1 t\tilde{s}_0\tilde{s}_1},$$

$$h_{s_1s_2s_0s_2,C} = \sum_{t\in Z'} T_{\tilde{s}_2^{-1}\tilde{s}_1\tilde{s}_2\tilde{s}_1 t\tilde{s}_0\tilde{s}_2} = \sum_{t\in Z'} T_{(\tilde{s}_2^{-1}\tilde{s}_1\tilde{s}_2\tilde{s}_1 t\tilde{s}_2^{-1}\tilde{s}_1^{-1})\tilde{s}_1\tilde{s}_2\tilde{s}_0\tilde{s}_2},$$

*where $\tilde{s}_1^{-1}\tilde{s}_2^{-1}\tilde{s}_1\tilde{s}_2\tilde{s}_1 t\tilde{s}_2^{-1}, \tilde{s}_2^{-1}\tilde{s}_1\tilde{s}_2\tilde{s}_1 t\tilde{s}_2^{-1}\tilde{s}_1^{-1}$ are elements in $Z$. So the elements $\tilde{s}_2\tilde{s}_0\tilde{s}_2\tilde{s}_1(\tilde{s}_1^{-1}\tilde{s}_2^{-1}\tilde{s}_1\tilde{s}_2\tilde{s}_1 t\tilde{s}_2^{-1})$ and $(\tilde{s}_2^{-1}\tilde{s}_1\tilde{s}_2\tilde{s}_1 t\tilde{s}_2^{-1}\tilde{s}_1^{-1})\tilde{s}_1\tilde{s}_2\tilde{s}_0\tilde{s}_2$ are indeed liftings of $s_2s_0s_2s_1$ and $s_1s_2s_0s_2$ respectively.*

*Since $s_1s_2s_1, s_0s_1s_2, s_0s_2s_1 < s_0s_1s_2s_1; s_1s_0s_1, s_1s_0s_2 < s_1s_0s_1s_2;$ $s_2s_0s_2 < s_2s_0s_2s_1; s_1s_2s_0, s_2s_1s_0 < s_1s_2s_1s_0; s_2s_0s_1 < s_2s_1s_0s_1$, we have*

$$h_{s_1s_2s_1,C} = r_{s_1s_2s_1,s_0s_1s_2s_1}(h_{s_0s_1s_2s_1,C}), h_{s_1s_0s_1,C} = r_{s_1s_0s_1,s_1s_0s_1s_2}(h_{s_1s_0s_1s_2,C}),$$

$$h_{s_2s_0s_2,C} = r_{s_2s_0s_2,s_2s_0s_2s_1}(h_{s_2s_0s_2s_1,C}), h_{s_0s_1s_2,C} = r_{s_0s_1s_2,s_0s_1s_2s_1}(h_{s_0s_1s_2s_1,C}),$$

$$h_{s_0s_2s_1,C} = r_{s_0s_2s_1,s_0s_1s_2s_1}(h_{s_0s_1s_2s_1,C}), h_{s_1s_0s_2,C} = r_{s_1s_0s_2,s_1s_0s_1s_2}(h_{s_1s_0s_1s_2,C}),$$

$$h_{s_1s_2s_0,C} = r_{s_1s_2s_0,s_1s_2s_1s_0}(h_{s_1s_2s_1s_0,C}), h_{s_2s_1s_0,C} = r_{s_2s_1s_0,s_1s_2s_1s_0}(h_{s_1s_2s_1s_0,C}),$$

$$h_{s_2s_0s_1,C} = r_{s_2s_0s_1,s_2s_1s_0s_1}(h_{s_2s_1s_0s_1,C}).$$

*Since $s_0s_1, s_0s_2, s_1s_2, s_2s_1 < s_0s_1s_2s_1; s_1s_0, s_2s_0 < s_1s_2s_1s_0$, we have*

$$h_{s_0s_1,C} = r_{s_0s_1,s_0s_1s_2s_1}(h_{s_0s_1s_2s_1,C}), h_{s_0s_2,C} = r_{s_0s_2,s_0s_1s_2s_1}(h_{s_0s_1s_2s_1,C}),$$

$$h_{s_1s_2,C} = r_{s_1s_2,s_0s_1s_2s_1}(h_{s_0s_1s_2s_1,C}), h_{s_2s_1,C} = r_{s_2s_1,s_0s_1s_2s_1}(h_{s_0s_1s_2s_1,C}),$$

$$h_{s_1s_0,C} = r_{s_1s_0,s_1s_2s_1s_0}(h_{s_1s_2s_1s_0,C}), h_{s_2s_0,C} = r_{s_2s_0,s_1s_2s_1s_0}(h_{s_1s_2s_1s_0,C}).$$

*Since* $s_0, s_1, s_2 < s_0s_1s_2s_1$, *we have*

$$h_{s_0,C} = r_{s_0,s_0s_1s_2s_1}(h_{s_0s_1s_2s_1,C}), h_{s_1,C} = r_{s_1,s_0s_1s_2s_1}(h_{s_0s_1s_2s_1,C}),$$

$$h_{s_2,C} = r_{s_2,s_0s_1s_2s_1}(h_{s_0s_1s_2s_1,C}).$$

*Since* $1 < s_0s_1s_2s_1$, *we have*

$$h_{1,C} = r_{1,s_0s_1s_2s_1}(h_{s_0s_1s_2s_1,C}).$$

*Therefore the corresponding central element is*

$$h_C = \sum_{t\in Z'} T_{\tilde{s}_0\tilde{s}_1\tilde{s}_2\tilde{s}_1t} + T_{\tilde{s}_1t\tilde{s}_0\tilde{s_1}\tilde{s}_2} + T_{\tilde{s}_2\tilde{s}_0\tilde{s}_2\tilde{s}_1(\tilde{s}_1^{-1}\tilde{s}_2^{-1}\tilde{s}_1\tilde{s}_2\tilde{s}_1t\tilde{s}_2^{-1})}$$

$$+ T_{\tilde{s}_1\tilde{s}_2\tilde{s}_1t\tilde{s}_0} + T_{\tilde{s}_2\tilde{s}_1t\tilde{s}_0\tilde{s}_1} + T_{(\tilde{s}_2^{-1}\tilde{s}_1\tilde{s}_2\tilde{s}_1t\tilde{s}_2^{-1}\tilde{s}_1^{-1})\tilde{s}_1\tilde{s}_2\tilde{s}_0\tilde{s}_2}$$

$$- c_{\tilde{s}_0}T_{\tilde{s}_1\tilde{s}_2\tilde{s}_1t} - T_{\tilde{s}_1t\tilde{s}_0\tilde{s_1}}c_{\tilde{s}_2} - T_{\tilde{s}_2\tilde{s}_0\tilde{s}_2}c_{\tilde{s}_1(\tilde{s}_1^{-1}\tilde{s}_2^{-1}\tilde{s}_1\tilde{s}_2\tilde{s}_1t\tilde{s}_2^{-1})}$$

$$- T_{\tilde{s}_0\tilde{s}_1\tilde{s}_2}c_{\tilde{s}_1t} - T_{\tilde{s}_0}c_{\tilde{s}_1}T_{\tilde{s}_2\tilde{s}_1t} - T_{\tilde{s}_1t\tilde{s}_0}c_{\tilde{s_1}}T_{\tilde{s}_2}$$

$$- T_{\tilde{s}_1\tilde{s}_2}c_{\tilde{s}_1t}T_{\tilde{s}_0} - c_{\tilde{s}_1}T_{\tilde{s}_2\tilde{s}_1t\tilde{s}_0} - T_{\tilde{s}_2}c_{\tilde{s}_1t}T_{\tilde{s}_0\tilde{s}_1}$$

$$+ T_{\tilde{s}_0\tilde{s}_1}c_{\tilde{s}_2}c_{\tilde{s}_1t} + T_{\tilde{s}_0}c_{\tilde{s}_1}T_{\tilde{s}_2}c_{\tilde{s}_1t} + c_{\tilde{s}_0}T_{\tilde{s}_1\tilde{s}_2}c_{\tilde{s}_1t} + c_{\tilde{s}_0}c_{\tilde{s}_1}T_{\tilde{s}_2\tilde{s}_1t}$$

$$+ T_{\tilde{s}_1}c_{\tilde{s}_2}c_{\tilde{s}_1t}T_{\tilde{s}_0} + c_{\tilde{s}_1}T_{\tilde{s}_2}c_{\tilde{s}_1t}T_{\tilde{s}_0} - T_{\tilde{s}_0}c_{\tilde{s}_1}c_{\tilde{s}_2}c_{\tilde{s}_1t}$$

$$- c_{\tilde{s}_0}c_{\tilde{s}_1}c_{\tilde{s}_2}T_{\tilde{s}_1t} - c_{\tilde{s}_0}c_{\tilde{s}_1}T_{\tilde{s}_2}c_{\tilde{s}_1t} + c_{\tilde{s}_0}c_{\tilde{s}_1}c_{\tilde{s}_2}c_{\tilde{s}_1t}.$$

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