ABSTRACT

Title of Dissertation: ESSAYS ON CONSUMER CHOICE
AND SOCIAL PREFERENCE:

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People frequently engage in daily activities with two distinct roles: as a consumer, and as a social being. These two are the major parts of human lives where we can find some of the most prominent and fundamental questions: How do consumers makes decision under resource scarcity and imperfect cognition? What are the considerations underneath social interactions such as allocative decision and reciprocal behavior? This dissertation, *Essays on Consumer Choice and Social Preference*, focuses on uncovering the mechanisms that drives choice behavior by employing a choice-theoretic approach.

Chapter 1: Recommendations play an undeniable role in decision-making. The empirical literature argues that recommendation can influence demand through two distinct channels: i) by enlarging awareness (attention channel), or ii) by altering

preferences (utility channel). In this paper, we develop a framework to study these two channels. We illustrate when and how one can distinguish through which channel the recommendations affect choices. We offer both deterministic and probabilistic models. While deterministic models aim to identify the basic observable behavioral differences between these two channels, our probabilistic models are suitable for econometric estimation, which is crucial for studying aggregate behavior used in empirical work. Our parametric models offer unique identification under minimal data requirements. This enables us to make out-of-sample predictions for counter factual analysis for policy design purposes. In addition, we offer simple and intuitive behavioral postulates characterizing each model so that one can test our models.

Chapter 2: Consumer has attention scarcity. When there are more products in the market, it is natural to expect that each product gets less attention due to the competition over attention. We call such phenomenon *attention overload*. However, existing models fall short of capturing attention overload. Interestingly, all existing models on attention fall short of capturing attention overload. Therefore, we propose a model of random attention, *Attention Overload Model*. Under this assumption, we develop theories on revealed preference and, also, revealed attention – uncovering the upper and lower bounds for the attention an alternative received. We present the characterization results and show that the model reconciles Choice

Overload. Lastly, we demonstrate how an additional condition over attention at binaries enhances welfare judgement for policy maker.

Chapter 3: We propose and axiomatize the inequality-averse model with rank-dependent (dis-)utility under risk and uncertainty. The model highlights an important linkage, *Guilt Moderation*, between different other-regarding behaviors: when choices are risky, decision maker feels *less* guilt by assigning more weight to the fairer outcomes, creating a tendency to exhibit self-centered (or altruistic) behavior when outcomes are mixed with a fairer (or unfairer) outcome. Our model provides a unifying explanation for two seemingly distinct *reversal* behaviors known in the literature as *moral wiggle room* and *ex-ante fairness for you* that put into question the consistency of attitudes towards inequality in the presence of uncertainty. Moreover, we characterize guilt moderation with the reversal behaviors and risk preference for others. Lastly, the model sheds light on self-other risk attitudes gap and increased envy in wage transparency.

ESSAYS ON CONSUMER CHOICE AND SOCIAL PREFERENCE

by

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Table of Contents

List of Tables

List of Figures

Chapter 1: Decision Making with Recommendation

Co-authored with Yusufcan Masatlioglu

1.1 Introduction

Recommendation is one of the key determinants in decision-making. For instance, we constantly rely on recommendations from our friends, consumer reports and mass media when selecting a movie to see; a book to read; a car to buy; a school to send our children.[1] Many internet sites such as Amazon.com, Netflix, YouTube, LinkedIn, Spotify, Tripadvisor, and Facebook incorporate recommendation tools to help their customers with the burden of choice. "*Frequently bought together*", "*Products related to this item* " and "*Best-selling*" sections of Amazon.com and "*the Top Picks*" of Netflix are just a few examples of these recommendation tools.[2]

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[1] Some online service websites such as Angi, HomeAdvisor, Houzz, Thumbtack recommend services for consumers' specific projects, which shows that people are willing to pay for recommendations.

[2] According to the 2013 data released by McKinsey & Company, recommendation systems drive 35% of purchases at Amazon. Similarly, 75% of what people watch on Netflix is initiated by their product recommendations.

Individual choices are directly influenced by what is recommended to them. The evidence on recommendations influencing choice behavior is conclusive across a wide spectrum of economic activities (e.g. for labour market, Horton [4]; for hospitality and tourism, Litvin et al. [5]; for music streaming service, Li et al. [6], Adomavicius et al. [7]; for e-commerce (of commodities or goods), Senecal and Nantel [8], Häubl and Trifts [9], Rowley [10], Vijayasarathy and Jones [11], Goodman et al. [12]). We know that recommendations affect choices, but there is not a theoretical framework through which we can understand how recommendations affect choices.

It has been shown that recommendations can influence choices through two distinct channels: attention and preferences. The attention channel works as follows: Consumers are unaware of some of the feasible products, and recommending a particular product makes the customer aware of the product. Therefore, recommendations' primary role is to inform customers about the existence and the availability of the products (see Goodman et al. [12], Chen et al. [13], Gupta and Harris [14], Mayzlin and Shin [15]). Recommendations working through the attention channel lead consumers into picking products that they otherwise would not have noticed. On the other hand, recommendations might also affect choices by influencing consumers' preferences (see Adomavicius et al. [7], Cosley et al. [16], Kawaguchi et al. [17], Adomavicius et al. [18]). This is what we refer to as the preference channel. In the preference channel, the decision maker might not choose a product unless it is recommended, even though she is aware of the product. Since an increase in the sales of the recommended alternatives can be attributed to either preference or attention, it is not clear how to distinguish between these two channels. In this chapter, we ask whether it is possible to determine whether a recommendation is informational (attention channel) or persuasive (preference channel) from observed choices. Distinguishing these two channels is essential for policy makers/firms designing effective recommendation, especially when contextual factors may influence the effects of recommendations on preference and attention in a distinct way.

In this chapter, each decision problem represents a different set of recommendations, while the set of feasible alternatives are always fixed.[3] Since the outside observer might be endowed with different types of data, we answer the above question for both deterministic and probabilistic choice environments. In either environment, we propose two models in which recommendation operates either through attention or preferences. The model capturing the attention channel is called *Informational Recommendation (IR)* and the model operating through the preference channel is called *Persuasive Recommendation (PR).* We then provide behavioral characterizations of these models to determine whether they are behaviorally distinct from each

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[3] As opposed to the standard choice model, there is no choice set variation in this model. What could vary is the set of recommended alternatives.

other.

To model the preference channel, we consider a model in which the recommendations only affect choices by increasing the valuation of the product. We assume that being recommended only increases the ranking of the recommended alternative in the decision maker's preferences (positive recommendation).[4] In the deterministic PR model, a decision maker (henceforth, DM) is aware of all of the products, but the relative ranking of alternatives may be affected by recommendation. As in classical choice theory, the DM picks the best alternative according to her altered preferences.

In the deterministic IR model, we consider a DM who only pays attention to a subset of the available alternatives. We assume that the recommendations enlarge the awareness set of the DM. Formally, each DM can be identified with a pre-recommendation choice and a fixed preference ordering. The pre-recommendation choice represents the most preferred alternative in the DM's awareness set according to her preferences before receiving any recommendation. It is natural that the pre-recommendation choice is picked if there is no recommendation since there are no changes in the DM's decision environment. If some alternatives are recommended, the DM picks the more preferred option between the recommended alternatives and

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[4] This formulation eliminates any strategic consideration due to trust concerns about the source and intention of recommendation. Nevertheless, our framework is rich enough to study other channels that recommendation might be operational. The areas of further research should include developing more elaborate and complex models of recommendation.

her pre-recommendation choice according to her preferences. Therefore, recommendations cannot hurt the DM since she can always ignore the recommendation if it is not preferred to her pre-recommendation choice.

Our first result identifies the simple postulate underneath these two models, which is Independence of Irrelevant Recommended Alternatives (IIRA). IIRA states that removing an alternative from the recommendation set does not affect the choice if the removed item were not initially chosen. This postulate is normatively appealing since it mimics Independence of Irrelevant Alternative (IIA), which is the cornerstone of classical choice theory.[5] One can show that any PR model must satisfies IIRA. Moreover, any choice behavior compatible with IIRA can be represented by a PR model. Hence, IIRA is the only behavioral implication of the PR model.

While IIRA is a necessary condition for the IR model, it is not sufficient. The discrepancy in the behavioral pattern between the IR and PR models can be captured in another postulate - the Sandwich Property. If two nested recommendations result in the same choice, then the decision maker chooses the same option when she receives any recommendation between these recommendations We show that the IR model is fully characterized by IIRA and Sandwich Property. These results imply that when the Sandwich Property fails, the attention channel cannot explain choices. It is surprising since in the standard domain, once choices satisfy IIA (the counter part of

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[5] IIA states that choices should not depend on the presence of unchosen alternatives.

IIRA), we are not able to distinguish whether choices can be attributed to attention or preference.

In many examples, the data is given in the form of the aggregate behavior, which can be captured by probabilistic choice. Therefore, we extend our intuition of the deterministic model to this environment. The probabilistic choice can be interpreted in two ways: It can be interpersonal, in which we observed choice made by different individual; or intrapersonal, in which the same individual makes choices under potentially different circumstances. As in the deterministic environment, we ask whether two channels - attention or preference - can be distinguished in the probabilistic environment. In Section 1.3, we introduce two parametric models of recommendations capturing the two channels based on the intuition from the deterministic models. Each model has a finite set of alternative-specific parameters. In both models, we incorporate the randomness in preferences through the idea of Luce model.[6] Therefore, we call them IR-Luce and PR-Luce.

The PR-Luce model captures the idea of persuasive recommendations through a boost in preference intensity when some alternatives are recommended. Firstly, when items are not recommended, we assume there is a crude measure of the baseline utility value. When items are recommended, the baseline utility is weakly increasing,

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[6] In the traditional economics wisdom, there could be *unobserved* underlying factors (e.g. weather) changing the preference.

which is captured by an alternative-specific recommendation factor $r(x)$, such that an item now has the utility of $w(x)r(x)$ instead of $w(x)$. In the deterministic model, the decision maker compares every product before making a decision. The PR-Luce maintains this feature of the model such that the choice probability of an alternative is going to be the share of the utility weight it has relative to the total utility weight in the choice set, where some utility weight are increased due to recommendation. Formally, the choice probability of x given recommendation R under PR channel is given by

$$\rho^{PR}(x, R) = \begin{cases} \dfrac{w(x)r(x)}{\sum\limits_{y \in R} w(y)r(y) + \sum\limits_{z \notin R} w(z)} & \text{if } x \in R \\ \dfrac{w(x)}{\sum\limits_{y \in R} w(y)r(y) + \sum\limits_{z \notin R} w(z)} & \text{otherwise} \end{cases}$$

We show that this model can be fully characterized by two simple behavioral postulates. The first postulate, called General Luce-IIA, mimics the famous Luce-IIA in our framework. Indeed, it implies that the choice ratio between recommended items stays constant (Recommended Luce-IIA). This postulate also implies that the choice ratio between non-recommended items stays constant. The second postulate is R-Regularity, which says that a non-recommended alternative must be chosen more often when the recommendation set gets smaller. Our characterization results does not require the knowledge of full data. We show that if the choice data includes all recommendation sets with size two or less, the characterization holds. In addition, we can uniquely pin down the recommendation factors, and the utility weights up to a scaling factor.

The IR-Luce model assumes that alternatives have a fixed utility value, denoted by u. Besides the randomness in preference, IR-Luce also captures the randomness in pre-recommendation choice, which is probabilistic on the aggregate level as people do not necessarily make the same choices when they do not receive any recommendation. To capture this randomness, we include an additional parameter, $d(x) \geq 0$, which represents the likelihood of x being the pre-recommendation choice. Therefore, $\sum_{x \in X} d(x) = 1$. Given a pre-recommendation choice, not recommended alternatives are chosen with zero probability except for the pre-recommendation choice. The probabilities of choosing a recommended item depends on its own weight proportional to the total weight of recommended alternatives and the pre-recommendation choice. Since being the pre-recommendation choice is also probabilistic, the odds of selecting a not recommended alternative, $x \notin R$, is the probability of being the pre-recommendation choice times the choice probability when it is the pre-recommendation choice. Formally, the choice probability of x given recommendation R under IR channel is given by

$$\rho^{IR}(x,R)=\begin{cases}\displaystyle\sum_{z\in X} d(z)\frac{u(x)}{\sum\limits_{y\in R\cup z} u(y)} & \text{if } x\in R\\[2ex] \displaystyle d(x)\frac{u(x)}{\sum\limits_{y\in R\cup x} u(y)} & \text{otherwise}\end{cases}$$

We show that this model can be fully characterized by two axioms. The first axiom is the Recommended Luce-IIA, in which the choice ratio between two recommended items stay constant across recommendation sets. Due to the characterizations in the deterministic model, one might suspect that a similar Luce-IIA property applies to not recommended items. It turns out that that property does not hold for non-recommended alternatives. Instead, not recommended items satisfy another well-known property called conditional choice, which states that $\rho(a,R)\rho(R\cup a,X)$ is independent of R. In the classical domain, this axiom is equivalent to Luce IIA. However, in our framework, this axiom does not hold for recommended alternatives, it only holds for not recommended alternatives.[7] We also demonstrate how one is able to identify the parameters of the model from data. In particular, we show that we can uniquely identify the parameters of the IR-Luce model with only two data points.[8] This is helpful for real-life applications since it offers a frugal way to make out-of-sample prediction for the effect of recommending different sets of items.

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[7] A caveat is that we need a stronger axiom when there is limited data, which will be elaborated in the corresponding section.

[8] While d's are uniquely identified, the weights u's are unique up to a scale factor.

Given two characterization theorems, we identify when one can distinguish between the attention channel and the preference channel. In the deterministic framework, any behavior explained by the attention channel can be accommodated by the preference channel. This no longer holds in the probabilistic world. Our characterization results show that IR-Luce and PR-Luce are independent. In other words, there are choice behaviors that are only explained by IR-Luce but not PR-Luce, and vice versa. Therefore, these two channels can be distinguished by observing richer data.

Models applying the intuition of the Luce model rely on the independence structure in the preference. However, one would also like to relax the independent structure for situational needs. Besides independence within preference, IR-Luce also imposes independence between pre-recommendation choice and preference.[9] To achieve this relaxation, we introduce a random utility (or types) structure into the models. For both the persuasive (resp. informational) channels, we assume the probabilistic choice data is given by underlying group of individual, where each of the individual is fully characterized by a PR (resp. IR) decision marker. In other words, the choice probability of an alternative is the sum of probabilities of choice functions which select that alternative. Due to the randomness structure, we call the

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[9] One may suspect there could be potential dependency between the two. For example, people whose default option is coke might be more likely to prefer soft drinks over sparkling water.

models as PR-RUM and IR-RUM.

We identify all the behavioral implications of IR-RUM and PR-RUM. The empirical content of the standard RUM is investigated by Falmagne [19], Barbera and Pattanaik [20] and McFadden and Richter [21] who show that the non-negativity of the Block-Marshak polynomials (BM) is equivalent to the choice data being generated by RUM. In our environment, the classical BM can be defined for both recommended and non-recommended alternatives. Our first axiom is the non-negativity of these BM polynomials. It turns out that PR-RUM is fully characterized this assumption. For IR-RUM, however, non-negativity of BMs is necessary but not sufficient. Since being recommended has a greater marginal effect in IR-Luce model, the BM of recommended alternative is always greater than the corresponding BM of non-recommendation, which we call Positive Marginal Recommendation. It turns out IR-RUM is fully characterized by these two properties.

Our method of proof is constructive: It offers two algorithms to construct a rationalizing distribution of choice types from the observed choice probabilities for each model. The uniqueness properties of IR-RUM and PR-RUM can be recovered by the algorithm. The uniqueness result of the models is similar to that of RUM, which is a weaker form of ordinal uniqueness. Given two possible representations, certain marginal distributions of the preferences are essentially unique in our model.

Literature Review

In all of our models, we utilize the idea that modelers can observe consumers' choice data as a function of their recommendation set. The idea of a recommendation set is a distinct notion from that of a choice set as choices can be outside of a recommendation set. Hence our choice function is both conceptually and mathematically different from the classical choice function. Therefore, this chapter, naturally separates itself from other theories in the standard domain in terms of modeling strategy. Moreover, it enables us to take a fresh perspective into choice data with some of our traditional economic intuition.

We would like mention a recent paper by Ke et al. [22]. While the focus of their paper is distinct from us, their primitive enjoys a similar feature to our choice function. They propose a belief updating model where a DM receives information from an unknown source. Their model differs from other updating rules where the posterior always belongs to the information set. Similar to our primitive, they allow posterior being outside of the information set.

The classical Luce model attracts a variety of scholarly attention into developing different generalizations of it (e.g. [23, 24, 25, 26, 27, 28, 29, 30]). All of these models involve different relaxations of the Luce IIA axiom. In our IR-Luce and PR-Luce models, we employ the Luce IIA axiom for the recommended alternatives to provide a simple applicable and tractable parametric model. On the other hand, there are several different strands of research departing from choice-set variation in the standard model. For example, some studies utilize list variation to study choices (e.g. Ishii et al. [31], Guney [32]) and approval rates (Manzini et al. [33]). Natenzon [34] and Guney et al. [35] study how non-choosable phantom options affect choices. In a similar spirit to our model, these lines of research are also augmenting the standard choice environment to enhance our understanding of human behavior.

1.2 Deterministic

As it was discussed in the introduction, our deterministic model is intended to be simple so that we can study the aggregate behavior of such consumers in a tractable way. However, our framework is flexible enough to study more involved models of recommendation capturing different aspects of the recommendation process.

In this chapter, the set of available alternatives is fixed, denoted by X (e.g., all documentaries available at Netflix or all 65 Inch Smart TVs sold at Amazon.com).[10] Hence there is no variation in terms of the availability of alternatives in our model.

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[10] We impose this assumption to capture possible data restrictions in observed data. Having said that, this framework is flexible enough to allow set variations by assuming $c(R, S) \in S$ and $R \in X$ where S represents the set of feasible alternatives and R is the recommended set. This formulation allows the recommended alternatives being unavailable in order to capture "out of stock" concept when $R \setminus S \neq \emptyset$.

The decision maker receives a recommendation in the form of a set of alternatives, say R, which is the source of variation in our model. Any subset of X could constitute a decision problem, including the empty set which corresponds to not receiving any recommendations. While recommendations can influence choices, it does not constrain it. To capture this, we define a choice rule c to be a function of recommendation set, R, but allow for $c(R)$ to be outside of R. Hence, the only restriction we impose is $c(R) \in X$. Notice that our choice function is different from the one used in the choice theory literature where $c(R)$ is always in R. This distinction will be important when we introduce behavioral properties. Additionally, we allow c to be observable only for some recommendation sets but not others. This assumption aims to capture some real world environments in which the collection of recommended set is just a fraction of the entire product space. Let $\mathcal{D} \subset 2^X$ denote all possible recommendation sets we have the data for. For instance, $\mathcal{D}$ can be consistent of single alternative recommendations. The following definition captures the choice rule under this framework.

Definition 1. A deterministic choice rule c on domain $\mathcal{D}$ is a mapping from $\mathcal{D}$ to X such that $c(R) \in X$ for all $R \in \mathcal{D}$.

Notice that if the DM always maximizes according to true underlying preference with full consideration, then recommendations should not affect choices even if the DM receives different recommendations. In other words, $c(R)$ is a constant function. However, as discussed in the introduction, there are plentiful of evidence that recommendation indeed affects choice. In particular, there are two different channels through which recommendations can affect choice behavior: attention and utility. Here, we would like to model these channels from two starting points. The reasons are two folded. Firstly, oftentimes attention and preference can be intertwined in explaining choice data. For example, in the attention literature, if choices satisfy IIA, there is no room to distinguish whether DM is choosing her most preferred alternative or she only pays attention to the chosen alternative. We would like to know whether we can distinguish between choice driven by attention and choice driven by utility from choice data involving recommendations. Secondly, given that the two different channels can be relatively more prominent than the other in different circumstances, studying the different models can also help us to identify which could be the better descriptive models in explaining behavior in different contexts. All of these require us to first have a deeper understanding into the behavioral implications of each of the models.

1.2.1 Persuasive Recommendation

We now discuss the model that captures the preference channel. We refer to this model as the *Persuasive Recommendation*. Since recommendations are persuasive, we assume that being recommended elevates the ranking of an alternative in the DM's preferences. To model this, we introduce a preference relation, which compares the same alternative when it is recommended and not recommended. In this setting, we consider two versions of each alternative, x and x^*, where x^* refers to the product x when it is recommended. We denote the choice set with recommended versions of the alternatives as X^*. In order for the DM to compare across recommended and non-recommended item, we assume that the decision maker has an underlying preference ordering $\succ^\star$ on $X \cup X^*$. Due to the assumption of recommendations being persuasive, for each alternative x, it must be that $x^* \succ^\star x$: Recommendations can only improve an alternative's ranking. R^c denotes the complement of R (*i.e.*, $X \setminus R$). Define

$$\begin{aligned} \max{}^*(R^* \cup R^c, \succ^\star) = \{x \in X |\ & \text{either } x \in R \text{ and } x^* \succ^\star \sigma \text{ for every } \sigma \in R^* \cup R^c \setminus x^*, \\ & \text{or } x \notin R \text{ and } x \succ^\star \sigma \ \text{ for every } \sigma \in R^* \cup R^c \setminus x\} \end{aligned}$$

This max operator depends on two components: the recommended set R and the preference $\succ^\star$. Notice that, the notation is a little involved although the intuition behind is simple. It is because we need to specify that the chosen alternative is in X even though the winning element can actually be in X^*. With this definition, we can state the model as follows.

Definition 2. A deterministic choice rule c has a persuasive recommendation (PR) representation on $\mathcal{D}$ if there exists a preference $\succ^\star$ on $X \cup X^*$ such that

$$c(R) = \max{}^{*}(R^* \cup R^c, \succ^\star)$$

for all $R \in \mathcal{D}$.

In this model, like the standard rational choice model, the DM considers every alternative in the choice set. Therefore, the only departure we introduce here is that recommendation can change the ranking of recommended items.

Notice that there is one special case of this model worth mentioning. We can control the elevation in preference to be so small so that the recommended version will never surpass other alternatives in the ranking *i.e.* there does not exists σ such that $x^* \succ^\star \sigma \succ^\star x$. In this case, the recommendation has no effect on choice and the DM always chooses the best alternative.

1.2.2 Informational Recommendation

We then move on to the attention channel, which we call *Informational Recommendation*. In this model, each decision maker is identified with a pre-recommendation choice and a preference ordering $(a, \succ)$, where $\succ$ is a linear order. The pre-recommendation choice might have different interpretations. It could be a last purchased item, easily identified alternative, a previously obtained recommendation, and/or a default option.[11] In addition to these interpretations, the pre-recommendation choice can be seen as the best alternative in DM's initial awareness set according to her preferences before receiving any recommendation. Hence, it is natural that the pre-recommendation choice option is picked if there is no recommendation. We use the terms default option and pre-recommendation choice interchangeably throughout the chapter. In this model, when the DM receives a recommendation, R, the DM picks the best option among $R \cup a$ according to her preferences, where the DM considers all recommended alternatives before making a final decision. Formally,

Definition 3. A deterministic choice rule c has an informational recommendation (IR) representation on $\mathcal{D}$ if there exist a preference $\succ$ and a default option a such

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[11] Default options help consumers to reduce the need of engaging deliberative decisions. They also eliminate difficult trade-offs (Thaler & Sunstein, 2008). The use of default options is also related decision task complexity (Redelmeier & Shafir, 1995), conflict (Tversky & Shafir, 1992), and/or emotionally difficult decisions (Luce, 1998).

that

$$c(R) = \max(R \cup a, \succ) := c_{(a,\succ)}(R)$$

for all $R \in \mathcal{D}$.[12]

In this model, the DM is both willing to and capable of considering all the recommended alternatives. This feature of the model is shared by the classical rational choice model in which the DM maximizes her preferences among all available alternatives. Therefore, The only relaxation we made in this environment is that the DM might not be aware of the best alternative to begin with. With that said, notice that the "rationality" requirement is not as severe as the classical model due to the fact that we require that only a limited number of alternatives are recommended in our data.

Two special cases of this model are worth mentioning. The first one is when the pre-recommendation choice is the best alternative according to the DM's preference. In that case, $c_{(a,\succ)}(R) = a$ for all $R \in \mathcal{D}$. In other words, choices are not influenced by recommendations, which gives the rational representation. The second one is when the pre-recommendation choice is the worst alternative according to preference. In that case, the decision-maker behaves as if she is a classical prefer-

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[12] One might interpret $R \cup a$ as the consideration set. We would like to highlight that the actual consideration set could be larger than $R \cup a$. For example, the union of R and the lower counter set of a with respect to $\succ$ could be the actual consideration set of the decision maker.

ence maximizer: $c_{(a,\succ)}(R) = \max(R, \succ) \in R$ where the recommendation dictates the chosen alternative.

1.2.3 Behavioral Characterization

We would like to know whenever we can distinguish the two models from the observed choices. This requires us to look into the behavioral implications of the two models and see if we can set them apart from the observed choices.

The first behavioral postulate is key axiom in this environment, which states that removing some of the unchosen alternatives from the recommendation set does not influence the final choice. Note that it shares a similar flavor as the famous IIA axiom (*i.e.* Independence of Irrelevant Alternative) in the choice theory literature.[13] We call this the Independence of Irrelevant Recommended Alternatives, abbreviated by IIRA.

IIRA. *If* $c(R) \notin R \setminus R'$ *and* $R' \subseteq R$*, then* $c(R) = c(R')$.

Since this axiom looks similar to the standard IIA, one might think that they are equivalent. While the standard IIA applies to classical choice functions in which the winner always belongs to the choice set, our postulate applies to new choice

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[13] This property is also known as Sen's α axiom ([36]), Postulate 4 of Chernoff [37], C3 of Arrow [38], the Heritage property of Aĭzerman and Aleskerov [39], or the Heredity property of Aleskerov et al. [40].

objects in which the winner can be a recommended or non-recommended item. This seemingly small distinction has important implications. Indeed, if we include "$c(R)$ belongs R'" in the premise of IIRA, then IIRA becomes the standard IIA, which is stated below as IIRA(1). However, IIRA(1) is strictly weaker than IIRA.

IIRA(1). *If* $c(R) \in R'$ *and* $R' \subseteq R$, *then* $c(R) = c(R')$.

In other words, seemingly similar axioms have distinct implications in our framework. Hence the existence results in the literature on the standard framework might not be valid in this framework. Since IIRA(1) is weaker than IIRA, we would like to state the counterpart of IIRA(1). To this, we include "$c(R)$ is not in R" into the premise and call it as IIRA(2). It states that the winner will persist outside of the recommendation set when there are less recommended alternatives.

IIRA(2). *If* $c(R) \notin R$ *and* $R' \subseteq R$, *then* $c(R) = c(R')$.

It is easy to see that IIRA(1) and IIRA(2) are equivalent to IIRA. Notice that both IR and PR models satisfy IIRA. To see why, first consider IIRA(1), so $c(R) \in R'$. Then, the IR model says that $c(R)$ must be better than all the other recommended alternatives in R and the default option. Hence, it stays the best alternative if we remove some of the unchosen alternatives from the recommended set. On the other hand, the PR model says that $c(R)$ is evaluated according to the recommended version, and it must be better than all the other recommended and non-recommended alternative. Therefore, the evaluated ranking of $c(R)$ stays the best alternative even if some alternatives are downgraded.

Now, consider the second postulate, IIRA(2), so $c(R) \notin R$. Then, the IR model requires that the chosen alternative must be the default option, which has to be preferred to all recommended alternatives in R. Hence, it must be also be better than all alternatives in R' since R' is a smaller set. On the other hand, PR model says that the non-recommended item $c(R)$ is already better than the recommended item in R and other non-recommended item. Hence, the non-evaluated version of $c(R)$ is still superior than other alternatives when some other alternatives are not evaluated anymore.

We now consider a new property in this framework– the Sandwich Property. As the name suggests, if two nested recommendations result in the same choice, then the decision maker chooses the same option when she receives any recommendation between these recommendations.

Sandwich Property. *If $R' \subseteq R \subseteq R''$ and $c(R'') = c(R')$, then $c(R) = c(R')$.*

Note that, in the standard choice domain where $c(R) \in R$, the sandwich property is implied by IIA.[14] Since $c(R'') \in R \subset R''$, IIA would imply $c(R) = c(R'')$.

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[14] A version of the sandwich property is appeared in Masatlioglu and Nakajima [41], which is stronger than Weak-WARP of Manzini and Mariotti [42].

However, in our framework, the sandwich property has distinct implications than IIA, due exactly to the fact that choices can be outside of the recommendation set. IR model satisfies the sandwich property. To see this, if $c(R') = c(R'')$, then the chosen item must be preferred to every recommended item in R'' and the default option. Hence, it must be still be better than any recommended item for any in-between recommendation set. At last but not least, it turns out that Sandwich Property is the only behavior choice pattern where IR and PR differ.

Theorem 1 (Characterization)**.** Let $\mathcal{D}$ includes all recommendation sets with $|R| \leq 3$. Then,

1) c has a PR representation if and only if c satisfies IIRA;

2) c has a IR representation if and only if c satisfies IIRA and Sandwich Property.

As far as we know, this is the first characterizations of decision-making under recommendation. As we stated before, these two models capture two distinct channels of recommendation. The theorem shows that these models are nested. It is a surprising result that persuasive channel fully encapsulates the explanatory power of the recommendation channel, while retaining room for us to distinguish the PR model from the IR model. In the standard environment, if the choice behavior satisfies the classical IIA, choices can be attributed both to limited attention or to preferences. Here, it turns out that we can actually distinguish them. Therefore, our intuition based on the standard framework may mislead us here.

Theorem 1 also informs us how to distinguish between these two channels. Given these results, if Sandwich property fails, we can confidently reject the attention channel. A simplest example of failing Sandwich property is given as follows.

Example 1 (Violating Sandwich Property)**.** Let $c(\emptyset) = x$, $c(\{y\}) = y$ and $c(\{x, y\}) = x$. Notice that it is immediate that this example violates Sandwich Property. Also, one can check that it cannot explained by the IR model. Suppose it does. Note that since $c(\emptyset) = x$, it reveals that IR-DM's default option is x. Then, $c(\{y\}) = y$ reveals that y is better than x. Therefore, we must have $y = c(\{x, y\})$. A contradiction. On the other hand, PR-DM can explain this behavior. One can consider $x^* \succ^\star y^* \succ^\star x \succ^\star y$, where all alternatives are elevated accordingly. ⌟

We now discuss the identification and uniqueness properties of our model. For the IR model, it is easy to see that the pre-recommendation choice must be the choice when there is no recommendation $a := c(\emptyset)$. Moreover, if x is chosen when R is recommended, then x is revealed to be preferred to every alternative in R. If x is different from $c(\emptyset)$, x is also revealed to be preferred to the pre-recommendation choice. Lastly, although the lower contour set of a is the same, the preference relation among those alternatives cannot be pinned down. The next proposition states all the uniqueness properties of the IR model. Throughout the chapter, we denote the lower contour set of a with respect to preference $\succ$ as $L_{\succ}(a)$.

Proposition 1 (Uniqueness)**.** If $(a_1, \succ_1)$ and $(a_2, \succ_2)$ represents the same choice rule, then

i) $a_1 = a_2 := a$,

ii) $L_{\succ_1}(a) = L_{\succ_2}(a) := L_a$, and

iii) $x \succ_1 y$ if and only if $x \succ_2 y$ for all $x, y \in X \setminus L_a$.

Secondly, one can show a similar result for the PR model. Notice that when there is no recommendation, alternatives are evaluated according to the non-elevated version. Therefore, the best alternative in the X according to the two preference must be the same. Also, if x is different from $c(\emptyset)$, x^* is also revealed to be preferred to the best alternative in X. Lastly, the lower contour set of the best alternative in X must be the same.

Proposition 2 (Uniqueness)**.** If ${\succ^{\star}}_1$ and ${\succ^{\star}}_2$ represents the same choice rule, then

i) $\max(X, {\succ^{\star}}_1) = \max(X, {\succ^{\star}}_2) := a$,

ii) $L_{{\succ^{\star}}_1}(a) = L_{{\succ^{\star}}_2}(a) := L_a$, and

iii) $x \, {\succ^{\star}}_1 \, y$ if and only if $x \, {\succ^{\star}}_2 \, y$ for all $x, y \in X \cup X^* \setminus L_a$.

1.3 Parametric Probabilistic Choice

In the last section, we built a robust microfoundation in which recommendations affect deterministic choices through two different channels. In this section, we extend these ideas to probabilistic choice. One interpretation of randomness in choice data is that there is a group of individuals with unknown types and we can only observe their aggregate behavior, this is the interpersonal interpretation. Another interpretation is the choices of a single individual in different situations, this is the intrapersonal interpretation. Our model can be interpreted as both intrapersonal and interpersonal probabilistic choice. In the classical choice-set variation framework, probabilistic choice data is well-studied. However, recommendations cannot be studied within this framework, because the DM can choose outside of the recommendation R. In order to incorporate this idea, we extend the definition of a (probabilistic) choice rule just as we did in our deterministic setting. Formally,

Definition 4. A choice rule ρ is a mapping from $X \times \mathcal{D}$ to $[0, 1]$ such that $\sum_{x \in X} \rho(x, R) = 1$.[15]

We first study two parametric models of recommendation for the two different channels with probabilistic data based on the Luce model. All the information about

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[15] This is different from the classical probabilistic choice model where $\rho(x, R) = 0$ for $x \notin R$.

the models' predictions can be summarized by a finite set of parameters depending only on the alternatives. As is typically the case with parametric models, our parametric models offers three advantage over a more general version we study in the next section. First, the model is very tractable, which is a desirable property for applications. Second, the model possesses strong uniqueness properties. Third, it has sharp identification results for application purposes.

To foster comparison between the two channels in the parametric models, we make the following positivity assumption throughout this section. Notice that this positivity property, as argued by McFadden [43], cannot be refuted based on any finite data set.

Assumption 1. For every x and R, $\rho(x, R) > 0$.

1.3.1 Persuasive Recommendation

In the persuasive recommendation framework, recommended items have elevated ranking. In terms of the Luce model, we assume that for each available alternative x, there are two alternative-specific parameters $w(x) > 0$ and $r(x) > 1$. Similar to Luce model, w represents the crude measure of utility value. To avoid confusion with the IR-Luce model, we use different notation for the utility function. On the other hand, $r(x)$ captures the boost when alternative x is recommended.

Since items have an elevated ranking when recommended, it must be that $r(x)$ is greater than or equal to 1. We sometimes use (w, r) as the primitive of the model for convenience. For notational simplicity, we write $v(x) := w(x)r(x)$ Also, similar to last section, we write $v(A)$ and $u(A)$ as shorthand for $\sum_{x \in A} v(A)$ and $\sum_{x \in A} u(A)$ respectively. In the persuasive recommendation framework, we assume the DM pays attention to every alternative. Therefore, the probability that a recommended alternative is chosen is equal to its elevated weight $v(x)$ divided by the total weight in the choice set. The probability that an alternative that is outside of the recommendation set is chosen is equal to its weight $w(x)$ divided by the total weight in the choice set. We summarize this into the following definition.

Definition 5. A choice rule ρ has a persuasive Luce recommendation representation (PR-Luce) if there exists functions $w : X \to R_{++}$ and $v : X \to R_{++}$ with $v(x) \geq w(x)$ such that for $x \in X$,

$$\rho(x, R) = \begin{cases} \dfrac{v(x)}{v(R) + w(R^c)} & \text{if } x \in R \\ \dfrac{w(x)}{v(R) + w(R^c)} & \text{otherwise} \end{cases}$$

for all $R \in \mathcal{D}$.

1.3.2 Informational Recommendation

In this model, each alternative x is represented by two parameters: $d(x) > 0$ and $u(x) > 0$.[16] The function d represents the likelihood of x being the pre-recommendation choice, hence $\sum_{x \in X} d(x) = 1$. The function u is a crude measure of the utility value, which extends preferences in the deterministic IR model to the probabilistic setting. An alternative with a high u value is chosen more often than an alternative with a low u value. We write $u(A)$ instead of $\sum_{x \in A} u(A)$. We follow the Luce model to describe the choice probabilities given a default. To encompass our deterministic model, for a fixed default option, all non-recommended alternatives are chosen with zero probability except the default option. The probabilities of choosing a recommended item depends on its own weight proportional to the total weight of recommended alternatives and the default option. Hence, we first define choice probabilities given a fixed default. The choices for a given pre-recommendation choice, a, can be expressed as:

$$U_a(x, A) = \begin{cases} \dfrac{u(x)}{u(A \cup a)} & \text{if } x \in A \cup a \\ \\ 0 & \text{otherwise} \end{cases}$$

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[16] We can relax to $d(x) \geq 0$ in the model. However, positivity assumption implies $d(x) > 0$ in our model.

Note that U_a is itself a parametric choice model where $\sum\limits_{x \in X} U_a(x, A) = 1$. According to this formulation, only the recommended alternatives and the pre-recommendation choice are chosen with positive probability. U_a captures the randomness in preferences as in the Luce model. Note that the deterministic model of Section 1.2 is a limit case of this model since deterministic model has no randomness in preferences.[17] We expand on this intuition by also assuming that being the default option is probabilistic. Since $d(x)$ is the probability of x being the default option, the choice probability of any alternative is defined as a mathematical expectation: The probabilities of a given default option times the conditional choice probability given that default option. Formally, we have the following.

Definition 6. A choice rule ρ has an informational Luce recommendation representation (IR-Luce) if there exists functions $u : X \to R_{++}$ and $d : X \to R_{++}$ with $\sum\limits_{x \in X} d(x) = 1$ such that for any $x \in X$,

$$\rho(x, R) = \sum_{a \in X} d(a) U_a(x, R)$$

for all $R \in \mathcal{D}$.

IR-Luce model has inherently two types of randomness. While d captures the

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[17] To see how W_a reduces (or, approaches) to deterministic case where DM's type is $(a, \succ)$, we first enumerate all alternatives according to $\succ$, $x_1 \succ x_2 \cdots \succ x_n$. We assign $w(x_i) = \epsilon^i$ for $i > 1$ and $w(x_1) = 1 - \sum_{i=2}^{n} \epsilon^i$. By taking ϵ to *zero*, $(a, \succ)$ becomes the limit case of W_a.

randomness in pre-recommendation choice, u represents the randomness in preferences. Note that the odds of selecting a non-recommended alternative, $x \notin R$, is the probability of being default times the conditional choice probability when it is the default, that is, for any $x \notin R$

$$\rho(x, R) = \frac{d(x)u(x)}{u(R \cup x)} = d(x)U_x(x, R)$$

. When $x \notin R$, $\rho(x, R)$ is always zero in the standard random utility model since R represents the set of feasible alternatives. However, in our model, the set of alternatives is fixed and R represents the recommended alternatives. The effective weight of x becomes $d(x)u(x)$, which is strictly less than $u(x)$ if $d(x) < 1$. This can be interpreted as the value of non-recommended alternatives is discounted while the recommended ones stay the same. For recommended alternatives, $x \in R$, our formula is more involved:

$$\rho(x, R) = \sum_{z \in X} \frac{d(z)u(x)}{u(R \cup z)}$$

1.3.3 Behavioral Characterization

In this section, we discuss the behavioral characterization of the IR-Luce and PR-Luce models. This is because each model has its own distinct characterization in this environment.

Characterization of IR-Luce

Since we extend the Luce model, as anticipated, some version of the Luce's IIA appears in our characterization. Recall that Luce's IIA says that the odds of choosing one alternative over another do not depend on the feasible set. In the IR-Luce model, Luce's IIA holds for all recommended alternatives:

Recommended Luce-IIA. *For* $x, y \in R \cap R'$,

$$\frac{\rho(x,R)}{\rho(y,R)} = \frac{\rho(x,R')}{\rho(y,R')}$$

One would suspect that a similar axiom must hold for not recommended alternatives. It turns out that this property does not hold for non-recommended alternatives. Instead, another well-known property is satisfied with a caveat. The property is known as Luce's Choice Axiom:

$$\rho(a,R)\rho(R,R') = \rho(a,R')$$

This property says that the probability of choosing a from R' is equal to the conditional probability that a is chosen from R given that the choice from R' belongs to

R. A modified version of the Luce's Choice Axiom appears in our characterization. Our property says that, for $x \notin R$ and $R \cup x \subseteq R'$, the probability of choosing x first when R is recommended and then choosing $R \cup x$ when R' is recommended is independent of R. We call this axiom R-Path Independence.

R-Path Independence. *For $x \notin R$ and $R \cup x \subseteq R'$, $\rho(x, R)\rho(R \cup x, R')$ is independent of R.*

As long as probabilities are strictly greater than zero, Luce's IIA and Luce's Choice Axiom are equivalent in the usual choice domain where $\rho(R, R) = 1$ for all R. Surprisingly, this equivalence does not hold in our domain since $\rho(R, R)$ could be strictly less than 1 for $R \neq X$. This discussion highlights that equivalence of these two properties depends on the domain to which they apply. Here, we show that recommended and non-recommended alternatives obey different rules. These two properties summarize the entire empirical content of the IR model.

Theorem 2 (Characterization)**.** Assume $\mathcal{D} = 2^X$. Then, ρ has a IR-Luce representation if and only if ρ satisfies Recommended Luce-IIA and R-Path Independence.

We now demonstrate that we can provide a similar characterization of this model under data restrictions. Suppose $\mathcal{D}$ includes all possible recommendation sets with $|R| \leq 2$. Under limited data, we need to impose a stronger axiom on the off-recommendation data. The reason why we need this axiom is because R-Path

Independence is not strong enough in this limited domain. This concern does not exist if we have full data. To see why we need a stronger axiom, consider the following example in Table 1.1 with $X = \{a, b, c, d\}$. Notice that Recommended Luce-IIA and R-Path Independence are immediately satisfied.[18] However, none of these axioms restrict how $\rho(c, \{a, b\})$ behaves: R-Path Independence puts restriction on $\rho(c, \{a, b\})$ only if we also observe data on some other recommendations that includes all a, b and c. Nonetheless, we might not observe a recommendation including all three alternatives due to the limited data assumption. Hence, to check for the validity of the model under limited data, we need to impose a stronger axiom.

	$\{a,b\}$	$\{a,c\}$	$\{a,d\}$	$\{b,c\}$	$\{b,d\}$	$\{c,d\}$	$\{a\}$	$\{b\}$	$\{c\}$	$\{d\}$	$\emptyset$
a	5/12	5/12	5/12	1/12	1/12	1/12	3/8	1/8	1/8	1/8	1/4
b	5/12	1/12	1/12	5/12	5/12	1/12	1/8	3/8	1/8	1/8	1/4
c	1/120	5/12	1/12	5/12	1/12	5/12	1/8	1/8	3/8	1/8	1/4
d	19/120	1/12	5/12	1/12	5/12	5/12	1/8	1/8	1/8	3/8	1/4

Table 1.1: Probabilistic choice data satisfying Recommended Luce-IIA and R-Path Independence but does not have a R-Luce representation under limited data.

The R-Path Independence axiom revolves around the fact that the choice probability of a not recommended alternative under recommendation set R is tightly related to its choice probability when recommended with the set R. It turns out that this dependency can be made more explicit. We define the following shorthand:

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[18] Since the data is symmetric, we consider a. Note that $\rho(a, \emptyset)\rho(a, \{a, b\}) = \rho(a, \emptyset)\rho(a, \{a, c\}) = \rho(a, \emptyset)\rho(a, \{a, d\}) = \rho(a, \{b\})\rho(\{a, b\}, \{a, b\}) = \rho(a, \{c\})\rho(\{a, c\}, \{a, c\}) = \rho(a, \{d\})\rho(\{a, d\}, \{a, d\}) = \frac{5}{48}$. Hence it satisfies Axiom .

$r(z,x) := \frac{\rho(z,A)}{\rho(x,A)}$ for some A including x and z. Note that Recommended Luce-IIA guarantees that $r(z,x)$ is well-defined. Each $r(z,x)$ captures exactly the choice ratio of x and z when both of them are recommended.

R-Independence. *For* $x \notin R$, $\rho(x,R)(1+\sum_{z\in R} r(z,x))$ *is independent of* R.

The intuition behind this axiom is that, within our model, there is a fixed amount for how often x can be chosen when x is not recommended. Notice that as recommendation set R grows, the markup $\sum_{z\in R} r(z,x)$ increases and in turn $\rho(x,R)$ decreases. Therefore, this axiom dictates that x must be chosen less as more alternatives are recommended when x is not recommended. Also, the rate that $\rho(x,R)$ decreases also depends on how "likable" x is when both x and his rival z are both recommended *i.e.* it depends on the $r(z,x)$. In particular, it decreases less if x is chosen much more often than z when both of them are recommended.

It is clear that the above example violates R-Independence. To see why, consider the choice probability of c under the recommendation set $\{a,b\}$ and $\emptyset$, we have $\rho(c,\emptyset) = \frac{1}{4} \neq \frac{1}{40} = \rho(c,\{a,b\})(1+r(a,c)+r(b,c))$. Moreover, we can also see that Recommended Luce-IIA and R-Independence imply R-Path Independence. Since $r(z,x)$ can be represented with any recommendation set as long as they include both z and x, we consider arbitrary R' such that $x \cup R \subseteq R'$. Then, by simplification, we can get $1+\sum_{z\in R} r(z,x) = \rho(R\cup x, R')/\rho(x,R')$, which basically implies R-Path

Independence.

Theorem 3. Let $\mathcal{D}$ includes all recommendation sets with $|R|\leq 2$. Then, ρ has a IR-Luce representation if and only if ρ satisfies Recommended Luce-IIA and R-Independence.

Theorem 3 provides a similar characterization with limited data. Thus, Theorem 3 weakens the data requirements of Theorem 2. Indeed, many models in decision theory require a similar rich data set, typically choices from all decision problems.

Characterization of PR-Luce

We now turn to the characterization of the PR-Luce model. Notice that the PR-Luce model is similar to the standard model except that the weights are multiplied by a factor if it is recommended. Therefore, the PR-Luce model satisfies a Luce axiom that is stronger than the Recommended Luce-IIA, which is the General Luce-IIA axiom.

General Luce-IIA. *For* $x, y \in R \cap R'$, $t, z \notin R \cup R'$,

$$\frac{\rho(x,R)}{\rho(x,R')} = \frac{\rho(y,R)}{\rho(y,R')} = \frac{\rho(t,R)}{\rho(t,R')} = \frac{\rho(z,R)}{\rho(z,R')}$$

Note that this property implies every possible pair of Luce-IIA axiom. Firstly, we can see that it implies the Recommended Luce-IIA from x and y. On the other hand, this axiom implies another IIA axiom when both t and z are not recommended. Lastly, this axiom implies another IIA axiom where one of them is recommended and the other one is not recommended (e.g. y and t).

One might suspect that this axiom alone is sufficient for the PR-representation. It turns out that we need an additional behavioral postulate. The idea is simple: Any not recommended item is chosen weakly more when the set of recommended items gets smaller. We call this property as R-Regularity.

R-Regularity. *For $x \notin R$, $\rho(x, R) \leq \rho(x, R \setminus y)$.*

Notice that in the standard Luce model, this axiom is trivially satisfied since both sides of the inequality are equal to zero. With this axiom, we are able to state the characterization result.

Theorem 4. Let $\mathcal{D}$ includes all recommendation sets with $|R| \leq 2$. Then, ρ has a PR-Luce representation if and only if ρ satisfies General Luce-IIA and R-Regularity.

1.3.4 Behavioral Distinction and Similarity

In the deterministic framework, the IR and PR models can be distinguished only in one way, but not the other way. In other words, we can only distinguish PR model from the IR model when we observe violation of sandwich property. In the probabilistic framework, surprisingly, the distinction becomes two way. This is due to the more structured model in this environment. It is easy to construct an example in which one probabilistic choice function has an IR-Luce representation but does not have PR-Luce representation and vice versa.

On the other hand, although the two models share different characterizations, the two models share some common behavioral properties. Firstly, as one might immediately expect, both IR-Luce and PR-Luce satisfy Recommended Luce-IIA. This is due to the fact that General Luce-IIA implies Recommended Luce-IIA. Secondly, as we discussed in the previous section, we already learn that R-Independence implies R-Regularity. In fact, one can even check that R-Path Independence also implies R-Regularity. To see this, by R-Path Independence, we know $\rho(x,\emptyset)\rho(x,\{x,y\}) = \rho(x,\{y\})\rho(\{x,y\},\{x,y\})$. Hence, we know that $\frac{\rho(x,\emptyset)}{\rho(x,\{y\})} = \frac{\rho(\{x,y\},\{x,y\})}{\rho(x,\{x,y\})} \geq 1$. One can apply similar argument to other set.

1.3.5 Uniqueness and Identification

In this section, we discuss the identification of the models. To achieve this, we first establish the fact that the parameters are unique in each models. Our first result shows that in the IR-Luce model, we can uniquely pin down the default option probabilities. Also, the utility measure is unique up to a scaling factor.

Proposition 3 (Uniqueness of IR-Luce)**.** If (u_1, d_1) and (u_2, d_2) represent the same choice rule ρ, then

i) $d_1 = d_2$, and

ii) $u_1 = \alpha u_2$ for some $\alpha > 0$.

Similar to the IR-Luce model, PR model also uniquely pins down the recommendation factor and the utility measure is again unique up to a scaling factor.

Proposition 4 (Uniqueness of PR-Luce)**.** If (w_1, r_1) and (w_2, r_2) represent the same choice rule ρ, then

i) $r_1 = r_2$, and

ii) $w_1 = \alpha w_2$ for some $\alpha > 0$.

We now turn to the identification of the parameters. We first consider the IR-Luce model. Our first result illustrates that observations from only two recommendation sets are sufficient for unique identification.

Proposition 5 (Identification of IR-Luce)**.** Suppose ρ is IR-Luce. Let $\mathcal{D}$ includes recommendation sets $\emptyset$ and $\{a\}$ for some $a \in X$, then we can fully identify the parameters of the model.

Note that IR-Luce requires only two data points for full identification of the model. The underlying reason behind the identification is as follows. Firstly, having no recommendation data allow us to identify $d(x)$. Secondly, suppose we have the recommendation data for one alternative. Since IR-Luce requires that the DM will compare each of the default option with the recommended item, we can fully recover every w. Since the procedure is simple enough, we demonstrate the identification here in the main text. Firstly, let $d(x) := \rho(x, \emptyset)$ for every $x \in X$. Then, we identify u. Note that u is unique up to a scaling factor, we let $u(A) = 1$. Note that, for $x \in X \setminus a$, we define $u(x) := \frac{\rho(x,\{a\})}{\rho(x,\emptyset)-\rho(x,\{a\})}$. Then, the parameters are fully identified.

On the other hand, PR-Luce requires more data to work with. Notice that the recommendation factor for an alternative can be "observed" only if the alternative is recommended in some occasions. Therefore, to retrieve this parameter, we must have data in which each alternative is recommended in some sets.

Proposition 6 (Identification of PR-Luce)**.** Suppose ρ is PR-Luce. Let $\mathcal{D}$ includes all recommendation sets with $|R| \leq 1$, then we can fully identify the parameters of the model.

To see how identification in our model works, let $w(x) := \rho(x, \emptyset)$ for every $x \in X$. Then, for an arbitrary $z \neq x$, we let $r(x) := \frac{\rho(x,\{x\})}{\rho(z_0,\{x\}} \frac{\rho(z_0,\emptyset)}{\rho(x,\emptyset)}$. Hence, we can fully the parameters. Notice that, in the above proposition, it is possible to obtain the same result by replacing the singleton recommendation data with doubletons where there is no overlap in recommended item across the doubletons. This requires less data for identification.

1.4 General Probabilistic Choice

In the last section, we focused on two parametric models in which the randomness of types are described by alternative-specific parameters. In this section, to enhance the explanatory power of the models, we introduce the two most general models in our framework. In the standard Random Utility Model, the data is assumed to be generated from a group of different individuals making deterministic choice. Notice that Luce model can be captured in the Random Utility Model by assuming independence in preference. *i.e.* the fraction of people who prefer, say, a over b is the same regardless of their preference on another alternative c. By introducing RUM version for our models in this probabilistic environment, we are also relaxing this independence. We discuss the generalization for each models in the following.

1.4.1 Persuasive Recommendation

In this framework, we assume that each DM in the population can differ on their preferences for recommended and non-recommended alternatives. Therefore, it relaxes the independence structure of preference in the PR-Luce model. To define the preference space, let $\mathcal{P}^\star$ be the set of all (linear order) preference on $X \cup X^*$, where

$x^* \succ^\star x$ for all $x \in X, x^* \in X^*$. Here, we are interested in a probability measure μ on $\mathcal{P}^\star$ such that $\sum_{\succ^\star \in \mathcal{P}^\star} \tau_{\succ^\star} = 1$, where $\mu_{\succ^\star} := \mu(\{\succ^\star\})$. Now we are ready to state the model.

Definition 7. A choice rule ρ has a persuasive recommendation representation under random utility (PR-RUM) if there exists a probability measure μ on $\mathcal{P}$ such that for every $x \in X$,

$$\rho(x, R) = \mu(\{\succ^* | c^{PR}_{\succ^*}(R) = x\})$$

for every $R \in \mathcal{D}$.

1.4.2 Informational Recommendation

In this framework, we assume that each DM in the population can differ on both their pre-recommendation choices and their preferences. In other words, this general model allows for potential dependence between preferences and pre-recommendation choice, whereas our parametric R-Luce model assumes independence. Also, by allowing different type in the model, we also relax the independence structure in the preferences. To define the type space, firstly, let $\mathcal{P}$ be the set of all (linear order) preferences and let $\mathcal{T}$ be the set of all pairs of $(a, \succ)$ where $\succ \in \mathcal{P}$ and $a \in X$, with an arbitrary type typically denoted by $t = (a, \succ)$. Here, we consider a probability measure τ on $\mathcal{T}$ such that $\sum_{t \in \mathcal{T}} \tau_t = 1$, where we denote $\tau_t := \tau(\{t\})$. We now state the definition of the model.

Definition 8. A choice rule ρ has a informational recommendation representation under random utility (IR-RUM) if there exists a probability measure τ on $\mathcal{T}$ such that for every $x \in X$,

$$\rho(x, R) = \tau(\{t|\ c_t^{IR}(R) = x\})$$

for every $R \in \mathcal{D}$.

1.4.3 Behavioral Characterization

Within the random utility (RUM) framework, Falmagne [19] answered the question whether individual preference maximization has any implications for aggregate data. For the characterization of the RUM, Falmagne [19] utilizes a well-known concept in the literature: the Block-Marschak polynomials, named after Block and Marschak [44]'s seminal work on the RUM. It has been shown that probabilistic choice data has a RUM representation if and only if its Block-Marschak polynomials are non-negative. The necessity was originally obtained by Block and Marschak [44]. Falmagne [19] showed that the non-negativity of the polynomials is also sufficient.

In our framework, we also utilize Block-Marschak (BM) polynomials. Let

$q_\rho(a, R)$ denote the BM polynomials *i.e.* for any $a \in R$,

$$q_\rho(a, R) := \sum_{B \supseteq R} (-1)^{|B \setminus R|} \rho(a, B)$$

The Block-Marschak polynomials are defined with respect to the choice data ρ. Throughout the chapter, we mostly skip denoting ρ and write $q(a, R)$ unless specified otherwise. Interestingly, this definition can be applied to off-recommendation data as well and, as we shall see, it has an important role in both IR-RUM and PR-RUM. We define for $a \notin R$,

$$y_\rho(a, R) := \sum_{a \notin B \supseteq R} (-1)^{|B \setminus R|} \rho(a, B)$$

Again, we skip denoting ρ and write $y(a, R)$ unless specified otherwise.

Figure 1.1 generalizes the classical network representation of partial order sets for our purposes. Each node represents a subset of the set of alternatives. Each black solid line indicates a subset relationship among subsets. The Block-Marschak polynomials can be thought of as the amount of flow on each black line. In the original network of this Hasse diagram, the degree of each node is equal to the number of alternatives, and inflow and outflow of black lines are always equal for each node.

$$\sum_{a\in R} q_\rho(a, R) = \sum_{b\notin R} q_\rho(b, R\cup b)$$

In RUM, each preference ranking corresponds a path starting from the empty set and ending at the grand set. For example, $\emptyset - \{c\} - \{b, c\} - \{a, b, c\}$ (with path $q(c, \{c\}) \to q(b, \{b, c\}) \to q(a, \{a, b, c\})$) represents $a \succ_1 b \succ_1 c$.[19] We first highlight that the above equality is no longer true in our model. But we discuss below how to recover a similar equality and provide a similar visual representation for types in R-RUM.

As opposed to RUM, IR-RUM and PR-RUM have two sets of BM conditions: one for recommended alternatives, q, and one for non-recommended alternatives, y. To represent the new BM conditions, y's, we introduce new flows, which are represented by dashed red lines. These are always the outflows (or "leakages") from the network. We abuse notation and denote both nodes and the flows with the same notation. $y_\rho(a, \{c\})$ denotes both the phantom node and the flow to the that node. Interestingly, if we also take into account y's, we recover the equality of inflow and outflow of all black and red lines.[20] That is,

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[19] One can refer to Fiorini [45] for a network analysis of RUM.

[20] This result is stated as Lemma 2, which is a generalization of Falmagne [19]'s Theorem 3. We believe that this lemma could be of independent interest since it is model-free. We provide the proof for it in the Appendix.

$$\sum_{a\in R} q_\rho(a,R) + \sum_{a\notin R} y_\rho(a,R) = \sum_{b\notin R} q_\rho(b, R\cup b)$$

Given this equality, we can represent each type in IR-RUM and PR-RUM by a path in the new Hasse diagram. Firstly, we consider IR-RUM. Each type corresponds to a path starting from a phantom node and ending at the grand set. For example, the path $y(c,\emptyset) \to q(c,\{c\}) \to q(b,\{b,c\}) \to q(a,\{a,b,c\})$ represents $a \succ_1 b \succ_1 c$ with c being the pre-recommendation choice, hence the type is $(c,\succ_1)$. Note that in this example, the pre-recommendation choice is the least preferred alternative according to $\succ_1$. Similarly, $y(b,\{c\}) \to q(b,\{b,c\}) \to q(a,\{a,b,c\})$ represents type $(b,\succ_1)$. Each of these two paths corresponds a unique type. However, the path $y(a,\{b,c\}) \to q(a,\{a,b,c\})$ corresponds to two types $(a,\succ_1)$ and $(a,\succ_2)$ where $a \succ_1 b \succ_1 c$ and $a \succ_2 c \succ_2 b$. Notice that these two types cannot be distinguished because they always choose a. This is a trivial non-uniqueness of the IR-RUM. Note that throughout these examples, every leakage path y follows its immediate preceding q. *i.e.* $y(a,A) \to q(a, A\cup a)$ for $a \notin A$. It turns out that every type in IR-RUM can be represented by such paths.

On the contrary, in PR-RUM, every path can be taken. Firstly, it captures the path which is permitted in IR-RUM. For example, $y(c,\emptyset) \to q(c,\{c\}) \to q(b,\{b,c\}) \to q(a,\{a,b,c\})$ represents the two preferences $a^* \succ^\star b^* \succ^\star c^* \succ^\star c \succ^\star a \succ^\star b$ and $a^* \succ^\star b^* \succ^\star c^*$

$\succ^\star c \succ^\star b \succ^\star a$ where the two preferences have the same ordering above c and share the same lower contour set below c. Notice these two preferences are inherently indistinguishable even in the deterministic model. Thus, in the following, we will use the notation $a^* \succ^\star b^* \succ^\star c^* \succ^\star c \succ^\star \{b, a\}$ to refer to both preference. Secondly, it captures path that is not permitted in IR-RUM. For instance, PR-RUM allows the path $y(b, \emptyset) \to q(c, \{c\}) \to q(b, \{b, c\}) \to q(a, \{a, b, c\})$, which represents the preferences $a^* \succ^\star b^* \succ^\star c^* \succ^\star b \succ^\star \{a, c\}$. Also, the path $y(a, \{c\}) \to q(b, \{b, c\}) \to q(a, \{a, b, c\})$ is also available in PR-RUM, which represents the preference $a^* \succ^\star b^* \succ^\star a \succ^\star \{b, c^*, c\}$.

In the following, we introduce the axioms of the model. The first axiom is the non-negativity of BM polynomials in the recommendation domain. This non-negativity is closely related to the non-negativity of BM polynomials in the standard domain. In the standard domain, the non-negativity of the BM polynomials means that item must be chosen marginally more if there are less available items. Here, the non-negativity of $y's$ and $q's$ means that both recommended and non-recommended alternatives must be chosen marginally more if there are less recommended products.

Non-negativity of BM. *For* $a \in R$, $q(a, R) \geq 0$ *and* $y(a, R \setminus a) \geq 0$.

There is another important property in this domain. Notice that the positive recommendation assumption suggests that the difference between $\rho(a, R)$ and $\rho(a, R \setminus a)$ must be weakly positive. Here, we look into a finer detail of this effect. Notice

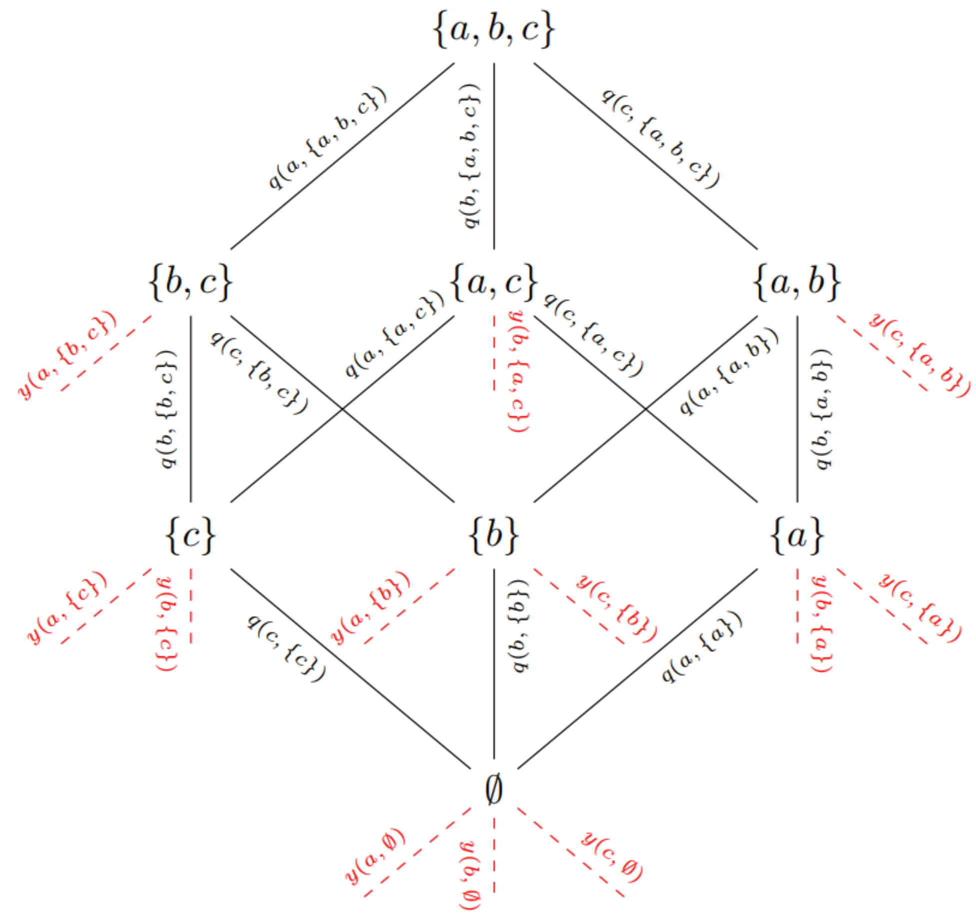

Figure 1.1: Hasse Diagram of R-RUM for three alternatives

that, the difference $q(a,R) - y(a,R \setminus a)$ captures the idea of the marginal change in the magnitude of positive recommendation effect. To see this, one can interpret this axiom as the comparison between the marginal gain from regularity in choice probabilities of alternatives when they are recommended and not recommended. In particular, this behavioral postulate says the marginal gain will be always higher if it is recommended.

Positive Marginal Recommendation. *For $a \in R$, $q(a,R) \geq y(a,R \setminus a)$.*

It turns out that non-negativity of BM polynomials alone fully characterizes PR-RUM. To characterize IR-RUM, we need the additional Positive Marginal Recommendation. Notice that by requiring qs' greater than ys', the non-negativity of qs' is immediately implied. Since the two models are general, we need a rich data set for characterization. Therefore, we assume that $\mathcal{D} = 2^X$ for the next theorem.[21]

Theorem 5 (Characterization)**.** Let $\mathcal{D} = 2^X$. Then,

a) ρ has a PR-RUM representation if and only if ρ satisfies Non-negativity of BM;

b) ρ has a IR-RUM representation if and only if ρ satisfies Non-negativity of BM and Positive Marginal Recommendation.

The sufficiency proof of the theorem is constructive. We provide an algorithm

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[21] This full data assumption is standard for RUM models.

to compute a full distribution of types in the IR-RUM and PR-RUM representation. The algorithm is certainly helpful in applications if one would like to have an estimate of the type distribution. However, just as the RUM in the standard choice domain, in general, there is not a unique distribution over types that can explain the choice data. To see this, we first look at the uniqueness result for IR. Let $L_{\succ}(a)$ represent strict lower contour set of a according to $\succ$. We state the uniqueness result in the following proposition.

Proposition 7 (Uniqueness and identification of IR-RUM)**.** If τ^1 and τ^2 represent the same choice rule ρ, then for every $A \subset X$, $b \notin A$ and $i = 1, 2$,

i) $y_\rho(b, A) = \tau^i\Big(\{(b, \succ) | A = L_{\succ}(b)\}\Big)$

ii) $q_\rho(b, A \cup b) = \tau^i\Big(\{(a, \succ) | A = L_{\succ}(b), a \in A \cup b\}\Big)$

The first property in Proposition 7 says that, $y_\rho(b, A)$ must be equal to the probability that b is the default option while b is exactly and only better than the set of alternatives A. On the other hand, the second property says that $q_\rho(b, A)$ must be equal to the probability that b is exactly and only better than the set of alternatives A and the default is within $A \cup b$. Note that (i) and (ii) imply that $q_\rho(b, A \cup b) - y_\rho(b, A) = \tau^i\Big(\{(a, \succ) | A = L_{\succ}(b), a \in A\}\Big)$, where the LHS is exactly the object from Positive Marginal Recommendation. From here, one can immediately see that $q's$ must be greater than its respective $y's$. Notice that this difference captures the fraction of people who rank b exactly above A while their default is in A. Surely, this fraction of people would switch to b if b is included in the recommendation set.

We now look the uniqueness result for PR.

Proposition 8 (Uniqueness and identification of PR-RUM)**.** If μ^1 and μ^2 represent the same choice rule ρ, then for every $R \subseteq X$, $b \in X \setminus R$ and $i = 1, 2$,

i) $y_\rho(b, R) = \mu^i\Big(\{\succ^\star | (X \setminus b) \cup R^* = L_{\succ^\star}(b)\}\Big)$

ii) $q_\rho(b, R \cup b) = \mu^i\Big(\{\succ^\star | X \cup R^* = L_{\succ^\star}(b^*)\}\Big)$

Notice that the result here is closely related to the uniqueness result of the IR-RUM model. Firstly, here $y_\rho(b, R)$ represents the preferences who rank $(X \setminus R)^*$ above b. Notice that $(X \setminus R)^*$ also include b^*. Therefore, it also captures different possible elevations of b in the preference. Note that since b is the first alternative after recommended alternatives, it is the best alternative in the set X. It shares the similar favor as the default option in the IR-RUM model. On the other hand, $q_\rho(b, R \cup b)$ represents the preference who rank $(X \setminus (R \cup b))^*$ above b^*. Therefore, it also captures different possibility of elevation of b^* above b. We can see from this result why Positive Marginal Recommendation is not satisfied. Let $X = \{a, b\}$. Consider we have $\mu_{\succ^\star_1} = 0.5$ for $a^* \succ^\star_1 b^* \succ^\star_1 a \succ^\star_1 b$ and $\mu_{\succ^\star_2} = 0.5$ for $b^* \succ^\star_2 a^* \succ^\star_2 a \succ^\star_2 b$. Then, $y_\rho(a, \emptyset) = \mu(\{\succ^\star_1, \succ^\star_2\}) = 1$ and $q_\rho(a, \{a\}) = \mu(\{\succ^\star_2\}) = 0.5$. Therefore, $y's$

can captures types that its corresponding q cannot capture.

1.4.4 Relationship to IR-Luce and PR-Luce

It is a well-known fact that the Luce model is a special case of the random utility model. In the recommendation environment, one might wonder whether IR-Luce belongs to IR-RUM and/or PR-Luce belongs to PR-RUM. It turns out that it is, which we state in the following.

Theorem 6.

a) Every choice rule ρ that has an IR-Luce representation has a IR-RUM representation.

b) Every choice rule ρ that has a PR-Luce representation has a PR-RUM representation.

1.5 Conclusion

Recommendations are abundant and prevalent in our lives. In this chapter, we consider two different channels through which recommendations can affect choices: informational recommendation (IR) and persuasive recommendation (PR). We first consider two deterministic models separately capturing the the informational and persuasive recommendations. We show that the deterministic models share the same behavioral trait: Independence of Irrelevant Recommended Alternatives. Moreover, one can distinguish PR from IR if they observe choice behavior that involves violation of Sandwich Property. Supported by our deterministic model of behavior, we introduce probabilistic choice models which can be interpreted as coming from aggregate choice data. We propose parametric versions of the models for tractability and applicability, namely the IR-Luce and PR-Luce models. We show that we can fully distinguish these two models from characterization. Lastly, to enhance explanatory power of the model, we incorporate the idea of Random Utility Model and propose the IR-RUM and PR-RUM model. We fully characterize these two models and show that each has a close connection to the classic well-known standard probabilistic choice models.

While our models investigate two important channels through which a recommendation can affect choice, we expect that they can be subject to refinement or generalization according to specific needs under different circumstances. Our framework is rich enough to study other channels that recommendation might be operational. The areas of further research should includes developing more elaborate and complex models of recommendation including strategic recommendations, limited consideration, status quo, behavioral search, satisficing, and temptation. Therefore, we believe that this chapter also paves a palpable path for fruitful future research and applications where we can apply the economic wisdom that have accumulated throughout the years for the standard models to this setting.

Chapter 2: Attention Overload

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2.1 Introduction

Decision-making is becoming a bustling task for consumers due to the abundance of options. For example, Amazon US sells more than 606 million products (87 million products in Home & Kitchen and 62 million Books). This phenomenon is also witnessed in other domains such as healthcare plans, car insurance, or financial services. It is without doubt that consumer cannot pay attention to all products: some are going to be more appealing than others, while some are completely unnoticed. The proliferation of options forces each product must compete with each other for consumers' attention.[1] Chernev et al. [53] argue that a plethora of options

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[1] The limited attention has been illustrated in different markets: investment decisions ([46]), school choice ([47]), job search ([48]), household grocery consumption ([49]), PC purchases ([50]), university choice ([51]; [47]), and airport choice ([52]).

leads to sub-optimal choices which is known as “choice overload” in the psychology literature.

Market competitions over consumer’s attention are fierce. According to Statista, the US has spent over \$253 billion dollars in advertisement in 2019. This fierce competition makes consumers’ attention problem more difficult. According to recent Ipsos eye-tracking research, the majority of TV advertising time (55%) is not paid attention to due to multitasking, switching channels, and fast-forwarding. In other words, it is possible that the same customer may attend to different products in the same environment on different occasions. The random attention idea has been utilized by new theoretical models [54, 55, 56, 57, 58]. In this stochastic environment, we define the amount of attention a product receives as the *attention frequency*.

Visschers et al. [59] suggests that the competition for attention gets more aggressive when the number of alternatives gets larger. For example, if a product grabs the consumer’s consideration in a large supermarket, then it will grab her attention in a small convenience store with fewer rivals. Hence the attention frequency for each alternative (weakly) decrease as the number of rivals increases (see Reutskaja and Hogarth [60], Reutskaja et al. [61], Geng [62]). We call this property as *attention overload*, which is the key ingredient of our model.

Choice overload are usually identified by indirect measures like the probability of choice deferral ([63]) or the probability of reversing the initial choice ([64]). While these indirect measures might be enough for providing evidence for choice overload, they are not satisfactory from a policy-making and welfare perspective. It is vital to identify whether and when larger choice sets lead to choice overload. Since choice overload is sub-optimal with respect to unobservable preference under limited attention, we must first reveal the preference of the individual. Our aim is to uncover preferences solely from observed choices when attention is limited and the products compete for consumers' attention. Following the traditional insight of economics, we assume that consumers have a complete and transitive preference over the alternatives. Consumer are also assumed to pick the best alternative in their consideration sets. The key assumption is that consideration is random and unobservable for the modeler or the econometrician. With our non-parametric attention overload assumption, we can reveal preference based solely on choice behavior. Our identification informs policy makers the nature of choice overload from observed choices.

Interestingly, existing models cannot capture attention overload. Manzini and Mariotti [54] considers an attention model with independent consideration where each alternative have a constant attention frequency even when there are more alternatives. Hence, the model cannot provide insight on consumer behavior when attention frequency is strictly monotonic. Aguiar [55]'s model of random categorization also shares the same feature. On the other hand, Cattaneo et al. [58] and Brady and Rehbeck [56] fail to satisfy the attention overload assumption, in which the attention frequency is non-monotonic. Hence, it is possible that an alternative is getting less attention even when the choice set gets smaller.

As we mentioned above, one of the behavioral consequences of choice overload is the likelihood of deferring choice. That is, when faced with too many unfamiliar choices, consumers choose the outside option (Iyengar and Lepper [63]). To be able to capture this notion, we introduce an outside option in our model. We show that our model is compatible with the fact that the outside option is chosen more often when the choice set size increases. The intuition is simple: the competition between alternatives get more fierce as the decision problems get bigger, the decision maker (henceforth, DM) tends not to consider any alternative and choose the outside option. Interestingly, existing models of random attention rule, e.g. Manzini and Mariotti [54], Brady and Rehbeck [56] and Cattaneo et al. [58], predict the other way.[2]

There are a number of insightful special cases of this model. Firstly, the idea of competition filter in Lleras et al. [65] is a special case of the model in the deterministic environment, which says that any item which wins the consumer's attention would

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[2] i.e. the outside option is chosen more often in the smaller set. For Manzini and Mariotti [54] and Brady and Rehbeck [56], the restriction lies in formation of attention rule. For Cattaneo et al. [58], the restriction comes from the monotonic attention rule.

also prevail in the smaller set. Secondly, attention models with constant attention frequency, e.g. Manzini and Mariotti [54] and Aguiar [55], are knife-edge special cases of the model, since attention overload assumes only weak inequality. Notions of rationalization (e.g. [66]), categorization (e.g. Manzini and Mariotti [67]) and narrowing down (e.g. Lleras et al. [65]) are nested in the model. In the words, this chapter subsumes the models above. It also means that any revealed preference in the using the simple non-parametric restriction would also hold in their environment. We discuss it further in Section 2.5.

However, due to the generality of the model, for some given data set, some preference over alternative may not be all identified. In other words, the revealed preference needs not be complete. In order to complement this property and reinforce revealed preference, we also investigate non-parametric restriction over binaries. We shows that it aids in revealed preference and we provide characterization result for the joint non-parametric restriction. Furthermore, we show that the condition is weaker (i.e. more general) than the one considered in existing literature Cattaneo et al. [58].

In the next section, we introduce the setup and the model. We discuss the revealed preference and characterization in section 2.3. Applications of the model are explored in Section 2.4. Lastly, we follow with a discussion on related literature in Section 2.5. Conclusion is in Section 2.6.

2.2 Choice under Attention Overload

We denote the grand alternative set as X, which are held fixed throughout the chapter A typical element of X is denoted by a and its cardinality is $|X|= N$. We let $\mathcal{X}$ denote the set of all non-empty subsets of X. Each member of $\mathcal{X}$ defines a choice problem.

Definition 9 (Choice Rule)**.** A choice rule is a map $\pi : X \times \mathcal{X} \to [0, 1]$ such that for all $S \in \mathcal{X}$, $\pi(a|S) \geq 0$ for all $a \in S$, $\pi(a|S) = 0$ for all $a \notin S$, and $\sum\limits_{a\in S} \pi(a|S) = 1$.

Therefore, $\pi(a|S)$ represents the probability that the DM chooses alternative a from the choice problem S. To see how the formulation allows deterministic choice rules. Take $\pi(a|S)$ as either 0 or 1, then choices are deterministic. The key ingredient of our model is probabilistic consideration sets. Given a choice problem S, each non-empty subset of S could be a consideration set with certain probability. It is natural to assume that each frequency is between 0 and 1 and that the total frequency adds up to 1. Formally,

Definition 10 (Attention Rule)**.** An attention rule is a map $\mu : \mathcal{X} \times \mathcal{X} \to [0, 1]$

such that for all $S \in \mathcal{X}$, $\mu(T|S) \geq 0$ for all $T \subset S$, $\mu(T|S) = 0$ for all $T \not\subset S$, and $\sum\limits_{T \subseteq S} \mu(T|S) = 1$.

Thus, $\mu(T|S)$ represents the probability of paying attention to the consideration set $T \subset S$ when the choice problem is S. This formulation also allows for deterministic attention rules. For example, $\mu(S|S) = 1$ represents an agent with full attention.

Another economically important variable we would like to keep track of is the amount of attention each *alternative* captures for a given μ. We can observe this information simply from μ by summing up the frequency of consideration sets containing the alternative. That is, for a given μ, the probability that a attracts attention in S is defined as:

$$\phi_\mu(a|S) := \sum_{a \in T \subseteq S} \mu(T|S)$$

When μ is clearly defined, we omit μ for convenience. Hence, $\phi(a|S)$ is the measure of attention for a in S. In deterministic attention model, the attention that one alternative receive is either zero or one. i.e. whether it is being considered or not. Yet, in stochastic environment, attention is probabilistic. This also means that the attention one alternative receives does not necessary be a zero-or-one dichotomy. We regard it as the *attention frequency* of an alternative.

When consumers are overwhelmed by the abundance of options, every product competes for consumers' attention. This implies that as the number of the alternatives increases, the competition gets more severe. That is, the frequency of attending to a product should weakly decrease when the set of available alternatives is expanded by adding more options to it. We call this property *Attention Overload*.

Assumption 2 (Attention Overload)**.** For any $a \in T \subseteq S$, $\phi_\mu(a|S) \leq \phi_\mu(a|T)$.[3]

We say that an attention rule satisfies attention overload if its corresponding ϕ is monotonic. Several models listed in Introduction satisfy this assumption. Take Manzini and Mariotti [54] as an example, the attention frequency is held fixed for each alternative. Having this non-parametric restriction, the choice rule can be defined accordingly. A DM who follows AOM maximizes his utility according to a preference ordering $\succ$ under each realized consideration set.

Definition 11. A choice rule π has a *attention overload representation* in $\succ$ if there exists a preference ordering $\succ$ over X and an attention rule satisfying attention overload such that

$$\pi(a|S) = \sum_{T \subseteq S} (a \text{ is } \succ\text{-best in } T) \cdot \mu(T|S)$$

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[3] If we allow the consideration set to be empty, then we should also require that the frequency of paying attention to nothing decreases when the choice set shrinks to capture choice overload. We discuss this in further detail in Section 2.4.1.

for all $a \in S$ and $S \in \mathcal{X}$. In this case, we say π is represented by $(\succ, \mu)$. We also say π is a *Attention Overload Model* (AOM).

2.3 Revealed Preference and Characterization

Given choice data that satisfies the attention overload model, is it possible to identify consumer's underlying preference? We show how it can be done in the following. To achieve this, we exploit the fact that attention frequency satisfies attention overload in attention overload model. Note that in attention overload model, each alternative gets more attention when choice set is smaller. Hence, it is natural to *expect* that each alternative would be more likely to be picked in a smaller choice set. However, if we observe the counterfactual, i.e. an alternative has a lower probability of being selected, we can deduce that there must be something better than it in the smaller choice set. Let us first define what it means to be revealed preference in the model.

Definition 12 (Revealed Preference)**.** Let $\{(\succ_j, \mu_j)\}_{j \in J}$ be all attention overload representations of π. We say that b is *revealed to be preferred* to a if $b \succ_j a$ for all j.

This reveal preference definition checks all possible representation for choice data π and concludes that a is revealed to be preferred to b only if all possible representations agree. This conservative specification is employed in most (if not all) limited consideration behavioral models; see Section **??** for references and further discussion.

Our first observation on the identification of preference is that *regularity violations* at binary choice problems reveal the DM's preference. More specifically, we illustrate that if $a, b \in S$ and $\pi(a|S) > \pi(a|\{a, b\})$, then b must be preferred to a. To reach such a conclusion, assume the contrary: there exists $(\succ, \mu)$ representing π such that $a \succ b$ and μ satisfies attention overload. First note that attention is a perquisite for choice. To be able to choose an alternative, the DM first must pay attention to it. Hence, the attention frequency is always greater (or equal) to the choice probability for any alternative and in any choice set $\pi(a|S) \leq \phi(a|S)$. In addition, they are equal for the best alternative in any choice set: a is $\succ$-best in S implies $\pi(a|S) = \phi(a|S)$. Given $a \succ b$, we have $\phi(a|\{a, b\}) = \pi(a|\{a, b\}) < \pi(a|S) \leq \phi(a|S)$. This contradicts with our attention overload assumption. The next lemma formally states this observation.

Lemma 1. Let π be a AOM. If $\pi(a|S) > \pi(a|\{a, b\})$, then b is revealed to be preferred to a.

To gain some insight, we also present below a more direct proof. Start with the attention overload assumption, then $\phi(a|\{a,b\}) \geq \phi(a|S)$ but this implies:

$$\Rightarrow \quad \sum_{a\in J\subseteq\{a,b\}} \mu(J|\{a,b\}) \geq \sum_{a\in J\subseteq S} \mu(J|S) \qquad \text{By definition}$$

$$\Rightarrow \quad \pi(a|\{a,b\}) + \sum_{\substack{a\in J\subseteq\{a,b\}\\ a \text{ is not } \succ\text{-best}}} \mu(J|\{a,b\}) \geq \pi(a|S) + \sum_{\substack{a\in J\subseteq S\\ a \text{ is not } \succ\text{-best}}} \mu(J|S)$$

$$\Rightarrow \quad \sum_{\substack{a\in J\subseteq\{a,b\}\\ a \text{ is not } \succ\text{-best}}} \mu(J|\{a,b\}) \geq \pi(a|S) - \pi(a|\{a,b\}) > 0 \qquad \text{By the hypothesis}$$

From the last line, since the consideration set that a is not $\succ$-best has positive probability, we can deduce that there *must be* something that is better than a in the set $\{a,b\}$. Hence, we can conclude that $b \succ a$. Now, the key question is whether we can generalize from $\{a,b\}$ to an arbitrary set $T \subseteq S$. To see why the answer is not straightforward, consider following example.

Consider the following choice data.

$\pi(.\|S)$	a	b	c	d
$S = \{a,b,c,d\}$	0	0.2	0.3	0.5
$\{b,c,d\}$		0.25	0	0.75

Notice that $\pi(c|\{a,b,c,d\}) > \pi(c|\{b,c,d\})$, and by the same operation, we have

$$\sum_{\substack{c\in J\subseteq\{b,c,d\}\\ c \text{ is not } \succ\text{-best}}} \mu(J|\{b,c,d\}) \geq \pi(c|\{a,b,c,d\}) - \pi(c|\{b,c,d\}) = 0.3 > 0.$$

This implies that there exists something better than c in the set $\{b, c, d\}$. However, we could not identify whether this alternative is b or c (might be both). $\square$

As the above example demonstrates, identification may not be clear when there are more than two alternatives in the smaller set. Therefore, we introduce a new perspective on choice data so that more revealed preference can be achieved: we consider the pair $(\pi, \succ)$, i.e., we pair choice data with a possible preference ordering over alternatives. With a slight abuse of notation, define

$$\pi(U_{\succ}(a)|S) = \sum_{b \in S, b \succ a} \pi(b|S),$$

which is the probability of choosing an alternative from the strict upper contour set of a in S, where $U_{\succ}(a)$ is the strict upper contour set of a. Similarly, let $\pi(U_{\succeq}(a)|S) = \pi(U_{\succ}(a)|S) + \pi(a|S)$, where $U_{\succeq}(a)$ is the weak upper contour set of a.

Definition 13 (Attention Compensation)**.** $(\pi, \succ)$ satisfies Attention Compensation (AC) if for all $a \in T \subseteq S$, $\pi(U_{\succeq}(a)|T) \geq \pi(a|S)$

There are two possible cases for the sign of $\pi(a|S) - \pi(a|T)$. If $\pi(a|S) - \pi(a|T) \leq 0$ for all $a \in T \subseteq S$ (i.e., there is no regularity violation), then AC automatically holds. A direct consequence is that if π satisfies regularity, then π also satisfies AC (for any preference ordering). Given this observation, we note that several important models also satisfy AC such as the random utility model. However, as there is no regularity violation in a standard random utility model, AC alone does not help identify the underlying preference.

To investigate the empirical content of AC, we define a revealed preference over this property, which is similar in spirit to the revealed preference in Definition 12:

$$bP_\pi a \quad \text{if } b \succ a \text{ for all preference orderings such that } (\pi, \succ) \text{ satisfies AC.} \tag{2.1}$$

To see how this definition aids revealed preference, suppose $\pi(a|S) - \pi(a|T) > 0$ and $T \subset S$. Then,

$$\pi(a|T) < \pi(a|S) \leq \pi(U_{\succeq}(a)|T) \quad \Rightarrow \quad \pi(U_{\succ}(a)|T) > 0,$$

which means there must be something that is better than a in the set T. Also, if the smaller set contains only two elements, we can immediately claim P_π from this property.

Corollary 1. If $(\pi, \succ)$ satisfies AC and $\pi(a|S) - \pi(a|T) > 0$ for some $T \subseteq S$, then there exists $b \in T$ such that $b \succ a$. In particular, if $T = \{a, b\}$, then $bP_\pi a$.

There exists a interesting connection between Lemma 1 and the second part of Corollary 1. Notice that AOM and AC are both able to claim revealed preference when they observe *regularity violation*. In addition, while Lemma 1 does not say anything when the smaller set contains more than two elements, AC gives a condition where one can make further exploration. In particular, we demonstrate that one can conclude some P_π in the setting of Example 2.3.

In this example, we have $\pi(c|\{b,c,d\}) - \pi(c|\{a,b,c,d\}) > 0$. Hence, from Corollary 1, we know that there exists $y \in \{b,c,d\}$ such that $y \succ c$. To look further into the condition for AC, we can see that $\pi(b|\{b,c,d\})$ alone is not able to account for the decrease of the choice probability of c. i.e. $\pi(b|\{b,c,d\}) < \pi(c|\{b,c,d\}) - \pi(c|\{a,b,c,d\})$. Hence, for π to satisfy AC, it must be either $d \succ c \succ b$ or $b \succ d \succ c$ or $d \succ b \succ c$ so that we have $\pi(d|\{b,c,d\}) \geq \pi(c|\{b,c,d\}) - \pi(c|\{a,b,c,d\})$ or $\pi(b|\{b,c,d\}) + \pi(d|\{b,c,d\}) \geq \pi(c|\{b,c,d\}) - \pi(c|\{a,b,c,d\})$. We can then conclude that $dP_\pi c$. $\square$

We state below the key characterization theorem that justifies the use of AC for revealed preference purposes when the choice rule is from an AOM. We provide the "only if" part of the proof here, which is similar in spirit to the proof for Lemma 1. The "if" part, which relies on Farkas's Lemma, is quite involved and is provided in the Appendix.

Theorem 7 (Characterization)**.** The pair $(\pi, \succ)$ is an AOM if and only if it satisfies AC.

Proof of the "only if" part. To start, note that

$$\phi(a|T) = \pi(a|T) + \sum_{\substack{a \in J \subseteq T \\ a \text{ is not } \succ\text{-best in } J}} \mu(J|T) \leq \pi(a|T) + \pi(U_{\succ}(a)|T) = \pi(U_{\succeq}(a)|T).$$

Since $\phi(a|T) \geq \phi(a|S)$, we then have $\pi(U_{\succeq}(a)|T) \geq \phi(a|S) \geq \pi(a|S)$. $\square$

With this characterization theorem, it follows directly that Definition 12 and (2.1) agree:

Corollary 2. Let π be an AOM. Then, b is revealed preferred to a if and only if $bP_{\pi}a$.

2.3.1 Revealed Attention

Since attention frequency is the key measure under the attention overload phenomenon, it is important to ask whether we can give a estimation of it from the choice data. In the following, we show how it can be done. In particular, we can recover a range of ϕ for each alternative under each choice set.

Proposition 9 (Revealed Attention)**.** Suppose $(\pi, \succ)$ satisfies AC. Then, for every AOM representation $(\succ, \mu)$ of π,

$$\max_{R \supseteq S} \pi(a|R) \leq \phi_{\mu}(a|S) \leq \min_{T \subseteq S} \pi(U_{\succeq}(a)|T)$$

The proof is immediately given from the following observation, since for all $T \subseteq S \subseteq R$, we must have, due to attention overload

$$\pi(a|R) \leq \phi(a|R) \leq \phi(a|S) \leq \phi(a|T) \leq U_{\succeq}(a)|T)$$

Note that the lower bound is a universal bound for regardless of preference while the upper bound is preference-specific. To learn the biggest upper bound, one can simply take the maximum across all $\succ$. The next concern is whether the bounds are sharp. i.e. given a choice rule that satisfies AC, one may wonder whether it is possible construct μ which hits the lower or upper bound. Interestingly, it is almost immediately given by the "if" proof from Theorem 7. Hence, we state the following corollary.

Corollary 3. Suppose $(\pi, \succ)$ satisfies AC, then there exists μ_1 and μ_2 such that

$$\phi_{\mu_1}(a|S) = \max_{R \supseteq S} \pi(a|R) \text{ and } \phi_{\mu_2}(a|S) = \min_{T \subseteq S} U_{\succeq}(a)$$

where π is represented by both $(\succ, \mu_1)$ and $(\succ, \mu_2)$.

Hence, if a choice rule satisfies AC, we can construct corresponding attention rule such that it always hits the upper bound or it always hits the lower bound. It may be tempting to conclude that for any value between the upper and lower bound, there exists a corresponding attention rule for the choice rule. Yet, there is a caveat: it is true if and only if we choose the value of ϕ such that it satisfies attention overload property.

2.4 Extentions

2.4.1 Default option and choice overload

Several existing literature considers the default option, e.g. Manzini and Mariotti [54], Brady and Rehbeck [56] and Echenique et al. [24]. To provide an accurate comparison to these models, we extend AOM to accommodate an outside option. Let a^* be the default option. In the model with the default option, we will allow an empty set consideration. Hence, now $\mu(.|S)$ is defined over all subsets of S including the empty set. The default option is always available and can be interpret as choosing nothing whenever the consideration set is empty. Let $X^* = X \cup \{a^*\}$ and $S^* = S \cup \{a^*\}$ for all $S \in \mathcal{X}$. We require that the choice rule satisfy $\sum_{a \in S^*} \pi(a|\pi) = 1$ and $\pi(a|S) \geq 0$ for all $a \in S^*$. Thus, $\pi(a^*|S) = \mu(\emptyset|S)$.

There is *a priori* no restriction on $\mu(\emptyset|S)$ from the attention overload model. In fact, for any choice data on default option, as long as the rest of the data satisfies Attention Compensation, the characterization still holds. Formally, we say that a choice rule π has a attention overload representation in $\succ$ with a default option if there exists a overloaded attention rule μ such that for each $a \in S$, $\pi(a|S) =$

$\sum_{T \subseteq S}(a \text{ is } \succ\text{-best in } T) \cdot \mu(T|S)$ and $\pi(a^*|S) = \mu(\emptyset|S)$. It is straight-forward to see that the characterization with the property AC is necessary and sufficient.

Remark 1. A choice rule π has a AOM presentation in $\succ$ with a default option if and only if $(\pi, \succ)$ satisfies compensating attention. ⌟

The next question is whether the outside options are chosen *more* or *less* often when the decision problem S gets bigger. The choice overload phenomenon, e.g. Iyengar and Lepper [63], suggests that people would tend to choose outside option *more* often if the sets get bigger. As mentioned in Section 2.1, existing models of random attention rule, e.g. Manzini and Mariotti [54], Brady and Rehbeck [56] and Cattaneo et al. [58], predict the other way.[4] In stark contrast, the intuition of our model is compatible with choice overload: while the competition between alternatives get more fierce as the decision problems get bigger, the DM tends not to consider any alternative and choose the outside option. Also, by the above characterization result, we know that choice overload can be explained by AOM if the rest of the data satisfies AC.

Lastly, we investigate how choice overload *affects* the explanatory power of AOM in the following scenario. Suppose an econometrican wasn't able to get the

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[4] i.e. the outside option is chosen more often in the smaller set. For Manzini and Mariotti [54] and Brady and Rehbeck [56], the restriction lies in formation of attention rule. For Cattaneo et al. [58], the restriction comes from the monotonic attention rule.

default option data but a new technology enables him to have access to such data. He also finds that the outside option data satisfies choice overload. One natural question to ask is that, if the original data satisfies AOM, does the new choice data under normalization still satisfies AOM? The answer is affirmative. The reason is that it is easier to satisfy AC under the new normalization.[5] In other words, the existence of outside option in choice overload *enhances* the explanatory power of AOM. We put this observation in the following remark. We denote π^* as the re-normalization of π with outside option satisfying choice overload.

Remark 2. If π is AOM, then π^* is AOM with a default option which satisfies choice overload $\lrcorner$

2.4.2 Attentive at binaries

When an alternative is chosen *frequently enough* in a binary choice set, a policy maker may want to conclude that the alternative is better than the other in the binary choice set. It is up to the choice of policy maker to decide what frequency is sufficient. We first denote this threshold frequency as η.

Definition 14. $(\pi, \succ)$ satisfies η-constrained revealed preference if $a \succ b$ whenever $\pi(a|\{a, b\}) > \eta$.

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[5] However, the converse of the statement is not true due exactly to the opposite reason.

As discussed before, by considering $(\pi, \succ)$, we are matching a preference ordering to a choice data. The above definition fulfills exactly our needs in revealing the preference from choice data. We can see that η measures how *cautious* the policy maker is when making welfare judgement. If η is 1, the policy would conclude nothing from the choice data. If η is lower than 0.5, the policy maker may get a cyclic $\succ$ which does not help at all with policy making. Hence, the question is, given choice data, how do we know we can safely impose this definition of revealed preference? The answer is straightforward. If we put a sufficient restriction on the choice generating process, i.e. the choice rule that is generated by attention rule, we can use the above definition and make the claim of revealed preference.

Assumption 3 (η-attentive at Binaries)**.** For any $a, b \in S$,

$$\eta \geq \max\{\mu(\{a\}|\{a,b\}), \mu(\{b\}|\{a,b\})\}$$

The above assumption fulfills our need. The condition is simple and intuitive. Consider η is 0.4. Given that the singleton consideration sets are bounded above by 0.4, if we observe that a is chosen more frequently than 0.4, we know that there must be some occasions where about a and b are considered together, i.e. $\mu(\{a,b\}|\{a,b\}) > 0$, and the consumer prefers a to b. Our assumption is weaker than the assumption proposed by Cattaneo et al. [58], meaning that any attention rule that satisfies their assumption would automatically satisfies our assumption. We then present the joint characterization results. The result guarantees us two things. Firstly, the definition 14 indeed gives us revealed preference by making the assumption. Secondly, given choice data, we know "when" these data can be represented by the AOM with the assumption of η-attentive at Binaries.

Theorem 8 (characterization). π has a attention overload representation in $\succ$, where the attention satisfies assumption 1 and 2, if and only if $(\pi, \succ)$ satisfies attention compensation and η-constrained revealed preference.

2.5 Related Literature

2.5.1 Attention Rule

The *attention overload model* (AOM) is a crucial missing piece of the puzzle in the limited consideration models. We can see this by its close connection to the *random attention model* (RAM) proposed by Cattaneo et al. [58]. While RAM generalises the *attention filter* in Masatlioglu et al. [68], AOM generalizes the *competition filter* in Lleras et al. [65]. Note that AOM and RAM are independent. Attention Overload assumption is orthogonal to Monotonic Attention assumption of Cattaneo et al. [58]. We provide two attention rules to highlight their differences. The first one satisfies Monotonic Attention but not Attention Overload. According this attention rule, DM consider everything in a larger set but she only considers singleton consideration sets for smaller sets.

$\mu(T\|S)$	$\{a,b,c\}$	$\{a,b\}$	$\{a,c\}$	$\{b,c\}$	$\{a\}$	$\{b\}$	$\{c\}$
$\{a,b,c\}$	1	0	0	0	0	0	0
$\{a,b\}$		0			1/2	1/2	
$\{a,c\}$			0		1/2		1/2
$\{b,c\}$				0		1/2	1/2

The second one satisfies Attention Overload but not Monotonic Attention. This attention rule highlights the idea of "less is more". DM cannot deal with larger choice sets, hence she only consider singleton consideration sets when she faces larger sets but she considers everything in smaller sets.

$\mu(T\|S)$	$\{a,b,c\}$	$\{a,b\}$	$\{a,c\}$	$\{b,c\}$	$\{a\}$	$\{b\}$	$\{c\}$
$\{a,b,c\}$	0	0	0	0	1/3	1/3	1/3
$\{a,b\}$		1			0	0	
$\{a,c\}$			1		0		0
$\{b,c\}$				1		0	0

These two attention rule make it clear the distinction between two assumptions on attention rule. However, since the attention rule is not observable, one might wonder whether these two models are behaviorally different. The example below illustrates that there are some choice data which have an AOM representation but not RAM representation, and also the other way around.

(Explanatory Power: AOM vs. RAM)

$\pi_1(.\|.)$	a	b	c
$\{a,b,c\}$	0.4	0.3	0.3
$\{a,b\}$	0.8	0.2	
$\{a,c\}$	0.8		0.2

In RAM, revealed preference says that $b \succ c$ and $c \succ b$. Hence, it could not be explained by RAM. In AOM, revealed preference says $a \succ c$ and $a \succ b$. Interestingly, there does not exists an example of three alternatives where RAM can explain but AOM cannot.[6] We need to go for four alternatives cases to achieve this.

$\pi_2(.\|.)$	a	b	c	d
$\{a,b,c,d\}$	1/2	1/2	0	0
$\{a,b,c\}$	0	2/3	1/3	
$\{a,b\}$	1/2	1/2		

---

[6] In other words, AOM explains *more* choice data than RAM in 3-alternative cases.

In AOM, revealed preference says that $a \succ b$ and $b \succ a$. Hence, it could not be explained by AOM. In RAM, revealed preference says $a \succ d$ and $b \succ c$.

In deterministic environment, both Competition filter and Attention filter give revealed preference by considering taking away an element from the choice set, but the intuition behind is distinctively different. When we observe choice reversal of alternative, attention filter says that the item is *better* than another item in the *bigger* choice set, while competition filter says that the item is *worse* than another item in the *smaller* choice set. Note that these two ideas naturally extend itself into the probabilistic environments in RAM and AOM: When one observes a violation of regularity of an alternative, RAM says the alternatives is *better* than another alternative in the bigger choice set, and AOM says the alternative is *worse* than another alternative in the smaller choice set. This is illustrated in Example above.

As discussed in Section 2.1, Manzini and Mariotti [54] is a special case of the model. Manzini and Mariotti [54] assumes that each alternative a has a fixed probability, $\gamma(a)$, to be considered. It is equivalent to say that the attention frequency, $\phi_{MM}(a|S)$, in their world is held fixed across different decision problem S. i.e. for all S

$$\phi_{MM}(a|S) = \gamma(a) \text{ for all } S$$

whenever $a \in S$. Since attention overload requires only weak inequality, the model falls into AOM. The other model which (implicitly) assumes constant attention frequency is Aguiar [55]. In his model, each category D has a fixed probability $m(D)$. If the category is *available*, the DM picks the best alternative out of it. If not, the DM chooses the default option. Let the set of all category be $\mathcal{D}$. Therefore, the attention frequency is given by, whenever $a \in S$ and for all S

$$\phi_{Aguiar}(a|S) = \sum_{a \in D \in \mathcal{D}} m(D)$$

where the right-hand side does not depend on S.

Lastly, we put our attention on random competition filter (RCF), which is a major special case of the model. Let $\Gamma_i(.)$ be consideration set mapping which satisfies competition filter and $\sum_{i=1}^{n} \alpha_i = 1$. A random competition filter model is specified by the following attention rule with the respective attention frequency,

$$\mu_{RCF}(T|S) = \sum_{i=1}^{n} \alpha_i (\Gamma_i(S) = T)$$
$$\phi_{RCF}(a|S) = \sum_{i: a \in \Gamma_i(S)} \alpha_i$$

One can immediately see that random competition filter satisfies attention overload, since $a \in \Gamma_i(T)$ for all $T \subseteq S$ if $a \in \Gamma_i(S)$. Random attention filter nests two others model: Bounded Rationalization and Imprecise Narrowing Down. Bounded Rationalization is a straightforward and meaningful generalization of Cherepanov et al. [66]. It states that the DM does not always stick to the same set of rationale. Hence, it is *as if* the DM assigns a probability distribution over sets of rationale. Since Cherepanov et al. [66] is a special case of Lleras et al. [65], it immediately follows that Bounded Rationalization model is a special case of random competition filter. Imprecise Narrowing Down shares a similar idea: The DM does not necessarily follow the same procedure on setting up criteria, which implies a probability distribution over the set of all possible procedures. It makes Imprecise Narrow Down a special case of random competition filter.

2.5.2 Regularity

In the following, we consider models which do not utilize attention rule. Notice that a number of models in this aspect respect regularity. The seminal Random Utility Model (RUM) is one of those. By previous discussion, the condition AC is automatically satisfied when models satisfy regularity. Hence, any RUM is AOM. Note that there are a number of models are included in RUM. For example, Gul et al. [26] considers attribute rule in which the DM first draw an attribute and then pick an alternative which contains such attribute. They show that every attribute rule is a RUM. Hence, every attribute rule is a AOM. On the other hand, Fudenberg et al. [23] introduces the additive perturbed utility model where the DM intentionally randomized as deterministic choices can be costly. Since the choices in their model always satisfies regularity, any choice rule in APU has a representation of AOM.

2.5.3 Others

There are several other stochastic choice model which are compatible with regularity violation. Intriguingly, we can show that some of them are AOM by directly checking the condition AC. Echenique et al. [24] considers priorities in alternatives *before* the DM applies the Luce rule, which is called the Perception-adjusted Luce model (PALM). In the model, DM impose a weak order $\succeq$ over alternatives as priority and attach Luce weight $u(x)$ to each of them. To explain their model, We take their primitive $\succeq$ as our preference $\succ$ and consider an arbitrary tie-breaking rule. We put the observation in the following.

Proposition 10. Any PALM satisfies AC.

Echenique and Saito [25] proposes a model, General Luce Model (GLM), where a deterministic consideration set mapping is applied before the DM use Luce rule over alternatives. Notice GLM reduces to the standard Luce model when every alternative is chosen with positive probability. We can construct example where every alternative has positive probability but does not satisfy Luce rule. Hence, GLM does not include AOM. On the other hand, AOM does not include GLM because the restriction-free consideration set mapping in GLM allows for cyclic P in our model. However, the Threshold GLM, which is a special case of GLM, is nested in AOM. In threshold GLM, alternatives with too low a Luce weight, $u(x)$, would not be considered. We take their primitive $u(x)$ and construct $a \succ b$ if $u(a) < u(b)$, and take an arbitrary tie-breaking rule if $u(a) = u(b)$.

Proposition 11. Any threshold GLM satisfies AC.

2.6 Conclusions

As there are more products while attention is limited, it is likely that consumer's attention span on each product decreases due to competition, which we call "attention overload". In this chapter, we develop the notion of attention frequency and propose the *Attention Overload Model* to capture attention overload. We show that the condition, Attention Compensation, is key to checking whether the data is consistent with the model. We also show that several existing models fall under this model, but at the same time, the richness of the model allows us to explain more different phenomena such as Choice Overload. We show how policy marker can draw inference over revealed preference by purely observing choice data. A more stringent requirement, attention at binaries, is proposed to provide more information to the policy marker for welfare judgement. The research on AOM opens up a path intelligent enquiry into consumer's attention under competition. For example, one may ask, what would be the stylistic parametric model of attention overload given existing parametric model of attention fall short of capturing attention overload? A useful parametric model can definitely further benefit the level of grip over consumer's behavior from the policy-making point of view. On the other hand, along the line of non-parametric restriction, one may be interested to find out what is the intersection between the RAM from Cattaneo et al. [58] and AOM. Further researches in this agenda are encouraged.

Chapter 3: Guilt Moderation

3.1 Introduction

Social decision making is complicated. The seminal works of Fehr and Schmidt [69] and Bolton and Ockenfels [70] emphasize how the inequality motives explain human behavior against the standard pure self-interest model. At the same time, scholars have also explored various forms of outcome-based other-regarding preferences.[1] When nature does not play a role, these social preference models give insightful explanations into social decision making such as in the dictator game, ultimatum game, gift exchange game, bargaining and so on. Yet, when nature is involved, social decision making can have intriguing interactions with chances. Such interactions can cause puzzling "reversals" in other-regarding behavior.

The first type of reversal, which we call *negative reversal*, captures the behavior where mixing with a fair outcome reverses an altruistic choice. An example of negative reversal can be found in the *moral wiggle room* experiments in Dana et al. [1]:

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[1] For a brief summary, see Fudenberg and Levine [2].

A significant fraction of subjects, while preferring $(\$5, \$5)$ to $(\$6, \$1)$, also prefer the mixture of $(\$5, \$5)$ and $(\$6, \$1)$ to $(\$5, \$5)$, where $(\$6, \$1)$ represents an allocation of \$6 for me and \$1 for others.[2] Here, the altruistic behavior of choosing $(\$5, \$5)$ over $(\$6, \$1)$ is *reversed* in the mixture when the outcomes are mixed with $(\$5, \$5)$. The behavior is listed in Figure 3.1.

$$(5,5) \succeq (6,1) \text{ and } (5,5) \prec$$

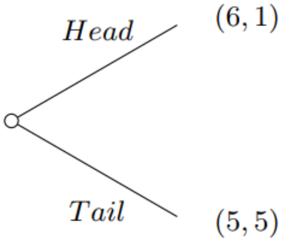

Figure 3.1: Experiments in Dana et al. [1]

The second type of reversal, which we call *positive reversal*, is exactly the opposite, where mixing with an unfairer outcome reverses a self-centered choice. Fudenberg and Levine [2] discussed an example which motivated the notion of *ex-ante fairness for you*.[3] Consider a decision maker (henceforth, DM), while preferring $(\$8, \$5)$ over $(\$7, \$7)$, *reverses* his preference when both outcomes are mixed with

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[2] Here, we are comparing the results from the baseline game and the Plausible deniability treatment. In the latter treatment, those who were cut off are effectively choosing a lottery. In this between-subject design, 74% of subjects in the baseline game choose the fair outcome and only 34% of subjects in the Plausible deniability treatment choose the fair outcome. Note that the increase in the proportion of unfair choice can only partially account for the .4 decrease in the choice of fair outcome. In fact, out of the .4 decrease, subtracting the increase in unfair choice, 62.5% $(= \frac{.4-(.41-.26)}{.4})$ choose the lottery.

[3] One key intuition from their discussion is that the decision maker behaves *as if* he considers certain ideas of fairness at the ex-ante decision making stage. Here, we do not assume that the decision maker has a ex-ante fairness concept in mind.

(\$1000, \$0). The example behavior is listed in Figure 3.2.

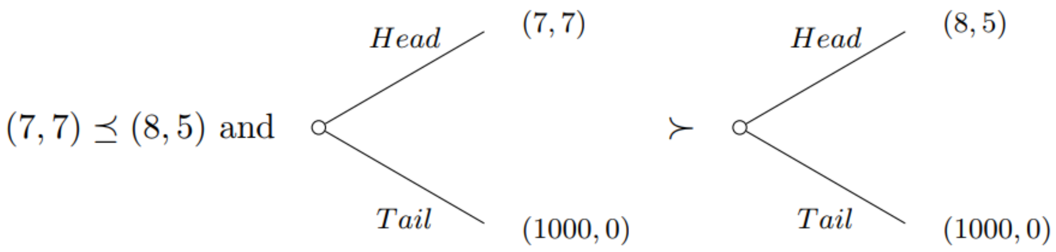

Figure 3.2: The leading example in Fudenberg and Levine [2]

Notice that these behaviors contradict both purely altruistic concern and purely selfish preferences because in either case the DM should not reverse his choice when given the mixture. In fact, any outcome-based social preference models cannot tackle either reversal by using expected utility since these are violations of the independence axiom.[4] Also, one should not expect existing social preference models motivated with *equality of opportunity* (e.g. Saito [71]) to produce negative reversal behavior. The reason is that the essence of the negative reversal behavior is about one being more selfish when outcomes are probabilistic, which, intuitively, should not be driven by the concern of being more fair in terms of opportunity.[5] One candidate hypothesis would be of *delegation of responsibility*. For instance, the DM could reckon that part of the responsibility for unfair outcomes is on *nature* rather than him, such

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[4] For an elaborated discussion, see Fudenberg and Levine [2].

[5] A more detailed comparison across models is provided in Section 3.6

that the decision disutility for picking the mixture is substantially less.[6] In this regard, this notion can induce the negative reversal behavior. However, such motives will nonetheless fail the positive reversal. Take the above example, there is even a stronger reason for the DM to choose the mixture of $(\$8, \$5)$ and $(\$1000, \$0)$ since the responsibility of the DM for the unfair outcome $(\$8, \$5)$ in the mixture is even less than the deterministic choice of $(\$8, \$5)$ over $(\$7, \$7)$.

We propose a simple idea to capture both positive and negative reversal, which is called *guilt moderation*. Here, guilt means disutility from advantageous inequality. The core idea is that when the choices are risky, the DM reallocates weights in disutility according to *fairness* distribution in the mixture. In particular, he puts less (or more) weight on unfairer (or fairer) outcomes. A result of such reallocation is that the DM will experience *less* guilt when compared with no reallocation of weight, and be more likely to exhibit self-centered (or altruistic) behavior when outcomes are mixed with a fairer (or unfairer) outcome.

Note that guilt moderation immediately captures both reversal behaviors discussed above. For the negative reversal: Disproportionally less weight is allocated to the guilt of $(\$6, \$1)$ in the mixture and hence the selfish motive is relatively *stronger* compared to the guilt. For the positive reversal: when mixed with $(\$1000, \$0)$, the

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[6] Bartling and Fischbacher [72]'s experimental result shows that subject is punished less when she delegates the choice of fair and unfair outcome to a randomizing device. The studies suggest that the delegation *reduces* responsibility.

fairer outcomes are $(\$8, \$5)$ and $(\$7, \$7)$. Guilt moderation allocates more weight to the guilt comparison of $(\$8, \$5)$ and $(\$7, \$7)$ and hence the selfish motive is relatively *weaker* compared to the social disutility.

It turns out that the notion can be implemented by utilizing existing models. We propose and discuss the inequality-averse model with rank-dependent disutility in Section 3.2 and investigate how guilt moderation characterizes the two reversal behaviors. Then, in Section 3.3, we take a closer examination of such a DM by studying the axioms leading to the RIA model. In section 3.4, we look at evidence on risk preference for others, which provides an *out-of-sample* test for guilt moderation. Applications of the model are studied in Section 3.5, which includes self-other risk-attitudes gap and the increased envy in wage transparency. Related literature is discussed in Section 3.6, and we conclude in Section 3.7.

3.2 The Model

As suggested, disutility over advantageous inequality is the key for understanding the reversal behaviors. Hence, we assume DM to behave as an inequality-averse agent in the deterministic environment with only the guilt component.[7] For risky and uncertain environments, we assume that the DM's disutility is regulated by

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[7] We relax this assumption in the Section 3.2.1 in order to prepare to study wider range of phenomenon later.

Rank-Dependent Utility (RDU), which reallocates weight according to the fairness ranking determined by the inequality aversion model. In the following, we opt for a more general framework of state space rather than lottery space. It is because the motivation of the chapter is about how events are weighted differently across utility component under uncertainty rather than how specifically probabilities are distorted. The general formulation allows us to capture both positive and negative reversal behavior even if an objective probability is not observable or the DM does not agree with an observable objective probability.[8]

Let Ω be a finite set of states with Σ denoting its power set. Let $v \in \Delta(\Omega)$ be a (subjective) probability measure over the states. Let $\mathcal{F}$ be the set of all acts with typical element $f : \Omega \to R^2$, where f_i represents the materialistic payoff of the ith player, where $i = 1, 2$. We denote the DM as the 1st player. Before we go into our model, we introduce the general class of models which nests the models in this field:

$$W(f) := \underbrace{E(u(f_1))}_{\text{Expected (selfish) utility}} + \underbrace{SC(f)}_{\text{Social utility component}}$$

where $E(.)$ is the expectation operator over the probability measure v, $u(.)$ is the standard Bernoulli utility and $SC(.)$ is the social utility component. There are two components in this general form: the first component captures the usual selfish

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[8] We introduce objective probability in Section 3.4 for risk preference for others.

expected utility from one's own payoff; while the second component captures social utility by considering joint social consequences.[9] Since the key insight of the examples is not about risk-preference of the DM, we assume $u(x) = x$ but adopt a novel social utility component.[10] In particular, the *Rank-Dependent Guilt* (RG) model assumes:

$$SC_{RG}(f) = -\underbrace{\beta \int_{\Omega} \max\{f_1(\omega) - f_2(\omega), 0\} dv_g}_{\text{Guilt}}$$

where $v_g : \Sigma \to [0,1]$ is an non-additive probability or capacity and β is the guilt parameter. Hence, the integral defined for the non-additive probability is the Choquet integral.[11] Therefore, each RG DM is specified by (β, v, v_g). A well-known property of RDU is that one can over-weight or under-weight utility through conditions over the non-additive probabilities.[12] Therefore, the reallocation of weight for the guilt term can be achieved by the condition $v_g(E) \leq v(E)$, such that DM will always

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[9] Several existing models of social preference are subclass of this model by assuming the probability measure follows objective probability: Expected inequality-averse (EIA) model from Saito [71] and the inequality-aversion with independence axiom model (IAI) discussed in Fudenberg and Levine [2], and the moral hypocrisy (HR) model from Borah [73]. We discuss them in more details in Section 3.6.

[10] We relax risk neutrality assumption in Section 3.5.1 to explain the self-other risk attitude gap.

[11] We follow Schmeidler [74]'s definition of non-additive probability that a function is said to be non-additive probability if it satisfies the normalization conditions $v(\emptyset) = 0$ and $v(S) = 1$, and monotonicity, i.e. for all E and G in Σ: $E \subset G$ implies $v(E) \leq v(G)$. Also, we define $\int_\Omega a dv := \sum_{i=1}^{k} (\alpha_i - \alpha_{i+1}) v(\cup_{j=1}^{i} E_j)$ for v an non-additive probability, $a(x) = \sum_{i=1}^{k} \alpha_i 1_{x \in E_i}$ a finite step function with $\alpha_1 > \alpha_2 > ... > \alpha_k$ and $(E_i)_{i=1}^{k}$ a partition of Ω, where $\alpha_{k+1} := 0$.

[12] See, for example, Wakker [75] for the discussion of the property of pessimism and optimism in RDU.

have less *guilt* compared to the expected utility counterpart. Hence, we make the following definition.

Definition 15. (guilt moderation) A (β, v, v_g) RG DM is said to be guilt moderating if $v_g(E) \leq v(E)$ for all $E \subseteq \Omega$.

Also, since the non-additive measure v_g is directly comparable to the probability measure v used in evaluating own payoff, one important interpretation is that excess or incremental payoffs will be weighted more than incremental "guilt" payoffs. The weighting of disutilities over unfair and fair events is illustrated in Example 2, which shows how the overweighting and underweighting of events can be evaluated with respect to $v(E)$.

Example 2. Consider the mixture of $(6, 1)$ and $(5, 5)$ in Figure 3.1. Let $\Omega = \{Head, Tail\}$, $v(Head) = v(Tail) = 0.5$ and $v_g(Head) = 0.3$. Then, the guilt term equals to

$$v_g(Head) * 5 + (1 - v_g(Head)) * 0$$
$$= v(Head) * \frac{v_g(Head)}{v(Head)} * 5 + (1 - v(Head))\frac{(1-v_g(Head))}{(1-v(Head))} * 0$$

Hence, compared to expected utility, $\frac{v_g(Head)}{v(Head)} = \frac{0.3}{0.5} = 0.6$ is the underweighting factor for the disutility in the unfairer event, and $\frac{(1-v_g(Head))}{(1-v(Head))} = \frac{0.7}{0.5} = 1.4$ is the overweighting factor for the disutility in the fairer event. ┘

In the following, one can see that the behavioral implication of guilt moderation is that it leads to a propensity to exhibit self-centered (or altruistic) behavior when outcomes are mixed with a fairer (or unfairer) outcome.[13] We first consider the reversal behaviors from the introduction. Let $\Omega = \{Head, Tail\}$. By a simple calculation, the conditions to explain the negative reversal behavior in Dana et al. [1] is

$$\frac{v(Head)}{5v_g(Head)} > \beta \geq \frac{1}{5},$$

which implies $v_g(Head) \leq v(Head)$; and the conditions for the positive reversal behavior in Fudenberg and Levine [2] is

$$\frac{1}{3} \geq \beta > \frac{1 - v(Tail)}{3(1 - v_g(Tail))},$$

which implies $v_g(Tail) \leq v(Tail)$. If we assume state independence, in which the DM exhibits the same behavior even if we change the name "Head" to "Tail", the implication of each of these behaviors is exactly guilt moderation. In other words, guilt moderation is a *necessary* condition for reversal behavior. Therefore, we define

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[13] One might also might also think of alternative way to characterize guilt moderation. Indeed, a helpful anonymous referee proposes a property based on an idea of compensation: for all $x < 0$, if $(-1, -1) \sim (0, x) \Rightarrow \forall E, (1, 1 + x)E^* \succeq (0, 0)$. Here, the first indifference condition determines the guilt parameter β, and the second preference relation suggests that the tolerance over advantageous inequality is exactly compensated by the guilt moderation and the corresponding guilt parameter. One can also further show that this property is equivalent to guilt moderation under RIA, and can be treated as a model-free definition of guilt moderation.

the following axiom. In the following, we use the notation fE^*g for the act which gives $f(\omega)$ for the state ω in the event $E \subseteq \Omega$ and g otherwise.

Negative Reversal. *For all non-empty $E \subset \Omega$, there exists $x_2 > x_1 > x_0 \geq 0$ and x_3 such that*

$$(x_1, x_1) \succeq (x_2, x_0) \textit{ and } (x_1, x_1)E^*(x_3, x_3) \preceq (x_2, x_0)E^*(x_3, x_3);$$

Positive Reversal. *For all non-empty $E \subset \Omega$, there exists $x_3 > x_2 > x_1 > x_0 \geq 0$ such that*

$$(x_1, x_1) \preceq (x_2, x_0) \textit{ and } (x_1, x_1)E^*(x_3, 0) \succeq (x_2, x_0)E^*(x_3, 0).$$

In fact, guilt moderation is also *sufficient* for the existence of reversal behaviors, as we will show below. Nevertheless, this sufficiency is weak in the sense that it does not tell us exactly *when* the DM would exhibit the reversal behavior.[14] This is reasonable because the exact occurrence of the phenomena still depends on the individual guilt parameter β and subjective probabilities, which are unobservable to the modeler.[15] Yet, if the modeler can observe indifference between two social

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[14] Note that we can also define strict versions of the reversals behaviors where $\succeq$'s are replaced by $\succ$'s for the lottery comparison. That axiom is equivalent to the strict version of guilt moderation where $v_g(E) < v(E)$ for all non-empty $E \subset \Omega$.

[15] While not being the main thesis of the chapter, one might conjecture reversal behavior could

allocations, guilt moderation gives a stronger prediction. It predicts that mixing with a fair (or unfair) outcome *will* lead to a self-centered (or altruistic) behavior.[16] We call these behaviors as *negative shift* and *positive shift* and they are stated below.[17] Therefore, the characterization of guilt moderation is stated in Proposition 12, which characterizes how guilt moderation generates the tendency to exhibit self-centered (or altruistic) behavior when outcomes are mixed with a fairer (or unfairer) outcome.[18]

Negative Shift. *For all non-empty* $E \subset \Omega$, *for every* $x_2 > x_1 > x_0 \geq 0$ *and* x_3,

$$(x_1, x_1) \sim (x_2, x_0) \text{ implies } (x_1, x_1)E^*(x_3, x_3) \preceq (x_2, x_0)E^*(x_3, x_3);$$

Positive Shift. *For all non-empty* $E \subset \Omega$, *for every* $x_3 > x_2 > x_1 > x_0 \geq 0$,

$$(x_1, x_1) \sim (x_2, x_0) \text{ implies } (x_1, x_1)E^*(x_3, 0) \succeq (x_2, x_0)E^*(x_3, 0).$$

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depend on the size of event. In particular, one might anticipate a bigger event might be more likely to induce reversal behavior. In fact, it can be accommodated in the model. Consider again the example for negative reversal behavior. We first let $E \subseteq E'$. Then, we compare the acts $(6,1)E^*(5,5)$ and $(6,1)E'^*(5,5)$ and their relations to the constant act $(5,5)$. The desired behavior is that $(6,1)E^*(5,5) \preceq (5,5)$ but $(6,1)E'^*(5,5) \succeq (5,5)$ and $(5,5) \succeq (6,1)$, which gives $\frac{v(E)}{v_g(E)} \leq 5\beta \leq \frac{v(E')}{v_g(E')}$ and $\beta \geq 0.2$. Intuitively, it means that the magnitude of guilt moderation has to be small for smaller event E and the magnitude of guilt moderation is larger for bigger event E'. To get a numerical example, take $\beta = 0.3$, $\frac{v(E)}{v_g(E)} = 1.2$ and $\frac{v(E')}{v_g(E')} = 1.8$ and the inequality becomes $1.2 \leq 1.5 \leq 1.8$ and it holds.

[16] I thank an anonymous referee for this suggestion.

[17] Similar to footnote 14, strict versions of shift behaviors where $\succeq$'s is replaced by $\succ$'s are equivalent to the strict version of guilt moderation.

[18] All proofs are in the Appendix.

Proposition 12. (characterization of guilt moderation) For a (β, v, v_g) RG DM, the following statements are equivalent:

(i) The DM is guilt moderating.

(ii) The DM exhibits negative reversal.

(iii) The DM exhibits positive reversal.

(iv) The DM satisfies negative shift.

(v) The DM satisfies positive shift.

Furthermore, one can get a direct test of guilt moderation if we assume the DM's subjective probability coincides with an observable objective probability. We discuss that in detail in Section 3.4.

3.2.1 The General Model

In the last section, we study the the RG model where we only consider the disutility over advantageous inequality. Nonetheless, the idea of moderation can also be carried over to the envy domain (i.e. disutility over disadvantageous inequality), which also have an important implication over different economic phenomena in the risk and uncertainty domain as we will discussed in Section 3.4 and 3.5. Here, we provide the most general model in this domain, namely *Rank-Dependent Inequality-Averse* (RIA) Model, which encompasses both envy and guilt component.

In particular, the social utility component of RIA can be written as

$$SC_{RIA}(f) = \underbrace{-\alpha \int_{\Omega} \max\{f_2(\omega) - f_1(\omega), 0\} dv_e}_{\text{Envy}} + SC_{RG}(f)$$

where $v_e : \Sigma \to [0,1]$ is again a non-additive probability used with the Choquet integral and $\alpha \geq 0$ is the envy parameter. Therefore, each RIA DM is specified by $(\alpha, \beta, \mathbf{v})$, where $\mathbf{v} := (v, v_g, v_e)$. There are several things to note regarding this model. Firstly, from this formulation, one can easily see RG is a special case of RIA when α is zero. Secondly, when there is no disadvantageous inequality, RIA is behaviorally equivalent to RG. Therefore, the entire analysis in the last section will still hold under RIA. Thirdly, note that the RDU is applied separately to guilt or envy. In other words, the RIA DM groups the possible events based on whether they are guilt-prone or envy-prone before the RDU applies. In this aspect, RIA resembles the cumulative prospect theory in Tversky and Kahneman [76], where the RDU is applied to loss and gain separately. Lastly, to allow for envy moderation, one can simply assume $v_e(E) \leq v(E)$ for all $E \subset \Omega$.

3.3 Axiomatization

In this section, we discuss the axioms that lead to the RIA and RG representation in detail. As we shall see, the two models share very similar characterizations in terms of axiomatization since RG is a special case of RIA. We define a *state-wise* mixture of acts with the "+" notation , i.e. $\forall\omega, (\alpha f + (1-\alpha)g)(\omega) := \alpha f(\omega) + (1-\alpha)g(\omega) \in R^2$. To avoid excessive notation, we sometimes use $f(\omega)$ to denote the constant act which gives the payoff $f(\omega) \in R^2$ for every state. Also, we write fE^* as a shorthand for $fE^*(0,0)$. The first axiom is standard as in Saito [71].

Axiom 1. (rationality) $\succeq$ is complete, transitive, continuous, and monotonic in constant equal allocation.[19]

The next axiom is specific towards the functional form of Fehr and Schmidt [69]'s inequality aversion model under the RIA model. This is a generalized version of the inequality aversion axiom in Saito [71] in the sense that the preference for inequality aversion is consistent across states where $(0,0)$ is always preferred to either $(0,-1)$ or $(0,1)$.[20]

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[19] The continuity here is: For every $f, g, h \in \mathcal{F}$, if $f \succ g$ and $g \succ h$, then there are $\alpha, \beta \in (0,1)$ such that $\alpha f + (1-\alpha)h \succ g$ and $g \succ \beta f + (1-\beta)h$. The monotonicity in constant equal allocation is simply: $x, y \in R$, $(x,x) \succeq (y,y) \iff x \geq y$.

[20] Interestingly, monotonicity in constant equal allocation and inequality aversion serve the role of the monotonicity axiom in Schmeidler [74] in this domain.

Axiom 2a. (inequality aversion) For $E \subseteq F \subseteq \Omega$, $(0,-1)E^* \succeq (0,-1)F^*$ and $(0,1)E^* \succeq (0,1)F^*$

We mentioned earlier RG is a special case of RIA when the DM does not care about (or indifferent to) disadvantageous inequality. As we shall see later, it turns out that this is the only behavioral property that they differ. The following axiom assumes that such indifference is consistent across states. One can easily see that Axiom 2b is a stronger axiom than Axiom 2a.

Axiom 2b. (only advantageous inequality aversion) For $E \subseteq F \subseteq \Omega$, $(0,-1)E^* \succeq (0,-1)F^*$ and $(0,1)E^* \sim (0,1)F^*$

In the following, to capture the behavioral content of rank-dependent utility, we use the notion of comonotonicity from Schmeidler [74]. Based on this intuition, we develop two related notions: within-comonotonicity and between-comonotonicity.[21]

Definition 16. (comonotonicity) $A \subset \mathcal{F}$ is said to be comonotonic if there are no states ω, ω' and $f, g \in A$ such that $f(\omega) \succ f(\omega')$ and $g(\omega) \prec g(\omega')$.

Definition 17. (within-comonotonicity) $A \subset \mathcal{F}$ is said to be within-comonotonic if there is no state ω and $f, g \in A$ such that $f_1(\omega) > f_2(\omega)$ and $g_1(\omega) < g_2(\omega)$.

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[21] One can consider more players in the model, then within-comonotonicity coincidences with an extended version of the quasi-comonotonicity introduced in Saito (2013) in which the states are involved.

Definition 18. (between-comonotonicity) $A \subset \mathcal{F}$ is said to be between-comonotonic if there are no states ω, ω' and $f, g \in A$ such that $f_1(\omega) - f_2(\omega) > f_1(\omega') - f_2(\omega')$ and $g_1(\omega) - g_2(\omega) < g_1(\omega') - g_2(\omega')$.

Each of these notions imposes different state-dependent considerations of acts: Within-comonotonicity restricts acts to not have opposing emotion (i.e. envy and guilt) in the same state; Between-comonotonicity restricts acts to give the same order of *valence* of guilt and envy across states.[22] Lastly, as in Schmeidler [74], comonotonicity restricts acts to give the same ranking of utility across states. Therefore, we introduce the weak comonotonic independence axiom. This axiom is weaker than comonotonic independence in Schmeidler [74] in this domain since within-comonotonicity and between-comonotonicity are also required.

Axiom 3. (weak comonotonic independence) If $\{f, h, g\}$ is within-comonotonic, between-comonotonic and comonotonic, then for every $\delta \in (0, 1]$,

$$f \succeq g \iff \delta f + (1-\delta)h \succeq \delta g + (1-\delta)h$$

To see the importance of the requirement of *within-comonotonicity* in the axiom, we assume the RIA model is correct and try to test the axiom in the absence of *within-comonotonicity*. Consider an example in Table 3.1, where a coin flip de-

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[22] *Valence* is a psychological term for the *magnitude* of emotion.

termines the state. To fix idea, we consider the usual assumption that $\alpha \geq \beta$. To check the premises, note that $\{f, g, h\}$ is comonotonic in that the state *Head* gives a weakly higher utility to player one,[23] $\{f, g, h\}$ is between-comonotonic in that the state *Head* gives a weakly higher difference in materialistic payoff, and $\{f, g\}$ and $\{f, h\}$ are within-comonotonic in that both states give weakly the same emotion for the two pairs of acts. So, the only thing that is missing is that $\{g, h\}$ is not within-comonotonic. Note that due to inequality aversion, $f \succeq g$. However, after mixing with h, due exactly to the missing pair of within-comonotonicity in $\{g, h\}$, we cannot apply weak comonotonic independence to get $\frac{1}{2}f + \frac{1}{2}h \succeq \frac{1}{2}g + \frac{1}{2}h$. In fact, in this example, the preference relation is exactly the other way round, i.e. $\frac{1}{2}f + \frac{1}{2}h \preceq \frac{1}{2}g + \frac{1}{2}h$, since the latter mixture gives a *fair* allocation in the state *Head*. Hence, within-comonontonicity is crucial to the DM as some of those mixtures can cancel out unfair allocation.

Act \ State	*Head*	*Tail*
f	(0,0)	(0,1)
g	(0,1)	(0,1)
h	(0,-1)	(0,1)
$\frac{1}{2}f + \frac{1}{2}h$	(0,-.5)	(0,1)
$\frac{1}{2}g + \frac{1}{2}h$	(0,0)	(0,1)

Table 3.1: An example where unfairness in g and h are cancelled out due to state-wise mixture

The next definition, anti-comonotonicity, prepares us for Axiom 4, which basi-

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[23] By assuming the model is correct and $\alpha > \beta$, we have $W(0, -1) = -\beta \geq -\alpha = W(0, 1)$.

cally requires a pair of acts to be in different domains (one in guilt and the other in envy).

Definition 19. (anti-comonotonicity) $A \subset \mathcal{F}$ is said to be anti-comonotonic if there are no states ω, ω' and $f, g \in A$ such that $(f_1(\omega) - f_2(\omega))(g_1(\omega') - g_2(\omega')) > 0$.

In Axiom 4 below, both anti-comonotonicity and within-comonotonicity are required. These two together provide the further constraint that whenever one act gives an unequal allocation in one state, the other act must give equal allocation. Therefore, Axiom 4 allows the DM to break each act into two groups - the guilt outcomes and the envy outcomes - and perform reallocation of weight on each group. In this sense, this is a descriptive behavioral assumption regarding *categorization* of similar events. Surprisingly, this axiom imposes a restriction on the third dimension of utility other than guilt or envy, i.e. selfish concerns, that the DM has to use an additive measure for the materialistic payoff. One interpretation is that the materialistic payoff serves as a bridge between guilt and envy, such that it must not be distorted in the first place as the DM reallocates the weight. The main result is stated in Theorem 9.

Axiom 4. (envy-guilt independence) For every $f, h, g \in \mathcal{F}$ where $\{f, h\}$ and $\{g, h\}$

are anti-comonotonic and within-comonotonic,

$$f \succeq g \iff \frac{1}{2}f + \frac{1}{2}h \succeq \frac{1}{2}g + \frac{1}{2}h$$

Theorem 9 (characterization of RIA)**.** A binary relation $\succeq$ satisfies Axiom 1, 2a, 3 and 4 if and only if there exists $\alpha, \beta \geq 0$, non-additive measures v_g, v_e and additive measure v such that $\succeq$ is represented by

$$W(f) := \int_\Omega f_1(\omega) dv - \alpha \int_\Omega \max\{f_2(\omega) - f_1(\omega), 0\} dv_e - \beta \int_\Omega \max\{f_1(\omega) - f_2(\omega), 0\} dv_g$$

In addition, if $(\alpha, \beta, \mathbf{v})$ and $(\alpha', \beta', \mathbf{v}')$ represent $\succeq$, then $(\alpha, \beta, \mathbf{v}) = (\alpha', \beta', \mathbf{v}')$.

The proof for sufficiency is as follows: We first construct the Fehr and Schmidt utility functional form for constant act. Then, we demonstrate that the DM can decompose acts across two types of acts with certain comonotonic properties. In addition, the probability measure is given by the decomposition while the non-additive measure is given by the inequality aversion. Lastly, the decomposition of each act guarantees that the non-additive measures satisfy the rank-dependent property. The necessity part of the theorem almost immediately follows from the representation. Note that in this representation, the additive measure is endogenous to the model so that the additive measure defined does not need to follow an objective probability.

If one wants the additive measure to conform to an observable objective probability $s : \Sigma \to [0,1]$, we can assume certain behavioral regularities.[24]

Lastly, as one can see immediately from the representation that if we assume $(0,0) \sim (0,1)$, we can shut down the envy channel. i.e. $\alpha = 0$. Interestingly, Axiom 2a and $(0,0) \sim (0,1)$ are equivalent to Axiom 2b, due to the monotonicity with respect to set inclusion in events. Therefore, we write the following corollary.[25]

Corollary 1 (characterization of RG)**.** A binary relation $\succeq$ satisfies Axiom 1, 2b, 3 and 4 if and only if there exists $\beta \geq 0$, non-additive measures v_g and additive measure v such that $\succeq$ is represented by

$$W(f) := \int_\Omega f_1(\omega) dv - \beta \int_\Omega \max\{f_1(\omega) - f_2(\omega), 0\} dv_g$$

In addition, if (β, v, v_g) and (β', v', v_g') represent $\succeq$, then $(\beta, v, v_g) = (\beta', v', v_g')$.

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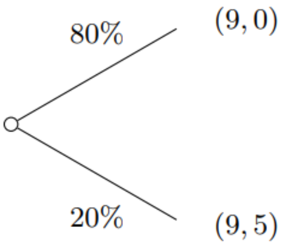
[24] A traditional axiom can be stated as: For every E, $s(E) \geq s(F)$ implies $(1,1)E^* \succeq (1,1)F^*$. This implies that there is a increasing function π such that $v(E) = \pi(s(E))$. Since both v and s are probabilities, π must be the identity function. I thank an anonymous referee for this suggestion.

[25] One may wonder the role of Axiom 4 in the RG. Note that although events of disadvantageous inequality are indifferent to the DM as long as they pay the same the DM, the state-wise mixture still needs to be taken carefully since they can cancel out advantageous inequality. In this sense, the DM is still grouping events based on whether it is guilt-prone or not guilt-prone.

3.4 Risk Preference for Others

In this section, we investigate the risk preference of the RIA DM by assuming that the objective risk is observable and the DM behaves as if he agrees with it. One motivation of this is that it provides an "out-of-sample" test of guilt moderation, as we derive another characterization for it in this context. There are two sets of evidence that we will be focusing on.

The first one is the intriguing phenomena found in Brock et al. [3]. It reports that some DM would buy a lottery for the recipient, but they would not directly give the amount to the recipient. The experimental result is listed in Figure 3.3, where for the same \$1, the DM would rather use it to buy a lottery for the recipient than keep it himself, and would rather keep it himself than give it out. Notice that the documented behavior is neither positive reversal nor negative reversal behavior as defined earlier.



$$\succ (10,0) \succeq (9,1)$$

Figure 3.3: Experiments in Brock et al. [3]

Secondly, there is a strand of emerging literature in economics and psychology that investigates self-vs-others differences in terms of risk attitudes. The meta-analysis by Polman and Wu [77] found that on average there is a significant (though small) difference in favor of a risky shift when people choose for others. In here, we examine the implication of guilt and envy moderation on the risk preference for others.

To formalize, we first assume that for every $E \in \Sigma$, $v_g(E) = \pi_g(s(E))$ and $v_e(E) = \pi_e(s(E))$, where $\pi_g, \pi_e : [0,1] \to [0,1]$.[26] Thus, the DM transforms the objective probability with the function π_g, π_e, i.e., the rank-dependent probability weighting function. Hence, each RIA DM is specified by $(\alpha, \beta, \mathbf{v}^*)$, where $\mathbf{v}^* = (\pi_g, \pi_e)$.[27] In this specification, the behavior in Figure 3.3 can be accommodated in the RIA model: a simple calculation yields the following conditions,

$$\frac{1}{2} \geq \beta > \frac{1}{6 - 5\pi_g(0.8)}$$

which implies $\pi_g(0.8) \leq 0.8$. Note that this example is again a confirmation that the DM is guilt moderating. From the outset, by assuming transitivity, the DM

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[26] This assumption can be achieved by a consistent condition similar to the axiom in footnote 24. For example, if $s(E) \geq s(F)$, then $(0,1)F^* \succeq (0,1)E^*$ for envy and $s(E) \geq s(F)$, then $(0,-1)F^* \succeq (0,-1)E^*$ for guilt. I thank an anonymous referee for this suggestion.

[27] Since the primitive is slightly different, we rewrite the definition of guilt moderating in the lottery domain, i.e. An $(\alpha, \beta, \mathbf{v}^*)$ RIA DM is said to be guilt moderating if $\pi_g(p) \leq p$ for $p \in [0,1]$.

prefers the lottery of $(9,0)$ and $(9,5)$ to $(9,1)$, which says that the DM prefers the *mean-preserving* spread for others. As we shall also see, guilt moderation predicts the DM always prefers the mean-preserving for others, which provides a direct test of guilt moderation.

In light of this, we conceptualize the notion of risk preference *for others* while holding the own payoff constant.[28] Let $l = (1, (a, b))$ be a degenerated lottery which gives (a, b) for sure and $l' = (p_1, (a, k_1); p_2, (a, k_2); ...; p_n, (a, k_n))$ be a lottery which gives (a, k_i) with probability p_i, where $\sum_{i=1}^{n} k_i p_i = b$ and $\sum_{i=1}^{n} p_i = 1$. Hence, l' is a mean-preserving spread of l in terms of the other's payoff. We define two notions of risk aversion for others in which the second is weaker than the first. In the following definition, we subsume the universal quantifiers for ease of notation.[29]

Definition 20. (risk aversion for others) A DM is said to be risk averse for others at wealth level a if $l \succeq l'$; A DM is said to be risk averse for others if the DM is risk averse for others at every wealth level.

Definition 21. (risk aversion for others in the guilt/envy domain) A DM is said to be risk averse (resp. risk loving) for others in the guilt domain at wealth level a if

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[28] In this chapter, we consider a notion of risk preference by comparison a lottery to the degenerate expected outcome. In the rank-dependent utility literature, it is studied as weak risk version (e.g. Chateauneuf et al. [78]). Therefore, the results presented here speak only to this particular notion of risk preference. For other notion of risk preference, contrary to what happens in the expected utility model, they need not coincide in this environment.

[29] That is "for every $n, b, \{k_i\}_{i=1}^{n}$ and $\{p_i\}_{i=1}^{n}$".

$a - k_i \geq 0$ and $l \succeq l'$ (resp. $l \preceq l'$); A DM is said to be risk averse (resp. risk loving) for others in the guilt domain if the DM is risk averse (resp. risk loving) for others in the guilt domain at every wealth level.[30]

The requirement of $a - k_i \geq 0$ gives a restriction on the lottery of having only disutility from guilt and hence the name *in the guilt domain*. Note that the lottery l' being in one domain also restricts the degenerate lottery l to be in the same domain since $\sum_{i=1}^{n} p_i k_i = b$. We state the key result regarding risk aversion for others in the guilt/envy domain in Proposition 13, which is then followed by a characterization result in Corollary 1.[31]

Proposition 13. An $(\alpha, \beta, \mathbf{v}^*)$ RIA DM is risk averse (resp. risk loving) for others in the guilt domain if and only if $\pi_g(p) \geq p$ (resp. $\pi_g(p) \leq p$) for all $p \in (0,1)$. An $(\alpha, \beta, \mathbf{v}^*)$ RIA DM is for risk averse (resp. risk loving) for others in the envy domain if and only if $\pi_e(p) \geq p$ (resp. $\pi_e(p) \leq p$) for all $p \in (0,1)$.

Corollary 2. (moderation and risk loving for others) For an $(\alpha, \beta, \mathbf{v}^*)$ RIA DM, the following statements are equivalent:

(i) The DM is risk loving for others in the guilt (resp. envy) domain.

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[30] The definition of risk aversion for others in the envy domain is analogous.

[31] The result is closely related to the result in Yaari [79]. However, it differs in two ways. The first obvious one is that Yaari [79] considers choice for oneself, but not for the other person, such that it would not have the natural separation of two terms (guilt and envy). The second and more subtle one is that for simplicity and application purposes we consider a finite support lottery rather than a continuum of state-space. It leads to a less restrictive requirement on the function $\pi_g(.)$.

(ii) The DM is guilt (resp. envy) moderating.

There are two ways to view this result. Firstly, moderation creates a self-other risk attitude gap where the DM maker is risk-neutral for himself but risk loving for others; Secondly, it provides a direct test of guilt and envy moderation in the RIA model using a simple lottery comparison. However, we should note that this simple model would not survive the test well: The first thing being that risk neutrality on the own payoff can easily be refuted by studies in risk aversion; and secondly, studies on decisions for others often found risk aversion for others as well. Therefore, we consider a *non-linear* extension in Section 3.5.1 to introduce risk aversion for oneself and others. As we shall see, guilt moderation and envy moderation are still the key in creating the wedge in self-other risk attitudes.

The last result in this section is that the notions of risk aversion for others in both the envy and guilt domain are sufficient for the general notion of risk aversion for others to hold. However, note that a similar result cannot be obtained for risk loving preferences for others. The reason why it holds for risk aversion for others but not risk loving for others is because inequality-averse DM is naturally averse to risk in other's wealth.

Proposition 14. For an $(\alpha, \beta, \mathbf{v}^*)$ RIA DM, the following statements are equivalent:

(i) The DM is risk averse for others.

(ii) The DM is risk averse for others in both the envy and guilt domain.

3.5 Applications

In this section, we study the applications of the model. Special cases and extensions of the models are also considered. In Section 3.5.1, we assume a non-linear extension of the RIA model and utilize a special form of probability weighting function for tractability of the results. Also, in Section 3.5.2, we look into an empirical puzzle of wage transparency by applying the notion of envy moderation.

3.5.1 Self-other Risk Aversion Gap

In the literature, there are often mixed findings regarding risk attitude on decision for others. There are ongoing discussions about when and why people exhibit risky or cautious shift.[32] In this section, we provide a mechanism for risky shift through guilt moderation. To fix ideas, we focus on one simple setup: the investment game from Gneezy and Potters [80]. This game has been applied to different experiments to study investment decisions for others. In particular, Pollmann et al. [81] found that subjects would invest more in risky assets for others than for them-

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[32] For an overview and mate-analysis, one can refer to Polman and Wu [77].

selves.[33] In the investment game, a DM needs to allocate x out of W dollars into a risky asset. The return of the risky asset is $+250\%$ with probability $\frac{2}{3}$ and -100% with probability $\frac{1}{3}$. The return of the safe asset is always zero. Note that a risk-neutral DM would invest all W dollars in the risky asset. In the following, echoing the analysis of guilt and envy domain in Section 3.4, we consider cases where one always has a higher or lower wealth than others.

We use CARA utility for the selfish payoff and the guilt and envy terms. In particular, we assume $u(x) = \frac{\left(1-e^{-\lambda_{me}x}\right)}{\lambda_{me}}$, where $\lambda_{me} > 0$ for own risk aversion. For the guilt and envy functions, we assume $m_i(x) = \frac{e^{\lambda_i x}}{\lambda_i}$ for $i = e, g$, where $\lambda_i > 0$ for risk aversion for others. In this sense, the social component utility becomes $-\alpha \int_\Omega m_e(\max\{f_2(\omega) - f_1(\omega), 0\})dv_e - \beta \int_\Omega m_g(\max\{f_1(\omega) - f_2(\omega), 0\})dv_g$. Also, for tractability, we assume $\pi_g(p) = p^{\frac{1}{k_g}}$ and $\pi_e(p) = p^{\frac{1}{k_e}}$, where $k_g \leq 1$ immediately captures guilt moderation. Hence, each *extended* RIA DM can be summarized by $(\alpha, \beta, k, \lambda)$, where $k = (k_e, k_g)$ and $\lambda = (\lambda_{me}, \lambda_e, \lambda_g)$. Due to the convexity in disutility, one immediate implication is that the RIA DM would invest a positive amount into the safe asset for others both in the guilt or envy domain for sufficiently low λ_i.

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[33] There are two studies utilizing investment game and they found cautious shift behaviors: Eriksen and Kvaløy [82] employed investment games and focuses on the idea of making decision for *clients*. On the other hand, Füllbrunn and Luhan [83] considered a version of it where the subject is making investment decision simultaneously for six other subjects. Note that the setting or instructions of these studies could enhance responsibility or accountability effect which our model does not intend to capture. For a detailed discussion for when to expect risky or cautious shift, one can refer to Polman and Wu [77].

Then, we state the key result.

Proposition 15. (investment decision for the richer and poorer) Assume $\lambda_{me} = \lambda_e = \lambda_g$. An extended $(\alpha, \beta, k, \lambda)$ RIA DM invests more in risky asset for others

(i) in the guilt domain if and only if $k_g \leq 1$;

(ii) in the envy domain if and only if $k_e \leq \frac{\ln(1/3)}{\ln(8/33)} (\leq 1)$.

There are three things to note about this result. Firstly, recall that $k_g \leq 1$ and $k_e \leq 1$ represent guilt and envy moderation. Hence, we can conclude that guilt moderation and (sufficient) envy moderation are still the key in creating a wedge in risk attitudes between oneself and others for the investment decision. Secondly, perhaps surprisingly, the symmetry between guilt and envy seems to be broken here since a stronger condition is required for k_e. However, we should keep in mind that the risky asset is a *better-than-fair* gamble. Note that for a DM in the envy domain, the motivation of inequality aversion actually drives him to *prevent* advantageous gambles for others (which is exactly the opposite for the guilt domain). Hence, the DM must be sufficiently *envy moderating* to invest into the risky gamble for others. Lastly, from an empirical standpoint, λ_i, $i = g, e$ allows for more variety in risk attitudes for others. One does not need to stick with the assumption that $\lambda_{me} = \lambda_e = \lambda_g$. The proposition here is to reinforce our understanding of how guilt and envy moderation can create the self-other risk attitudes gap.

3.5.2 Envy and Wage Transparency

Mas [84] studied a policy adopted in California that requires chief executive officers ("city managers") to disclose their salaries to the public. The finding is that there is significant drop in wages that cannot be explained by efficiency wage or accountability mechanisms. To explain the data, the author argues that *"...segments of the public are inequality averse and demand lower wages at the top of the distribution"*. We would like to expand on this idea and see how it can be explained through our framework.

We assume that DM has *correct* belief on average, meaning that his subjective probability distributions over a city managers' wage has a correct mean. Note that the lower income segments of society would always be in the envy domain compared to those in the top. In the model, for an envy moderating RIA DM, the subjective expected envy level is always lower than the expected envy level. Also, by our analysis of risk-preference for others in Section 3.4, the DM would find a more dispersed belief more desirable, meaning exactly that the DM would feel *less* envy had he or she not known the true state. This gives a plausible reason why wage transparency brings an increased level of envy from the lower income segments of the society.

3.6 Related Literature

In the riskless domain, Fehr and Schmidt [69]'s model of inequality aversion was first axiomatized by Rohde [85]. Later, Fudenberg and Levine [2] and Saito [71] brought the model into the risk domain, and use it to motivate a notion of equality of opportunity where the DM has inequality aversion in expected allocation. To write Saito [71]'s model under the decomposition in Section 3.2, one can set $u(x) = x$ and the social utility component as

$$\begin{aligned} SC_{EIA}(f) := & -(1-\gamma)[E(\beta \max\{f_1 - f_2, 0\} + \alpha \max\{f_2 - f_1, 0\})] \\ & - \gamma[\beta \max\{E(f_1) - E(f_2), 0\} + \alpha \max\{E(f_2) - E(f_1), 0\}] \end{aligned}$$

Here, the term following $(1-\gamma)$ captures ex-post fairness concern while the term following γ capture the ex-ante fairness concern. As shown in this chapter, in order for ex-post fairness concern to deliver the reversal behaviors, the DM must exhibit guilt moderation. Hence, the only remaining ingredient that might capture the reversals is the ex-ante fairness concern. However, as discuss in the introduction, intuitively, one should not expect a model motivated with ex-ante fairness to produce negative reversal behavior. This intuition is true and, interestingly, EIA cannot accommodate positive reversal as well. The reason why the EIA model cannot explain both reversal behaviors is that both positive and negative reversal are only in the guilt domain. In that regard, the terms capturing ex-ante and ex post fairness concern will collapse into one term such that the expected utility of guilt would equal the guilt of the expected allocation. Since the DM does not exhibit guilt moderation, it cannot capture the reversal behaviors.

One other model for social preference models in the risk domain is from Borah [73]. The chapter introduces the HR model focusing on treatments on the marginal distribution of the opponent. The model consider two Bernoulli utilities for player 1 and player 2, namely $u(x)$ and $v(x)$. The social utility component can be stated as $SC_{HR}(f) := E(\max\{v(f_2), E(v(f_2))\})$. In essence, the model assumes good outcome shades bad outcome for others, such that bad outcome which gives player 2 less-than-average payoff will be evaluated exactly at the original expected utility. Interestingly, while the model is capable to produce instances of reversals behaviors, the model satisfies positive shift but not negative shift, due to its focuses on marginal distribution on the opponent.[34]

Some social preference models focus on menu selection problem. Noor and Ren [86]'s theory of guilt and temptation focuses on menu choice such that the DM can avoid the normative *tempting* option by avoiding a menu altogether. We should note the model makes no specification for the second period utility. In this regard, our

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[34] One can see this if we choose $x_3 = x_0$ from the definition of negative shift.

model focuses on an entirely different domain where the object of choice is still the act/lottery itself. On the other hand, Dillenberger and Sadowski [87] and Saito [88] consider menu choice in an environment which distinguishes the ex-ante private stage and the ex-post public stage to study other social emotions such as *shame* and *pride*. In contrast, our model does not distinguish private or public stage and focuses on the alleviation of *guilt* and *envy* in risky choice.

Lastly, our theory uses insights from the rank-dependence (e.g. Quiggin [89], Schmeidler [90], Schmeidler [74] and Yaari [79]) and sign-dependence (e.g. Wakker and Tversky [91], Tversky and Kahneman [76]) literatures. Our model shares some similarities to the cumulative prospect theory in Wakker and Tversky [91]. One may even find the notion of *co-signed* prospect to be similar to the notion of *within-comonotonicity* in our model. Nonetheless, as noted by Schmeidler [74], one important technical difference is that *"(an earlier work of Wakker) substituted a connected topological space for the linear structure"*. In our model, the linear structure (the payoff space R^2) is innate to the problem so abstraction is not necessary.[35]

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[35] Note that R^2 is also connected so that one might also try to utilize the technique from Wakker and Tversky [91] to axiomatize the model. More broadly, a more abstract connected space might help us potentially to capture allocations outside of materialistic payoff.

3.7 Conclusion

Inequality aversion is central to social decision making. Meanwhile, rank-dependent utility has proven to be a powerful tool in explaining human behavior. This chapter serves to provide a new model with the intersection of these two traditional insights from behavioral economics. In particular, the model tells us that the idea of *guilt moderation* can tackle the reversal behaviors from the moral wiggle room experiment and the idea of “ex-ante fairness for you”. Moreover, two systemic violations of the independence axiom, i.e. *negative shift* and *positive shift*, are provided for characterization of guilt moderation. In addition, the model sheds light on risky shift in the self-other risk attitudes gap and increased envy in wage transparency. Lastly, we should reiterate that the main ingredient of the social preference considered here is the concern for inequality aversion: i.e. the DM only cares about how much he has relative to the other person and does not inherently take other’s allocation into account (i.e. no *unconditional* altruistic preference). Yet, one might want to extend the model by including an extra additive altruistic Bernoulli utility, which can be *activated* when inequality aversion is not the prominent issue in the mind of the DM. This extension may allow us to capture the scenario that the DM is genuinely concerned about the risk preferences of others rather than having risk aversion driven by inequality aversion. Either way, more exploration is needed for researchers to figure out which motive is more salient under what circumstances. Further research is greatly encouraged in this direction.

Appendix A: Proofs for Chapter 1

In this Appendix, we will provide proofs for chapter 1. For ease of referencing, we would spell out each behavioral property in axiom number such that one can have an easier time referring to the specific behavioral element.

Axiom 5 (IIRA(1))**.** If $c(R) \in R' \subset R$ then $c(R) = c(R')$.

Axiom 6 (IIRA(2))**.** If $c(R) \notin R$ then $c(R') \notin R'$ for all $R' \subset R$.

Axiom 7 (Sandwich Property)**.** If $R' \subseteq R \subseteq R''$ and $c(R'') = c(R')$, then $c(R) = c(R')$.

Axiom 8 (Recommended Luce-IIA)**.** For $x, y \in R \cap R'$, $\frac{\rho(x,R)}{\rho(y,R)} = \frac{\rho(x,R')}{\rho(y,R')}$

Axiom 9 (R-Path Independence)**.** For $x \notin R$ and $R \cup x \subseteq R'$, $\rho(x,R)\rho(R \cup x, R')$ is independent of R

Axiom 10 (R-Independence)**.** For $x \notin R$, $\rho(x,R)(1+\sum_{z\in R} r(z,x))$ is independent of R

Axiom 11 (General Luce-IIA)**.** For $x, t \in R \cap R' \cap R''$, $y, z \notin R \cup R' \cup R''$, $\frac{\rho(x,R)}{\rho(x,R')}\frac{\rho(t,R')}{\rho(t,R'')} = \frac{\rho(y,R)}{\rho(y,R')}\frac{\rho(z,R')}{\rho(z,R'')}$

Axiom 12 (R-Regularity)**.** For $x \notin R$, $\rho(x, R) \leq \rho(x, R \setminus y)$.

Axiom 13 (Non-negativity of BM(1))**.** For $a \in R$, $q(a, R) \geq 0$.

Axiom 14 (Non-negativity of BM(2))**.** For $a \notin R$, $y(a, R) \geq 0$.

Axiom 15 (Positive Marginal Recommendation)**.** For $a \in R$, $q(a, R) \geq y(a, R \setminus a)$.

Proof of Theorem 1(a)

Proof. We use σ as an arbitrary element of $X \cup X^*$. For every set R, we define R_x as x^* if x belongs to R, otherwise R_x is simply x. We now define a binary relation on $X \cup X^*$ for any two distinct alternatives x and y:

$$(R_x, R_y) \in P \text{ if } x = c(R).$$

Claim 1. P is asymmetric.

Proof. For two distinct σ_1 and σ_2, if $(\sigma_1, \sigma_2) \in P$ then $(\sigma_2, \sigma_1) \notin P$. Assume not, there exists R_1 and R_2 such that $x = c(R_1)$ and $y = c(R_2)$. There are four cases depending on whether x or y is recommended. If x is recommended, then $\sigma_1 = x^*$, x otherwise. Similarly, σ_2 is y if it is not recommended, y^* otherwise.

First we assume that $x, y \in R_1 \cap R_2$, then by Axiom 5, we must have $x = c(\{x, y\}) = y$, a contradiction. Now assume $x \in R_1 \cap R_2$ and $y \notin R_1 \cup R_2$, then Axiom 5 implies $x = c(\{x, y\})$ whereas Axiom 6 implies $y = c(\{x, y\})$, a contradiction. The third case where $x \notin R_1 \cap R_2$ and $y \in R_1 \cup R_2$ is identical to the the second case. Finally, assume $x, y \notin R_1 \cap R_2$. Then applying Axiom 6 twice yields $x = c(\{x, y\}) = y$, a contradiction. Therefore, P is asymmetric. $\square$

Let a be $c(\emptyset)$. Given definition P, we reveal that a is preferred to any alternative in $X \setminus a$. Moreover, we cannot reveal the relative ranking of these alternatives. Formally, for $x, y \neq a$, aPx and aPy then $(x, y) \notin P$ or $(y, x) \notin P$. Notice that a might be revealed to be better than alternatives in X^*. Especially, if $a = c(R)$, then aPy^* for all $y \in R$. The same intuition applies for these alternatives too. Hence, P is incomplete for the lower contour set of a.

On the other hand, P is complete in the upper contour set of a. To show this, assume x^*Pa and y^*Pa for $x, y \neq a$. In other words, there exist $R \ni x$ and $R' \ni y$ such that $x = c(R)$ and $y = c(R')$ where $a \notin R \cup R'$. If $x \in R'$ or $y \in R$, we would reveal y^*Px^* or x^*Py^*, respectively. Assume not. Then consider $\{x, y\}$ as the recommended set. First, $c(\{x, y\}) \in \{x, y\}$. If not, $a = c(\{x\})$ by Axiom 6. Then by Axiom 5, x cannot be chosen from $c(R)$. Hence, either y^*Px^* or x^*Py^*, P is complete for the upper contour set of a.

Claim 2. If x^*Py^* and $x^* \neq a$ then x^*Pa.

Proof. x^*Py^* implies that there exists R such that $x, y \in R$ and $x = c(R)$. Since $\{x\} \subset R$, Axiom 5 implies $c(\{x\}) = x$ implying x^*Pa. $\square$

Note that our definition of P does not relate a and a^* since it requires two distinct alternatives. However, it is possible that we can reveal $a^* P a$. To see this, assume $a = c(R)$ where $a, x \in R$ and $x = c(R')$ where $a \notin R'$. The former implies $a^* P x^*$ and the former yields $x^* P a$. Hence we infer that a is ranked strictly higher when it is recommended.

Claim 3. P is transitive in the strict upper contour set set of a.

Proof. Assume that $x^* P y^* P z^*$ for three distinct x, y, and z. There exists R and R' such that (i) $x = c(R)$ and $y = c(R')$, and ii) $\{x, y\} \subset R$ and $\{y, z\} \subset R'$. Axiom 5 implies that $c(\{x, y\}) = x$ and $c(\{y, z\}) = y$. Then we consider $c(\{x, y, z\})$. We must have $c(\{x, y, z\}) \in \{x, y, z\}$ by Axiom 6. It cannot be z since Axiom 5 implies $z = c(\{y, z\})$. It cannot be y since Axiom 5 implies $y = c(\{x, y\})$ which is a contradiction. Hence, we have $x = c(\{x, y, z\})$, which gives $x^* P z^*$. □

Take any completion of P, say $\succ$. It is routine to show that the representation holds. □

Proof of Theorem 1(b)

Proof. We first identify the default option a. We set $a := c(\emptyset)$. If $c(R) \notin R$ then by Axiom 6, we have $c(R) = a$. Hence, a is unique. For every distinct $x, y \in R \cup a$, we write

$$xPy \text{ if } x = c(R)$$

Claim 4. P is asymmetric

Proof. For two distinct x and y, if $(x, y) \in P$ then $(y, x) \notin P$. Assume not. Then there exists R and R' such that $x = c(R)$ and $y = c(R')$ and $\{x, y\} \subset (R \cap R') \cup a$. If $x, y \neq a$, then by Axiom 5, we must have $x = c(\{x, y\}) = y$. On the other hand, if $x = a$, then $y \neq a$. By Axiom 5, we have $c(\{y\}) = y$. However, by Axiom 7, since $\emptyset \subseteq \{y\} \subseteq R$, we must have $x = c(\{y\})$. A contradiction must arise regardless. Therefore, P is asymmetric. $\square$

Note that if the default option is preferred to two distinct alternatives, we cannot reveal the relative ranking of these alternatives. In other words, for $x, y \neq a$, aPx and aPy then $(x, y) \notin P$ or $(y, x) \notin P$. While P is incomplete for the lower counter set of a, P is complete in the upper counter set of a. To show this, assume xPa and yPa. In other words, there exist R and R' such that $x = c(R)$ and $y = c(R')$. If $x \in R'$ or $y \in R$, we would reveal yPx or xPy, respectively. Assume not. Then consider $\{x, y\}$ as the recommended set. First, $c(\{x, y\}) \in \{x, y\}$. If not, $a = c(\{x\})$ by Axiom 7. Then by Axiom 5, x cannot be chosen from $c(R)$. Hence, either xPy or yPx, P is complete for the upper counter set of a.

Claim 5. If xPy and $x \neq a$ then xPa.

Proof. xPy implies that there exists R such that $x, y \in R$ and $x = c(R)$. Since $\{x\} \subset R$, Axiom 5 implies $c(\{x\}) = x$ implying xPa. ☐

Claim 6. If $xPyPz$ then xPz.

Proof. First note that x cannot be a since P is silent for the lower counter set of a. If z is equal to a, by Claim 5, we have xPz. If $y = a$, then $c(\{x\})$ is equal to x and $c(\{z\})$ is equal to a. Hence $c(\{x, z\})$ must be x by Axiom 5, 7. To see this, suppose $c(\{x, z\}) \notin \{x, z\}$, then $a = c(\{x, z\})$ by Axiom 6. However, by Axiom 7, we must have $c(\{x\}) = a$. On the other hand, if $z = c(\{x, z\})$, by Axiom 5, we must have $c(\{z\}) = z$. Contradictions in either way. Finally, we assume that x, y, z are distinct from a. Then we consider $c(\{x, y, z\})$. We must have $c(\{x, y, z\}) \in \{x, y, z\}$ by Axiom 7. It cannot be z since Axiom 5 implies $z = c(\{y, z\})$ which contradicts Claim 4. Finally, it cannot be y since Axiom 5 implies $y = c(\{x, y\})$ which contradicts Claim 4. Hence, we have $x = c(\{x, y, z\}$, which gives xPz. ☐

Take any completion of P, say $\succ$. It is routine to show that $c = c_{(a,\succ)}$. ☐

Proof of Proposition 1

Proof. $(a_1, \succ_1)$ and $(a_2, \succ_2)$ represents the same choice rule. For i), note that it is immediate that $a_1 = c(\emptyset) = a_2$. For ii), suppose not, there exists b such that

$b \in L_{\succ_1}(a)$ but $b \notin L_{\succ_2}(a)$. Then, we know that $c_{a,\succ_1}(\{b\}) = a \neq b = c_{a,\succ_2}(\{b\})$. Contradiction arises. For iii), suppose not, there exists $x, y \in X \setminus L_a$ such that $x \succ_1 y$ but $y \succ_2 x$. Then, we have $c_{a,\succ_1}(\{x, y\}) = x \neq y = c_{a,\succ_2}(\{x, y\})$. Contradiction arises. $\square$

Proof of Proposition 2

Proof. ${\succ^\star}_1$ and ${\succ^\star}_2$ represents the same choice rule. For i), note that it is immediate that $\max(X, {\succ^\star}_1) = c(\emptyset) = \max(X, {\succ^\star}_2)$. We call it a. For ii), suppose not, there exists b such that $b \in L_{{\succ^\star}_1}(a)$ but $b \notin L_{{\succ^\star}_2}(a)$. Then, we know that $c_{{\succ^\star}_1}(\{b\}) = a \neq b = c_{{\succ^\star}_2}(\{b\})$. Contradiction arises. For iii), suppose not, there exists $x, y \in X \cup X^* \setminus L_a$ such that $x \succ_1 y$ but $y \succ_2 x$. Then, we have $c_{{\succ^\star}_1}(\{x, y\}) = x \neq y = c_{{\succ^\star}_2}(\{x, y\})$. Contradiction arises. $\square$

Proof of Theorem 2

Proof. We first prove the necessity of the axioms. Suppose the model is correct. We prove the necessity of Axiom 8.

Note that, for $x, y \in B$

$$\begin{aligned}\frac{\rho(x,B)}{\rho(y,B)} =& \frac{\left[\sum_{z\in B}\frac{d(z)}{u(B)}u(x)+\sum_{z\notin B}\frac{d(z)}{u(B\cup z)}u(x)\right]}{\left[\sum_{z\in B}\frac{d(z)}{u(B)}u(y)+\sum_{z\notin B}\frac{d(z)}{u(B\cup z)}u(y)\right]}\\ =&\frac{u(x)\left[\sum_{z\in B}\frac{d(z)}{u(B)}+\sum_{z\notin B}\frac{d(z)}{u(B\cup z)}\right]}{u(y)\left[\sum_{z\in B}\frac{d(z)}{u(B)}+\sum_{z\notin B}\frac{d(z)}{u(B\cup z)}\right]}=\frac{u(x)}{u(y)}\end{aligned}$$

Since B is arbitrary, it immediately implies Axiom 8. We then prove the necessity of Axiom 9. We make the following claim.

Claim 7. For every $x \in B$ and $x \notin A \subseteq B$, we have $\rho(x,A)\rho(A\cup x,B) = \rho(x,\emptyset)\rho(x,B)$.

Proof.

$$\begin{aligned}\rho(x,A)\rho(A\cup x,B)-\rho(x,\emptyset)\rho(x,B) =& d(x)\left[\frac{u(x)}{u(A\cup x)}[\rho(x,B)+\rho(A,B)]-\rho(x,B)\right]\\ =&\frac{d(x)}{u(A\cup x)}[u(x)\rho(A,B)-u(A)\rho(x,B)]\end{aligned}$$

Note that where

$$u(x)\rho(A,B) = u(x)\left[\sum_{y\in B}\frac{d(y)}{u(B)}u(A)+\sum_{y\notin B}\frac{d(y)}{w(B\cup y)}u(A)\right]$$

$$=u(A)\left[\sum_{y\in B}\frac{d(y)}{u(B)}u(x)+\sum_{y\notin B}\frac{d(y)}{w(B\cup y)}u(x)\right]=u(A)\rho(x,B)$$

Hence, since A is arbitrary, the above claim immediately implies Axiom . $\square$

For sufficiency, we define

$$d(x):=\rho(x,\emptyset)\geq 0 \text{ and } u(x):=\rho(x,X)>0$$

First, Axiom 9 implies that

$$\rho(x,A)\rho(A\cup x,X)=\rho(x,\emptyset)\rho(x,X)$$

Hence we have representation for off-recommendation, *i.e.* $x\notin A$:

$$\rho(x,A)=\rho(x,\emptyset)\frac{\rho(x,X)}{\rho(A\cup x,X)}=d(x)\frac{u(x)}{\sum\limits_{z\in A\cup x}u(z)}=d(x)W_x(x,A)$$

For on-recommendation alternative, we first make the following claim.

Claim 8. Axiom 9 implies that for $\emptyset\neq A\neq X$,

$$\rho(A,A)-\rho(A,\emptyset)=\rho(A,X)\sum_{y\notin A}\frac{\rho(y,A)}{\rho(y,X)}$$

Proof. To prove this, fix a A, we first consider $x \notin X \setminus A$. By Axiom , we have, for every $x \notin X \setminus A$,

$$\begin{aligned}
\rho(x, A)(\rho(x, X) + \rho(A, X)) &= \rho(x, \emptyset)\rho(x, X) \\
\rho(x, A) + \frac{\rho(x, A)}{\rho(x, X)}\rho(A, X) &= \rho(x, \emptyset)
\end{aligned}$$

Summing all $x \notin A$, we have

$$\begin{aligned}
\sum_{x \notin A}\left(\rho(x, A) + \frac{\rho(x, A)}{\rho(x, X)}\rho(A, X)\right) &= \sum_{x \notin A}\rho(x, \emptyset) \\
1 - \rho(A, A) + \rho(A, X)\sum_{x \notin A}\frac{\rho(x, A)}{\rho(x, X)} &= 1 - \rho(A, \emptyset) \\
\rho(A, A) - \rho(A, \emptyset) &= \rho(A, X)\sum_{y \notin A}\frac{\rho(y, A)}{\rho(y, X)}
\end{aligned}$$

□

By Axiom 8, if $x \in A$ then

$$\frac{\rho(y, A)}{\rho(x, A)} = \frac{\rho(y, X)}{\rho(x, X)}$$

By summing all $y \in A$, we have

$$\frac{\rho(A,A)}{\rho(x,A)} = \frac{\rho(A,X)}{\rho(x,X)}$$

then

$$\rho(x,A) = \frac{\rho(A,A)\rho(x,X)}{\rho(A,X)}$$

Hence, for $x \in A$, by Claim 8 and $\rho(x,A) = \frac{\rho(A,A)\rho(x,X)}{\rho(A,X)}$, we have

$$\begin{aligned}
\rho(x,A) &= \frac{\rho(x,X)}{\rho(A,X)}\Big[\rho(A,\emptyset) + \rho(A,X)\sum_{y\notin A}\frac{\rho(y,A)}{\rho(y,X)}\Big] \\
&= \frac{\rho(x,X)}{\rho(A,X)}\Big[\rho(A,\emptyset) + \rho(A,X)\sum_{y\notin A}\frac{\rho(y,\emptyset)}{\rho(A\cup y,X)}\Big] \\
&\qquad\qquad \text{Since } \rho(y,A) = \rho(y,\emptyset)\tfrac{\rho(y,X)}{\rho(A\cup y,X)} \text{ for } y \notin A \\
&= \rho(x,X)\Big[\frac{\rho(A,\emptyset)}{\rho(A,X)} + \sum_{y\notin A}\frac{\rho(y,\emptyset)}{\rho(A\cup y,X)}\Big] \\
&= \sum_{y\in A}\frac{d(y)u(x)}{\sum\limits_{z\in A}u(z)} + \sum_{y\notin A}\frac{d(y)u(x)}{\sum\limits_{z\in A\cup y}u(z)} \qquad \text{By construction of } d \text{ and } u \\
&= \sum_{a\in X} d(a)W_a(x,A)
\end{aligned}$$

The proof is complete. □

Proof of Theorem 3

Proof. The proof for necessity of Axiom 9 is proven in Theorem 2. We prove necessity of Axiom 10. Suppose the model is correct, let $x \notin A$,

$$\begin{aligned} &\frac{\rho(x,\emptyset)}{\sum_{z\in A\cup x} r(z,x)} \\ =&\frac{d(x)}{\sum_{z\in A\cup x}\frac{u(z)}{u(x)}} && \text{By the necessity proof of Axiom 8} \\ =&\frac{d(x)u(x)}{\sum_{z\in A\cup x} u(z)} = \rho(x,A) \end{aligned}$$

Since A is arbitrary, it is proven.

We then prove the sufficiency. We first let, for every $x \in X$, $d(x) := \rho(x,\emptyset)$. We arbitrarily designate $z_0 \in X$ as an "anchored" element. And let $u(z_0) = 1$. Since we have all the binary recommendation sets in our data, we let for every $x \in X$,

$$u(x) = \frac{\rho(x,\{x,z_0\})}{\rho(z_0,\{x,z_0\})}$$

Then, we prove the following claim.

Claim 9. For any $x, y \in A$ with $|A| \leq k-1$, we have $\frac{\rho(x,A)}{\rho(y,A)} = \frac{u(x)}{u(y)}$. And similarly, for any $x \in A$ and $B \subseteq A$ we have $\frac{\rho(x,A)}{\rho(B,A)} = \frac{u(x)}{u(B)}$

Proof. Note that, firstly, for any set $A \supseteq \{x, z_0\}$, we have $u(x) = \frac{\rho(x,A)}{\rho(z_0,A)}$ by Axiom 8. Then, for any $x, y \in A$, we have

$$\begin{aligned} \frac{\rho(x,A)}{\rho(y,A)} &= \frac{\rho(x, A \cup z_0)}{\rho(y, A \cup z_0)} && \text{By Axiom 8} \\ &= \frac{\rho(x, A \cup z_0)\rho(z_0, A \cup z_0)}{\rho(z_0, A \cup z_0)\rho(y, A \cup z_0)} = \frac{u(x)}{u(y)} \end{aligned}$$

Hence, the first part is proven. The second part is immediate. □

We first show that the representation holds for off-recommendation set. Let $x \notin A$, by Axiom 10, we have

$$\begin{aligned} \rho(x, A) = \frac{\rho(x, \emptyset)}{1 + \sum_{z \in A} r(z, x)} &= \frac{d(x)}{\sum_{z \in A \cup x} \frac{u(z)}{u(x)}} && \text{By Claim 9} \\ &= \frac{d(x)u(x)}{u(A \cup x)} \end{aligned}$$

Hence, the representation holds for off-recommendation alternative.

Then, we show that the representation holds for on-recommendation set (*i.e.* $x \in A$). The representation for $\rho(x, A)$ is immediately proven if $|A| = 1$. Let $|A| \geq 2$, then for every $x \in A$, we have

$$\begin{aligned}
\rho(x,A) + \sum_{y \in A\setminus x} \rho(y,A) + \sum_{y \in X\setminus A} \rho(y,A) &= 1 \\
\rho(x,A) + \sum_{y \in A\setminus x} \rho(x,A)\frac{\rho(y,\{x,y\})}{\rho(x,\{x,y\})} &= 1 - \sum_{y \in X\setminus A} \rho(y,A) && \text{By Axiom 8} \\
\rho(x,A) + \sum_{y \in A\setminus x} \rho(x,A)\frac{u(y)}{u(x)} &= 1 - \sum_{y \in X\setminus A} \rho(y,A) && \text{By Claim 9} \\
\rho(x,A)\frac{u(A)}{u(x)} &= 1 - \sum_{y \in X\setminus A} \rho(y,A) \\
\rho(x,A) &= \frac{u(x)}{u(A)}\left[1 - \sum_{y \in X\setminus A} \frac{d(y)u(y)}{u(A\cup y)}\right] \\
& && \text{By construction of } \rho(y,A) \text{ for } y \notin A
\end{aligned}$$

Then, we prove the following claim.

Claim 10. For $x \in A$ with $|A| \geq 2$

$$\frac{u(x)}{u(A)}\left[1 - \sum_{y \in X\setminus A} \frac{d(y)u(y)}{u(A\cup y)}\right] = u(x)\left[\frac{d(A)}{u(A)} + \sum_{y \in X\setminus A} \frac{d(y)}{u(A\cup y)}\right]$$

Proof.

$$\frac{u(x)}{u(A)}\left[1 - \sum_{y \in X\setminus A} \frac{d(y)u(y)}{u(A\cup y)}\right] - u(x)\left[\frac{d(A)}{u(A)} + \sum_{y \in X\setminus A} \frac{d(y)}{u(A\cup y)}\right]$$

$$
\begin{aligned}
=&u(x)\left[\frac{1-d(A)}{u(A)} - \sum_{y\in X\setminus A}[\frac{d(y)u(y)}{u(A)u(A\cup y)} + \frac{d(y)}{u(A\cup y))}]\right] \\
=&u(x)\left[\sum_{y\in X\setminus A} d(y)[\frac{1}{u(A)} - \frac{u(y)}{u(A)u(A\cup y)} - \frac{1}{u(A\cup y)}]\right] = 0
\end{aligned}
$$

Hence, this claim is proven. □

Hence, we have shown that

$$\rho(x, A) = u(x)\left[\frac{d(A)}{u(A)} + \sum_{y\in X\setminus A} \frac{d(y)}{u(A\cup y)}\right]$$

By re-arrangement, one can see that it is the representation for $\rho(x, A)$ where $x \in A$. It is proven. Firstly, we consider A such that $|A|= k$.

Hence, it holds for $|A|= k$. Then, let A such that $|A|\le k-1$. Firstly, we prove the following claim.

Claim 11. For any A with $|A|\le k-1$, we have

$$\rho(A, A) = \rho(A, \emptyset) + \sum_{z\notin A} \frac{\rho(z, A)\rho(A, A\cup z)}{\rho(z, A\cup z)}$$

Proof. For $z \notin A$, we have, by Axiom ,

$$\begin{aligned}
\rho(z,A)\rho(A\cup z, A\cup z) &= \rho(z,\emptyset)\rho(z,A\cup z) \\
\rho(z,A)\Big[\rho(z,A\cup z)+\rho(A,A\cup z)\Big] &= \rho(z,\emptyset)\rho(z,A\cup z) \\
\rho(z,A)+\rho(z,A)\frac{\rho(A,A\cup z)}{\rho(z,A\cup z)} &= \rho(z,\emptyset) \\
1-\rho(A,A)+\sum_{z\notin A}\frac{\rho(z,A)\rho(A,A\cup z)}{\rho(z,A\cup z)} &= 1-\rho(A,\emptyset) \qquad \text{Summing over } z\notin A
\end{aligned}$$

which gives our result. □

Let $x \in A$ with $|A| \leq k-1$. Note that since the representation is correct for $\rho(y,A)$ for $y \notin A$. The representation for $\rho(x,A)$ is immediately proven if $|A| = 1$. Let $|A| \geq 2$. Then, by Axiom 8, we have for $x, y \in A$ and $u \notin A$

$$\begin{aligned}
\frac{\rho(y,A)}{\rho(x,A)} &= \frac{\rho(y,A\cup u)}{\rho(x,A\cup u)} \\
\frac{\rho(A,A)}{\rho(x,A)} &= \frac{\rho(A,A\cup u)}{\rho(x,A\cup u)} \qquad \text{By summing over } y\in A
\end{aligned}$$

Hence, we have, for $x \in A$ and $u \notin A$

$$\rho(x,A) = \frac{\rho(x,A\cup u)}{\rho(A,A\cup u)}\rho(A,A)$$

$$= \frac{\rho(x, A \cup u)}{\rho(A, A \cup u)} \left[\rho(A, \emptyset) + \sum_{z \notin A} \frac{\rho(z, A)\rho(A, A \cup z)}{\rho(z, A \cup z)} \right]$$

$$= \sum_{y \in A} \frac{u(x)d(x)}{u(A)} + \frac{u(x)}{u(A)} \sum_{z \notin A} \frac{u(A)}{u(z)} \rho(z, A) \qquad \text{By Claim 9}$$

$$= \sum_{y \in A} \frac{u(x)d(x)}{u(A)} + \sum_{z \notin A} \frac{u(x)}{u(z)} \frac{d(z)u(z)}{w(A \cup z)}$$

By the representation for off recommendation

$$= \sum_{y \in A} \frac{u(x)d(x)}{u(A)} + \sum_{z \notin A} \frac{d(z)u(x)}{w(A \cup z)}$$

Hence, it is proven.

□

Proof of Theorem 4

Proof. The necessity proof is straightforward. For the sufficiency, we let $w(x) := \rho(x, \emptyset)$. We designate an anchored z_0 such that for every $x \in X \setminus z_0$, we let

$$v(x) := \frac{\rho(x, R)}{\rho(z_0, R)} \rho(z_0, \emptyset)$$

for some R such that $x \in R$ and $z_0 \notin R$. Firstly, note that denominator is greater than 0 by the assumption that it is choice probability is positive. Secondly, note that by Axiom 11, $\frac{\rho(x,R)}{\rho(z_0,R)}$ is independent of R as long as $x \in R$ and $z_0 \notin R$. Hence, the definition is valid. What's more, note that by Axiom 11, for every $y \in X$ and $y \notin R'$, we have $\frac{\rho(x,R')}{\rho(y,R')}\rho(y,\emptyset) = \frac{\rho(x,R)}{\rho(z_0,R)}\rho(z_0,\emptyset) = v(x)$. In other words, the definition of $v(x)$ is independent of the anchored z_0. Therefore, we can define $v(z_0)$ which any other arbitrary anchor.

Claim 12. 1) For $x, y \notin R$, $\frac{\rho(x,R)}{\rho(y,R)} = \frac{w(x)}{w(y)}$. 2) For $x, y \in R$, $\frac{\rho(x,R)}{\rho(y,R)} = \frac{v(x)}{v(y)}$. 3) For $x \in R$ and $y \notin R$, $\frac{\rho(x,R)}{\rho(y,R)} = \frac{v(x)}{w(y)}$.

Proof. To see 1) holds, note that Axiom 11 implies $\frac{\rho(x,R)}{\rho(y,R)} = \frac{\rho(x,\emptyset)}{\rho(y,\emptyset)} = \frac{w(x)}{w(y)}$. To see 2) holds, note that for some $z \notin R$, Axiom 11 implies $\frac{\rho(x,R)}{\rho(y,R)} = \frac{\rho(x,R)}{\rho(y,R)}\frac{\rho(z,\emptyset)}{\rho(z,\emptyset)}\frac{\rho(z,R)}{\rho(z,R)} = \frac{v(x)}{y(x)}$. To see 3) holds, Axiom 11 implies $\frac{\rho(x,R)}{\rho(y,R)} = \frac{\rho(x,R)}{\rho(y,R)}\frac{\rho(y,\emptyset)}{\rho(y,\emptyset)} = \frac{v(x)}{w(y)}$. The proof is complete. □

Lastly, for $x \in R$, taking summation over all choice ratio of every elements in X with $\rho(x, R)$ being at the denominator, we have

$$\sum_{y \in X} \frac{\rho(y,R)}{\rho(x,R)} = \frac{\sum_{y \in R} v(x) + \sum_{y \in R^c} w(x)}{v(x)}$$

$$\rho(x,R) = \frac{v(x)}{\sum_{y \in R} v(x) + \sum_{y \in R^c} w(x)}$$

One can apply the same argument analogously to the case that $x \notin R$.

Claim 13. $v(x) \geq w(x)$ for every $x \in X$.

Proof. We check $\rho(y, \{x\})$ and $\rho(y, \emptyset)$ for some y. It is easy to show that $v(x) \geq w(x)$ if and only if

$$\rho(y, \{x\}) = \frac{w(y)}{\sum_{z \in X \setminus x} w(z) + v(x)} \geq \frac{w(y)}{\sum_{z \in X \setminus x} w(z) + w(x)} = \rho(y, \emptyset)$$

which is given by Axiom 12.

□

Due to this claim, we define $r(x) := \frac{v(x)}{w(x)}$. The proof is complete.

□

Proof of Proposition 3

Proof. Suppose that (u_1, d_1) and (u_2, d_2) represent the same choice rule. Then, by definition, for every $x \in X$, $d_1(x) = \rho(x, \emptyset) = d_2(x)$. Also, for every $x \in X$, we have

$$\frac{u_1(x)}{\sum_{x \in X} u_1(x)} = \rho(x, X) = \frac{u_2(x)}{\sum_{x \in X} u_2(x)}$$

Hence, $u_1 = \frac{\sum_{x \in X} u_1(x)}{\sum_{x \in X} u_2(x)} u_2$, where $\frac{\sum_{x \in X} u_1(x)}{\sum_{x \in X} u_2(x)} > 0$ by definition. □

Proof of Proposition 4

Proof. Suppose that (w_1, r_1) and (w_2, r_2) represent the same choice rule. Then, by definition, for every $x \in X$, $\frac{w_1(x)}{w_1(X)} = \rho(x, \emptyset) = \frac{w_2(x)}{w_2(X)}$. Therefore, we get $w_1(x) = \frac{w_1(X)}{w_2(X)} w_2(x)$, where $\frac{w_1(X)}{w_2(X)} > 0$. Also, for every $x \in X$, we can write for some $z \neq x$

$$r_1(x) = \frac{\rho(x, \{x\}}{\rho(z, \{x\}} \frac{\rho(z, \emptyset)}{\rho(x, \emptyset)} = r_2(x)$$

Therefore, $r_1 = r_2$. $\square$

Proof of Lemma 2

To prove Theorem 5, we need to first prove the following Lemma.

Lemma 2. For $R \subset X$ and choice rule ρ,

$$\sum_{a \in R} q_\rho(a, R) + \sum_{a \notin R} y_\rho(a, R) = \sum_{b \notin R} q_\rho(b, R \cup b)$$

Proof. We need to show that, for every $R \subset X$,

$$\sum_{a \in R} q(a, R) + \sum_{a \notin R} y(a, R) = \sum_{b \notin R} q(b, R \cup b)$$

We prove by strong induction by "stepping down". For $R = X \setminus \{x\}$, we have

$$\begin{aligned}
\text{RHS} &= q(x, X) = \rho(x, X) \\
\text{LHS} &= \sum_{a \in X \setminus \{x\}} q(a, X \setminus \{x\}) + \sum_{a \notin X \setminus \{x\}} y(a, X \setminus \{x\}) \\
&= \sum_{a \in X \setminus \{x\}} q(a, X \setminus \{x\}) + \rho(x, X \setminus \{x\}) \\
&= \sum_{a \in X \setminus \{x\}} \Big[\rho(a, X \setminus \{x\}) - \rho(a, X) \Big] + \rho(x, X \setminus \{x\}) \\
&= \sum_{a \in X} \rho(a, X \setminus \{x\}) - \sum_{a \in X \setminus \{x\}} \rho(a, X) \\
&= 1 - \sum_{a \in X \setminus \{x\}} \rho(a, X) \\
&= \rho(x, X)
\end{aligned}$$

Suppose that equality holds for size of $k+1, k+2, ...N-1$. Let $|R| = k$,

$$\begin{aligned}
&\text{LHS} - \text{RHS} \\
=& \sum_{a \in R} q(a, R) + \sum_{a \notin R} y(a, R) - \sum_{b \notin R} q(b, R \cup b) \\
=& \sum_{a \in R} \left(\rho(a, R) - \sum_{B \supset R} q(a, B) \right) + \sum_{a \notin R} \left(\rho(a, R) - \sum_{a \notin B \supset R} y(a, B) \right) - \sum_{b \notin R} q(b, R \cup b)
\end{aligned}$$

$$= \sum_{a \in X} \rho(a, R) - \left[\sum_{a \in R} \sum_{B \supset R} q(a, B) + \sum_{a \notin R} \sum_{a \notin B \supset R} y(a, B) + \sum_{b \notin R} q(b, R \cup b) \right]$$

Since $\sum\limits_{a \in X} \rho(a, R) = 1$, it remains to show that the latter term in the above expression equals 1. We denote $\mathcal{D}_R(i)$ as the collection of superset of R with i element. Hence, we can rewrite

$$\sum_{a \in R} \sum_{B \supset R} q(a, B) + \sum_{a \notin R} \sum_{a \notin B \supset R} y(a, B) + \sum_{b \notin R} q(b, R \cup b)$$

$$= \sum_{i=|R|+1}^{N} \sum_{B \in \mathcal{D}_R(i)} \left[\sum_{a \in R} q(a, B) + \sum_{a \notin B} y(a, B) \right] + \sum_{B \in \mathcal{D}_R(|R|+1)} \sum_{a \notin R} q(a, B)$$

By rearrangement

$$= \sum_{i=|R|+2}^{N} \sum_{B \in \mathcal{D}_R(i)} \left[\sum_{a \in R} q(a, B) + \sum_{a \notin B} y(a, B) \right] + \sum_{B \in \mathcal{D}_R(|R|+1)} \left[\sum_{a \in B} q(a, B) + \sum_{a \notin B} y(a, B) \right]$$

By taking $i = |R|{+}1$ from the 1st term and summing it to the second term

$$= \sum_{i=|R|+2}^{N} \sum_{B \in \mathcal{D}_R(i)} \left[\sum_{a \in R} q(a, B) + \sum_{a \notin B} y(a, B) \right] + \sum_{B \in \mathcal{D}_R(|R|+1)} \sum_{a \notin B} q(a, B \cup a)$$

By induction hypothesis

$$= \sum_{i=|R|+2}^{N} \sum_{B \in \mathcal{D}_R(i)} \left[\sum_{a \in R} q(a, B) + \sum_{a \notin B} y(a, B) \right] + \sum_{B \in \mathcal{D}_R(|R|+2)} \sum_{a \notin R} q(a, B)$$

By rearrangement

$$
\begin{aligned}
&=\ldots\text{(repetitively applying induction hypothesis)}\\
&=\sum_{i=N}^{N}\sum_{B\in\mathcal{D}_R(i)}\left[\sum_{a\in R}q(a,B)+\sum_{a\notin B}y(a,B)\right]+\sum_{B\in\mathcal{D}_R(|N|)}\sum_{a\notin R}q(a,B)\\
&=\sum_{a\in X}q(a,X)\\
&=\sum_{a\in X}\rho(a,X) && \text{By definition of } q\\
&=1
\end{aligned}
$$

Hence, it is proven.

□

Proof of Theorem 5(a)

Proof. For the necessity proof, we suppose the data follows the model. We introduce the following notation, for $a \in X$ and $A \subseteq X \setminus a$

$$
\begin{aligned}
M_y(b,A) &:= \mu\Big(\{\succ^\star|(X\setminus b)\cup A^* = L_{\succ^\star}(b)\}\Big)\\
M_q(b,A\cup b) &:= \mu\Big(\{\succ^\star|X\cup A^* = L_{\succ^\star}(b^*)\}\Big)
\end{aligned}
$$

Claim 14. For $a \in X$ and $A \subseteq X \setminus a$,

i) $M_y(b, A) = y(b, A)$

ii) $M_q(b, A) = q(b, A \cup b)$

Proof. We prove by strong induction by "stepping down". For $A = X \setminus \{x\}$, we have

$$\begin{aligned} y(x, X \setminus \{x\}) &= \rho(x, X \setminus \{x\}) \\ &= \mu\Big(\{\succ^\star | (X \setminus \{x\}) \cup (X \setminus \{x\})^* = L_{\succ^\star}(x)\}\Big) \\ &= M_y(x, X \setminus \{x\}) \end{aligned}$$

So, i) is true for size of A equals to $|X|-1$. Suppose i) is true for size of $k+1$, $k+2$,...,$|N|-1$. Let $|A|= k$

$$\begin{aligned} y(x, A) &= \rho(x, A) - \sum_{x \notin B \supset A} y(x, B) \\ &= \sum_{x \notin B \supseteq A} M_y(x, B) - \sum_{x \notin B \supset A} M_y(x, B) \end{aligned}$$

by Definition and induction hypothesis

$$= M_y(x, A)$$

Hence, the proof is complete for i). For ii), we prove again by strong induction by “stepping down”. For $A = X \setminus \{x\}$, we have

$$\begin{aligned} q(x, X) &= \rho(x, X) \\ &= \mu\Big(\{\succ^\star | X \cup (X \setminus \{x\})^* = L_{\succ^\star}(x)\}\Big) \\ &= M_q(x, X \setminus \{x\}) \end{aligned}$$

So, i) is true for size of A equals to $|X|-1$. Suppose i) is true for size of $k+1$, $k+2$,...,$|N|-1$. Let $|A|= k$

$$\begin{aligned} q(x, A) &= \rho(x, A) - \sum_{x \notin B \supset A} q(x, B) \\ &= \sum_{x \notin B \supseteq A} M_q(x, B) - \sum_{x \notin B \supset A} M_q(x, B) \\ &\qquad\qquad \text{by Definition and induction hypothesis} \\ &= M_q(x, A) \end{aligned}$$

Hence, the proof is complete for ii). $\square$

From this claim, we immediately show that Axiom 13 and 14, since μ are by definition positive.

For the sufficiency proof, we need to introduce new notation. For any $R^* \subseteq X^*$, we write Π_{R^*} for the set of $|R^*|!$ permutations on R, with typical element π_{R^*}. We will note $\pi_{R^*}\sigma$ as the extended sequence for some $\sigma \in X \cup X^*$. we first construct $G(a^*) := q(a, X)$ for all $a^* \in X^*$. Then, we construct, recursively,

$$G(\pi_{R^*}\sigma) = \begin{cases} G(\pi_{R^*})\dfrac{y(z, X \setminus R)}{\sum_{\pi'_{R^*}\in\Pi_{R^*}} G(\pi_{R^*})} & \text{if } \sigma = z \in R \\ G(\pi_{R^*})\dfrac{q(z, X \setminus R)}{\sum_{\pi'_{R^*}\in\Pi_{R^*}} G(\pi_{R^*})} & \text{if } \sigma = z^* \in (X \setminus R)^* \end{cases}$$

It is obvious that $G \geq 0$. We then prove the following properties for G.

Claim 15. For $R \subseteq X$, we have

i) $\sum_{\pi_{R^*}\in\Pi_{R^*}} G(\pi_{R^*}\sigma) = y(x, X \setminus R)$ for $\sigma = x \in R$,

ii) $\sum_{\pi_{R^*}\in\Pi_{R^*}} G(\pi_{R^*}\sigma) = q(x, X \setminus R)$ for $\sigma = x^* \in (X \setminus R)^*$.

iii) $\sum_{\pi_{R^*}\in\Pi_{R^*}} G(\pi_{R^*}) = \sum_{x\in R} q(x, x \cup (X \setminus R))$

iv) $G(\pi_{R^*}) = \sum_{\sigma\in R\cup(X\setminus R)^*} G(\pi_{R^*}\sigma)$ for all π_{R^*},

Proof. Note that i) and ii) are straight forward. For iii), we have

$$\begin{aligned} LHS = \sum_{\pi_{R^*}\in\Pi_{R^*}} G(\pi_{R^*}) &= \sum_{z\in R} \sum_{\pi_{(R\setminus z)^*}\in\Pi_{(R\setminus z)^*}} G(\pi_{(R\setminus z)^* z^*}) \\ &= \sum_{z\in R} q(z, x \cup (X \setminus R)) = RHS \qquad \text{By ii)} \end{aligned}$$

For iv), we have

$$RHS = \sum_{\sigma \in R \cup (X\setminus R)^*} G(\pi_{R^*}\sigma) = \frac{G(\pi_{R^*})}{\sum_{\pi'_{R^*} \in \Pi_{R^*}} G(\pi'_{R^*})}(\sum_{z\in R} y(z, X\setminus R) + \sum_{z \in X\setminus R} q(z, X \setminus R))$$

$$= \frac{G(\pi_{R^*})}{\sum_{\pi'_{R^*} \in \Pi_{R^*}} G(\pi'_{R^*})}(\sum_{z\in R} q(z, z \cup (X \setminus R))$$

By Lemma 2

$$= G(\pi_{R^*}) = LHS \qquad \text{By (iii)}$$

□

We then define each individual weight. We will consider permutation on $X \cup X^*$. We let π denote each permutation in $X \cup X^*$. Note that we only consider permutation where x^* comes before x. Then, we perform a "truncation" on π. Firstly, we truncate π up to the first element in X appears in the sequence. For example, for $X = \{x, y, z\}$, we will truncate the following sequence in the following way.

$$\underbrace{x^*y^*x}_{\pi^t} z^*zy$$

Here, since the type space of PR-RUM is huge and many types are inherently indistinguishable from other, we mainly assign the weights on a group of type whose behavior are the same. Then, we allocate the weight evenly across the group. We define, for every $\pi_{R^*} \in \Pi_{R^*}$ and $x \in X$, for every π such that $\pi^t = \pi_{R^*}x$

$$\hat{\mu}_\pi := G(\pi_{R^*}x) * \frac{1}{|\{\pi : \pi^t = \pi_{R^*}x\}|}$$

Firstly, note that $\hat{\mu} \geq 0$ due to the fact that $G \geq 0$. Claim 15 ensures that $\hat{\mu}$ is additive. In the following, we introduce the following notation. Let $L_\pi(\sigma)$ be the lower contour set of σ according to π.

$$\begin{aligned}
\hat{M}_y(b, A) &:= \hat{\mu}\Big(\{\pi|(X \setminus b) \cup A^* = L_\pi(b)\}\Big) \\
\hat{M}_q(b, A \cup b) &:= \hat{\mu}\Big(\{\pi|X \cup A^* = L_\pi(b^*)\}\Big)
\end{aligned}$$

Since the weights are constructed, for $x \notin R$

$$\begin{aligned}
\hat{\rho}(x, R) &= \sum_{x \notin B \supseteq R} \hat{M}_y(x, B) && \text{by definition} \\
&= \sum_{x \notin B \supseteq R} \sum_{\pi_{(X \setminus B)^*} \in \Pi_{(X \setminus B)^*}} G(\pi_{(X \setminus B)^*}x) && \text{by construction} \\
&= \sum_{x \notin B \supseteq R} y(x, B) && \text{By Claim 15(i)} \\
&= \rho(x, R) && \text{By definition of } y
\end{aligned}$$

On the other hand, for $x \in R$

$$\begin{aligned}
\hat{\rho}(x, R) &= \sum_{A \supseteq R} \hat{M}_q(x, A) && \text{by definition} \\
&= \sum_{A \supseteq R} \sum_{\pi_{(X \setminus A)^*} \in \Pi_{(X \setminus A)^*}} G(\pi_{(X \setminus A)^*} x) && \text{by construction} \\
&= \sum_{A \supseteq R} q(x, A) && \text{By Claim 15(ii)} \\
&= \rho(x, R) && \text{By definition of } q
\end{aligned}$$

Hence, the constructed weights explain the data. Since it explains the data, it is immediately that $\sum_{\pi \in \Pi} \hat{\mu}_\pi = 1$. The sufficiency proof is complete.

$\square$

Proof of Theorem 5(b)

Proof. For the necessity proof, we suppose the data follows the model. We introduce the notation, for $a, b \in X$ and $b \notin A$,

$$T(a, b, A) := \tau\Big(\{(a, \succ) \in \mathcal{T} : A = L_\succ(b)\}\Big)$$

where $L_\succ(a)$ is the strict lower contour set of a according to $\succ$.

Claim 16. For $x \notin A$,

i) $T(x,x,A) = y(x,A)$

ii) $\sum_{a \in A \cup \{x\}} T(a,x,A) = q(x, A \cup \{x\})$

Proof. For *i*), we prove by strong induction by "stepping down". For $A = X \setminus \{x\}$, we have,

$$\begin{aligned} y(x, X \setminus \{x\}) &= \rho(x, X \setminus \{x\}) && \text{By definition of } y \\ &= \tau\Big(\{(x, \succ) : X \setminus \{x\} = L_{\succ}(x)\}\Big) \\ &= T(x,x,X \setminus \{x\}) \end{aligned}$$

So, i) is true for size of A equals to $|X|-1$. Suppose i) is true for size of $k+1$, $k+2$,...,$|N|-1$. Let $|A|= k$

$$\begin{aligned} y(x,A) &= \rho(x,A) - \sum_{x \notin B \supset A} y(x,B) \\ &= \sum_{x \notin B \supseteq A} T(x,x,B) - \sum_{x \notin B \supset A} T(x,x,B) \\ & \qquad \text{by Definition and induction hypothesis} \\ &= T(x,x,A) \end{aligned}$$

Hence, the proof is complete for i).

For ii), we also prove by strong induction by "stepping down". For $A = X\backslash\{x\}$, we have,

$$\begin{aligned} q(x, X) &= \rho(x, X) && \text{By definition} \\ &= \sum_{a \in A \cup \{x\}} \tau\Big(\{(a, \succ) : X \setminus \{x\} = L_{\succ}(x)\}\Big) \\ &= \sum_{a \in X} T(a, x, X \setminus \{x\}) \end{aligned}$$

So, ii) is true for size of A equals to $|X|-1$. Suppose i) is true for size of $k+1$, $k+2$,...,$N-1$. Let $|A|= k$

$$\begin{aligned} q(x, A \cup \{x\}) &= \rho(x, A \cup \{x\}) - \sum_{B \supset A \cup \{x\}} q(x, B) \\ &= \sum_{B \supseteq A \cup \{x\}} \sum_{a \in B} T(a, x, B \setminus \{x\}) - \sum_{B \supset A \cup \{x\}} \sum_{a \in B} T(a, x, B \setminus \{x\}) \\ & \qquad \text{by Definition and induction hypothesis} \\ &= \sum_{a \in A \cup \{x\}} T(a, x, A) \end{aligned}$$

Hence, the proof is complete for ii). □

From this claim, we immediately show Axiom 14 and Axiom 15 since for $x \notin A$, we have

$$y(x, A) = T(x, x, A) \geq 0$$

$$q(x, A \cup \{x\}) - y(x, A) = \sum_{a \in A \cup \{x\}} T(a, x, A) - T(x, x, A) = \sum_{a \in A} T(a, x, A) \geq 0$$

The necessity proof is complete.

For the sufficiency proof, we make the following claim for later use.

Claim 17. Axiom 15 and Lemma 2 implies for all a, $q(a, a) = y(a, \emptyset)$.

Proof. One can prove by contradiction. Suppose not, there exists a such that $q(a, a) > y(a, \emptyset)$. Due to Lemma 2, then there exists b such that $q(b, b) < y(b, \emptyset)$. $\square$

Then, we need to introduce the following notation. For any $R \subseteq X$, we write Π_R for the set of $|R|!$ permutations on R, with typical element π_R. We write Π for Π_X. Let $\pi_R(i)$ refers to the ith element on the permutation. The type space is now instead specified by $X \times \Pi$, with element (a, π), where $a \in X$ and $\pi \in \Pi$.

For the sufficiency proof, we first construct $F(a, a, A) := y(a, A)$ for $A \subseteq X \setminus \{a\}$. In the following, for every π_R, we denote $a\pi_R$ as the lengthen element of π_R in $\Pi_{R \cup \{a\}}$ where a is inserted at the beginning of the permutation, and similarly, we denote $\pi_R b$ as the lengthen permutation of π_R where b is inserted at the end of the permutation. Anaglously, we denote $a\pi_R b$ where a and b are inserted at the beginning and the end, respectively. Then, we construct, recursively, for $a, b \notin R \cup A$, $R \cap A = \emptyset$ and $b\pi_R a \in \Pi_{R\cup\{a,b\}}$

$$F(a, b\pi_R a, A) =$$
$$\begin{cases} \dfrac{F(a, \pi_R a, A)(q(b, R \cup A \cup \{a, b\}) - y(b, R \cup A \cup \{a\}))}{\sum_{x\in R\cup A\cup\{a\}} q(x, R \cup A \cup \{a\})} & \text{if denominator is non-zero} \\ 0 & \text{otherwise} \end{cases}$$

Firstly, note that $F \geq 0$ by Axiom 14 and 15.

Claim 18. For every a, A, $F(a, \pi_R a, A) = \sum_{b\in X\setminus R\cup A\cup\{a\}} F(a, b\pi_R a, A)$

Proof.

$$\begin{aligned} &\sum_{b\in X\setminus A\cup R\cup\{a\}} F(a, b\pi_R a, A) \\ &= \sum_{b\in X\setminus R\cup A\cup\{a\}} \frac{F(a, \pi_R a, A)(q(b, R \cup A \cup \{a, b\}) - y(b, R \cup A \cup \{a\}))}{\sum_{x\in R\cup A\cup\{a\}} q(x, R \cup A \cup \{a\})} \\ &= F(a, \pi_R a, A) \frac{\sum_{b\in X\setminus R\cup A\cup\{a\}}(q(b, R \cup A \cup \{a, b\}) - y(b, R \cup A \cup \{a\}))}{\sum_{x\in R\cup A\cup\{a\}} q(x, R \cup A \cup \{a\})} \\ &= F(a, \pi_R a, A) && \text{by Lemma 2} \end{aligned}$$

□

We show two more properties of F.

Claim 19. For every non-empty A and $x \notin A$,

i) $\sum_{a\in A} \sum_{B\subseteq A\setminus\{a\}} \sum_{\pi_B\in\Pi_B} F(a, \pi_B a, A \setminus (B \cup a)) = \sum_{b\in A} q(b, A)$

ii) $\sum_{a\in A} \sum_{B\subseteq A\setminus\{a\}} \sum_{\pi_B\in\Pi_B} F(a, x\pi_B a, A \setminus (B \cup a)) = q(x, A \cup \{x\}) - y(x, A)$

Proof. Note that by expanding the LHS of i) and ii) with the definition of F, one can show that

$$\sum_{a\in A} \sum_{B\subseteq A\setminus\{a\}} \sum_{\pi_B\in\Pi_B} F(a, \pi_B a, A \setminus (B \cup a)) = \sum_{a\in A} F(a, a, A \setminus \{a\}) +$$

$$\sum_{\substack{C\subseteq A\\ |C|=|A|-1}} \left\{ \sum_{a\in C} \sum_{B\subseteq C\setminus\{a\}} \sum_{\pi_B\in\Pi_B} F(a, c\pi_B a, C \setminus (B \cup a)) : \{c\} = A \setminus C \right\}$$

.....(*)

and also

$$\sum_{a\in A} \sum_{B\subseteq A\setminus\{a\}} \sum_{\pi_B\in\Pi_B} F(a, x\pi_B a, A \setminus (B \cup a))$$

$$= \frac{q(x, A \cup \{x\}) - y(x, A)}{\sum_{b\in A} q(b, A)} \sum_{a\in A} \sum_{B\subseteq A\setminus\{a\}} \sum_{\pi_B\in\Pi_B} F(a, \pi_B a, A \setminus (B \cup a)) \qquad .....(**)$$

Based on this observation, we prove i) and ii) together by induction by the size of A.

For $|A|= 1$, we let $A = \{a\}$. Then, for i), we have

$$\begin{aligned}
\sum_{a\in A}\sum_{B\subseteq A\setminus\{a\}}\sum_{\pi_B\in\Pi_B} F(a, \pi_B a, A\setminus(B\cup a)) &= F(a, a, \emptyset) \\
&= y(a, \emptyset) && \text{by construction} \\
&= q(a, a) && \text{by Claim 17}
\end{aligned}$$

For ii), we have

$$\begin{aligned}
\sum_{a\in A}\sum_{B\subseteq A\setminus\{a\}}\sum_{\pi_B\in\Pi_B} F(a, x\pi_B a, A\setminus(B\cup a)) &= F(a, xa, \emptyset) \\
&= \frac{F(a, a, \emptyset)(q(x, \{x, a\}) - q(x, \{a\})}{q(a, a)} \\
& && \text{by construction} \\
&= q(x, \{x, a\}) - q(x, \{a\}) && \text{by Claim 17}
\end{aligned}$$

Hence, i) and ii) are true for size of A equals 1. Suppose i) and ii) are true for size of $k-1$. Let $|A|= k$. Then, for i), we have, by using $(*)$

$$\begin{aligned}
&\sum_{a\in A}\sum_{B\subseteq A\setminus\{a\}}\sum_{\pi_B\in\Pi_B} F(a,\pi_B a, A\setminus(B\cup a))\\
=&\sum_{a\in A} F(a,a,A\setminus\{a\}) + \sum_{\substack{C\subseteq A\\ |C|=|A|-1}} \left\{\sum_{a\in C}\sum_{B\subseteq C\setminus\{a\}}\sum_{\pi_B\in\Pi_B} F(a,c\pi_B a, C\setminus(B\cup a)) : \{c\} = A\setminus C\right\}\\
=&\sum_{a\in A} y(a,A\setminus\{a\}) + \sum_{\substack{C\subseteq A\\ |C|=|A|-1}} \left\{q(c,C\cup\{c\}) - y(c,C) : \{c\} = A\setminus C\right\}
\end{aligned}$$

by construction and induction hypotheses where $|C|= k-1$

$$\begin{aligned}
=&\sum_{a\in A} y(a,A\setminus\{a\}) + \sum_{a\in A}(q(a,A) - y(a,A\setminus\{a\}))\\
=&\sum_{a\in A} q(a,A)
\end{aligned}$$

Hence, it is confirmed that $i)$ is true for size of k. Then, for $ii)$, by using $(**)$

$$\begin{aligned}
&\sum_{a\in A}\sum_{B\subseteq A\setminus\{a\}}\sum_{\pi_B\in\Pi_B} F(a,x\pi_B a, A\setminus(B\cup a))\\
=&\frac{q(x,A\cup\{x\}) - y(x,A)}{\sum_{b\in A} q(b,A)}\sum_{a\in A}\sum_{B\subseteq A\setminus\{a\}}\sum_{\pi_B\in\Pi_B} F(a,\pi_B a, A\setminus(B\cup a))\\
=&\frac{q(x,A\cup\{x\}) - y(x,A)}{\sum_{b\in A} q(b,A)}\sum_{b\in A} q(b,A) \qquad \text{since } i) \text{ is true for size of A to be } k\\
=&q(x,A\cup\{x\}) - y(x,A)
\end{aligned}$$

Hence, by induction, i) and ii) hold. □

We then define each individual weight. For $\pi \in \Pi$, we first write $\pi^{t(a)}$ as the "truncated" sequence of π up to a and does not include a. Also, we write $L_\pi(a)$ as the strict lower contour set of a according to π. Hence, IR-RUM type space is big and many types are behaviorally indistinguishable from other types. We main assign weight on a group of types, and distribute even weight across them. We define

$$\hat{\tau}_{a,\pi} := F(a, \pi^{t(a)}a, L_\pi(a)) \frac{1}{|L_\pi(a)|!}$$

Firstly, note that $\hat{\tau}_{a,\pi} \geq 0$ due to the fact that $F \geq 0$. Claim 18 ensures that $\hat{\tau}$ is additive.

In the following, we introduce the notation, for $a, b \in X$ and $b \notin A$,

$$\hat{M}(a, b, A) = \sum \{\hat{\tau}_{a,\pi} : \ A = L_\pi(b)\}$$

Claim 20. For $x \notin A$,

i) $\hat{M}(x, x, A) = y(x, A)$

ii) $\sum_{a \in A} \hat{M}(a, x, A) = q(x, A \cup \{x\}) - y(x, A)$

Proof. For i), by using Claim 18, one can show that

$$\hat{M}(x, x, A) = F(x, x, A)$$

Hence, by construction of F, it is proven. For ii), by expanding and using Claim 18, one can see that, for $a \neq x$,

$$\hat{M}(a, x, A) = \sum_{B \subseteq A \setminus \{a\}} \sum_{\pi_B \in \Pi_B} F(a, x\pi_B a, A \setminus (B \cup a))$$

Hence, by putting $\sum\limits_{a \in A}$ on both side and applying Claim 19(i), it is proven. □

Since the weight are constructed, we have, for $x \in R$,

$$\begin{aligned}
\hat{\rho}(x, R) &= \sum_{A \supseteq R} \sum_{a \in A} \hat{M}(a, x, A \setminus \{x\}) && \text{by Definition} \\
&= \sum_{A \supseteq R} \left[\hat{M}(x, x, A \setminus \{x\}) + \sum_{a \in A \setminus \{x\}} \hat{M}(a, x, A \setminus \{x\}) \right] \\
&= \sum_{A \supseteq R} \left[y(x, A \setminus \{x\}) + q(x, A) - y(x, A \setminus \{x\}) \right] && \text{By Claim 20} \\
&= \sum_{A \supseteq R} q(x, A) \\
&= \rho(x, R) && \text{By definition of q}
\end{aligned}$$

for $x \notin R$,

$$\begin{aligned}
\hat{\rho}(x,R) &= \sum_{a \notin B \supseteq R} \hat{M}(x,x,B) && \text{by Definition} \\
&= \sum_{a \notin B \supseteq R} y(x,B) && \text{By Claim 20(i)} \\
&= \rho(x,R) && \text{By definition of y}
\end{aligned}$$

Hence, the constructed weights explain the data. Since it explains the data, it is immediately that $\sum\limits_{a \in X} \sum\limits_{\pi \in \Pi} \hat{\tau}_{a,\pi} = 1$. The sufficiency proof is complete.

□

Proof of Theorem 6(a)

Proof. It suffices to show that it satisfies Axiom 13 and Axiom 14. We first state the following fact, due to the the well known fact that any Luce is RUM.

Let $k_i \geq 0$ for $i = 1, ..., N$, and we denote $\mathcal{N} = \{1, .., N\}$ Then, for any $0 \leq \frac{1}{K} \leq 1$, we have

$$\sum_{A \subseteq \mathcal{N}} (-1)^{|A|} \frac{1}{K + \sum_{i \in A} k_i} \geq 0$$

We first consider $y(x,R)$. Note that we can write $v(x) = w(x) + (r(x) - 1)w(x)$.

Therefore, we can see that

$$\begin{aligned} y(x,R) &= \sum_{x\notin B\supseteq R} (-1)^{|B\setminus R|}\rho(x,B) \\ &= \sum_{x\notin B\supseteq R} (-1)^{|B\setminus R|}\frac{w(x)}{w(B^c)+v(B)} \\ &= w(x)\sum_{x\notin B\supseteq R} (-1)^{|B\setminus R|}\frac{1}{w(B^c)+v(B)} \end{aligned}$$

Then we use the above fact. Let $w(R^c)+v(R)=K$ and each i corresponds to each element in $B\setminus R$ such that $k_i=(r(x)-1)w(x)\geq 0$ for some x. Then, it is immediate that $y(x,R)\geq 0$. Analogously, one can also see that $q(x,R)\geq 0$. □

Proof of Theorem 6(b)

Proof. It suffices to show that it satisfies Axiom 14 to 15. In the following, for notational ease, we denote $u(A):=\sum_{x\in A}u(x)$. For Axiom 14, we prove it by using standard results from the relationship between RUM and Luce model. Notice that, for $x \notin R$,

$$
\begin{aligned}
y(x,R) &= \sum_{x\notin B\supseteq R} (-1)^{|B\setminus R|}\rho(x,B) \\
&= \sum_{x\notin B\supseteq R} (-1)^{|B\setminus R|} d(x)\frac{u(x)}{w(B\cup x)} \\
&= d(x) \sum_{A\supseteq R\cup x} (-1)^{|A\setminus (R\cup x)|}\frac{u(x)}{u(A)}
\end{aligned}
$$

It is a well-known result that every Luce model has a RUM representation. Hence, the term $\sum\limits_{A\supseteq R\cup x} (-1)^{|A\setminus (R\cup x)|}\frac{u(x)}{u(A)}$ is guaranteed to be non-negative since it is exactly the standard block Marschak polynomials of a Luce model.

For Axiom 15, we first make the following auxiliary claim.

Claim 21. For every $A \subset X$ and $z \notin A$, $\sum\limits_{B\supseteq A} (-1)^{|B\setminus A|}\frac{1}{u(B\cup z)} = 0$

Proof.

$$
\begin{aligned}
&\sum_{B\supseteq A} (-1)^{|B\setminus A|}\frac{1}{u(B\cup z)} \\
&= \sum_{B\supseteq A} (-1)^{|B\setminus A|}\frac{1}{u(B\cup z)} \\
&= \sum_{z\notin B\supseteq A} (-1)^{|B\setminus A|}\frac{1}{u(B\cup z)} + \sum_{z\in B\supseteq A} (-1)^{|B\setminus A|}\frac{1}{u(B\cup z)} \\
&= \sum_{C\supseteq A\cup z} (-1)^{|C\setminus A\cup z|}\frac{1}{w(C)} + \sum_{B\supseteq A\cup z} (-1)^{|B\setminus (A\cup z)|+1}\frac{1}{u(B)}
\end{aligned}
$$

$$=0$$

$\square$

Then, we make the following claim. To prove this, we utilize the following expression of $\rho(x, A)$ for $x \in A$,

$$\rho(x, A) = \rho(x, A \setminus x) + \sum_{z \in A \setminus x} \frac{u(x)}{u(z)} \rho(z, A \setminus z) + \sum_{z \notin A} \frac{u(x)}{u(z)} \rho(z, A)$$

Claim 22. For every $x \in A$,

$$q(x, A) = y(x, A \setminus x) + \sum_{z \in A \setminus x} \frac{u(x)}{u(z)} y(z, A \setminus z)$$

Proof.

$$\begin{aligned} q(x, A) &= \sum_{B \supseteq A} (-1)^{|B \setminus A|} \rho(a, B) && \text{By definition} \\ &= \sum_{B \supseteq A} (-1)^{|B \setminus A|} \left[\rho(x, B \setminus x) + \sum_{z \in B \setminus x} \frac{u(x)}{u(z)} \rho(z, B \setminus z) + \sum_{z \notin B} \frac{u(x)}{u(z)} \rho(z, B) \right] \\ & && \text{by the above expression} \\ &= \sum_{B \supseteq A} (-1)^{|B \setminus A|} \left[\rho(x, B \setminus x) + \sum_{z \in A \setminus x} \frac{u(x)}{u(z)} \rho(z, B \setminus z) \right. \end{aligned}$$

$$+\sum_{z\in B\setminus A}\frac{u(x)}{u(z)}\rho(z,B\setminus z)+\sum_{z\notin B}\frac{u(x)}{u(z)}\rho(z,B)\Big]$$

$$= y(x,A\setminus x)+\sum_{z\in A\setminus x}\frac{u(x)}{u(z)}y(z,A\setminus z)$$

$$+\sum_{B\supseteq A}(-1)^{|B\setminus A|}\Big[\sum_{z\in B\setminus A}\frac{u(x)}{u(z)}\rho(z,B\setminus z)+\sum_{z\notin B}\frac{u(x)}{u(z)}\rho(z,B)\Big]$$

By applying the definition of y on the first two terms

It remains to show that the last sum is zero. Note that

$$\sum_{B\supseteq A}(-1)^{|B\setminus A|}\Big[\sum_{z\in B\setminus A}\frac{u(x)}{u(z)}\rho(z,B\setminus z)+\sum_{z\notin B}\frac{u(x)}{u(z)}\rho(z,B)\Big]$$

$$=u(x)\sum_{B\supseteq A}(-1)^{|B\setminus A|}\sum_{z\in X\setminus A}\frac{1}{u(z)}\rho(z,B\setminus z)$$

$$=u(x)\sum_{B\supseteq A}(-1)^{|B\setminus A|}\sum_{z\in X\setminus A}\frac{1}{u(z)}\frac{u(z)d(z)}{w(B\cup z)}$$ By the definition of $\rho(z,B)$

$$=u(x)\sum_{z\in X\setminus A}d(z)\sum_{B\supseteq A}(-1)^{|B\setminus A|}\frac{1}{w(B\cup z)}$$ By switching summation sign

$$=0$$ by Claim 21

□

Hence, Claim 22 is proven. Hence, to show that Axiom 15 is satisfied, we have

$$q(x, A) - y(x, A \setminus x) = \sum_{z \in A \setminus x} \frac{u(x)}{u(z)} y(z, A \setminus z) \geq 0$$

The proof is complete. □

Proof of Proposition 7

Proof. It is proven in Claim 16. □

Proof of Proposition 8

Proof. It is proven in Claim 14. □

Appendix B: Proofs for Chapter 2

In this Appendix, we will provide proofs for chapter 2.

Proof for the if-part in Theorem 7

Proof. The idea of the proof: The proof is mainly divided into two parts idea-wise. The first part sets up the system of linear equation which pins down the μ that satisfies the desired property. Some algebraic operations are devoted into lining up the system in a way to prepare for the second part. The second part shows how we can utilize the Farkas's Lemma for proving the existence of a solution to the system for any parameter value which satisfies the property AC. Claim 4 concludes the proof.

Assume $(\pi(.), \succ)$ satisfies property AC. For every S and $x \in S$, we set $\sum_{\substack{x \in J \subseteq S \\ x \text{ is } \succ\text{-best}}} \mu(J|S) = \pi(x|S)$ and $\phi(x|S) = \max_{R \supseteq S} \pi(x|R)$.[1] It is immediate to

---

[1] In here, we set $\phi(x|S) = \max\limits_{R \supseteq S} \pi(x|R)$. Yet, we should note that if we set $\phi(x|S) = \min\limits_{T \subseteq S} \sum_{S \cap \bar{U}_{\succ}(a)} \pi(a|T)$, the proof would also go through. In fact, for any value between the two, the proof still go through if we choose ϕ such that it satisfies attention overload.

see that the attention rule satisfies the desired non-parametric properties. What remains is to show that there exists a solution to the system of linear equation. Let $x_1 \succ x_2 \succ ...x_n$. Then, we have for $i = 1, ...n$

$$\sum_{\substack{x_i \in J \subseteq S \\ x_i \text{ is } \succ\text{-best}}} \mu(J|S) = \pi(x_i|S) \qquad .....(\text{denoted by } \mathcal{P}_i)$$

$$\phi(x_i|S) = \max_{R \supseteq S} \pi(x_i|R) \qquad .....(\text{denoted by } \mathcal{M}_i)$$

Note that for x_1, $\max_{R \supseteq S} \pi(x_i|R) = \pi(x_i|S)$ (or it violates property AC). Also, $\sum_{\substack{x_i \in J \subseteq S \\ x_i \text{ is } \succ\text{-best}}} \mu(J|S) = \phi(x_i|S)$. Hence, $(\mathcal{P}_1) = (\mathcal{M}_1)$. On the other hand, $\mathcal{P}_n$ is $\pi(x_n|S) = \mu(\{x_n\}|S)$, which immediate gives the solution to the "unknown" $\mu(\{x_n\}|S)$. Hence, we are left with $(\mathcal{P}_i)$, $i = 1, ..n-1$ and $(\mathcal{M}_i)$, $i = 2, ..n$. Then, we create $\mathcal{M}'_i \equiv \sum_{j \leq i} (\mathcal{P}_j) - (\mathcal{M}_i)$ for every $i = 2, ...n$, i.e.

$$\sum_{j<i} \sum_{\substack{x_i \notin J \subseteq S \\ x_j \text{ is } \succ\text{-best}}} \mu(J|S) = \sum_{j \leq i} \pi(x_j|S) - \max_{R \supseteq S} \pi(x_i|R) \qquad .....(\text{denoted by } \mathcal{M}'_i)$$

where $\sum_{j \leq i} \pi(x_j|S) - \max_{R \supseteq S} \pi(x_i|R) \geq 0$ for $i = 2, ...n$ by Property K. Lastly, we create $(\mathcal{P}'_1) \equiv (\mathcal{P}_1) - \sum_{j>1} (\mathcal{M}_j)$. Hence, we are left with $(\mathcal{P}'_1), (\mathcal{P}_i), i = 2, ..n-1$ and $(\mathcal{M}'_1)$,$i = 2, ...n$. We utilize the Farkas's Lemma to prove the existence of solution to the above system of linear equations. Note that the system is straightforward when $n < 3$. Hence, we focus only the case that $n \geq 3$.

Farkas's Lemma : Let $A \in R^{m \times n}$ and $b \in R^m$. Then exactly one of the following is true:

1. There exists an $x \in R^n$ such that $Ax = b$ and $x \geq 0$.
2. There exists a $y \in R^m$ such that $yA \geq 0$ and $yb < 0$.

We let A be the matrix and b be the vector which represents the above system of linear equations by $A\mu = b$. i.e. $A := \Big(r_1, r_2, ..., r_{2n-2}\Big)^T$, and $b := (b_1, b_2, ...b_{2n-2})^T$, where r_j's are row vector. In particular, we let r_1 and b_1 correspond to the LHS and RHS of $\mathcal{P}_1'$ respectively; r_j and b_j correspond to the LHS and RHS of $\mathcal{M}_{n+2-j}'$ respectively for $j = 2, ..., n$; r_j and b_j correspond to the LHS and RHS of $\mathcal{P}_{-n+1+j}$ respectively for $j = n + 1, ..., 2n - 2$.

For shorthand, we write, for all i, whenever the RHS is defined, $m_i \equiv \max_{R \supseteq S} \pi(x_i|R)$, $\pi_i \equiv \pi(x_i|S)$ and $k_i \equiv \sum_{j \leq i} \pi(x_j|S) - \max_{R \supseteq S} \pi(x_i|R) = \sum_{j \leq i} \pi_j - m_i$. We let the set B as the set of b which is generated by a data set that satisfies AC . i.e.

$$B = \Big\{ b \in R^{2n-2} : b_1 = \pi_1 - k_n - k_{n-1} - .... - k_2,$$

$$b_i = k_{n-i+2}, \text{ for } i = 2, .., n,$$

$$b_i = \pi_{i+1-n} \text{ for } i = n+1, ..., 2n-2,$$

$$\text{where } \pi(.|S) \text{ satisfies AC.} \Big\}$$

We would show that there does not exist $y = (y_1, y_2, y_3, ...y_{2n-2}) \in R^{2n-2}$ such that $yA \geq 0$ and $yb < 0$ for all $b \in B$. We define the set $Y(A)$ as the set of y which satisfies $yA \geq 0$. Hence, it suffices to show that for all $b \in B$, $\min_{y \in Y(A)} yb \geq 0$. Note that except b_1, all b_j are positive for all possible $\pi(.|S)$ that satisfies AC. Hence, the key insight in the following proof is to show that how we can guarantee $yb \geq 0$ despite the possibility of b_1 being negative.

Claim 23. For all $y \in Y(A)$, $y \geq 0$.

Proof. It is due to the fact that A admits a reduced row-echelon form by construction. To see this, first note that $yA \geq 0$. Since A admits a reduced row-echelon form, the leading entry is 1 and the leading entry in each row is the only non-zero entry in its column. It gives us $y_j \geq 0$ for all j. □

With claim 1, we can see that if $b_1 \geq 0$, the proof is trivially done. We then state claim 2, which is a special case of claim 3.

Claim 24. For all $y \in Y(A)$ and for $i = 2, ...n-1$, we have $y_i + y_j \geq y_1$ for $j = i+1, ..., 2n-i$.

Let P_n be the power set of the set $\{2, 3, ...n-1\}$

Claim 25. (A generalization of claim 2) For all $y \in Y(A)$ and for all $P \in P_n$, $\sum_{i \in P} y_i + y_j \geq |P| {*} y_1$, for $j = \max_{i \in P} i + 1, \max_{i \in P} i + 2, ..., 2n - \max_{i \in P} i$.

Proof. For any set P, we get

$$\sum_{i \in P} y_i + y_j \geq |P| {*} y_1 \text{ from the column of } \mu(S - \cup_{i \in P \cup \{j\}} \{x_{i+n-2}\} | S)$$

for any $j \in \{\max_{i \in P} i + 1, i+2, ...n\}$. For the LHS: it is because for any $i \in P$, the row vector r_{n-i+2} has the coefficient of 1 in the column of $\mu(S - \cup_{i \in P \cup \{j\}} \{x_{i+n-2}\} | S)$ by construction. For the RHS: it is because the row vector r_1 has the coefficient of $|P|$ in the same column by construction. Also, we get

$$\sum_{i \in P} y_i + y_j \geq |P| {*} y_1 \text{ from the column of } \mu(S - \cup_{i \in P} \{x_{i+n-2}\} - \cup_{i < j-n} \{x_i\} | S)$$

for any $j \in \{n+1, n+2, ...2n - \max_{i \in P} i\}$. For the LHS: it is because for any $i \in P$, the row vector r_{n-i+2} has the coefficient of 1 in the column of $\mu(S - \cup_{i \in P} \{x_{i+n-2}\} -$

$\cup_{i<j-n}\{x_i\}|S)$ by construction. For the RHS: it is because the row vector r_1 has the coefficient of $|P|$ in the same column by construction. Hence, we have covered any j in $\{\max_{i\in P} i+1, \max_{i\in P} i+2, ..., 2n-\max_{i\in P}\}$. The proof is complete. $\square$

We need to show an auxiliary minimization problem to complete the proof. Let $\mathbf{c}_n$ and $\mathbf{z}_n$ be two vectors. To be consistent with the above in notation, both vectors start with subscript 2 and end with $2n-2$. i.e. $\mathbf{c}_n = (c_2, c_3, ...c_{2n-2})$

Claim 26. For all $n \geq 3$,

$$\min_{\mathbf{c}_n\in\mathbf{C}_n,\mathbf{z}_n\in\mathbf{Z}_n} \mathbf{c}_n \cdot \mathbf{z}_n \geq 1$$

where

$$\mathbf{C}_n = \{\mathbf{c}_n \in R_+^{2n-3} | \sum_{i=2}^{n+1-j} c_i + \sum_{i=n}^{n-3+j} (c_{i+1} + c_{i+3-j}) \geq 1, j = 2, 3, ...n\}$$

$$\mathbf{C}_n = \{\mathbf{c}_n \in R_+^{2n-3} | \sum_{i=2}^{n+1-j} c_i + \sum_{i=n+3-j}^{n} c_i + \sum_{i=n+1}^{n-2+j} c_i \geq 1, j = 2, 3, ...n\}$$

$$\mathbf{Z}_n = \{\mathbf{z}_n \in R_+^{2n-3} | \sum_{i\in P} z_i + z_j \geq |P|, \forall P \in P_n, j = \max_{i\in P} i+1, \max_{i\in P} i+2, ..., 2n - \max_{i\in P} i\}$$

Proof. We prove by induction. Consider n=3. We have

$$\mathbf{C}_3 = \{\mathbf{c}_3 | c_2 \geq 1, c_4 + c_3 \geq 1\}$$

$$\mathbf{Z}_3 = \{\mathbf{z}_3 | z_2 + z_3 \geq 1, z_2 + z_4 \geq 1\}$$

It is straight-forward to see that minimum must be attained by binding constraint $c_4 + c_3 = 1$.

$$\begin{aligned}
\mathbf{c}_3 \cdot \mathbf{z}_3 &= z_2 c_2 + z_3 c_3 + z_4 c_4 \\
&\geq z_2 + c_3(1 - z_2) + c_4(1 - z_2) \\
&= z_2 + (c_3 + c_4) * (1 - z_2) = 1
\end{aligned}$$

Suppose $n = k - 1$ is true. Consider $n = k$. We set up the Lagrangian minimization problem and assigns Lagrangian multiplier $\lambda_i's$ to the constraints in $\mathbf{C}_n$. For notational convenience, we adopt a slightly different notational convention for the λ. We give each multiplier the subscript as a set of all the subscript involved in the constraint. Take $n = 3$ as an example, we would have multiplier λ_2 for $c_2 \geq 1$ and $\lambda_{3,4}$ for $c_4 + c_3 \geq 1$. It is simple to check that each constraint has it own unique respect set of subscript. One advantage of using this subscript convention is that it is more informative than giving natural number to the constraints. We collect all possible subscript of λ and name it Λ_k. The Lagrangian multiplier for the constraints in $\mathbf{Z}_n$ is not impactful in the proof.

First order condition of the Lagrangian equation gives:

$$\frac{\partial L}{\partial c_i} = z_i - \sum_{i \in S \in \Lambda_k} \lambda_S \geq 0; \qquad (z_i - \sum_{i \in S \in \Lambda_k} \lambda_S) c_i = 0, i = 2, 3, ...2k-2$$

By plugging in first order condition, we can get the following,

$$\begin{aligned} \mathbf{c}_n \cdot \mathbf{z}_n &\geq \sum_{i=2}^{2k-2} c_i (\sum_{i \in S \in \Lambda_k} \lambda_S) \\ &= \sum_{S \in \Lambda_k} \lambda_S \sum_{i \in S} c_i \\ &\geq \sum_{S \in \Lambda_k} \lambda_S \end{aligned}$$

Note that if all $c_i \neq 0$ for all i, it is straight-forward to see that $\sum_{S \in \Lambda_k} \lambda \geq 1$. For example, if $c_{2k-2} \neq 0$ and $c_2 \neq 0$, we can get from Z_k,

$$\begin{aligned} z_2 + z_{2n-2} &\geq 1 \\ \sum_{2 \in S \in \Lambda_k} \lambda_S + \sum_{2n-2 \in S \in \Lambda_k} \lambda_S &\geq 1 \\ \sum_{S \in \Lambda_k} \lambda_S &\geq 1 \end{aligned}$$

In fact, it is straight-forward to check that as long as

$$(c_2 \neq 0 \text{ and } c_{2k-2} \neq 0) \text{ or}$$

$$(c_2 \neq 0, c_3 \neq 0 \text{ and } c_{2k-3} \neq 0) \text{ or}$$

$$....$$

$$(c_2 \neq 0, c_3 \neq 0, ...c_{k-1} \neq 0 \text{ and } c_{k+1} \neq 0)$$

$$(c_2 \neq 0, c_3 \neq 0, ...c_{k-1} \neq 0 \text{ and } c_k \neq 0)$$

then $\sum_{S \in \Lambda} \lambda_S \geq 1$. For cases outside the above, we check sequentially and apply mathematical induction in each cases:

Case 1: $c_2 = 0$. By re-numbering some of the variables, in particular, write $z'_i = z_{i+1}$ and $c'_i = c_{i+1}$ for $i = 2, 3, ..2(k-1) - 2$. We name this set of constraint as C_k where both c'_i and c_j for some i, j co-exist. We perform the same procedure on and Z_k. Then, by restricting attention only at c'_i and z'_i, it is straightforward to see that $C_k \subset \mathbf{C'}_{k-1}$ and $Z_k \subset \mathbf{Z}'_{k-1}$, where $\mathbf{C}'_{k-1}$ is the same set as $\mathbf{C}_{k-1}$ by just renaming c to c'. Hence, in this case, by induction hypothesis,

$$\min_{\mathbf{c}_k \in C_k, \mathbf{z}_k \in Z_k} \mathbf{c}_k \cdot \mathbf{z}_k \geq \min_{\mathbf{c}_{k-1} \in \mathbf{C}'_{k-1}, \mathbf{z}_{k-1} \in \mathbf{Z}'_{k-1}} \mathbf{c}_{k-1} \cdot \mathbf{z}_{k-1} \geq 1$$

Case 2: $c_3 = 0$ and $c_{2k-2} = 0$. We re-number the variable, in particular, write

$c'_i = c_i$ for $i = 2$, write $c'_i = c_{i+1}$ for $i = 3, ..., 2(k-1) - 3$ and write $c'_i = c_{i+2}$ for $i = 2(k-1) - 2$. Analogously, we do the same for z. By a similar argument. We show $\min_{\mathbf{c}_k \in C_k, \mathbf{z}_k \in Z_k} \mathbf{c}_k \cdot \mathbf{z}_k \geq 1$.

....

Case $k-3$: $c_{k-1} = 0$ and $c_{k+2} = ... = c_{2k-3} = c_{2k-2} = 0$. Write $c'_i = c_i$ for $i = 2, ..k-3$, write $c'_i = c_{i+1}$ for $i = k-3, ..., k$, write $c'_i = i + 2$ for $i = k+1, .., 2(k-1) - 2$.

Last Case: $c_k = c_{k+1} = ... = c_{2k-2} = 0$. This case needs its special attention. We need to prove a auxiliary claim to finish this proof.

Claim 27. For all $n \geq 4$

$$\min_{\mathbf{d}_n \in \mathbf{D}_n, \mathbf{w}_n \in \mathbf{W}_n} \mathbf{d}_n \cdot \mathbf{w}_n \geq 1$$

where

$$\mathbf{D}_n = \{\mathbf{d}_n \in R_+^{n-2} | \sum_{i=2}^{n+1-j} d_i + \sum_{i=n}^{n-4+j} d_{i+3-j} \geq 1, j = 3, ...n\}$$

$$\mathbf{W}_n = \{\mathbf{w}_n \in R_+^{n-2} | \sum_{i \in P} w_i + w_j \geq |P|, \forall P \in P_n, j = \max_{i \in P} i + 1, ..., n\}$$

Proof. For $n = 4$. We have

$$\mathbf{D}_4 = \{\mathbf{d}_4 | d_2 \geq 1, d_3 \geq 1\}$$

$$\mathbf{W}_4 = \{\mathbf{w}_4 | w_2 + w_3 \geq 1\}$$

Hence, we have

$$\begin{aligned}\mathbf{d_4} \cdot \mathbf{w_4} &= d_2 w_2 + d_3 w_3 \\ &\geq w_2 + w_3 \\ &\geq 1\end{aligned}$$

Suppose $n = k-1$ is true. Consider $n = k$. We apply the same technique for naming the Lagrangian multiplier. We assign Lagrangian multiplier $\omega_i' s$ to the constraint in $\mathbf{D}_k$, and collect the subscript of those multiplier in the set Ω_k. The first order condition of the Lagrangian equation gives:

$$\frac{\partial L}{\partial d_i} = w_i - \sum_{i \in S \in \Omega_k} \omega_S \geq 0; \qquad (w_i - \sum_{i \in S \in \Omega_k} \omega_S) d_i = 0, i = 2, 3, ..., k-2$$

By plugging in the first order condition, we can get

$$\begin{aligned}\mathbf{d}_n \cdot \mathbf{w}_n &\geq \sum_{i=2}^{k-2} (d_i \sum_{i \in S \in \Omega_k} \omega_k) \\ &= \sum_{S \in \Omega_k} (\omega_S \sum_{i \in S} d_i)\end{aligned}$$

$$\geq \sum_{S\in\Omega_k} \omega_S$$

Note that if all $d_i \neq 0$ for all i, it is straight-forward to see that $\sum_{S\in\Omega_k} \omega_S \geq 1$. Since, by the constraint in $\mathbf{W}_k$, we have

$$\sum_{i=2}^{k-1} z_i \geq k-3$$

$$(k-3)\sum_{S\in\Omega_k} \omega_S \geq k-3$$

$$\sum_{S\in\Omega_k} \omega_S \geq 1$$

If any of the $d_i = 0$, the problem reduces to the minimization problem for $k-1$. Hence, by mathematical induction, we proved claim 5. □

Hence, in the last case, we can apply claim 5. Hence, we finish the proof of claim 4. □

Recall the previous problem that if $b_1 \geq 0$, the proof is trivially done. If not, i.e. $b_1 < 0$ then we can apply claim 4 by setting

$$c_i = -\frac{b_i}{b_1} \text{ for } i = 2, ..., 2n-2$$

$$z_i = \frac{y_i}{y_1} \text{ for } i = 2, ..., 2n-2$$

Hence, the statement that all $b \in B$, $\min_{y \in Y(A)} yb \geq 0$ is equivalent to the statement that $\min_{\mathbf{c}_n \in \mathbf{C}_n, \mathbf{z}_n \in \mathbf{Z}_n} \mathbf{c}_n \cdot \mathbf{z}_n \geq 1$. It completes the proof. □

Proof for Theorem 8

Proof. The only-if part is immediate. For the if-part, we need to be concerned about constructing the μ. For non-binaries choice set, we follow the technique in the proof for Theorem 7. For binaries choice set, WLOG, we first let $x \succ y$ throughout the proof. We assume

$$\mu(\{x,y\}|\{x,y\}) = \pi(x|\{x,y\})$$

$$\mu(\{y\}|\{x,y\}) = \pi(y|\{x,y\})$$

To check that it fulfills assumption 3. Suppose not. i.e. there exists y s.t. $\eta < \mu(y|\{x,y\}$. Hence, by definition 14, we know that $y \succ x$, which violates the fact that $x \succ y$.

To check that it fulfills assumption 2. Note that $\phi(x|\{x,y\}) = \pi(x|\{x,y\}) \geq \pi(x|S)$ for all S (or there would be a contradiction that $y \succ x$). Hence, $\phi(x|\{x,y\}) \geq$

$\phi(x|S)$ for all S since $\phi(x|S) = \max_{R \supseteq S} \pi(x|R)$. On the other hand, $\phi(y|\{x, y\}) = 1$, which automatically satisfies assumption 2.

□

Proof for Proposition 10

It suffices to show the following claim.

Claim 28. For any $n \in N$ and $M, u_1, u_2, ..., u_n \in R_+$, we have, for $k = 2.., n$,

$$\frac{u_1}{\sum_{i=1}^n u_i} + \sum_{s=1}^{k-1} \prod_{j=1}^{s} (1 - \frac{u_j}{\sum_{i=1}^n u_i}) \frac{u_{s+1}}{\sum_{i=1}^n u_i} \geq \prod_{j=1}^{k-1} (1 - \frac{u_j}{\sum_{i=1}^n u_i + M}) \frac{u_k}{\sum_{i=1}^n u_i + M}$$

Proof. Fix n, M, we prove by induction. Let $k = 2$, we have

$$\begin{aligned} LHS - RHS &= \frac{u_1}{\sum_{i=1}^n u_i} + (1 - \frac{u_1}{\sum_{i=1}^n u_i}) \frac{u_2}{\sum_{i=1}^n u_i} - (1 - \frac{u_1}{\sum_{i=1}^n u_i + M}) \frac{u_2}{\sum_{i=1}^n u_i + M} \\ &\geq \frac{u_1}{\sum_{i=1}^n u_i} + (1 - \frac{u_1}{\sum_{i=1}^n u_i}) \frac{u_2}{\sum_{i=1}^n u_i} - (1 - \frac{u_1}{\sum_{i=1}^n u_i + M}) \frac{u_2}{\sum_{i=1}^n u_i} \\ &\geq \frac{u_1}{\sum_{i=1}^n u_i} - \frac{u_1}{\sum_{i=1}^n u_i} \frac{u_2}{\sum_{i=1}^n u_i} \frac{M}{\sum_{i=1}^n u_i + M} \geq 0 \end{aligned}$$

Let the statement be true for $k - 1$.

$$\begin{aligned} LHS - RHS = \frac{u_1}{\sum_{i=1}^n u_i} + \sum_{s=1}^{k-1} \prod_{j=1}^{s} (1 - \frac{u_j}{\sum_{i=1}^n u_i}) \frac{u_{s+1}}{\sum_{i=1}^n u_i} - \prod_{j=1}^{k-1} (1 - \frac{u_j}{\sum_{i=1}^n u_i + M}) \frac{u_k}{\sum_{i=1}^n u_i + M} \\ \geq \frac{u_1}{\sum_{i=1}^n u_i} + (1 - \frac{u_1}{\sum_{i=1}^n u_i}) \prod_{j=2}^{k-1} (1 - \frac{u_j}{\sum_{i=1}^n u_i + M}) \frac{u_k}{\sum_{i=1}^n u_i + M} - \end{aligned}$$

$$\prod_{j=1}^{k-1}(1-\frac{u_j}{\sum_{i=1}^{n}u_i+M})\frac{u_k}{\sum_{i=1}^{n}u_i+M}$$

$$\geq \frac{u_1}{\sum_{i=1}^{n}u_i}-\frac{u_1}{\sum_{i=1}^{n}u_i}\frac{M}{\sum_{i=1}^{n}u_i+M}\prod_{j=2}^{k-1}(1-\frac{u_j}{\sum_{i=1}^{n}u_i+M})\frac{u_k}{\sum_{i=1}^{n}u_i+M}\geq 0$$

where the second step is by induction hypothesis. □

Proof for Proposition 11

It suffices to show the following claim.

Claim 29. For any $n \in N$ and $M, u_1, u_2, ..., u_n \in R_+$, where $u_i < u_{i+1}$ we have, for $k = 1, 2, ..., n$ and $l = 1, 2..., k$ with $M > u_l$

$$\frac{u_1}{\sum_{i=1}^{n}u_i}+\frac{u_2}{\sum_{i=1}^{n}u_i}+...+\frac{u_k}{\sum_{i=1}^{n}u_i}\geq\frac{u_k}{\sum_{i=l}^{n}u_i+M}$$

Proof. It suffices to show that for any $s, A, B > 0$, the fraction $\frac{s+A}{s+A+B}$ is increasing in A, which is straightforward. □

Appendix C: Proofs for Chapter 3

In this Appendix, we will provide proofs for chapter 3.

Proof for Proposition 12

Proof. To show that $(i) \iff (iv)$, assume that for some E, $x_2 > x_1 > x_0$, we have $(x_1, x_1) \sim (x_2, x_0)$ which is equivalent to $x_1 = x_2 - \beta(x_2 - x_0)$. Then, consider

$$\begin{aligned}
&W((x_1, x_1)E^*(x_3, x_3)) - W((x_2, x_0)E^*(x_3, x_3)) \\
=&x_1 v(E) + x_3(1 - v(E)) - (x_2 v(E) + x_3(1 - v(E)) - \beta v_g(E)(x_2 - x_0)) \\
=&(x_1 - x_2)v(E) - (x_1 - x_2)v_g(E) \qquad \text{substituting the above equality} \\
=&(x_1 - x_2)(v(E) - v_g(E))
\end{aligned}$$

Hence, $W((x_1, x_1)E^*(x_3, x_3)) \leq W((x_2, x_0)E^*(x_3, x_3))$ if and only if $(x_1 - x_2)(v(E) - v_g(E)) \leq 0$ if and only if $(v(E) - v_g(E)) \geq 0$ if and only if $v(E) \geq v_g(E)$. Hence, it is proven.

To show that $(ii) \Rightarrow (i)$, we utilize the same setup above but instead we have $(x_1, x_1) \succeq (x_2, x_0)$ which is equivalent to $x_1 \geq x_2 - \beta(x_2 - x_0)$. Hence, $W((x_1, x_1)E^*(x_3, x_3)) \leq W((x_2, x_0)E^*(x_3, x_3))$ simplifies to $(x_1 - x_2)v(E) + \beta v_g(E)(x_2 - x_0) \leq 0$. Plugging in the inequality, we get $-\beta(x_2 - x_0)v(E) + \beta(x_2 - x_0)v_g(E) \leq 0$ which gives $v(E) \geq v_g(E)$.

To show that $(i) \iff (v)$, assume that for some E, $x_3 > x_2 > x_1 > x_0 \geq 0$ and, we have $(x_1, x_1) \sim (x_2, x_0)$ which is equivalent to $x_1 = x_2 - \beta(x_2 - x_0)$. Then, consider

$$\begin{aligned}
&W((x_1, x_1)E^*(x_3, 0)) - W((x_2, x_0)E^*(x_3, 0)) \\
=&x_1 v(E) + x_3(1 - v(E)) - \beta v_g(\Omega \setminus E)x_3 \\
&- (x_2 v(E) + x_3(1 - v(E)) - \beta(1 - v_g(\Omega \setminus E))(x_2 - x_0) - \beta v_g(\Omega \setminus E)x_3) \\
=&(x_1 - x_2)v(E) - (x_1 - x_2)(1 - v_g(\Omega \setminus E)) \qquad \text{substituting the above equality} \\
=&(x_1 - x_2)(-v(\Omega \setminus E) + v_g(\Omega \setminus E))
\end{aligned}$$

Hence, $W((x_1, x_1)E^*(x_3, 0)) \geq W((x_2, x_0)E^*(x_3, 0))$ if and only if $(x_1 - x_2)(-v(\Omega \setminus E) + v_g(\Omega \setminus E)) \geq 0$ if and only if $-v(\Omega \setminus E) + v_g(\Omega \setminus E) \leq 0$ if and only if $v(\Omega \setminus E) \geq v_g(\Omega \setminus E)$. Hence, it is proven.

To show that $(iii) \Rightarrow (i)$, we utilize the same setup above but instead we have $(x_1, x_1) \preceq (x_2, x_0)$ which is equivalent to $x_1 \leq x_2 - \beta(x_2 - x_0)$. Hence, $W((x_1, x_1)E^*(x_3, 0)) \geq W((x_2, x_0)E^*(x_3, 0))$ simplifies to $(x_1 - x_2)v(E) - \beta(1 - v_g(\Omega \setminus E))(x_2 - x_0) \geq 0$. Plugging in the inequality, we get $-\beta(x_2 - x_0)v(E) + \beta(x_2 - x_0)(1 - v_g(\Omega \setminus E)) \geq 0$ which gives $v(\Omega \setminus E) \geq v_g(\Omega \setminus E)$.

For (i) implying (ii), we first normalize $x_0 = 0$ and set $x_3 = x_1 = 1$. Then, there exists x_2 such that $(x_1, x_1) \sim (x_2, x_0)$ by setting $x_2 = \frac{1}{1-\beta}$, which implies $(x_1, x_1) \succeq (x_2, x_0)$. Then, since (i) implies (iv), (iv) guarantees that $(x_1, x_1)E^*(x_3, x_3) \preceq (x_2, x_0)E^*(x_3, x_3)$.

For (i) implying (iii), we first normalize $x_0 = 0$ and set $x_1 = 1$. Then, there exists x_2 such that $(x_1, x_1) \sim (x_2, x_0)$ by setting $x_2 = \frac{1}{1-\beta}$, which implies $(x_1, x_1) \preceq (x_2, x_0)$. We set $x_3 = x_2 + 1$. Then, since (i) implies (v), (v) guarantees that $(x_1, x_1)E^*(x_3, 0) \succeq (x_2, x_0)E^*(x_3, 0)$.

□

Proof for Theorem 9

In the following, we denote $c(\mathcal{F})$ as the set of constant acts, and $e(\mathcal{F})$ as the set of constant acts which give equal allocation across players. The proof for sufficiency is as follows: We first construct the Fehr and Schmidt utility functional form for constant act. Then, we demonstrate that the DM can decompose acts across two types of acts with certain comonotonicity properties. In addition, the probability measure is given by the decomposition, while the non-additive measure is given by the inequality aversion. Lastly, the decomposition of each acts guarantee that the probability measures satisfy the rank-dependent property. The necessity part almost immediately follows from the representation.

Proof. We proceed in a sequence of lemma.

For $f \in c(\mathcal{F})$, $\succeq$ is represented by Fehr and Schmit utility function. i.e. there exists $W : c(\mathcal{F}) \to \mathbb{R}$ which represents $\succeq$ such that $W(f) = f_1 - \alpha \max\{f_2 - f_1, 0\} - \beta \max\{f_1 - f_2, 0\}$ for some $\alpha, \beta \in R_+$.

Proof. We first define two regions, which represent two sets of constant acts. $R_1 := \{(x_1, x_2) \in \mathbb{R}^2 : x_2 - x_1 \geq 0\}$ and $R_2 := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 - x_2 \geq 0\}$. With a slight abuse of notation, we have $R_1 \cup R_2 = c(\mathcal{F})$ and $R_1 \cap R_2 = e(\mathcal{F})$. Since each region R_i is within-comonotonic, between-comonotonic and comonotonic, weak comonotonic independence applies to R_i. Hence, R_i is a mixture space and satisfies the mixture space axioms. By the mixture space theorem, for each $i = 1, 2$, we have $U_i : R_i \to \mathbb{R}$ such that $\forall f, g \in R_i$, $f \succeq g \iff U_i(f) \geq U_i(g)$, and $\forall \lambda \in [0, 1]$, $U_i(\lambda f + (1 - \lambda)g) = \lambda U_i(f) + (1 - \lambda)U_i(g)$, where U_i's are unique up to affine transformation. Because each U_i is affine, we normalize $U_1(1, 1) = U_2(1, 1) = 1$ and $U_1(-1, -1) = U_2(-1, -1) = -1$.

Claim 30. For every $x \in R$ and $i = 1, 2$, $U_i(x, x) = x$.

Proof. It is immediate for $x \in [-1, 1]$ by the representation. Let $x > 1$, then $\exists!\, \lambda \in (0, 1)$ such that $\lambda(x, x) + (1 - \lambda)(-1, -1) = (1, 1)$. In particular, $\lambda = \frac{2}{x+1}$. By the above, we get for each $i = 1, 2$, $1 = U_i(1, 1) = \lambda U_i(x, x) + (1 - \lambda)U_i(-1, -1) = \lambda U_i(x, x) + (1 - \lambda)(-1)$. Hence, $U_i(x, x) = x$. Similar argument holds for $x < -1$. $\square$

Claim 31. $U_i(\lambda f) = \lambda U_i(f)$ for $\lambda \in R_+$.

Proof. since $U_i(0, 0) = 0$, we have $U_i(\lambda f) = \lambda U_i(f)$ for $\lambda \in [0, 1]$. Also, for $\lambda > 1$, since $U_i(f) = U(\frac{1}{\lambda}\lambda f + (1 - \frac{1}{\lambda})(0, 0))$, we have $U_i(\lambda f) = \lambda U_i(f)$. $\square$

By monotonicity in constant equal allocation, we know that for $i = 1, 2$, $\forall f \in R_i$, there exists a unique $f^* \in R_1 \cap R_2$ such that $f^* \sim f$. Hence, there exists unique x_1^*, x_2^* such that $(0, 1) \sim (x_1^*, x_1^*)$ and $(0, -1) \sim (x_2^*, x_2^*)$. By Inequality aversion and monotonicity in equal allocation, we know that $x_1^*, x_2^* < 0$. Let $\alpha := -x_1^*$ and $\beta := -x_2^*$.

Claim 32. $U_1(f) = x_1 - \alpha(x_2 - x_1), U_2(f) = x_1 - \beta(x_1 - x_2)$.

Proof. $U_1(f) = 2U_1(\frac{1}{2}f) = 2U_1(\frac{1}{2}(x_1, x_1) + \frac{1}{2}(0, x_2 - x_1)) = U_1(x_1, x_1) + U_1(0, x_2 - x_1) = x_1 + (x_2 - x_1)U_1(0, 1) = x_1 - \alpha(x_2 - x_1)$. Similar arguments hold for $U_2(f)$. $\square$

Hence, the following function represents $\succeq$ in $c(\mathcal{F})$

$$W(f) := \begin{cases} U_1(f) = x_1 - \alpha(x_2 - x_1) & \text{if } f \in R_1 \\ \\ U_2(f) = x_1 - \beta(x_1 - x_2) & \text{if } f \in R_2 \end{cases}$$

In other words, $W(f) = x_1 - \alpha(x_2 - x_1)1_{f \in R_1} - \beta(x_1 - x_2)1_{f \in R_2} = x_1 - \alpha \max\{x_2 - x_1, 0\} - \beta \max\{x_1 - x_2, 0\}$. $\square$

The function W can be extended to $W : F \to R$ such that for every $\{f, h\}$ which is comonotonic, within-comonotonic and between-comonotonic, we have for all $\alpha \in [0, 1]$, $W(\alpha f + (1 - \alpha)g) = \alpha W(f) + (1 - \alpha)W(g)$. In addition, $W(.)$ is homogeneous of degree 1.

Proof. For an arbitrary f, we define

$$M_f := \{g \in F : \{f, g\} \text{ is comonotonic, within-comonotonic and between-comonotonic}\}$$

It is easy to check that M_f is non-empty and that $e(\mathcal{F}) \subseteq M_f$. On the other hand, one can check that M_f is convex. i.e. for $g, h \in M_f$ and $\alpha \in [0, 1]$, $\alpha g + (1 - \alpha)h \in M_f$. Checking both within-comonotonicity, between-comonotonicity is straight-forward. For comonotonicity, suppose not, i.e. there exists $g, h \in M_f, \beta \in [0, 1], \omega_1, \omega_2$ such that $f(\omega_1) \succ f(\omega_2)$ but $\beta g(\omega_1) + (1 - \beta)h(\omega_1) \prec \beta g(\omega_2) + (1 -$

$\beta)h(\omega_2)$. Note that since $\{g,f\}$ and $\{h,f\}$ are within-comonotonic, $\{g,h\}$ are within-comonotonic. Hence, there exists $i_j = 1,2$, $j = 1,2$ such that $g(\omega_1), h(\omega_1) \in R_{i_1}$ and $g(\omega_2), h(\omega_2) \in R_{i_2}$. Hence, by Lemma C, for $j = 1,2$, we have $W(\beta g(\omega_j) + (1-\beta)h(\omega_j)) = \beta W(g(\omega_j)) + (1-\beta)W(h(\omega_j))$. Yet, since $\{g,f\}$ and $\{h,f\}$ are comonotonic, as $f(\omega_1) \succ f(\omega_2)$, then it must be that $W(g(w_1)) \geq W(g(w_2))$ and $W(h(w_1)) \geq W(h(w_2))$, and hence it gives $\beta W(g(\omega_1)) + (1-\beta)W(h(\omega_1)) \geq \beta W(g(\omega_2)) + (1-\beta)W(h(\omega_2)) \iff \beta g(\omega_1) + (1-\beta)h(\omega_1) \succeq \beta g(\omega_2) + (1-\beta)h(\omega_2)$. Contradiction arises. Hence, M_f is a mixture space and satisfies the mixture space axioms since weak comonotonic independence applies to M_f. Therefore, there exists a function $J_f : M_f \to \mathbb{R}$ such that we have for all $g, h \in M_f$, $\alpha \in [0,1]$, $J_f(\alpha g + (1-\alpha)h) = \alpha J_f(g) + (1-\alpha)J_f(h)$. We normalised the function such that $J_f(1,1) = 1$ and $J_f(-1,-1) = -1$. Under the same argument in Claim 30, we get $J_f(x,x) = x$. Note that if $h \in M_f \cap M_g$, $J_f(h) = J_g(h)$. To see this, by the Monotonicity in constant equal allocation, there exist $(x_h, x_h) \in e(\mathcal{F})$ such that $(x_h, x_h) \sim h$. Since $e(\mathcal{F}) \subseteq M_f \cap M_g$, we then get $J_f(h) = J_f(x_h, x_h) = x_h = J_g(x_h, x_h) = J_g(h)$. Hence, we write $W(f) = J_f(f)$. The first part is complete. For the second part of the lemma, since an act f is comonotonic, within-comonotonic and between-comonotonic with constant act $(0,0)$, a similar argument as in Claim 31 applies. $\square$

For every $\{f,g\}$ which is anti-comonotonic and within-comonotonic, we have

$W(f+g) = W(f) + W(g)$.

Proof. In light of homogeneity of $W(.)$, it suffices to show $W(\frac{1}{2}f + \frac{1}{2}g) = \frac{1}{2}W(f) + \frac{1}{2}W(g)$ for every $\{f, g\}$ which is anti-comonotonic and within-comonotonic. By Monotonicity in constant equal allocation, for every f, there exists $c_f \in e(\mathcal{F})$ such that, $c_f \sim f$. Note that for every act f in $\mathcal{F}$ and every act $c \in e(f)$, $\{c, f\}$ is anti-comonotonic and within-comonotonic. By Envy-Guilt independence, we have, for every $\{f, g\}$ which is anti-comonotonic and within-comonotonic, $\frac{1}{2}f + \frac{1}{2}g \sim \frac{1}{2}c_f + \frac{1}{2}g$. Hence, with the representation, $W(\frac{1}{2}f + \frac{1}{2}g) = W(\frac{1}{2}c_f + \frac{1}{2}g) = \frac{1}{2}W(c_f) + \frac{1}{2}W(g) = \frac{1}{2}W(f) + \frac{1}{2}W(g)$. The second to last equality is by Lemma C and due to the fact that $\{c_f, g\}$ is comonotonic, within-comonotonic and between-comonotonic for all $g \in \mathcal{F}$. □

Then, we define the function $v : \Omega \to [0,1]$ such that for each $E \in \Sigma$, $v(E) := W((1,1)E^*)$.

We define the function $v_g : \Omega \to [0,1]$ where $v_g(E) := \frac{W((0,-1)E^*)}{W((0,-1)\Omega^*)}$ if $\beta \neq 0$ and also define function $v_e : \Omega \to [0,1]$ where $v_e(E) := \frac{W((0,1)E^*)}{W((0,1)\Omega^*)}$ if $\alpha \neq 0$.

$v(.)$ is a probability measure i.e. 1) for every $E, F \in \Sigma$ with $E \cap F = \emptyset$, $v(E \cup F) = v(E) + v(F)$, and 2) $v(\Omega) = 1$. In addition, $v_g(.)$, $v_e(.)$ are "non-additive" probability measures. i.e. for $i = g, e$ 1) for every $E \subseteq F \in \Sigma$, we have $v_i(E) \leq v_i(F)$, and 2) $v_i(\Omega) = 1$.

Proof. Firstly we show $v(.)$ is a probability measure. We first observe that $v(\Omega) = W((1,1)\Omega^*) = 1$ (by Lemma C). Let $E, F \in \Sigma$ with with $E \cap F = \emptyset$, $v(E \cup F) = W((1,1)(E \cup F)^*) = W((1,1)E^* + (1,1)F^*) = W((1,1)E^*) + W((1,1)F^*) = v(E) + v(F)$. The second to last equality is due to the fact that $\{(1,1)E^*, (1,1)F^*\}$ is anti-comonotonic and within-comonotonic and hence by Lemma C. Therefore, $v(.)$ is a probability measure. On the other hand, consider $v_g(.)$ (hence, $\beta \neq 0$). Note that $v_g(\Omega) = 1$ is obvious. Let $E \subseteq \mathcal{F}$, then by inequality aversion, we have, $(0,-1)E^* \succeq (0,-1)F^* \Rightarrow W((0,-1)E^*) \geq W((0,-1)F^*) \Rightarrow \frac{W((0,-1)E^*)}{W((0,-1)\Omega^*)} \leq \frac{W((0,-1)F^*)}{W((0,-1)\Omega^*)} \Rightarrow v_g(E) \leq v_g(F)$. The second to last inequality is because $W((0,-1)\Omega^*) = -\beta < 0$. Similar reasoning is true for $v_e(.)$. □

For every f, if $\alpha \neq 0$ and $\beta \neq 0$, then

$$W(f) = \int_\Omega f_1(\omega) dv - \alpha \int_\Omega \max\{f_2(\omega) - f_1(\omega), 0\} dv_e - \beta \int_\Omega \max\{f_1(\omega) - f_2(\omega), 0\} dv_g$$

Also, if $\alpha = 0$ and $\beta \neq 0$, $W(f) = \int_\Omega f_1(\omega) dv - \beta \int_\Omega \max\{f_1(\omega) - f_2(\omega), 0\} dv_g$; if $\beta = 0$ and $\alpha \neq 0$, $W(f) = \int_\Omega f_1(\omega) dv - \beta \int_\Omega \max\{f_1(\omega) - f_2(\omega), 0\} dv_g$; if $\alpha = \beta = 0$, $W(f) = \int_\Omega f_1(\omega) dv$.[1]

Proof. We only prove the first case since the last three cases are immediate. We

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[1] Superficially, the last three statements might seem redundant. However, rigorously speaking, if $\alpha = 0$, then v_e is undefined from the construction, the expression in the first statement will be undefined.

think of act f as "step" function with its value in R^2 and its indicator function with E^* such that it has a canonical representation in that $E_i \cap E_j = \emptyset$ and the values associated with each E_i are distinct. We prove by induction with the number of steps different from zero that the f have.

For $n = 1$, $L.H.S. = W((\delta_1, \delta_2)E^*) = W((\delta_1, \delta_1)E^* + (0, \delta_2 - \delta_1)E^*) = W((\delta_1, \delta_1)E^*) + W((0, \delta_2 - \delta_1)E^*)$

$$= \begin{cases} \delta_1 W((1,1)E^*) + (\delta_1 - \delta_2)W((0,-1)E^*) \text{ if } \delta_1 > \delta_2 \\ \delta_1 W((1,1)E^*) + (\delta_2 - \delta_1)W((0,1)E^*) \text{ if } \delta_1 < \delta_2 \\ \delta_1 W((1,1)E^*) \text{ otherwise} \end{cases}$$

$$= \begin{cases} \delta_1 v(E) - \beta(\delta_1 - \delta_2)v_g(E) \text{ if } \delta_1 > \delta_2 \\ \delta_1 v(E) - \alpha(\delta_2 - \delta_1)v_e(E) \text{ if } \delta_1 < \delta_2 \\ \delta_1 v(E) \text{ otherwise} \end{cases}$$

$= R.H.S.$ The third equality is by Lemma C. Then, assume it is true for $n = k - 1$. Let f have k steps different from zero. There are two cases. We first define the following notion which is helpful for the following proof: f is said to be self-comonotonic if there is no ω, ω', such that $f_1(\omega) > f_2(\omega)$ and $f_1(\omega') < f_2(\omega')$.

Case 1: f is not self-comonotonic. In this case, f is a "sum" of two acts, where

$g = fE^*(0,0)$ and $h = (0,0)E^*f$, such that $\{g,h\}$ is anti-comonotonic and within-comonotonic. By Lemma C, we have $W(f) = W(g) + W(h)$. Note that since both g and h have less than k steps away from zero, by induction hypothesis, we have the representation for each $W(g)$ and $W(h)$ and the induction completes.

Case 2: f is self-comonotonic. assume first that $f_1(\omega) > f_2(\omega)$ for all ω. Let $f = \sum_i^k (\delta_1^i, \delta_2^i)E_i^*$, and we rank them such that $\delta_1^1 - \delta_2^1 > \delta_1^2 - \delta_2^2 > ... > \delta_k^1 - \delta_2^k$. Hence, we have $f = g + h$ where $g = \sum_i^{k-1}(\delta_1^i - \delta_1^k, \delta_2^i - \delta_2^k)E_i^*$ and $h = (\delta_1^k, \delta_2^k)(\cup_{i=1}^n E_i)^*$, and

$$\begin{aligned} W(g) = &\sum_{i=1}^{k-1}(\delta_1^i - \delta_1^k)v(E_i) - \beta\sum_{i=1}^{k-1}\left(\left((\delta_1^i - \delta_1^k) - (\delta_2^i - \delta_2^k)\right) - \right. \\ &\left.\left((\delta_1^{i+1} - \delta_1^k) - (\delta_2^{i+i} - \delta_2^k)\right)\right)v_g(\cup_{j=1}^i E_j) \\ = &\sum_{i=1}^{k-1}(\delta_1^1 - \delta_1^k)v(E_i) - \beta\sum_{i=1}^{k-1}\left((\delta_1^i - \delta_2^i) - (\delta_1^{i+1} - \delta_2^{i+1}))\right)v_g(\cup_{j=1}^i E_j) \end{aligned}$$

and $W(h) = \delta_1^k v(\cup_{i=1}^k E_i) - \beta(\delta_1^k - \delta_2^k)v_g(\cup_{i=1}^k E_i)$

Note that since $\{g,h\}$ is comonotonic, within-comonotonic and between-comonotonic, by lemma C, $W(f) = W(g) + W(h)$. The other case $f_1(\omega) < f_2(\omega)$ is similar. Hence, the induction is complete. Since the number of states in Ω is finite, it completes the lemma. □

The last lemma completes the sufficient part of the proof. The necessary part and uniqueness are straight-forward. $\square$

Proof for Proposition 13

Proof. We first prove risk aversion for others in the guilt domain. Order the lottery l' such that $k_1 < ... < k_n$ and hence we have $a-k_1 > ... > a-k_n$. We prove the "only if" part first. Fix p, we consider the lottery $l' = \Big(p, (a, k_1); 1-p, (a, k_2)\Big)$. Note that by risk aversion for others in the guilt domain, we have $W(l) \geq W(l') \iff a - \beta(a-b) \geq a - \beta((a-k_1)-(a-k_2))\pi_g(p) - \beta(a-k_2)\pi_g(1) \iff \pi_g(p)(k_1-k_2) \leq b - k_2 \iff \pi_g(p) \geq p$. The second to last inequality is because $pk_1 + (1-p)k_2 = b$. The reverse inequality is also true for risk loving for others in the guilt domain. Hence the "only if" part is complete since p is arbitrary. For the "if" part, we prove by induction on the number of supports of the lottery l'. We first consider the case of risk aversion for others in the guilt domain. Note that it is trivially true for $n = 1$. Assume it is true for $n = t-1$.

$$\begin{aligned}
W(l') &= a - \beta[(a-k_1)-(a-k_2)]\pi_g(p_1) - \beta[(a-k_2)-(a-k_3)]\pi_g(p_1+p_2)... - \beta(a-k_t) \\
&\leq a - \beta[(a-k_1)-(a-k_2)]\pi_g(p_1) - \beta[a - [(p_1+p_2)k_2 + p_3k_3 + ... + p_tk_t]] \\
&= a - \beta[(a-k_1)-(a-k_2)]\pi_g(p_1) - \beta[a-k_2] - \beta k_2(1-p_1) + \beta(p_2k_2 + p_3k_3... + p_tk_t) \\
&\leq a - \beta(a - p_1k_1 - (1-p_1)k_2) - \beta k_2(1-p_1) + \beta(p_2k_2 + p_3k_3... + p_tk_t)
\end{aligned}$$

$$= a - \beta(a - \sum_{i=1}^{t} p_i k_i) = W(l)$$

Note that the less-than inequalities from induction hypothesis can be reversed for the case of risk loving for others in the guilt domain. Hence the "if" part is complete. On the other hand, the proof for risk aversion/risk loving for others in the envy domain is similar. The proof is complete. $\square$

Proof for Proposition 14

Proof. (i) implying (ii) is by definition since risk aversion for others is a stronger condition. For (ii) implying (i), we first consider the case that $a > b$. We know every l' can be written as

$$l' = \Big(p_1, (a, k_1); p_2, (a, k_2); ...; p_n, (a, k_n); q_1, (a, h_1); q_2, (a, h_2); ...; q_m, (a, h_m)\Big)$$

, where $a - k_i > 0$ and $h_i - a > 0$ for all i and $k_1 < ... < k_n$ and $h_1 > ... > h_m$. Hence, for the sake of notation, defining $(a-k)_{n+1} = 0$, $(a-k)_i = a - k_i$ for all $i = 1, ..., n$ and $(h-a)_{m+1} = 0$, $(h-a)_i = a - h_i$ for all $i = 1, ..., m$, we have

$$W(l') = a - \beta \sum_{i=1}^{n} [(a-k)_i - (a-k)_{i+1}] \pi_g (\sum_{j=1}^{n} p_j) - \alpha \sum_{i=1}^{m} [(h-a)_i - (h-a)_{i+1}] \pi_e (\sum_{j=1}^{m} q_j)$$

$$\leq a - \beta(a - \sum_{i=1}^{n} p_i k_i) - \alpha(\sum_{i=1}^{m} q_i h_i - a) \text{ by Proposition 13}$$
$$\leq a - \beta(a - \sum_{i=1}^{n} p_i k_i) \leq a - \beta(a - \sum_{i=1}^{n} p_i k_i) - \beta(a - \sum_{i=1}^{m} q_i h_i) = W(l)$$

The case for $a < b$ is similar. The proof is complete. ☐

Proof for Proposition 15

Proof. Let the wealth of mine be w_m and the wealth of you be w_y. One need to solve the following maximization problem. The investment problem for oneself is given by $\max_{x \in [0, w_m]} \frac{2}{3} u(w_m - x) + \frac{1}{3} u(w_m + 2.5x)$. By solving, one get $x_{me} = \frac{1}{3.5\lambda_{me}} \ln \frac{5}{4}$. On the other hand, the investment for others in the guilt domain is given by $\max_{x \in [0, m_y]} -\pi_g(\frac{2}{3}) m_g(w_m - (w_y - x)) - (1 - \pi_g(\frac{2}{3})) m_g(w_m - (w_y + 2.5x))$, where $\pi_g(x) = x^{\frac{1}{k_g}}$. By solving, we get $x_g = \frac{1}{3.5\lambda_g} \ln(2.5 * \frac{1 - \frac{2}{3}^{\frac{1}{k}}}{\frac{2}{3}^{\frac{1}{k}}})$. It can be easily shown that $x_g \geq x_m$ if and only if $k_g \leq 1$. Lastly, the investment problem for others in the envy domain is given by $\max_{x \in [0, w_y]} -(1 - \pi_e(\frac{1}{3})) m_e((w_y - x) - w_m) - \pi_e(\frac{1}{3}) m_e((w_y + 2.5x) - w_m)$, where $\pi_e(x) = x^{\frac{1}{k_e}}$. By solving, we get $x_e = \frac{1}{3.5\lambda_e} \ln(\frac{1}{2.5} * \frac{1 - \frac{1}{3}^{\frac{1}{k}}}{\frac{1}{3}^{\frac{1}{k}}})$. It can be easily shown that $x_e \geq x_m$ if and only if $k_e \leq \frac{\ln(1/3)}{\ln(8/33)}$. ☐

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