ABSTRACT

Title of Dissertation: LARGE-SCALE BEHAVIOR OF SOME STOCHASTIC PDES

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In this dissertation, we include four projects that study the large-scale behavior of some stochastic PDEs. The stochastic PDEs that we study are the Kardar-Parisi-Zhang (KPZ) equation and the stochastic heat equation (SHE).

The Kardar-Parisi-Zhang equation is a nonlinear stochastic PDE, known as the default model for random interface growth in physics. To study the Kardar-Parisi-Zhang equation, we use the approach of Hopf-Cole transform, which relates the KPZ equation to the stochastic heat equation. In spatial dimension one, the KPZ equation and the stochastic heat equation have received extensive studies, but the equations that are boundary driven or in higher dimensions remain more mysterious.

In the first part of this dissertation, we consider the stochastic heat equation and KPZ equation in spatial dimension two, which is a critical dimension, and the solutions can only be made sense as scaling limits. In the first project, we consider a nonlinear version of the stochastic heat equation and prove its limiting second-order Gaussian fluctuation. This result has been published

in paper [1]. In the second project, we show a mesoscopic averaging phenomenon for the local averages of the two-dimensional KPZ equation. This result has been published in [2].

The second major part of this dissertation is a project that considers the KPZ equation in a one-dimensional half-space domain with a Neumann boundary condition. This half-space KPZ model exhibits a “depinning” phase transition as the boundary condition changes. We study the half-space KPZ equation starting from stationary Brownian initial data. We obtain its optimal fluctuation exponents in both the subcritical and critical regimes of the phase transition, and give an optimal upper bound for the fluctuation exponents in the supercritical regime. We also compute the average growth rate as a function of the boundary parameter. This result has been submitted for publication in paper [3].

In the last part of this dissertation, we also include a work that studies the time-dependent spatial averages of a long-range critically correlated stochastic heat equation in spatial dimension three or higher. These spatial averages have different limits under different space-time scales. This result has been included in preprint [4].

LARGE-SCALE BEHAVIOR OF SOME STOCHASTIC PDES

by

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Chapter 1: Introduction

The *Kardar-Parisi-Zhang (KPZ) equation* is a nonlinear stochastic partial differential equation (SPDE), expressed in dimensionless units and after scaling as

$$\partial_t h(t,x) = \frac{1}{2}\Delta h(t,x) + \frac{1}{2}|\nabla h(t,x)|^2 + \xi(t,x), \tag{1.0.1}$$

with ξ denoting the standard space-time white noise, which is a distribution-valued Gaussian field with a formal correlation function

$$\mathbb{E}\left[\xi(t,x)\xi(t',x')\right] = \delta_0(t-t')\delta_0(x-x'),$$

with δ_0 being the Dirac mass.

The equation (1.0.1) models a height function of the interface so that its growth is a combination of the smoothing effect from the heat kernel, a slope-dependent mechanism governed by the quadratic term, and a random forcing modeling the impurities in nature. It is the default random interface growth model in statistical physics, first introduced by Kardar, Parisi, and Zhang in 1986 [5] and is a singular SPDE due to the roughness of the noise. Mathematically, this equation (1.0.1) is ill-posed. The main problem comes from the nonlinear term ∇h. With roughness from the noise ξ, for any spatial dimension $d \geq 1$, the gradient of h is not function-valued, and thus its square is not mathematically well-defined.

In 1995, Bertini and Giacomin [6] proposed that one way to have (1.0.1) well-defined in the $(1+1)$-dimension (i.e. $(t,x)\in[0,\infty)\times\mathbb{R}$) is through the Hopf-Cole transform, which means that one obtains a physical solution to (1.0.1) by letting $h(t,x)=\log u(t,x)$, where $u(t,x)$ is the solution to the *stochastic heat equation (SHE)*

$$\partial_t u(t,x)=\frac{1}{2}\Delta u(t,x)+u(t,x)\xi(t,x). \tag{1.0.2}$$

It is not hard to see that at least formally, ignoring the Itô correction, $\log u(t,x)$ satisfies (1.0.1). In fact, the Hopf-Cole solution arises as the weak noise limit of discrete and continuum models in the KPZ universality class.

Through the Hopf-Cole transform, the height function $h(t,x)$ can also be interpreted as the free energy of the *continuum directed random polymer* model, which is also a model in the KPZ universality class.

Over the last decade, much progress has been made on studying the KPZ equation and the KPZ universality class in noticable works such as [7,8]. Around the same time, Martin Hairer first proposed the theory of regularity structures [9,10] to construct and analyze solutions to (1.0.1) in dimension $(1+1)$ bypass the Hopf-Cole transformation. Since then, there have also been other direct approaches, such as through paracontrolled distributions [11,12], energy solutions [13,14], and more recently, flow equation [15]. Nevertheless, the study of KPZ equation (1.0.1) via the Hopf-Cole transformation is still powerful and it is our approach in this dissertation.

Despite the extensive research in the literature, KPZ systems that are boundary driven in dimension $(1+1)$ or stochastic heat equations and KPZ equations in higher dimensions remain more mysterious. In this dissertation, we study the large-scale behavior of several stochastic heat equations and KPZ equations that fall in those two categories.

In the first part of this dissertation, we present two works [1,2] that are related to the two-dimensional ($(t,x) \in [0,\infty) \times \mathbb{R}^2$) stochastic heat equation and KPZ equations in Chapter 2 and 3. A short summary of these works can be found in Section 1.1 below.

In the second part of this dissertation, we study the stationary KPZ equation in half-space ($(t,x) \in [0,\infty) \times [0,\infty)$) with Neumann boundary condition at $x = 0$. This is based on a joint work [3] with Yu Gu and is presented in Chapter 4. A detailed overview of this work can be found in Section 1.2 below.

Finally, in the last part of this dissertation, we also include a work [4] with Sefika Kuzgun in Chapter 5 that studies a time-dependent spatial average of a critical long-range stochastic heat equation in dimension $d \geq 3$. We refer to Section 1.3 for an overview.

Before we proceed, we would like to mention that the main tools in our research are various probabilistic tools, including but not limited to the Malliavin calculus with Stein's method, Gaussian integration by parts, chaos expansions, concentration inequalities, hypercontractivity, and Karlin-McGregor type arguments.

1.1 SHE and KPZ in $d = 2$

We consider (1.0.1) and (1.0.2) in the $(1+d)$-dimensional full-space domain: $(t,x) \in [0,+\infty) \times \mathbb{R}^d$. By a computation with the d-dimensional heat kernel, one can easily show that the stochastic heat equation (1.0.2) is only well-posed for $d = 1$. For dimension $d \geq 2$, (1.0.2) does not have function-valued solutions, and thus the Hopf-Cole transformation $h(t,x) = \log u(t,x)$

will not make sense. This makes the problem of stochastic heat equation and KPZ equation in higher dimensions naturally harder than in one dimension.

Among these higher dimensions, the dimension $d = 2$ is a critical case as it also differs from $d \geq 3$. To see a nontrivial evolution, one needs to take an exponential time scale in $d = 2$, while in contrast, one can consider an arbitrarily long diffusive scale for $d \geq 3$. We will focus on the dimension $d = 2$ in the following. Analogous results for $d \geq 3$ are in literature [16, 17].

One of the first fundamental work for the $(1+2)$ dimension SHE is conducted by Caravenna, Sun and Zygouras in [18], in which they considered the limit of the following regularized equation as $\varepsilon \to 0$

$$\partial_t u^\varepsilon(t,x) = \frac{1}{2}\Delta u^\varepsilon(t,x) + \frac{\beta}{\sqrt{\log \varepsilon^{-1}}} u^\varepsilon(t,x)\xi_\varepsilon(t,x). \tag{1.1.1}$$

Here $\beta > 0$ is a constant. The function $\phi \in C_c^\infty(\mathbb{R}^2)$ is a non-negative smooth mollifier with $\int \phi \mathrm{d}x = 1$. The regularized noise $\xi_\varepsilon(t,x) = \phi^\varepsilon \star \xi(t,x)$ is the space-time white noise ξ convoluted in space with $\phi^\varepsilon(x) := \frac{1}{\varepsilon^2}\phi(\frac{x}{\varepsilon})$. The equation (1.1.1) is then well-posed, and the $\frac{1}{\sqrt{\log \varepsilon^{-1}}}$ scaling of the disorder strength is crucial in $d = 2$.

In [18], the authors proved the following result:

Theorem 1.1.1 ([18], Theorem 2.15)**.** *For any $t > 0$ and $x \in \mathbb{R}^2$, if $u^\varepsilon(0,x) = u_0(x) = 1$, $\beta \in (0, \sqrt{2\pi})$, $u^\varepsilon(t,x)$ converges in distribution to the log-normal distribution $e^{Y - \frac{1}{2}\mathbb{E}(Y)^2}$, where $Y = (\log \frac{2\pi}{2\pi - \beta^2})Z$ and Z is a standard Gaussian, as $\varepsilon \to 0$. If $\beta \geq \sqrt{2\pi}$, $u^\varepsilon(t,x)$ converges to 0 in probability as $\varepsilon \to 0$.*

Now we can use a Hopf-Cole transformation to investigate the two-dimensional KPZ equation. If we consider the mollified KPZ equation

$$\partial_t h^\varepsilon(t,x) = \frac{1}{2}\Delta h^\varepsilon(t,x) + \frac{1}{2}|\nabla h^\varepsilon(t,x)|^2 + \frac{\beta}{\sqrt{\log \varepsilon^{-1}}}\xi_\varepsilon(t,x) - C_\varepsilon, \tag{1.1.2}$$

with initial condition $h^\varepsilon(0,x) = h_0(x) = 0$ and $C_\varepsilon = \frac{\beta^2}{\varepsilon^2 \log \varepsilon^{-1}}\|\phi\|_{L^2(\mathbb{R}^2)}$, we have $h^\varepsilon(t,x) = \log u^\varepsilon(t,x)$.

Another result in [18] is on the fluctuation of the two-dimensional SHE. The authors showed that in the regime when $\beta < \sqrt{2\pi}$, the solution to the two-dimensional SHE, as a random field in $\mathbb{R}^2$, has a Gaussian fluctuation:

Theorem 1.1.2 ([18], Theorem 2.17)**.** *Let $u^\varepsilon(t,x)$ be as in Theorem 1.1.1 with $\beta < \sqrt{2\pi}$. Fix $t > 0$, for any fixed Schwartz function $g \in C_c^\infty(\mathbb{R}^2)$, the random variable*

$$\sqrt{\log \varepsilon^{-1}} \int_{\mathbb{R}^2} [u^\varepsilon(t,x) - 1]\, g(x) \mathrm{d}x$$

converges in distribution to a Gaussian distribution

$$\int_{\mathbb{R}^2} U(t,x) g(x) \mathrm{d}x$$

as $\varepsilon \to 0$. The random distribution U is the solution to the Edwards-Wilkinson equation in dimension two:

$$\partial_t U(t,x) = \frac{1}{2}\Delta U(t,x) + \beta\sqrt{\frac{2\pi}{2\pi - \beta^2}}\xi(t,x), \qquad U(0,x) = 0.$$

1.1.1 Fluctuation of nonlinear SHE

In Chapter 2, we are interested in an analog result to the above Theorem 1.1.2 for a nonlinear SHE. Namely, we consider a stochastic heat equation where the noise term is multiplied by $\sigma(u^\varepsilon(t,x))$ instead of $u^\varepsilon(t,x)$:

$$\partial_t u^\varepsilon(t,x) = \frac{1}{2}\Delta u^\varepsilon(t,x) + \frac{\beta}{\sqrt{\log \varepsilon^{-1}}}\sigma(u^\varepsilon(t,x))\xi_\varepsilon(t,x). \tag{1.1.3}$$

Here $\sigma : [0,\infty) \to [0,\infty)$ is a globally Lipschitz function with $\sigma(0) = 0$, $\sigma(1) \neq 1$ and a Lipschitz constant σ_{Lip}. The motivation behind this problem is that, by studying a nonlinear SHE, we hope it would shed light on the study of more general KPZ-type stochastic partial differential equations.

In Chapter 2, we present the work [1], in which we proved the following theorem.

Theorem 1.1.3 ([1]). *There exists some $\beta_0 \in (0, \frac{\sqrt{2\pi}}{\sigma_{\mathrm{Lip}}})$ such that if $\beta < \beta_0$, for any fixed $t > 0$ and any fixed Schwartz function $g \in C_c^\infty(\mathbb{R}^2)$, the random variable*

$$\mathrm{X}^{\varepsilon,t}(g) := \sqrt{\log \varepsilon^{-1}} \int_{\mathbb{R}^2} [u^\varepsilon(t,x) - 1]\, g(x)\mathrm{d}x$$

converges in law to a Gaussian distribution

$$\mathrm{X}^t(g) := \int_{\mathbb{R}^2} U(t,x)g(x)\mathrm{d}x$$

as $\varepsilon \to 0$. Here U is the random distribution that solves the Edwards-Wilkinson equation in *dimension two:*

$$\partial_t U = \frac{1}{2}\Delta U + \beta\sqrt{\mathbb{E}\sigma\left(\Xi_{1,2}(2)\right)^2}\mathrm{d}W(t,x), \qquad U(0,x) = 0,$$

where $\Xi_{1,2}(\cdot)$ is a random process defined as in [19] by a forward-backward stochastic differential equation (FBSDE):

Let $\{B(q)\}_{q\geq 0}$ be a 1D standard Brownian motion with the natural filtration $\{\mathcal{G}_q\}_{q\geq 0}$. Then $\Xi_{1,2}(\cdot)$ is the (unique) solution to the FBSDE:

$$\mathrm{d}\Xi_{1,2}(q) = \frac{\beta}{2\sqrt{\pi}}\left(\mathbb{E}[\sigma^2(\Xi_{1,2}(2)) \mid \mathcal{G}_q]\right)^{1/2}\mathrm{d}B(q), \qquad q \in (0,2]; \quad \Xi_{1,2}(0) \quad = 1.$$

Our proof was based on the one-point and multi-point convergence results for the nonlinear SHE in [19]. The methods once used in [18] to obtain Theorem 1.1.2, namely the Feynman-Kac formula and the Wiener Chaos expansion, are no longer valid in the nonlinear setting. Instead, the Malliavin-Stein's method inspired by work [17] is used here.

1.1.2 Mesoscopic averaging of KPZ

In Chapter 3, we are interested in the two-dimensional KPZ equation, which is the limit of $h^\varepsilon(t,x)$ in (1.1.2) as $\varepsilon \to 0$ in the regime $\beta \in (0,\sqrt{2\pi})$. By Theorem 1.1.1 and the continuous mapping theorem, one can easily obtain a multi-scale point-wise asymptotic limit of $h^\varepsilon(t,x)$, and in [20], the authors proved a second-order Edwards-Wilkinson fluctuation for the two-dimensional KPZ equation (as a random field on $\mathbb{R}^2$).

Inspired by Chatterjee's work for the KPZ equation in $d \geq 3$ [21], we investigated the limit of a local average of the mollified KPZ equation (1.1.2) when $\beta \in (0, \sqrt{2\pi})$ with a general initial condition h_0. We showed that when $h^\varepsilon(t,x)$ is averaged in space in a proper scale, its limit would equal in distribution to the sum of the solution to a deterministic KPZ equation and a Gaussian random variable. The deterministic part only depends on the initial condition h_0. The Gaussian random variable only depends on the scale of averaging. Our result shows a "mesoscopic" averaging phenomenon that appears exclusively in dimension two. This result is presented in [2] as the following:

Theorem 1.1.4 ([2]). *Let $h^\varepsilon(t,x)$ be the solution to the mollified two-dimensional KPZ equation (1.1.2) with the initial condition $h_0 : \mathbb{R}^2 \to \mathbb{R}$ being bounded and Lipschitz continuous. Let $0 < \beta < \sqrt{2\pi}$ and $r_\varepsilon = \varepsilon^{1-\gamma}$ for some $0 \le \gamma \le 1$. We use $B(x, r_\varepsilon)$ to denote a ball with center x and radius r_ε.*

For any fixed $t > 0$, $x \in \mathbb{R}^2$, the local average of h^ε over $B(x, r_\varepsilon)$ converges as follows

$$\begin{aligned}&\frac{1}{|B(x,r_\varepsilon)|}\int_{|y-x|\le r_\varepsilon} h^\varepsilon(t,y)\mathrm{d}y\\ &\xrightarrow{d} \mathcal{N}(0,\sigma_\gamma^2) - \frac{1}{2}\log\left(\frac{2\pi}{2\pi-\beta^2}\right) + \begin{cases}\bar{h}(t,x), & \text{if } 0\le\gamma<1,\\ \frac{1}{|B(x,1)|}\int_{|y-x|\le 1}\bar{h}(t,y)\mathrm{d}y, & \text{if } \gamma = 1,\end{cases}\end{aligned} \tag{1.1.4}$$

as $\varepsilon \to 0$, where $\bar{h}(t,x)$ is the solution to the two-dimensional deterministic KPZ equation

$$\partial_t \bar{h}(t,x) = \frac{1}{2}\Delta\bar{h}(t,x) + \frac{1}{2}|\nabla\bar{h}(t,x)|^2, \quad \bar{h}(0,x) = h_0(x), \tag{1.1.5}$$

and $\mathcal{N}(0,\sigma_\gamma^2)$ is a normal distribution with mean 0 and variance $\sigma_\gamma^2 = \log\left(\frac{2\pi-\beta^2\gamma}{2\pi-\beta^2}\right)$. When $0 \le \gamma < 1$, we have $\sigma_\gamma^2 > 0$. We treat $\mathcal{N}(0,0)$ as constant zero, when $\gamma = 1$ and $\sigma_\gamma^2 = 0$.

If $\gamma = 0$, the averaging was performed over a ball with radius $r_\varepsilon = \varepsilon$. The limit in distribution is

$$\bar{h}(t,x) + \mathcal{N}(0, \log \frac{2\pi}{2\pi - \beta^2}) - \frac{1}{2}\log\left(\frac{2\pi}{2\pi - \beta^2}\right),$$

which is a generalization of the point-wise limit of KPZ equation to bounded and Lipschitz continuous initial conditions. The case $\gamma = 0$ can be viewed as a "microscopic" averaging result.

If $\gamma = 1$, the averaging was performed over a ball with radius $r_\varepsilon = 1$. The limit is then deterministic. It equals to a spatial average of the deterministic KPZ equation over the ball of radius 1, plus a height shift $-\frac{1}{2}\log\left(\frac{2\pi}{2\pi-\beta^2}\right)$. Due to the independence in the limit of h^ε for distinct points, taking a local average over a fixed-radius ball eliminates the randomness as a result of the law of large numbers. The height shift $-\frac{1}{2}\log\left(\frac{2\pi}{2\pi-\beta^2}\right)$ is the mean term remained. The case $\gamma = 1$ can be viewed as a "macroscopic" averaging result.

If $0 < \gamma < 1$, the averaging was performed over a ball with radius $r_\varepsilon = \varepsilon^{1-\gamma}$, where $\varepsilon \ll r_\varepsilon \ll 1$. The limit in distribution is the deterministic KPZ equation with the same height shift, plus a Gaussian random variable. The variance of the Gaussian, σ_γ^2, is dependent solely on γ. This is an interpolation between the limiting case $\gamma = 0$ and $\gamma = 1$ and is a "mesoscopic" averaging result.

The main idea of this mesoscopic averaging phenomenon and its proof stemmed from the following observation: the random part in the limit of $h^\varepsilon(t,x)$ depends only on the white noise ξ in an infinitesimal time window $[t - o(1), t]$ as $\varepsilon \to 0$, and it is independent of the initial condition h_0. Thus, we can split $h^\varepsilon(t,x)$ into two components: a "random" quantity which is the solution to the KPZ equation on time window $[t - o(1), t]$ with zero initial condition, and an "almost deterministic" quantity that concentrates to a deterministic value when $\varepsilon \to 0$. The idea of this decomposition was first mentioned in [18] and later applied in [21].

1.2 Half-space KPZ in $d = 1$

In Chapter 4 of the dissertation, we study the half-space variant of the KPZ equations. This is the KPZ equation (1.0.1) restricted to the half line $(t, x) \in [0, +\infty) \times [0, \infty)$, with a Neumann boundary condition with parameter $u \in \mathbb{R}$ imposed on $x = 0$:

$$\begin{aligned} \partial_t H_u(t,x) &= \frac{1}{2}\partial_x^2 H_u(t,x) + \frac{1}{2}\left(\partial_x H_u(t,x)\right)^2 + \xi(t,x), \\ \partial_x H_u(t,x)\big|_{x=0} &= u. \end{aligned} \tag{1.2.1}$$

The solution to equation (1.2.1) can be interpreted as a half-space KPZ height function, as well as the free energy to a half-space continuum directed random polymer model. The continuum directed random polymer model is a model in the KPZ universality class.

The half-space KPZ models are related to a growing interface near an attractive wall, which arises naturally from the studies of wetting and entropic repulsion phenomena. In his pioneering work [22], Kardar introduced the half-space KPZ models to describe such interface growth and he predicted a "depinning" phase transition as the wall attraction weakens for these half-space KPZ models. The underlying concept is as follows: when the wall exerts a strong attractive force, which is also referred to as in the subcritical regime or the bound phase, the half-space directed random polymer would be "pinned" to the wall due to the force; as this attractive force decreases below a certain threshold, the polymer becomes entropically repelled away from the wall. It is expected that when "pinned", the polymer endpoint has $O(1)$ transversal fluctuation in a window around the wall and the free energy of the polymer fluctuates with an exponent $1/2$. After being "unpinned", the polymer exhibits KPZ behaviors: the free energy fluctuates with exponent $1/3$ and the endpoints have transversal fluctuation exponent $2/3$. We are interested to demonstrate this phase transition for (1.2.1).

The boundary conditions in (1.2.1) do not make a priori sense as the solutions of KPZ equations are not differentiable. We take care of this through a Hopf-Cole transform.

Let $Z_\mu(t,x)$ with $(t,x) \in [0,\infty) \times [0,\infty)$ be the solution to the half-space stochastic heat equation with Robin boundary condition. More precisely, $Z_\mu(t,x)$ is defined through a mild formulation

$$Z_\mu(t,x) = \int_{\mathbb{R}_{\geq 0}} \mathscr{P}_\mu^R(0,y;t,x) Z_\mu(0,y) \mathrm{d}y + \int_0^t \int_{\mathbb{R}_{\geq 0}} \mathscr{P}_\mu^R(s,y;t,x) Z_\mu(s,y) \xi(s,y) \mathrm{d}y \mathrm{d}s, \quad (1.2.2)$$

where the stochastic integral is in Itô sense. Here the kernel $\mathscr{P}_\mu^R(s,y;t,x)$ is the Robin heat kernel on half-line with parameter μ:

$$\mathscr{P}_\mu^R(s,y;t,x) = p_{t-s}(x-y) + p_{t-s}(x+y) - 2\mu \int_0^{+\infty} p_{t-s}(x+y+z) e^{-\mu z} \mathrm{d}z, \quad (1.2.3)$$

where $p_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/(2t)}$ is the standard continuous heat kernel for the heat operator $\partial_t - \frac{1}{2}\partial_x^2$ on $\mathbb{R}$. We then define the solution to (1.2.1) for $(t,x) \in [0,\infty) \times [0,\infty)$ by letting

$$H_u(t,x) := \log Z_{u-\frac{1}{2}}(t,x).$$

While this model is of great interest to physicists and mathematicians, less literature is available on it until recently. One may refer to [23–27], etc. for some recent research on this topic. In the language of the polymer, the half-line KPZ equation is equivalent to a continuum directed polymer on a half-space with a wall at the origin $x = 0$, which is repulsive for $u - \frac{1}{2} > 0$, and attractive for $u - \frac{1}{2} < 0$. A remarkable feature of the half-space KPZ equation is the existence of a critical value u_c of the parameter u at which a phase transition occurs. This phase transition was first predicted by Kardar in 1985 [22] for half-space KPZ models. Recently, in [28], the authors proved the occurrence of the phase transition at $u = 0$ for (1.2.1) when the initial condition is set to equal to the logarithm of the Dirac mass δ.

We will focus on the half-space KPZ equations (1.2.1) in stationary. In [26], the authors constructed a one-parameter family of stationary measures for (1.2.1) for each choice of boundary parameter u. In particular, when the initial condition is

$$H_u(0,x) = W_u(x) := W(x) + ux,$$

where $W(x)$ is a Brownian motion on $[0,\infty)$ starting from 0 and independent of ξ, our (1.2.1) is starting at a stationary measure. By stationary, we mean that the increment process $x \mapsto H_u(t,x) - H_u(t,0)$ is stationary. While the KPZ height function $H_u(t,x)$ is not stationary in t, when $H_u(0,x) = W_u(x)$, for all $t > 0$, $H_u(t,x) - H_u(t,0)$ is equal to $H_u(0,x) - H_u(0,0) = W_u(x)$ in law, as a process of $x \in [0,\infty)$.

In the work [3] with Yu Gu, we utilize this stationarity and the Brownian structure in the initial condition to deploy an integration by part technique previously used in [29]. As a result, we derived a variance identity that links the fluctuations of the height function to the transversal fluctuations of a half-space polymer model.

Let $p^{\mathrm{R}}_{u-\frac{1}{2}}(y|t,0)$ be defined as the quenched density of the half-space continuum directed polymer starting from $(t,0)$ and running backward in time with boundary $\exp(W_u(\cdot))$:

$$p^{\mathrm{R}}_{u-\frac{1}{2}}(y|t,x) := \frac{\mathcal{Z}_{u-\frac{1}{2}}(t,x|0,y)\exp\left(W_u(y)\right)}{\int_0^{+\infty}\mathcal{Z}_{u-\frac{1}{2}}(t,x|0,y')\exp\left(W_u(y')\right)\mathrm{d}y'}.$$

Here $\mathcal{Z}_{u-\frac{1}{2}}(t,x|s,y)$ is the Green's function of the half-space stochastic heat equation with boundary parameter $u-\frac{1}{2}$. In particular, $\mathcal{Z}_{u-\frac{1}{2}}(t,x|s,y)$ satisfies the following mild formulation:

$$\mathcal{Z}_{u-\frac{1}{2}}(t,x|s,y) = \mathscr{P}^R_{u-\frac{1}{2}}(t,x|s,y) + \int_s^t \int_0^\infty \mathscr{P}^R_{u-\frac{1}{2}}(t,x|r,z)\mathcal{Z}_{u-\frac{1}{2}}(r,z|s,y)\xi(r,z)\mathrm{d}z\,\mathrm{d}r.$$

In [3], we are able to show the following variance identity (see Section 4.4 with some different notations):

Theorem 1.2.1 ([3]). *For any $u\in\mathbb{R}, t\geq 0$,*

$$\mathbb{V}ar\left[H_u(t,0)\right] = 2\mathbb{E}\int_0^{+\infty} y\cdot p^R_{u-\frac{1}{2}}(y|t,0)\mathrm{d}y - ut. \tag{1.2.4}$$

With the above variance identity, one can observe that there would be a phase transition at $u=0$, as what was previously predicted in the physics literature [25, 30].

Furthermore, one can obtain the fluctuation exponents for the height function by estimating the annealed mean of the polymer endpoints. In particular, we prove an optimal bound to the transversal endpoint when $u<0$ as the following (see Section 4.5):

Theorem 1.2.2 ([3]). *Assume $u<0$. We have*

$$0\leq \mathbb{E}\int_0^{+\infty} y\cdot p^R_{u-\frac{1}{2}}(y|t,0)\mathrm{d}y \leq \psi'(-2u), \tag{1.2.5}$$

where ψ is the digamma function and as $z\to 0^+, \psi'(z)\sim\frac{1}{z^2}$.

Combing (1.2.4) and (1.2.5), we obtain the optimal fluctuation exponent $1/2$ for the height function $H_u(t,0)$ in the subcritical regime $u < 0$.

Furthermore, through some scaling, an additional symmetry argument in Section 4.6, and an additional comparison argument in Section 4.7, we are able to extend the analysis beyond the subcritical regime and cover the critical and supercritical regime.

As a result, we have that when $u \neq 0$,

$$|u|t \leq \mathbb{V}ar[H_u(t,0)] \leq 2\psi'(2|u|) + |u|t,$$

and the free energy fluctuation exponent is $1/2$ for $u \neq 0$. If we scale $u = ct^{-\alpha}$ for some $c \neq 0$, $\alpha \in [0, \frac{1}{3}]$, we have

$$\mathbb{V}ar[H_{ct^{-\alpha}}(t,0)] \asymp t^{1-\alpha}, \text{ as } t \to \infty.$$

In particular, when u is in the so-called critical regime $u = ct^{-1/3}$ for any $c \neq 0$, the transversal fluctuation exponent is $2/3$ and the free energy fluctuation exponent is $1/3$.

When $u = ct^{-\alpha}$ with $c \in \mathbb{R}, \alpha > 1/3$, we are in the so-called supercritical regime of the phase transition. For this regime, we show that for any $t \geq 1$,

$$\mathbb{E}\int_0^{+\infty} y \cdot p^{\mathrm{R}}_{u-\frac{1}{2}}(y|t,0)\mathrm{d}y \leq Ct^{2/3},$$

for some $C > 0$ depending only on c, and

$$\mathbb{V}ar[H_u(t,0)] \leq Ct^{2/3},$$

for some $C > 0$ depending only on c.

Besides, we also compute the average growth rate ($\mathbb{E}H_u(t, 0)$ divided by time t) as a function of the boundary parameter in Section 4.3.

In conclusion, the above results from [3] confirm the aforementioned prediction for the large-scale behavior of the half-space KPZ equation. Our results shows a continuous interpolation of the fluctuation exponents from the bound phase to the critical regime. In the bound phase and the critical regime, we obtain the exact exponents for the fluctuation size. We show that when the polymer is "pinned" to the wall, the endpoint has $O(1)$ transversal fluctuation and the free energy fluctuates with exponent $1/2$. This is consistent with the prediction that the free energy exhibits a Gaussian limit in the bound phase. We also show that in the critical regime of the phase transition, the polymer exhibits KPZ behavior, with transversal fluctuation exponent $2/3$ and free energy fluctuation exponent $1/3$. Finally, we give the $2/3$ upper bound to the transversal fluctuation exponents and $1/3$ upper bound to the free energy fluctuation exponents in the supercritical regime.

The main tools used in this section are the Gaussian integration by parts for obtaining the variance identity, and two different stochastic comparison arguments for obtaining the bounds on the transversal fluctuation exponents. A Karlin-McGregor type argument is used for one of the stochastic comparisons.

1.3 Long-range correlated SHE in $d \geq 3$

Consider (1.0.2) in dimension $(1+d)$ with $d \geq 3$ and ξ replaced by F, a mean zero Gaussian space-time noise with a special covariance function

$$\mathbb{E}[F(t,x)F(s,y)] = \delta(t-s)|x-y|^{-2}.$$

With this noise, (1.0.2) is a boarderline critical stochastic PDE with long-range correlation. To make sense of this equation, one puts a coupling constant κ before the noise F. When in the subcritical regime $0 < \kappa < \frac{d-2}{2}$, [31] defines the (martingale problem) solution to this equation and shows that such solutions exist (through Wiener chaos expansion) and are unique in law. Moreover, [31] also shows that this solution is a singular measure.

In the work with Sefika Kuzgun [4], we study the time-dependent spatial averages of this critical stochastic PDE. We show that the time-dependent spatial averages over a ball of radius R at time t have different limits under different space-time scales. In particular, when $t \ll R^2$, the central limit theorem holds; when $t = R^2$, the spatial average is a non-Gaussian random variable; when $t \gg R^2$, the spatial average becomes extinct.

For more detail, we refer to Chapter 5. The main tool of this chapter is the central limit theorem based on chaos expansion.

Chapter 2: Fluctuations of Nonlinear SHE in $d = 2$

2.1 Introduction

2.1.1 Main results

Consider the following parameterized two-dimensional nonlinear stochastic heat equation (SHE) with constant initial data on $(t, x) \in [0, +\infty) \times \mathbb{R}^2$:

$$\partial_t u^\varepsilon(t,x) = \frac{1}{2}\Delta u^\varepsilon(t,x) + \frac{\beta}{\sqrt{\log \varepsilon^{-1}}}\sigma(u^\varepsilon(t,x))\mathrm{d}W_{\phi^\varepsilon}(t,x); \tag{2.1.1}$$

$$u^\varepsilon(0,x) = 1. \tag{2.1.2}$$

Here $\beta > 0$ is a constant; $\sigma : [0,\infty) \to [0,\infty)$ is a globally Lipschitz function satisfying $\sigma(0) = 0$, $\sigma(1) \neq 0$, and $|\sigma(x) - \sigma(y)| \leq \sigma_{\mathrm{Lip}}|x - y|$ for all $x, y \in \mathbb{R}^2$ with $\sigma_{\mathrm{Lip}} > 0$ fixed. We define

$$\mathrm{d}W_{\phi^\varepsilon}(t,x) = \phi^\varepsilon * \mathrm{d}W(t,x), \tag{2.1.3}$$

where $\mathrm{d}W(t,x)$ is a space-time white noise on $[0,+\infty) \times \mathbb{R}^2$, $\phi \in C_c^\infty(\mathbb{R}^2)$ is a non-negative mollifier with $\int \phi \mathrm{d}x = 1$ and $\phi^\varepsilon(x) = \frac{1}{\varepsilon^2}\phi(\frac{x}{\varepsilon})$, and $*$ denotes convolution in space.

The noise $\mathrm{d}W_{\phi^\varepsilon}(t,x)$ is a centered Gaussian noise, white in time and homogeneously colored in space. The spatial correlation length is at scale ε. Formally, the covariance operator of $\mathrm{d}W_{\phi}^{\varepsilon}$ is given by

$$\mathbb{E}\left[\mathrm{d}W_{\phi}^{\varepsilon}(t,x)\mathrm{d}W_{\phi}^{\varepsilon}(t',x')\right] = \delta_0(t-t')\tfrac{1}{\varepsilon^2}R(\tfrac{x-x'}{\varepsilon}).$$

Here δ_0 is the Dirac delta measure with unit mass at zero. $R(x)$ is non-negative and non-negative definite, given by

$$R(x) = \int_{\mathbb{R}^2}\phi(x+y)\phi(y)\mathrm{d}y \in C_c^{\infty}(\mathbb{R}^2).$$

For $\varepsilon > 0$, it is well-known that the initial value problem (2.1.1)–(2.1.2) is well-posed and has a mild formulation

$$u^{\varepsilon}(t,x) = 1 + \frac{\beta}{\sqrt{\log\varepsilon^{-1}}}\int_0^t\int_{\mathbb{R}^2}G_{t-s}(x-y)\sigma\left(u^{\varepsilon}(s,y)\right)\mathrm{d}W_{\phi^{\varepsilon}}(s,y). \tag{2.1.4}$$

Here $G_t(x) := \frac{1}{2\pi t}\mathrm{e}^{-|x|^2/(2t)}$ denotes the two-dimensional heat kernel. The stochastic integral in (2.1.4) is interpreted in the Itô-Walsh sense.

Our main result is the following central limit theorem (CLT):

Theorem 2.1.1. *There exists some $\beta_0 \in (0, \frac{\sqrt{2\pi}}{\sigma_{\mathrm{Lip}}})$ such that if $\beta < \beta_0$, for any fixed $T > 0$ and any fixed Schwartz function $g \in C_c^{\infty}(\mathbb{R}^2)$, the random variable*

$$\mathrm{X}^{\varepsilon,T}(g) := \sqrt{\log\varepsilon^{-1}}\int_{\mathbb{R}^2}\left[u^{\varepsilon}(T,x)-1\right]g(x)\mathrm{d}x \tag{2.1.5}$$

converges in law to a Gaussian distribution

$$\mathrm{X}^{T}(g) := \int_{\mathbb{R}^2}U(T,x)g(x)\mathrm{d}x \tag{2.1.6}$$

as $\varepsilon \to 0$. Here U is the random distribution that solves the Edwards-Wilkinson equation in dimension two:

$$\partial_t U = \frac{1}{2}\Delta U + \beta\sqrt{\mathbb{E}\sigma\left(\Xi_{1,2}(2)\right)^2}\mathrm{d}W(t,x), \qquad U(0,x) = 0. \tag{2.1.7}$$

$\Xi_{1,2}(\cdot)$ *is a random process defined by the following as in [19, p.3]:*

Let $\{B(q)\}_{q\geq 0}$ be a 1D standard Brownian motion with the natural filtration $\{\mathcal{G}_q\}_{q\geq 0}$. Then $\Xi_{1,2}(\cdot)$ is the (unique) solution to a forward-backward stochastic differential equation (FBSDE):

$$\mathrm{d}\Xi_{1,2}(q) = \frac{\beta}{2\sqrt{\pi}}\Big(\mathbb{E}[\sigma^2(\Xi_{1,2}(2)) \mid \mathcal{G}_q]\Big)^{1/2}\mathrm{d}B(q), \qquad q \in (0,2]; \tag{2.1.8}$$

$$\Xi_{1,2}(0) = 1. \tag{2.1.9}$$

The existence and uniqueness of solutions to (2.1.8)–(2.1.9) are given in [19, Theorem 1.1]. To keep the consistency of notations, the subscript $_{1,2}$ denotes that the initial data equals to 1 constantly and the terminal time is at $q = 2$.

From [19, Theorem 1.1], we have that the coefficient in (2.1.7) satisfies

$$\beta\sqrt{\mathbb{E}\sigma\left(\Xi_{1,2}(2)\right)^2} = 2\sqrt{\pi}J(2,1),$$

where $J^2(q,a)$ is a viscosity solution to the quasilinear heat equation

$$\partial_q J^2 = \frac{1}{2}J^2\partial_{aa}J^2;$$
$$J^2(0,a) = \frac{\beta^2}{4\pi}\sigma^2(a).$$

We also prove the following functional version of the CLT:

Theorem 2.1.2. *Let $\beta < \beta_0$. For any $T > 0$ and $g \in C_c^\infty(\mathbb{R}^2)$. We have convergence in law in the space of continuous functions $C([0,T])$,*

$$(\mathrm{X}^{\varepsilon,t}(g))_{t\in[0,T]} \to \left(\beta\sqrt{\mathbb{E}\sigma\left(\Xi_{1,2}(2)\right)^2}\int_0^t\int_{\mathbb{R}^2} G_{t-s} * g(x)\mathrm{d}W(s,x)\right)_{t\in[0,T]},$$

as $\varepsilon \to 0$. As above, $\mathrm{d}W$ is a space-time white noise on $[0,+\infty)\times\mathbb{R}^2$.

2.1.2 Context

The Gaussian fluctuations of many SHE and KPZ equations have been widely studied. If $\sigma(x) = x$, our SHE (2.1.1)–(2.1.2) becomes a linear one:

$$\partial_t v^\varepsilon(t,x) = \frac{1}{2}\Delta v^\varepsilon(t,x) + \frac{\beta}{\sqrt{\log\varepsilon^{-1}}} v^\varepsilon(t,x)\mathrm{d}W_{\phi^\varepsilon}(t,x); \tag{2.1.10}$$

$$v^\varepsilon(0,x) = 1. \tag{2.1.11}$$

It is known that there is a phase transition at $\beta = \sqrt{2\pi}$ for this multiplicative SHE in dimension two. In [18], Caravenna, Sun, and Zygouras showed that for any fixed $T > 0$ and $x \in \mathbb{R}^2$, if $\beta \geq \sqrt{2\pi}$, $v^\varepsilon(T,x)$ converges to 0 in probability as $\varepsilon \to 0$; if $\beta < \sqrt{2\pi}$ (known as β in the ”subcritical” regime), $v^\varepsilon(T,x)$ converges in law to a log-normal random variable as $\varepsilon \to 0$. They also proved that (see [18, Theorem 2.17]) if $\beta < \sqrt{2\pi}$, for any fixed time $T > 0$ and any fixed Schwartz function $g \in C_c^\infty(\mathbb{R}^2)$, the random variable

$$\sqrt{\log\varepsilon^{-1}}\int_{\mathbb{R}^2}\left[v^\varepsilon(T,x) - 1\right]g(x)\mathrm{d}x$$

converges in law to a Gaussian distribution. In other words, the solution random field to (2.1.10)–(2.1.11), after centering and rescaling, converges (in distribution) to the solution to an Edwards-Wilkinson equation.

We are interested in obtaining an analog result for our nonlinear SHE (2.1.1)–(2.1.2). In [18], the authors used the Feynman–Kac formula, which is not available for nonlinear case. Therefore, new methods should be used for this problem. In [19], Dunlap and Gu characterized the local statistics for the limiting solution field to nonlinear SHE (2.1.1)–(2.1.2) when $\beta\sigma_{\text{Lip}} < \sqrt{2\pi}$. (See Proposition 2.2.10 below or [19, Theorem 1.2].) They showed that $u^\varepsilon(T,x)$ converges in law to a specific random variable $\Xi_{1,2}(2)$ as $\varepsilon \to 0$. They also gave the multipoint statistics in their work.

Based on their findings, we want to study the asymptotics of the (centered and rescaled) solution field to the nonlinear SHE as $\varepsilon \to 0$. One may conjecture an Edwards-Wilkinson limit as in the linear case. In Theorem 2.1.1, we gave a proof to this limit for a regime $\beta < \beta_0 < \frac{\sqrt{2\pi}}{\sigma_{\text{Lip}}}$. Whether or not this result can be extended to the entire "subcritical" regime $\beta \in (0, \frac{\sqrt{2\pi}}{\sigma_{\text{Lip}}})$ remains unknown. We will discuss it in more detail below.

We proved Theorem 2.1.1 by using the mild formulation (2.1.4) and the Malliavin-Stein's method. The use of the Malliavin-Stein's method in proving the Gaussian fluctuations for SHE is inspired by works [17, 32–35]. We also use a multivariate version to prove a functional CLT in Theorem 2.1.2.

Through a Hopf-Cole transformation $h = \log u$, the linear SHE (2.1.10)–(2.1.11) is related to a KPZ-type equation in $d = 2$. The Gaussian fluctuations of the KPZ equation in $d = 2$ in the subcritical regime is proved in [20] and [36]. In [37], the authors proved the Gaussian fluctuations of the linear SHE and KPZ equations in dimension two with general initial conditions. One of the reasons why we are interested in the nonlinear SHE is that we hope it will shed lights on the study of more general Hamilton–Jacobi SPDEs:

$$\partial_t h(t,x) = \frac{1}{2}\Delta h(t,x) + H(\nabla h(t,x)) + \beta \mathrm{d}W_\phi(t,x), \tag{2.1.12}$$

where the Hamiltonian H is not necessarily quadratic. The only result in this direction that we are aware of is the study of a two-dimensional anisotropic KPZ equation in [38]. While the nonlinear SHE (2.1.1)–(2.1.2) is more complicated than the linear one, it is still much more approachable than (2.1.12) because we can make use of the mild formulation (2.1.4).

[17] presents analog Gaussian fluctuations of the nonlinear SHE in $d \geq 3$. Gaussian fluctuations of the linear SHE and KPZ equations in $d \geq 3$ are studied in [39–42] and [43].

There are a few other things we would like to remark before we proceed.

2.1.2.1 β-region

In general, for SHE with multiplicative noise and KPZ equations in $d \geq 2$, the noise-strength parameter β plays a noticeable role.

For linear SHE (2.1.10)–(2.1.11), if $v^\varepsilon(t,x)$ converges to 0 in probability as $\varepsilon \to 0$ for any $t > 0, x \in \mathbb{R}^2$, we say that the system is in the *strong disorder* regime. If $v^\varepsilon(t,x)$ converges in distribution to a non-degenerate random variable as $\varepsilon \to 0$ for any $t > 0, x \in \mathbb{R}^2$, we say that the system is in the *weak disorder* or *subcritical* regime.

As stated above, in [18], the authors proved that $\beta_c = \sqrt{2\pi}$ is the exact critical threshold where the departure from weak disorder to strong disorder takes place. They also proved the Edwards-Wilkinson limit for linear SHE in the entire subcritical regime $\beta \in (0, \sqrt{2\pi})$. (Work [36] gives an alternative proof to this limiting result for a smaller regime $\beta < \beta_0$, where $\beta_0 < \sqrt{2\pi}$.) When $\beta \approx \sqrt{2\pi}$, the solution random field $v^\varepsilon(t,x)$ no longer needs any centering and rescaling to obtain a nontrivial limiting structure. We refer to [44–46] for the study in this direction.

Here we are to prove the Edwards-Wilkinson limit (Theorem 2.1.1) for nonlinear SHE (2.1.1)–(2.1.2) in a regime $\beta < \beta_0 < \frac{\sqrt{2\pi}}{\sigma_{\mathrm{Lip}}}$ smaller than the entire subcritical regime $\beta \in (0, \frac{\sqrt{2\pi}}{\sigma_{\mathrm{Lip}}})$. In fact, our proof below ensures that Theorem 2.1.1 would hold for $\beta_0 = \frac{1}{2\sqrt{6}}\frac{\sqrt{2\pi}}{\sigma_{\mathrm{Lip}}}$ (see Remark 2.2.6 for detail). While we don't believe that this is an optimal β-region, whether or not (and how) Theorem 2.1.1 can be extended to the entire subcritical regime remains open.

For SHE in $d \geq 3$, the readers may refer to [16] and [40] on the weak/strong disorder regions of β. In particular, [40] proved the Edwards-Wilkinson limit for linear SHE in $d \geq 3$ in the entire subcritical regime. For nonlinear SHE in $d \geq 3$, [17] also restricted β in a smaller region.

2.1.2.2 General initial condition

Following the same methods as in [19], we can generate the results from [19, Theorem 1.2] to nonlinear SHE (2.1.1) with general initial condition

$$u^\varepsilon(0,x) = u_0(x), \qquad x \in \mathbb{R}^2, \tag{2.1.13}$$

if we assume u_0 satisfies

$$0 < \inf_{x\in\mathbb{R}^2} u_0(x) \leq \sup_{x\in\mathbb{R}^2} u_0(x) < +\infty.$$

In fact, it can be shown that $u^\varepsilon(T,x) \to \bar{u}(T,x)\Xi_{1,2}(2)$ in law as $\varepsilon \to 0$ where $\bar{u}(T,x) = G_T * u_0(x)$ is a deterministic function. One can now go through the exactly same arguments as in the proof of Theorem 2.1.1 here to obtain the Gaussian fluctuations of the SHE (2.1.1) with initial condition (2.1.13) as $\varepsilon \to 0$. We claim, without proof, that Theorem 2.1.1 still holds, except that the Edwards-Wilkinson equation (2.1.7) should be replaced by a new SPDE:

$$\partial_t U = \frac{1}{2}\Delta U + \beta\sqrt{\mathbb{E}\sigma\left(\Xi_{1,2}(2)\right)^2}\bar{u}(t,x)\mathrm{d}W(t,x), \qquad U(0,x) = 0.$$

This result aligns with the one in [37, Theorem 1.3, Remark 1.4] for linear SHE in dimension two with general initial condition.

2.1.3 Outline

In Section 2.2, we first scale our SHE (2.1.1)–(2.1.2) to microscopic variables. Then we introduce the basics of the Malliavin-Stein's method and prove some uniform moment bounds to use in the sequel. We will also state [19, Theorem 1.2] in Proposition 2.2.10 for convenience. In Section 2.3 and 2.4, we prove Theorem 2.1.1 and 2.1.2 respectively.

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2.2 Preliminaries

2.2.1 Scaling

We are going to use the Malliavin calculus for Gaussian spaces. In order to simplify the calculation, we want to fix a specific Gaussian space. For this purpose, we introduce another representation of the SHE (2.1.1)–(2.1.2).

We study the following equation in microscopic variables on $(t, x) \in [0, +\infty) \times \mathbb{R}^2$:

$$\partial_t \mathcal{V}^\varepsilon(t, x) = \frac{1}{2} \Delta \mathcal{V}^\varepsilon(t, x) + \frac{\beta}{\sqrt{\log \varepsilon^{-1}}} \sigma\left(\mathcal{V}^\varepsilon(t, x)\right) \mathrm{d}W_\phi(t, x); \tag{2.2.1}$$

$$\mathcal{V}^\varepsilon(0, x) = 1. \tag{2.2.2}$$

Here $W_\phi(t, x) := W_{\phi^1}(t, x) = \int_{\mathbb{R}^2} \phi(x - y) W(t, y) dy$ is defined as in (2.1.3) with $\varepsilon = 1$. By the scaling property of the space-time white noise, we have

$$u^\varepsilon(\cdot, \cdot) = \mathcal{V}^\varepsilon\left(\frac{\cdot}{\varepsilon^2}, \frac{\cdot}{\varepsilon}\right) \quad \text{jointly in law.}$$

Since we are only interested in proving convergence in law, it is sufficient to prove Theorem 2.1.1 with $u^\varepsilon(T, x)$ substituted by $\mathcal{V}^\varepsilon(\frac{T}{\varepsilon^2}, \frac{x}{\varepsilon})$. In fact, as the mild formulation of $\mathcal{V}^\varepsilon(t, x)$ is

$$\mathcal{V}^\varepsilon(t, x) = 1 + \frac{\beta}{\sqrt{\log \varepsilon^{-1}}} \int_0^t \int_{\mathbb{R}^2} G_{t-s}(x - y) \sigma\left(\mathcal{V}^\varepsilon(s, y)\right) \mathrm{d}W_\phi(s, y), \tag{2.2.3}$$

we have

$$
\begin{aligned}
\mathrm{X}^{\varepsilon,T}(g) &\overset{\text{law}}{=} \mathrm{Y}^{\varepsilon,T}(g) := \sqrt{\log\varepsilon^{-1}}\int_{\mathbb{R}^2}\Big[\mathcal{V}^\varepsilon(\frac{T}{\varepsilon^2},\frac{x}{\varepsilon})-1\Big]g(x)\mathrm{d}x \\
&= \beta\int_{\mathbb{R}^2}\int_0^{\frac{T}{\varepsilon^2}}\int_{\mathbb{R}^2} G_{\frac{T}{\varepsilon^2}-s}(\frac{x}{\varepsilon}-y)\sigma\left(\mathcal{V}^\varepsilon(s,y)\right)\mathrm{d}W_\phi(s,y)g(x)\mathrm{d}x.
\end{aligned}
\tag{2.2.4}
$$

For $0 \le t_1 < \cdots < t_m \le T$, we have

$$
\left(X^{\varepsilon,t_1}(g),\ldots,X^{\varepsilon,t_m}(g)\right) \overset{\text{law}}{=} \left(Y^{\varepsilon,t_1}(g),\ldots,Y^{\varepsilon,t_m}(g)\right).
$$

From now on, we will study $Y^{\varepsilon,T}(g)$ instead of $X^{\varepsilon,T}(g)$, etc.

Remark 2.2.1. As discussed in [19, Section 1.1], in the subcritical regime $\beta\sigma_{\text{Lip}} < \sqrt{2\pi}$, the 2-dimensional stochastic heat equation (2.2.1)–(2.2.2) evolves on an exponential time scale with respect to the strength of the random noise. The exponential time scale explains why we need the tuning $\frac{1}{\sqrt{\log\varepsilon^{-1}}}$ before the noise term in our SHE (2.1.1)–(2.1.2).

This scaling is different from the $d \ge 3$ cases. When $d \ge 3$, if the strength of the noise is small enough so that the system is in the weak disorder regime, the SHE (2.2.1)–(2.2.2) evolves in an 'arbitrarily long' diffusion scale $(\frac{t}{\lambda},\frac{x}{\lambda^2})$ where λ is independent of the strength of the noise. See e.g. [16].

2.2.2 The Malliavin calculus

Now we state the Gaussian space in which we will be working with and give some elementary results from the Malliavin calculus theory. A detailed discussion should be referred to the monograph [47].

Consider a complete probability space $(\Omega,\mathscr{F},\mathbb{P})$ generated by the Gaussian noise $\mathrm{d}W_\phi$ as defined above. Let $\mathcal{H}$ be the Hilbert space given by the closure of $C_c^\infty(\mathbb{R}_{\geq 0} \times \mathbb{R}^2)$ with respect to the inner product

$$\langle f, g \rangle_{\mathcal{H}} := \int_0^{+\infty} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f(s,x) g(s,y) R(x-y) \mathrm{d}x \mathrm{d}y \mathrm{d}s.$$

Define an operator $W : \mathcal{H} \to L^2(\Omega)$ as

$$W(h) = \int_0^{+\infty} \int_{\mathbb{R}^2} h(t,x) \mathrm{d}W_\phi(t,x).$$

Then W is a linear isometry and the Gaussian space $\{W(h) : h \in \mathcal{H}\}$ is an isonormal Gaussian process over $\mathcal{H}$.

Following the conventional notations, we denote the Malliavin derivative operator for the isonormal Guassian process $\{W(h) : h \in \mathcal{H}\}$ by D. Let $\mathbb{D}^{k,p}, k, p \in \mathbb{Z}^+$ denote the associated Gaussian Sobolev spaces.

For any $t \geq 0$, let $\mathscr{F}_t$ be the σ-algebra generated by $\{W_\phi([0,s] \times A) : 0 \leq s \leq t, A \in \mathscr{B}_b(\mathbb{R}^2)\}$ and the null sets of $\mathscr{F}$. Here $\mathscr{B}_b(\mathbb{R}^2)$ denotes the bounded Borel subsets of $\mathbb{R}^2$. A random field $v = \{v(t,x), t \geq 0, x \in \mathbb{R}^2\}$ is adapted if it is $\mathscr{F}_t$-measurable for each $t \geq 0$ and $x \in \mathbb{R}^2$.

Let δ denote the divergence operator. δ is the adjoint operator of D. If $v(t,x)$ is an adapted and measurable random field on $[0,+\infty) \times \mathbb{R}^2$ such that

$$\int_0^{+\infty} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \mathbb{E}\left(v(t,x) v(t,y)\right) R(x-y) \mathrm{d}x \mathrm{d}y \mathrm{d}t < +\infty,$$

then $v \in \mathrm{Dom}(\delta)$ and

$$\delta(v) = \int_0^{+\infty} \int_{\mathbb{R}^2} v(t,x) \mathrm{d}W_\phi(t,x),$$

where the last integral is well-defined in the Itô-Walsh sense with the isometry

$$\mathbb{E}\left(\delta(v)\right)^2 = \int_0^{+\infty} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \mathbb{E}\left(v(t,x)v(t,y)\right) R(x-y) \mathrm{d}x \mathrm{d}y \mathrm{d}t$$

By [47, Proposition 1.3.8], if $v \in \mathrm{Dom}(\delta)$ and is an adapted process in $\mathbb{D}^{1,2}$, then

$$D_{r,z}\delta(v) = v(r,z) + \int_0^{+\infty} \int_{\mathbb{R}^2} D_{r,z} v(s,y) \mathrm{d}W_\phi(s,y).$$

A standard Picard's iteration scheme shows that for all $\varepsilon > 0$ and $(t,x) \in [0,+\infty) \times \mathbb{R}^2$, $\mathcal{V}^\varepsilon(t,x) \in \mathbb{D}^{1,p}$ for all $p \geq 1$. Since $\mathcal{V}^\varepsilon(t,x)$ has the mild formulation (2.2.3), the Malliavin derivative $D_{r,z}\mathcal{V}^\varepsilon(t,x)$ satisfies the equation

$$\begin{aligned} D_{r,z}\mathcal{V}^\varepsilon(t,x) &= \frac{\beta}{\sqrt{\log \varepsilon^{-1}}} \mathbb{1}_{[0,t]}(r) G_{t-r}(x-z) \sigma(\mathcal{V}^\varepsilon(r,z)) \\ &\quad + \frac{\beta}{\sqrt{\log \varepsilon^{-1}}} \int_r^t \int_{\mathbb{R}^2} G_{t-s}(x-y) D_{r,z} \sigma(\mathcal{V}^\varepsilon(s,y)) \mathrm{d}W_\phi(s,y) \\ &= \frac{\beta}{\sqrt{\log \varepsilon^{-1}}} \mathbb{1}_{[0,t]}(r) G_{t-r}(x-z) \sigma(\mathcal{V}^\varepsilon(r,z)) \\ &\quad + \frac{\beta}{\sqrt{\log \varepsilon^{-1}}} \int_r^t \int_{\mathbb{R}^2} G_{t-s}(x-y) \Sigma(s,y) D_{r,z} \mathcal{V}^\varepsilon(s,y) \mathrm{d}W_\phi(s,y). \end{aligned} \tag{2.2.5}$$

Here Σ is an adapted process bounded by the Lipschitz constant σ_{Lip}. If we further assume that $\sigma \in C^1([0,+\infty))$, then $\Sigma(s,y) = \sigma'(\mathcal{V}^\varepsilon(s,y))$.

In addition, we will need the following version of the Clark-Ocone formula in the proof of

Theorem 2.1.1.

Proposition 2.2.2 (Clark-Ocone Formula)**.** *For $X \in \mathbb{D}^{1,2}$,*

$$X = \mathbb{E}X + \int_0^{+\infty} \int_{\mathbb{R}^2} \mathbb{E}(D_{r,z}X|\mathscr{F}_r)\mathrm{d}W_\phi(r,z) \qquad \mathbb{P}-\textit{almost surely}.$$

Proof. See [48, Proposition 6.3]. □

2.2.3 Moment bounds

We will use the following two uniform moment bounds in the proof of Theorem 2.1.1.

Note that in both lemmas we are not able to cover the entire subcritical regime $0 < \beta < \frac{\sqrt{2\pi}}{\sigma_{\mathrm{Lip}}}$. In fact, the extension of Lemma 2.2.3 to the entire subcritical regime should be true and its counterpart in the 2D directed polymer environment is proved in [49]. However, it remains unknown whether Lemma 2.2.5 can be extended to the entire subcritical regime.

We first give a uniform moment bound on the mild solution to the SHE.

Lemma 2.2.3. *For all $p \geq 2$, there exists $0 < \beta_0(p) \leq \frac{\sqrt{2\pi}}{\sigma_{\mathrm{Lip}}}$ depending only on p, such that for all $T > 0$ and $\beta < \beta_0(p)$, there exists $\varepsilon_0 > 0$ (depending on ϕ, T, $\beta\sigma_{\mathrm{Lip}}$ and p) and $C > 0$ (depending on $\beta\sigma_{\mathrm{Lip}}$ and p) such that*

$$\sup_{t\in[0,T],x\in\mathbb{R}^2} \mathbb{E}\left|u^\varepsilon(t,x)\right|^p \leq C^p, \tag{2.2.6}$$

for all $\varepsilon < \varepsilon_0$.

Proof. First we notice that, for any $\varepsilon, t > 0$, $u^\varepsilon(t,\cdot)$ is stationary in the x-variable, so we only need to prove for $u^\varepsilon(t,0)$.

By the mild formulation (2.1.4), the triangle inequality and the inequality $2ab \le (ca)^2 + (\frac{b}{c})^2$ for all $c > 1$, we have that for any arbitrary $\alpha > 1$,

$$
\begin{aligned}
&(\mathbb{E}\,|u^\varepsilon(t,0)|^p)^{\frac{2}{p}} \\
&\le \frac{\alpha}{\alpha-1} + \frac{\alpha\beta^2}{\log\varepsilon^{-1}}\left(\mathbb{E}\left(\int_0^t\int_{\mathbb{R}^2} G_{t-s}(y)\sigma\left(u^\varepsilon(s,y)\right)\mathrm{d}W_{\phi^\varepsilon}(s,y)\right)^p\right)^{\frac{2}{p}}
\end{aligned}
$$

Apply the Burkholder-Davis-Gundy inequality, we have that with $c_p = \frac{p(p-1)}{2}$,

$$
\begin{aligned}
&(\mathbb{E}\,|u^\varepsilon(t,0)|^p)^{\frac{2}{p}} \\
&\le \frac{\alpha}{\alpha-1} + c_p\frac{\alpha\beta^2}{\log\varepsilon^{-1}} \\
&\cdot\int_0^t\left(\mathbb{E}\left(\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left|\prod_{i=1}^2 G_{t-s}(y_i)\sigma(u^\varepsilon(s,y_i))\right|\frac{1}{\varepsilon^2}R(\frac{y_1-y_2}{\varepsilon})\mathrm{d}y_1\mathrm{d}y_2\right)^{\frac{p}{2}}\right)^{\frac{2}{p}}\mathrm{d}s
\end{aligned}
$$

The Hölder's inequality gives that

$$
\begin{aligned}
\left(\mathbb{E}\,|\sigma(u^\varepsilon(t,x))\sigma(u^\varepsilon(t,y))|^{\frac{p}{2}}\right)^{\frac{2}{p}} &\le (\mathbb{E}\,|\sigma(u^\varepsilon(t,x))|^p\,\mathbb{E}\,|\sigma(u^\varepsilon(t,y))|^p)^{1/p} \\
&= (\mathbb{E}\,|\sigma(u^\varepsilon(t,x))|^p)^{\frac{2}{p}} \le \sigma_{\mathrm{Lip}}^2\,(\mathbb{E}\,|u^\varepsilon(t,0)|^p)^{\frac{2}{p}}
\end{aligned}
\tag{2.2.7}
$$

Combined the above results. Use the Minkowski inequality and a change of variable $y_1 - y_2 \mapsto y$, we obtain

$$
\begin{aligned}
&\left(\mathbb{E}\left|u^{\varepsilon}(t,0)\right|^{p}\right)^{\frac{2}{p}} \\
&\leq \frac{\alpha}{\alpha-1}+c_p\frac{\alpha\beta^2}{\log\varepsilon^{-1}}\int_0^t\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left(\mathbb{E}\left|\prod_{i=1}^{2}G_{t-s}(y_i)\sigma\left(u^{\varepsilon}(s,y_i)\right)\right|^{\frac{p}{2}}\right)^{\frac{2}{p}}\frac{1}{\varepsilon^2}R(\frac{y_1-y_2}{\varepsilon})\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}s \\
&\leq \frac{\alpha}{\alpha-1}+c_p\frac{\alpha\beta^2\sigma_{\mathrm{Lip}}^2}{\log\varepsilon^{-1}}\int_0^t\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}G_{t-s}(y_1)G_{t-s}(y_2)\left(\mathbb{E}\left|u^{\varepsilon}(t,0)\right|^{p}\right)^{\frac{2}{p}}\frac{1}{\varepsilon^2}R(\frac{y_1-y_2}{\varepsilon})\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}s \\
&= \frac{\alpha}{\alpha-1}+c_p\frac{\alpha\beta^2\sigma_{\mathrm{Lip}}^2}{\log\varepsilon^{-1}}\int_0^t\int_{\mathbb{R}^2}G_{2(t-s)}(y)\left(\mathbb{E}\left|u^{\varepsilon}(s,0)\right|^{p}\right)^{\frac{2}{p}}\frac{1}{\varepsilon^2}R(\frac{y}{\varepsilon})\mathrm{d}y\mathrm{d}s.
\end{aligned}
$$

It is easy to see that with $R(\cdot)\leq\|\phi\|_\infty$ and $\int_{\mathbb{R}^2}\frac{1}{\varepsilon^2}R(\frac{y}{\varepsilon})dy=1$, we have the elementary inequality

$$\int_{\mathbb{R}^2}G_{2(t-s)}(y)\frac{1}{\varepsilon^2}R(\frac{y}{\varepsilon})\mathrm{d}y\leq\frac{1}{4\pi(t-s)}\wedge\frac{\|\phi\|_\infty}{\varepsilon^2}.$$

By the Lemma 2.2.4 below, if for some $\alpha>1$,

$$c_p\alpha\beta^2\sigma_{\mathrm{Lip}}^2<2\pi, \tag{2.2.8}$$

there exists an $\varepsilon_0>0$ and $C_0=C_0(c_p,\alpha,\beta\sigma_{\mathrm{Lip}})>0$ such that for all $\varepsilon<\varepsilon_0$

$$\left(\mathbb{E}\left|u^{\varepsilon}(t,0)\right|^{p}\right)^{\frac{2}{p}}\leq\frac{\alpha}{\alpha-1}C_0.$$

Since $\alpha>1$ is arbitrary, condition (2.2.8) is valid if and only if

$$\beta<\beta_0(p)=\frac{\sqrt{2\pi}}{\sqrt{c_p}\sigma_{\mathrm{Lip}}}.$$

In particular, $\beta_0(2) = \frac{\sqrt{2\pi}}{\sigma_{\mathrm{Lip}}}$ and $\beta_0(4) = \frac{1}{\sqrt{6}}\frac{\sqrt{2\pi}}{\sigma_{\mathrm{Lip}}}$. We can choose an appropriate α according to p and $\beta\sigma_{\mathrm{Lip}}$. $\square$

As an immediate corollary, we have that under the same condition as in Lemma 2.2.3,

$$\sup_{t\in[0,T],x\in\mathbb{R}^2} \mathbb{E}\left|\mathcal{V}^\varepsilon(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon})\right|^p \leq C^p \quad \text{for all } \varepsilon < \varepsilon_0. \tag{2.2.9}$$

Lemma 2.2.4. *Fix constants $0 < b < \sqrt{2\pi}$ and $a, c, T > 0$. There exists an $\varepsilon_0 > 0$ (depending on b, c and T), such that for all $\varepsilon \in (0, \varepsilon_0)$ the following holds:*

If $f^\varepsilon : [0,T] \to [0,\infty)$ is a function such that for all $t \in [0,T]$,

$$f^\varepsilon(t) \leq a + \frac{b^2}{\log\varepsilon^{-1}} \int_0^t f^\varepsilon(s)\left(\frac{1}{4\pi(t-s)} \wedge \frac{c}{\varepsilon^2}\right) \mathrm{d}s,$$

then for all $t \in [0,T]$, we have

$$f^\varepsilon(t) \leq aC, \tag{2.2.10}$$

for some uniform constant $C > 0$ depending only on b.

Proof. Define $[0,t]^j_< = \{(s_1,\ldots,s_j) \in [0,t]^j \mid s_1 \leq \cdots \leq s_j\}$, we have

$$\begin{aligned}
f^\varepsilon(t) &\leq a\sum_{j=0}^\infty \frac{b^{2j}}{(4\pi\log\varepsilon^{-1})^j} \int_{[0,t]^j_<} \prod_{k=1}^j \left(\frac{1}{s_{k+1}-s_k} \wedge \frac{4\pi c}{\varepsilon^2}\right) \mathrm{d}s_1 \cdots \mathrm{d}s_j \\
&\leq a\sum_{j=0}^\infty \frac{b^{2j}}{(4\pi\log\varepsilon^{-1})^j} \int_{[0,t]^j} \prod_{k=1}^j \left(\frac{1}{r_j} \wedge \frac{4\pi c}{\varepsilon^2}\right) \mathrm{d}r_1 \cdots \mathrm{d}r_j \\
&= a\sum_{j=0}^\infty \frac{b^{2j}}{(4\pi\log\varepsilon^{-1})^j} \left[\int_0^t \left(\frac{1}{s} \wedge \frac{4\pi c}{\varepsilon^2}\right) \mathrm{d}s\right]^j
\end{aligned} \tag{2.2.11}$$

Notice that when $t > 0$,

$$\int_0^t \left(\frac{1}{s} \wedge \frac{4\pi c}{\varepsilon^2}\right) \mathrm{d}s = \int_0^{\frac{\varepsilon^2}{4\pi c}} \frac{4\pi c}{\varepsilon^2} \mathrm{d}s + \int_{\frac{\varepsilon^2}{4\pi c}}^t \frac{1}{s} \mathrm{d}s = \log \frac{4\pi c t}{\varepsilon^2} + 1,$$

and

$$\lim_{\varepsilon \to 0} \frac{b^2}{4\pi \log \varepsilon^{-1}} \left(\log \frac{4\pi c T}{\varepsilon^2} + 1\right) = \frac{b^2}{2\pi}.$$

Therefore, if $b < \sqrt{2\pi}$, there exists an $\varepsilon_0 > 0$ depending on b, c and T, such that for all $\varepsilon \in (0, \varepsilon_0)$, the sum of the series in (2.2.11) converges for all $t \in [0, T]$ and is uniformly bounded by some constant C depending only on b. $\square$

Next we give the uniform moment bounds for the Malliavin derivatives.

Lemma 2.2.5. *For all $p \geq 2$, there exists $0 < \beta_0(p) \leq \frac{\sqrt{2\pi}}{\sigma_{\mathrm{Lip}}}$ depending only on p, such that for all $T > 0$ and $\beta < \beta_0(p)$, there exists $0 < \varepsilon_0' \leq \varepsilon_0$ (depending on ϕ, T, $\beta\sigma_{\mathrm{Lip}}$ and p) and $C' > 0$ (depending on $\beta\sigma_{\mathrm{Lip}}$ and p) such that for all $(t, x) \in (0, \frac{T}{\varepsilon^2}] \times \mathbb{R}^2$ and all $(r, z) \in [0, t) \times \mathbb{R}^2$,*

$$(\mathbb{E}|D_{r,z}\mathcal{V}^\varepsilon(t, x)|^p)^{\frac{1}{p}} \leq \frac{C'}{\sqrt{\log \varepsilon^{-1}}} G_{t-r}(x - z), \quad \textit{for all } \varepsilon < \varepsilon_0'.$$

Proof. We follow the idea from [17, Lemma 2.2].

Let $\alpha > 1$ be arbitrary. By (2.2.5), the triangle inequality and the inequality $2ab \leq (ca)^2 +$

$(\frac{b}{c})^2$ for all $c > 0$, for all $0 \le r < t \le \frac{T}{\varepsilon^2}$, $x, z \in \mathbb{R}^2$,

$$
\begin{aligned}
&\left(\mathbb{E}\,|D_{r,z}\mathcal{V}^\varepsilon(t,x)|^p\right)^{\frac{2}{p}} \\
&\le \frac{\alpha}{\alpha-1}\frac{\beta^2}{\log\varepsilon^{-1}} G_{t-r}(x-z)^2 \left(\mathbb{E}|\sigma(\mathcal{V}^\varepsilon(r,z))|^p\right)^{\frac{2}{p}} \\
&+ \frac{\alpha\beta^2}{\log\varepsilon^{-1}}\left(\mathbb{E}\left|\int_r^t\int_{\mathbb{R}^2} G_{t-s}(x-y)\Sigma(s,y)D_{r,z}\mathcal{V}^\varepsilon(s,y)\mathrm{d}W_\phi(s,y)\right|^p\right)^{\frac{2}{p}}
\end{aligned}
$$

Applying the Burkholder-Davis-Gundy inequality with $c_p = \frac{p(p-1)}{2}$, we have

$$
\begin{aligned}
&\left(\mathbb{E}\,|D_{r,z}\mathcal{V}^\varepsilon(t,x)|^p\right)^{\frac{2}{p}} \le \frac{\alpha}{\alpha-1}\frac{\beta^2\sigma_{\mathrm{Lip}}^2}{\log\varepsilon^{-1}} G_{t-r}(x-z)^2 \left(\mathbb{E}|\mathcal{V}^\varepsilon(r,z)|^p\right)^{\frac{2}{p}} \\
&+ c_p\frac{\alpha\beta^2}{\log\varepsilon^{-1}}\int_r^t\Bigg[\mathbb{E}\Bigg(\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} G_{t-s}(x-y_1)G_{t-s}(x-y_2)\Sigma(s,y_1)\Sigma(s,y_2) \\
&\cdot D_{r,z}\mathcal{V}^\varepsilon(s,y_1)D_{r,z}\mathcal{V}^\varepsilon(s,y_2)R(y_1-y_2)\mathrm{d}y_1\mathrm{d}y_2\Bigg)^{\frac{p}{2}}\Bigg]^{\frac{2}{p}}\mathrm{d}s
\end{aligned}
$$

Using the Minkowski inequality, (2.2.9) and the boundness of Σ, we further obtain

$$
\begin{aligned}
&\left(\mathbb{E}\,|D_{r,z}\mathcal{V}^\varepsilon(t,x)|^p\right)^{\frac{2}{p}} \\
&\le \frac{\alpha}{\alpha-1}\frac{\beta^2\sigma_{\mathrm{Lip}}^2C^2}{\log\varepsilon^{-1}} G_{t-r}(x-z)^2 + c_p\frac{\alpha\beta^2\sigma_{\mathrm{Lip}}^2}{\log\varepsilon^{-1}}\int_r^t\int_{\mathbb{R}^4} R(y_1-y_2) \\
&\cdot G_{t-s}(x-y_1)G_{t-s}(x-y_2)\left[\mathbb{E}|D_{r,z}\mathcal{V}^\varepsilon(s,y_1)|^p\right]^{\frac{1}{p}}\left[\mathbb{E}|D_{r,z}\mathcal{V}^\varepsilon(s,y_2)|^p\right]^{\frac{1}{p}}\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}s.
\end{aligned}
$$

If we set $t = \theta + r$ and $x = \eta + z$, let

$$
\mathcal{K}(\theta,\eta) := \left(\mathbb{E}\,|D_{r,z}\mathcal{V}^\varepsilon(\theta+r,\eta+z)|^p\right)^{\frac{1}{p}} = \left(\mathbb{E}\,|D_{r,z}\mathcal{V}^\varepsilon(t,x)|^p\right)^{\frac{1}{p}},
$$

then we can rewrite the above inequality as

$$\mathcal{K}(\theta,\eta)^2 \leq \frac{\alpha}{\alpha-1}\frac{\beta^2\sigma_{\mathrm{Lip}}^2 C^2}{\log\varepsilon^{-1}}G_\theta(\eta)^2 + c_p\frac{\alpha\beta^2\sigma_{\mathrm{Lip}}^2}{\log\varepsilon^{-1}}\int_0^\theta\int_{\mathbb{R}^4}R(y_1-y_2)$$
$$\cdot G_{\theta-s}(\eta-y_1)G_{\theta-s}(\eta-y_2)\mathcal{K}(s,y_1)\mathcal{K}(s,y_2)\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}s.$$

Applying [50, Lemma 2.2], we have

$$\mathcal{K}(\theta,\eta) \leq \sqrt{\frac{\alpha}{\alpha-1}}\frac{\beta\sigma_{\mathrm{Lip}}C}{\sqrt{\log\varepsilon^{-1}}}G_\theta(\eta)\cdot H\left(\theta, 2c_p\frac{\alpha\beta^2\sigma_{\mathrm{Lip}}^2}{\log\varepsilon^{-1}}\right)^{\frac{1}{2}} \tag{2.2.12}$$

where $H(t,\gamma)$ is defined for all $\gamma \geq 0$ as

$$H(t,\gamma) = \sum_{n=0}^{\infty}\gamma^n h_n(t).$$

Here $h_0(t) := 1$ and for $n \geq 1$,

$$h_n(t) := \int_0^t h_{n-1}(s)\int_{\mathbb{R}^2}G_{t-s}(z)R(z)\mathrm{d}z\mathrm{d}s.$$

It is easy to see that with $R(\cdot) \leq \|\phi\|_\infty$ and $\int_{\mathbb{R}^2}R(y)dy = 1$, we have the elementary inequality

$$\int_{\mathbb{R}^2}G_{t-s}(z)R(z)\mathrm{d}z \leq \frac{1}{2\pi(t-s)} \wedge \|\phi\|_\infty \quad \text{for all } 0 \leq s \leq t.$$

Since $0 \le \theta \le t \le \frac{T}{\varepsilon^2}$,

$$
\begin{aligned}
h_1(\theta) &= \int_0^\theta \int_{\mathbb{R}^2} G_{\theta-s}(z)R(z)\mathrm{d}z\mathrm{d}s \le \int_0^\theta \left(\frac{1}{2\pi(\theta-s)} \wedge \|\phi\|_\infty\right)\mathrm{d}s \\
&= \int_0^\theta \left(\frac{1}{2\pi s} \wedge \|\phi\|_\infty\right)\mathrm{d}s \le \int_0^{\frac{T}{\varepsilon^2}} \left(\frac{1}{2\pi s} \wedge \|\phi\|_\infty\right)\mathrm{d}s \\
&= \int_0^T \left(\frac{1}{2\pi s} \wedge \frac{\|\phi\|_\infty}{\varepsilon^2}\right)\mathrm{d}s = \int_0^{\frac{\varepsilon^2}{2\pi\|\phi\|_\infty}} \frac{\|\phi\|_\infty}{\varepsilon^2}\mathrm{d}s + \int_{\frac{\varepsilon^2}{2\pi\|\phi\|_\infty}}^T \frac{1}{2\pi s}\mathrm{d}s \\
&= \frac{1}{2\pi} + \frac{1}{2\pi}\log\frac{2\pi\|\phi\|_\infty T}{\varepsilon^2}.
\end{aligned}
$$

Thus for all $n \ge 1$,

$$
h_n(\theta) \le \left[\frac{1}{2\pi} + \frac{1}{2\pi}\log\frac{2\pi\|\phi\|_\infty T}{\varepsilon^2}\right]^n.
$$

Notice that we have

$$
\lim_{\varepsilon\to 0}\left(2c_p\frac{\alpha\beta^2\sigma_{\mathrm{Lip}}^2}{\log\varepsilon^{-1}}\right)\cdot\left(\frac{1}{2\pi} + \frac{1}{2\pi}\log\frac{2\pi\|\phi\|_\infty T}{\varepsilon^2}\right) = \frac{2c_p\alpha\beta^2\sigma_{\mathrm{Lip}}^2}{\pi}. \tag{2.2.13}
$$

The limit is independent of ϕ and T. Therefore, if for some $\alpha > 1$,

$$
2c_p\alpha\beta^2\sigma_{\mathrm{Lip}}^2 < \pi, \tag{2.2.14}
$$

then there exists an $\varepsilon_0' > 0$, such that when $\varepsilon \le \varepsilon_0'$,

$$
H\left(\theta, 2c_p\frac{\alpha\beta^2\sigma_{\mathrm{Lip}}^2}{\log\varepsilon^{-1}}\right) \le C_0'
$$

for some $C_0' = C_0'(\alpha, c_p, \beta\sigma_{\mathrm{Lip}}) > 0$. By (2.2.12), there exists a uniform $C' = C'(\alpha, c_p, \beta\sigma_{\mathrm{Lip}}) > 0$

such that

$$\mathcal{K}(\theta,\eta) \le \frac{C'}{\sqrt{\log \varepsilon^{-1}}} G_\theta(\eta).$$

We may choose $\alpha > 1$ according to p and $\beta\sigma_{\mathrm{Lip}}$. Condition (2.2.14) is valid if and only if

$$\beta < \beta_0(p) := \frac{\sqrt{2\pi}}{2\sqrt{c_p}\sigma_{\mathrm{Lip}}}.$$

In particular, $\beta_0(2) = \frac{1}{2}\frac{\sqrt{2\pi}}{\sigma_{\mathrm{Lip}}}$ and $\beta_0(4) = \frac{1}{2\sqrt{6}}\frac{\sqrt{2\pi}}{\sigma_{\mathrm{Lip}}}$. □

Remark 2.2.6. In Section 2.3 below, we only need the results from Lemma 2.2.3 and Lemma 2.2.5 for $p = 2, 4$. As a result, we can let $\beta_0 = \frac{1}{2\sqrt{6}}\frac{\sqrt{2\pi}}{\sigma_{\mathrm{Lip}}}$ in Theorem 2.1.1. Compared with the linear SHE case, we don't believe that it is an optimal value. The main difficulty of extension came from the restriction of β in Lemma 2.2.5. The coefficient $\frac{1}{\sqrt{6}}$ came from the BDG-inequality and $\frac{1}{2}$ came from (2.2.12). One may find elementary ways to improve the BDG-inequality here, but while we believe that (2.2.12) is not sharp, it remains unclear how we can improve it.

2.2.4 Stein's method

The following propositions are key ingredients in our proof of the central limit theorems Theorem 2.1.1 and Theorem 2.1.2. They are derived from the Stein's method for normal approximations and are also used in [17] and [33]. A more detailed discussion on the Stein's method with Malliavin calculus should be referred to the monograph [51].

Proposition 2.2.7. *Let X be a random variable such that $X = \delta(v)$ for $v \in \mathrm{Dom}\delta$. Assume $X \in \mathbb{D}^{1,2}$. Let Z be a centered Gaussian random variable with variance Σ. For any C^2-function*

$h : \mathbb{R} \to \mathbb{R}$ *with a bounded second order derivative,*

$$|\mathbb{E}h(X) - \mathbb{E}h(Z)| \le \frac{1}{2}\|h''\|_\infty \sqrt{\mathbb{E}\left|\Sigma - \langle DX, v\rangle_{\mathcal{H}}\right|^2}.$$

Proof. This is a special case of the Proposition 2.2.9 below with $m = 1$. □

Remark 2.2.8. The class of C^2 functions with bounded second order derivatives is a sufficient class of test functions $h(\cdot)$ to show convergence in law. With Proposition 2.2.7, if we can prove that $\mathbb{E}\left|\Sigma - \langle DX_n, v_n\rangle_{\mathcal{H}}\right|^2 \to 0$ as $n \to \infty$ for a series of random variables $X_n \in \mathbb{D}^{1,2}$ with $X_n = \delta(v_n)$, then we have $X_n \Rightarrow Z$ in law.

To prove Theorem 2.1.2, we need the following multivariate version.

Proposition 2.2.9. *Let $F = (F^{(1)}, \dots, F^{(m)})$ be a random vector such that $F^{(i)} = \delta(v_i)$ for $v_i \in \mathrm{Dom}\delta$, $i = 1, \dots, m$. Assume $F^{(i)} \in \mathbb{D}^{1,2}$ for $i = 1, \dots, m$. Let Z be an m-dimensional centered Gaussian vector with covariance matrix $(C_{i,j})_{1\le i,j\le m}$. For any C^2-function $h : \mathbb{R}^m \to \mathbb{R}$ with bounded second order derivatives,*

$$|\mathbb{E}h(F) - \mathbb{E}h(Z)| \le \frac{1}{2}\|h''\|_\infty \sqrt{\sum_{i,j=1}^{m} \mathbb{E}\left|C_{i,j} - \langle DF^{(i)}, v^{(j)}\rangle_{\mathcal{H}}\right|^2},$$

with

$$\|h''\|_\infty = \max_{1\le i,j\le m} \sup_{x\in\mathbb{R}^m} \left|\frac{\partial^2 h}{\partial x_i \partial x_j}(x)\right|.$$

Proof. See [33, Proposition 2.3]. □

2.2.5 Local statistics of the limit

We need the following local statistics of the limiting random field $u^\varepsilon(t,x)$ as $\varepsilon \to 0$.

Proposition 2.2.10. *If $\beta\sigma_{\mathrm{Lip}} < \sqrt{2\pi}$, for any $t > 0$ and $x \in \mathbb{R}^2$, we have*

$$u^\varepsilon(t,x) \overset{law}{\to} \Xi_{1,2}(2) \qquad \text{as } \varepsilon \to 0.$$

The random process $\Xi_{1,2}(\cdot)$ is defined as in (2.1.8)–(2.1.9).

Proof. See [19, Theorem 1.2]. □

We also need the following spatial regularity statement for $u^\varepsilon(t,\cdot)$.

Proposition 2.2.11. *For fixed $T > 0$, we have*

$$\lim_{\varepsilon\to 0} \sup_{\substack{t\in[0,T] \\ x_1,x_2\in\mathbb{R}^2}} \frac{\left(\mathbb{E}(u^\varepsilon(t,x_1) - u^\varepsilon(t,x_2))^2\right)^{1/2}}{1+\varepsilon^{-1}|x_1 - x_2|} = 0.$$

Proof. See [19, Corollary 7.2]. □

2.3 Proof of Theorem 2.1.1

For fixed $T > 0$ and $g \in C_c^\infty(\mathbb{R}^2)$, let

$$v_{\varepsilon,T}(g)(s,y) = \beta \mathbb{1}_{[0,\frac{T}{\varepsilon^2}]}(s)\sigma\left(\mathcal{V}^\varepsilon(s,y)\right) \int_{\mathbb{R}^2} G_{\frac{T}{\varepsilon^2}-s}\left(\frac{x}{\varepsilon} - y\right) g(x)\mathrm{d}x.$$

By the formulation of $Y^{\varepsilon,T}$ in (2.2.4) and [47, Proposition 1.3.8], we have

$$\mathrm{Y}^{\varepsilon,T}(g) = \delta(v_{\varepsilon,T}(g)) \in \mathbb{D}^{1,2},$$

and

$$\begin{aligned}
&D_{s,y}\left(\mathrm{Y}^{\varepsilon,T}(g)\right) \qquad (2.3.1)\\
&= v_{\varepsilon,T}(g)(s,y)\\
&+ \beta\int_s^{\frac{T}{\varepsilon^2}}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} G_{\frac{T}{\varepsilon^2}-r}(\frac{x}{\varepsilon}-z)g(x)\mathrm{d}x D_{s,y}\sigma\left(\mathcal{V}^\varepsilon(r,z)\right)\mathrm{d}W_\phi(r,z)\\
&= v_{\varepsilon,T}(g)(s,y)\\
&+ \beta\int_s^{\frac{T}{\varepsilon^2}}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} G_{\frac{T}{\varepsilon^2}-r}(\frac{x}{\varepsilon}-z)g(x)\mathrm{d}x\Sigma(r,z)D_{s,y}\mathcal{V}^\varepsilon(r,z)\mathrm{d}W_\phi(r,z).
\end{aligned}$$

Here $\Sigma(r,z)$ is the same random process as in (2.2.5).

Our goal is to prove that $Y^{\varepsilon,T}(g)$ converges in law to $\mathrm{X}^T(g)$ as defined in (2.1.6) when $\varepsilon\to 0$. Since U is the random distribution that solves the Edwards-Wilkinson equation (2.1.7), we have that $\mathrm{X}^T(g)$ is of normal distribution $N(0,\Sigma_g^T)$ where

$$\begin{aligned}
\Sigma_g^T &:= \mathbb{V}ar\left(\int_{\mathbb{R}^2} U(T,x)g(x)\mathrm{d}x\right) \qquad (2.3.2)\\
&= \beta^2\mathbb{E}\sigma\left(\Xi_{1,2}(2)\right)^2\int_0^T\int_{\mathbb{R}^{2\times 3}} G_{T-s}(x-z)G_{T-s}(y-z)g(x)g(y)\mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}s\\
&= \beta^2\mathbb{E}\sigma\left(\Xi_{1,2}(2)\right)^2\int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} G_{2(T-s)}(x-y)g(x)g(y)\mathrm{d}x\mathrm{d}y\mathrm{d}s\\
&= \beta^2\mathbb{E}\sigma(\Xi_{1,2}(2))^2\int_0^T\int_{\mathbb{R}^2}|G_{T-s}*g(y)|^2\mathrm{d}y\mathrm{d}s.
\end{aligned}$$

Note that the last integral is finite for any $g\in C_c^\infty(\mathbb{R}^2)$, but is not finite if $g=\delta_0$.

By Proposition 2.2.7, the proof of Theorem 2.1.1 is reduced to showing that

$$\mathbb{E}\left|\Sigma_g^T - \langle DY^{\varepsilon,T}(g), v_{\varepsilon,T}(g)\rangle_{\mathcal{H}}\right|^2 \to 0, \qquad \text{as } \varepsilon \to 0. \tag{2.3.3}$$

We follow the procedure in [17, Section 4]. From (2.3.1), we have

$$\langle DY^{\varepsilon,T}(g), v_{\varepsilon,T}(g)\rangle_{\mathcal{H}} = A_{1,\varepsilon} + A_{2,\varepsilon},$$

where

$$\begin{aligned}
A_{1,\varepsilon} &= \langle v_{\varepsilon,T}(g), v_{\varepsilon,T}(g)\rangle_{\mathcal{H}} \\
&= \beta^2 \int_0^{\frac{T}{\varepsilon^2}} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \sigma(\mathcal{V}^\varepsilon(s,y_1))\sigma(\mathcal{V}^\varepsilon(s,y_2)) R(y_1 - y_2) \\
&\cdot \left(\int_{\mathbb{R}^2} G_{\frac{T}{\varepsilon^2}-s}(\frac{x_1}{\varepsilon} - y_1) g(x_1) \mathrm{d}x_1\right)\left(\int_{\mathbb{R}^2} G_{\frac{T}{\varepsilon^2}-s}(\frac{x_2}{\varepsilon} - y_2) g(x_2) \mathrm{d}x_2\right) \mathrm{d}y_1 \mathrm{d}y_2 \mathrm{d}s.
\end{aligned} \tag{2.3.4}$$

and

$$\begin{aligned}
A_{2,\varepsilon} &= \langle DY^{\varepsilon,T}(g) - v_{\varepsilon,T}(g), v_{\varepsilon,T}(g)\rangle_{\mathcal{H}} \\
&= \beta^2 \int_0^{\frac{T}{\varepsilon^2}} \int_{\mathbb{R}^{2\times 2}} \sigma(\mathcal{V}^\varepsilon(s,y_1)) \left(\int_{\mathbb{R}^2} G_{\frac{T}{\varepsilon^2}-s}(\frac{x_1}{\varepsilon} - y_1) g(x_1) \mathrm{d}x_1\right) \\
&\cdot \left(\int_s^{\frac{T}{\varepsilon^2}} \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} G_{\frac{T}{\varepsilon^2}-r}(\frac{x_2}{\varepsilon} - z) g(x_2) \mathrm{d}x_2\right) \Sigma(r,z) D_{s,y_2}\mathcal{V}^\varepsilon(r,z) \mathrm{d}W_\phi(r,z)\right) \\
&\cdot R(y_1 - y_2) \mathrm{d}y_1 \mathrm{d}y_2 \mathrm{d}s.
\end{aligned} \tag{2.3.5}$$

We notice that

$$\mathbb{E}\left|\Sigma_g^T - \langle DY^{\varepsilon,T}(g), v_{\varepsilon,T}(g)\rangle_{\mathcal{H}}\right|^2 \le 2\mathbb{E}\left|\Sigma_g^T - A_{1,\varepsilon}\right|^2 + 2\mathbb{E}\left|A_{2,\varepsilon}\right|^2. \tag{2.3.6}$$

Thus it is sufficient to prove the following two lemmas.

Lemma 2.3.1. *If* $\beta < \beta_0$*,*

$$\mathbb{E}\left|\Sigma_g^T - A_{1,\varepsilon}\right|^2 \to 0 \qquad \textit{as } \varepsilon \to 0.$$

Lemma 2.3.2. *If* $\beta < \beta_0$*,*

$$\mathbb{E}\left|A_{2,\varepsilon}\right|^2 \to 0 \qquad \textit{as } \varepsilon \to 0.$$

To simplify the equations, we use the convolution notation

$$G_t * g(x) := \int_{\mathbb{R}^2} G_t(x-y)g(y)\mathrm{d}y, \quad \text{for all } (t,x) \in (0,+\infty)\times\mathbb{R}^2.$$

Then by the properties of heat kernel, we have for all $t > 0$,

$$G_t * g \in C^\infty(\mathbb{R}^2);$$

$$\|G_t * g\|_{L^\infty(\mathbb{R}^2)} \le \|g\|_{L^\infty(\mathbb{R}^2)}; \tag{2.3.7}$$

$$\|G_t * g\|_{L^1(\mathbb{R}^2)} \le \|g\|_{L^1(\mathbb{R}^2)}; \tag{2.3.8}$$

and for all $x \in \mathbb{R}^2$,

$$\lim_{t\to 0} G_t * g(x) = g(x).$$

We may rewrite

$$\int_{\mathbb{R}^2} G_{\frac{T}{\varepsilon^2}-s}(\frac{x}{\varepsilon}-y)g(x)\mathrm{d}x = \varepsilon^2\int_{\mathbb{R}^2} G_{T-\varepsilon^2 s}(x-\varepsilon y)g(x)\mathrm{d}x = \varepsilon^2 G_{T-\varepsilon^2 s}*g(\varepsilon y).$$

By a change of variable $\varepsilon^2 s\mapsto s$, $\varepsilon y_1\mapsto y_1$, $y_2-y_1\mapsto y_2$, we have

$$A_{1,\varepsilon} = \beta^2\int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\sigma\left(\mathcal{V}^\varepsilon(\frac{s}{\varepsilon^2},\frac{y_1}{\varepsilon})\right)\sigma\left(\mathcal{V}^\varepsilon(\frac{s}{\varepsilon^2},\frac{y_1}{\varepsilon}+y_2)\right)R(y_2) \qquad (2.3.9)$$
$$\cdot G_{T-s}*g(y_1)G_{T-s}*g(y_1+\varepsilon y_2)\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}s,$$

and

$$A_{2,\varepsilon} = \beta^2\int_0^T\int_{\mathbb{R}^{2\times 2}}\sigma\left(\mathcal{V}^\varepsilon(\frac{s}{\varepsilon^2},\frac{y_1}{\varepsilon})\right)\cdot\Bigg(\int_{\frac{s}{\varepsilon^2}}^{\frac{T}{\varepsilon^2}}\int_{\mathbb{R}^2} G_{T-s}*g(y_1)R(y_2) \qquad (2.3.10)$$
$$G_{T-\varepsilon^2 r}*g(\varepsilon z)\Sigma(r,z)D_{\frac{s}{\varepsilon^2},\frac{y_1}{\varepsilon}+y_2}\mathcal{V}^\varepsilon(r,z)\mathrm{d}W_\phi(r,z)\Bigg)\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}s.$$

Without loss of generality, we assume that $g\geq 0$ when we estimate the integrals involving g in the part (ii) of the proof of Lemma 2.3.1 and the proof of Lemma 2.3.2. Observing that $|G_t*g(x)|\leq G_t*|g|(x)$ for any $t>0$ and $x\in\mathbb{R}^2$, one can easily check for general $g\in C_c^\infty(\mathbb{R}^2)$.

Proof of Lemma 2.3.1. To prove $\mathbb{E}\left|\Sigma_g^T - A_{1,\varepsilon}\right|^2\to 0$ as $\varepsilon\to 0$, we show that (i) $\mathbb{E}(A_{1,\varepsilon})\to\Sigma_g^T$ as $\varepsilon\to 0$, and (ii) $\mathbb{V}ar(A_{1,\varepsilon})\to 0$ as $\varepsilon\to 0$.

Part (i): By (2.3.9),

$$\mathbb{E}(A_{1,\varepsilon}) = \beta^2\int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\mathbb{E}\left(\sigma\left(\mathcal{V}^\varepsilon(\frac{s}{\varepsilon^2},\frac{y_1}{\varepsilon})\right)\sigma\left(\mathcal{V}^\varepsilon(\frac{s}{\varepsilon^2},\frac{y_1}{\varepsilon}+y_2)\right)\right)R(y_2) \qquad (2.3.11)$$
$$\cdot G_{T-s}*g(y_1)G_{T-s}*g(y_1+\varepsilon y_2)\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}s.$$

By Proposition 2.2.10, 2.2.11 and Lemma 2.2.3, we have that, for any $s \in (0,T)$ and $y_1, y_2 \in \mathbb{R}^2$,

$$\mathbb{E}\left(\sigma\left(\mathcal{V}^\varepsilon(\frac{s}{\varepsilon^2},\frac{y_1}{\varepsilon})\right)\sigma\left(\mathcal{V}^\varepsilon(\frac{s}{\varepsilon^2},\frac{y_1}{\varepsilon}+y_2)\right)\right) \to \mathbb{E}\sigma(\Xi_{1,2}(2))^2, \quad \text{as } \varepsilon \to 0.$$

The Lipschitz condition and Lemma 2.2.3 gives the uniform integrability to pass to the limit in (2.3.11) and conclude that

$$\begin{aligned}\mathbb{E}(A_{1,\varepsilon}) &\to \beta^2 \int_0^T \int_{\mathbb{R}^4} \mathbb{E}\sigma(\Xi_{1,2}(2))^2 R(y_2)\cdot|G_{T-s}*g(y_1)|^2 \mathrm{d}y_1\mathrm{d}y_2\mathrm{d}s\\ &= \beta^2\mathbb{E}\sigma(\Xi_{1,2}(2))^2 \int_0^T\int_{\mathbb{R}^2}|G_{T-s}*g(y_1)|^2\mathrm{d}y_1\mathrm{d}s\\ &= \Sigma_g^T, \qquad \text{as } \varepsilon\to 0.\end{aligned}$$

Part (ii): As in [17, Lemma 4.1], let

$$\Lambda_\varepsilon(s,y_1,y_2) = \sigma\left(\mathcal{V}^\varepsilon(\frac{s}{\varepsilon^2},\frac{y_1}{\varepsilon})\right)\sigma\left(\mathcal{V}^\varepsilon(\frac{s}{\varepsilon^2},\frac{y_1}{\varepsilon}+y_2)\right).$$

Then by the chain rule, for all $0 \le r < \frac{s}{\varepsilon^2}$ and $z \in \mathbb{R}^2$,

$$\begin{aligned}D_{r,z}\Lambda_\varepsilon(s,y_1,y_2) = {}& \Sigma\left(\frac{s}{\varepsilon^2},\frac{y_1}{\varepsilon}\right)D_{r,z}\mathcal{V}^\varepsilon(\frac{s}{\varepsilon^2},\frac{y_1}{\varepsilon})\sigma\left(\mathcal{V}^\varepsilon(\frac{s}{\varepsilon^2},\frac{y_1}{\varepsilon}+y_2)\right)\\ &+\sigma\left(\mathcal{V}^\varepsilon(\frac{s}{\varepsilon^2},\frac{y_1}{\varepsilon})\right)\Sigma\left(\frac{s}{\varepsilon^2},\frac{y_1}{\varepsilon}+y_2\right)D_{r,z}\mathcal{V}^\varepsilon(\frac{s}{\varepsilon^2},\frac{y_1}{\varepsilon}+y_2).\end{aligned}$$

Use Hölder inequality and apply the uniform moment bounds from Lemma 2.2.3 and 2.2.5 (with

$p = 4$). We obtain

$$\left(\mathbb{E}\left[D_{r,z}\Lambda_\varepsilon(s,y_1,y_2)\right]^2\right)^{\frac{1}{2}} \tag{2.3.12}$$
$$\leq \frac{C}{\sqrt{\log\varepsilon^{-1}}}\left(G_{\frac{s}{\varepsilon^2}-r}\left(\frac{y_1}{\varepsilon}-z\right)+G_{\frac{s}{\varepsilon^2}-r}\left(\frac{y_1}{\varepsilon}+y_2-z\right)\right)$$

Now, by (2.3.9) and using that $\sqrt{\mathrm{Var}\left(\int_0^t \Phi_s \mathrm{d}s\right)} \leq \int_0^t \sqrt{\mathrm{Var}\left(\Phi_s\right)}\mathrm{d}s$ for any process $\Phi = \{\Phi(s), s \in [0,t]\}$ with $\sqrt{\mathrm{Var}\left(\Phi_s\right)}$ being integrable on $[0,t]$, we obtain

$$\sqrt{\mathbb{V}ar(A_{1,\varepsilon})} \leq \beta^2 \int_0^T \Bigg(\int_{\mathbb{R}^8} \mathrm{Cov}\left[\Lambda_\varepsilon(s,y_1,y_2), \Lambda_\varepsilon(s,y_1',y_2')\right] R(y_2)R(y_2') \tag{2.3.13}$$
$$\cdot G_{T-s} * g(y_1) G_{T-s} * g(y_1+\varepsilon y_2) G_{T-s} * g(y_1') G_{T-s} * g(y_1'+\varepsilon y_2')$$
$$\mathrm{d}y_1 \mathrm{d}y_1' \mathrm{d}y_2 \mathrm{d}y_2' \Bigg)^{\frac{1}{2}} \mathrm{d}s.$$

By the Clark-Ocone formula (Proposition 2.2.2),

$$\Lambda_\varepsilon(s,y_1,y_2) = \mathbb{E}\left(\Lambda_\varepsilon(s,y_1,y_2)\right) + \int_0^{\frac{s}{\varepsilon^2}} \int_{\mathbb{R}^2} \mathbb{E}\left(D_{r,z}\Lambda_\varepsilon(s,y_1,y_2)|\mathscr{F}_r\right) \mathrm{d}W_\phi(r,z).$$

Using the Hölder's inequality and the Jensen's inequality, we have

$$
\begin{aligned}
&|\mathrm{Cov}\left[\Lambda_\varepsilon(s,y_1,y_2),\Lambda_\varepsilon(s,y_1',y_2')\right]| \\
&= \int_0^{\frac{s}{\varepsilon^2}}\int_{\mathbb{R}^4}\mathbb{E}\left|\mathbb{E}\left(D_{r,z}\Lambda_\varepsilon(s,y_1,y_2)|\mathscr{F}_r\right)\mathbb{E}\left(D_{r,z'}\Lambda_\varepsilon(s,y_1',y_2')|\mathscr{F}_r\right)\right| \\
&\cdot R(z-z')\mathrm{d}z\mathrm{d}z'\mathrm{d}r \\
&\le \int_0^{\frac{s}{\varepsilon^2}}\int_{\mathbb{R}^4}\left(\mathbb{E}\left[D_{r,z}\Lambda_\varepsilon(s,y_1,y_2)\right]^2\right)^{\frac12}\left(\mathbb{E}\left[D_{r,z'}\Lambda_\varepsilon(s,y_1',y_2')\right]^2\right)^{\frac12} \\
&\cdot R(z-z')\mathrm{d}z\mathrm{d}z'\mathrm{d}r \\
&\le \frac{C^2}{\log\varepsilon^{-1}}\int_0^{\frac{s}{\varepsilon^2}}\int_{\mathbb{R}^4}\left(G_{\frac{s}{\varepsilon^2}-r}(\frac{y_1}{\varepsilon}-z)+G_{\frac{s}{\varepsilon^2}-r}(\frac{y_1}{\varepsilon}+y_2-z)\right) \\
&\cdot\left(G_{\frac{s}{\varepsilon^2}-r}(\frac{y_1'}{\varepsilon}-z')+G_{\frac{s}{\varepsilon^2}-r}(\frac{y_1'}{\varepsilon}+y_2'-z')\right)R(z-z')\mathrm{d}z\mathrm{d}z'\mathrm{d}r. \\
&= \frac{C^2}{\log\varepsilon^{-1}}\int_0^{s}\int_{\mathbb{R}^4}\left(G_{s-r}(y_1-z)+G_{s-r}(y_1+\varepsilon y_2-z)\right) \\
&\cdot\left(G_{s-r}(y_1'-z')+G_{s-r}(y_1'+\varepsilon y_2'-z')\right)\frac{1}{\varepsilon^2}R(\frac{z-z'}{\varepsilon})\mathrm{d}z\mathrm{d}z'\mathrm{d}r.
\end{aligned}
$$

We applied (2.3.12) and do a change of variable in the last two steps.

If we do a change of variable $z-z'\mapsto z$ and integrate in z', the above expression equals

$$
\begin{aligned}
&\frac{C^2}{\log\varepsilon^{-1}}\int_0^s\int_{\mathbb{R}^2}G_{2s-2r}(y_1-y_1'-z)\frac{1}{\varepsilon^2}R(\frac{z}{\varepsilon})+G_{2s-2r}(y_1+\varepsilon y_2-y_1'-z)\frac{1}{\varepsilon^2}R(\frac{z}{\varepsilon}) \\
&+G_{2s-2r}(y_1-y_1'-\varepsilon y_2'-z)\frac{1}{\varepsilon^2}R(\frac{z}{\varepsilon})+G_{2s-2r}(y_1+\varepsilon y_2-y_1'-\varepsilon y_2'-z)\frac{1}{\varepsilon^2}R(\frac{z}{\varepsilon})\mathrm{d}z\mathrm{d}r.
\end{aligned}
\tag{2.3.14}
$$

Denote $R^\varepsilon(\cdot):=\frac{1}{\varepsilon^2}R(\frac{\cdot}{\varepsilon})$. We have $\|R^\varepsilon\|_{L^1(\mathbb{R}^2)}=1$ and for all $t>0$, it holds that

$$
\|G_t * g * R^\varepsilon\|_{L^\infty(\mathbb{R}^2)} \le \|g * R^\varepsilon\|_{L^\infty(\mathbb{R}^2)} \le \|g\|_{L^\infty(\mathbb{R}^2)}, \tag{2.3.15}
$$

and

$$\|G_t * g * R^\varepsilon\|_{L^1(\mathbb{R}^2)} \le \|g * R^\varepsilon\|_{L^1(\mathbb{R}^2)} \le \|g\|_{L^1(\mathbb{R}^2)} . \tag{2.3.16}$$

Go back to (2.3.13) with a change of variable $\varepsilon y_2 \mapsto y_2$ and $\varepsilon y_2' \mapsto y_2'$, we have

$$\begin{aligned}
&\sqrt{\mathbb{V}ar(A_{1,\varepsilon})} \\
&\le \frac{\beta^2 C}{\sqrt{\log \varepsilon^{-1}}} \int_0^T \bigg(\int_{\mathbb{R}^8} \int_0^s \big(G_{2s-2r} * R^\varepsilon(y_1 - y_1') \\
&+ G_{2s-2r} * R^\varepsilon(y_1 + y_2 - y_1') + G_{2s-2r} * R^\varepsilon(y_1 - y_1' - y_2') \\
&+ G_{2s-2r} * R^\varepsilon(y_1 + y_2 - y_1' - y_2')\big) G_{T-s} * g(y_1) G_{T-s} * g(y_1 + y_2) \\
&\cdot G_{T-s} * g(y_1') G_{T-s} * g(y_1' + y_2') R^\varepsilon(y_2) R^\varepsilon(y_2') \mathrm{d}r \mathrm{d}y_1 \mathrm{d}y_1' \mathrm{d}y_2 \mathrm{d}y_2' \bigg)^{\frac{1}{2}} \mathrm{d}s \\
&\le \frac{\beta^2 C}{\sqrt{\log \varepsilon^{-1}}} \int_0^T \bigg(\int_0^s \int_{\mathbb{R}^8} \big(G_{2s-2r} * R^\varepsilon(y_1 - y_1') \\
&+ G_{2s-2r} * R^\varepsilon(y_1 + y_2 - y_1') + G_{2s-2r} * R^\varepsilon(y_1 - y_1' - y_2') \\
&+ G_{2s-2r} * R^\varepsilon(y_1 + y_2 - y_1' - y_2')\big) G_{T-s} * g(y_1) \cdot R^\varepsilon(y_2) R^\varepsilon(y_2') \mathrm{d}y_1 \mathrm{d}y_1' \mathrm{d}y_2 \mathrm{d}y_2' \mathrm{d}r \bigg)^{\frac{1}{2}} \mathrm{d}s \\
&= \frac{\beta^2 C}{\sqrt{\log \varepsilon^{-1}}} \int_0^T \bigg(\int_0^s \int_{\mathbb{R}^6} \big(G_{T+s-2r} * R^\varepsilon * g(y_1') + G_{T+s-2r} * R^\varepsilon * g(y_1' - y_2) \\
&+ G_{T+s-2r} * R^\varepsilon * g(y_1' + y_2') + G_{T+s-2r} * R^\varepsilon * g(y_1' + y_2' - y_2)\big) \\
&\cdot R^\varepsilon(y_2) R^\varepsilon(y_2') \mathrm{d}y_1' \mathrm{d}y_2 \mathrm{d}y_2' \mathrm{d}r \bigg)^{\frac{1}{2}} \mathrm{d}s
\end{aligned}$$

If we integrate each term in the order of y_1, y_1', y_2, y_2' and use the inequalities (2.3.7) and (2.3.16), the above is bounded by

$$\frac{\beta^2 C}{\sqrt{\log \varepsilon^{-1}}} \int_0^T \left(\int_0^s \int_{\mathbb{R}^4} R^\varepsilon(y_2) R^\varepsilon(y_2') \mathrm{d}y_2 \mathrm{d}y_2' \mathrm{d}r \right)^{\frac{1}{2}} \mathrm{d}s \le \frac{\beta^2 C}{\sqrt{\log \varepsilon^{-1}}}$$

The constant C may vary at each occurrence. The last C depend on T and $\beta\sigma_{\mathrm{Lip}}$. □

Proof of Lemma 2.3.2. By Minkowski inequality and (2.3.10),

$$
\begin{aligned}
&\left(\mathbb{E}|A_{2,\varepsilon}|^2\right)^{\frac{1}{2}} \qquad (2.3.17)\\
&\leq \beta^2 \int_0^T \Bigg(\mathbb{E}\bigg|\int_{\mathbb{R}^{2\times 2}} \sigma(\mathcal{V}^\varepsilon(\frac{s}{\varepsilon^2},\frac{y_1}{\varepsilon}))G_{T-s} * g(y_1)R(y_2)\\
&\cdot \int_{\frac{s}{\varepsilon^2}}^{\frac{T}{\varepsilon^2}} \int_{\mathbb{R}^2} G_{T-\varepsilon^2 r} * g(\varepsilon z)\Sigma(r,z) D_{\frac{s}{\varepsilon^2},\frac{y_1}{\varepsilon}+y_2}\mathcal{V}^\varepsilon(r,z)\mathrm{d}W_\phi(r,z)\mathrm{d}y_1\mathrm{d}y_2\bigg|^2\Bigg)^{\frac{1}{2}}\mathrm{d}s\\
&= \beta^2 \int_0^T \Bigg(\int_{\mathbb{R}^{2\times 2}}\int_{\mathbb{R}^{2\times 2}} G_{T-s} * g(y_1)R(y_2)G_{T-s} * g(y_1')R(y_2')\\
&\cdot \mathbb{E}\bigg[\sigma(\mathcal{V}^\varepsilon(\frac{s}{\varepsilon^2},\frac{y_1}{\varepsilon}))\sigma(\mathcal{V}^\varepsilon(\frac{s}{\varepsilon^2},\frac{y_1'}{\varepsilon}))\\
&\cdot \int_{\frac{s}{\varepsilon^2}}^{\frac{T}{\varepsilon^2}} \int_{\mathbb{R}^2} G_{T-\varepsilon^2 r} * g(\varepsilon z)\Sigma(r,z) D_{\frac{s}{\varepsilon^2},\frac{y_1}{\varepsilon}+y_2}\mathcal{V}^\varepsilon(r,z)\mathrm{d}W_\phi(r,z)\\
&\cdot \int_{\frac{s}{\varepsilon^2}}^{\frac{T}{\varepsilon^2}} \int_{\mathbb{R}^2} G_{T-\varepsilon^2 r'} * g(\varepsilon z')\Sigma(r',z') D_{\frac{s}{\varepsilon^2},\frac{y_1'}{\varepsilon}+y_2'}\mathcal{V}^\varepsilon(r',z')\mathrm{d}W_\phi(r',z')\bigg]\\
&\mathrm{d}y_1\mathrm{d}y_1'\mathrm{d}y_2\mathrm{d}y_2'\Bigg)^{\frac{1}{2}}\mathrm{d}s.
\end{aligned}
$$

Using Itô isometry, we have

$$
\begin{aligned}
&\mathbb{E}\bigg[\sigma(\mathcal{V}^\varepsilon(\frac{s}{\varepsilon^2},\frac{y_1}{\varepsilon}))\sigma(\mathcal{V}^\varepsilon(\frac{s}{\varepsilon^2},\frac{y_1'}{\varepsilon}))\\
&\cdot \int_{\frac{s}{\varepsilon^2}}^{\frac{T}{\varepsilon^2}} \int_{\mathbb{R}^2} G_{T-\varepsilon^2 r} * g(\varepsilon z)\Sigma(r,z) D_{\frac{s}{\varepsilon^2},\frac{y_1}{\varepsilon}+y_2}\mathcal{V}^\varepsilon(r,z)\mathrm{d}W_\phi(r,z)\\
&\cdot \int_{\frac{s}{\varepsilon^2}}^{\frac{T}{\varepsilon^2}} \int_{\mathbb{R}^2} G_{T-\varepsilon^2 r'} * g(\varepsilon z')\Sigma(r',z') D_{\frac{s}{\varepsilon^2},\frac{y_1'}{\varepsilon}+y_2'}\mathcal{V}^\varepsilon(r',z')\mathrm{d}W_\phi(r',z')\bigg]\\
&= \int_{\frac{s}{\varepsilon^2}}^{\frac{T}{\varepsilon^2}} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} R(z-z')G_{T-\varepsilon^2 r} * g(\varepsilon z)G_{T-\varepsilon^2 r} * g(\varepsilon z')\mathbb{E}\bigg[\sigma(\mathcal{V}^\varepsilon(\frac{s}{\varepsilon^2},\frac{y_1}{\varepsilon}))\\
&\cdot \sigma(\mathcal{V}^\varepsilon(\frac{s}{\varepsilon^2},\frac{y_1'}{\varepsilon}))\Sigma(r,z)D_{\frac{s}{\varepsilon^2},\frac{y_1}{\varepsilon}+y_2}\mathcal{V}^\varepsilon(r,z)\Sigma(r,z')D_{\frac{s}{\varepsilon^2},\frac{y_1'}{\varepsilon}+y_2'}\mathcal{V}^\varepsilon(r,z')\bigg]\mathrm{d}z\mathrm{d}z'\mathrm{d}r.
\end{aligned}
$$

By applying (2.2.9), Lemma 2.2.5 (with $p=4$) and the Cauchy-Schwarz inequality, there exists a uniform $C>0$ such that when ε is small enough,

$$
\begin{aligned}
&\mathbb{E}\Big[\sigma(\mathcal{V}^\varepsilon(\frac{s}{\varepsilon^2},\frac{y_1}{\varepsilon}))\sigma(\mathcal{V}^\varepsilon(\frac{s}{\varepsilon^2},\frac{y_1'}{\varepsilon}))\Sigma(r,z)D_{\frac{s}{\varepsilon^2},\frac{y_1}{\varepsilon}+y_2}\mathcal{V}^\varepsilon(r,z)\Sigma(r,z')D_{\frac{s}{\varepsilon^2},\frac{y_1'}{\varepsilon}+y_2'}\mathcal{V}^\varepsilon(r,z')\Big] \\
&\le \frac{C^2}{\log\varepsilon^{-1}}G_{r-\frac{s}{\varepsilon^2}}(z-\frac{y_1}{\varepsilon}-y_2)G_{r-\frac{s}{\varepsilon^2}}(z'-\frac{y_1'}{\varepsilon}-y_2').
\end{aligned}
$$

Substitute back into (2.3.17) and integrate in y_1, y_1', we have

$$
\begin{aligned}
&\big(\mathbb{E}|A_{2,\varepsilon}|^2\big)^{\frac12} \le \frac{\beta^2 C}{\sqrt{\log\varepsilon^{-1}}}\int_0^T\Bigg(\int_{\mathbb{R}^{2\times2}}\int_{\mathbb{R}^{2\times2}}G_{T-s}*g(y_1)R(y_2)G_{T-s}*g(y_1')R(y_2') \qquad (2.3.18)\\
&\int_{\frac{s}{\varepsilon^2}}^{\frac{T}{\varepsilon^2}}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}R(z-z')G_{T-\varepsilon^2 r}*g(\varepsilon z)G_{T-\varepsilon^2 r}*g(\varepsilon z')\\
&\cdot G_{r-\frac{s}{\varepsilon^2}}(z-\frac{y_1}{\varepsilon}-y_2)G_{r-\frac{s}{\varepsilon^2}}(z'-\frac{y_1'}{\varepsilon}-y_2')\mathrm{d}z\mathrm{d}z'\mathrm{d}r\mathrm{d}y_1\mathrm{d}y_1'\mathrm{d}y_2\mathrm{d}y_2'\Bigg)^{\frac12}\mathrm{d}s\\
&=\frac{\beta^2 C}{\sqrt{\log\varepsilon^{-1}}}\int_0^T\Bigg(\varepsilon^2\int_s^T\int_{\mathbb{R}^8}R(y_2)R(y_2')R(z-z')G_{T-r}*g(\varepsilon z)G_{T-r}*g(\varepsilon z')\\
&\cdot G_{T+r-2s}*g(\varepsilon z-\varepsilon y_2)G_{T+r-2s}*g(\varepsilon z'-\varepsilon y_2')\mathrm{d}r\mathrm{d}z\mathrm{d}z'\mathrm{d}y_2\mathrm{d}y_2'\Bigg)^{\frac12}\mathrm{d}s\\
&=\frac{\beta^2 C}{\sqrt{\log\varepsilon^{-1}}}\int_0^T\Bigg(\int_s^T\int_{\mathbb{R}^8}R^\varepsilon(y_2)R^\varepsilon(y_2')R^\varepsilon(z-z')G_{T-r}*g(z)G_{T-r}*g(z')\\
&\cdot G_{T+r-2s}*g(z-y_2)G_{T+r-2s}*g(z'-y_2')\mathrm{d}r\mathrm{d}z\mathrm{d}z'\mathrm{d}y_2\mathrm{d}y_2'\Bigg)^{\frac12}\mathrm{d}s.
\end{aligned}
$$

Here we do a change of variable $\varepsilon y_2\mapsto y_2$, $\varepsilon y_2'\mapsto y_2'$, $\varepsilon z\mapsto z$ and $\varepsilon z'\mapsto z'$. We integrate the last integral of (2.3.18) in the order of y_2, y_2', z,z'. Together with (2.3.8) and (2.3.15), we obtain that the last expression is bounded by

$$\begin{aligned}&\frac{\beta^2 C}{\sqrt{\log\varepsilon^{-1}}}\int_0^T\left(\int_s^T\int_{\mathbb{R}^4}R^\varepsilon(z-z')G_{T-r}*g(z)G_{T-r}*g(z')\mathrm{d}r\mathrm{d}z\mathrm{d}z'\right)^{\frac{1}{2}}\mathrm{d}s\\&\le\frac{\beta^2 C}{\sqrt{\log\varepsilon^{-1}}},\end{aligned}$$

The constant C may vary at each occurrence. The last C would depend on T and $\beta\sigma_{\mathrm{Lip}}$. □

2.4 Proof of Theorem 2.1.2

It suffices to prove the convergence of the finite-dimensional distributions and the tightness. To give the tightness, we use the following proposition.

Proposition 2.4.1. *For $p \ge 2$, $T > 0$, $g \in C_c^\infty(\mathbb{R}^2)$, let $\beta_0(p)$ and ε_0 be defined as in Lemma 2.2.3. If $\beta < \beta_0(p)$, there exists a uniform constant $C > 0$ (depending only on $\beta\sigma_{\mathrm{Lip}}$, p and g), such that for any $0 \le s < t \le T$ and $\varepsilon < \varepsilon_0$, $\mathbb{E}|X^{\varepsilon,t}(g) - X^{\varepsilon,s}(g)|^p \le C(t-s)^{\frac{p}{2}}$.*

Proof. By the mild formulation of u^ε in (2.1.4) and the Burkholder-Davis-Gundy inequality, let

$c_p = \frac{p(p-1)}{2}$, we have

$$
\begin{aligned}
&\left(\mathbb{E}\left|X^{\varepsilon,t}(g) - X^{\varepsilon,s}(g)\right|^p\right)^{\frac{2}{p}} = \left(\mathbb{E}\left|\sqrt{\log\varepsilon^{-1}}\int_{\mathbb{R}^2}\left[u^\varepsilon(t,x) - u^\varepsilon(s,x)\right]g(x)\mathrm{d}x\right|^p\right)^{\frac{2}{p}} \\
&= \Bigg(\mathbb{E}\Bigg|\beta\int_{\mathbb{R}^2}\Bigg[\int_0^t\int_{\mathbb{R}^2} G_{t-r}(x-y)\sigma(u^\varepsilon(r,y))\mathrm{d}W_{\phi^\varepsilon}(r,y) \\
&- \int_0^s\int_{\mathbb{R}^2} G_{s-r}(x-y)\sigma(u^\varepsilon(r,y))\mathrm{d}W_{\phi^\varepsilon}(r,y)\Bigg]g(x)\mathrm{d}x\Bigg|^p\Bigg)^{\frac{2}{p}} \\
&= \beta^2\left(\mathbb{E}\left|\int_0^T\int_{\mathbb{R}^2}\left(G_{t-r}*g(y)\mathbb{1}_{[0,t]}(r) - G_{s-r}*g(y)\mathbb{1}_{[0,s]}(r)\right)\sigma(u^\varepsilon(r,y))\mathrm{d}W_{\phi^\varepsilon}(r,y)\right|^p\right)^{\frac{2}{p}} \\
&\le c_p\beta^2\int_0^T\Bigg(\mathbb{E}\Bigg(\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left(G_{t-r}*g(y_1)\mathbb{1}_{[0,t]}(r) - G_{s-r}*g(y_1)\mathbb{1}_{[0,s]}(r)\right) \\
&\cdot\left(G_{t-r}*g(y_2)\mathbb{1}_{[0,t]}(r) - G_{s-r}*g(y_2)\mathbb{1}_{[0,s]}(r)\right) \\
&\cdot\left(\sigma(u^\varepsilon(r,y_1))\sigma(u^\varepsilon(r,y_2))\right)\frac{1}{\varepsilon^2}R(\frac{y_1-y_2}{\varepsilon})\mathrm{d}y_1\mathrm{d}y_2\Bigg)^{\frac{p}{2}}\Bigg)^{\frac{2}{p}}\mathrm{d}r.
\end{aligned}
$$

Apply the Minkowski inequality, inequality (2.2.7) and Lemma 2.2.3, the above is bounded by

$$
\begin{aligned}
&c_p\beta^2\sigma_{\mathrm{Lip}}^2\int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left(G_{t-r}*g(y_1)\mathbb{1}_{[0,t]}(r) - G_{s-r}*g(y_1)\mathbb{1}_{[0,s]}(r)\right) \\
&\cdot\left(G_{t-r}*g(y_2)\mathbb{1}_{[0,t]}(r) - G_{s-r}*g(y_2)\mathbb{1}_{[0,s]}(r)\right)\left(\mathbb{E}|u^\varepsilon(r,0)|^p\right)^{\frac{2}{p}}\frac{1}{\varepsilon^2}R(\frac{y_1-y_2}{\varepsilon})\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}r \\
&\le C\beta^2\sigma_{\mathrm{Lip}}^2\int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left(G_{t-r}*g(y_1)\mathbb{1}_{[0,t]}(r) - G_{s-r}*g(y_1)\mathbb{1}_{[0,s]}(r)\right) \\
&\cdot\left(G_{t-r}*g(y_2)\mathbb{1}_{[0,t]}(r) - G_{s-r}*g(y_2)\mathbb{1}_{[0,s]}(r)\right)\frac{1}{\varepsilon^2}R(\frac{y_1-y_2}{\varepsilon})\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}r \\
&\le C\beta^2\sigma_{\mathrm{Lip}}^2 \\
&\int_0^T\left(\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left(G_{t-r}*g(y_1)\mathbb{1}_{[0,t]}(r) - G_{s-r}*g(y_1)\mathbb{1}_{[0,s]}(r)\right)^2\frac{1}{\varepsilon^2}R(\frac{y_1-y_2}{\varepsilon})\mathrm{d}y_1\mathrm{d}y_2\right)^{\frac{1}{2}} \\
&\cdot\left(\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left(G_{t-r}*g(y_2)\mathbb{1}_{[0,t]}(r) - G_{s-r}*g(y_2)\mathbb{1}_{[0,s]}(r)\right)\frac{1}{\varepsilon^2}R(\frac{y_1-y_2}{\varepsilon})\mathrm{d}y_1\mathrm{d}y_2\right)^{\frac{1}{2}}\mathrm{d}r \\
&= C\beta^2\sigma_{\mathrm{Lip}}^2\int_0^T\int_{\mathbb{R}^2}\left(G_{t-r}*g(y)\mathbb{1}_{[0,t]}(r) - G_{s-r}*g(y)\mathbb{1}_{[0,s]}(r)\right)^2\mathrm{d}y\mathrm{d}r.
\end{aligned}
$$

Here we used the Hölder inequality and the fact that $\|R^\varepsilon\|_{L^1} = 1$ in the last two steps. Denote the Fourier transform of g by $\hat{g}$. Adopt a similar procedure as in [33, Proposition 4.1], we have

$$
\begin{aligned}
&\int_0^T \int_{\mathbb{R}^2} \left(G_{t-r} * g(y)\mathbb{1}_{[0,t]}(r) - G_{s-r} * g(y)\mathbb{1}_{[0,s]}(r)\right)^2 \mathrm{d}y\mathrm{d}r \\
&= C\int_0^T \int_{\mathbb{R}^2} \left(e^{-\frac{t-r}{2}|\xi|^2}\mathbb{1}_{[0,t]}(r) - e^{-\frac{s-r}{2}|\xi|^2}\mathbb{1}_{[0,s]}(r)\right)^2 \hat{g}(\xi)^2 \mathrm{d}\xi\mathrm{d}r \\
&= C\int_0^s \int_{\mathbb{R}^2} e^{-(s-r)|\xi|^2}\left(e^{-\frac{t-s}{2}|\xi|^2} - 1\right)^2 \hat{g}(\xi)^2 \mathrm{d}\xi\mathrm{d}r + C\int_s^t \int_{\mathbb{R}^2} e^{-(t-r)|\xi|^2}\hat{g}(\xi)^2 \mathrm{d}\xi\mathrm{d}r \\
&= C\int_{\mathbb{R}^2} \frac{1}{|\xi|^2}\left(1 - e^{-s|\xi|^2}\right)\left(e^{-\frac{t-s}{2}|\xi|^2} - 1\right)^2 \hat{g}(\xi)^2 \mathrm{d}\xi + C\int_{\mathbb{R}^2} \frac{1}{|\xi|^2}\left(1 - e^{-(t-s)|\xi|^2}\right)\hat{g}(\xi)^2 \mathrm{d}\xi \\
&\le C\int_{\mathbb{R}^2} \frac{1}{|\xi|^2}\left(1 - e^{-s|\xi|^2}\right)\frac{t-s}{2}|\xi|^2 \hat{g}(\xi)^2 \mathrm{d}\xi + C\int_{\mathbb{R}^2} \frac{1}{|\xi|^2}(t-s)|\xi|^2 \hat{g}(\xi)^2 \mathrm{d}\xi \\
&\le C(t-s)\int_{\mathbb{R}^2} \hat{g}(\xi)^2 \mathrm{d}\xi.
\end{aligned}
$$

In the last steps, we applied inequalities $|1-e^{-a}| \le \sqrt{a}$ and $(1-e^{-a}) \le a$ for all $a \ge 0$. The constant C may vary at each occurrence.

In the proof, the conditions $\beta < \beta_0$ and $\varepsilon < \varepsilon_0$ are given only to validate the use of Lemma 2.2.3. Therefore, we can choose $\beta_0(p)$ and ε_0 to be the same value as in Lemma 2.2.3. □

Choose $p = 4$. By [52, Problem 4.11], the measure induced by $X^\varepsilon(g) = \{X^{\varepsilon,t}(g); 0 \le t \le T\}$ on $(C([0,T]), \mathscr{B}(C([0,T])))$ is tight. By [52, Theorem 4.15], to prove Theorem 2.1.2, we only need to show the convergence of the finite-dimensional distribution.

Lemma 2.4.2. *Fix $0 \le t_1 < \cdots < t_m \le T$. As $\varepsilon \to 0$, the sequence of random vectors*

$$
(X^{\varepsilon,t_1}(g), \ldots, X^{\varepsilon,t_m}(g))
$$

converges in law to the m-dimensional Gaussian centered vector $Z = (Z^1, \dots, Z^m)$ with covariance

$$C_{i,j} := \mathbb{E}(Z^i Z^j) = \beta^2 \mathbb{E}\sigma\left(\Xi_{1,2}(2)\right)^2 \int_0^{t_i \wedge t_j} \int_{\mathbb{R}^2} G_{t_i - s} * g(y) G_{t_j - s} * g(y) \mathrm{d}y \mathrm{d}s.$$

Proof. Recall that we have shown

$$\left(X^{\varepsilon, t_1}(g), \dots, X^{\varepsilon, t_m}(g)\right) \overset{\text{law}}{=} \left(Y^{\varepsilon, t_1}(g), \dots, Y^{\varepsilon, t_m}(g)\right),$$

and

$$\mathrm{Y}^{\varepsilon, t_i}(g) = \delta(v_{\varepsilon, t_i}(g)) \in \mathbb{D}^{1,2}.$$

By Proposition 2.2.9, it suffices to show that for each i, j,

$$\mathbb{E}|C_{i,j} - \langle DY^{\varepsilon, t_i}(g), v_{\varepsilon, t_j}(g)\rangle_{\mathcal{H}}|^2 \to 0.$$

Use (2.3.1), we have $\langle DY^{\varepsilon, t_i}(g), v_{\varepsilon, t_j}(g)\rangle_{\mathcal{H}} = A^{i,j}_{1,\varepsilon} + A^{i,j}_{2,\varepsilon}$, where

$$\begin{aligned}
A^{i,j}_{1,\varepsilon} &= \langle v_{\varepsilon, t_i}(g), v_{\varepsilon, t_j}(g)\rangle_{\mathcal{H}} \\
&= \beta^2 \int_0^{\frac{t_i \wedge t_j}{\varepsilon^2}} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \sigma(\mathcal{V}^\varepsilon(s, y_1)) \sigma(\mathcal{V}^\varepsilon(s, y_2)) R(y_1 - y_2) \\
&\cdot \left(\int_{\mathbb{R}^2} G_{\frac{t_i}{\varepsilon^2} - s}(\frac{x_1}{\varepsilon} - y_1) g(x_1) \mathrm{d}x_1\right) \left(\int_{\mathbb{R}^2} G_{\frac{t_j}{\varepsilon^2} - s}(\frac{x_2}{\varepsilon} - y_2) g(x_2) \mathrm{d}x_2\right) \mathrm{d}y_1 \mathrm{d}y_2 \mathrm{d}s.
\end{aligned}$$

and

$$
\begin{aligned}
A_{2,\varepsilon}^{i,j} &= \langle DY^{\varepsilon,t_i}(g) - v_{\varepsilon,t_i}(g), v_{\varepsilon,t_j}(g)\rangle_{\mathcal{H}} \\
&= \beta^2 \int_0^{\frac{t_i \wedge t_j}{\varepsilon^2}} \int_{\mathbb{R}^{2\times 2}} \sigma(\mathcal{V}^\varepsilon(s,y_1)) \left(\int_{\mathbb{R}^2} G_{\frac{t_j}{\varepsilon^2}-s}(\frac{x_1}{\varepsilon} - y_1) g(x_1) \mathrm{d}x_1 \right) \\
&\cdot \left(\int_s^{\frac{t_i}{\varepsilon^2}} \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} G_{\frac{t_i}{\varepsilon^2}-r}(\frac{x_2}{\varepsilon} - z) g(x_2) \mathrm{d}x_2 \right) \Sigma(r,z) D_{s,y_2} \mathcal{V}^\varepsilon(r,z) \mathrm{d}W_\phi(r,z) \right) \\
&\cdot R(y_1 - y_2) \mathrm{d}y_1 \mathrm{d}y_2 \mathrm{d}s.
\end{aligned}
$$

Following the same arugments as in Section 2.3, we can show

$$
\mathbb{E}\left|C_{i,j} - A_{1,\varepsilon}^{i,j}\right|^2 \to 0 \qquad \text{as } \varepsilon \to 0,
$$

and

$$
\mathbb{E}\left|A_{2,\varepsilon}^{i,j}\right|^2 \to 0 \qquad \text{as } \varepsilon \to 0.
$$

□

Chapter 3: Mesoscopic averaging of $2d$ KPZ

3.1 Introduction

3.1.1 Main results

We are interested in the two dimensional KPZ equation driven by a mollified space-time white noise

$$\partial_t h^\varepsilon(t,x) = \frac{1}{2}\Delta h^\varepsilon(t,x) + \frac{1}{2}|\nabla h^\varepsilon(t,x)|^2 + \frac{\beta}{\sqrt{\log \varepsilon^{-1}}}\xi_\varepsilon(t,x) - C_\varepsilon, \tag{3.1.1}$$

with initial condition $h^\varepsilon(0,x) = h_0(x)$ and $C_\varepsilon = \frac{\beta^2}{2\varepsilon^2 \log \varepsilon^{-1}}\|\phi\|^2_{L^2(\mathbb{R}^2)}$. Here $\beta > 0$ is a constant. We define $\xi_\varepsilon(t,x) = \phi^\varepsilon \star \xi(t,x)$, where ξ is the space-time white noise, and $\phi \in C_c^\infty(\mathbb{R}^2)$ is a non-negative, smooth, symmetric mollifier with $\int \phi \mathrm{d}x = 1$ and $\phi^\varepsilon(x) = \frac{1}{\varepsilon^2}\phi(\varepsilon^{-1}x)$. We use $\star$ to denote convolution in space. The noise ξ_ε is white in time and colored in space, with spatial correlation length in the scale of ε.

Inspired by Chatterjee's recent work [21], we will investigate the limit of a *local average* of the mollified KPZ equation (3.1.1) when $\beta \in (0, \sqrt{2\pi})$. We will show that when $h^\varepsilon(t,x)$ is averaged in space in a proper scale, its limit would equal in distribution to the sum of the solution to a deterministic KPZ equation and a Gaussian random variable. The deterministic part only depends on the initial condition h_0. The Gaussian random variable only depends on the scale of averaging. Our result shows a "mesoscopic" averaging phenomenon that appears exclusively in dimension two.

The following theorem is the main result of this section.

Theorem 3.1.1. *Let $h^\varepsilon(t,x)$ be the solution to the mollified 2-dimensional KPZ equation (3.1.1) with the initial condition $h_0:\mathbb{R}^2\to\mathbb{R}$ being bounded and Lipschitz continuous. Let $0<\beta<\sqrt{2\pi}$ and $r_\varepsilon=\varepsilon^{1-\gamma}$ for some $0\le\gamma\le 1$. We use $B(x,r_\varepsilon)$ to denote a ball with center x and radius r_ε.*

For any fixed $t>0$, $x\in\mathbb{R}^2$, the local average of h^ε over $B(x,r_\varepsilon)$ converges as follows

$$\begin{aligned}&\frac{1}{|B(x,r_\varepsilon)|}\int_{|y-x|\le r_\varepsilon} h^\varepsilon(t,y)\mathrm{d}y\\ &\xrightarrow{d}\mathcal{N}(0,\sigma_\gamma^2)-\frac12\log\left(\frac{2\pi}{2\pi-\beta^2}\right)+\begin{cases}\bar h(t,x), & \text{if } 0\le\gamma<1,\\[2ex] \frac{1}{|B(x,1)|}\int_{|y-x|\le 1}\bar h(t,y)\mathrm{d}y, & \text{if } \gamma=1,\end{cases}\end{aligned}\tag{3.1.2}$$

as $\varepsilon\to0$, where $\bar h(t,x)$ is the solution to the 2-dimensional deterministic KPZ equation

$$\partial_t\bar h(t,x)=\frac12\Delta\bar h(t,x)+\frac12|\nabla\bar h(t,x)|^2,\quad \bar h(0,x)=h_0(x),\tag{3.1.3}$$

and $\mathcal{N}(0,\sigma_\gamma^2)$ is a normal distribution with mean 0 and variance $\sigma_\gamma^2=\log\left(\frac{2\pi-\beta^2\gamma}{2\pi-\beta^2}\right)$. When $0\le\gamma<1$, we have $\sigma_\gamma^2>0$. We treat $\mathcal{N}(0,0)$ as constant zero, when $\gamma=1$ and $\sigma_\gamma^2=0$.

We first make the following remarks.

If $\gamma=0$, the averaging was performed over a ball with radius $r_\varepsilon=\varepsilon$. The limit in distribution is

$$\bar h(t,x)+\mathcal{N}(0,\log\frac{2\pi}{2\pi-\beta^2})-\frac12\log\left(\frac{2\pi}{2\pi-\beta^2}\right).$$

If we take $h_0 = 0$, then $\bar{h}(t,x) = 0$ and the limit coincides with the point-wise limit of $h^\varepsilon(t,x)$ as $\varepsilon \to 0$ obtained in [18] (see Theorem 3.1.5 below). In fact, our result generalizes the point-wise limit of KPZ equation with flat initial data to KPZ equation with more general (bounded and Lipschitz continuous) initial conditions. The case $\gamma = 0$ can be viewed as a "microscopic" averaging result.

If $\gamma = 1$, the averaging was performed over a ball with radius $r_\varepsilon = 1$. As shown in (3.1.2), the Gaussian term equals zero as $\sigma_\gamma^2 = 0$. The limit would be deterministic. It equals to a spatial average of the deterministic KPZ equation over the ball of radius 1, plus a height shift $-\frac{1}{2}\log\left(\frac{2\pi}{2\pi-\beta^2}\right)$.

The randomness has disappeared due to the independence in the limit of h^ε for distinct points. As we will see in Theorem 3.1.5 below, a result of Caravenna-Sun-Zygouras, for any finite set of distinct points $(x_i)_{1\le i\le n}$, the random variables $(h^\varepsilon(t,x_i))_{1\le i\le n}$ converge to independent Gaussians as $\varepsilon \to 0$. Thus taking a local average over a fixed-radius ball eliminates the randomness as a result of the law of large numbers. The height shift $-\frac{1}{2}\log\left(\frac{2\pi}{2\pi-\beta^2}\right)$ is the mean of the Gaussians. The case $\gamma = 1$ can be viewed as a "macroscopic" averaging result.

When $\gamma = 1$, to study the next order random fluctuations, one should look at the error $h^\varepsilon(t,x) - \mathbb{E}h^\varepsilon(t,x)$ with amplification. [53] started such studies. [20] (as well as [36] for a smaller regime of β) proved that, after proper rescaling, the error converges to an Edwards-Wilkinson limit. [37] later enhanced this result for more general initial conditions and multi-dimensional parameters. Although we both derive Gaussian limits, our work here differs from previous studies. Our study focuses on the first order term $\int_{B(0,r_\varepsilon)} h^\varepsilon(t,x)\mathrm{d}x$, whereas the previous works all studied the second order fluctuations $\sqrt{\log\varepsilon^{-1}}\int_{B(0,1)} h^\varepsilon(t,x) - \mathbb{E}h^\varepsilon(t,x)\mathrm{d}x$.

If $0 < \gamma < 1$, the averaging was performed over a ball with radius $r_\varepsilon = \varepsilon^{1-\gamma}$, where $\varepsilon \ll r_\varepsilon \ll 1$. The limit in distribution is the deterministic KPZ equation with the same height shift, plus a

Gaussian random variable. The variance of the Gaussian, σ_γ^2, is dependent solely on γ. This is an interpolation between the limiting case $\gamma = 0$ and $\gamma = 1$ and is a "mesoscopic" averaging result.

The choice of radius $r_\varepsilon = \varepsilon^{1-\gamma}$ is motivated by the work of Chatterjee [21] in dimensions three and higher and also by the multi-dimensional limit of $h^\varepsilon(t,x)$ in Theorem 3.1.5 below. For a finite collection of space-time points $(t, x_\varepsilon^{(i)})_{1\le i\le n}$, $(h^\varepsilon(t, x_\varepsilon^{(i)}))_{1\le i\le n}$ converges in joint distribution to a multi-dimensional normal distribution with a covariance matrix depending on the power scales of $\{|x_\varepsilon^{(i)} - x_\varepsilon^{(j)}| : 1 \le i < j \le n\}$. When $|x_\varepsilon^{(i)} - x_\varepsilon^{(j)}| = \varepsilon^{1-\gamma+o(1)}$, $\mathrm{Cov}(h^\varepsilon(t, x_\varepsilon^{(i)}), h^\varepsilon(t, x_\varepsilon^{(j)})) \to \sigma_\gamma^2$ as $\varepsilon \to 0$. Since an average of Gaussians would remain a Gaussian, the limit of h^ε averaged over a ball of radius $\varepsilon^{1-\gamma}$ would be a Gaussian with variance σ_γ^2.

The "height shift" term $-\frac{1}{2}\log\left(\frac{2\pi}{2\pi-\beta^2}\right)$ equals to $\lim_{\varepsilon\to0}\mathbb{E}h^\varepsilon(t,x)$ with initial condition $h_0 = 0$ for any $t > 0$ and $x \in \mathbb{R}^2$. This term appears because we take logarithm in solving the KPZ equation and it is also derived in the point-wise limit in Theorem 3.1.5.

Remark 3.1.2. Another interpretation of Theorem 3.1.1 is to view the convergence from the perspective of generalized random fields (also known as *random distribution* in the literature) in microscopic variables. For any $(t,x) \in [0,+\infty) \times \mathbb{R}^2$, we consider the following microscopic equation

$$\partial_t \tilde{h}^\varepsilon(t,x) = \frac{1}{2}\Delta\tilde{h}^\varepsilon(t,x) + \frac{1}{2}|\nabla\tilde{h}^\varepsilon(t,x)|^2 + \frac{\beta}{\sqrt{\log\varepsilon^{-1}}}\xi_1(t,x) - \frac{\beta^2}{2\log\varepsilon^{-1}}\|\phi\|_{L^2(\mathbb{R}^2)}^2,$$

with initial condition $\tilde{h}^\varepsilon(0,x) = h_0(\varepsilon x)$. The above ξ_1 is defined as in (3.1.1) with $\varepsilon = 1$. By the scaling property of the space-time white noise,

$$h^\varepsilon(\cdot,\cdot) = \tilde{h}^\varepsilon(\frac{\cdot}{\varepsilon^2},\frac{\cdot}{\varepsilon}) \qquad \text{jointly in law.}$$

The proof of Theorem 3.1.1 can be modified to generalize the following result:

Fix $t > 0$ and $x \in \mathbb{R}^2$. Let $0 \le \gamma < 1$ and $0 < \beta < \sqrt{2\pi}$. For any smooth and compactly supported test function $g \in C_c^\infty(\mathbb{R}^2)$,

$$\int_{\mathbb{R}^2} \tilde{h}^\varepsilon(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}+\frac{y}{\varepsilon^\gamma})g(y)\mathrm{d}y \xrightarrow{d} [\mathcal{N}(0,\sigma_\gamma^2) - \frac{1}{2}\log\left(\frac{2\pi}{2\pi-\beta^2}\right) + \bar{h}(t,x)] \int_{\mathbb{R}^2} g(y)\mathrm{d}y,$$

as $\varepsilon \to 0$.

By [54, Corollary 2.4], as a generalized random field, $\tilde{h}^\varepsilon(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}+\frac{\cdot}{\varepsilon^\gamma})$ converges in law to the constant Gaussian random field $[\mathcal{N}(0,\sigma_\gamma^2) - \frac{1}{2}\log\left(\frac{2\pi}{2\pi-\beta^2}\right) + \bar{h}(t,x)]\mathbb{1}_{\mathbb{R}^2}(\cdot)$. This constant Gaussian field on $\mathbb{R}^2$ has a fixed spatial covariance σ_γ^2 independent of the spatial variable. This limit is distinct from other Gaussian random field limits found in the literature on KPZ equation.

One can also consider the locally averaged KPZ equation as a generalized random field on $\mathbb{R}^2$. Let $0 \le \gamma < 1$. Define

$$\mathfrak{h}^{\varepsilon,\gamma}(t,x) := \frac{1}{|B(x,r_\varepsilon)|}\int_{|y-x|\le r_\varepsilon} h^\varepsilon(t,y)\mathrm{d}y, \quad \text{with } r_\varepsilon = \varepsilon^{1-\gamma}.$$

The following corollary shows that, as a generalized random field, $\mathfrak{h}^{\varepsilon,\gamma}(t,\cdot)$ converges to the 2-dimensional deterministic KPZ limit $\bar{h}(t,\cdot) - \frac{1}{2}\log\left(\frac{2\pi}{2\pi-\beta^2}\right)$ as $\varepsilon \to 0$.

Corollary 3.1.3. *Take any smooth and compactly supported test function $g \in C_c^\infty(\mathbb{R}^2)$. For any*

$t > 0$ and $0 \le \gamma < 1$, as $\varepsilon \to 0$, the random variable $\int_{\mathbb{R}^2} \mathfrak{h}^{\varepsilon,\gamma}(t,x) g(x) \mathrm{d}x$ converges in probability to $\int_{\mathbb{R}^2} \left[\bar{h}(t,x) - \frac{1}{2} \log \left(\frac{2\pi}{2\pi - \beta^2} \right) \right] g(x) \mathrm{d}x$.

Remark 3.1.4. A natural question to ask next is to investigate the second order fluctuations of $\mathfrak{h}^{\varepsilon,\gamma}(t,\cdot)$ after proper rescaling. Inspired by the results in [20, 36] and [37], one may expect that $\sqrt{\log \varepsilon^{-1}} \int_{\mathbb{R}^2} [\mathfrak{h}(t,x) - \mathbb{E}\mathfrak{h}(t,x)] g(x) \mathrm{d}x$ has a non-degenerate Gaussian limiting distribution as $\varepsilon \to 0$. We do not pursue further in this direction, as proving such results may require techniques that are beyond the scope of this section.

We now conclude the section with a note on the significance of this study. In Theorem 3.1.1, we proved that the local average of the KPZ equation in dimension two shows a specific mesoscopic averaging phenomenon. In Remark 3.1.2, when the local average is understood in a microscopic scale, we obtained a generalized random field converge in law to a constant Gaussian random field, which is novel to the literature. Additionally, we showed in Corollary 3.1.3 that the locally averaged KPZ equation, as a generalized random field, converges to a deterministic KPZ limit in dimension two. Our work studies the two-dimensional KPZ equation from a new perspective and it improves the understanding on this subject.

3.1.2 Context

On $(t,x) \in [0,+\infty) \times \mathbb{R}^d$, the KPZ equation is an SPDE formally given by

$$\partial_t h(t,x) = \frac{1}{2} \Delta h(t,x) + \frac{1}{2} |\nabla h(t,x)|^2 + \xi(t,x). \tag{3.1.4}$$

Here ξ denotes the space-time white noise which is the distribution valued Gaussian field in spatial dimension d with the covariance function

$$\mathbb{E}\left[\xi(t,x)\xi(t',x')\right] = \delta_0(t-t')\delta_0(x-x').$$

Here δ_0 is the Dirac mass.

The KPZ equation was first introduced in 1986 by Kardar, Parisi and Zhang [5] and has since become the standard random interface growth model in physics. However, mathematically, the KPZ equation is ill-posed due to the non-linear term ∇h being a generalized function, which makes the interpretation of $|\nabla h|^2$ unclear.

In 1990's, Bertini and Giacomin [55] formulated the Hopf-Cole solution of (3.1.4) in $d = 1$ by a transformation $h(t,x) = \log u(t,x)$. In fact, let $u(t,x)$ be the solution to the d-dimensional stochastic heat equation (SHE) on $(t,x) \in [0,+\infty) \times \mathbb{R}^d$

$$\partial_t u(t,x) = \frac{1}{2}\Delta u(t,x) + u(t,x)\xi(t,x). \tag{3.1.5}$$

Then at least formally, if ignoring the Itô correction, $\log u(t,x)$ satisfies (3.1.4).

However, the stochastic heat equation (3.1.5) is only well-posed for $d = 1$. For dimension $d \geq 2$, (3.1.5) does not have function or distribution valued solutions. This makes the problem of stochastic heat equation and KPZ equation in higher dimensions more challenging to solve.

In [18], Caravenna, Sun and Zygouras introduced a space-regularized equation with a scaling on the disorder strength to address the well-posedness issue in dimension $d = 2$. The equation is given by

$$\partial_t u^\varepsilon(t,x) = \frac{1}{2}\Delta u^\varepsilon(t,x) + \frac{\beta}{\sqrt{\log \varepsilon^{-1}}} u^\varepsilon(t,x)\xi_\varepsilon(t,x), \qquad u^\varepsilon(0,x) = u_0(x). \tag{3.1.6}$$

Through the Hopf-Cole transformation $h^\varepsilon(t,x) = \log u^\varepsilon(t,x)$, h^ε solves the mollified KPZ equation (3.1.1) with $h_0 = \log u_0$.

In [18], the following multi-scale point-wise asymptotic limit of $h^\varepsilon(t,x)$ with initial condition $h_0(x) = 0$ was derived.

Theorem 3.1.5 ([18], Theorem 2.15, Remark 2.16)**.** *Let $h_0(x) = 0$. Fix $t > 0$. Consider a finite collection of space-time points $(t_\varepsilon, x_\varepsilon^{(i)})_{1\le i\le n}$, where $t_\varepsilon > 0$, $x_\varepsilon^{(i)} \in \mathbb{R}^2$, such that as $\varepsilon \to 0$, $t_\varepsilon = \varepsilon^{o(1)} t$ and*

$$\forall i,j \in \{1,\ldots,n\}: \quad |x_\varepsilon^{(i)} - x_\varepsilon^{(j)}| \le \varepsilon^{(1-\zeta_{i,j})+o(1)} \qquad \textit{for some } \zeta_{i,j} \in [0,1].$$

Then if $\beta \in (0,\sqrt{2\pi})$, we have that $(h^\varepsilon(t_\varepsilon, x_\varepsilon^{(i)}))_{1\le i\le n}$ converges in joint distribution to the multi-dimensional normal distribution $(Y_i - \frac{1}{2}\mathrm{Var}[Y_i])_{1\le i\le n}$ as $\varepsilon \to 0$, where $(Y_i)_{1\le i\le n}$ are jointly Gaussian random variables with

$$\mathbb{E}[Y_i] = 0, \qquad \mathrm{Cov}[Y_i, Y_j] = \log\left(\frac{2\pi - \beta^2\zeta_{i,j}}{2\pi - \beta^2}\right).$$

If $\beta \ge \sqrt{2\pi}$, $h^\varepsilon(t_\varepsilon, x_\varepsilon^{(i)})$ converges to $-\infty$ in probability for all $1 \le i \le n$, as $\varepsilon \to 0$.

If $\beta \in (0,\sqrt{2\pi})$, $\mathrm{Var}[Y_i] = \log\left(\frac{2\pi}{2\pi-\beta^2}\right)$ as $\zeta_{i,i} = 0$ for any $1 \le i \le n$. For any finite set of distinct points $(x_i)_{1\le i\le n}$, the random variables $(h^\varepsilon(t_\varepsilon, x_i))_{1\le i\le n}$ converge to independent

Gaussians as $\varepsilon \to 0$. This is because $\zeta_{i,j} = 1$ for all $1 \le i, j \le n$ with $i \ne j$, implying independence in the limit.

Remark 3.1.6. The value $\beta_c = \sqrt{2\pi}$ is critical here as there is a phase transition in Theorem 3.1.5. The interval $(0, \sqrt{2\pi})$ is known as the *subcritical regime* for 2-dimensional KPZ equation. In our paper, we will only focus on this regime. The research on the 2-dimensional SHE (3.1.5) in a critical window around β_c initiated with [56], and notable recent advancements have been made in [44–46, 57–59].

The directed polymer model in the random environment in $2+1$ dimension is related to the two-dimensional KPZ equation. The solution to the KPZ equation equals in law to the logarithm of the partition function of a continuum directed polymer. (See (3.1.10) below.)

In a prior work, Chatterjee [21] proved that a growing random surface generated by discrete directed polymers in $d \ge 3$ converges to a deterministic KPZ equation. It is an analogue of our Theorem 3.1.1 for directed polymers in dimensions $d+1$ with $d \ge 3$. The same result as in [21] is expected to hold in the continuum setting, i.e. for the analogue KPZ equation in dimension $d \ge 3$ defined as

$$\partial_t \hat{h}_\varepsilon(t,x) = \frac{1}{2}\Delta \hat{h}_\varepsilon(t,x) + \frac{1}{2}|\nabla \hat{h}_\varepsilon(t,x)|^2 + \beta \varepsilon^{\frac{d-2}{2}} \hat{\xi}_\varepsilon(t,x) - \frac{\beta^2}{2\varepsilon^2}\|\hat{\phi}\|^2_{L^2(\mathbb{R}^d)}, \quad \hat{h}_\varepsilon(0,x) = \hat{h}_0(x),$$

where $\hat{\xi}_\varepsilon = \xi(t,x) * \hat{\phi}^\varepsilon$ is the space-time white noise in dimension $d+1$ spatially smoothened at scale ε, $\hat{\phi} \in C_c^\infty(\mathbb{R}^d)$ is a non-negative, smooth, symmetric mollifier with $\int \hat{\phi} \mathrm{d}x = 1$ and $\hat{\phi}^\varepsilon(x) = \frac{1}{\varepsilon^d}\hat{\phi}(\varepsilon^{-1}x)$, $\hat{h}_0$ is bounded and Lipschitz continuous, and β is sufficiently small.

We note that there is a significant difference between the $d = 2$ KPZ equation and the $d \ge 3$ KPZ equation. In dimension $d \ge 3$, there is no "mesoscopic" averaging phenomenon. In fact, in

$d \geq 3$, by [21, Theorem 2.2], we expect that, by taking the average of $\hat{h}_\varepsilon(t,x)$ over a ball $B(x,r_\varepsilon)$ with radius $\varepsilon \ll r_\varepsilon \ll 1$, the limit would be the corresponding deterministic KPZ equation with a height shift. The "mesoscopic" averaging result (when $\varepsilon \ll r_\varepsilon \ll 1$ in our Theorem 3.1.1) is exclusive to dimension $d = 2$.

The difference arises as follows. For $d \geq 3$, if $x_\varepsilon^{(1)}, x_\varepsilon^{(2)} \in \mathbb{R}^d$ and $\varepsilon \ll |x_\varepsilon^{(1)} - x_\varepsilon^{(2)}| \ll 1$, then as $\varepsilon \to 0$, $\hat{h}_\varepsilon(t, x_\varepsilon^{(1)})$ and $\hat{h}_\varepsilon(t, x_\varepsilon^{(2)})$ always become asymptotically independent, regardless of the scale of $|x_\varepsilon^{(1)} - x_\varepsilon^{(2)}|$. Taking a spatial average over a ball with radius $\varepsilon \ll r_\varepsilon \ll 1$ eliminates the "randomness" due to the law of large numbers. However, in dimension $d = 2$, as noted in Theorem 3.1.5, when $\zeta_{i,j} \neq 0$, there is a nontrivial multi-scale correlation in the limit of $(h^\varepsilon(t,x_i))_{1\leq i \leq n}$.

We shall finish this section by mentioning some other relevant recent works. The second order fluctuations of KPZ equation in dimension $d = 2$, i.e. the limit of $\sqrt{\log \varepsilon^{-1}}[h^\varepsilon(t,\cdot) - \mathbb{E}h^\varepsilon(t,\cdot)]$ as $\varepsilon \to 0$, was studied in the aforementioned works [20, 36, 37, 53]. The second order fluctuations of SHE and nonlinear SHE in $d = 2$ were studied in [18, 37] and [1] respectively. [60] studied the macroscopic-level limit of the polymer paths in dimension $2 + 1$. In $d \geq 3$, the pointwise limit of SHE and KPZ equation were studied in [16, 61]. The second order fluctuations of SHE and KPZ equation were studied in [39–43, 62, 63]. The second order fluctuations of nonlinear SHE was studied in [17].

Studies of the deterministic KPZ equation and deterministically growing surfaces were recently conducted in [64, 65]. By dropping the random noise from the environment, the deterministic KPZ equation is a much simpler object and is well-defined in all dimensions. The hope of such studies was that it could lead to an understanding of the universal nature of KPZ growth with noise in the dimensions greater than one.

3.1.3 Sketch of proof

Before beginning the proof of Theorem 3.1.1, let us first outline its central idea.

To study h^ε, we analyze $u^\varepsilon = e^{h^\varepsilon}$. To do so, we employ the Feynman-Kac formula from [6], which can be easily adapted to any dimension.

Let $0 < \inf_{x\in\mathbb{R}^2} u_0(x) \le \sup_{x\in\mathbb{R}^2} u_0(x) < \infty$, which is equivalent to the requirement that $h_0(x)$ is bounded. The Feynman-Kac formula states that the solution to the 2-dimensional stochastic heat equation (3.1.6) is given by:

$$u^\varepsilon(t,x) = \mathrm{E}_x\left[u_0(B_t)\exp\left\{\beta_\varepsilon\int_0^t \xi_\varepsilon(t-r,B_r)\mathrm{d}r - \frac{\beta_\varepsilon^2}{2}\mathbb{E}\left[\left(\int_0^t \xi_\varepsilon(t-r,B_r)\mathrm{d}r\right)^2\right]\right\}\right], \quad (3.1.7)$$

where $\beta_\varepsilon := \frac{\beta}{\sqrt{\log\varepsilon^{-1}}}$, E_x is the expectation w.r.t. $(B_r)_{r\ge0}$, $(B_r)_{r\ge0}$ is a standard Brownian motion in $\mathbb{R}^2$ starting from $B_0 = x$ and $\mathbb{E}$ denotes the expectation with respect to the space-time white noise in the environment. Later, we will also make use of the Brownian bridge from $(0,x)$ to (s,y) with $s>0$ and $x,y\in\mathbb{R}^2$. We denote the expectation w.r.t. such Brownian bridge by $\mathrm{E}_{0,x}^{s,y}$.

By a time reversal in ξ_ε, a scaling property of the space-time white noise in dimension $d=2$, and the scaling invariance of Brownian motion ($\varepsilon B_{\varepsilon^{-2}r} \overset{law}{=} B_r$), we find that $\{u^\varepsilon(t,x)\}_{x\in\mathbb{R}^2}$ in (3.1.7) has the same joint distribution as $\{\tilde{u}^\varepsilon(t,x)\}_{x\in\mathbb{R}^2}$, where

$$\begin{aligned}
\tilde{u}^\varepsilon(t,x) &= \mathrm{E}_x\Big[u_0(B_t)\exp\Big\{\beta_\varepsilon\int_0^t \xi_\varepsilon(r,B_r)\mathrm{d}r - \frac{\beta_\varepsilon^2}{2}\mathbb{E}\Big(\int_0^t \xi_\varepsilon(r,B_r)\mathrm{d}r\Big)^2\Big\}\Big] \\
&= \mathrm{E}_x\Big[u_0(B_t)\exp\Big\{\beta_\varepsilon\int_0^t\int_{\mathbb{R}^2}\phi^\varepsilon(B_r-y)\xi(r,y)\mathrm{d}y\mathrm{d}r - \frac{\beta_\varepsilon^2}{2}t\|\phi^\varepsilon\|^2_{L^2(\mathbb{R}^2)}\Big\}\Big] \\
&= \mathrm{E}_{\varepsilon^{-1}x}\Big[u_0(\varepsilon B_{\varepsilon^{-2}t})\exp\Big\{\beta_\varepsilon\int_0^{\varepsilon^{-2}t}\int_{\mathbb{R}^2}\phi(B_{\tilde{r}}-\tilde{y})\tilde{\xi}(\tilde{r},\tilde{y})\mathrm{d}\tilde{y}\mathrm{d}\tilde{r} - \frac{\beta_\varepsilon^2}{2}\varepsilon^{-2}t\|\phi\|^2_{L^2(\mathbb{R}^2)}\Big\}\Big].
\end{aligned}$$

In the last step, we made the change of variable $(\varepsilon\tilde{y},\varepsilon^2\tilde{r}) := (y,r)$. The scaling property of the space-time white noise implies that

$$\tilde{\xi}(\tilde{r},\tilde{y})\mathrm{d}\tilde{y}\mathrm{d}\tilde{r} := \varepsilon^{-2}\xi(\varepsilon^2\tilde{r},\varepsilon\tilde{y})\mathrm{d}(\varepsilon\tilde{y})\mathrm{d}(\varepsilon^2\tilde{r})$$

is another space-time white noise in $\mathbb{R}^2$.

Now since $h^\varepsilon = \log u^\varepsilon$ and $h_0 = \log u_0$, we have

$$\begin{aligned} &h^\varepsilon(t,x) \\ &\overset{law}{=} \log \mathrm{E}_{\varepsilon^{-1}x}\Big[\exp\Big\{h_0(\varepsilon B_{\varepsilon^{-2}t}) + \beta_\varepsilon \int_0^{\varepsilon^{-2}t}\int_{\mathbb{R}^2}\phi(B_r - y)\xi(r,y)\mathrm{d}y\mathrm{d}r - \frac{1}{2}\beta_\varepsilon^2\varepsilon^{-2}t\|\phi\|^2_{L^2(\mathbb{R}^2)}\Big\}\Big]. \end{aligned} \tag{3.1.8}$$

For any $s \geq 0$ and $f \in C(\mathbb{R}^2)$ being bounded and Lipschitz continuous, define

$$\Psi_s^{\beta_\varepsilon,f} = \Psi_s^{\beta_\varepsilon,f}(B,\xi) = \exp\Big[f(\varepsilon B_s) + \beta_\varepsilon\int_0^s\int_{\mathbb{R}^2}\phi(y-B_r)\xi(r,y)\mathrm{d}y\mathrm{d}r - \frac{1}{2}\beta_\varepsilon^2\|\phi\|^2_{L^2}s\Big]. \tag{3.1.9}$$

Then (3.1.8) can be rewritten as

$$h^\varepsilon(t,x) \overset{law}{=} \log \mathrm{E}_{\varepsilon^{-1}x}\Big[\Psi^{\beta_\varepsilon,h_0}_{\varepsilon^{-2}t}\Big]. \tag{3.1.10}$$

In the following sections, in order to prove Theorem 3.1.1 for $h^\varepsilon(t,x)$, we analyze the limit of $\log \mathrm{E}_{\varepsilon^{-1}x}\Big[\Psi^{\beta_\varepsilon,h_0}_{\varepsilon^{-2}t}\Big]$ as $\varepsilon \to 0$.

The main idea behind the proof for Theorem 3.1.1 is the following decomposition:

$$\log \mathrm{E}_{\varepsilon^{-1}x}\Big[\Psi^{\beta_\varepsilon,h_0}_{\varepsilon^{-2}t}\Big] = \log\frac{\mathrm{E}_{\varepsilon^{-1}x}\Big[\Psi^{\beta_\varepsilon,h_0}_{\varepsilon^{-2}t}\Big]}{\mathrm{E}_{\varepsilon^{-1}x}\Big[\Psi^{\beta_\varepsilon,0}_{\varepsilon^{-2(1-a_\varepsilon)}t}\Big]} + \log \mathrm{E}_{\varepsilon^{-1}x}\Big[\Psi^{\beta_\varepsilon,0}_{\varepsilon^{-2(1-a_\varepsilon)}t}\Big] \tag{3.1.11}$$

for some $a_\varepsilon = o(1)$ satisfying $\varepsilon^{2a_\varepsilon} = o(1)$ that we will define later.

This decomposition is based on the following observation: the random part in the limit of $h^\varepsilon(t,x)$ depends only on the white noise ξ in an infinitesimal time window $[t-o(1),t]$ as $\varepsilon \to 0$, and it is independent of the initial condition h_0. Thus, we can split $h^\varepsilon(t,x)$ into two components: a "random" quantity which is the solution to the KPZ equation on time window $[t-o(1),t]$ with zero initial condition, and an "almost deterministic" quantity that concentrates to a deterministic value when $\varepsilon \to 0$. The right-hand-side of (3.1.11) demonstrates this decomposition, with the first term being the "almost deterministic" quantity and the second term being the "random" quantity.

The remaining proof consists of two parts. In the first part, we show that the "almost deterministic" quantity converges to the solution of the deterministic KPZ equation with original initial data h_0. This is proved in Proposition 3.3.2 below, with the assistance of Proposition 3.3.1. In the second part, we prove that the local average of the "random" quantity converges in law to the Gaussian random variable $\mathcal{N}(0,\sigma_\gamma^2) - \frac{1}{2}\log\left(\frac{2\pi}{2\pi-\beta^2}\right)$. This is shown in Proposition 3.3.3.

We should note that the idea of this decomposition was previously mentioned in [18] Remark 2.18 for 2-dimensional SHE with general initial conditions. It has since been utilized in various recent works, including [19, 20, 63], with a particular emphasis in [21].

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3.2 Moment bounds

In this section, we will discuss some properties of the random variables $\mathrm{E}_{\varepsilon^{-1}x}\Big[\Psi_s^{\beta_\varepsilon,0}\Big]$, $\mathrm{E}_{0,\varepsilon^{-1}x}^{s,\varepsilon^{-1}y}\Big[\Psi_s^{\beta_\varepsilon,0}\Big]$ and $\log \mathrm{E}_{\varepsilon^{-1}x}\Big[\Psi_s^{\beta_\varepsilon,0}\Big]$, where $\Psi_s^{\beta_\varepsilon,0}$ is defined as in (3.1.9) with f being a zero function. These properties will be used in the proof of Theorem 3.1.1 later.

Hereafter, for any $p > 0$, we use the notation $\|\cdot\|_{L^p(\Omega)}$ to denote the $L^p(\Omega)$ norm of the probability space $(\Omega, \mathscr{F}, \mathbb{P})$ where the space-time white noise ξ is built on.

3.2.1 Second moments

We will first show that $\mathrm{E}_{\varepsilon^{-1}x}\Big[\Psi_s^{\beta_\varepsilon,0}\Big]$ and $\mathrm{E}_{0,\varepsilon^{-1}x}^{s,\varepsilon^{-1}y}\Big[\Psi_s^{\beta_\varepsilon,0}\Big]$ have bounded second moments. Let

$$V(x) = \int_{\mathbb{R}^2} \phi(x-y)\phi(y)\mathrm{d}y. \tag{3.2.1}$$

We first remark that for any $x, y \in \mathbb{R}^2$,

$$\begin{aligned}
&\mathbb{E}\Big(\mathrm{E}_{\varepsilon^{-1}x}\Big[\Psi_s^{\beta_\varepsilon,0}\Big]\mathrm{E}_{\varepsilon^{-1}y}\Big[\Psi_s^{\beta_\varepsilon,0}\Big]\Big)\\
&= \mathrm{E}_{\varepsilon^{-1}x}\otimes \mathrm{E}_{\varepsilon^{-1}y}\otimes \mathbb{E}\Big(\exp\Big[\beta_\varepsilon \int_0^s \int_{\mathbb{R}^2}[\phi(y-B_r^1)+\phi(y-B_r^2)]\xi(r,y)\mathrm{d}y\mathrm{d}r - \beta_\varepsilon^2\|\phi\|_{L^2}^2 s\Big]\Big)\\
&= \mathrm{E}_{\varepsilon^{-1}x}\otimes \mathrm{E}_{\varepsilon^{-1}y}\Big(\exp\Big[\frac{\beta_\varepsilon^2}{2}\int_0^s \int_{\mathbb{R}^2}[\phi(y-B_r^1)+\phi(y-B_r^2)]^2\mathrm{d}y\mathrm{d}r - \beta_\varepsilon^2\|\phi\|_{L^2}^2 s\Big]\Big)\\
&= \mathrm{E}_{\varepsilon^{-1}x}\otimes \mathrm{E}_{\varepsilon^{-1}y}\Big(\exp\Big[\beta_\varepsilon^2\int_0^s \int_{\mathbb{R}^2}\phi(y-B_r^1)\phi(y-B_r^2)\mathrm{d}y\mathrm{d}r\Big]\Big)\\
&= \mathrm{E}_{\varepsilon^{-1}x}\otimes \mathrm{E}_{\varepsilon^{-1}y}\Big(\exp\Big[\beta_\varepsilon^2\int_0^s V(B_r^1-B_r^2)\mathrm{d}r\Big]\Big)\\
&= \mathrm{E}_{\varepsilon^{-1}\frac{x-y}{\sqrt{2}}}\Big(\exp\Big[\beta_\varepsilon^2\int_0^s V(\sqrt{2}B_r)\mathrm{d}r\Big]\Big)\\
&= 1+\sum_{n=1}^\infty \beta_\varepsilon^{2n}\int_{0<s_1<\dots<s_n<s}\int_{\mathbb{R}^{2n}}\prod_{i=1}^n V(\sqrt{2}x_i)\rho_{s_i-s_{i-1}}(x_{i-1},x_i)\mathrm{d}s_1\dots\mathrm{d}s_n\mathrm{d}x_1\dots\mathrm{d}x_n
\end{aligned} \tag{3.2.2}$$

where we set $x_0 = \varepsilon^{-1}\frac{x-y}{\sqrt{2}}$, $s_0 = 0$ and let $\rho_t(x) = (2\pi t)^{-1}e^{-\frac{|x|^2}{2t}}$ be the heat kernel in $d = 2$ such that $\rho_t(x,y) := \rho_t(x-y)$. We use $\mathrm{E}_x \otimes \mathrm{E}_y$ to denote the expectation in two independent Brownian motions B^1 and B^2 starting from x and y respectively.

Since

$$\sup_{x\in\mathbb{R}^2}\int_{\mathbb{R}^2} V(\sqrt{2}y)\rho_r(x,y)\mathrm{d}y \le \frac{1}{4\pi r}\wedge \|V\|_\infty, \tag{3.2.3}$$

where $\|V\|_\infty$ denotes the supremum norm of V on $\mathbb{R}^2$, (3.2.2) is bounded from above by

$$1+\sum_{n=1}^\infty \beta_\varepsilon^{2n}\Big(\int_0^s \frac{1}{4\pi r}\wedge\|V\|_\infty \mathrm{d}r\Big)^n = 1+\sum_{n=1}^\infty \beta_\varepsilon^{2n}\Big(\frac{\log 4\pi\|V\|_\infty s}{4\pi}+\frac{1}{4\pi}\Big)^n.$$

Now for any fixed $t > 0$, we have that as $\varepsilon \to 0$,

$$\beta_\varepsilon^2\Big(\frac{\log 4\pi\|V\|_\infty\varepsilon^{-2}t}{4\pi}+\frac{1}{4\pi}\Big) = \frac{\beta^2}{\log\varepsilon^{-1}}\Big(\frac{\log\varepsilon^{-2}}{4\pi}+\frac{1+\log 4\pi\|V\|_\infty t}{4\pi}\Big) \to \frac{\beta^2}{2\pi}.$$

Hence when $s \le \varepsilon^{-2}t$ and $0 < \beta < \sqrt{2\pi}$, we have

$$\limsup_{\varepsilon\to 0} \mathbb{E}\Big(\mathrm{E}_{\varepsilon^{-1}x}\Big[\Psi_s^{\beta_\varepsilon,0}\Big]\mathrm{E}_{\varepsilon^{-1}y}\Big[\Psi_s^{\beta_\varepsilon,0}\Big]\Big) \le \sum_{n=0}^{\infty}\Big(\frac{\beta^2}{2\pi}\Big)^n \le \frac{2\pi}{2\pi-\beta^2}. \tag{3.2.4}$$

We also have a second moment bound for the expectation w.r.t. Brownian bridge.

Lemma 3.2.1. *Let $0 < \beta < \sqrt{2\pi}$, $t > 0$ and $x, y \in \mathbb{R}^2$. There exists $C_\beta < \infty$ independent of t, x, y, such that*

$$\sup_{\varepsilon\le 1}\sup_{s\le\varepsilon^{-2}t}\left\|\mathrm{E}_{0,\varepsilon^{-1}x}^{s,\varepsilon^{-1}y}\left[\Psi_s^{\beta_\varepsilon,0}\right]\right\|_{L^2(\Omega)} < C_\beta. \tag{3.2.5}$$

Proof. By the shear invariance of the environment, for any $\varepsilon > 0$ and $x, y \in \mathbb{R}^2$,

$$\left\|\mathrm{E}_{0,\varepsilon^{-1}x}^{s,\varepsilon^{-1}y}\left[\Psi_s^{\beta_\varepsilon,0}\right]\right\|_{L^2(\Omega)} = \left\|\mathrm{E}_{0,0}^{s,0}\left[\Psi_s^{\beta_\varepsilon,0}\right]\right\|_{L^2(\Omega)}.$$

(3.2.5) is then derived by applying [37, (2.5)]. □

3.2.2 Higher moments

Next we present the boundedness of some higher moments of

$$\mathrm{E}_{\varepsilon^{-1}x}\Big[\Psi_s^{\beta_\varepsilon,0}\Big] \quad \text{and} \quad \mathrm{E}_{0,\varepsilon^{-1}x}^{s,\varepsilon^{-1}y}\Big[\Psi_s^{\beta_\varepsilon,0}\Big].$$

This is proved via hypercontractivity.

Lemma 3.2.2. *Let $0 < \beta < \sqrt{2\pi}$, $t > 0$ and $x, y \in \mathbb{R}^2$. There exists some $p_\beta > 2$ such that*

$\forall 2 \le p < p_\beta$, $\exists C_{\beta,p} < \infty$ *independent of* t, x, y, *such that*

$$\sup_{\varepsilon \le 1} \sup_{s \le \varepsilon^{-2} t} \left\| \mathrm{E}_{\varepsilon^{-1} x} \left[\Psi_s^{\beta_\varepsilon, 0} \right] \right\|_{L^p(\Omega)} \le C_{\beta,p}, \tag{3.2.6}$$

and

$$\sup_{\varepsilon \le 1} \sup_{s \le \varepsilon^{-2} t} \left\| \mathrm{E}_{0,\varepsilon^{-1} x}^{s,\varepsilon^{-1} y} \left[\Psi_s^{\beta_\varepsilon, 0} \right] \right\|_{L^p(\Omega)} \le C_{\beta,p}. \tag{3.2.7}$$

Proof. (3.2.6) is exactly [20, (5.11)] with $p_\beta = 1 + 2\pi/\beta^2$.

(3.2.7) can be proved in a similar way, as $\mathrm{E}_{0,\varepsilon^{-1} x}^{s,\varepsilon^{-1} y}\left[\Psi_s^{\beta_\varepsilon, 0}\right]$ admits a similar Wiener chaos expansion and is bounded in L^2. A discrete version of (3.2.7) for the partition function of point-to-point polymer was proved in [60, Corollary 2.8 (i)]. Here we present a proof in the continuum setting.

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be the probability space on which the space-time white noise ξ is built. Let $\{T_v, v \ge 0\}$ be the Ornstein-Uhlenbeck semigroup on $L^2(\Omega)$ (see e.g. [47] for a reference). Let $\tilde{\xi}$ be an independent copy of ξ built on the probability space $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbb{P}})$. By Mehler's formula,

$$\begin{aligned}
T_v\left(\mathrm{E}_{0,\varepsilon^{-1} x}^{s,\varepsilon^{-1} y}\left[\Psi_s^{\beta_\varepsilon, 0}\right]\right) &= T_v\left(\mathrm{E}_{0,\varepsilon^{-1} x}^{s,\varepsilon^{-1} y}\left[\exp\left[\beta_\varepsilon \int_0^s \int_{\mathbb{R}^2} \phi(y - B_r)\xi(r,y)\mathrm{d}y\mathrm{d}r - \frac{1}{2}\beta_\varepsilon^2 \|\phi\|_{L^2}^2 s\right]\right]\right) \\
&= \tilde{\mathbb{E}}\mathrm{E}_{0,\varepsilon^{-1} x}^{s,\varepsilon^{-1} y}\left[\exp\left[\beta_\varepsilon \int_0^s \int_{\mathbb{R}^2} \phi(y - B_r)\left(e^{-v}\xi(r,y) + \sqrt{1 - e^{-2v}}\tilde{\xi}(r,y)\right)\mathrm{d}y\mathrm{d}r - \frac{1}{2}\beta_\varepsilon^2 \|\phi\|_{L^2}^2 s\right]\right],
\end{aligned}$$

where $\tilde{\mathbb{E}}$ is the expectation w.r.t. $\tilde{\mathbb{P}}$. Since

$$\begin{aligned}
&\tilde{\mathbb{E}}\left[\exp\left[\beta_\varepsilon \int_0^s \int_{\mathbb{R}^2} \phi(y - B_r)\left(e^{-v}\xi(r,y) + \sqrt{1 - e^{-2v}}\tilde{\xi}(r,y)\right)\mathrm{d}y\mathrm{d}r - \frac{1}{2}\beta_\varepsilon^2 \|\phi\|_{L^2}^2 s\right]\right] \\
&= \exp\left[\beta_\varepsilon \int_0^s \int_{\mathbb{R}^2} \phi(y - B_r)e^{-v}\xi(r,y)\mathrm{d}y\mathrm{d}r - \frac{1}{2}e^{-2v}\beta_\varepsilon^2 \|\phi\|_{L^2}^2 s\right],
\end{aligned}$$

we have

$$
\begin{aligned}
&T_v\left(\mathrm{E}_{0,\varepsilon^{-1}x}^{s,\varepsilon^{-1}y}\left[\Psi_s^{\beta_\varepsilon,0}\right]\right)\\
&= \mathrm{E}_{0,\varepsilon^{-1}x}^{s,\varepsilon^{-1}y}\left[\exp\left[\beta_\varepsilon \int_0^s \int_{\mathbb{R}^2} \phi(y-B_r)e^{-v}\xi(r,y)\mathrm{d}y\mathrm{d}r - \frac{1}{2}e^{-2v}\beta_\varepsilon^2\|\phi\|_{L^2}^2 s\right]\right] \qquad (3.2.8)\\
&= \mathrm{E}_{0,\varepsilon^{-1}x}^{s,\varepsilon^{-1}y}\left[\Psi_s^{(e^{-v}\beta_\varepsilon),0}\right].
\end{aligned}
$$

By the hypercontractivity property of $\{T_v, v \geq 0\}$, for any $v > 0$, if $p(v) = e^{2v} + 1$, we have

$$
\left\|T_v\left(\mathrm{E}_{0,\varepsilon^{-1}x}^{s,\varepsilon^{-1}y}\left[\Psi_s^{\beta_\varepsilon,0}\right]\right)\right\|_{L^{p(v)}(\Omega)} \leq \left\|\mathrm{E}_{0,\varepsilon^{-1}x}^{s,\varepsilon^{-1}y}\left[\Psi_s^{\beta_\varepsilon,0}\right]\right\|_{L^2(\Omega)}.
$$

With $0 < \beta < \sqrt{2\pi}$, we have $e^v\beta < \sqrt{2\pi}$ when $v < \log\frac{\sqrt{2\pi}}{\beta}$. Thus by (3.2.8) and Lemma 3.2.1, if we restrict $p < p_\beta = 1 + 2\pi/\beta^2$ and let $v = \frac{1}{2}\log(p-1)$ accordingly, then

$$
\sup_{\varepsilon\leq 1}\sup_{s\leq\varepsilon^{-2}t}\left\|\mathrm{E}_{0,\varepsilon^{-1}x}^{s,\varepsilon^{-1}y}\left[\Psi_s^{\beta_\varepsilon,0}\right]\right\|_{L^p(\Omega)} \leq \sup_{\varepsilon\leq 1}\sup_{s\leq\varepsilon^{-2}t}\left\|\mathrm{E}_{0,\varepsilon^{-1}x}^{s,\varepsilon^{-1}y}\left[\Psi_s^{(e^v\beta_\varepsilon),0}\right]\right\|_{L^2(\Omega)} < C_{\beta,v},
$$

for some constant $C_{\beta,v}$. □

3.2.3 Negative moments

We next bound the negative moments of $\mathrm{E}_{\varepsilon^{-1}x}\left[\Psi_s^{\beta_\varepsilon,0}\right]$. The following lemma is proved in [20] via concentration inequality.

Lemma 3.2.3 ([20], (5.13))**.** *Let $0 < \beta < \sqrt{2\pi}$, $t > 0$ and $x \in \mathbb{R}^2$. For any $p > 0$, there exists $C_{p,\beta} < \infty$ independent of t and x, such that*

$$
\sup_{\varepsilon\leq 1}\sup_{s\leq\varepsilon^{-2}t}\mathbb{E}\left[\left(\mathrm{E}_{\varepsilon^{-1}x}\left[\Psi_s^{\beta_\varepsilon,0}\right]\right)^{-p}\right] \leq C_{p,\beta}.
$$

A direct corollary is the following moment bounds of the logarithm.

Corollary 3.2.4. *Let* $0 < \beta < \sqrt{2\pi}$*,* $t > 0$ *and* $x \in \mathbb{R}^2$*. For any* $p > 0$*, there exists* $C_{p,\beta} < \infty$ *independent of* t *and* x*, such that*

$$\sup_{\varepsilon \le 1} \sup_{s \le \varepsilon^{-2} t} \mathbb{E}\Big(\Big| \log \mathrm{E}_{\varepsilon^{-1} x}\Big[\Psi_s^{\beta_\varepsilon, 0}\Big]\Big|^p \Big) \le C_{p,\beta}.$$

Proof. For any $p > 0$, there exists $C_p > 0$ such that $|x|^p \le C_p(e^x + e^{-x})$ for any $x \in \mathbb{R}$. Apply (3.2.4) and Lemma 3.2.3. □

If $h_0 : \mathbb{R}^2 \to \mathbb{R}$ is bounded, (3.2.4) and Lemma 3.2.3 remain valid for $\Psi_s^{\beta_\varepsilon, h_0}$ after we multiply some constants to the bounds. As a result, Corollary 3.2.4 also holds for $\Psi_s^{\beta_\varepsilon, h_0}$.

3.3 Proof of Theorem 3.1.1

The proof here is inspired by [21].

We first state the following propositions that we will prove later.

Proposition 3.3.1. *Let* $0 < \beta < \sqrt{2\pi}$ *and* $h_0 \in C(\mathbb{R}^2)$ *be bounded and Lipschitz continuous. Let* $a_\varepsilon = o(1)$*. There exist* $C_\beta > 0$ *and* $\varepsilon_0 > 0$*, such that when* $0 < \varepsilon < \varepsilon_0$*, for any* $t > 0$*,* $x \in \mathbb{R}^2$*, we have*

$$\mathbb{E}\left[\left|\frac{\mathrm{E}_{\varepsilon^{-1}x}\Big[\Psi^{\beta_\varepsilon, h_0}_{\varepsilon^{-2}t}\Big]}{\mathrm{E}_{\varepsilon^{-1}x}\Big[\Psi^{\beta_\varepsilon, 0}_{\varepsilon^{-2(1-a_\varepsilon)}t}\Big]} - \exp[\bar{h}(t - \varepsilon^{2a_\varepsilon} t, x)]\right|^2\right] \le C_\beta(\varepsilon^{2a_\varepsilon} t + a_\varepsilon^{1/3}), \tag{3.3.1}$$

where $\bar{h}(t,x)$ *is defined as in* (3.1.3).

Proposition 3.3.2. *With the same assumptions as in Proposition 3.3.1, there exist* $C_\beta > 0$ *and*

$\varepsilon_0 > 0$, *such that when* $0 < \varepsilon < \varepsilon_0$, *for any* $t > 0$, $x \in \mathbb{R}^2$, *we have*

$$\mathbb{E}\left[\left|\log \frac{\mathrm{E}_{\varepsilon^{-1}x}\left[\Psi^{\beta_\varepsilon,h_0}_{\varepsilon^{-2}t}\right]}{\mathrm{E}_{\varepsilon^{-1}x}\left[\Psi^{\beta_\varepsilon,0}_{\varepsilon^{-2(1-a_\varepsilon)}t}\right]} - \bar{h}(t-\varepsilon^{2a_\varepsilon}t, x)\right|^2\right] \le C_\beta(\varepsilon^{2a_\varepsilon}t + a_\varepsilon^{1/3})^{1/8}. \tag{3.3.2}$$

Proposition 3.3.3. *Let* $0 < \beta < \sqrt{2\pi}$, $r_\varepsilon = \varepsilon^{1-\gamma}$ *for some* $0 \le \gamma \le 1$ *and* $a_\varepsilon = (\log \varepsilon^{-1})^{-1/2}$. *For any* $t > 0$, $x \in \mathbb{R}^2$, *as* $\varepsilon \to 0$,

$$\frac{1}{|B(x,r_\varepsilon)|}\int_{|y-x|\le r_\varepsilon} \log \mathrm{E}_{\varepsilon^{-1}y}\left[\Psi^{\beta_\varepsilon,0}_{\varepsilon^{-2(1-a_\varepsilon)}t}\right]\mathrm{d}y \xrightarrow{d} \mathcal{N}(0,\sigma_\gamma^2) - \frac{1}{2}\log\left(\frac{2\pi}{2\pi-\beta^2}\right), \tag{3.3.3}$$

where $\mathcal{N}(0,\sigma_\gamma^2)$ *is a normal distribution with mean 0 and variance* $\sigma_\gamma^2 := \log\left(\frac{2\pi-\beta^2\gamma}{2\pi-\beta^2}\right)$. *When* $0 \le \gamma < 1$, $\sigma_\gamma^2 > 0$. *When* $\gamma = 1$, $\sigma_\gamma^2 = 0$.

We first use Proposition 3.3.2 and Proposition 3.3.3 to prove Theorem 3.1.1.

Let

$$a_\varepsilon = (\log \varepsilon^{-1})^{-1/2}. \tag{3.3.4}$$

Then we have $a_\varepsilon = o(1)$ and $\varepsilon^{a_\varepsilon} = o(1)$. In particular, as $\varepsilon \to 0$,

$$C_\beta(\varepsilon^{2a_\varepsilon}t + a_\varepsilon^{1/3})^{1/8} \to 0. \tag{3.3.5}$$

By applying (3.1.8) and the decomposition in (3.1.11), we have

$$\begin{aligned}
&\frac{1}{|B(x,r_\varepsilon)|}\int_{|y-x|\le r_\varepsilon} h^\varepsilon(t,y)\mathrm{d}y \overset{law}{=} \frac{1}{|B(x,r_\varepsilon)|}\int_{|y-x|\le r_\varepsilon} \log \mathrm{E}_{\varepsilon^{-1}y}\left[\Psi^{\beta_\varepsilon,h_0}_{\varepsilon^{-2}t}\right]\mathrm{d}y\\
&= \frac{1}{|B(x,r_\varepsilon)|}\int_{|y-x|\le r_\varepsilon} \log \frac{\mathrm{E}_{\varepsilon^{-1}y}\left[\Psi^{\beta_\varepsilon,h_0}_{\varepsilon^{-2}t}\right]}{\mathrm{E}_{\varepsilon^{-1}y}\left[\Psi^{\beta_\varepsilon,0}_{\varepsilon^{-2(1-a_\varepsilon)}t}\right]} + \log \mathrm{E}_{\varepsilon^{-1}y}\left[\Psi^{\beta_\varepsilon,0}_{\varepsilon^{-2(1-a_\varepsilon)}t}\right]\mathrm{d}y\\
&= \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3,
\end{aligned}$$

where

$$\mathcal{I}_1 := \frac{1}{|B(x,r_\varepsilon)|}\int_{|y-x|\le r_\varepsilon}\log\frac{\mathrm{E}_{\varepsilon^{-1}y}\big[\Psi^{\beta_\varepsilon,h_0}_{\varepsilon^{-2}t}\big]}{\mathrm{E}_{\varepsilon^{-1}y}\big[\Psi^{\beta_\varepsilon,0}_{\varepsilon^{-2(1-a_\varepsilon)}t}\big]}\mathrm{d}y - \frac{1}{|B(x,r_\varepsilon)|}\int_{|y-x|\le r_\varepsilon}\bar{h}(t-\varepsilon^{2a_\varepsilon}t,y)\mathrm{d}y,$$

$$\mathcal{I}_2 := \frac{1}{|B(x,r_\varepsilon)|}\int_{|y-x|\le r_\varepsilon}\log \mathrm{E}_{\varepsilon^{-1}y}\big[\Psi^{\beta_\varepsilon,0}_{\varepsilon^{-2(1-a_\varepsilon)}t}\big]\mathrm{d}y,$$

$$\mathcal{I}_3 := \frac{1}{|B(x,r_\varepsilon)|}\int_{|y-x|\le r_\varepsilon}\bar{h}(t-\varepsilon^{2a_\varepsilon}t,y)\mathrm{d}y.$$

By the Minkowski's integral inequality, Proposition 3.3.2 and (3.3.5), when $\varepsilon < \varepsilon_0$,

$$\begin{aligned}
&\|\mathcal{I}_1\|_{L^2(\Omega)}\\
&= \left\| \frac{1}{|B(x,r_\varepsilon)|}\int_{|y-x|\le r_\varepsilon}\log\frac{\mathrm{E}_{\varepsilon^{-1}y}\big[\Psi^{\beta_\varepsilon,h_0}_{\varepsilon^{-2}t}\big]}{\mathrm{E}_{\varepsilon^{-1}y}\big[\Psi^{\beta_\varepsilon,0}_{\varepsilon^{-2(1-a_\varepsilon)}t}\big]}\mathrm{d}y - \frac{1}{|B(x,r_\varepsilon)|}\int_{|y-x|\le r_\varepsilon}\bar{h}(t-\varepsilon^{2a_\varepsilon}t,y)\mathrm{d}y\right\|_{L^2(\Omega)}\\
&\le \frac{1}{|B(x,r_\varepsilon)|}\int_{|y-x|\le r_\varepsilon}\left\|\log\frac{\mathrm{E}_{\varepsilon^{-1}y}\big[\Psi^{\beta_\varepsilon,h_0}_{\varepsilon^{-2}t}\big]}{\mathrm{E}_{\varepsilon^{-1}y}\big[\Psi^{\beta_\varepsilon,0}_{\varepsilon^{-2(1-a_\varepsilon)}t}\big]} - \bar{h}(t-\varepsilon^{2a_\varepsilon}t,y)\right\|_{L^2(\Omega)}\mathrm{d}y\\
&\le \frac{1}{|B(x,r_\varepsilon)|}\int_{|y-x|\le r_\varepsilon} C_\beta^{1/2}(\varepsilon^{2a_\varepsilon}t + a_\varepsilon^{1/3})^{1/16}\mathrm{d}y\\
&= C_\beta^{1/2}(\varepsilon^{2a_\varepsilon}t + a_\varepsilon^{1/3})^{1/16} \to 0, \quad \text{as } \varepsilon \to 0.
\end{aligned}$$

In particular, $\mathcal{I}_1$ converges to 0 in distribution.

By Proposition 3.3.3, $\mathcal{I}_2$ converges to $\mathcal{N}(0,\sigma_\gamma^2) - \frac{1}{2}\log\left(\frac{2\pi}{2\pi-\beta^2}\right)$ in distribution for all $0 \le \gamma \le 1$.

The last term $\mathcal{I}_3$ is deterministic. By Lebesgue's dominated convergence theorem, if $0 \le \gamma < 1, \mathcal{I}_3 \to \bar{h}(t,x)$ as $\varepsilon \to 0$. If $\gamma = 1$,

$$\mathcal{I}_3 \to \frac{1}{|B(x,1)|}\int_{|y-x|\le 1}\bar{h}(t,y)\mathrm{d}y, \quad \text{as } \varepsilon \to 0.$$

Therefore, by proving Proposition 3.3.2 and Proposition 3.3.3, we would have proved Theorem 3.1.1. In order to prove Proposition 3.3.2, we shall first prove Proposition 3.3.1.

3.3.1 Proof of Proposition 3.3.1

Hereafter, we always assume $0 < \beta < \sqrt{2\pi}$. By the Markov property, for any $0 < s < \varepsilon^{-2}t$,

$$\begin{aligned}
&\mathrm{E}_{\varepsilon^{-1}x}\Big[\Psi^{\beta_\varepsilon,h_0}_{\varepsilon^{-2}t}\Big] \\
&= \mathrm{E}_{\varepsilon^{-1}x}\Big[\exp\Big\{h_0(\varepsilon B_{\varepsilon^{-2}t}) + \beta_\varepsilon\int_0^{\varepsilon^{-2}t}\int_{\mathbb{R}^2}\phi(B_r - y)\xi(r,y)\mathrm{d}y\mathrm{d}r - \frac{1}{2}\beta_\varepsilon^2\varepsilon^{-2}t\|\phi\|^2_{L^2(\mathbb{R}^2)}\Big\}\Big] \\
&= \mathrm{E}_{\varepsilon^{-1}x}\Big[\exp\Big\{h_0(\varepsilon B_{\varepsilon^{-2}t}) + \beta_\varepsilon\int_s^{\varepsilon^{-2}t}\int_{\mathbb{R}^2}\phi(B_r - y)\xi(r,y)\mathrm{d}y\mathrm{d}r \\
&\quad + \beta_\varepsilon\int_0^{s}\int_{\mathbb{R}^2}\phi(B_r - y)\xi(r,y)\mathrm{d}y\mathrm{d}r - \frac{1}{2}\beta_\varepsilon^2\varepsilon^{-2}t\|\phi\|^2_{L^2(\mathbb{R}^2)}\Big\}\Big] \\
&= \int_{z\in\mathbb{R}^2}\rho_s(\varepsilon^{-1}x,z)\mathrm{E}^{s,z}_{0,\varepsilon^{-1}x}\Big[\exp\Big\{\beta_\varepsilon\int_0^{s}\int_{\mathbb{R}^2}\phi(B^B_r - y)\xi(r,y)\mathrm{d}y\mathrm{d}r - \frac{1}{2}\beta_\varepsilon^2 s\|\phi\|^2_{L^2(\mathbb{R}^2)}\Big\}\Big] \\
&\mathrm{E}_z\Big[\exp\Big\{h_0(\varepsilon\tilde{B}_{\varepsilon^{-2}t-s}) \\
&+ \beta_\varepsilon\int_0^{\varepsilon^{-2}t-s}\int_{\mathbb{R}^2}\phi(\tilde{B}_r - y)\xi(r+s,y)\mathrm{d}y\mathrm{d}r - \frac{1}{2}\beta_\varepsilon^2(\varepsilon^{-2}t-s)\|\phi\|^2_{L^2(\mathbb{R}^2)}\Big\}\Big]\mathrm{d}z \\
&= \int_{z\in\mathbb{R}^2}\rho_s(\varepsilon^{-1}x,z)\mathrm{E}^{s,z}_{0,\varepsilon^{-1}x}\Big[\Psi^{\beta_\varepsilon,0}_s(B^B,\xi)\Big]\mathrm{E}_z\Big[\Psi^{\beta_\varepsilon,h_0}_{\varepsilon^{-2}t-s}(\tilde{B},\xi')\Big]\mathrm{d}z.
\end{aligned}$$

Here ξ' is a time shift of ξ by s, i.e. $\xi'(r,\cdot) = \xi(s+r,\cdot)$, $\{B^B_r : 0 \le r \le s\}$ is a Brownian bridge starting from $\varepsilon^{-1}x$ at time 0 and ending at z at time s, and $\{\tilde{B}_r : 0 \le r \le \varepsilon^{-2}t - s\}$ is a Brownian motion starting from z, where B^B and $\tilde{B}$ are independent.

For any $s > 0$,

$$\mathrm{E}_{\varepsilon^{-1}x}\Big[\Psi^{\beta_\varepsilon,0}_s\Big] = \int_{z\in\mathbb{R}^2}\rho_s(\varepsilon^{-1}x,z)\mathrm{E}^{s,z}_{0,\varepsilon^{-1}x}\Big[\Psi^{\beta_\varepsilon,0}_s(B^B,\xi)\Big]\mathrm{d}z. \tag{3.3.6}$$

In the rest of this section, we will always set

$$s = \varepsilon^{-2(1-a_\varepsilon)}t,$$

where a_ε is defined as in (3.3.4) and $\varepsilon^2 s = \varepsilon^{2a_\varepsilon} = o(1)$. By (3.3.6), the ratio expression in (3.3.1) satisfies that

$$\frac{\mathrm{E}_{\varepsilon^{-1}x}\Big[\Psi^{\beta_\varepsilon,h_0}_{\varepsilon^{-2}t}\Big]}{\mathrm{E}_{\varepsilon^{-1}x}\Big[\Psi^{\beta_\varepsilon,0}_{\varepsilon^{-2(1-a_\varepsilon)}t}\Big]} = \frac{\int_{z\in\mathbb{R}^2}\rho_s(\varepsilon^{-1}x,z)\mathrm{E}^{s,z}_{0,\varepsilon^{-1}x}\Big[\Psi^{\beta_\varepsilon,0}_s(B^B,\xi)\Big]\mathrm{E}_z\Big[\Psi^{\beta_\varepsilon,h_0}_{\varepsilon^{-2}t-s}(\tilde{B},\xi')\Big]\mathrm{d}z}{\int_{z\in\mathbb{R}^2}\rho_s(\varepsilon^{-1}x,z)\mathrm{E}^{s,z}_{0,\varepsilon^{-1}x}\Big[\Psi^{\beta_\varepsilon,0}_s(B^B,\xi)\Big]\mathrm{d}z}. \tag{3.3.7}$$

We could thus interpret the ratio $\frac{\mathrm{E}_{\varepsilon^{-1}x}\left[\Psi^{\beta_\varepsilon,h_0}_{\varepsilon^{-2}t}\right]}{\mathrm{E}_{\varepsilon^{-1}x}\left[\Psi^{\beta_\varepsilon,0}_{\varepsilon^{-2(1-a_\varepsilon)}t}\right]}$ as a (randomly) weighted average of

$$\mathrm{E}_z\Big[\Psi^{\beta_\varepsilon,h_0}_{\varepsilon^{-2}t-s}(\tilde{B},\xi')\Big]$$

over $z\in\mathbb{R}^2$.

Let $\mathscr{F}_s$ be the σ-algebra generated by the environment $\{\xi(r,\cdot):0\le r<s\}$. Let $\mathbf{E}^{\mathscr{F}_s}$, $\mathbf{P}^{\mathscr{F}_s}$, $\mathbf{Var}^{\mathscr{F}_s}$, $\mathbf{Cov}^{\mathscr{F}_s}$ denote the conditional expectation, conditional probability, conditional variance and conditional covariance given $\mathscr{F}_s$, respectively. In particular, the noise (with a time shift by s) ξ' is independent of the filtration $\mathscr{F}_s$.

We first show that the conditional expectation of the ratio $\frac{\mathrm{E}_{\varepsilon^{-1}x}\left[\Psi^{\beta_\varepsilon,h_0}_{\varepsilon^{-2}t}\right]}{\mathrm{E}_{\varepsilon^{-1}x}\left[\Psi^{\beta_\varepsilon,0}_{\varepsilon^{-2(1-a_\varepsilon)}t}\right]}$ in (3.3.7) w.r.t. $\mathscr{F}_s$ is approximately deterministic as $\varepsilon\to 0$.

Lemma 3.3.4. *Let $\bar{h}(t,x)$ be as in (3.1.3) and $a_\varepsilon<1$. There exists some $C_\beta>0$ such that for any*

$t > 0$, $x \in \mathbb{R}^2$,

$$\left\| \mathbf{E}^{\mathscr{F}_s} \left\{ \frac{\mathrm{E}_{\varepsilon^{-1}x}\left[\Psi^{\beta_\varepsilon, h_0}_{\varepsilon^{-2}t}\right]}{\mathrm{E}_{\varepsilon^{-1}x}\left[\Psi^{\beta_\varepsilon, 0}_{\varepsilon^{-2(1-a_\varepsilon)}t}\right]} \right\} - \exp\left[\bar{h}(t - \varepsilon^{2a_\varepsilon}t, x)\right] \right\|_{L^2(\Omega)} \leq C_\beta \varepsilon^{a_\varepsilon} \sqrt{t}. \tag{3.3.8}$$

We leave the proof of Lemma 3.3.4 to Appendix 3.5.1.

We next show that the expectation of the conditional variance

$$\mathbb{E}\left[\mathbf{Var}^{\mathscr{F}_s} \left\{ \frac{\mathrm{E}_{\varepsilon^{-1}x}\left[\Psi^{\beta_\varepsilon, h_0}_{\varepsilon^{-2}t}\right]}{\mathrm{E}_{\varepsilon^{-1}x}\left[\Psi^{\beta_\varepsilon, 0}_{\varepsilon^{-2(1-a_\varepsilon)}t}\right]} \right\} \right]$$

converges to 0 as $\varepsilon \to 0$. This result is basically saying that the randomness from the shifted white noise ξ' is not contributing to the randomness in the ratio $\frac{\mathrm{E}_{\varepsilon^{-1}x}\left[\Psi^{\beta_\varepsilon, h_0}_{\varepsilon^{-2}t}\right]}{\mathrm{E}_{\varepsilon^{-1}x}\left[\Psi^{\beta_\varepsilon, 0}_{\varepsilon^{-2(1-a_\varepsilon)}t}\right]}$ as $\varepsilon \to 0$. In fact, we have the following bound and we leave its proof to Appendix 3.5.2.

Lemma 3.3.5. *Let $a_\varepsilon = o(1)$. There exists some $C_\beta > 0$ and $\varepsilon_0 > 0$ such that for any $x \in \mathbb{R}^2$ and $t > 0$, when $0 < \varepsilon < \varepsilon_0$,*

$$\mathbb{E}\left[\mathbf{Var}^{\mathscr{F}_s} \left\{ \frac{\mathrm{E}_{\varepsilon^{-1}x}\left[\Psi^{\beta_\varepsilon, h_0}_{\varepsilon^{-2}t}\right]}{\mathrm{E}_{\varepsilon^{-1}x}\left[\Psi^{\beta_\varepsilon, 0}_{\varepsilon^{-2(1-a_\varepsilon)}t}\right]} \right\} \right] \leq C_\beta a_\varepsilon^{1/3}.$$

Now by the law of total expectation, we have

$$\begin{aligned}
&\mathbb{E}\left[\frac{\mathrm{E}_{\varepsilon^{-1}x}\left[\Psi^{\beta_\varepsilon, h_0}_{\varepsilon^{-2}t}\right]}{\mathrm{E}_{\varepsilon^{-1}x}\left[\Psi^{\beta_\varepsilon, 0}_{\varepsilon^{-2(1-a_\varepsilon)}t}\right]} - \exp\left[\bar{h}(t - \varepsilon^{2a_\varepsilon}t, x)\right] \right]^2 \\
&= \mathbb{E}\left[\left(\mathbf{E}^{\mathscr{F}_s} \left\{ \frac{\mathrm{E}_{\varepsilon^{-1}x}\left[\Psi^{\beta_\varepsilon, h_0}_{\varepsilon^{-2}t}\right]}{\mathrm{E}_{\varepsilon^{-1}x}\left[\Psi^{\beta_\varepsilon, 0}_{\varepsilon^{-2(1-a_\varepsilon)}t}\right]} \right\} - \exp\left[\bar{h}(t - \varepsilon^{2a_\varepsilon}t, x)\right] \right)^2 \right] + \mathbb{E}\left[\mathbf{Var}^{\mathscr{F}_s} \left\{ \frac{\mathrm{E}_{\varepsilon^{-1}x}\left[\Psi^{\beta_\varepsilon, h_0}_{\varepsilon^{-2}t}\right]}{\mathrm{E}_{\varepsilon^{-1}x}\left[\Psi^{\beta_\varepsilon, 0}_{\varepsilon^{-2(1-a_\varepsilon)}t}\right]} \right\} \right],
\end{aligned}$$

which makes (3.3.1) a direct result of Lemma 3.3.4 and Lemma 3.3.5.

3.3.2 Proof of Proposition 3.3.2

We then use Proposition 3.3.1 to prove Proposition 3.3.2. We first start with the $L^{1/2}(\Omega)$ norm.

Lemma 3.3.6. *Under the same assumptions as in Proposition 3.3.2, there exist some $C_\beta > 0$ and $\varepsilon_0 > 0$, such that when $0 < \varepsilon < \varepsilon_0$, we have*

$$\mathbb{E}\left[\left|\log \frac{\mathrm{E}_{\varepsilon^{-1}x}\left[\Psi^{\beta_\varepsilon,h_0}_{\varepsilon^{-2}t}\right]}{\mathrm{E}_{\varepsilon^{-1}x}\left[\Psi^{\beta_\varepsilon,0}_{\varepsilon^{-2(1-a_\varepsilon)t}}\right]} - \bar{h}(t-\varepsilon^{2a_\varepsilon}t,x)\right|^{1/2}\right] \le C_\beta(\varepsilon^{2a_\varepsilon}t + a_\varepsilon^{1/3})^{1/4}.$$

Proof. For any $a,b>0$, we have $|\log a - \log b|^{1/2} \le \sqrt{|1-\frac{a}{b}|} + \sqrt{|1-\frac{b}{a}|} = \sqrt{|a-b|}(a^{-1/2}+b^{-1/2})$. Thus

$$\begin{aligned}
&\mathbb{E}\left[\left|\log \frac{\mathrm{E}_{\varepsilon^{-1}x}\left[\Psi^{\beta_\varepsilon,h_0}_{\varepsilon^{-2}t}\right]}{\mathrm{E}_{\varepsilon^{-1}x}\left[\Psi^{\beta_\varepsilon,0}_{s}\right]} - \bar{h}(t-\varepsilon^{2a_\varepsilon}t,x)\right|^{\frac{1}{2}}\right] \\
&\le \mathbb{E}\left\{\left|\frac{\mathrm{E}_{\varepsilon^{-1}x}\left[\Psi^{\beta_\varepsilon,h_0}_{\varepsilon^{-2}t}\right]}{\mathrm{E}_{\varepsilon^{-1}x}\left[\Psi^{\beta_\varepsilon,0}_{s}\right]} - \exp[\bar{h}(t-\varepsilon^{2a_\varepsilon}t,x)]\right|^{\frac{1}{2}}\left[\left(\frac{\mathrm{E}_{\varepsilon^{-1}x}\left[\Psi^{\beta_\varepsilon,h_0}_{\varepsilon^{-2}t}\right]}{\mathrm{E}_{\varepsilon^{-1}x}\left[\Psi^{\beta_\varepsilon,0}_{s}\right]}\right)^{-\frac{1}{2}} + \exp[-\frac{1}{2}\bar{h}(t-\varepsilon^{2a_\varepsilon}t,x)]\right]\right\} \\
&\le \sqrt{2}\left[\mathbb{E}\left|\frac{\mathrm{E}_{\varepsilon^{-1}x}\left[\Psi^{\beta_\varepsilon,h_0}_{\varepsilon^{-2}t}\right]}{\mathrm{E}_{\varepsilon^{-1}x}\left[\Psi^{\beta_\varepsilon,0}_{s}\right]} - \exp[\bar{h}(t-\varepsilon^{2a_\varepsilon}t,x)]\right|\right]^{\frac{1}{2}} \\
&\cdot \left[\mathbb{E}\left(\frac{\mathrm{E}_{\varepsilon^{-1}x}\left[\Psi^{\beta_\varepsilon,h_0}_{\varepsilon^{-2}t}\right]}{\mathrm{E}_{\varepsilon^{-1}x}\left[\Psi^{\beta_\varepsilon,0}_{s}\right]}\right)^{-1} + \exp[-\bar{h}(t-\varepsilon^{2a_\varepsilon}t,x)]\right]^{\frac{1}{2}}.
\end{aligned}$$

Note that

$$\mathbb{E}\left(\frac{\mathrm{E}_{\varepsilon^{-1}x}\left[\Psi^{\beta_\varepsilon,h_0}_{\varepsilon^{-2}t}\right]}{\mathrm{E}_{\varepsilon^{-1}x}\left[\Psi^{\beta_\varepsilon,0}_{s}\right]}\right)^{-1} \le \left\|\mathrm{E}_{\varepsilon^{-1}x}\left[\Psi^{\beta_\varepsilon,0}_{s}\right]\right\|_2 \left\|\frac{1}{\mathrm{E}_{\varepsilon^{-1}x}\left[\Psi^{\beta_\varepsilon,h_0}_{\varepsilon^{-2}t}\right]}\right\|_2. \tag{3.3.9}$$

By (3.2.4) and Lemma 3.2.3, together with the assumption that h_0 is bounded, (3.3.9) is bounded by some constant C_β. Lemma 3.3.6 is then the consequence of the Cauchy-Schwarz inequality and Proposition 3.3.1. □

Now to prove Proposition 3.3.2, we apply the Cauchy-Schwarz inequality to the following expression:

$$\begin{aligned}&\mathbb{E}\left[\left|\log\frac{\mathrm{E}_{\varepsilon^{-1}x}\left[\Psi^{\beta_\varepsilon,h_0}_{\varepsilon^{-2}t}\right]}{\mathrm{E}_{\varepsilon^{-1}x}\left[\Psi^{\beta_\varepsilon,0}_{s}\right]}-\bar{h}(t-\varepsilon^{2a_\varepsilon}t,x)\right|^2\right]\\&=\mathbb{E}\left[\left|\log\frac{\mathrm{E}_{\varepsilon^{-1}x}\left[\Psi^{\beta_\varepsilon,h_0}_{\varepsilon^{-2}t}\right]}{\mathrm{E}_{\varepsilon^{-1}x}\left[\Psi^{\beta_\varepsilon,0}_{s}\right]}-\bar{h}(t-\varepsilon^{2a_\varepsilon}t,x)\right|^{1/4}\left|\log\frac{\mathrm{E}_{\varepsilon^{-1}x}\left[\Psi^{\beta_\varepsilon,h_0}_{\varepsilon^{-2}t}\right]}{\mathrm{E}_{\varepsilon^{-1}x}\left[\Psi^{\beta_\varepsilon,0}_{s}\right]}-\bar{h}(t-\varepsilon^{2a_\varepsilon}t,x)\right|^{7/4}\right],\end{aligned}$$

and bound the $L^{7/2}(\Omega)$ term by using Corollary 3.2.4 and the discussion after it. Then there exists some constant C_β such that the above is bounded by

$$C_\beta\left\|\log\frac{\mathrm{E}_{\varepsilon^{-1}x}\left[\Psi^{\beta_\varepsilon,h_0}_{\varepsilon^{-2}t}\right]}{\mathrm{E}_{\varepsilon^{-1}x}\left[\Psi^{\beta_\varepsilon,0}_{\varepsilon^{-2(1-a_\varepsilon)}t}\right]}-\bar{h}(t-\varepsilon^{2a_\varepsilon}t,x)\right\|^{1/4}_{L^{1/2}(\Omega)}.$$

We now apply Lemma 3.3.6.

3.3.3 Proof of Proposition 3.3.3

We shall prove that

$$\frac{1}{|B(x,r_\varepsilon)|}\int_{|y-x|\le r_\varepsilon}\log\mathrm{E}_{\varepsilon^{-1}y}\left[\Psi^{\beta_\varepsilon,0}_{\varepsilon^{-2(1-a_\varepsilon)}t}\right]\mathrm{d}y$$

converges to the Gaussian random variable $\mathcal{N}(0,\sigma_\gamma^2)-\frac{1}{2}\log\left(\frac{2\pi}{2\pi-\beta^2}\right)$ as $\varepsilon\to0$. To do so, we prove the convergence of all its moments.

By the shear invariance of the space-time white noise, we can set $x = 0$ without loss of generality.

We first show the convergence of the mean.

By Corollary 3.2.4, for any $t > 0$, $y \in \mathbb{R}^2$, $\log \mathrm{E}_{\varepsilon^{-1}y}\left[\Psi_s^{\beta_\varepsilon,0}\right]$ is uniformly integrable. With $a_\varepsilon = o(1)$, by Theorem 3.1.5, its expectation converges to $-\frac{1}{2}\log\left(\frac{2\pi}{2\pi-\beta^2}\right)$ as $\varepsilon \to 0$. By a change of variable $\frac{y}{r_\varepsilon} \to \tilde{y}$,

$$\mathbb{E}\left(\frac{1}{|B(0,r_\varepsilon)|}\int_{|y|\le r_\varepsilon}\log \mathrm{E}_{\varepsilon^{-1}y}\big[\Psi_s^{\beta_\varepsilon,0}\big]\mathrm{d}y\right) = \mathbb{E}\left(\frac{1}{|B(0,1)|}\int_{|\tilde{y}|\le 1}\log \mathrm{E}_{\varepsilon^{-1}\tilde{y}r_\varepsilon}\big[\Psi_s^{\beta_\varepsilon,0}\big]\mathrm{d}\tilde{y}\right).$$

By Lebesgue's dominated convergence theorem,

$$\mathbb{E}\left(\frac{1}{|B(0,1)|}\int_{|\tilde{y}|\le 1}\log \mathrm{E}_{\varepsilon^{-1}\tilde{y}r_\varepsilon}\big[\Psi_s^{\beta_\varepsilon,0}\big]\mathrm{d}\tilde{y}\right) \to -\frac{1}{2}\log\left(\frac{2\pi}{2\pi-\beta^2}\right), \qquad \text{as } \varepsilon \to 0.$$

For the second moment, with $\frac{y}{r_\varepsilon} \to \tilde{y}$ and $\frac{y'}{r_\varepsilon} \to \tilde{y}'$,

$$\begin{aligned}
&\mathrm{Var}\left(\frac{1}{|B(0,r_\varepsilon)|}\int_{|y|\le r_\varepsilon}\log \mathrm{E}_{\varepsilon^{-1}y}\big[\Psi_s^{\beta_\varepsilon,0}\big]\mathrm{d}y\right)\\
&= \frac{1}{|B(0,r_\varepsilon)|^2}\int_{|y|\le r_\varepsilon}\int_{|y'|\le r_\varepsilon}\mathrm{Cov}\left(\log \mathrm{E}_{\varepsilon^{-1}y}\big[\Psi_s^{\beta_\varepsilon,0}\big], \log \mathrm{E}_{\varepsilon^{-1}y'}\big[\Psi_s^{\beta_\varepsilon,0}\big]\right)\mathrm{d}y\mathrm{d}y'\\
&= \frac{1}{|B(0,1)|^2}\int_{|\tilde{y}|\le 1}\int_{|\tilde{y}'|\le 1}\mathrm{Cov}\left(\log \mathrm{E}_{\varepsilon^{-1}\tilde{y}r_\varepsilon}\big[\Psi_s^{\beta_\varepsilon,0}\big], \log \mathrm{E}_{\varepsilon^{-1}\tilde{y}'r_\varepsilon}\big[\Psi_s^{\beta_\varepsilon,0}\big]\right)\mathrm{d}\tilde{y}\mathrm{d}\tilde{y}'.
\end{aligned}$$

Again, Corollary 3.2.4 gives the uniform integrability. By Theorem 3.1.5 and the continuous mapping theorem, for any $\tilde{y}, \tilde{y}' \in \mathbb{R}^2$, as $\varepsilon \to 0$,

$$\mathrm{Cov}\left(\log \mathrm{E}_{\varepsilon^{-1}\tilde{y}r_\varepsilon}\big[\Psi_s^{\beta_\varepsilon,0}\big], \log \mathrm{E}_{\varepsilon^{-1}\tilde{y}'r_\varepsilon}\big[\Psi_s^{\beta_\varepsilon,0}\big]\right) \to \mathrm{Cov}[Y_1, Y_2],$$

where (Y_1, Y_2) are the jointly Gaussian random variables defined in Theorem 3.1.5 with $\zeta_{1,2} = \gamma$. In particular, since $\mathrm{Cov}[Y_1, Y_2] = \log\left(\frac{2\pi-\beta^2\gamma}{2\pi-\beta^2}\right) =: \sigma_\gamma^2$, we have

$$\mathrm{Var}\left(\frac{1}{|B(0,r_\varepsilon)|}\int_{|y|\le r_\varepsilon}\log \mathrm{E}_{\varepsilon^{-1}y}\left[\Psi_s^{\beta_\varepsilon,0}\right]\mathrm{d}y\right) \to \sigma_\gamma^2, \qquad \text{as } \varepsilon \to 0.$$

The convergence of any higher moments can be proved following a same procedure. In fact, for any $p \ge 1$, with $\frac{y_i}{r_\varepsilon} \to \tilde{y}_i$ for all $1 \le i \le p$,

$$\begin{aligned}
&\mathbb{E}\left(\left[\frac{1}{|B(0,r_\varepsilon)|}\int_{|y|\le r_\varepsilon}\log \mathrm{E}_{\varepsilon^{-1}y}\left[\Psi_s^{\beta_\varepsilon,0}\right]\mathrm{d}y - \mathbb{E}\left(\frac{1}{|B(0,r_\varepsilon)|}\int_{|y|\le r_\varepsilon}\log \mathrm{E}_{\varepsilon^{-1}y}\left[\Psi_s^{\beta_\varepsilon,0}\right]\mathrm{d}y\right)\right]^p\right)\\
&= \frac{1}{|B(0,r_\varepsilon)|^p}\int_{|y_1|\le r_\varepsilon}\cdots\int_{|y_p|\le r_\varepsilon}\mathbb{E}\left[\prod_{i=1}^p\left(\log \mathrm{E}_{\varepsilon^{-1}y_i}\left[\Psi_s^{\beta_\varepsilon,0}\right] - \mathbb{E}\log \mathrm{E}_{\varepsilon^{-1}y_i}\left[\Psi_s^{\beta_\varepsilon,0}\right]\right)\right]\mathrm{d}y_1\ldots\mathrm{d}y_p\\
&= \frac{1}{|B(0,1)|^p}\int_{|\tilde{y}_1|\le 1}\cdots\int_{|\tilde{y}_p|\le 1}\mathbb{E}\left[\prod_{i=1}^p\left(\log \mathrm{E}_{\varepsilon^{-1}\tilde{y}_i r_\varepsilon}\left[\Psi_s^{\beta_\varepsilon,0}\right] - \mathbb{E}\log \mathrm{E}_{\varepsilon^{-1}\tilde{y}_i r_\varepsilon}\left[\Psi_s^{\beta_\varepsilon,0}\right]\right)\right]\mathrm{d}\tilde{y}_1\ldots\mathrm{d}\tilde{y}_p.
\end{aligned} \tag{3.3.10}$$

The uniform integrability is guaranteed by Corollary 3.2.4. Using Theorem 3.1.5, the continuous mapping theorem and the uniform integrability, for any $(\tilde{y}_i)_{1\le i\le p}$,

$$\mathbb{E}\left[\prod_{i=1}^p\left(\log \mathrm{E}_{\varepsilon^{-1}\tilde{y}_i r_\varepsilon}\left[\Psi_s^{\beta_\varepsilon,0}\right] - \mathbb{E}\log \mathrm{E}_{\varepsilon^{-1}\tilde{y}_i r_\varepsilon}\left[\Psi_s^{\beta_\varepsilon,0}\right]\right)\right] \to \mathbb{E}\left[\prod_{i=1}^p Y_i\right], \qquad \text{as } \varepsilon \to 0,$$

where $(Y_i)_{1\le i\le p}$ are again the jointly Gaussian random variables defined in Theorem 3.1.5 with $\zeta_{i,j} = \gamma$ for any $i \ne j$.

Let P_p^2 be the set of all the pairings of $\{1, \ldots, p\}$. By Wick's probability theorem,

$$\mathbb{E}\left[\prod_{i=1}^p Y_i\right] = \sum_{s\in P_p^2}\prod_{\{i,j\}\in s}\mathrm{Cov}[Y_i, Y_j].$$

In particular,

$$\mathbb{E}\left[\prod_{i=1}^{p} Y_i\right] = \begin{cases} 0, & \text{if } p \text{ is odd,} \\ \\ \sigma_\gamma^p (p-1)!!, & \text{if } p \text{ is even.} \end{cases}$$

Here we use $n!!$ to denote the double factorial.

By Lebesgue's dominated convergence theorem, (3.3.10) converges to $\mathbb{E}\left[\prod_{i=1}^{p} Y_i\right]$ as $\varepsilon \to 0$. Thus for any $p \geq 1$, the p-th moment of

$$\frac{1}{|B(0, r_\varepsilon)|} \int_{|y| \leq r_\varepsilon} \log \mathrm{E}_{\varepsilon^{-1}y}\left[\Psi_s^{\beta_\varepsilon, 0}\right] \mathrm{d}y$$

converges to the p-th moment of the Gaussian distribution $\mathcal{N}(0, \sigma_\gamma^2) - \frac{1}{2} \log\left(\frac{2\pi}{2\pi - \beta^2}\right)$ as $\varepsilon \to 0$. This implies (3.3.3).

3.4 Proof of Corollary 3.1.3

By the equality in law (3.1.10), we can prove Corollary 3.1.3 by showing that as $\varepsilon \to 0$,

$$\mathbb{E}\left[\int_{\mathbb{R}^2} \left(\frac{1}{|B(x, r_\varepsilon)|} \int_{|y-x| \leq r_\varepsilon} \log \mathrm{E}_{\varepsilon^{-1}y}\left[\Psi_{\varepsilon^{-2}t}^{\beta_\varepsilon, h_0}\right] \mathrm{d}y - \left[\bar{h}(t, x) - \frac{1}{2} \log\left(\frac{2\pi}{2\pi - \beta^2}\right)\right]\right) g(x) \mathrm{d}x\right]^2 \to 0. \tag{3.4.1}$$

In order to prove (3.4.1), we prove that as $\varepsilon \to 0$,

$$\begin{aligned} &\left\| \int_{\mathbb{R}^2} \left(\frac{1}{|B(x, r_\varepsilon)|} \int_{|y-x| \leq r_\varepsilon} \log \mathrm{E}_{\varepsilon^{-1}y}\left[\Psi_{\varepsilon^{-2}t}^{\beta_\varepsilon, h_0}\right] \mathrm{d}y \right.\right. \\ &\left.\left. - \mathbb{E}\left[\frac{1}{|B(x, r_\varepsilon)|} \int_{|y-x| \leq r_\varepsilon} \log \mathrm{E}_{\varepsilon^{-1}y}\left[\Psi_{\varepsilon^{-2}t}^{\beta_\varepsilon, h_0}\right] \mathrm{d}y\right]\right) g(x) \mathrm{d}x \right\|_{L^2(\Omega)} \to 0, \end{aligned} \tag{3.4.2}$$

and

$$\int_{\mathbb{R}^2}\left(\mathbb{E}\left[\frac{1}{|B(x,r_\varepsilon)|}\int_{|y-x|\le r_\varepsilon}\log \mathrm{E}_{\varepsilon^{-1}y}\left[\Psi^{\beta_\varepsilon,h_0}_{\varepsilon^{-2}t}\right]\mathrm{d}y\right]-\left[\bar{h}(t,x)-\frac{1}{2}\log\left(\frac{2\pi}{2\pi-\beta^2}\right)\right]\right)g(x)\mathrm{d}x\to 0. \tag{3.4.3}$$

Let $\tilde{\mathfrak{h}}^{\varepsilon,\gamma}(t,x) := \frac{1}{|B(x,r_\varepsilon)|}\int_{|y-x|\le r_\varepsilon}\log \mathrm{E}_{\varepsilon^{-1}y}\left[\Psi^{\beta_\varepsilon,h_0}_{\varepsilon^{-2}t}\right]\mathrm{d}y$. By using again the decomposition (3.1.11), we can prove that for any $x_1\neq x_2$, $x_1,x_2\in\mathbb{R}^2$, as $\varepsilon\to 0$,

$$\begin{aligned}
&\mathrm{Cov}\left[\tilde{\mathfrak{h}}^{\varepsilon,\gamma}(t,x_1),\tilde{\mathfrak{h}}^{\varepsilon,\gamma}(t,x_2)\right]\\
&=\mathbb{E}\left(\prod_{i=1,2}\left[\frac{1}{|B(x_i,r_\varepsilon)|}\int_{|y_i-x_i|\le r_\varepsilon}\left(\log \mathrm{E}_{\varepsilon^{-1}y_i}\left[\Psi^{\beta_\varepsilon,h_0}_{\varepsilon^{-2}t}\right]-\mathbb{E}\log \mathrm{E}_{\varepsilon^{-1}y_i}\left[\Psi^{\beta_\varepsilon,h_0}_{\varepsilon^{-2}t}\right]\right)\mathrm{d}y_i\right]\right)\\
&=\mathbb{E}\left(\prod_{i=1,2}\left[\frac{1}{|B(x_i,1)|}\int_{|\tilde{y}_i|\le 1}\left(\log \mathrm{E}_{\varepsilon^{-1}(\tilde{y}_i r_\varepsilon+x_i)}\left[\Psi^{\beta_\varepsilon,h_0}_{\varepsilon^{-2}t}\right]-\mathbb{E}\log \mathrm{E}_{\varepsilon^{-1}(\tilde{y}_i r_\varepsilon+x_i)}\left[\Psi^{\beta_\varepsilon,h_0}_{\varepsilon^{-2}t}\right]\right)\mathrm{d}\tilde{y}_i\right]\right)\\
&\to 0.
\end{aligned}$$

More precisely, we can apply (3.1.11) to split $\tilde{\mathfrak{h}}^{\varepsilon,\gamma}(t,x_i)$ into an "almost deterministic" part on $[0,t-o(1))$ and a "random" part on $[t-o(1),t]$ again. We can then use the Lebesgue's dominated convergence theorem to prove that the covariance of the "random" part converges to zero as $\varepsilon\to 0$, where we appeal to Corollary 3.2.4 with $p=1$ for the uniform integrability. If $h_0=0$, the above convergence is a direct consequence of Theorem 3.1.5 and Corollary 3.2.4.

We can now prove (3.4.2). In fact,

$$\begin{aligned}
&\mathbb{E}\left[\int_{\mathbb{R}^2}\left(\frac{1}{|B(x,r_\varepsilon)|}\int_{|y-x|\le r_\varepsilon}\log \mathrm{E}_{\varepsilon^{-1}y}\left[\Psi^{\beta_\varepsilon,h_0}_{\varepsilon^{-2}t}\right]-\mathbb{E}\log \mathrm{E}_{\varepsilon^{-1}y}\left[\Psi^{\beta_\varepsilon,h_0}_{\varepsilon^{-2}t}\right]\mathrm{d}y\right)g(x)\mathrm{d}x\right]^2\\
&=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\mathrm{Cov}\left[\tilde{\mathfrak{h}}^{\varepsilon,\gamma}(t,x_1),\tilde{\mathfrak{h}}^{\varepsilon,\gamma}(t,x_2)\right]g(x_1)g(x_2)\mathrm{d}x_1\mathrm{d}x_2.
\end{aligned} \tag{3.4.4}$$

Again by using Corollary 3.2.4, we can apply Lebesgue's dominated convergence theorem to show that (3.4.4) converges to zero as $\varepsilon \to 0$.

To prove (3.4.3), we note that Corollary 3.2.4 also guarantees the uniform integrability of $\frac{1}{|B(x,r_\varepsilon)|}\int_{|y-x|\le r_\varepsilon} \log \mathrm{E}_{\varepsilon^{-1}y}\Big[\Psi^{\beta_\varepsilon,h_0}_{\varepsilon^{-2}t}\Big]\mathrm{d}y$ for any fixed $t > 0, x \in \mathbb{R}^2$. Thus for any fixed $x \in \mathbb{R}^2$, as $\varepsilon \to 0$,

$$\mathbb{E}\left[\frac{1}{|B(x,r_\varepsilon)|}\int_{|y-x|\le r_\varepsilon} \log \mathrm{E}_{\varepsilon^{-1}y}\Big[\Psi^{\beta_\varepsilon,h_0}_{\varepsilon^{-2}t}\Big]\mathrm{d}y\right] \to \bar{h}(t,x) - \frac{1}{2}\log\left(\frac{2\pi}{2\pi-\beta^2}\right).$$

Now by using again Corollary 3.2.4 with $p = 1$, we can apply Lebesgue's dominated convergence theorem to obtain (3.4.3).

3.5 Appendix

3.5.1 Proof of Lemma 3.3.4

From (3.3.7), we have

$$\begin{aligned}
&\mathbf{E}^{\mathscr{F}_s}\left\{\frac{\mathrm{E}_{\varepsilon^{-1}x}\Big[\Psi^{\beta_\varepsilon,h_0}_{\varepsilon^{-2}t}\Big]}{\mathrm{E}_{\varepsilon^{-1}x}\Big[\Psi^{\beta_\varepsilon,0}_{\varepsilon^{-2(1-a_\varepsilon)}t}\Big]}\right\} - \exp\left[\bar{h}(t-\varepsilon^{2a_\varepsilon}t,x)\right] \\
&= \mathbf{E}^{\mathscr{F}_s}\left\{\frac{\int_{z\in\mathbb{R}^2}\rho_s(\varepsilon^{-1}x,z)\mathrm{E}^{s,z}_{0,\varepsilon^{-1}x}\Big[\Psi^{\beta_\varepsilon,0}_s(B^B,\xi)\Big]\Big\{\mathrm{E}_z\Big[\Psi^{\beta_\varepsilon,h_0}_{\varepsilon^{-2}t-s}(\tilde{B},\xi')\Big]-\exp[\bar{h}(t-\varepsilon^{2a_\varepsilon}t,x)]\Big\}\mathrm{d}z}{\int_{z\in\mathbb{R}^2}\rho_s(\varepsilon^{-1}x,z)\mathrm{E}^{s,z}_{0,\varepsilon^{-1}x}\Big[\Psi^{\beta_\varepsilon,0}_s(B^B,\xi)\Big]\mathrm{d}z}\right\} \\
&= \int_{z\in\mathbb{R}^2} \\
&\mathbf{E}^{\mathscr{F}_s}\left\{\frac{\rho_s(\varepsilon^{-1}x,z)\mathrm{E}^{s,z}_{0,\varepsilon^{-1}x}\Big[\Psi^{\beta_\varepsilon,0}_s(B^B,\xi)\Big]\Big\{\mathrm{E}_z\Big[\Psi^{\beta_\varepsilon,h_0}_{\varepsilon^{-2}t-s}(\tilde{B},\xi')\Big]-\exp[\bar{h}(t-\varepsilon^{2a_\varepsilon}t,x)]\Big\}}{\int_{z\in\mathbb{R}^2}\rho_s(\varepsilon^{-1}x,z)\mathrm{E}^{s,z}_{0,\varepsilon^{-1}x}\Big[\Psi^{\beta_\varepsilon,0}_s(B^B,\xi)\Big]\mathrm{d}z}\right\}\mathrm{d}z.
\end{aligned} \tag{3.5.1}$$

By the definition above,

$$\xi'(r,\cdot) = \xi(s+r,\cdot)$$

is a time shift of ξ by s. Thus $\{\xi'(r,\cdot)\}_{r\geq 0}$ is independent of the filtration $\mathscr{F}_s$. Since B^B and $\tilde{B}$ are independent, and $\mathrm{E}^{s,z}_{0,\varepsilon^{-1}x}\Big[\Psi^{\beta_\varepsilon,0}_s(B^B,\xi)\Big]$ is measurable w.r.t. $\mathscr{F}_s$, the above (3.5.1) equals to

$$\int_{z\in\mathbb{R}^2}\rho_s(\varepsilon^{-1}x,z)\mathbf{E}^{\mathscr{F}_s}\left\{\frac{\mathrm{E}^{s,z}_{0,\varepsilon^{-1}x}\Big[\Psi^{\beta_\varepsilon,0}_s(B^B,\xi)\Big]}{\int_{z\in\mathbb{R}^2}\rho_s(\varepsilon^{-1}x,z)\mathrm{E}^{s,z}_{0,\varepsilon^{-1}x}\Big[\Psi^{\beta_\varepsilon,0}_s(B^B,\xi)\Big]\mathrm{d}z}\right\}$$

$$\mathbb{E}\Big\{\mathrm{E}_z\Big[\Psi^{\beta_\varepsilon,h_0}_{\varepsilon^{-2}t-s}(\tilde{B},\xi')\Big]-\exp[\bar{h}(t-\varepsilon^{2a_\varepsilon}t,x)]\Big\}\,\mathrm{d}z$$

$$=\int_{z\in\mathbb{R}^2}\rho_s(\varepsilon^{-1}x,z)\frac{\mathrm{E}^{s,z}_{0,\varepsilon^{-1}x}\Big[\Psi^{\beta_\varepsilon,0}_s(B^B,\xi)\Big]}{\mathrm{E}_{\varepsilon^{-1}x}\Big[\Psi^{\beta_\varepsilon,0}_s\Big]}\mathbb{E}\Big\{\mathrm{E}_z\Big[\Psi^{\beta_\varepsilon,h_0}_{\varepsilon^{-2}t-s}(\tilde{B},\xi')\Big]-\exp[\bar{h}(t-\varepsilon^{2a_\varepsilon}t,x)]\Big\}\,\mathrm{d}z,$$

where we applied (3.3.6) to the denominator. Since $\mathbb{E}\left[\Psi^{\beta_\varepsilon,0}_s(\tilde{B},\xi')\right]=1$, through scaling invariance of the Brownian motion $\tilde{B}$, the above equals to

$$\int_{z\in\mathbb{R}^2}\rho_s(\varepsilon^{-1}x,z)\frac{\mathrm{E}^{s,z}_{0,\varepsilon^{-1}x}\Big[\Psi^{\beta_\varepsilon,0}_s(B^B,\xi)\Big]}{\mathrm{E}_{\varepsilon^{-1}x}\Big[\Psi^{\beta_\varepsilon,0}_s\Big]}$$

$$\mathbb{E}\left\{\mathrm{E}_z\left[\exp\left[h_0(\varepsilon\tilde{B}_{\varepsilon^{-2}t-s})\right]\Psi^{\beta_\varepsilon,0}_{\varepsilon^{-2}t-s}(\tilde{B},\xi')\right]-\exp[\bar{h}(t-\varepsilon^{2a_\varepsilon}t,x)]\right\}\mathrm{d}z$$

$$=\int_{z\in\mathbb{R}^2}\rho_s(\varepsilon^{-1}x,z)\frac{\mathrm{E}^{s,z}_{0,\varepsilon^{-1}x}\Big[\Psi^{\beta_\varepsilon,0}_s(B^B,\xi)\Big]}{\mathrm{E}_{\varepsilon^{-1}x}\Big[\Psi^{\beta_\varepsilon,0}_s\Big]}\Big(\mathrm{E}_{\varepsilon z}\left[\exp[h_0(\tilde{B}_{t-\varepsilon^{2a_\varepsilon}t})]\right]-\exp[\bar{h}(t-\varepsilon^{2a_\varepsilon}t,x)]\Big)\,\mathrm{d}z$$

$$=\int_{z'\in\mathbb{R}^2}\rho_s(\varepsilon^{-1}x,\varepsilon^{-1}z')\frac{\mathrm{E}^{s,\varepsilon^{-1}z'}_{0,\varepsilon^{-1}x}\Big[\Psi^{\beta_\varepsilon,0}_s(B^B,\xi)\Big]}{\mathrm{E}_{\varepsilon^{-1}x}\Big[\Psi^{\beta_\varepsilon,0}_s\Big]}$$

$$\Big(\mathrm{E}_{z'}\left[\exp[h_0(\tilde{B}_{t-\varepsilon^{2a_\varepsilon}t})]\right]-\exp[\bar{h}(t-\varepsilon^{2a_\varepsilon}t,x)]\Big)\,\varepsilon^{-2}\mathrm{d}z'.$$

We applied a change of variable $\varepsilon z\to z'$ in the last equation.

Using Minkowski's integral inequality and the above computation, we have

$$\begin{aligned}
&\left\| \mathbf{E}^{\mathscr{F}_s} \left\{ \frac{\mathrm{E}_{\varepsilon^{-1}x}\left[\Psi^{\beta_\varepsilon, h_0}_{\varepsilon^{-2}t}\right]}{\mathrm{E}_{\varepsilon^{-1}x}\left[\Psi^{\beta_\varepsilon,0}_s\right]} \right\} - \exp\left[\bar{h}(t - \varepsilon^{2a_\varepsilon} t, x)\right] \right\|_{L^2(\Omega)} \\
&\le \int_{z' \in \mathbb{R}^2} \rho_s(\varepsilon^{-1}x, \varepsilon^{-1}z') \left| \mathrm{E}_{z'}\left[\exp[h_0(\tilde{B}_{t-\varepsilon^{2a_\varepsilon}t})]\right] - \exp[\bar{h}(t - \varepsilon^{2a_\varepsilon}t, x)] \right| \qquad (3.5.2) \\
&\quad \left\| \frac{\mathrm{E}^{s,\varepsilon^{-1}z'}_{0,\varepsilon^{-1}x}\left[\Psi^{\beta_\varepsilon,0}_s(B^B, \xi)\right]}{\mathrm{E}_{\varepsilon^{-1}x}\left[\Psi^{\beta_\varepsilon,0}_s\right]} \right\|_{L^2(\Omega)} \varepsilon^{-2} \mathrm{d}z'.
\end{aligned}$$

By Hölder's inequality, for any $p, q > 2$ and $1/p + 1/q = 1/2$,

$$\left\| \frac{\mathrm{E}^{s,\varepsilon^{-1}z'}_{0,\varepsilon^{-1}x}\left[\Psi^{\beta_\varepsilon,0}_s(B^B, \xi)\right]}{\mathrm{E}_{\varepsilon^{-1}x}\left[\Psi^{\beta_\varepsilon,0}_s\right]} \right\|_{L^2(\Omega)} \le \left\| \mathrm{E}^{s,\varepsilon^{-1}z'}_{0,\varepsilon^{-1}x}\left[\Psi^{\beta_\varepsilon,0}_s(B^B, \xi)\right] \right\|_{L^p(\Omega)} \left\| \frac{1}{\mathrm{E}_{\varepsilon^{-1}x}\left[\Psi^{\beta_\varepsilon,0}_s\right]} \right\|_{L^q(\Omega)}.$$

By Lemma 3.2.2 and Lemma 3.2.3, if $p < p_\beta$, there exists some $\tilde{C}_\beta > 0$ such that for any $z' \in \mathbb{R}^2$ and $0 < \varepsilon \le 1$,

$$\left\| \frac{\mathrm{E}^{s,\varepsilon^{-1}z'}_{0,\varepsilon^{-1}x}\left[\Psi^{\beta_\varepsilon,0}_s(B^B, \xi)\right]}{\mathrm{E}_{\varepsilon^{-1}x}\left[\Psi^{\beta_\varepsilon,0}_s\right]} \right\|_{L^2(\Omega)} \le \tilde{C}_\beta.$$

Therefore, (3.5.2) is bounded by

$$\begin{aligned}
&\tilde{C}_\beta \int_{z' \in \mathbb{R}^2} \rho_s(\varepsilon^{-1}x, \varepsilon^{-1}z') \left| \mathrm{E}_{z'}\left[\exp[h_0(\tilde{B}_{t-\varepsilon^{2a_\varepsilon}t})]\right] - \exp[\bar{h}(t - \varepsilon^{2a_\varepsilon}t, x)] \right| \varepsilon^{-2} \mathrm{d}z' \\
&= \tilde{C}_\beta \int_{z' \in \mathbb{R}^2} \rho_{\varepsilon^2 s}(x, z') \left| \mathrm{E}_{z'}\left[\exp[h_0(\tilde{B}_{t-\varepsilon^{2a_\varepsilon}t})]\right] - \exp[\bar{h}(t - \varepsilon^{2a_\varepsilon}t, x)] \right| \mathrm{d}z'.
\end{aligned}$$

By Feynman-Kac formula, $\exp[\bar{h}(t - \varepsilon^{2a_\varepsilon}t, x)] = \mathrm{E}_x\left[\exp[h_0(B_{t-\varepsilon^{2a_\varepsilon}t})]\right]$ for some Brownian motion B starting at x. Let B^0 be another Brownian motion starting at 0. For any $a, b \in \mathbb{R}$, $|e^a - e^b| \le (e^a + e^b)|a - b|$. Now since h_0 is bounded and Lipschitz continuous, there exists some

$C > 0$, such that

$$
\begin{aligned}
&\left|\mathrm{E}_{z'}\left[\exp\left[h_0(\tilde{B}_{t-\varepsilon^{2a_\varepsilon}t})\right]\right] - \exp\left[h(t-\varepsilon^{2a_\varepsilon}t, x)\right]\right| \\
&= \left|\mathrm{E}_{z'}\left[\exp[h_0(\tilde{B}_{t-\varepsilon^{2a_\varepsilon}t})]\right] - \mathrm{E}_x\left[\exp[h_0(B_{t-\varepsilon^{2a_\varepsilon}t})]\right]\right| \\
&\le \mathrm{E}_0\left|\exp[h_0(z' + B^0_{t-\varepsilon^{2a_\varepsilon}t})] - \exp[h_0(x + B^0_{t-\varepsilon^{2a_\varepsilon}t})]\right| \\
&\le C|z'-x|.
\end{aligned}
$$

Then since

$$
\int_{z'\in\mathbb{R}^2} \rho_{\varepsilon^2 s}(x,z')|z'-x|\mathrm{d}z' = 2\pi \int_0^{+\infty} \frac{r^2}{2\pi\varepsilon^2 s} e^{-\frac{r^2}{2\varepsilon^2 s}}\mathrm{d}r = \sqrt{\frac{\pi}{2}\varepsilon^2 s},
$$

and $s = \varepsilon^{-2(1-a_\varepsilon)}t$, we have proved (3.3.8).

3.5.2 Proof of Lemma 3.3.5

Again, we shall use the facts that $\{\xi'(r,\cdot)\}_{r\ge 0}$ is independent of the filtration $\mathscr{F}_s$, B^B and $\tilde{B}$ are independent, and $\mathrm{E}^{s,z}_{0,\varepsilon^{-1}x}\left[\Psi^{\beta_\varepsilon,0}_s\right] := \mathrm{E}^{s,z}_{0,\varepsilon^{-1}x}\left[\Psi^{\beta_\varepsilon,0}_s(B^B,\xi)\right]$ is measurable w.r.t. $\mathscr{F}_s$. With the same approach as in the proof of Lemma 3.3.4, we have

$$
\begin{aligned}
&\mathbf{Var}^{\mathscr{F}_s}\left\{\frac{\mathrm{E}_{\varepsilon^{-1}x}\left[\Psi^{\beta_\varepsilon,h_0}_{\varepsilon^{-2}t}\right]}{\mathrm{E}_{\varepsilon^{-1}x}\left[\Psi^{\beta_\varepsilon,0}_s\right]}\right\} = \mathbf{Var}^{\mathscr{F}_s}\left\{\frac{\int_{z\in\mathbb{R}^2}\rho_s(\varepsilon^{-1}x,z)\mathrm{E}^{s,z}_{0,\varepsilon^{-1}x}\left[\Psi^{\beta_\varepsilon,0}_s\right]\mathrm{E}_z\left[\Psi^{\beta_\varepsilon,h_0}_{\varepsilon^{-2}t-s}(\tilde{B},\xi')\right]\mathrm{d}z}{\int_{z\in\mathbb{R}^2}\rho_s(\varepsilon^{-1}x,z)\mathrm{E}^{s,z}_{0,\varepsilon^{-1}x}\left[\Psi^{\beta_\varepsilon,0}_s\right]\mathrm{d}z}\right\} \\
&= \frac{\int_{z\in\mathbb{R}^2}\int_{z'\in\mathbb{R}^2}\rho_s(\varepsilon^{-1}x,z)\rho_s(\varepsilon^{-1}x,z')\mathrm{E}^{s,z}_{0,\varepsilon^{-1}x}\left[\Psi^{\beta_\varepsilon,0}_s\right]\mathrm{E}^{s,z'}_{0,\varepsilon^{-1}x}\left[\Psi^{\beta_\varepsilon,0}_s\right]\mathcal{C}^\varepsilon_{z,z'}\mathrm{d}z\mathrm{d}z'}{\int_{z\in\mathbb{R}^2}\int_{z'\in\mathbb{R}^2}\rho_s(\varepsilon^{-1}x,z)\rho_s(\varepsilon^{-1}x,z')\mathrm{E}^{s,z}_{0,\varepsilon^{-1}x}\left[\Psi^{\beta_\varepsilon,0}_s\right]\mathrm{E}^{s,z'}_{0,\varepsilon^{-1}x}\left[\Psi^{\beta_\varepsilon,0}_s\right]\mathrm{d}z\mathrm{d}z'},
\end{aligned}
\tag{3.5.3}
$$

where

$$
\mathcal{C}^\varepsilon_{z,z'} := \mathbf{Cov}^{\mathscr{F}_s}\left(\mathrm{E}_z\left[\Psi^{\beta_\varepsilon,h_0}_{\varepsilon^{-2}t-s}(\tilde{B},\xi')\right], \mathrm{E}_{z'}\left[\Psi^{\beta_\varepsilon,h_0}_{\varepsilon^{-2}t-s}(\tilde{B},\xi')\right]\right).
$$

Since $\{\xi'(r,\cdot)\}_{r\geq 0}$ is independent of the filtration $\mathscr{F}_s$,

$$\begin{aligned}&\mathbf{Cov}^{\mathscr{F}_s}\left(\mathrm{E}_z\Big[\Psi^{\beta_\varepsilon,h_0}_{\varepsilon^{-2}t-s}(\tilde{B},\xi')\Big],\mathrm{E}_{z'}\Big[\Psi^{\beta_\varepsilon,h_0}_{\varepsilon^{-2}t-s}(\tilde{B},\xi')\Big]\right)\\ &=\mathrm{Cov}\left(\mathrm{E}_z\Big[\Psi^{\beta_\varepsilon,h_0}_{\varepsilon^{-2}t-s}(\tilde{B},\xi')\Big],\mathrm{E}_{z'}\Big[\Psi^{\beta_\varepsilon,h_0}_{\varepsilon^{-2}t-s}(\tilde{B},\xi')\Big]\right).\end{aligned}$$

For fixed $z,z'\in\mathbb{R}^2$ and $\varepsilon>0$, $\mathcal{C}^\varepsilon_{z,z'}$ is a deterministic value, and

$$\begin{aligned}\mathcal{C}^\varepsilon_{z,z'}&=\mathrm{Cov}\left(\mathrm{E}_z\Big[\Psi^{\beta_\varepsilon,h_0}_{\varepsilon^{-2}t-s}(\tilde{B},\xi')\Big],\mathrm{E}_{z'}\Big[\Psi^{\beta_\varepsilon,h_0}_{\varepsilon^{-2}t-s}(\tilde{B},\xi')\Big]\right)\\ &=\mathrm{Cov}\Big(\mathrm{E}_z\left[\exp\left[h_0(\varepsilon\tilde{B}^1_{\varepsilon^{-2}t-s})\right]\Psi^{\beta_\varepsilon,0}_{\varepsilon^{-2}t-s}(\tilde{B}^1,\xi')\right],\mathrm{E}_{z'}\left[\exp\left[h_0(\varepsilon\tilde{B}^2_{\varepsilon^{-2}t-s})\right]\Psi^{\beta_\varepsilon,0}_{s}(\tilde{B}^2,\xi')\right]\Big)\\ &=\mathbb{E}\Big\{\mathrm{E}_z\left[\exp\left[h_0(\varepsilon\tilde{B}^1_{\varepsilon^{-2}t-s})\right]\left(\Psi^{\beta_\varepsilon,0}_{\varepsilon^{-2}t-s}(\tilde{B}^1,\xi')-1\right)\right]\\ &\qquad \mathrm{E}_{z'}\left[\exp\left[h_0(\varepsilon\tilde{B}^2_{\varepsilon^{-2}t-s})\right]\left(\Psi^{\beta_\varepsilon,0}_{\varepsilon^{-2}t-s}(\tilde{B}^2,\xi')-1\right)\right]\Big\}\\ &=\mathrm{E}_z\otimes\mathrm{E}_{z'}\Big\{\exp\left[h_0(\varepsilon\tilde{B}^1_{\varepsilon^{-2}t-s})+h_0(\varepsilon\tilde{B}^2_{\varepsilon^{-2}t-s})\right]\\ &\mathbb{E}\left(\Psi^{\beta_\varepsilon,0}_{\varepsilon^{-2}t-s}(\tilde{B}^1,\xi')-1\right)\left(\Psi^{\beta_\varepsilon,0}_{\varepsilon^{-2}t-s}(\tilde{B}^2,\xi')-1\right)\Big\}\\ &=\mathrm{E}_z\otimes\mathrm{E}_{z'}\Big\{\exp\left[h_0(\varepsilon\tilde{B}^1_{\varepsilon^{-2}t-s})+h_0(\varepsilon\tilde{B}^2_{\varepsilon^{-2}t-s})\right]\Big(\exp\Big[\beta_\varepsilon^2\int_0^{\varepsilon^{-2}t-s}V(\tilde{B}^1_r-\tilde{B}^2_r)\mathrm{d}r\Big]-1\Big)\Big\},\end{aligned}$$

where $V(x)=\phi\star\phi(x)$ as defined in (3.2.1). Note that we can bound

$$\mathrm{E}_z\otimes\mathrm{E}_{z'}\left(\exp\Big[\beta_\varepsilon^2\int_0^{\varepsilon^{-2}t-s}V(\tilde{B}^1_r-\tilde{B}^2_r)\mathrm{d}r\Big]-1\right)$$

uniformly by some constant as in (3.2.2). Since h_0 is also bounded, there exists some $0<C^{(1)}_\beta<\infty$, such that

$$\sup_{\varepsilon\leq 1}\sup_{z,z'\in\mathbb{R}^2}\mathcal{C}^\varepsilon_{z,z'}\leq C^{(1)}_\beta. \tag{3.5.4}$$

From (3.5.3), $\mathbf{Var}^{\mathscr{F}_s}\left\{\frac{\mathrm{E}_{\varepsilon^{-1}x}\left[\Psi^{\beta_\varepsilon,h_0}_{\varepsilon^{-2}t}\right]}{\mathrm{E}_{\varepsilon^{-1}x}\left[\Psi^{\beta_\varepsilon,0}_{s}\right]}\right\}$ is a weighted average of $C^\varepsilon_{z,z'}$ over $z,z'\in\mathbb{R}^2$. Therefore, by (3.5.4), for $\varepsilon \le 1$,

$$\mathbf{Var}^{\mathscr{F}_s}\left\{\frac{\mathrm{E}_{\varepsilon^{-1}x}\Big[\Psi^{\beta_\varepsilon,h_0}_{\varepsilon^{-2}t}\Big]}{\mathrm{E}_{\varepsilon^{-1}x}\Big[\Psi^{\beta_\varepsilon,0}_{s}\Big]}\right\} \le C^{(1)}_\beta, \qquad \text{almost surely.} \tag{3.5.5}$$

Now we first assume that, for a particular realization of the noise field, $\mathrm{E}_{\varepsilon^{-1}x}\Big[\Psi^{\beta_\varepsilon,0}_{s}\Big] \le b_\varepsilon$ for some $b_\varepsilon > 0$. We will choose an appropriate b_ε later so that $b_\varepsilon = o(1)$.

By the Markov's inequality and Lemma 3.2.3, we have

$$\mathbb{P}\Big[\mathrm{E}_{\varepsilon^{-1}x}\Big[\Psi^{\beta_\varepsilon,0}_{s}\Big] \le b_\varepsilon\Big] = \mathbb{P}\Big[\Big(\mathrm{E}_{\varepsilon^{-1}x}\Big[\Psi^{\beta_\varepsilon,0}_{s}\Big]\Big)^{-1} \ge b_\varepsilon^{-1}\Big] \le \mathbb{E}\Big[\Big(\mathrm{E}_{\varepsilon^{-1}x}\Big[\Psi^{\beta_\varepsilon,0}_{s}\Big]\Big)^{-1}\Big] b_\varepsilon \le C^{(2)}_\beta b_\varepsilon, \tag{3.5.6}$$

for some $C^{(2)}_\beta > 0$. By (3.5.5) and (3.5.6) together, we have

$$\mathbb{E}\left[\mathbf{Var}^{\mathscr{F}_s}\left\{\frac{\mathrm{E}_{\varepsilon^{-1}x}\Big[\Psi^{\beta_\varepsilon,h_0}_{\varepsilon^{-2}t}\Big]}{\mathrm{E}_{\varepsilon^{-1}x}\Big[\Psi^{\beta_\varepsilon,0}_{s}\Big]}\right\}; \mathrm{E}_{\varepsilon^{-1}x}\Big[\Psi^{\beta_\varepsilon,0}_{s}\Big] \le b_\varepsilon\right] \le C^{(1)}_\beta C^{(2)}_\beta b_\varepsilon. \tag{3.5.7}$$

We then assume that for a particular realization of the noise field,

$$\mathrm{E}_{\varepsilon^{-1}x}\Big[\Psi^{\beta_\varepsilon,0}_{s}\Big] > b_\varepsilon.$$

In this case, by (3.5.3),

$$\begin{aligned}&\mathbf{Var}^{\mathscr{F}_s}\left\{\frac{\mathrm{E}_{\varepsilon^{-1}x}\Big[\Psi^{\beta_\varepsilon,h_0}_{\varepsilon^{-2}t}\Big]}{\mathrm{E}_{\varepsilon^{-1}x}\Big[\Psi^{\beta_\varepsilon,0}_{s}\Big]}\right\}\\ &\le b_\varepsilon^{-2}\int_{z\in\mathbb{R}^2}\int_{z'\in\mathbb{R}^2}\rho_s(\varepsilon^{-1}x,z)\rho_s(\varepsilon^{-1}x,z')\mathrm{E}^{s,z}_{0,\varepsilon^{-1}x}\Big[\Psi^{\beta_\varepsilon,0}_{s}\Big]\mathrm{E}^{s,z'}_{0,\varepsilon^{-1}x}\Big[\Psi^{\beta_\varepsilon,0}_{s}\Big]\mathcal{C}^{\varepsilon}_{z,z'}\mathrm{d}z\mathrm{d}z'.\end{aligned} \tag{3.5.8}$$

We bound the expectation of the right-hand-side of (3.5.8). By Hölder's inequality and

Lemma 3.2.1, there exists some $0 < C_\beta^{(3)} < \infty$, such that

$$\begin{aligned}
&b_\varepsilon^{-2}\mathbb{E}\int_{z\in\mathbb{R}^2}\int_{z'\in\mathbb{R}^2}\rho_s(\varepsilon^{-1}x,z)\rho_s(\varepsilon^{-1}x,z')\mathrm{E}_{0,\varepsilon^{-1}x}^{s,z}\Big[\Psi_s^{\beta_\varepsilon,0}\Big]\mathrm{E}_{0,\varepsilon^{-1}x}^{s,z'}\Big[\Psi_s^{\beta_\varepsilon,0}\Big]\mathcal{C}_{z,z'}^\varepsilon \mathrm{d}z\mathrm{d}z' \\
&= b_\varepsilon^{-2}\int_{z\in\mathbb{R}^2}\int_{z'\in\mathbb{R}^2}\rho_s(\varepsilon^{-1}x,z)\rho_s(\varepsilon^{-1}x,z')\mathbb{E}\Big\{\mathrm{E}_{0,\varepsilon^{-1}x}^{s,z}\Big[\Psi_s^{\beta_\varepsilon,0}\Big]\mathrm{E}_{0,\varepsilon^{-1}x}^{s,z'}\Big[\Psi_s^{\beta_\varepsilon,0}\Big]\Big\}\mathcal{C}_{z,z'}^\varepsilon \mathrm{d}z\mathrm{d}z' \\
&\le b_\varepsilon^{-2}C_\beta^{(3)}\int_{z\in\mathbb{R}^2}\int_{z'\in\mathbb{R}^2}\rho_s(\varepsilon^{-1}x,z)\rho_s(\varepsilon^{-1}x,z')\mathcal{C}_{z,z'}^\varepsilon \mathrm{d}z\mathrm{d}z' \\
&= b_\varepsilon^{-2}C_\beta^{(3)}\int_{y\in\mathbb{R}^2}\int_{y'\in\mathbb{R}^2}\rho_{\varepsilon^2 s}(x,y)\rho_{\varepsilon^2 s}(x,y')\mathcal{C}_{\varepsilon^{-1}y,\varepsilon^{-1}y'}^\varepsilon \mathrm{d}y\mathrm{d}y',
\end{aligned} \tag{3.5.9}$$

with a change of variable $\varepsilon z \to y$ and $\varepsilon z' \to y'$ in the last equation.

Since we have the term b_ε^{-2} in (3.5.9), the uniform bound of $\mathcal{C}_{\varepsilon^{-1}y,\varepsilon^{-1}y'}^\varepsilon$ solely will not give us a desired bound. We shall proceed as the following.

By the boundedness of h_0, there exists some constant $C_{h_0} > 0$ such that

$$\begin{aligned}
&\int_{y\in\mathbb{R}^2}\int_{y'\in\mathbb{R}^2}\rho_{\varepsilon^2 s}(x,y)\rho_{\varepsilon^2 s}(x,y')\mathcal{C}_{\varepsilon^{-1}y,\varepsilon^{-1}y'}^\varepsilon \mathrm{d}y\mathrm{d}y' \\
&= \int_{y\in\mathbb{R}^2}\int_{y'\in\mathbb{R}^2}\rho_{\varepsilon^2 s}(x,y)\rho_{\varepsilon^2 s}(x,y') \\
&\mathrm{E}_{\varepsilon^{-1}y}\otimes\mathrm{E}_{\varepsilon^{-1}y'}\Big\{\exp\big[h_0(\varepsilon\tilde{B}^1_{\varepsilon^{-2}t-s})+h_0(\varepsilon\tilde{B}^2_{\varepsilon^{-2}t-s})\big] \\
&\Big(\exp\big[\beta_\varepsilon^2\int_0^{\varepsilon^{-2}t-s}V(\tilde{B}^1_r-\tilde{B}^2_r)\mathrm{d}r\big]-1\Big)\Big\}\mathrm{d}y\mathrm{d}y' \\
&\le C_{h_0}\int_{y\in\mathbb{R}^2}\int_{y'\in\mathbb{R}^2}\rho_{\varepsilon^2 s}(x,y)\rho_{\varepsilon^2 s}(x,y') \\
&\mathrm{E}_{\varepsilon^{-1}y}\otimes\mathrm{E}_{\varepsilon^{-1}y'}\Big(\exp\big[\beta_\varepsilon^2\int_0^{\varepsilon^{-2}t-s}V(\tilde{B}^1_r-\tilde{B}^2_r)\mathrm{d}r\big]-1\Big)\mathrm{d}y\mathrm{d}y' \\
&= C_{h_0}\int_{y\in\mathbb{R}^2}\int_{y'\in\mathbb{R}^2}\rho_{\varepsilon^2 s}(x,y)\rho_{\varepsilon^2 s}(x,y')\mathrm{E}_{\varepsilon^{-1}\frac{y-y'}{\sqrt{2}}}\Big(\exp\big[\beta_\varepsilon^2\int_0^{\varepsilon^{-2}t-s}V(\sqrt{2}B_r)\mathrm{d}r\big]-1\Big)\mathrm{d}y\mathrm{d}y' \\
&= C_{h_0}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\rho_{\varepsilon^2 s}(y)\rho_{\varepsilon^2 s}(y')\mathrm{E}_{\varepsilon^{-1}\frac{y-y'}{\sqrt{2}}}\Big(\exp\big[\beta_\varepsilon^2\int_0^{\varepsilon^{-2}t-s}V(\sqrt{2}B_r)\mathrm{d}r\big]-1\Big)\mathrm{d}y\mathrm{d}y' \\
&= C_{h_0}\int_{\mathbb{R}^2}\rho_{2\varepsilon^2 s}(y)\mathrm{E}_{\varepsilon^{-1}\frac{y}{\sqrt{2}}}\Big(\exp\big[\beta_\varepsilon^2\int_0^{\varepsilon^{-2}t-s}V(\sqrt{2}B_r)\mathrm{d}r\big]-1\Big)\mathrm{d}y.
\end{aligned} \tag{3.5.10}$$

By Taylor expansion, if we set $x_0 = \varepsilon^{-1}\frac{y}{\sqrt{2}}$ and $s_0 = 0$, the last line equals to

$$C_{h_0}\int_{\mathbb{R}^2}\rho_{2\varepsilon^2 s}(y)$$
$$\cdot\Big[\sum_{n=1}^{\infty}\beta_\varepsilon^{2n}\int_{0<s_1<\cdots<s_n<\varepsilon^{-2}t-s}\int_{\mathbb{R}^{2n}}\prod_{i=1}^{n}V(\sqrt{2}x_i)\rho_{s_i-s_{i-1}}(x_{i-1},x_i)\mathrm{d}s_1\ldots\mathrm{d}s_n\mathrm{d}x_1\ldots\mathrm{d}x_n\Big]\mathrm{d}y. \tag{3.5.11}$$

With $\int_{\mathbb{R}^2}\varepsilon^{-2}V(\varepsilon^{-1}x)\mathrm{d}x = 1$, we bound the $n = 1$ term by the following:

$$\begin{aligned}
&\int_{\mathbb{R}^2}\rho_{2\varepsilon^2 s}(y)\Big[\beta_\varepsilon^2\int_0^{\varepsilon^{-2}t-s}\int_{\mathbb{R}^2}V(\sqrt{2}x_1)\rho_{s_1}(x_1,\varepsilon^{-1}\frac{y}{\sqrt{2}})\mathrm{d}s_1\mathrm{d}x_1\Big]\mathrm{d}y\\
&= \beta_\varepsilon^2\int_0^{\varepsilon^{-2}t-s}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\rho_{2\varepsilon^2 s}(y)2\varepsilon^2\rho_{2\varepsilon^2 s_1}(\sqrt{2}\varepsilon x_1,y)V(\sqrt{2}x_1)\mathrm{d}x_1\mathrm{d}y\mathrm{d}s_1\\
&= \beta_\varepsilon^2\int_0^{\varepsilon^{-2}t-s}\int_{\mathbb{R}^2}2\varepsilon^2\rho_{2\varepsilon^2(s+s_1)}(\sqrt{2}\varepsilon x_1)V(\sqrt{2}x_1)\mathrm{d}x_1\mathrm{d}s_1\\
&= \beta_\varepsilon^2\int_0^{\varepsilon^{-2}t-s}\int_{\mathbb{R}^2}\rho_{2\varepsilon^2(s+s_1)}(x)V(\varepsilon^{-1}x)\mathrm{d}x\mathrm{d}s_1 = \beta_\varepsilon^2\int_{\varepsilon^2 s}^{t}\int_{\mathbb{R}^2}\rho_{2r}(x)\frac{1}{\varepsilon^2}V(\varepsilon^{-1}x)\mathrm{d}x\mathrm{d}r\\
&\le \beta_\varepsilon^2\int_{\varepsilon^2 s}^{t}\frac{1}{4\pi r}\mathrm{d}r = \frac{\beta_\varepsilon^2}{4\pi}\log\frac{t}{\varepsilon^2 s} = \frac{\beta^2\log\varepsilon^{-2a_\varepsilon}}{4\pi\log\varepsilon^{-1}} = \frac{\beta^2}{2\pi}a_\varepsilon.
\end{aligned}$$

For higher order terms, we have

$$\begin{aligned}
&\int_{\mathbb{R}^2}\rho_{2\varepsilon^2 s}(y)\Big[\beta_\varepsilon^{2n}\int_{0<s_1<\cdots<s_n<\varepsilon^{-2}t-s}\int_{\mathbb{R}^{2n}}\prod_{i=1}^{n}V(\sqrt{2}x_i)\rho_{s_i-s_{i-1}}(x_{i-1},x_i)\mathrm{d}s_1\ldots\mathrm{d}s_n\mathrm{d}x_1\ldots\mathrm{d}x_n\Big]\mathrm{d}y\\
&= \int_{\mathbb{R}^2}\rho_{2\varepsilon^2 s}(y)\Big[\beta_\varepsilon^{2n}\int_{0<s_1<\cdots<s_n<\varepsilon^{-2}t-s}\int_{\mathbb{R}^{2n}}\prod_{i=1}^{n}V(\sqrt{2}x_i)\\
&\quad(2\varepsilon^2)^n\rho_{2\varepsilon^2(s_i-s_{i-1})}(\sqrt{2}\varepsilon x_{i-1},\sqrt{2}\varepsilon x_i)\mathrm{d}s_1\ldots\mathrm{d}s_n\mathrm{d}x_1\ldots\mathrm{d}x_n\Big]\mathrm{d}y\\
&= \int_{\mathbb{R}^2}\rho_{2\varepsilon^2 s}(y)\\
&\Big[\beta_\varepsilon^{2n}\int_{0<s_1<\cdots<s_n<\varepsilon^{-2}t-s}\int_{\mathbb{R}^{2n}}\prod_{i=1}^{n}V(\frac{\tilde{x}_i}{\varepsilon})\rho_{2\varepsilon^2(s_i-s_{i-1})}(\tilde{x}_{i-1},\tilde{x}_i)\mathrm{d}s_1\ldots\mathrm{d}s_n\mathrm{d}\tilde{x}_1\ldots\mathrm{d}\tilde{x}_n\Big]\mathrm{d}y\\
&= \int_{\mathbb{R}^2}\rho_{2\varepsilon^2 s}(y)\\
&\Big[\beta_\varepsilon^{2n}\int_{0<\tilde{s}_1<\cdots<\tilde{s}_n<t-\varepsilon^2 s}\int_{\mathbb{R}^{2n}}\prod_{i=1}^{n}\frac{1}{\varepsilon^2}V(\frac{\tilde{x}_i}{\varepsilon})\rho_{2(\tilde{s}_i-\tilde{s}_{i-1})}(\tilde{x}_{i-1},\tilde{x}_i)\mathrm{d}\tilde{s}_1\ldots\mathrm{d}\tilde{s}_n\mathrm{d}\tilde{x}_1\ldots\mathrm{d}\tilde{x}_n\Big]\mathrm{d}y,
\end{aligned}$$

with $\tilde{s}_0 = 0$ and $\tilde{x}_0 = y$ by a change of variable.

Using again $\int_{\mathbb{R}^2} \frac{1}{\varepsilon^2} V(\varepsilon^{-1}x)\mathrm{d}x = 1$, for any $\varepsilon, u_1, u_2 > 0$ and $z_1, z_2 \in \mathbb{R}^2$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^2} \rho_{u_1}(z_1)\rho_{u_2}(z_2, z_1)\frac{1}{\varepsilon^2}V(\varepsilon^{-1}z_1)\mathrm{d}z_1 \\
&= \int_{\mathbb{R}^2} \frac{1}{4\pi^2 u_1 u_2} \exp\left(-\frac{|z_1|^2}{2u_1} - \frac{|z_1 - z_2|^2}{2u_2}\right)\frac{1}{\varepsilon^2}V(\varepsilon^{-1}z_1)\mathrm{d}z_1 \\
&= \rho_{u_1+u_2}(z_2)\int_{\mathbb{R}^2} \frac{2\pi(u_1+u_2)}{4\pi^2 u_1 u_2} \exp\left(\frac{|z_2|^2}{2(u_1+u_2)} - \frac{|z_1|^2}{2u_1} - \frac{|z_1 - z_2|^2}{2u_2}\right)\frac{1}{\varepsilon^2}V(\varepsilon^{-1}z_1)\mathrm{d}z_1 \qquad (3.5.12) \\
&= \rho_{u_1+u_2}(z_2)\int_{\mathbb{R}^2} \frac{2\pi(u_1+u_2)}{4\pi^2 u_1 u_2} \exp\left(-\frac{u_1+u_2}{2u_1u_2}(z_1 - \frac{u_1}{u_1+u_2}z_2)^2\right)\frac{1}{\varepsilon^2}V(\varepsilon^{-1}z_1)\mathrm{d}z_1 \\
&\le \rho_{u_1+u_2}(z_2)\int_{\mathbb{R}^2} \frac{(u_1+u_2)}{2\pi u_1 u_2}\frac{1}{\varepsilon^2}V(\varepsilon^{-1}z_1)\mathrm{d}z_1 = \frac{u_1+u_2}{2\pi u_1 u_2}\rho_{u_1+u_2}(z_2).
\end{aligned}$$

Since V is bounded by $\|V\|_\infty$, we also have that

$$\int_{\mathbb{R}^2} \rho_{u_1}(z_1)\rho_{u_2}(z_2, z_1)\frac{1}{\varepsilon^2}V(\varepsilon^{-1}z_1)\mathrm{d}z_1 \le \frac{\|V\|_\infty}{\varepsilon^2}\rho_{u_1+u_2}(z_2). \qquad (3.5.13)$$

Now if we integrate y first, and then $x_1, \ldots, x_n$ successively utilizing the bounds (3.5.12) and (3.5.13), we will have that

$$\begin{aligned}
&\int_{\mathbb{R}^2} \rho_{2\varepsilon^2 s}(y) \\
&\left[\beta_\varepsilon^{2n} \int_{0<s_1<\cdots<s_n<t-\varepsilon^2 s} \int_{\mathbb{R}^{2n}} \prod_{i=1}^{n} \frac{1}{\varepsilon^2}V(\varepsilon^{-1}x_i)\rho_{2(s_i-s_{i-1})}(x_{i-1}, x_i)\mathrm{d}s_1 \ldots \mathrm{d}s_n \mathrm{d}x_1 \ldots \mathrm{d}x_n\right]\mathrm{d}y \\
&\le \left(\frac{\beta_\varepsilon^2}{2\pi}\right)^n \int_{0<s_1<\cdots<s_n<t-\varepsilon^2 s} \frac{1}{2s_n + 2\varepsilon^2 s} \prod_{i=1}^{n-1}\left[\left(\frac{1}{2(s_{i+1}-s_i)} + \frac{1}{2s_i + 2\varepsilon^2 s}\right) \wedge \frac{2\pi\|V\|_\infty}{\varepsilon^2}\right]\mathrm{d}s_1 \ldots \mathrm{d}s_n.
\end{aligned}$$
(3.5.14)

To bound the last integral, we integrate $s_1, \ldots, s_n$ successively. For any $1 \le i \le n-1$, we have

$$\begin{aligned}
&\int_0^{s_{i+1}} \left(\frac{1}{2s_{i+1}-2s_i}+\frac{1}{2s_i+2\varepsilon^2 s}\right)\wedge \frac{2\pi\|V\|_\infty}{\varepsilon^2}\mathrm{d}s_i\\
&\le \int_0^{s_{i+1}} \frac{1}{2s_{i+1}-2s_i}\wedge \frac{2\pi\|V\|_\infty}{\varepsilon^2}\mathrm{d}s_i+\frac{1}{2}\int_0^{t-\varepsilon^2 s}\frac{1}{s_i+\varepsilon^2 s}\mathrm{d}s_i\\
&\le \int_0^{s_{i+1}} \frac{1}{2s_i}\wedge \frac{2\pi\|V\|_\infty}{\varepsilon^2}\mathrm{d}s_i+\frac{1}{2}\int_0^{t-\varepsilon^2 s}\frac{1}{s_i+\varepsilon^2 s}\mathrm{d}s_i\\
&\le \int_0^{t} \frac{1}{2s_i}\wedge \frac{2\pi\|V\|_\infty}{\varepsilon^2}\mathrm{d}s_i+\frac{1}{2}\int_0^{t-\varepsilon^2 s}\frac{1}{s_i+\varepsilon^2 s}\mathrm{d}s_i\\
&=\frac{1}{2}\log\frac{4\pi\|V\|_\infty t}{\varepsilon^2}+\frac{1}{2}+\frac{1}{2}\log\frac{t}{\varepsilon^2 s}\\
&=\frac{1}{2}\log(4\pi\|V\|_\infty t)+\frac{1}{2}+\log\varepsilon^{-1}+a_\varepsilon\log\varepsilon^{-1}.
\end{aligned}$$

Thus (3.5.14) is bounded by

$$\begin{aligned}
&\left(\frac{\beta_\varepsilon^2}{2\pi}\right)^n\int_0^{t-\varepsilon^2 s}\frac{1}{2s_n+2\varepsilon^2 s}\left(\frac{1}{2}\log(4\pi\|V\|_\infty t)+\frac{1}{2}+\log\varepsilon^{-1}+a_\varepsilon\log\varepsilon^{-1}\right)^{n-1}\mathrm{d}s_n\\
&=\left(\frac{\beta_\varepsilon^2}{2\pi}\right)^n\left(\frac{1}{2}\log(4\pi\|V\|_\infty t)+\frac{1}{2}+\log\varepsilon^{-1}+a_\varepsilon\log\varepsilon^{-1}\right)^{n-1}a_\varepsilon\log\varepsilon^{-1}\\
&=\left(\frac{\beta^2}{2\pi}\right)^n\left(1+a_\varepsilon+\frac{C_0}{\log\varepsilon^{-1}}\right)^{n-1}a_\varepsilon,
\end{aligned}\tag{3.5.15}$$

with some constant $C_0\in\mathbb{R}$ depends only on V and t.

We can now bound (3.5.9) by (3.5.10)–(3.5.11) and (3.5.14)–(3.5.15). We have that

$$\begin{aligned}
&b_\varepsilon^{-2}C_\beta^{(3)}\int_{y\in\mathbb{R}^2}\int_{y'\in\mathbb{R}^2}\rho_{\varepsilon^2 s}(x,y)\rho_{\varepsilon^2 s}(x,y')\mathcal{C}^\varepsilon_{\varepsilon^{-1}y,\varepsilon^{-1}y'}\mathrm{d}y\mathrm{d}y'\\
&\le b_\varepsilon^{-2}C_\beta^{(3)}C_{h_0}a_\varepsilon\sum_{n=1}^\infty\left(\frac{\beta^2}{2\pi}\right)^n\left(1+a_\varepsilon+\frac{C_0}{\log\varepsilon^{-1}}\right)^{n-1}.
\end{aligned}$$

Since $a_\varepsilon\to 0$ and $\frac{C_0}{\log\varepsilon^{-1}}\to 0$ as $\varepsilon\to 0$, the above sum is bounded when ε is sufficiently small and $\beta<\sqrt{2\pi}$. In particular, there exist constants $C_\beta^{(4)}>0$ and $\varepsilon_0>0$, such that when $0<\varepsilon<\varepsilon_0$, the infinite sum equals to

$$A_\varepsilon := \sum_{n=1}^{\infty} \left(\frac{\beta^2}{2\pi}\right)^n \left(1 + a_\varepsilon + \frac{C_0}{\log \varepsilon^{-1}}\right)^{n-1} = \frac{\frac{\beta^2}{2\pi}}{1 - (\frac{\beta^2}{2\pi})(1 + a_\varepsilon + \frac{C_0}{\log \varepsilon^{-1}})} \le C_\beta^{(4)}.$$

Now by (3.5.8) and (3.5.9), we have

$$\mathbb{E}\left[\mathbf{Var}^{\mathscr{F}_s}\left\{\frac{\mathrm{E}_{\varepsilon^{-1}x}\left[\Psi_{\varepsilon^{-2}t}^{\beta_\varepsilon,h_0}\right]}{\mathrm{E}_{\varepsilon^{-1}x}\left[\Psi_s^{\beta_\varepsilon,0}\right]}\right\}; \mathrm{E}_{\varepsilon^{-1}x}\left[\Psi_s^{\beta_\varepsilon,0}\right] > b_\varepsilon\right] \le C_{h_0} C_\beta^{(3)} C_\beta^{(4)} a_\varepsilon b_\varepsilon^{-2}.$$

Together with (3.5.7), we conclude that when $\varepsilon < \varepsilon_0$,

$$\begin{aligned}
&\mathbb{E}\left[\mathbf{Var}^{\mathscr{F}_s}\left\{\frac{\mathrm{E}_{\varepsilon^{-1}x}\left[\Psi_{\varepsilon^{-2}t}^{\beta_\varepsilon,h_0}\right]}{\mathrm{E}_{\varepsilon^{-1}x}\left[\Psi_s^{\beta_\varepsilon,0}\right]}\right\}\right] \\
&= \mathbb{E}\left[\mathbf{Var}^{\mathscr{F}_s}\left\{\frac{\mathrm{E}_{\varepsilon^{-1}x}\left[\Psi_{\varepsilon^{-2}t}^{\beta_\varepsilon,h_0}\right]}{\mathrm{E}_{\varepsilon^{-1}x}\left[\Psi_s^{\beta_\varepsilon,0}\right]}\right\}; \mathrm{E}_{\varepsilon^{-1}x}\left[\Psi_s^{\beta_\varepsilon,0}\right] \le b_\varepsilon\right] \\
&+ \mathbb{E}\left[\mathbf{Var}^{\mathscr{F}_s}\left\{\frac{\mathrm{E}_{\varepsilon^{-1}x}\left[\Psi_{\varepsilon^{-2}t}^{\beta_\varepsilon,h_0}\right]}{\mathrm{E}_{\varepsilon^{-1}x}\left[\Psi_s^{\beta_\varepsilon,0}\right]}\right\}; \mathrm{E}_{\varepsilon^{-1}x}\left[\Psi_s^{\beta_\varepsilon,0}\right] > b_\varepsilon\right] \\
&\le C_\beta^{(1)} C_\beta^{(2)} b_\varepsilon + C_{h_0} C_\beta^{(3)} C_\beta^{(4)} a_\varepsilon b_\varepsilon^{-2}.
\end{aligned}$$

By choosing $b_\varepsilon = a_\varepsilon^{1/3}$, Lemma 3.3.5 is proved.

Chapter 4: Half-space KPZ at stationarity

4.1 Introduction

4.1.1 Main results

The Kardar-Parisi-Zhang (KPZ) equation in half-space with a Neumann boundary condition at the origin models stochastic interface growth in contact with a boundary [22, 66]. Indexed by the boundary condition parameter $u \in \mathbb{R}$, this equation is the stochastic PDE:

$$\begin{aligned} &\partial_t H_u(t,x) = \frac{1}{2}\partial_x^2 H_u(t,x) + \frac{1}{2}\left(\partial_x H_u(t,x)\right)^2 + \xi(t,x), \quad t \geq 0, x \geq 0, \\ &\partial_x H_u(t,x)\big|_{x=0} = u, \end{aligned} \tag{4.1.1}$$

where ξ is a (1+1) space-time white noise on $(t,x) \in \mathbb{R}^2$. Throughout this section, we assume that the initial condition to the half-space KPZ equation (4.1.1) is

$$H_u(0,x) = W_u(x) := W(x) + ux, \tag{4.1.2}$$

with $W(\cdot)$ being a standard Brownian motion starting from 0 and independent of ξ. We consider the solution to (4.1.1) by the Hopf-Cole transform

$$H_u(t,x) := \log Z_{u-\frac{1}{2}}(t,x),$$

where $Z_\mu(t,x)$ solves the half-space stochastic heat equation (SHE) with a Robin boundary condition for $\mu \in \mathbb{R}$:

$$\begin{aligned} \partial_t Z_\mu(t,x) &= \frac{1}{2}\partial_x^2 Z_\mu(t,x) + Z_\mu(t,x)\xi(t,x), \quad t \geq 0, x \geq 0, \\ \partial_x Z_\mu(t,x)\big|_{x=0} &= \mu Z_\mu(t,0), \end{aligned} \tag{4.1.3}$$

with initial condition given by $Z_\mu(0,x) = \exp(H_{\mu+\frac{1}{2}}(0,x))$.

The boundary conditions in (4.1.1) or (4.1.3) are not immediately meaningful, as the solutions to the KPZ equations or the stochastic heat equations are not differentiable. We defer precise definitions to Section 4.2. We also note that the parametrization here follows the convention as in [26]. Under this convention, $H_u(t,x)$ is the logarithm of a half-space SHE with Robin boundary parameter $u - \frac{1}{2}$ and starting at initial condition $Z_{u-\frac{1}{2}}(0,x) = \exp(W(x) + ux)$. As it will become clear later, the shift of $-\frac{1}{2}$ introduces a certain symmetry over $u = 0$.

It was shown in [26] that, with the initial data in (4.1.2), the evolution of the height function H_u is at stationarity, namely, for any $t > 0$,

$$H_u(t,x) - H_u(t,0) \overset{\text{law}}{=} H_u(0,x) - H_u(0,0) = W(x) + ux, \text{ as a process in } x \in [0,\infty). \tag{4.1.4}$$

The goal of this section is to study the statistics of the height function in this stationary setting, including the mean and the variance, for different choices of the boundary conditions.

Let $(\Omega, \mathscr{F}, \mathbf{P})$ be a probability space that supports all possible randomness in the environment and the initial data. We use $\mathbf{E}, \mathbf{Var}, \mathbf{Cov}$ to denote the expectation, variance and covariance, respectively, under this probability space. Given two functions $a(t)$ and $b(t)$, we use notation $a(t) \sim b(t)$ when $\lim_{t\to\infty} a(t)/b(t) = 1$, and use notation $a(t) \asymp b(t)$ when $C^{-1}b(t) \le a(t) \le Cb(t)$ for some constants $C > 0$ independent of $t \ge 1$.

Associated with equations (4.1.1) and (4.1.3), one can construct a half-space continuum directed random polymer (half-space CDRP) measure $\mathbb{Q}_0^{u,t}$. We defer the precise definitions of $\mathbb{Q}_0^{u,t}$ to Definition 4.2.9. One can think of it as a random Gibbs measure on the path space $\mathcal{C}([0,t],[0,\infty))$ with Borel σ-algebra. The paths starts from $(t,0)$ and runs backwards in time, and it is restricted to the nonnegative half-space, attracted or repulsed by the boundary 0. Along the trajectory, it collects energy $\int_0^t \xi(t-s, X_s)ds$, and the terminal tilt is given by $\exp(W_u(\cdot))$ at time t. In particular, the endpoint of the polymer path induces a measure $\mathbb{Q}_0^{u,t}(X_t \in \mathrm{d}x)$ on the half-space $[0,\infty)$ equipped with Borel σ-algebra $\mathcal{B}([0,\infty))$. Throughout this section, we use $\mathbb{E}_0^{\mathbb{Q}^{u,t}}$ as the quenched expectation over this polymer measure.

To describe the result, we will use the digamma function ψ

$$\psi(z) = \partial_z \log\Gamma(z) = -\gamma_0 + \sum_{n=0}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n+z}\right), \qquad z \in (0,\infty), \tag{4.1.5}$$

with γ_0 being the Euler–Mascheroni constant. Its derivative ψ' (the trigamma function) is a strictly decreasing function on $(0,\infty)$

$$\psi'(z) = \sum_{n=0}^{\infty} \frac{1}{(n+z)^2}. \tag{4.1.6}$$

In particular, the leading behavior as $z \to 0^+$ is

$$\psi'(z) \sim \frac{1}{z^2}. \tag{4.1.7}$$

Our first result below establishes a relation between the variance of $H_u(t,0)$ and the annealed mean of the endpoint displacement of the polymer paths under $\mathbb{Q}_0^{u,t}$. This identity holds true for arbitrary $u \in \mathbb{R}, t \geq 0$.

Theorem 4.1.1. *For any* $u \in \mathbb{R}, t \geq 0$*,*

$$\mathbf{Var}[H_u(t,0)] = 2\mathbf{E}\mathbb{E}_0^{\mathbb{Q}^{u,t}}[X_t] - ut, \tag{4.1.8}$$

Next, we provide an estimate on $\mathbf{E}\mathbb{E}_0^{\mathbb{Q}^{u,t}}[X_t]$ under the contraint of $u < 0$. The bound in turn leads to the bounds on the fluctuations of the height function.

Theorem 4.1.2. *Assume* $u < 0$*. For any* $t \geq 0$*, we have*

$$0 \leq \mathbf{E}\mathbb{E}_0^{\mathbb{Q}^{u,t}}[X_t] \leq \psi'(2|u|). \tag{4.1.9}$$

As a result,

$$|u|t \leq \mathbf{Var}[H_u(t,0)] \leq 2\psi'(2|u|) + |u|t. \tag{4.1.10}$$

In particular, for any constants $\alpha \in [0, \frac{1}{3})$ *and* $c < 0$*, with* $u = ct^{-\alpha}$*, we have*

$$\mathbf{Var}[H_{ct^{-\alpha}}(t,0)] \sim |c|t^{1-\alpha}, \quad \textit{as } t \to \infty. \tag{4.1.11}$$

For $\alpha = \frac{1}{3}$ *and* $c < 0$*, with* $u = ct^{-1/3}$*, we have*

$$\mathbf{Var}[H_{ct^{-1/3}}(t,0)] \asymp t^{\frac{2}{3}}. \tag{4.1.12}$$

As a consequence, we obtain the tightness for the annealed polymer endpoint measure.

Corollary 4.1.3. *For any constants* $\alpha \in [0, \frac{1}{3}]$ *and* $c < 0$*, with* $u = ct^{-\alpha}$*, if we denote the annealed law of the scaled endpoint displacement* $\frac{X_t}{t^{2\alpha}}$ *on* $[0,\infty)$ *by* $\hat{\mathbf{P}}^{\mathbb{Q}}_{u,t}$*, (i.e. for any* $t > 0$ *and Borel subset* $A \subset [0,\infty)$*, we let* $\hat{\mathbf{P}}^{\mathbb{Q}}_{u,t}(\frac{X_t}{t^{2\alpha}} \in A) = \mathbf{E}\mathbb{E}^{\mathbb{Q}^{u,t}}_0[\mathbb{1}_A(X_t/t^{2\alpha})]$*,) then the sequence of measures* $\{\hat{\mathbf{P}}^{\mathbb{Q}}_{u,t}\}_{t>0}$ *is tight.*

Independent of the above results, we also provide a formula for the mean of $H_u(t,x)$ when $u \le 0$.

Theorem 4.1.4. *For any* $u \le 0$*,* $t \ge 0$*, and* $x \ge 0$*,*

$$\mathbf{E}[H_u(t,x)] = \left(\frac{1}{2}u^2 - \frac{1}{24}\right)t + ux. \tag{4.1.13}$$

The next results concern the cases when $u > 0$. For this part, we rely on the following symmetry result inspired by [30].

Proposition 4.1.5. *For any* $u > 0$ *and* $t \ge 0$*, we have*

$$H_u(t,0) \overset{law}{=} H_{-u}(t,0).$$

Consequently, (4.1.10)–(4.1.12) *and Theorem 4.1.4 remain valid when* $u > 0$ *(or correspondingly* $c > 0$*).*

Combining Theorem 4.1.1 and (4.1.10), we can also obtain the order of the polymer endpoint displacement when $u > 0$.

Corollary 4.1.6. *For any* $u > 0, t \geq 0$*, we have*

$$ut \leq \mathbf{E}\mathbb{E}_0^{\mathbb{Q}^{u,t}}[X_t] \leq ut + \psi'(2u). \tag{4.1.14}$$

In particular, for any $\alpha \in [0, \frac{1}{3}]$ *and* $c > 0$*, with* $u = ct^{-\alpha}$*, if we denote the annealed law of the scaled endpoint displacement* $\frac{X_t}{t^{1-\alpha}}$ *on* $[0, \infty)$ *by* $\tilde{\mathbf{P}}^{\mathbb{Q}}_{u,t}$*, (i.e. for any* $t > 0$ *and Borel subset* $A \subset [0, \infty)$*, we let* $\tilde{\mathbf{P}}^{\mathbb{Q}}_{u,t}(\frac{X_t}{t^{1-\alpha}} \in A) = \mathbf{E}\mathbb{E}_0^{\mathbb{Q}^{u,t}}[\mathbb{1}_A(X_t/t^{1-\alpha})]$*,) then the sequence of measures* $\{\tilde{\mathbf{P}}^{\mathbb{Q}}_{u,t}\}_{t>0}$ *is tight.*

We further prove the following upper bounds for the polymer endpoint displacement, as well as the fluctuation of the height function, when $u = ct^{-\alpha}$ for any $c \in \mathbb{R}$ and $\alpha > 1/3$.

Theorem 4.1.7. *Assume that* $u = ct^{-\alpha}$ *with some* $\alpha > 1/3$ *and* $c \in \mathbb{R}$*. For any* $t \geq 1$*,*

$$0 \leq \mathbf{E}\mathbb{E}_0^{\mathbb{Q}^{u,t}}[X_t] \leq Ct^{2/3}, \tag{4.1.15}$$

for some $C > 0$ *depending only on* c*.*

As a result, for any $t \geq 1$*, we also have*

$$\mathbf{Var}[H_u(t, 0)] \leq Ct^{2/3}, \tag{4.1.16}$$

for some $C > 0$ *depending only on* c*.*

The tightness of the annealed polymer endpoint measure in the supercritical regime is the following.

Corollary 4.1.8. *For any $\alpha > 1/3$ and $c \in \mathbb{R}$, with $u = ct^{-\alpha}$, if we denote the annealed law of the scaled endpoint displacement $\frac{X_t}{t^{2/3}}$ on $[0, \infty)$ by $\bar{\mathbf{P}}^{\mathbb{Q}}_{u,t}$, (i.e. for any $t > 0$ and Borel subset $A \subset [0, \infty)$, $\bar{\mathbf{P}}^{\mathbb{Q}}_{u,t}\left(\frac{X_t}{t^{2/3}} \in A\right) = \mathbf{E}\mathbb{E}_0^{\mathbb{Q}^{u,t}}[\mathbb{1}_A(X_t/t^{2/3})]$,) then the sequence of measures $\{\bar{\mathbf{P}}^{\mathbb{Q}}_{u,t}\}_{t\geq 1}$ is tight.*

4.1.2 Context

The exploration of a growing interface near an attractive wall arises naturally from the studies of wetting and entropic repulsion phenomena. In his pioneering work [22], Kardar introduced the half-space KPZ models to describe such interface growth and he predicted a "depinning" phase transition as the wall attraction weakens. The underlying concept is as follows: when the wall exerts a strong attractive force, which is also referred to as in the subcritical regime or the bound phase, Kardar conjectured that the half-space directed random polymer is "pinned" to the wall due to the force; as this attractive force decreases below a certain threshold, the polymer becomes entropically repelled away from the wall. It is further expected that when "pinned", the polymer endpoint has $O(1)$ transversal fluctuation in a window around the wall and the free energy of the polymer fluctuates with an exponent $1/2$. After being "unpinned", the polymer exhibits KPZ behaviors: the free energy fluctuates with exponent $1/3$ and the endpoints have transversal fluctuation exponent $2/3$.

The mathematical studies on the KPZ models in half-space started with a series of works on half-space last passage percolation (LPP) models [67–70]. For studies on the half-space LPP models starting from stationary initial conditions, we refer to [71–73]. For models with positive temperatures, significant breakthroughs were only made in recent studies.

The phase transition predicted in [22] in a positive temperature model was proved through studying the free energy statistics. In [74], the authors used a combinatorial identity to connect the free energy of the point-to-line half-space log-gamma (HSLG) polymer and a point-to-point full-space log-gamma polymer model. For the point-to-point HSLG polymer free energy, the phase transition was demonstrated by [28] using a novel correspondence between the HSLG polymer model and free boundary Schur measures. They also established the asymptotic behavior and the corresponding phase transition for the height function of the half-space KPZ equation at the boundary, starting from the droplet initial condition, by taking the weak noise scaling limit of the point-to-point HSLG polymer free energy [26, 75].

Beyond free energy statistics, progress has also been made in studying the "depinning" phase transition through the transversal fluctuations of polymer measures. In two consecutive works— [27] for the unbound phase and [76] for the bound phase—the authors derived this transition for the transversal fluctuations in the HSLG polymer model. Their arguments rely on combinatorial identities and the solvability of the HSLG model, utilizing inputs from the HSLG line ensemble [27]. Additionally, as applications of the HSLG line ensemble, [76] and [77] demonstrated convergence to stationary measures for the point-to-point HSLG polymer in the bound and unbound phases, respectively.

Other related works are [78–81]. In [78], the author examines discrete point-to-point half-space directed polymer models where the attractive force from the wall is strong enough to increase the free energy macroscopically. [78] demonstrated that this condition is sufficient to pin the polymer to the wall (resulting in order-one transversal polymer fluctuations) and for the free energy to exhibit diffusive fluctuations and follow an asymptotically Gaussian distribution (with a fluctuation exponent of 1/2). [79, 80] study the half-space ASEP and six vertex model and obtain the corresponding phase transitions. In [81], the author solves the half-space TASEP with a general deterministic initial condition and obtain the transition probability for the half-space KPZ fixed point.

Our work investigates the half-space KPZ equation (4.1.1) starting from the stationary Brownian initial condition (4.1.2), which differs from the setups discussed above. Through a (formal) Feynman-Kac representation, the results are interpreted in terms of the point-to-stationary-measure half-space continuum directed random polymer (CDRP). Previous studies in the physics literature on the half-space KPZ equation with Brownian initial conditions include [25,30]. Other related works include [24, 82, 83]. We discuss our approaches and the main contributions in the following sections.

4.1.2.1 Variance identity

In Theorem 4.1.1, we establish a relation between the variance of the height function and the expectation of the polymer endpoint displacement, which serves as the foundation of our analysis. Similar identities have previously appeared in several full-space models at stationarity, such as log-gamma polymer [84], geometric LPP [85], the O'Connell-Yor polymer [86], the asymmetric simple exclusion process [87], interacting diffusion [88], the KPZ fixed point [89], and the KPZ equation [29], among others. Our derivation of the variance identity is based on Gaussian integration by parts, inspired by [29, 89]. Such variance identities play a pivotal role in establishing fluctuation exponents: given a family of stationary measures parameterized by a certain variable, one can proceed with a coupling argument involving a small perturbation of this parameter, and by comparing the two processes corresponding to different parameters and leveraging the quadratic form of the average free energy, it is possible to derive the correct fluctuation exponent. This approach has been developed and extensively explored by Seppäläinen and collaborators, leading to breakthroughs [87, 90] and numerous further applications.

With the variance identity (4.1.8) and the quadratic form of the averaging free energy given in (4.1.13), it seems plausible to apply the coupling argument for the half-space KPZ equation and prove the fluctuation exponents for all values of $u \in \mathbb{R}$. However, an immediate challenge arises: the family of *Brownian* stationary measures is parameterized by the Neumann boundary condition of the equation, meaning that perturbing the initial data while maintaining stationarity necessitates perturbing the equation itself. This creates extra technical difficulties for the half-space model compared to the full-space setting, and we use different comparison arguments to cover various ranges of $u \in \mathbb{R}$.

4.1.2.2 Transversal fluctuation

With the variance identity, the analysis of height function fluctuations reduces to studying the transversal fluctuations of the polymer endpoint. For the point-to-stationary-measure half-space continuum directed random polymer (CDRP) $\mathbb{Q}_0^{u,t}$, our result (4.1.9) provides the conjectured optimal upper bound for the endpoint displacement in the bound phase $u < 0$. Specifically, for a polymer path starting at $(t, 0)$ and running backward in time, when t is sufficiently large, the endpoint reaches equilibrium in the subcritical regime. The term $\psi'(2|u|)$ on the right-hand side of (4.1.9) represents the endpoint position $\mathbf{E}\mathbb{E}_0^{\mathbb{Q}^{u,t}}[X_t]$ for $t \gg 1$. This can also be interpreted as the localization length of the model, which was conjectured by [22] to diverge quadratically and is confirmed in our analysis (see (4.1.7)). In essence, the result follows from the explicit stationary measure for the half-space KPZ equation constructed in [26], the Dufresne identity [91], and a stochastic dominance argument, where we compare the polymer path starting at the origin with the one starting from equilibrium.

The explicit calculation for the stationary endpoint position yields $\mathbf{E}\mathbb{E}_0^{\mathbb{Q}^{u,t}}[X_t] \approx \psi'(2|u|) \sim |u|^{-2}$ in the bound phase, which allows us to extend the analysis beyond the subcritical regime and cover the critical and supercritical regime. Using the symmetry result in Proposition 4.1.5, as well as another comparison argument, we obtain the conjectured optimal upper bounds $t^{2/3}$ for the transversal fluctuations in the critical and supercritical regime $u \sim t^{-\alpha}$ with $\alpha \geq 1/3$, in Corollary 4.1.3, Corollary 4.1.6, and Theorem 4.1.7.

4.1.2.3 Height function statistics

The variance estimates of the height function follow directly from the variance identity and the bounds obtained for the polymer endpoint.

In [30], the authors conjectured that for any fixed $u \neq 0$, $H_u(t,0)$ would have a large time behavior as

$$H_u(t,0) \simeq \left(\frac{1}{2}u^2 - \frac{1}{24}\right)t + t^{1/2}\chi,$$

where χ is an $O(1)$ Gaussian random variable. For u in a scaling window around 0, the authors later stated in [25] that for the entire critical regime $u = ct^{-1/3}$ with $c \in \mathbb{R}$, one should expect

$$\lim_{t\to\infty} \mathbf{P}\left(\frac{H_{ct^{-1/3}}(0,t) + \frac{t}{24}}{t^{1/3}} \leqslant s\right) = F_c(s),$$

where $F_c(\cdot)$ admits an explicit expression computed through the Fredholm determinant.

Our results confirm the predictions regarding the mean and fluctuation exponent of the height function $H_u(t,0)$ in both the subcritical and critical regimes. Theorem 4.1.2 further illustrates how the boundary parameter u influences the fluctuation scale for any $u \neq 0$ at any finite time $t \geq 0$, which was not immediately evident in earlier predictions. As expected, the method we employ does not yield the exact asymptotic fluctuation distributions.

For the supercritical regime $u \sim t^{-\alpha}$ with $\alpha > 1/3$, (including $u = 0$,) the variance identity and the optimal upper bound for the endpoint displacement in (4.1.15) derives an optimal upper bound with order $t^{2/3}$ for the variance of the height function in (4.1.16). Our method does not lead to the lower bound for the variance of the height function in the supercritical regime, which is conjectured to be of order $t^{2/3}$ as well.

Remark 4.1.9*.* The upper bounds in (4.1.9) and (4.1.10) are expected to be sharp as $t \to \infty$ for any fixed $u < 0$. This expectation stems from a conjecture, yet to be proven, that when $u < 0$, the half-space KPZ equation starting from the droplet initial condition would converge weakly (modulo height shift) to the stationary measure – a Brownian motion with drift u as $t \to \infty$. The upper bound in (4.1.9) corresponds to the annealed mean of endpoint displacement under this conjectured limit. For further discussions of this conjecture, see [26, 92].

However, as shown by (4.1.15) and (4.1.16), the upper bounds in (4.1.9) and (4.1.10) are no longer sharp in the supercritical regime.

4.1.2.4 Organization of this section

In Section 4.2, we introduce some preliminary results and auxiliary lemmas for the half-space SHE. As most results are analogous to full-space SHE, we defer their proofs to Appendix 4.8.1. In Section 4.3, we compute the average growth rate and prove Theorem 4.1.4. In Section 4.4, we establish the variance identity in Theorem 4.1.1. Section 4.5 contains the proof of the upper bound for the annealed mean of polymer endpoint displacement, leading to Theorem 4.1.2 and Corollary 4.1.3. In Section 4.6, we provide the proofs of Proposition 4.1.5 and Corollary 4.1.6. Finally, in Section 4.7, we prove Theorem 4.1.7 and Corollary 4.1.8.

4.1.3 Notations

We use the following notations and conventions throughout this section.

(i) For two topological spaces E and F, we use $\mathcal{C}(E,F)$ to denote the space of continuous functions from E to F. We use $\mathcal{C}^\infty$ to denote smooth functions, $\mathcal{C}_b$ to denote the space of continuous and bounded functions, and $\mathcal{C}_c$ to denote the space of continuous and compactly supported functions.

(ii) We use $\mathbf{E}_\eta$ as the expectation on any specific noise η defined on the probability space $(\Omega, \mathscr{F}, \mathbf{P})$. The notation $\mathbf{E}$ is for the total expectation on $(\Omega, \mathscr{F}, \mathbf{P})$. The expectation on Brownian motions independent of $(\Omega, \mathscr{F}, \mathbf{P})$ will be denoted by $\mathbb{E}_B$. The expectation on the quenched polymer measures (for each fixed $\omega \in \Omega$) is denoted by $\mathbb{E}^{\mathbb{Q}_{\cdot,\cdot}}_{\cdot}$. We use $\|\cdot\|_p$ to denote the norm of $L^p(\Omega, \mathscr{F}, \mathbf{P})$ for any $p > 0$.

(iii) We use $C(\cdot,\cdot,...,\cdot)$ to denote any constant C that depends only on the parameters inside the parentheses. These constants may vary from line to line.

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4.2 Preliminaries

In this section, we review some basics of the half-space SHE with Robin boundary conditions, including definition, and moment estimates, among others. We also introduce the half-space CDRP. Most of the results presented here are fairly standard, so readers already familiar with the topic may choose to skip this section.

4.2.1 Robin heat kernel

For $\mu, s, t \in \mathbb{R}, t > s$ and $x, y \geq 0$, the Robin heat kernel on half-line with parameter μ, denoted by $\mathscr{P}_\mu^R(t,x|s,y)$, is the unique solution to the deterministic heat equation,

$$\begin{aligned}
&\partial_t \mathscr{P}_\mu^R(t,x|s,y) = \frac{1}{2}\partial_x^2 \mathscr{P}_\mu^R(t,x|s,y) \quad \text{for } (t,x) \in (s,\infty)\times[0,\infty),\\
&\partial_x \mathscr{P}_\mu^R(t,x|s,y)\big|_{x=0} = \mu \mathscr{P}_\mu^R(t,0|s,y),\\
&\lim_{t\to s} \mathscr{P}_\mu^R(t,x|s,y) = \delta_y(x) \quad \text{in a weak sense on } L^2([0,\infty)).
\end{aligned} \tag{4.2.1}$$

An explicit expression of $\mathscr{P}_\mu^R(t,x|s,y)$ for any $t > s$ is given by

$$\mathscr{P}_\mu^R(t,x|s,y) = p_{t-s}(x-y) + p_{t-s}(x+y) - 2\mu \int_0^\infty p_{t-s}(x+y+z)e^{-\mu z}\mathrm{d}z, \tag{4.2.2}$$

where $p_t(x) = \frac{1}{\sqrt{2\pi t}}e^{-x^2/(2t)}$ is the standard heat kernel. A proof can be found in [94, Section 4].

The heat kernel $\mathscr{P}_\mu^R(t,x|s,y)$ satisfies the semigroup property and is monotonically decreasing in μ (see (4.2.6) below). To obtain an upper bound on $\mathscr{P}_\mu^R$, first, when $x, y, z \geq 0$, $(x-y)^2 \leq (x+y)^2$, thus $p_{t-s}(x+y) \leq p_{t-s}(x-y)$. Also, since $(x+y)^2 + z^2 \leq (x+y+z)^2$, we have

$$\int_0^\infty p_{t-s}(x+y+z)e^{-\mu z}\mathrm{d}z \leq p_{t-s}(x+y)\int_0^\infty e^{-\frac{z^2}{2(t-s)}-\mu z}\mathrm{d}z.$$

It follows that for any $a, b, \mu_0 \in \mathbb{R}$, there exists a constant $C(a,b,\mu_0)$ such that for any $\mu \geq \mu_0, a \leq s < t \leq b$ and $x, y \in [0,\infty)$, we have

$$\mathscr{P}_\mu^R(t,x|s,y) \leq C(a,b,\mu_0)p_{t-s}(x-y) \leq C(a,b,\mu_0)(t-s)^{-1/2}. \tag{4.2.3}$$

Another useful result is the probabilistic representation of the Robin heat kernel, as it is related to the Feynman-Kac formula of the half-space SHE. We present the representation below, and one can find a proof via Itô-Tanaka formula in [95, Section 2.5].

Let B_t be a Brownian motion starting at $B_0 = x \in [0, \infty)$. The process $|B_t|$, defined by the absolute value of B_t, is known as a reflected Brownian motion (RBM). We use notations $\mathbb{P}_B^x$ and $\mathbb{E}_B^x$ to denote the probability and expectation with respect to B_t only, emphasizing the initial point x. For any $\mu \in \mathbb{R}$, $f \in \mathcal{C}_b(\mathbb{R}, \mathbb{R})$,

$$\int_0^\infty \mathscr{P}_\mu^R(t, x|s, y) f(y) \mathrm{d}y = \mathbb{E}_B^x[f(|B_{t-s}|) \exp(-\mu L_{t-s}^0)], \tag{4.2.4}$$

where

$$L_t^0 := \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t \mathbb{1}_{[-\varepsilon, \varepsilon]}(B_s) \mathrm{d}\langle B \rangle_s = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \text{Leb}(s : |B_s| \le \varepsilon, 0 \le s \le t)$$

is the Brownian local time of B_t at the origin with Leb representing the Lebesgue measure.

(To be more precise, [95, Section 2.5] says that

$$\int_0^\infty \mathscr{P}_\mu^R(t, x|s, y) f(y) \mathrm{d}y = \mathbb{E}_\Upsilon^x[f(\Upsilon_{t-s}) \exp(-\mu L_{t-s}^{0,\Upsilon})], \tag{4.2.5}$$

where $\Upsilon_\cdot$ is the reflected Brownian motion $\Upsilon_t := |B_t|$, $\mathbb{P}_\Upsilon^x$ and $\mathbb{E}_\Upsilon^x$ are the probability and expectation with respect to Υ, and $L_t^{0,\Upsilon}$ is the local time of Υ at zero. It is further known by [96] that

$$L_t^{0,\Upsilon} = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t \mathbb{1}_{[0,\varepsilon]}(\Upsilon_s) \, \mathrm{d}s.$$

In our case, (4.2.5) is equivalent to (4.2.4). We will use both of these two representations.)

Consequently, for any $x, y \in [0, \infty)$,

$$\mathscr{P}^R_\mu(t, x|s, y) = \mathscr{P}^N(t, x|s, y)\mathbb{E}^x_B\left[\exp(-\mu L^0_{t-s}) \mid |B_{t-s}| = y\right] \tag{4.2.6}$$

where $\mathscr{P}^N(t, x|s, y) := \mathscr{P}^R_0(t, x|s, y)$ is the transition density of the reflected Brownian motion $|B_t|$.

Using (4.2.6), one can check that for any $\mu \in \mathbb{R}$, there exist two positive constants $C_1(\mu, t - s), C_2(\mu, t - s)$ such that

$$C_1(\mu, t-s)\mathscr{P}^N(t, x|s, y) \le \mathscr{P}^R_\mu(t, x|s, y) \le C_2(\mu, t-s)\mathscr{P}^N(t, x|s, y). \tag{4.2.7}$$

The constants C_1, C_2 can be chosen uniformly for $(\mu, t-s)$ in any compact subsets of $\mathbb{R} \times (0, \infty)$.

4.2.2 Mild formulation and Feynman-Kac formula

For any $\mu \in \mathbb{R}$, we use $Z_\mu(t, x|s, \zeta)$ to denote the mild solution to the half-space SHE starting from an arbitrary time $s \in \mathbb{R}$ and from a broad class of positive Borel measure ζ:

$$\begin{aligned} \partial_t Z_\mu(t, x|s, \zeta) &= \frac{1}{2}\partial^2_x Z_\mu(t, x|s, \zeta) + Z_\mu(t, x|s, \zeta)\xi(t, x) \quad \text{for } (t, x) \in (s, \infty) \times [0, \infty), \\ \partial_x Z_\mu(t, x|s, \zeta)\big|_{x=0} &= \mu Z_\mu(t, 0|s, \zeta), \\ Z_\mu(s, \cdot|s, \zeta) &= \zeta(\cdot). \end{aligned} \tag{4.2.8}$$

We discuss this generalized half-space SHE because some estimates below are easier to obtain when the initial condition is deterministic and either localized (e.g., δ-type initial conditions) or spatially homogeneous (e.g., constant initial conditions). To avoid confusion, we use the notation $Z_\mu(t,x|s,\zeta)$ to specify the initial time and data. We only use notation $Z_\mu(t,x)$ for the solution of (4.1.3) when the initial time is $s = 0$ and the initial data is fixed as $\zeta(\mathrm{d}x) = \exp(W(x) + (\mu + \frac{1}{2})x)\mathrm{d}x$.

We first define the mild solution to (4.2.8) with an arbitrary starting time $s \in \mathbb{R}$ and a (potentially random) Borel measure initial condition. For any $s < t$, let $(\mathscr{F}_r^{\xi,\zeta})_{r\in[s,t]}$ be the natural filtration generated by ζ and ξ on the time interval $[s,t]$.

Definition 4.2.1. *Let $\mu, s \in \mathbb{R}$, ζ be some (potentially random) Borel measure that is independent of the white noise ξ and $\mathbf{P}$-almost surely supported on $[0,\infty)$. We say that a measurable process $Z_\mu(\cdot,\cdot|s,\zeta)$ on $(s,\infty)\times[0,\infty)$ is the mild solution to (4.2.8) with initial condition $\zeta(\cdot)$, if for all $t \in (s,\infty)$, $Z_\mu(t,\cdot|s,\zeta)$ is $(\mathscr{F}_r^{\xi,\zeta})_{r\in[s,t]}$-adapted and satisfies*

$$Z_\mu(t,x|s,\zeta) = \int_0^\infty \mathscr{P}_\mu^R(t,x|s,y)\zeta(\mathrm{d}y) + \int_s^t \int_0^\infty \mathscr{P}_\mu^R(t,x|r,y)Z_\mu(r,y|s,\zeta)\xi(\mathrm{d}y\mathrm{d}r), \tag{4.2.9}$$

for all $x \in [0,\infty)$, where the stochastic integral is interpreted in the Itô-Walsh sense.

The two hypotheses below regarding the initial condition ζ ensure the existence and uniqueness (under different assumptions) of the mild solution $Z_\mu(\cdot,\cdot|s,\zeta)$. All the initial conditions we deal with are included in either case. Throughout this section, we assume that ζ is not the zero measure and does not depend on s.

Hypothesis 1. *There exists a random variable f taking values in $\mathcal{C}([0,\infty),(0,\infty))$ and a constant $a > 0$ so that*

$$\zeta(\mathrm{d}x) = f(x)\mathrm{d}x \text{ and } \sup_{x\in[0,\infty)} e^{-ax}\mathbf{E}\left[f(x)^2\right] < \infty.$$

Hypothesis 2. *ζ is deterministic, does not depend on μ, and for any $s < \tau$,*

$$\sup_{s<t\le\tau}\ \sup_{x\in[0,\infty)} \sqrt{t-s}\int_0^\infty \mathscr{P}_\mu^R(t,x|s,y)\zeta(\mathrm{d}y) < \infty.$$

For any $\mu\in\mathbb{R}$, $\zeta(\mathrm{d}x) = Z_\mu(0,x)\mathrm{d}x = \exp(W(x)+(\mu+\frac{1}{2})x)\mathrm{d}x$ satisfies Hypothesis 1. The Dirac-δ initial condition $\zeta(\cdot) = \delta_y(\cdot) = \delta(\cdot - y)$ at any point $y\in[0,\infty)$ satisfy Hypothesis 2 (see (4.2.3)). The constant initial condition $\zeta(\mathrm{d}x) = \mathrm{d}x$ satisfies both. By (4.2.7), whenever ζ satisfies Hypothesis 2 for some $\mu\in\mathbb{R}$, it satisfies it for all $\mu\in\mathbb{R}$.

Proposition 4.2.2. *(1) When the initial data ζ satisfies Hypothesis 1, there exists a unique mild solution $Z_\mu(\cdot,\cdot|s,\zeta)\in\mathcal{C}([s,\infty)\times[0,\infty),\mathbb{R})$ as defined in Definition 4.2.1 that satisfies, for any terminal time $\tau > s$,*

$$\sup_{s\le t\le\tau}\ \sup_{x\in[0,\infty)} e^{-ax}\mathbf{E}\left[Z_\mu(t,x|s,\zeta)^2\right] < \infty. \tag{4.2.10}$$

(2) When the initial data ζ satisfies Hypothesis 2, there exists a unique mild solution $Z_\mu(\cdot,\cdot|s,\zeta)\in\mathcal{C}((s,\infty)\times[0,\infty),\mathbb{R})$ as defined in Definition 4.2.1 that satisfies, for any terminal time $\tau > s$,

$$\sup_{s<t\le\tau}\ \sup_{x\in[0,\infty)} \int_s^t\int_s^r\int_0^\infty\int_0^\infty \mathscr{P}_\mu^R(t,x|r,y)^2\mathscr{P}_\mu^R(r,y|w,z)^2\mathbf{E}\left[Z_\mu(w,z|s,\zeta)^2\right]\mathrm{d}z\mathrm{d}y\mathrm{d}w\mathrm{d}r < \infty. \tag{4.2.11}$$

The mild solution admits an explicit chaos series representation as

$$Z_\mu(t,x|s,\zeta) = \int_0^\infty \mathscr{P}_\mu^R(t,x|s,y)\zeta(\mathrm{d}y) + \sum_{k=1}^{\infty} \int_{[0,\infty)^{k+1}} \int_{\mathbb{R}^k} \zeta(\mathrm{d}y) \mathscr{P}_\mu^R(r_{1:k}, w_{1:k}|s,y;t,x)\, \xi(\mathrm{d}r_1\mathrm{d}w_1)\cdots\xi(\mathrm{d}r_k\mathrm{d}w_k), \tag{4.2.12}$$

where, with convention $r_0 = s, r_{k+1} = t, w_0 = y, w_{k+1} = x$, *we define*

$$\mathscr{P}_\mu^R(r_{1:k}, w_{1:k}|s,y;t,x) = \prod_{j=0}^{k} \mathscr{P}_\mu^R(r_{j+1}, w_{j+1}|r_j, w_j)\mathbb{1}_{(0,\infty)}(r_{j+1}-r_j).$$

Proof. [97, Proposition 4.2] has proved part (1). To prove part (2), one can follow the same argument as in [6], with the use of the iterative arguments as in the proof of Lemma 4.2.4 below. By iterative arguments, one can also show that the right-hand side of (4.2.12) is well-defined, $(\mathscr{F}_r^{\xi,\zeta})_{r\in[s,t]}$-adapted, and $\mathbf{P}$-almost surely satisfies (4.2.9) and (4.2.11). Then the uniqueness of the mild solution implies (4.2.12). □

Remark 4.2.3. A more general solution theory for the half-space SHE should be able to relex the assumptions on non-random initial conditions. Similar generalizations have been explored for the full-space SHE in [98, 99]. If such relaxations were made, we would only need our initial condition in (4.1.3) to meet the required hypotheses for $\mathbf{P}$-almost surely each realization of W. As the current hypotheses are sufficient for our purposes, we do not explore these extensions further.

We next give an estimate on positive moments for $Z_\mu(t,x|s,\zeta)$ with ζ under Hypothesis 2 and s,t in any arbitrary interval $[a,b] \subset \mathbb{R}$. Since the proof is through a rather standard iteration scheme, we defer it to Appendix 4.8.1.1.

Lemma 4.2.4. *Let $a < b, \mu_0 \in \mathbb{R}$ and $\zeta(\cdot)$ satisfy Hypothesis 2. For any $p \in [1,\infty)$, there exists a constant $C = C(a,b,\mu_0,p,\zeta)$ so that*

$$\|Z_\mu(t,x|s,\zeta)\|_p \le C(t-s)^{-1/4}\left(\int_0^\infty \mathscr{P}_\mu^R(t,x|s,y)\zeta(\mathrm{d}y)\right)^{1/2}, \tag{4.2.13}$$

for any $\mu \in [\mu_0,\infty)$, $a \le s < t \le b$ and $x \in [0,\infty)$.

It also helps to study the solution $Z_\mu(t,x|s,\zeta)$ through the Feynman-Kac formula. By the probabilistic representation of parabolic PDEs with Robin boundary condition (see e.g., [95, Section 2.5]), it is natural to expect that for any $\mu \in \mathbb{R}$ and deterministic $f \in \mathcal{C}_b([0,\infty),\mathbb{R})$, the half-space stochastic heat equation (4.2.8) started from $\zeta(\mathrm{d}x) = f(x)\mathrm{d}x$ (under Hypothesis 2) formally admits the Feynman–Kac type representation

$$\begin{aligned} Z_\mu(t,x|s,\zeta) &= \mathbb{E}_B^x\left[f(|B_{t-s}|)\exp(-\mu L_{t-s}^0) : \exp : \left\{\int_0^{t-s} \xi\,(t-r,|B_r|)\,dr\right\}\right] \\ &= \int_0^\infty \mathscr{P}^N(t,x|s,y)\mathbb{E}_B^x\left[\exp(-\mu L_{t-s}^0) : \exp : \left\{\int_0^{t-s} \xi\,(t-r,|B_r|)\,dr\right\}\middle||B_{t-s}| = y\right] f(y)\mathrm{d}y. \end{aligned} \tag{4.2.14}$$

Here $:\exp:$ denotes the Wick-ordered exponential and other notations are the same as in (4.2.4) and (4.2.6). This representation is only formal because the integral of the white noise over a Brownian path makes no sense. Meanwhile, analogous to the classical result [6] on full-space, we can rigorously approximate $Z_\mu(t,x|s,\zeta)$ by using the Feynman-Kac formulas with a smoothed noise. We defer this detailed approximation to Appendix 4.8.1.2.

A useful result that can be proved using the Feynman-Kac formula is the negative moments bound of the solutions.

Lemma 4.2.5 (Negative moments)**.** *Assume that the initial data ζ is either the constant $\zeta(\mathrm{d}x) =$*

$\mathrm{d}x$ *or the Dirac-δ initial data $\zeta = \delta_y$ for some $y \in [0,\infty)$. Let*

$$z_\mu(t,x|0,\zeta) := \int_0^\infty \mathscr{P}_\mu^R(t,x|0,y)\zeta(\mathrm{d}y).$$

Then for any $\mu \in \mathbb{R}, p \in [0,\infty), \tau > 0$, there exists a constant $C = C(\tau,\mu,p)$ such that for any $x \in [0,\infty)$, $t \in (0,\tau]$, and ζ being constant or Dirac-δ,

$$\mathbf{E}\left[Z_\mu(t,x|0,\zeta)^{-p}\right] \leq C z_\mu(t,x|0,\zeta)^{-p}. \tag{4.2.15}$$

The constant C can be chosen uniformly for μ in any compact subset of $\mathbb{R}$.

The bound of negative moments seems to be a well-established result, proven for SHEs in various contexts [100–103]. Since no proof has been written for the half-space setting, we prove Lemma 4.2.5 in Appendix 4.8.1.3.

4.2.3 Green's function

For any fixed $\mu, s \in \mathbb{R}$, when $\zeta(\cdot) = \delta_y(\cdot) = \delta(\cdot - y)$ for some $y \in [0,\infty)$, the unique mild solution $Z_\mu(\cdot,\cdot|s,\zeta)$ as described in Proposition 4.2.2 part (2) is known as the Green's function, or the fundamental solution, to the equation (4.2.8). This is a crucial object for the half-space SHE model which can be used to construct the half-space continuum directed random polymer (CDRP) measure.

In fact, using the chaos expansion (4.2.12), we can define a five-parameter random field in our common probability space $(\Omega, \mathscr{F}, \mathbf{P})$ that supports the space-time white noise ξ. Let

$$\mathcal{D} = \{(\mu, s, y, t, x) | \mu, s, t \in \mathbb{R} \text{ with } s < t \text{ and } x, y \in [0, \infty)\} \subset \mathbb{R}^5.$$

For $(\mu, s, y, t, x) \in \mathcal{D}$, define (with the same notation and convention as in (4.2.12))

$$\begin{aligned} \mathcal{Z}_\mu(t, x | s, y) &:= \mathscr{P}^R_\mu(t, x | s, y) \\ &\quad + \sum_{k=1}^{\infty} \int_{[0,\infty)^k} \int_{\mathbb{R}^k} \mathscr{P}^R_\mu(r_{1:k}, w_{1:k} | s, y; t, x)\, \xi(\mathrm{d}r_1 \mathrm{d}w_1) \cdots \xi(\mathrm{d}r_k \mathrm{d}w_k). \end{aligned} \tag{4.2.16}$$

It follows that, for any fixed $s, \mu \in \mathbb{R}$, $y \in [0, \infty)$, the field $\mathcal{Z}_\mu(\cdot, \cdot | s, y)$ is the unique mild solution to (4.2.8) satisfying (4.2.9) and (4.2.11) with initial conditon $\zeta = \delta_y$.

Corollary 4.2.6. *Let $a < b, \mu_0 \in \mathbb{R}$ and $p \in [1, \infty)$. For any $\mu \in [\mu_0, \infty)$, $a \leq s < t \leq b$, $x, y \in [0, \infty)$,*

$$\|\mathcal{Z}_\mu(t, x | s, y)\|_p \lesssim (t - s)^{-1/4} \mathscr{P}^R_\mu(t, x | s, y)^{1/2} \lesssim (t - s)^{-1/2} \exp(-(x - y)^2 / 4(t - s)), \tag{4.2.17}$$

where $\alpha \lesssim \beta$ denotes $\alpha \leq C\beta$ with some constant $C = C(a, b, \mu_0, p)$.

Proof. Substituting ζ with δ_y in Lemma 4.2.4 and using (4.2.3). Note that in the statement of Lemma 4.2.4, the prefactor C may depend on the initial data ζ, but one can follow the same proof and check that, with the Dirac-δ initial conditions, we can relax the bounds in the proof so that the constant C does not depend on $y \in [0, \infty)$. □

We record the following properties for the field $\mathcal{Z}$.

Proposition 4.2.7. *There exists a modification of the field $\mathcal{Z}_\mu(t, x | s, y)$ that is jointly continuous* *in all five variables* $(\mu, s, y, t, x) \in \mathcal{D}$*, and has the following properties:*

(i). For any $(\mu, s, y, t, x) \in \mathcal{D}$*,*

$$\mathbf{E}\left[\mathcal{Z}_\mu(t,x|s,y)\right] = \mathscr{P}_\mu^R(t,x|s,y). \tag{4.2.18}$$

(ii). For any fixed $s, \mu \in \mathbb{R}$ *and* $\zeta(\mathrm{d}x)$ *satisfying Hypothesis 1 (or Hypothesis 2), the random process defined by the convolution formula*

$$(t,x) \mapsto \int_0^\infty \mathcal{Z}_\mu(t,x|s,y)\zeta(\mathrm{d}y)$$

is the unique mild solution $Z_\mu(\cdot,\cdot|s,\zeta)$ *in Proposition 4.2.2 satisfying* (4.2.9) *and* (4.2.10) *(or* (4.2.11)*).*

(iii). (Positivity) For any $(\mu, s, y, t, x) \in \mathcal{D}$*, we have* $\mathcal{Z}_\mu(t,x|s,y) > 0$*.*

(iv). (Time stationarity) For any fixed $\mu, t_0 \in \mathbb{R}, x, y \in [0,\infty)$*,*

$$\{\mathcal{Z}_\mu(t+t_0,x|s+t_0,y)\}_{s,t\in\mathbb{R},s<t} \overset{law}{=} \{\mathcal{Z}_\mu(t,x|s,y)\}_{s,t\in\mathbb{R},s<t}.$$

(v). For any finite disjoint time intervals $\{(s_i,t_i]\}_{i=1}^n$ *and any* $x_i, y_i \in [0,\infty)$*, the random variables* $\{\mathcal{Z}_\mu(t_i,x_i|s_i,y_i)\}_{i=1}^n$ *are mutually independent.*

(vi). (Chapman-Kolmogorov identity) For any fixed $\mu \in \mathbb{R}$*, there exists an event* Ω_0 *with* $\mathbf{P}(\Omega_0) =$

1 *so that for any* $\omega \in \Omega_0$, $s, t \in \mathbb{R}$ *with* $s < t$, $x, y \in [0, \infty)$, *and* $r \in (s, t)$,

$$\mathcal{Z}_\mu(t, x|s, y) = \int_0^\infty \mathcal{Z}_\mu(t, x|r, w)\mathcal{Z}_\mu(r, w|s, y)\mathrm{d}w. \tag{4.2.19}$$

(vii). For any fixed $\mu, s, t \in \mathbb{R}, t > s$,

$$\{\mathcal{Z}_\mu(t, x|s, y)\}_{x,y\in[0,\infty)} \overset{law}{=} \{\mathcal{Z}_\mu(t, y|s, x)\}_{x,y\in[0,\infty)}. \tag{4.2.20}$$

Proof. The joint continuity of the field and properties (i)–(vi) closely mirror those of the Green's function field associated with the full-space stochastic heat equations in [104]. There are two primary differences from the full-space Green's function. First is that we have an additional parameter μ from the boundary condition in the half-space setting. Since the Robin heat kernel defined in (4.2.1) is continuous in μ, generalizing the joint continuity property to include this parameter μ by Kolmogorov continuity criteria is straightforward. The second difference is that the field no longer has stationarity in space, but this fact does not impact the properties above. Due to the presence of the boundary, for $x_0 \neq 0, x, y \geq -x_0$, we do not expect that $\mathcal{Z}_\mu(t, x|s, y)$ and $\mathcal{Z}_\mu(t, x + x_0|s, y + x_0)$ are equal in law.

To prove the Chapman-Kolmogorov identity (vi), we note that a detailed proof for the full-space Green's functions can be found in [93, Lemma 3.12]. One can adapt this proof to our half-space setting, except that we do not have a quenched level tail bound analogous to [93, Corollary 3.10]. Alternatively, one can use the Kolmogorov continuity criteria applied to the five variables (s, t, x, y, r) on the right-hand-side of (4.2.19) to conclude the proof.

The proof of property (vii) is similar to [105, Appendix B] and follows from an approximation of Feynman-Kac formulas. We defer it to Appendix 4.8.1.4. □

For any fixed $\mu \in \mathbb{R}$, we use $Z_\mu(\cdot,\cdot)$ to denote the unique mild solution to (4.1.3) in Proposition 4.2.2 part (1). By Proposition 4.2.7 (ii), when $\mu = u - \frac{1}{2}$, for any $t > 0$, $x \in [0,\infty)$,

$$Z_{u-\frac{1}{2}}(t,x) = \int_0^\infty \mathcal{Z}_{u-\frac{1}{2}}(t,x|0,y) \exp\left(W(y) + uy\right) \mathrm{d}y. \tag{4.2.21}$$

We finish this subsection by providing $L^p(\Omega)$-norm bounds to $Z_{u-\frac{1}{2}}(t,x)$ and $H_u(t,x)$. These bounds follow from Lemma 4.8.1 and Lemma 4.2.5 and will be very useful throughout this section. A detailed proof is given in Appendix 4.8.1.5.

Proposition 4.2.8. *For any fixed $u \in \mathbb{R}, t \geq 0, p \in [1,\infty)$, there exists a constant $C = C(u,t,p)$ so that the following holds for all $x \in [0,\infty)$:*

$$\|Z_{u-\frac{1}{2}}(t,x)\|_p \leq C\exp(Cx); \tag{4.2.22}$$

$$\|Z_{u-\frac{1}{2}}(t,x)^{-1}\|_p \leq C\exp(Cx). \tag{4.2.23}$$

It follows that for any fixed $u \in \mathbb{R}$, $\mathbf{P}$-almost surely,

$$0 < Z_{u-\frac{1}{2}}(t,x) < \infty, \quad \forall t \geq 0, x \geq 0. \tag{4.2.24}$$

The Hopf-Cole solution to (4.1.1) is thus well-defined:

$$H_u(t,x) = \log Z_{u-\frac{1}{2}}(t,x) = \log \int_0^\infty \mathcal{Z}_{u-\frac{1}{2}}(t,x|0,y)\exp(W(y)+uy)\mathrm{d}y, \tag{4.2.25}$$

and satisfies

$$\|H_u(t,x)\|_p \le C(u,t,p)\exp(C(u,t,p)x), \tag{4.2.26}$$

for any $p \in [1,\infty), t \ge 0, x \ge 0$. *All the constants above can be chosen uniformly over any compact subset of* $(u,t) \in \mathbb{R} \times [0,\infty)$.

4.2.4 Half-space CDRP

We can now use the Green's function $\mathcal{Z}_\mu(t,x|s,y)$ to construct the continuum directed random polymer (CDRP) measures in half-space. We then introduce the induced (quenched) endpoint measures on $([0,\infty),\mathcal{B}([0,\infty)))$ from these polymers.

For simplicity, we always set the initial time $s = 0$ and fix some $\mu \in \mathbb{R}$. To align with the previous conventions, we use the parameter $u = \mu + \frac{1}{2}$ to index these measures.

For any $t > 0$, we equip the path space $\mathcal{C}_{[0,t]} := \mathcal{C}([0,t],[0,\infty))$ with the Borel σ-algebra $\mathcal{B}(\mathcal{C}_{[0,t]})$. Let $X = (X_r)_{0\le r\le t}$ be the canonical variable on $\mathcal{C}_{[0,t]}$. A probability measure on this space is uniquely determined by its finite dimensional distributions. For our purposes, we are interested in the continuum directed polymer X on $\mathcal{C}([0,t],[0,\infty))$ that starts from a fixed point, runs backward in time in the environment of ξ, and has the terminal condition $\exp(W_u(\cdot))$ at $r = t$.

For each fixed $u \in \mathbb{R}, t > 0$ and $x \in [0,\infty)$, we define the following (quenched) point-to-measure polymer:

Definition 4.2.9. *For any* $\omega \in \Omega$ *such that* (4.2.24) *holds, define the (quenched, point-to-measure) half-space continuum directed random polymer (CDRP) measure* $\mathbb{Q}_x^{u,t}$ *as the probability measure* *on $\mathcal{C}_{[0,t]}$ with finite dimensional marginals given by*

$$\begin{aligned}&\mathbb{Q}_x^{u,t}(X_{t_1} \in \mathrm{d}x_1, \dots, X_{t_k} \in \mathrm{d}x_k)\\&\quad= \frac{\int_0^\infty \prod_{j=0}^k \mathcal{Z}_{u-\frac{1}{2}}(t-t_j, x_j | t-t_{j+1}, x_{j+1}) \exp(W_u(x_{k+1}))\mathrm{d}x_{k+1}}{\int_0^\infty \mathcal{Z}_{u-\frac{1}{2}}(t,x|0,y)\exp(W_u(y))\mathrm{d}y}\mathrm{d}x_1 \dots \mathrm{d}x_k,\end{aligned} \tag{4.2.27}$$

for $0 = t_0 < t_1 < \cdots < t_k < t_{k+1} = t$ *and with* $x_0 = x$*.*

We first address the well-definedness of the half-space CDRP $\mathbb{Q}_x^{u,t}$:

By the Chapman-Kolmogorov equation (4.2.19), the family of finite dimensional distributions in (4.2.27) is consistent. Thus by the Kolmogorov consistency theorem, for $\mathbf{P}$-almost every realization of (ξ, W), (4.2.27) gives a unique probability measure on $([0,\infty)^{[0,t]}, \mathcal{B}([0,\infty))^{[0,t]})$. It remains to show that the measure $\mathbb{Q}_x^{u,t}$ is $\mathbf{P}$-almost surely supported on paths in $\mathcal{C}_{[0,t]}$.

To prove this, we consider a (deterministic) probability measure $\mathbb{P}_{u,t,x}^{RB}$ that is defined on $([0,\infty)^{[0,t]}, \mathcal{B}([0,\infty))^{[0,t]})$ with finite-dimensional marginals

$$\begin{aligned}&\mathbb{P}_{u,t,x}^{RB}(X_{t_1} \in \mathrm{d}x_1, \dots, X_{t_k} \in \mathrm{d}x_k)\\&\quad= \frac{\mathbf{E}\left[\mathbb{Q}_x^{u,t}(X_{t_1} \in \mathrm{d}x_1, \dots, X_{t_k} \in \mathrm{d}x_k)\int_0^\infty \mathcal{Z}_{u-\frac{1}{2}}(t,x|0,y)\exp(W_u(y))\mathrm{d}y\right]}{\mathbf{E}[\int_0^\infty \mathcal{Z}_{u-\frac{1}{2}}(t,x|0,y)\exp(W_u(y))\mathrm{d}y]}.\end{aligned}$$

By (4.2.18), $\mathbb{P}_{u,t,x}^{RB}$ is a probability measure on reflected Brownian paths reweighted correspondingly by a local time term as well as the terminal boundary condition. (To see this, one can compute the expectation with respect to ξ and W explicitly, separately for the numerator and the denominator.) Let $Y : [0,\infty)^{[0,t]} \to \mathbb{R}$ be an indicator function such that $Y = 1$ if the paths are uniformly continuous when restricted to rational numbers and $Y = 0$ if not. By the regularity of the Brownian paths, $\mathbb{E}_{u,t,x}^{RB}[Y] = 1$, which suggests that $\mathbb{Q}_x^{u,t}(Y = 1) = 1$ $\mathbf{P}$-almost surely. It thus follows that the polymer path $X_\cdot$ has a continuous modification $\mathbf{P}$-almost surely.

We define $\mathbb{Q}_x^{u,0} := \delta_x$ as a measure on $([0,\infty), \mathcal{B}([0,\infty)))$. We use $\mathbb{E}_x^{\mathbb{Q}^{u,t}}$ to denote the expectation under $\mathbb{Q}_x^{u,t}$. We also define the following.

Definition 4.2.10. *Fix $u \in \mathbb{R}$ and $\omega \in \Omega$ such that (4.2.24) holds. For any $t > 0$, $x \in [0,\infty)$, we define the (quenched) density $\rho_u^R(\cdot|t,x)$ on $[0,\infty)$ as*

$$\rho_u^R(y|t,x) := \frac{\mathcal{Z}_{u-\frac{1}{2}}(t,x|0,y)\exp(W_u(y))}{\int_0^\infty \mathcal{Z}_{u-\frac{1}{2}}(t,x|0,y')\exp(W_u(y'))\mathrm{d}y'}, \quad \forall y \in [0,\infty). \tag{4.2.28}$$

It is clear that $\rho_u^R(\cdot|t,x)$ is well-defined for all $t > 0, x \in [0,\infty)$ simultaneously on a probability one event. It follows that for any fixed $t > 0, x \in [0,\infty)$, $\rho_u^R(\cdot|t,x)$ is the (quenched) density of the polymer endpoint on $[0,\infty)$ under $\mathbb{Q}_x^{u,t}$.

Proposition 4.2.11. *For any fixed $u \in \mathbb{R}, t > 0, x \in [0,\infty)$, $\mathbf{P}$-almost surely, we have*

$$\mathbb{Q}_x^{u,t}(X_t \in A) = \int_A \rho_u^R(y|t,x)\mathrm{d}y, \qquad \textit{for all } A \in \mathcal{B}([0,\infty)),$$

and the quenched expectation of the polymer endpoint is

$$\mathbb{E}_x^{\mathbb{Q}^{u,t}}[X_t] = \int_0^\infty y\rho_u^R(y|t,x)\mathrm{d}y.$$

Proof. By the path continuity of $X_\cdot$, the continuity of $Z_{u-\frac{1}{2}}(\cdot,\cdot)$ on $\mathbb{R}^2_{\geq 0}$, and (4.2.27), for any

$\varphi \in \mathcal{C}_c([0,\infty),\mathbb{R})$,

$$\begin{aligned}
\mathbb{E}_x^{\mathbb{Q}^{u,t}}[\varphi(X_t)] &= \lim_{r\to t} \mathbb{E}_x^{\mathbb{Q}^{u,t}}[\varphi(X_r)] \\
&= \lim_{r\to t} \frac{1}{Z_{u-\frac{1}{2}}(t,x)} \int_0^\infty \varphi(z)\mathcal{Z}_{u-\frac{1}{2}}(t,x|t-r,z)Z_{u-\frac{1}{2}}(t-r,z)\mathrm{d}z \\
&= \frac{1}{Z_{u-\frac{1}{2}}(t,x)} \int_0^\infty \varphi(y)\mathcal{Z}_{u-\frac{1}{2}}(t,x|0,y)\exp(W_u(y))\mathrm{d}y,
\end{aligned}$$

where the interchange of the limit and the integral is justified by the dominated convergence theorem. By Urysohn's lemma and the monotone convergence theorem, this extends to any function $\varphi = \mathbb{1}_O$ with O being an open set of $[0,\infty)$. By Dynkin's $\pi-\lambda$ theorem, we can further extend to $\mathbb{1}_A$ with $A \in \mathcal{B}([0,\infty))$. $\square$

Remark 4.2.12. We define the half-space CDRP $\mathbf{P}$-almost surely for each fixed $t > 0, x \geq 0$, but do not address its coupling over all $t > 0, x \geq 0$. Additionally, we do not prove some basic properties (Markov, Feller, etc.) for the half-space CDRP, analogous to those proved in the recent comprehensive work [93] for the full-space CDRP. One reason for this is that an adaption of [93] to half-space setting still needs some technical estimates similar to those obtained in [93, Appendix C] on the Robin heat kernels, yet we were not able to obtain them. On the other hand, these properties are not required for our purposes, as we will use an alternative method to establish the stochastic monotonicity needed in Section 4.5.

On a heuristic level, the proof in Section 4.5 is based on sampling the initial points of the CDRP $\mathbb{Q}_x^{u,t}$ from another density on $[0,\infty)$ to construct a measure-to-measure polymer $\mathbb{Q}_{\tilde{W}}^{u,t}$ for any fixed $\omega \in \Omega$. Without a simultaneous coupling, we do not justify that this construction defines a new measure on $\mathcal{C}[0,t]$ $\mathbf{P}$-almost surely. However, as we will show in Section 4.5, we can still apply this idea. In particular, we only need the quenched endpoint distribution for this polymer, which is well-defined through SHE mild solutions $\mathbf{P}$-almost surely. All of our $\mathbf{P}$-almost sure results are derived from the properties and continuity of mild solutions, making the coupling issue irrelevant for our purposes.

We do not attempt to couple the CDRP measures for all $u \in \mathbb{R}$ either, since it is unnecessary for our purposes.

Remark 4.2.13. By the formal Feynman-Kac representation (4.2.14), it is natural to interpret the above continuum directed polymer measure as a "Gibbs measure" of the reflected Brownian motion in half-space with start-point $x \in [0, \infty)$. The Radon-Nikodyn derivative with respect to the law of $\Upsilon_\cdot = |B_\cdot|$ should be expressed as

$$\exp(W(\Upsilon_t) + u\Upsilon_t) \exp\left(-(u-1/2)L_t^{0,\Upsilon}\right) : \exp : \left\{\int_0^t \xi\left(t-r, \Upsilon_r\right) dr\right\}.$$

The above expression is only formal when ξ is the spacetime white noise. In fact, we suspect that the half-space continuum directed polymer measure should be singular with respect to the measure of reflected Brownian motion in half-space, just as the full-space polymer measure studied in [104].

Finally, we remark that with the definition of $\mathbb{Q}_x^{u,t}$, the solution $Z_{u-\frac{1}{2}}(t,x)$ to the half-space stochastic heat equation (4.1.3) is the partition function of the half-space CDRP $\mathbb{Q}_x^{u,t}$. Thus, its logarithm, which is the solution $H_u(t,x)$ to the KPZ equation (4.1.1), is the free energy of the polymer.

4.3 Average growth rate: proof of Theorem 4.1.4

In this section, we study the average growth speed of H_u and prove Theorem 4.1.4. Together with Proposition 4.1.5, it proves the conjecture of the average growth speed in [30, (4.18)].

The method here is stochastic analysis, essentially an application of Itô's formula. The only explicit calculation we do relies on a nice conditional Laplace transform formula obtained for exponential functionals of a Brownian motion with a drift [106]. Similar calculations in the periodic setting can be found in [107, Section 2.2.3], based on Yor's formula [108]. An alternative proof for Theorem 4.1.4 can be pursued by using the half-space log-gamma polymer and making the claims led to [30, (4.18)] rigorous. Notably, the average growth rate has also been recently established for the open KPZ equation [109]; it would be interesting to investigate if the method presented here can be adapted to that context. This section can be read independently of the rest of this section.

It is well-known that, under appropriate assumptions, the partition function of the point-to-line directed polymer can be related to a positive martingale. Thus, to study the average of its logarithm, one can perform a semimartingale decomposition and only need to analyze the drift term. This trick has been used extensively in the study of disordered systems, see e.g. [110] for a spin glass model and [111, Chapter 5] for directed polymers in random environments.

We begin by presenting the following preliminary result, originally proved by [91]. A more detailed discussion on the study of exponential functionals of Brownian motions can be found in the review [112].

Lemma 4.3.1. *For any $u < 0$, the perpetual exponential functional of Brownian motion with drift* *satisfies*

$$\int_0^\infty \exp(W_u(x))\mathrm{d}x \overset{law}{=} \frac{2}{\gamma(-2u)}, \tag{4.3.1}$$

where $\gamma(-2u)$ is a gamma random variable with shape parameter $-2u$.

Now for any $u < 0, t \geq 0$, define

$$\mathcal{Y}_u(t) := \int_0^\infty Z_{u-\frac{1}{2}}(t,x)\mathrm{d}x,$$

which can be viewed as the ζ-to-line partition function of the underlying directed polymer, with the initial data $\zeta(\mathrm{d}x) = \exp(W_u(x))\mathrm{d}x$. $\mathcal{Y}_u(0)$ is the random variable appearing on the left-hand side of (4.3.1). We know that $0 < \mathcal{Y}_u(0) < \infty$ $\mathbf{P}$-almost surely.

As described in (4.1.4), the Brownian motion with drift u is stationary to the half-space KPZ equation with boundary parameter u, which implies that for each $t \geq 0$,

$$\left\{\frac{Z_{u-\frac{1}{2}}(t,y)}{Z_{u-\frac{1}{2}}(t,0)}\right\}_{y\in[0,\infty)} \overset{\text{law}}{=} \{\exp(W_u(y))\}_{y\in[0,\infty)}, \tag{4.3.2}$$

Thus, the random variable $\mathcal{Y}_u(t)/Z_{u-\frac{1}{2}}(t,0)$ satisfies

$$\frac{\mathcal{Y}_u(t)}{Z_{u-\frac{1}{2}}(t,0)} = \int_0^\infty Z_{u-\frac{1}{2}}(t,x)[Z_{u-\frac{1}{2}}(t,0)]^{-1}\mathrm{d}x \overset{\text{law}}{=} \int_0^\infty \exp(W_u(x))\mathrm{d}x = \mathcal{Y}_u(0). \tag{4.3.3}$$

Since $\mathcal{Y}_u(0)$ and $Z_{u-\frac{1}{2}}(t,0)$ are both $\mathbf{P}$-almost surely finite and positive, $\mathcal{Y}_u(t)$ is also $\mathbf{P}$-almost surely finite and positive.

The proof of Theorem 4.1.4 for $u < 0$ relies on the stationarity (4.3.2) as well as the identity in law (4.3.1). For $u \geq 0$, $\exp(W_u(y))\mathrm{d}y$ is not in $L^1([0,\infty))$. This is the key reason why our proof cannot be extended to $u \geq 0$.

Now we state a decomposition of $\mathcal{Y}_u(t)$ which follows from the mild formulation of the SHE. Due to the Robin boundary condition, there exists an extra drift term compared to the full space case.

Lemma 4.3.2. *Let $\mathscr{F}_t^{\xi,W}$ be the natural filtration generated by W and ξ on the time interval $[0,t]$. For any $u < 0$, $(\mathcal{Y}_u(t), \mathscr{F}_t^{\xi,W})$ is a continuous semimartingale, and $\mathcal{Y}_u(t)$ admits a decomposition*

$$\mathcal{Y}_u(t) = \mathcal{Y}_u(0) + M_u(t) + A_u(t), \tag{4.3.4}$$

where $M_u(t)$ is a continuous local martingale with

$$M_u(t) = \int_0^t \int_0^\infty Z_{u-\frac{1}{2}}(s,z)\xi(\mathrm{d}z\mathrm{d}s),$$

and $A_u(t)$ is a process of finite-variation

$$A_u(t) = -\frac{1}{2}\left(u - \frac{1}{2}\right)\int_0^t Z_{u-\frac{1}{2}}(v,0)\mathrm{d}v.$$

Proof. For any $t > 0$, by the convolution formula (4.2.21), we can use the Green's function $\mathcal{Z}$ to write

$$\begin{aligned}\mathcal{Y}_u(t) &= \int_0^\infty Z_{u-\frac{1}{2}}(t,x)\mathrm{d}x = \int_0^\infty \int_0^\infty \mathcal{Z}_{u-\frac{1}{2}}(t,x|0,y)\exp(W_u(y))\mathrm{d}y\mathrm{d}x \\ &= \int_0^\infty \int_0^\infty \mathcal{Z}_{u-\frac{1}{2}}(t,x|0,y)\mathrm{d}x\exp(W_u(y))\mathrm{d}y,\end{aligned} \tag{4.3.5}$$

where the last equality follows from Tonelli's theorem. By (4.2.9), for any $t > 0, y \geq 0$,

$$\int_0^\infty \mathscr{Z}_{u-\frac{1}{2}}(t,x|0,y)\mathrm{d}x = \int_0^\infty \mathscr{P}^R_{u-\frac{1}{2}}(t,x|0,y)\mathrm{d}x \\ + \int_0^\infty \int_0^t \int_0^\infty \mathscr{P}^R_{u-\frac{1}{2}}(t,x|s,z)\mathscr{Z}_{u-\frac{1}{2}}(s,z|0,y)\xi(\mathrm{d}z\mathrm{d}s)\mathrm{d}x. \tag{4.3.6}$$

By (4.2.17),

$$\int_0^\infty \left(\int_0^t \int_0^\infty \mathscr{P}^R_{u-\frac{1}{2}}(t,x|s,z)^2 \mathbf{E}_\xi\left[\mathscr{Z}_{u-\frac{1}{2}}(s,z|0,y)^2\right]\mathrm{d}z\mathrm{d}s\right)^{1/2}\mathrm{d}x < \infty, \tag{4.3.7}$$

so we can use the stochastic Fubini theorem ([113, Theorem 4.33]) to switch the order of integration in the last term of (4.3.6) and integrate $x \in [0,\infty)$ first.

Applying the fundamental theorem of calculus to the integral $\int_0^\infty \mathscr{P}^R_{u-\frac{1}{2}}(t,x|s,y)\mathrm{d}x$ and using the definition of the Robin heat kernel in (4.2.1), for any fixed $y \geq 0, s \in [0,t]$, we have

$$\begin{aligned}
&\int_0^\infty \mathscr{P}^R_{u-\frac{1}{2}}(t,x|s,z)\mathrm{d}x = 1 + \int_s^t \int_0^\infty \partial_v \mathscr{P}^R_{u-\frac{1}{2}}(v,x|s,z)\mathrm{d}x\mathrm{d}v \\
&= 1 + \int_s^t \int_0^\infty \frac{1}{2}\partial_x^2 \mathscr{P}^R_{u-\frac{1}{2}}(v,x|s,z)\mathrm{d}x\mathrm{d}v = 1 + \frac{1}{2}\int_s^t \left[-\partial_x \mathscr{P}^R_{u-\frac{1}{2}}(v,x|s,z)\Big|_{x=0}\right]\mathrm{d}v \\
&= 1 - \frac{1}{2}\int_s^t \left(u - \frac{1}{2}\right)\mathscr{P}^R_{u-\frac{1}{2}}(v,0|s,z)\mathrm{d}v.
\end{aligned}$$

Thus (4.3.6) equals to

$$\begin{aligned}
&1 - \frac{1}{2}\left(u-\frac{1}{2}\right)\int_0^t \mathscr{P}^R_{u-\frac{1}{2}}(v,0|0,y)\mathrm{d}v + \int_0^t \int_0^\infty \mathscr{Z}_{u-\frac{1}{2}}(s,z|0,y)\xi(\mathrm{d}z\mathrm{d}s) \\
&- \frac{1}{2}\left(u-\frac{1}{2}\right)\int_0^t \int_0^\infty \left[\int_s^t \mathscr{P}^R_{u-\frac{1}{2}}(v,0|s,z)\mathrm{d}v\right]\mathscr{Z}_{u-\frac{1}{2}}(s,z|0,y)\xi(\mathrm{d}z\mathrm{d}s).
\end{aligned}$$

Again by (4.2.17),

$$\int_0^t \left(\int_0^\infty \int_0^v \mathscr{P}^R_{u-\frac{1}{2}}(v,0|s,z)^2 \mathbf{E}_\xi \left[\mathcal{Z}_{u-\frac{1}{2}}(s,z|0,y)^2 \right] \mathrm{d}s\mathrm{d}z \right)^{1/2} \mathrm{d}v < \infty,$$

so we can use the stochastic Fubini theorem to switch the order of integrations as

$$\begin{aligned}
&\int_0^t \int_0^\infty \left[\int_s^t \mathscr{P}^R_{u-\frac{1}{2}}(v,0|s,z)\mathrm{d}v \right] \mathcal{Z}_{u-\frac{1}{2}}(s,z|0,y)\xi(\mathrm{d}z\mathrm{d}s) \\
&= \int_0^t \int_0^\infty \int_0^v \mathscr{P}^R_{u-\frac{1}{2}}(v,0|s,z)\mathcal{Z}_{u-\frac{1}{2}}(s,z|0,y)\xi(\mathrm{d}z\mathrm{d}s)\mathrm{d}v \\
&= \int_0^t \mathcal{Z}_{u-\frac{1}{2}}(v,0|0,y)\mathrm{d}v - \int_0^t \mathscr{P}^R_{u-\frac{1}{2}}(v,0|0,y)\mathrm{d}v.
\end{aligned}$$

Therefore, for each $y \in [0,\infty)$, (4.3.6) equals to

$$\begin{aligned}
&\int_0^\infty \mathcal{Z}_{u-\frac{1}{2}}(t,x|0,y)\mathrm{d}x = 1 + \int_0^t \int_0^\infty \mathcal{Z}_{u-\frac{1}{2}}(s,z|0,y)\xi(\mathrm{d}z\mathrm{d}s) \\
&- \frac{1}{2}\left(u - \frac{1}{2}\right) \int_0^t \mathcal{Z}_{u-\frac{1}{2}}(v,0|0,y)\mathrm{d}v.
\end{aligned}$$

Using again the stochastic Fubini theorem and Tonelli's theorem, we have

$$\begin{aligned}
\mathcal{Y}_u(t) &= \int_0^\infty \Bigg[1 + \int_0^t \int_0^\infty \mathcal{Z}_{u-\frac{1}{2}}(s,z|0,y)\xi(\mathrm{d}z\mathrm{d}s) \\
&\quad - \frac{1}{2}\left(u-\frac{1}{2}\right) \int_0^t \mathcal{Z}_{u-\frac{1}{2}}(v,0|0,y)\mathrm{d}v \Bigg] e^{W_u(y)}\mathrm{d}y \\
&= \mathcal{Y}_u(0) + \int_0^t \int_0^\infty Z_{u-\frac{1}{2}}(s,z)\xi(\mathrm{d}z\mathrm{d}s) - \frac{1}{2}\left(u-\frac{1}{2}\right) \int_0^t Z_{u-\frac{1}{2}}(v,0)\mathrm{d}v,
\end{aligned}$$

which is the decomposition (4.3.4). Note that for $\mathbf{P}$-almost surely all realization of W, by

(4.2.17),

$$\begin{aligned}
&\mathbf{E}_\xi \int_0^t \int_0^\infty Z_{u-\frac{1}{2}}(s,y)^2 \mathrm{d}y\mathrm{d}s \\
&= \int_0^\infty \int_0^\infty \int_0^t \int_0^\infty \mathbf{E}_\xi \left[\mathcal{Z}_{u-\frac{1}{2}}(s,y|0,z_1) \mathcal{Z}_{u-\frac{1}{2}}(s,y|0,z_2) \right] \mathrm{d}y\mathrm{d}s e^{W_u(z_1)} e^{W_u(z_2)} \mathrm{d}z_1 \mathrm{d}z_2 \\
&\le \int_0^\infty \int_0^\infty \int_0^t \int_0^\infty \| \mathcal{Z}_{u-\frac{1}{2}}(s,y|0,z_1)\|_2 \| \mathcal{Z}_{u-\frac{1}{2}}(s,y|0,z_2)\|_2 \mathrm{d}y\mathrm{d}s e^{W_u(z_1)} e^{W_u(z_2)} \mathrm{d}z_1 \mathrm{d}z_2 \\
&\le C(t,u) \left[\int_0^\infty e^{W_u(z)} \mathrm{d}z \right]^2,
\end{aligned} \tag{4.3.8}$$

so for any $t > 0$,

$$\int_0^t \int_0^\infty Z_{u-\frac{1}{2}}(s,y)^2 \mathrm{d}y\mathrm{d}s < \infty, \quad \mathbf{P}\text{-almost surely,}$$

and $M_u(t)$ is a continuous local martingale with quadratic variation

$$\langle M_u \rangle_t = \int_0^t \int_0^\infty Z_{u-\frac{1}{2}}(s,y)^2 \mathrm{d}y\mathrm{d}s.$$

The proof is complete. □

Remark 4.3.3. One needs to be cautious with the L^p norms after putting the Brownian motion W into the filtration $\mathscr{F}_t^{\xi,W}$. In fact, by (4.3.1), $\mathcal{Y}_u(0) = \int_0^\infty e^{W_u(x)} \mathrm{d}x \in L^p(\Omega)$ if and only if $-2u > p$. Consequentially, taking an expectation $\mathbf{E}_W$ on the last line of (4.3.8) only gives a bounded term when $u < -1$, so we do not claim that $\mathbf{E}\langle M_u \rangle_t$ is bounded for all $u < 0$.

On the other hand, if we consider each fixed realization of the initial data, we can think of $M_u(t)$ as a local martingale with respect to $\mathscr{F}_t^\xi$ – the filtration generated by ξ only. The bound in (4.3.8) shows that $M_u(t)$ is a square-integrable martingale w.r.t. $\mathscr{F}_t^\xi$ for $\mathbf{P}$-almost surely all realization of W.

A similar issue also occurs when we apply the stochastic Fubini theorem in the above proof.

In order to overcome it, we used (4.3.5) to separate the randomness from ξ and from W. The stochastic Fubini theorem is applied to processes that are adapted to $\mathscr{F}_t^{\xi}$, as shown in (4.3.7).

The following lemma provides the semimartingale decomposition for $\log \mathcal{Y}_u(t)$, using which we compute $\mathbf{E}\log \mathcal{Y}_u(t)$ which further leads to $\mathbf{E}H_u(t,0)$. The crucial part is that the drift term will be written as an additive functional of the process $\{Z_{u-\frac{1}{2}}(s,\cdot)/\mathcal{Y}_u(s)\}_{s\geq 0}$, which is at stationarity by (4.1.4).

Lemma 4.3.4. *For any $u < 0$ and $t \geq 0$, we have*

$$\mathbf{E}[H_u(t,0)] = \mathbf{E}\left[-\frac{1}{2}\left(u - \frac{1}{2}\right)\frac{1}{\mathcal{Y}_u(0)} - \frac{1}{2}\frac{\int_0^\infty \exp(2W_u(y))\mathrm{d}y}{\mathcal{Y}_u(0)^2}\right]t. \tag{4.3.9}$$

Proof. Recall that as defined in (4.1.1), $H_u(t,0) = \log Z_{u-\frac{1}{2}}(t,0)$. Taking the logarithm and then the expectation on both sides of (4.3.3), we have

$$\mathbf{E}[H_u(t,0)] = \mathbf{E}[\log \mathcal{Y}_u(t) - \log \mathcal{Y}_u(0)].$$

By Lemma 4.3.2 and Itô's formula [113, Theorem 4.32], $\log \mathcal{Y}_u(t)$ can be decomposed as

$$\begin{aligned}\log \mathcal{Y}_u(t) - \log \mathcal{Y}_u(0) &= \int_0^t \mathcal{Y}_u(s)^{-1}\mathrm{d}M_u(s) + \int_0^t \mathcal{Y}_u(s)^{-1}\mathrm{d}A_u(s) - \frac{1}{2}\int_0^t \mathcal{Y}_u(s)^{-2}\mathrm{d}\langle M_u\rangle_s \\ &= \int_0^t\int_0^\infty \frac{Z_{u-\frac{1}{2}}(s,y)}{\mathcal{Y}_u(s)}\xi(\mathrm{d}y\mathrm{d}s) - \frac{1}{2}\left(u-\frac{1}{2}\right)\int_0^t \frac{Z_{u-\frac{1}{2}}(s,0)}{\mathcal{Y}_u(s)}\mathrm{d}s - \frac{1}{2}\int_0^t\int_0^\infty \frac{Z_{u-\frac{1}{2}}(s,y)^2}{\mathcal{Y}_u(s)^2}\mathrm{d}y\mathrm{d}s.\end{aligned}$$

By (4.3.2), for any $u < 0$ and $s \geq 0$, we have

$$\begin{aligned}\mathbf{E}\frac{Z_{u-\frac{1}{2}}(s,0)}{\mathcal{Y}_u(s)} &= \mathbf{E}\left[\int_0^\infty Z_{u-\frac{1}{2}}(s,y)Z_{u-\frac{1}{2}}(s,0)^{-1}\mathrm{d}y\right]^{-1} \\ &= \mathbf{E}\left[\int_0^\infty \exp(W_u(y))\mathrm{d}y\right]^{-1} = \mathbf{E}\frac{1}{\mathcal{Y}_u(0)},\end{aligned}$$

and

$$\mathbf{E}\int_0^\infty \frac{Z_{u-\frac{1}{2}}(s,y)^2}{\mathcal{Y}_u(s)^2}\mathrm{d}y = \mathbf{E}\frac{\int_0^\infty Z_{u-\frac{1}{2}}(s,y)^2\mathrm{d}y}{\left[\int_0^\infty Z_{u-\frac{1}{2}}(s,y')\mathrm{d}y'\right]^2}$$
$$= \mathbf{E}\frac{\int_0^\infty Z_{u-\frac{1}{2}}(s,y)^2 Z_{u-\frac{1}{2}}(s,0)^{-2}\mathrm{d}y}{\left[\int_0^\infty Z_{u-\frac{1}{2}}(s,y')Z_{u-\frac{1}{2}}(s,0)^{-1}\mathrm{d}y'\right]^2}$$
$$= \mathbf{E}\frac{\int_0^\infty \exp(W_u(y))^2\mathrm{d}y}{\left[\int_0^\infty \exp(W_u(y'))\mathrm{d}y'\right]^2}$$
$$= \mathbf{E}\left[\mathcal{Y}_u(0)^{-2}\int_0^\infty \exp(2W_u(y))\mathrm{d}y\right].$$

Here we used the stationarity (4.3.2), so both expectations are time independent, and by (4.3.1) and (4.3.10) below, they are both finite. Thus the term

$$\int_0^t \int_0^\infty Z_{u-\frac{1}{2}}(s,y)\mathcal{Y}_u(s)^{-1}\xi(\mathrm{d}y\mathrm{d}s)$$

is a square-integrable martingale, and (4.3.9) follows. □

To complete the proof of Theorem 4.1.4, for any $u < 0$, it only remains to compute the expectation on the right hand side of (4.3.9).

Proof of Theorem 4.1.4. It follows from (4.3.1) that for any fixed $u < 0$,

$$\mathbf{E}\left[\frac{1}{\mathcal{Y}_u(0)}\right] = \frac{1}{2}\mathbf{E}[\gamma(-2u)] = -u.$$

Define

$$\mathcal{J} = \int_0^\infty \mathrm{e}^{2W(y)+2uy}\mathrm{d}y, \quad \text{and} \quad \mathcal{K} = \int_0^\infty \mathrm{e}^{W(y)+uy}\mathrm{d}y = \mathcal{Y}_u(0),$$

which are finite $\mathbf{P}$-almost surely since $u < 0$. It remains to compute

$$\mathbf{E}\left[\frac{\int_0^\infty \exp(2W_u(y))\mathrm{d}y}{\mathcal{Y}_u(0)^2}\right] = \mathbf{E}\frac{\int_0^\infty \exp(W_u(y))^2\mathrm{d}y}{\left[\int_0^\infty \exp(W_u(y'))\mathrm{d}y'\right]^2} = \mathbf{E}\left[\frac{\mathcal{J}}{\mathcal{K}^2}\right].$$

Note that the same Brownian motion appears in both the numerator and the denominator, which makes the calculation more involved. By [106, Example 2.4], for any $u < 0$, the conditional Laplace transform takes the form

$$\mathbf{E}\left(\mathrm{e}^{-\frac{1}{2}\lambda^2\mathcal{J}}\middle|\mathcal{K} = s\right)\mathbf{P}\left(\mathcal{K} \in \mathrm{d}s\right) = \frac{\lambda^{-2u+1}}{2\Gamma(-2u)}\frac{\mathrm{e}^{-\lambda\coth\left(\frac{\lambda s}{2}\right)}}{\left(\sinh\left(\frac{\lambda s}{2}\right)\right)^{-2u+1}}\mathrm{d}s, \quad \forall s > 0.$$

By differentiating both sides in λ^2 and taking the limit $\lambda \to 0$, we have

$$\begin{aligned}
&\mathbf{E}\left(-\frac{1}{2}\mathcal{J}\middle|\mathcal{K} = s\right)\mathbf{P}\left(\mathcal{K} \in \mathrm{d}s\right) = \lim_{\lambda\to 0}\frac{\mathrm{d}}{\mathrm{d}\lambda^2}\mathbf{E}\left(\mathrm{e}^{-\frac{1}{2}\lambda^2\mathcal{J}}\middle|\mathcal{K} = s\right)\mathbf{P}\left(\mathcal{K} \in \mathrm{d}s\right)\\
&= \lim_{\lambda\to 0}\frac{\mathrm{d}}{\mathrm{d}\lambda^2}\left(\frac{\lambda^{-2u+1}}{2\Gamma(-2u)}\frac{\mathrm{e}^{-\lambda\coth\left(\frac{\lambda s}{2}\right)}}{\left(\sinh\left(\frac{\lambda s}{2}\right)\right)^{-2u+1}}\right)\mathrm{d}s\\
&= \lim_{\lambda\to 0}\frac{1}{2\lambda}\left[\frac{\lambda^{-2u+1}}{2\Gamma(-2u)}\frac{e^{-\lambda\coth\left(\frac{\lambda s}{2}\right)}}{\left(\sinh\left(\frac{\lambda s}{2}\right)\right)^{-2u+1}}\right]\Psi(s|u,\lambda)\mathrm{d}s.
\end{aligned}$$

with

$$\Psi(s|u,\lambda) = \left[\frac{-2u+1}{\lambda} - \coth\left(\frac{\lambda s}{2}\right) + \frac{\lambda s}{2\left(\sinh\left(\frac{\lambda s}{2}\right)\right)^2} + \frac{(2u-1)s}{2}\coth\left(\frac{\lambda s}{2}\right)\right].$$

Taylor series expressions give that

$$\coth(x) = x^{-1} + \frac{x}{3} - \frac{x^3}{45} + O(x^5), \quad 0 < |x| < \pi,$$

and

$$\sinh(x)^{-2} = \operatorname{csch}(x)^2 = \left(x^{-1} - \frac{x}{6} + O(x^3)\right)^2 = x^{-2} - \frac{1}{3} + O(x^2), \quad 0 < |x| < \pi.$$

Thus

$$\lim_{\lambda\to 0} e^{-\lambda\coth\left(\frac{\lambda s}{2}\right)} = e^{-\frac{2}{s}}, \quad \lim_{\lambda\to 0} \frac{\lambda^{-2u+1}}{\left(\sinh\left(\frac{\lambda s}{2}\right)\right)^{-2u+1}} = \left(\frac{2}{s}\right)^{-2u+1},$$

and

$$\Psi(s|u,\lambda) = (-2u+1)\frac{1}{\lambda} - \frac{2}{s}\frac{1}{\lambda} - \frac{1}{3}\frac{\lambda s}{2} + \frac{\lambda s}{2}\left(\left(\frac{\lambda s}{2}\right)^{-2} - \frac{1}{3}\right) + (2u-1)\frac{s}{2}\left(\frac{2}{s}\frac{1}{\lambda} + \frac{1}{3}\frac{\lambda s}{2}\right) + O(\lambda^3)$$
$$= -\frac{1}{3}\lambda s + \frac{2u-1}{12}\lambda s^2 + O(\lambda^3).$$

These results imply that

$$\lim_{\lambda\to 0}\frac{1}{2\lambda}\left[\frac{\lambda^{-2u+1}}{2\Gamma(-2u)}\frac{e^{-\lambda\coth\left(\frac{\lambda s}{2}\right)}}{\left(\sinh\left(\frac{\lambda s}{2}\right)\right)^{-2u+1}}\right]\Psi(s|u,\lambda) = \frac{e^{-\frac{2}{s}}}{4\Gamma(-2u)}\left(\frac{2}{s}\right)^{-2u+1}\left(-\frac{1}{3}s + \frac{2u-1}{12}s^2\right),$$

and

$$\begin{aligned}
\mathbf{E}\left[\frac{\mathcal{J}}{\mathcal{K}^2}\right] &= \int_0^\infty \frac{-2}{s^2}\mathbf{E}\left(-\frac{1}{2}\mathcal{J}\middle|\mathcal{K}=s\right)\mathbf{P}\left(\mathcal{K}\in \mathrm{d}s\right)\\
&= \int_0^\infty \frac{-e^{-\frac{2}{s}}}{2s^2\Gamma(-2u)}\frac{2^{-2u+1}}{s^{-2u+1}}\left(-\frac{s}{3} + \frac{2u-1}{12}s^2\right)\mathrm{d}s\\
&= \frac{1}{2^{2u}\Gamma(-2u)}\int_0^\infty \frac{1}{3}e^{-\frac{2}{s}}s^{2u-2} - \frac{2u-1}{12}e^{-\frac{2}{s}}s^{2u-1}\mathrm{d}s \qquad (4.3.10)\\
&= \frac{1}{2^{2u}\Gamma(-2u)}\left(\frac{1}{3}\frac{\Gamma(-2u+1)}{2^{-2u+1}} - \frac{2u-1}{12}\frac{\Gamma(-2u)}{2^{-2u}}\right)\\
&= -\frac{u}{3} - \frac{2u-1}{12} = -\frac{u}{2} + \frac{1}{12}.
\end{aligned}$$

Now by (4.3.9), we have

$$\mathbf{E}[H_u(t,0)] = \left(-\frac{1}{2}\left(u-\frac{1}{2}\right)\mathbf{E}\left[\frac{1}{\mathcal{Y}_u(0)}\right] - \frac{1}{2}\mathbf{E}\left[\frac{\mathcal{J}}{\mathcal{K}^2}\right]\right)t$$
$$= \left[\frac{1}{2}\left(u-\frac{1}{2}\right)u - \frac{1}{2}\left(-\frac{u}{2}+\frac{1}{12}\right)\right]t = \left(\frac{u^2}{2}-\frac{1}{24}\right)t.$$

We next extend to $u = 0$. For any fixed $t > 0$, by Proposition 4.2.8, $H_u(t,0)$ is uniformly integrable for u in any compact subset of $\mathbb{R}$. Thus the convergence $\mathbf{E}\left[H_u(t,0)\right] \to \mathbf{E}\left[H_0(t,0)\right]$ as $u \to 0$ follows from the convergence in probability

$$Z_{u-\frac{1}{2}}(t,0) \overset{\text{prob}}{\to} Z_{-\frac{1}{2}}(t,0) \quad \text{as } u \to 0,$$

together with the continuous mapping theorem. To show the above convergence in probability, we note that by the Minkowski and Hölder's inequalities,

$$\|Z_{u-\frac{1}{2}}(t,0) - Z_{-\frac{1}{2}}(t,0)\|_2 \le \int_0^\infty \|\mathcal{Z}_{u-\frac{1}{2}}(t,0|0,y) - \mathcal{Z}_{-\frac{1}{2}}(t,0|0,y)\|_4 \|\exp(W(y)+uy)\|_4 \mathrm{d}y$$
$$+ \int_0^\infty \|\mathcal{Z}_{-\frac{1}{2}}(t,0|0,y)\|_4 \|\exp(W(y)+uy) - \exp(W(y)+y)\|_4 \mathrm{d}y.$$

By the $\mathbf{P}$-almost surely continuity of the Green's function $\mathcal{Z}_\cdot(\cdot,\cdot|\cdot,\cdot)$ in Proposition 4.2.7, the uniform moment bounds in (4.2.17), and the dominated convergence theorem, we have $\|Z_{u-\frac{1}{2}}(t,0) - Z_{-\frac{1}{2}}(t,0)\|_2 \to 0$ as $u \to 0$.

For any $x > 0$, the result follows from (4.1.4). □

4.4 Variance identity: proof of Theorem 4.1.1

To prove the identity (4.1.9), which relates the variance of the height function to the endpoint displacement of the directed polymer, we utilize an integration-by-parts formula in the Gaussian space generated by the Brownian motion $W(x)$. Although the model involves two sources of Gaussianity, we only rely on the one from the initial data. This method heavily relies on the fact that the Brownian motion with drift $W(x) + ux$ is the stationary measure of the increment process $H_u(t,x) - H_u(t,0)$ (see (4.1.4)).

In the following argument, we consider the four height increments along different directions: $A = H_u(t,0) - H_u(0,0)$, $B = H_u(t,x) - H_u(t,0)$, $C = H_u(t,x) - H_u(0,x)$ and $D = H_u(0,x) - H_u(0,0)$. The quantity of interest is A, while B has Gaussian distribution due to the choice of initial data, C shares the same distribution as the height increment in the full space case when $x \gg 1$, and $D = W(x)+ux$ is the Gaussian random variable through which we perform the integration by parts. Similar arguments appeared in various previous works, see [29, 85].

Proof of Theorem 4.1.1. Fix $u \in \mathbb{R}, t > 0$. Define the height increment of (4.1.1) in the time interval $[0,t]$ for any $x \in [0,\infty)$ as

$$\mathcal{H}_u(t,x) := H_u(t,x) - H_u(0,x) = \log Z_{u-\frac{1}{2}}(t,x) - W_u(x).$$

Then we have

$$H_u(t,x) - H_u(t,0) = \mathcal{H}_u(t,x) - \mathcal{H}_u(t,0) + W_u(x),$$

which implies that

$$\mathbf{Var}[H_u(t,x) - H_u(t,0)] = \mathbf{Var}[\mathcal{H}_u(t,x) - \mathcal{H}_u(t,0) + W_u(x)].$$

By (4.1.4), for any $t \geq 0$, $\mathbf{Var}[H_u(t,x) - H_u(t,0)] = \mathbf{Var}[W_u(x)] = x$. Thus we have an identity

$$
\begin{aligned}
0 &= \mathbf{Var}[\mathcal{H}_u(t,x) - \mathcal{H}_u(t,0)] + 2\mathbf{Cov}[\mathcal{H}_u(t,x) - \mathcal{H}_u(t,0), W_u(x)] \\
&= \mathbf{Var}[\mathcal{H}_u(t,x)] - 2\mathbf{Cov}[\mathcal{H}_u(t,x), \mathcal{H}_u(t,0)] + \mathbf{Var}[\mathcal{H}_u(t,0)] \\
&+ 2\mathbf{Cov}[\mathcal{H}_u(t,x) - \mathcal{H}_u(t,0), W_u(x)] \\
&= \mathbf{Var}[\mathcal{H}_u(t,0)] + (\mathbf{Var}[\mathcal{H}_u(t,x)] + 2\mathbf{Cov}[\mathcal{H}_u(t,x), W_u(x)]) \\
&- 2\mathbf{Cov}[\mathcal{H}_u(t,x), \mathcal{H}_u(t,0)] - 2\mathbf{Cov}[\mathcal{H}_u(t,0), W_u(x)].
\end{aligned}
$$

Since $H_u(t,0) = \mathcal{H}_u(t,0)$, we have $\mathbf{Var}[H_u(t,0)] = \mathbf{Var}[\mathcal{H}_u(t,0)]$. The above equation can be rewritten as

$$
\begin{aligned}
\mathbf{Var}[H_u(t,0)] = 2\mathbf{Cov}[H_u(t,0), W_u(x)] + 2\mathbf{Cov}[\mathcal{H}_u(t,x), \mathcal{H}_u(t,0)] \\
- (\mathbf{Var}[\mathcal{H}_u(t,x)] + 2\mathbf{Cov}[\mathcal{H}_u(t,x), W_u(x)]).
\end{aligned}
$$

Recalling that $\mathbb{E}_x^{\mathbb{Q}^{u,t}}[X_t]$ is the (quenched) expectation of the half-space CDRP endpoint. The endpoint has a quenched density $\rho_u^R(\cdot|t,x)$ as defined in (4.2.28). The proof is complete once we establish the following results:

$$
\mathbf{Cov}[H_u(t,0), W_u(x)] \to \mathbf{E}\int_0^\infty y\rho_u^R(y|t,0)\mathrm{d}y = \mathbf{E}\mathbb{E}_0^{\mathbb{Q}^{u,t}}[X_t], \quad \text{as } x \to \infty; \tag{4.4.1}
$$

$$
\mathbf{Var}[\mathcal{H}_u(t,x)] + 2\mathbf{Cov}[\mathcal{H}_u(t,x), W_u(x)] \to ut, \quad \text{as } x \to \infty; \tag{4.4.2}
$$

and

$$
\mathbf{Cov}[\mathcal{H}_u(t,x), \mathcal{H}_u(t,0)] \to 0, \quad \text{as } x \to \infty. \tag{4.4.3}
$$

We will prove equations (4.4.1) and (4.4.2) in the following subsections. The proof of (4.4.3) is very similar to the proof of the analogous result for full-space KPZ equation in [29], so we put it in Appendix 4.8.2. □

Remark 4.4.1. If using the above definitions for the four height increments A, B, C, D, the computation above is equivalent to taking the variance on both sides of the identity $B = D + C - A$. The value of interest is $\mathbf{Var}[A]$. With stationarity (4.1.4), $\mathbf{Var}[B] = \mathbf{Var}[D]$. We perform the integration-by-parts to prove the limit of $\mathbf{Cov}[A, D]$ as $x \to \infty$ in (4.4.1). For the terms $\mathbf{Var}[C] + 2\mathbf{Cov}[C, D]$, note that they only involve the height function $H_u(t, \cdot)$ evaluated at point x away from the boundary. Thus we use its full-space counterpart's value to approximate it when $x \gg 1$ in (4.4.2). The remaining term $\mathbf{Cov}[A, C]$ does not involve any KPZ behavior. It vanishes when $x \gg 1$ at any finite time t, which is stated in (4.4.3).

4.4.1 Proof of (4.4.1)

We prove (4.4.1) through an integration-by-parts in the Gaussian space generated by the Brownian motion $W(x)$. The following is analogous to the full-space result [29, Lemma 2.4], but a few details are different in the half-space setting. As discussed above after Proposition 4.2.7, due to the boundary, the Green's function $\mathcal{Z}_\mu(t, x|0, y)$ is not spatially stationary.

Let $\mathscr{W}$ be the spatial white noise associated with the Brownian motion $W(\cdot)$. Let $\mathcal{D}$ be the Malliavin derivative operator on the Gaussian probability space generated by $\mathscr{W}$.

For each $t, x, z \geq 0$ and each realization of ξ, we apply an integration-by-parts on the

Gaussian space generated by $\mathscr{W}$. Further taking an expectation over ξ, we obtain

$$\begin{aligned}
&\mathbf{Cov}[H_u(t,z), W_u(x)] = \mathbf{E}[H_u(t,z)W(x)] \\
&= \mathbf{E}[H_u(t,z)\int_0^\infty \mathbb{1}_{[0,x]}(r)\mathscr{W}(r)\mathrm{d}r] = \mathbf{E}\langle \mathcal{D}H_u(t,z), \mathbb{1}_{[0,x]}\rangle.
\end{aligned}$$

The integral above is the standard Wiener integral since $\mathbb{1}_{[0,x]}(r)$ is a deterministic function, and the bracket denotes the inner product in $L^2(\mathbb{R})$. By (4.2.25), for any $r \geq 0$,

$$\begin{aligned}
\mathcal{D}_r H_u(t,z) &= \mathcal{D}_r \log \int_0^\infty \mathcal{Z}_{u-\frac{1}{2}}(t,z|0,y)\exp(W_u(y))\mathrm{d}y \\
&= \frac{\mathcal{D}_r[\int_0^\infty \mathcal{Z}_{u-\frac{1}{2}}(t,z|0,y)\exp(W_u(y))\mathrm{d}y]}{\int_0^\infty \mathcal{Z}_{u-\frac{1}{2}}(t,z|0,y)\exp(W_u(y))\mathrm{d}y} \\
&= \frac{\int_0^\infty \mathcal{Z}_{u-\frac{1}{2}}(t,z|0,y)\exp(W_u(y))\mathbb{1}_{[0,y]}(r)\mathrm{d}y}{\int_0^\infty \mathcal{Z}_{u-\frac{1}{2}}(t,z|0,y)\exp(W_u(y))\mathrm{d}y} = \int_r^\infty \rho_u^R(y|t,z)\mathrm{d}y.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\mathbf{E}\langle \mathcal{D}H_u(t,z), \mathbb{1}_{[0,x]}\rangle = \mathbf{E}\int_0^x \int_r^\infty \rho_u^R(y|t,z)\mathrm{d}y\mathrm{d}r = \mathbf{E}\int_0^\infty \min(x,y)\rho_u^R(y|t,z)\mathrm{d}y \\
&= \mathbf{E}\int_0^x y\rho_u^R(y|t,z)\mathrm{d}y + x\mathbf{E}\int_x^\infty \rho_u^R(y|t,z)\mathrm{d}y.
\end{aligned}$$

Setting $z = 0$, we have

$$\mathbf{Cov}[H_u(t,0), W_u(x)] = \mathbf{E}\int_0^x y\rho_u^R(y|t,0)\mathrm{d}y + x\mathbf{E}\int_x^\infty \rho_u^R(y|t,0)\mathrm{d}y. \tag{4.4.4}$$

By Hölder's inequality, (4.2.17), (4.8.23) and (4.2.23), for any $t > 0, y \in [0, \infty)$, we have

$$\begin{aligned}&\mathbf{E}\rho_u^R(y|t,0) = \mathbf{E}\frac{\mathcal{Z}_{u-\frac{1}{2}}(t,0|0,y)\exp(W_u(y))}{\int_0^\infty \mathcal{Z}_{u-\frac{1}{2}}(t,0|0,y')\exp(W_u(y'))\mathrm{d}y'}\\&\le \|\mathcal{Z}_{u-\frac{1}{2}}(t,0|0,y)\|_4 \|\exp(W_u(y))\|_4 \left\|Z_{u-\frac{1}{2}}(t,0)^{-1}\right\|_2 \le C\exp(-y^2/C),\end{aligned} \tag{4.4.5}$$

for some constant $C = C(t,u)$. Taking the limit $x \to \infty$ on both sides of (4.4.4) gives (4.4.1).

4.4.2 Proof of (4.4.2)

The idea is that, for large $x \gg 1$ and fixed $t > 0$, the value $H_u(t,x)$ almost does not feel the boundary effects, hence the result should be close to the full space case. We proceed in two steps. First, we show that (4.4.2) holds with $H_u(t,x)$ replaced by $H_u^{fs}(t,x)$, where $H_u^{fs}(t,x)$ solves the KPZ equation on full-space starting from a two-sided Brownian motion with drift u. In the second step, we show that the error induced by the approximation vanishes as $x \to \infty$, namely, $\mathbf{Var}[\mathcal{H}_u^{fs}(t,x)] + 2\mathbf{Cov}[\mathcal{H}_u^{fs}(t,x), W_u^{fs}(x)]$ can be used to approximate the left-hand side of (4.4.2) when $x \to \infty$.

Let us begin with a few definitions. For $(t,x) \in [0,\infty) \times \mathbb{R}$ and $u \in \mathbb{R}$, let $Z_u^{fs}(t,x)$ be the solution to the (1+1)-dimensional stochastic heat equation on the full space

$$\begin{aligned}\partial_t Z_u^{fs}(t,x) &= \frac{1}{2}\partial_x^2 Z_u^{fs}(t,x) + Z_u^{fs}(t,x)\xi(t,x),\\ Z_u^{fs}(0,x) &= \exp(W_u^{fs}(x)),\end{aligned} \tag{4.4.6}$$

where ξ is the same space-time white noise on $(t,x) \in \mathbb{R}^2$ as in (4.1.3) and $W_u^{fs}(x) := W^{fs}(x) + ux$ with W^{fs} being a two-sided Brownian motion on $\mathbb{R}$. For simplicity, we assume that $W^{fs}(\cdot)$ is defined on our probability space $(\Omega, \mathscr{F}, \mathbf{P})$ and it coincides with $W(\cdot)$ on $[0,\infty)$ for each $\omega \in \Omega$.

The solution to the KPZ equation on the full space is given by the Hopf-Cole transform

$$H_u^{fs}(t,x) := \log Z_u^{fs}(t,x). \tag{4.4.7}$$

In particular, $H_u^{fs}(0,x) = W_u^{fs}(x)$. For $x \in [0,\infty)$, $H_u^{fs}(0,x) = H_u(0,x) = W_u(x)$. Note that the boundary parameter u appears in the half-space SHE (4.1.3) itself, while in (4.4.6), u is only parametrizing the initial condition. Thus one needs to be careful to include a shift term $1/2$ when referring to a proper half-space SHE. We have $Z_u^{fs}(0,x) = Z_{u-\frac{1}{2}}(0,x) = \exp(W_u(x))$ when $x \in [0,\infty)$.

It is a classical result (see [55]) that for each $u \in \mathbb{R}$, (4.4.6) has a unique mild solution satisfying

$$Z_u^{fs}(t,x) = \int_{\mathbb{R}} p_t(x-y)\exp(W_u^{fs}(y))\mathrm{d}y + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y) Z_u^{fs}(s,y)\xi(\mathrm{d}y\mathrm{d}s), \tag{4.4.8}$$

and for any $\tau > 0$, $\sup_{0\le t\le\tau}\sup_{x\in\mathbb{R}} e^{-a|x|}\mathbf{E}\left[Z_u^{fs}(t,x)^2\right] < \infty$ for some $a > 0$. It is also well-known (see [114]) that when $H_u^{fs}(\cdot,\cdot)$ starts from the initial data $W_u^{fs}(\cdot)$, the increment process $\{H_u^{fs}(t,\cdot) - H_u^{fs}(t,0)\}_{t\ge 0}$ is stationary in time.

We use $\mathcal{Z}^{fs}(t,x|s,y)$ to denote the Green's function of the (1+1)-dimensional full-space SHE, of which we refer to [93] for further properties. In particular, there is a convolution formula

$$Z_u^{fs}(t,x) = \int_{-\infty}^{\infty} \mathcal{Z}^{fs}(t,x|0,y)\exp(W_u^{fs}(y))\mathrm{d}y. \tag{4.4.9}$$

Now define $\mathcal{H}_u^{fs}(t,x) := H_u^{fs}(t,x) - H_u^{fs}(0,x)$. We first prove the following result, which is almost (4.4.2) but with $\mathcal{H}_u(t,x)$ substituted by $\mathcal{H}_u^{fs}(t,x)$.

Proposition 4.4.2. *For any $u \in \mathbb{R}, t > 0$,*

$$\mathbf{Var}[\mathcal{H}_u^{fs}(t,x)] + 2\mathbf{Cov}[\mathcal{H}_u^{fs}(t,x), W_u^{fs}(x)] \to ut, \quad \textit{as } x \to \infty.$$

Proof. Using that $W_u^{fs}(\cdot)$ is the stationary measure of the increment process

$$\{H_u^{fs}(t,\cdot) - H_u^{fs}(t,0)\}_{t\geq 0},$$

and the fact that for any fixed $t > 0$, $\{\mathcal{H}_u^{fs}(t,x)\}_{x\in\mathbb{R}}$ is stationary, following the argument for [29, Lemma 2.3], we derive that

$$\begin{aligned}
&\mathbf{Var}[\mathcal{H}_u^{fs}(t,x)] + 2\mathbf{Cov}[\mathcal{H}_u^{fs}(t,x), W_u^{fs}(x)] = \mathbf{Var}[\mathcal{H}_u^{fs}(t,0)] + 2\mathbf{Cov}[\mathcal{H}_u^{fs}(t,x), W_u^{fs}(x)] \\
&= \mathbf{Cov}[\mathcal{H}_u^{fs}(t,x) + \mathcal{H}_u^{fs}(t,0), W_u^{fs}(x)] + \mathbf{Cov}[\mathcal{H}_u^{fs}(t,x), \mathcal{H}_u^{fs}(t,0)].
\end{aligned}$$

For the second term on the right-hand side, by adapting the proof in [29, Appendix A] to include the drift term, we can show

$$\mathbf{Cov}[\mathcal{H}_u^{fs}(t,x), \mathcal{H}_u^{fs}(t,0)] \to 0 \text{ as } x \to \infty.$$

It remains to analyze the term $\mathbf{Cov}[\mathcal{H}_u^{fs}(t,x) + \mathcal{H}_u^{fs}(t,0), W_u^{fs}(x)]$.

Let $\rho_u^{fs}(y|t,x)$ be the quenched density of the (1+1)-dimensional continuum directed polymer model with boundary $\exp(W_u^{fs}(\cdot))$, starting from (t,x) and running backward to time zero:

$$\rho_u^{fs}(y|t,x) := \frac{\mathcal{Z}^{fs}(t,x|0,y)\exp(W_u^{fs}(y))}{\int_{\mathbb{R}} \mathcal{Z}^{fs}(t,x|0,y')\exp(W_u^{fs}(y'))\mathrm{d}y'}.$$

Using the same integration-by-parts method as in the proof of (4.4.1) and [29, Lemma 2.5], for any $t > 0$ and $x, z \geq 0$, we have

$$\begin{aligned}\mathbf{Cov}[H_u^{fs}(t,z), W_u^{fs}(x)] &= \mathbf{E}\int_0^\infty \rho_u^{fs}(y|t,z)\min(x,y)\mathrm{d}y \\ &= \mathbf{E}\int_{-z-ut}^\infty \rho_0^{fs}(y|t,0)\min(x,y+z+ut)\mathrm{d}y.\end{aligned}$$

Here, for the second "=", we used the fact that

$$\{\rho_u^{fs}(y+z+ut|t,z)\}_{y\in\mathbb{R}} \overset{\text{law}}{=} \{\rho_0^{fs}(y|t,0)\}_{y\in\mathbb{R}},$$

which follows from the spatial stationarity of $\mathcal{Z}^{fs}$ and the identity in law:

$$\left\{\mathcal{Z}^{fs}(t,ut+y|0,0)e^{uy+\frac{1}{2}u^2t}\right\}_{t>0,y\in\mathbb{R}} \overset{\text{law}}{=} \left\{\mathcal{Z}^{fs}(t,y|0,0)\right\}_{t>0,y\in\mathbb{R}}.$$

This relation was proved in [29, Proposition 2.7] and arises from the shear invariance of the space-time white noise ξ. It follows that for fixed $u \in \mathbb{R}, t > 0$ and any $x \geq 0$,

$$\begin{aligned}&\mathbf{Cov}[\mathcal{H}_u^{fs}(t,x) + \mathcal{H}_u^{fs}(t,0), W_u^{fs}(x)] \\ &= -x + \mathbf{E}\int_{-x-ut}^\infty \rho_0^{fs}(y|t,0)\min(x,y+x+ut)\mathrm{d}y + \mathbf{E}\int_{-ut}^\infty \rho_0^{fs}(y|t,0)\min(x,y+ut)\mathrm{d}y \\ &= -x + \mathbf{E}\int_{-x-ut}^{x-ut} \rho_0^{fs}(y|t,0)(y+x+ut)\mathrm{d}y + 2x\mathbf{E}\int_{x-ut}^\infty \rho_0^{fs}(y|t,0)\mathrm{d}y. \\ &= \mathbf{E}\int_{-x-ut}^{x-ut}(y+ut)\rho_0^{fs}(y|t,0)\mathrm{d}y + x\mathbf{E}\int_{x-ut}^\infty \rho_0^{fs}(y|t,0)\mathrm{d}y - x\mathbf{E}\int_{-\infty}^{-x-ut} \rho_0^{fs}(y|t,0)\mathrm{d}y.\end{aligned}$$

Since $\mathbf{E}\rho_0^{fs}(t,0;y)$ is an even probability density on $y \in \mathbb{R}$,

$$\mathbf{E}\int_{-x-ut}^{x-ut}(y+ut)\rho_0^{fs}(y|t,0)\mathrm{d}y \to ut, \qquad \text{when } x \to \infty.$$

By [29, Lemma 2.6], for any $t > 0$, $\mathbf{E}\rho_0^{fs}(y|t,0) \le C\exp(-y^2/C)$ for some positive constant $C = C(t)$. Thus we also have that

$$x\mathbf{E}\int_{x-ut}^{\infty}\rho_0^{fs}(y|t,0)\mathrm{d}y - x\mathbf{E}\int_{-\infty}^{-x-ut}\rho_0^{fs}(y|t,0)\mathrm{d}y \to 0, \qquad \text{when } x \to \infty.$$

The proof is complete. □

The next proposition is to justify the substitution. It shows that when $x \to \infty$, the statistics of $\mathcal{H}_u(t,x)$ can be approximated by the statistics of $\mathcal{H}_u^{fs}(t,x)$.

Proposition 4.4.3. *For any $u \in \mathbb{R}, t > 0$, we have*

$$\left|\mathbf{Var}[\mathcal{H}_u^{fs}(t,x)] - \mathbf{Var}[\mathcal{H}_u(t,x)]\right| \to 0, \quad \textit{as } x \to \infty; \tag{4.4.10}$$

and

$$\left|\mathbf{Cov}[\mathcal{H}_u^{fs}(t,x), W_u^{fs}(x)] - \mathbf{Cov}[\mathcal{H}_u(t,x), W_u(x)]\right| \to 0, \quad \textit{as } x \to \infty. \tag{4.4.11}$$

Proof. First, we claim that for any $u \in \mathbb{R}, t > 0$, there exists a positive constant $C = C(u,t)$ such that

$$\|\mathcal{H}_u^{fs}(t,x) - \mathcal{H}_u(t,x)\|_2 = \|H_u^{fs}(t,x) - H_u(t,x)\|_2 \le C\exp(-x^2/C), \qquad x \ge 0. \tag{4.4.12}$$

By a straightforward computation, we have

$$\mathbf{Var}[\mathcal{H}_u^{fs}(t,x)] - \mathbf{Var}[\mathcal{H}_u(t,x)] = \mathbf{E}\left([\mathcal{H}_u^{fs}(t,x) - \mathcal{H}_u(t,x)][\mathcal{H}_u^{fs}(t,x) + \mathcal{H}_u(t,x)]\right)$$

$$- \mathbf{E}[\mathcal{H}_u^{fs}(t,x) - \mathcal{H}_u(t,x)]\mathbf{E}[\mathcal{H}_u^{fs}(t,x) + \mathcal{H}_u(t,x)],$$

and it follows that

$$\left|\mathbf{Var}[\mathcal{H}_u^{fs}(t,x)] - \mathbf{Var}[\mathcal{H}_u(t,x)]\right| \le 2\|\mathcal{H}_u^{fs}(t,x) - \mathcal{H}_u(t,x)\|_2 \|\mathcal{H}_u^{fs}(t,x) + \mathcal{H}_u(t,x)\|_2.$$

When $x \ge 0$,

$$\|\mathcal{H}_u^{fs}(t,x) + \mathcal{H}_u(t,x)\|_2 \le \|H_u^{fs}(t,x)\|_2 + \|H_u^{fs}(0,x)\|_2 + \|H_u(t,x)\|_2 + \|H_u(0,x)\|_2,$$

with $H_u^{fs}(0,x) = H_u(0,x) = W_u(x)$. By (4.2.26), $\|H_u(t,x)\|_2 \le C_1 \exp(C_1 x)$ for some positive constant $C_1 = C_1(u,t)$. By a similar argument for the full-space KPZ equation, there exists another positive constant $C_2 = C_2(u,t)$ so that $\|H_u^{fs}(t,x)\|_2 \le C_2 \exp(C_2 x)$. Then the convergence in (4.4.10) follows from these bounds and (4.4.12).

Similarly, for the covariance, we have

$$\begin{aligned}
&\left|\mathbf{Cov}[\mathcal{H}_u^{fs}(t,x), W_u^{fs}(x)] - \mathbf{Cov}[\mathcal{H}_u(t,x), W_u(x)]\right| \\
&= |\mathbf{E}[\mathcal{H}_u^{fs}(t,x) W^{fs}(x)] - \mathbf{E}[\mathcal{H}_u(t,x) W(x)]| \\
&= \left|\mathbf{E}\left[\left(\mathcal{H}_u^{fs}(t,x) - \mathcal{H}_u(t,x)\right) W(x)\right]\right| \\
&\le \|\mathcal{H}_u^{fs}(t,x) - \mathcal{H}_u(t,x)\|_2 \|W(x)\|_2.
\end{aligned}$$

Thus (4.4.11) also follows from (4.4.12).

To complete the proof of the proposition, it remains to prove (4.4.12). For any positive real numbers $\alpha, \beta > 0$, we have the elementary inequality

$$|\log \alpha - \log \beta| \le \frac{|\alpha - \beta|}{\min(\alpha, \beta)} \le \frac{|\alpha - \beta|}{\alpha} + \frac{|\alpha - \beta|}{\beta}.$$

Therefore, for any $t > 0, x \ge 0$

$$\begin{aligned}
&\|H_u^{fs}(t,x) - H_u(t,x)\|_2 = \|\log Z_u^{fs}(t,x) - \log Z_{u-\frac{1}{2}}(t,x)\|_2 \\
&\le \left\| |Z_u^{fs}(t,x) - Z_{u-\frac{1}{2}}(t,x)| Z_u^{fs}(t,x)^{-1} + |Z_u^{fs}(t,x) - Z_{u-\frac{1}{2}}(t,x)| Z_{u-\frac{1}{2}}(t,x)^{-1} \right\|_2 \\
&\le \|Z_u^{fs}(t,x) - Z_{u-\frac{1}{2}}(t,x)\|_4 \|Z_u^{fs}(t,x)^{-1}\|_4 + \|Z_u^{fs}(t,x) - Z_{u-\frac{1}{2}}(t,x)\|_4 \|Z_{u-\frac{1}{2}}(t,x)^{-1}\|_4.
\end{aligned}$$

By (4.2.23) and a similar result for the full-space SHE, there exists a positive constant $C_3 = C_3(u,t)$ such that for any $x \ge 0$,

$$\|Z_{u-\frac{1}{2}}(t,x)^{-1}\|_4 \le C_3 \exp(C_3 x) \quad \text{and} \quad \|Z_u^{fs}(t,x)^{-1}\|_4 \le C_3 \exp(C_3 x).$$

Thus, the proof of (4.4.12) will be complete if we can show there exists a positive constant $C_4 = C_4(u,t)$ such that

$$\|Z_u^{fs}(t,x) - Z_{u-\frac{1}{2}}(t,x)\|_4 \le C_4 \exp(-x^2/C_4), \quad \forall x \ge 0. \tag{4.4.13}$$

Applying Lemma 4.4.4 below, we complete the proof. □

The following lemma shows that, for fixed t and large x, the solution to the SHE with Robin boundary condition does not "feel" the boundary.

Lemma 4.4.4. *For any $\tau > 0$, $u \in \mathbb{R}$, $p \geq 2$, there exists some positive constant $C = C(\tau, u, p)$ such that*

$$\|Z_u^{fs}(t,x) - Z_{u-\frac{1}{2}}(t,x)\|_p \leq C\exp(-x^2/C), \quad \text{for all } t \in [0,\tau] \text{ and } x \geq 0.$$

Proof. We use $\alpha \lesssim \beta$ to denote $\alpha \leq C\beta$ with any constant $C = C(\tau, u, p)$.

When $t = 0$, $Z_u^{fs}(0,x) = Z_{u-\frac{1}{2}}(0,x) = \exp(W_u(x))$ for any $x \geq 0$. When $t > 0$, we use the chaos expansion for the half-space SHE (4.1.3) and its full-space counterpart (4.4.6). For any $x \geq 0$, we write

$$Z_u^{fs}(t,x) - Z_{u-\frac{1}{2}}(t,x) = \sum_{k=0}^{\infty} z_k^{fs}(t,x) - \sum_{k=0}^{\infty} z_k(t,x),$$

with

$$\begin{aligned} z_0(t,x) &:= \int_0^\infty \mathscr{P}^R_{u-\frac{1}{2}}(t,x|0,y)\exp(W_u(y))\mathrm{d}y, \\ z_0^{fs}(t,x) &:= \int_{-\infty}^\infty p_t(x-y)\exp(W_u^{fs}(y))\mathrm{d}y, \end{aligned}$$

and for any $n \geq 0$,

$$\begin{aligned} z_{n+1}(t,x) &:= \int_0^t \int_0^\infty \mathscr{P}^R_{u-\frac{1}{2}}(t,x|s,w) z_n(s,w)\xi(\mathrm{d}w\mathrm{d}s), \\ z_{n+1}^{fs}(t,x) &:= \int_0^t \int_{-\infty}^\infty p_{t-s}(x-w) z_n^{fs}(s,w)\xi(\mathrm{d}w\mathrm{d}s). \end{aligned}$$

The above expansion of $Z_{u-\frac{1}{2}}$ can be obtained through the chaos expansion of the Greens function (4.2.16) together with the convolution formula (4.2.21) and the stochastic Fubini theorem. Similarly, the expansion of $Z_u^{fs}(t,x)$ follows from the convolution formula (4.4.9) together with the chaos expansion of the Green's function $\mathcal{Z}^{fs}$, as proved in [115].

We shall use the following inequalities later to estimate the difference between the integrals involving the Robin heat kernel and the ones involving the standard heat kernel. For any $t > 0$ and $x, y \geq 0$, we can bound $p_t(x+y) \leq p_t(x)e^{-\frac{y^2}{2t}}$. Also, for any $\tau > 0$, there exists some constant $C = C(\tau, u)$ so that for any $0 \leq s < t \leq \tau$ and $x, y \geq 0$,

$$\int_0^\infty p_{t-s}(x+y+z)e^{-(u-\frac{1}{2})z}\mathrm{d}z \leq p_{t-s}(x+y)\int_0^\infty e^{-\frac{z^2}{2(t-s)}}e^{-(u-\frac{1}{2})z}\mathrm{d}z \leq Cp_{t-s}(x+y)\sqrt{t-s}.$$

By the explicit expression (4.2.2) and the above estimates, we estimate the difference between z_0 and z_0^{fs} as

$$\begin{aligned}
&\|z_0(t,x) - z_0^{fs}(t,x)\|_p \\
&= \left\| \int_0^\infty \mathscr{P}^R_{u-\frac{1}{2}}(t,x|0,y)\exp(W_u(y))\mathrm{d}y - \int_{-\infty}^\infty p_t(x-y)\exp(W_u^{fs}(y))\mathrm{d}y \right\|_p \\
&= \Bigg\| \int_0^\infty \left(p_t(x+y) - (2u-1)\int_0^\infty p_t(x+y+z)e^{-(u-\frac{1}{2})z}\mathrm{d}z \right) e^{W_u(y)}\mathrm{d}y \\
&- \int_{-\infty}^0 p_t(x-y)e^{W_u^{fs}(y)}\mathrm{d}y \Bigg\|_p \\
&\leq \int_0^\infty \left| p_t(x+y) - (2u-1)\int_0^\infty p_t(x+y+z)e^{-(u-\frac{1}{2})z}\mathrm{d}z \right| \|e^{W_u(y)}\|_p \mathrm{d}y \\
&+ \int_{-\infty}^0 p_t(x-y)\|e^{W_u^{fs}(y)}\|_p\mathrm{d}y \\
&\leq p_t(x)(1+|2u-1|C\sqrt{t})\int_0^\infty e^{-\frac{y^2}{2t}}\|\exp(W_u(y))\|_p\mathrm{d}y + p_t(x)\int_0^\infty e^{-\frac{y^2}{2t}}\|\exp(W_u^{fs}(-y))\|_p\mathrm{d}y \\
&\lesssim p_t(x)\sqrt{t}.
\end{aligned} \tag{4.4.14}$$

Now for any $n \geq 0$,

$$
\begin{aligned}
&z_{n+1}(t,x) - z^{fs}_{n+1}(t,x) \\
&= \int_0^t \int_0^\infty \mathscr{P}^R_{u-\frac{1}{2}}(t,x|s,w) z_n(s,w)\xi(\mathrm{d}w\mathrm{d}s) - \int_0^t \int_{-\infty}^\infty p_{t-s}(x-w) z^{fs}_n(s,w)\xi(\mathrm{d}w\mathrm{d}s) \\
&= I_n(t,x) + J_n(t,x),
\end{aligned}
$$

with

$$
\begin{aligned}
I_n(t,x) &:= \int_0^t \int_0^\infty p_{t-s}(x-w)[z_n(s,w) - z^{fs}_n(s,w)]\xi(\mathrm{d}w\mathrm{d}s), \\
\text{and} \quad J_n(t,x) & \\
&:= \int_0^t \int_0^\infty \left(p_{t-s}(x+w) - (2u-1)\int_0^\infty p_{t-s}(x+w+z)e^{-(u-\frac{1}{2})z}\mathrm{d}z \right) z_n(s,w)\xi(\mathrm{d}w\mathrm{d}s) \\
&- \int_0^t \int_{-\infty}^0 p_{t-s}(x-w) z^{fs}_n(s,w)\xi(\mathrm{d}w\mathrm{d}s).
\end{aligned}
$$

While I_n is not a martingale in t, the process

$$
I_n(v) := \int_0^v \int_0^\infty p_{t-s}(x-w)[z_n(s,w) - z^{fs}_n(s,w)]\xi(\mathrm{d}w\mathrm{d}s)
$$

is a martingale in v so we could apply the Burkholder-Davis-Gundy inequality to obtain

$$
\|I_n(t,x)\|_p^2 \lesssim \int_0^t \int_0^\infty p_{t-s}(x-w)^2 \|z_n(s,w) - z^{fs}_n(s,w)\|_p^2 \mathrm{d}s\mathrm{d}w.
$$

Similarly, we can convert the two terms in J_n into martingales and apply the Burkholder-Davis-

Gundy inequality:

$$
\begin{aligned}
&\|J_n(t,x)\|_p^2 \lesssim \left\| \int_0^t \int_{-\infty}^0 p_{t-s}(x-w)^2 z_n^{fs}(s,w)^2 \mathrm{d}s\mathrm{d}w \right\|_{p/2} \\
&+ \left\| \int_0^t \int_0^\infty \left(p_{t-s}(x+w) - (2u-1) \int_0^\infty p_{t-s}(x+w+z) e^{-(u-\frac{1}{2})z} \mathrm{d}z \right)^2 z_n(s,w)^2 \mathrm{d}s\mathrm{d}w \right\|_{p/2} \\
&\le \int_0^t \int_0^\infty p_{t-s}(x+w)^2 \left\| z_n^{fs}(s,-w) \right\|_p^2 \mathrm{d}s\mathrm{d}w \\
&+ \int_0^t \int_0^\infty \left(p_{t-s}(x+w) - (2u-1) \int_0^\infty p_{t-s}(x+w+z) e^{-(u-\frac{1}{2})z} \mathrm{d}z \right)^2 \|z_n(s,w)\|_p^2 \mathrm{d}s\mathrm{d}w \\
&\lesssim \int_0^t \int_0^\infty p_{t-s}(x+w)^2 \|z_n(s,w)\|_p^2 \mathrm{d}s\mathrm{d}w + \int_0^t \int_0^\infty p_{t-s}(x+w)^2 \left\| z_n^{fs}(s,-w) \right\|_p^2 \mathrm{d}s\mathrm{d}w.
\end{aligned}
$$

Thus we have achieved an intermediate result

$$
\begin{aligned}
&\|z_{n+1}(t,x) - z_{n+1}^{fs}(t,x)\|_p^2 \le \left(\|I_n(t,x)\|_p + \|J_n(t,x)\|_p \right)^2 \\
&\lesssim \int_0^t \int_0^\infty p_{t-s}(x-w)^2 \|z_n(s,w) - z_n^{fs}(s,w)\|_p^2 \mathrm{d}s\mathrm{d}w \\
&+ \int_0^t \int_0^\infty p_{t-s}(x+w)^2 \|z_n(s,w)\|_p^2 \mathrm{d}s\mathrm{d}w + \int_0^t \int_0^\infty p_{t-s}(x+w)^2 \|z_n^{fs}(s,-w)\|_p^2 \mathrm{d}s\mathrm{d}w.
\end{aligned} \tag{4.4.15}
$$

We first estimate the last two terms in the right-hand side of (4.4.15). By a similar iteration as in the proof of Lemma 4.2.4, for any $s \in [0,\tau], w \ge 0$, there exists a constant $C = C(\tau, p, u)$ such that (with the convention of $(n/2)! = \Gamma(\frac{n}{2}+1)$),

$$
\|z_n(s,w)\|_p^2 \le \frac{(Cs)^{(n-1)/2}}{(n/2)!} e^{Cw} \int_0^\infty \mathscr{P}^R_{u-\frac{1}{2}}(s,w|0,y) \left\| \exp(W_u(y)) \right\|_p \mathrm{d}y \le \frac{(Cs)^{(n-1)/2}}{(n/2)!} e^{2Cw}.
$$

It then follows that

$$\int_0^t \int_0^\infty p_{t-s}(x+w)^2 \|z_n(s,w)\|_p^2 \mathrm{d}s\mathrm{d}w \le \int_0^t \int_0^\infty p_{t-s}(x+w)^2 \frac{(Cs)^{(n-1)/2}}{(n/2)!} e^{2Cw} \mathrm{d}s\mathrm{d}w$$
$$\le \frac{(Ct)^{(n-1)/2}}{(n/2)!} \int_0^t p_{t-s}(x)^2 \int_0^\infty e^{-\frac{w^2}{2(t-s)}} e^{2Cw} \mathrm{d}w\mathrm{d}s \lesssim \frac{(Ct)^{(n-1)/2}}{(n/2)!} \int_0^t p_{t-s}(x)^2 \sqrt{t-s}\mathrm{d}s$$
$$\lesssim \frac{(Ct)^{n/2}}{(n/2)!} \left[\sqrt{t} p_t(x)\right]^2.$$

A similar result can be obtained for z_n^{fs}. We record that, for $0 < t \le \tau$ and $x \ge 0$,

$$\int_0^t \int_0^\infty p_{t-s}(x+w)^2 \|z_n^{fs}(s,-w)\|_p^2 \mathrm{d}s\mathrm{d}w \lesssim \frac{(Ct)^{n/2}}{(n/2)!} \left[\sqrt{t} p_t(x)\right]^2,$$

for some positive constant $C = C(\tau, p, u)$. With these results, we could further bound (4.4.15) by

$$\|z_{n+1}(t,x) - z_{n+1}^{fs}(t,x)\|_p^2$$
$$\lesssim \frac{(Ct)^{n/2}}{(n/2)!} \left[\sqrt{t} p_t(x)\right]^2 + \int_0^t \int_0^\infty p_{t-s}(x-w)^2 \|z_n(s,w) - z_n^{fs}(s,w)\|_p^2 \mathrm{d}s\mathrm{d}w,$$

with some constant $C = C(\tau, p, u)$. Using the estimate (4.4.14), we iterate the above inequality to derive

$$\|z_{n+1}(t,x) - z_{n+1}^{fs}(t,x)\|_p^2 \le C(n+2) \frac{(Ct)^{n/2}}{(n/2)!} \left[\sqrt{t} p_t(x)\right]^2,$$

for some $C = C(\tau, p, u)$. Thus for any $t \in (0, \tau]$,

$$\|Z_u^{fs}(t,x) - Z_{u-\frac{1}{2}}(t,x)\|_p \le \sum_{k=0}^\infty \|z_k(t,x) - z_k^{fs}(t,x)\|_p \le \sum_{k=0}^\infty \left((k+2)C\frac{(Ct)^{k/2}}{(k/2)!}\right)^{1/2} \sqrt{t} p_t(x).$$

The proof is complete as the sum of the infinite series is bounded. □

4.5 Endpoint displacement: proof of Theorem 4.1.2

In this section, we prove (4.1.9), which provides an upper bound on the first moment of the polymer endpoint displacement in the bounded phase (when the boundary parameter $u < 0$). Combining with the variance identity, this leads to the estimates on the fluctuations of the height function. The main contribution here is to leverage stochastic dominance to derive an upper bound of the annealed mean of the CDRP endpoint displacement in the bounded phase (when $u < 0$). The upper bound is expected to be sharp since it is conjectured to be the limit of the annealed mean when $t \to \infty$ (see Remark 4.1.9).

The quantity of interest $\mathbf{E}\mathbb{E}_0^{\mathbb{Q}^{u,t}}[X_t]$ is the annealed average of the polymer endpoint. By our choice of the stationary initial data for KPZ, the polymer path $\{X_s\}_{s\in[0,t]}$, starting from $(t,0)$ and running backward in time, has a stationary terminal condition. Stochastic dominance ensures that the displacement of the endpoint in this case can be bounded from above by the endpoint of a different polymer path, one that starts from stationarity. In other words, we use the polymer measure with both endpoints sampled from stationarity as a comparison. The upper bound in $\mathbf{E}\mathbb{E}_0^{\mathbb{Q}^{u,t}}[X_t] \le \psi'(-2u)$ is derived from this stationarity-to-stationarity polymer measure.

For any $u \in \mathbb{R}, t > 0, x \in [0,\infty)$, the half-space CDRP $\mathbb{Q}_x^{u,t}$ defined in Definition 4.2.9 is a point-to-measure polymer with initial point x and terminal data $\exp(W_u(x))\,\mathrm{d}x$. When $u < 0$, the terminal data is $\mathbf{P}$-almost surely in $L^1([0,\infty))$, thus we can also think of $\mathbb{Q}_x^{u,t}$ as a polymer with the normalized terminal data

$$\frac{\exp(W_u(x))\,\mathrm{d}x}{\int_0^\infty \exp(W_u(x'))\,\mathrm{d}x'}. \tag{4.5.1}$$

Now let $\tilde{W}$ be another standard Brownian motion defined on probability space $(\Omega, \mathscr{F}, \mathbf{P})$, independent of ξ and W. When $u < 0$, for $\mathbf{P}$-almost surely all realization of $\tilde{W}$, we have $\int_0^\infty \exp\left(\tilde{W}_u(x)\right) \mathrm{d}x < \infty$ and one can sample the initial point of the polymer $\mathbb{Q}_x^{u,t}$ from the density

$$\frac{\exp\left(\tilde{W}_u(x)\right) \mathrm{d}x}{\int_0^\infty \exp\left(\tilde{W}_u(x')\right) \mathrm{d}x'} \tag{4.5.2}$$

to construct another polymer $\mathbb{Q}_{\tilde{W}_u}^{u,t}$. By (4.1.4) (and thus (4.3.2)), the polymer $\mathbb{Q}_{\tilde{W}_u}^{u,t}$ has the special property that the start-point and end-point distributions of the paths are both stationary but independent of each other as well as the random environment. For $\mathbf{P}$-almost sure all realization of ξ, W and $\tilde{W}$, the (quenched) endpoint density of this polymer is

$$\tilde{\rho}_u^R(y|t) := \frac{\int_0^\infty \mathcal{Z}_{u-\frac{1}{2}}(t,x|0,y) \exp(\tilde{W}_u(x)) \mathrm{d}x \exp(W_u(y))}{\int_0^\infty \int_0^\infty \mathcal{Z}_{u-\frac{1}{2}}(t,x'|0,y') \exp(\tilde{W}_u(x')) \mathrm{d}x' \exp(W_u(y')) \mathrm{d}y'}, \quad \forall y \in [0,\infty). \tag{4.5.3}$$

As discussed in Remark 4.2.12, since we do not show the simultaneous coupling of CDRP $\mathbb{Q}_x^{u,t}$ for all $x \in [0,\infty)$, it is not immediately clear that such a construction by sampling the start-point gives a measure-to-measure CDRP $\mathbb{Q}_{\tilde{W}_u}^{u,t}$ on an event of probability one. For our purpose, we do not need to check this. Throughout the proof, we only need the endpoint density formula (4.5.3) to be $\mathbf{P}$-almost surely well-defined, yet it is intuitive to understand this density from the "polymer measure" $\mathbb{Q}_{\tilde{W}_u}^{u,t}$.

We verify that the endpoint density (4.5.3) is $\mathbf{P}$-almost surely well-defined. By (4.3.2), the denominator in (4.5.3) multiplied by $Z_{u-\frac{1}{2}}(t,0)^{-1}$ is

$$Z_{u-\frac{1}{2}}(t,0)^{-1} \int_0^\infty Z_{u-\frac{1}{2}}(t,x) \exp(\tilde{W}_u(x)) \mathrm{d}x \overset{\text{law}}{=} \int_0^\infty \exp(W_u(x) + \tilde{W}_u(x)) \mathrm{d}x. \tag{4.5.4}$$

The right-hand side is a $\mathbf{P}$-almost surely positive and finite random variable when $u < 0$. Since $Z_{u-\frac{1}{2}}(t,0)$ is also $\mathbf{P}$-almost surely positive and finite by (4.2.24), the denominator in (4.5.3) is $\mathbf{P}$-almost surely positive and finite. The density $\tilde{\rho}_u^R(y|t)$ is thus well-defined.

Let us first briefly explain how to prove (4.1.9) through the density (4.5.3). As we shall see below in Corollary 4.5.2, the polymer measures with different starting points $\{\mathbb{Q}_x^{u,t}\}_{x\in[0,\infty)}$ satisfies a stochastic monotonicity inherited from their path continuity and the planar structure of half-space. Intuitively, for any fixed environment noise ξ and W, two independent polymer paths X^1, X^2 starting from points x_1, x_2 with $x_1 < x_2$ should satisfy $\rho^1 \le_{st} \rho^2$, where $\le_{st}$ refers to stochastic dominance and ρ^i is the endpoint density. Without an established strong Markov property for the half-space CDRP, we take a detour to prove this stochastic monotonicity through a Karlin-McGregor based argument. In particular, we analyze some determinants formed by four copies of Green's functions, working with a smoothed noise to utilize the Feynman-Kac formula before passing to the limit. This argument gives $\mathbf{P}$-almost sure comparison between products of Green's functions, which can be further used to prove the relation $\rho^1 \le_{st} \rho^2$.

We then compare the quenched densities $\rho_u^R(\cdot|t,0)$ and $\tilde{\rho}_u^R(\cdot|t)$ for $\mathbf{P}$-almost surely all $\omega \in \Omega$. We know that $\rho_u^R(\cdot|t,0)$ is the endpoint density of polymer paths starting from $x = 0$, while $\tilde{\rho}_u^R(\cdot|t)$ is the endpoint density of polymer paths with starting points sampled from the density (4.5.2) on $[0,\infty)$. It shall follow that $\rho_u^R(\cdot|t,0) \le_{st} \tilde{\rho}_u^R(\cdot|t)$ $\mathbf{P}$-almost surely, and as a corollary,

$$\mathbf{E}\mathbb{E}_0^{\mathbb{Q}^{u,t}}[X_t] = \mathbf{E}\int_0^\infty y\rho_u^R(y|t,0)\mathrm{d}y \le \mathbf{E}\int_0^\infty y\tilde{\rho}_u^R(y|t)\mathrm{d}y = \mathbf{E}\mathbb{E}_{\tilde{W}_u}^{\mathbb{Q}^{u,t}}[X_t].$$

On the other hand, since $\mathbb{Q}_{\tilde{W}_u}^{u,t}$ has the initial point sampled from the stationary density, we can compute exactly $\mathbf{E}\mathbb{E}_{\tilde{W}_u}^{\mathbb{Q}^{u,t}}(X_t)$ for any $t > 0$. These two facts combined together lead to the upper bound in (4.1.9). Since $\exp(W_u(x))\,\mathrm{d}x$ is not in $L^1([0,\infty))$ when $u \geq 0$, the proof in this section does not extend to $u \geq 0$.

We now prove the aforementioned results in sequence. We begin by showing that some specific determinants consisting of four copies of Green's functions are $\mathbf{P}$-almost surely nonnegative. This argument is an adaption of [116, Proposition 5.5] to the half-space setting.

Lemma 4.5.1. *Fix $\mu \in \mathbb{R}$ and $t > 0$. For any $0 \leq x_1 < x_2$ and $0 \leq y_1 < y_2$, we have*

$$\det[\mathcal{Z}_\mu(t, x_i|0, y_j)]_{i,j\in\{1,2\}} \geq 0 \quad \mathbf{P}\text{-almost surely}. \tag{4.5.5}$$

Proof. We will use an argument based on the Karlin-McGregor formula. We first prove the inequality (4.5.5) for the Green's functions of half-space SHE with a smoothed noise, using a Feynman-Kac representation. We then pass to the limit to obtain the same result for the white noise using Lemma 4.8.1.

Let $\xi_\delta(t,x) = \int_{\mathbb{R}} p_\delta(x-y)\xi(t,\mathrm{d}y)$ so that it is a Gaussian noise that is white in time and smooth in space, with the covariance function

$$Q_\delta(t_1,t_2,x_1,x_2) = \mathbf{E}\xi_\delta(t_1,x_1)\xi_\delta(t_2,x_2) = \delta_0(t_1-t_2)p_{2\delta}(x_1-x_2).$$

For any $\delta > 0$, $x, y \in [0,\infty)$, define

$$\mathcal{Z}_\mu^\delta(t,x|0,y) := \mathscr{P}^N(t,x|0,y)\mathbb{E}_B^x\left[\exp\left(-\mu L_t^0 + \int_0^t \xi_\delta(t-r,|B_r|)\mathrm{d}r - \frac{t}{2}p_{2\delta}(0)\right)\middle||B_t| = y\right].$$

As in (4.2.5), we rewrite the right-hand side in terms of the "reflected Brownian bridges". Let Υ_t be the reflected Brownian motion with $\Upsilon_t := |B_t|$ starting from $\Upsilon_0 = |B_0| = x$. For any $t > 0, y \geq 0$, there exists a unique probability measure $\mathbb{P}_{\Upsilon}^{x,y}$ on $\mathcal{C}([0,t],[0,\infty))$ such that $\mathbb{P}_{\Upsilon}^{x,y}$ equals to the conditional probability distribution $\mathbb{P}_{\Upsilon}^{x}(\cdot|\Upsilon_t = y)$. We call the canonical process associated with $\mathbb{P}_{\Upsilon}^{x,y}$ the "reflected Brownian bridge", starting from $\Upsilon_0 = x$ and ending at $\Upsilon_t = y$. The above display equals to

$$\mathscr{P}^N(t,x|0,y)\mathbb{E}_{\Upsilon}^{x,y}\left[\exp\left(-\mu L_t^{0,\Upsilon} + \int_0^t \xi_\delta(t-r,\Upsilon_r)\mathrm{d}r - \frac{t}{2}p_{2\delta}(0)\right)\right],$$

where we used $L_t^{0,\Upsilon}$ to denote the local time at zero:

$$L_t^{0,\Upsilon} := \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_0^t \mathbb{1}_{[0,\varepsilon]}(\Upsilon_s)\,\mathrm{d}s. \tag{4.5.6}$$

For discussions on reflected Brownian bridges, we refer to [117] in which the authors discussed general Markovian bridges, and also [118, 119] and the references therein.

To ease notations, we define

$$F_t^\delta(\Upsilon) := \exp\left(-\mu L_t^{0,\Upsilon} + \int_0^t \xi_\delta(t-r,\Upsilon_r)\mathrm{d}r - \frac{t}{2}p_{2\delta}(0)\right).$$

Then $F_t^\delta(\cdot)$ is an $\mathbf{P}$-almost surely continuous, strictly positive and multiplicative functional of Υ.

Next we prove that for any $\delta > 0$, $0 \leq x_1 < x_2$ and $0 \leq y_1 < y_2$, the following identity holds $\mathbf{P}$-almost surely:

$$\det[\mathcal{Z}_\mu^\delta(t,x_i|0,y_j)]_{i,j\in\{1,2\}} = \det[\mathscr{P}^N(t,x_i|0,y_j)]_{i,j\in\{1,2\}}\mathbb{E}_{\Xi^1,\Xi^2}^{\mathbf{x},\mathbf{y}}[F_t^\delta(\Xi^1)F_t^\delta(\Xi^2)], \tag{4.5.7}$$

where the expectation $\mathbb{E}^{\mathbf{x},\mathbf{y}}_{\Xi^1,\Xi^2}$ is over the two-dimensional reflected Brownian bridges (Ξ^1, Ξ^2) started from $\mathbf{x} = (x_1, x_2)$, ending at $\mathbf{y} = (y_1, y_2)$ at time t, and Ξ^1, Ξ^2 being non-intersecting on $[0, t]$.

To prove (4.5.7), by the continuity of the Green's function, it suffices to show that for arbitrary functions $\varphi_1, \varphi_2 \in \mathcal{C}_c([0,\infty), \mathbb{R})$ with coordinate-wise ordered supports (which means that for any $y_1 \in \operatorname{supp} \varphi_1$ and $y_2 \in \operatorname{supp} \varphi_2$, we have $y_1 < y_2$), there is the identity

$$\begin{aligned}&\int_0^\infty \int_0^\infty \varphi_1(y_1)\varphi_2(y_2) \det[\mathcal{Z}_\mu(t, x_i|0, y_j)]_{i,j\in\{1,2\}} \mathrm{d}y_1 \mathrm{d}y_2 \\ &= \int_0^\infty \int_0^\infty \varphi_1(y_1)\varphi_2(y_2) \det[\mathscr{P}^N(t, x_i|0, y_j)]_{i,j\in\{1,2\}} \mathbb{E}^{\mathbf{x},\mathbf{y}}_{\Xi^1,\Xi^2}[F^\delta_t(\Xi^1) F^\delta_t(\Xi^2)] \mathrm{d}y_1 \mathrm{d}y_2.\end{aligned} \tag{4.5.8}$$

By Karlin-McGregor formula [120], the determinant $\det[\mathscr{P}^N(t, x_i|0, y_j)]_{i,j\in\{1,2\}}$ is the transition density at time t of a two-dimensional reflected Brownian motion (Υ^1, Υ^2) in the domain $\Lambda = \{\mathbf{y} = (y_1, y_2) \in \mathbb{R}^2_{\geq 0} : y_1 \leq y_2\}$ killed when Υ^1 and Υ^2 first intersect. It then follows that the right-hand side of (4.5.8) equals to

$$\mathbb{E}^{\mathbf{x}}_{\Upsilon^1,\Upsilon^2}[\varphi_1(\Upsilon^1_t)\varphi_2(\Upsilon^2_t) F^\delta_t(\Upsilon^1) F^\delta_t(\Upsilon^2) \mathbb{1}(\tau > t)],$$

where the expectation $\mathbb{E}^{\mathbf{x}}_{\Upsilon^1,\Upsilon^2}$ is taken over the two-dimensional reflected Brownian motions (Υ^1, Υ^2) in $\mathbb{R}^2_{\geq 0}$ started at $\mathbf{x} = (x_1, x_2)$, and τ is the stopping time taking values in $[0, t] \cup \{\infty\}$ defined by

$$\tau = \inf \left\{ r \in [0, t] : \Upsilon^1_r = \Upsilon^2_r \right\}.$$

On the other hand, by expanding the determinant $\det[\mathcal{Z}_\mu(t, x_i|0, y_j)]_{i,j\in\{1,2\}}$, the left-hand side of (4.5.8) equals to

$$\mathbb{E}^{\mathbf{x}}_{\Upsilon^1,\Upsilon^2}[\varphi_1(\Upsilon^1_t)\varphi_2(\Upsilon^2_t)F^\delta_t(\Upsilon^1)F^\delta_t(\Upsilon^2)] - \mathbb{E}^{\mathbf{x}}_{\Upsilon^1,\Upsilon^2}[\varphi_2(\Upsilon^1_t)\varphi_1(\Upsilon^2_t)F^\delta_t(\Upsilon^1)F^\delta_t(\Upsilon^2)].$$

Recall that the reflected Brownian motions Υ^1, Υ^2 started from ordered initial points $x_1 < x_2$, and the support of φ_1, φ_2 are also ordered. If $\tau > t$, the paths of Υ^1, Υ^2 do not cross each other on $[0,t]$. Since the half-space domain is planar, the path continuity of the reflected Brownian motions forces that

$$\mathbb{E}^{\mathbf{x}}_{\Upsilon^1,\Upsilon^2}[\varphi_2(\Upsilon^1_t)\varphi_1(\Upsilon^2_t)F^\delta_t(\Upsilon^1)F^\delta_t(\Upsilon^2)\mathbb{1}(\tau > t)] = 0.$$

Combining the above results, the identity (4.5.8) is now equivalent to

$$\begin{aligned} &\mathbb{E}^{\mathbf{x}}_{\Upsilon^1,\Upsilon^2}[\varphi_1(\Upsilon^1_t)\varphi_2(\Upsilon^2_t)F^\delta_t(\Upsilon^1)F^\delta_t(\Upsilon^2)\mathbb{1}(\tau \le t)] \\ &- \mathbb{E}^{\mathbf{x}}_{\Upsilon^1,\Upsilon^2}[\varphi_2(\Upsilon^1_t)\varphi_1(\Upsilon^2_t)F^\delta_t(\Upsilon^1)F^\delta_t(\Upsilon^2)\mathbb{1}(\tau \le t)] = 0. \end{aligned} \tag{4.5.9}$$

We prove (4.5.9) by using a path-switching argument. Define a new two-dimensional reflected Brownian motion on $r \in [0,t]$ by

$$(\tilde{\Upsilon}^1_r, \tilde{\Upsilon}^2_r) = \begin{cases} (\Upsilon^1_r, \Upsilon^2_r) & r \le \tau \\ (\Upsilon^2_r, \Upsilon^1_r) & r \ge \tau. \end{cases}$$

The paths of $\tilde{\Upsilon}^1, \tilde{\Upsilon}^2$ intersect on $[0,t]$ if and only if the paths of Υ^1, Υ^2 intersect on $[0,t]$. Since the functional $F_t^\delta(\cdot)$ is multiplicative, for each realization of (Υ^1, Υ^2),

$$F_t^\delta(\tilde{\Upsilon}^1)F_t^\delta(\tilde{\Upsilon}^2) = F_t^\delta(\Upsilon^1)F_t^\delta(\Upsilon^2).$$

On the event of $\{\tau \le t\}$, $\varphi_1(\tilde{\Upsilon}_t^1)\varphi_2(\tilde{\Upsilon}_t^2) = \varphi_1(\Upsilon_t^2)\varphi_2(\Upsilon_t^1)$. It follows that

$$\begin{aligned}&\mathbb{E}^{\mathbf{x}}_{\tilde{\Upsilon}^1,\tilde{\Upsilon}^2}[\varphi_1(\tilde{\Upsilon}_t^1)\varphi_2(\tilde{\Upsilon}_t^2)F_t^\delta(\tilde{\Upsilon}^1)F_t^\delta(\tilde{\Upsilon}^2)\mathbb{1}(\tau \le t)]\\ &= \mathbb{E}^{\mathbf{x}}_{\Upsilon^1,\Upsilon^2}[\varphi_1(\Upsilon_t^2)\varphi_2(\Upsilon_t^1)F_t^\delta(\Upsilon^1)F_t^\delta(\Upsilon^2)\mathbb{1}(\tau \le t)].\end{aligned}$$

Moreover, as the reflected Brownian motion is strong Markov, $(\tilde{\Upsilon}^1, \tilde{\Upsilon}^2)$ has the same law as (Υ^1, Υ^2), which implies that

$$\begin{aligned}&\mathbb{E}^{\mathbf{x}}_{\Upsilon^1,\Upsilon^2}[\varphi_1(\Upsilon_t^1)\varphi_2(\Upsilon_t^2)F_t^\delta(\Upsilon^1)F_t^\delta(\Upsilon^2)\mathbb{1}(\tau \le t)]\\ &= \mathbb{E}^{\mathbf{x}}_{\tilde{\Upsilon}^1,\tilde{\Upsilon}^2}[\varphi_1(\tilde{\Upsilon}_t^1)\varphi_2(\tilde{\Upsilon}_t^2)F_t^\delta(\tilde{\Upsilon}^1)F_t^\delta(\tilde{\Upsilon}^2)\mathbb{1}(\tau \le t)].\end{aligned}$$

The above two identities prove (4.5.9).

Now we have proved (4.5.8) for any $\varphi_1, \varphi_2 \in \mathcal{C}_c([0,\infty),\mathbb{R})$ with coordinate-wise ordered supports. The identity (4.5.7) follows. When $0 \le x_1 < x_2$ and $0 \le y_1 < y_2$,

$$\det[\mathscr{P}^N(t, x_i|0, y_j)]_{i,j\in\{1,2\}}$$

is strictly positive. Thus for any $\delta > 0$, $0 \le x_1 < x_2$ and $0 \le y_1 < y_2$, we have

$$\det[\mathcal{Z}_\mu^\delta(t, x_i|0, y_j)]_{i,j\in\{1,2\}} > 0 \quad \mathbf{P}\text{-almost surely.}$$

By Lemma 4.8.1 (i) (with a mollification in space only), for any $t > 0, x, y \in [0, \infty)$, $p \in [1, \infty)$,

$$\mathcal{Z}_\mu^\delta(t, x|0, y) \to \mathcal{Z}_\mu(t, x|0, y) \quad \text{in } L^p(\Omega) \quad \text{as } \delta \to 0.$$

The result (4.5.5) is thus proved. □

With Lemma 4.5.1, we are ready to prove the stochastic monotonicity.

Corollary 4.5.2. *For any $u \in \mathbb{R}$, $t > 0$, $0 \le x_1 < x_2$, there exists an event Ω_0 with $\mathbf{P}(\Omega_0) = 1$ such that for all $\omega \in \Omega_0$ and $a > 0$,*

$$\mathbb{Q}_{x_1}^{u,t}(X_t \in [0, a]) = \int_0^a \rho_u^R(y|t, x_1)\mathrm{d}y \ge \int_0^a \rho_u^R(y|t, x_2)\mathrm{d}y = \mathbb{Q}_{x_2}^{u,t}(X_t \in [0, a]). \tag{4.5.10}$$

As a result, for any fixed $u < 0$, $t > 0$,

$$\mathbb{E}_0^{\mathbb{Q}^{u,t}}[X_t] = \int_0^\infty y\rho_u^R(y|t, 0)\mathrm{d}y \le \int_0^\infty y\tilde{\rho}_u^R(y|t)\mathrm{d}y, \quad \mathbf{P}\textit{-almost surely}, \tag{4.5.11}$$

where $\tilde{\rho}_u^R(\cdot|t)$ is as defined in (4.5.2).

Proof. By Lemma 4.5.1, for any fixed $u \in \mathbb{R}$, $t > 0$, $0 \le x_1 < x_2$, with $\mathbf{P}$ probability one,

$$\mathcal{Z}_{u-\frac{1}{2}}(t, x_1|0, y_1)\mathcal{Z}_{u-\frac{1}{2}}(t, x_2|0, y_2) \ge \mathcal{Z}_{u-\frac{1}{2}}(t, x_1|0, y_2)\mathcal{Z}_{u-\frac{1}{2}}(t, x_2|0, y_1),$$

for all $0 \le y_1 < y_2$ simultaneously (one can first consider rational y_1, y_2 then use the continuity of $\mathcal{Z}$). By (4.2.24), $\{\mathcal{Z}_{u-\frac{1}{2}}(t, x_i|0, y)\}_{i=1,2}$ are $\mathbf{P}$-almost surely integrable with respect to the density (4.5.1). Therefore, there exists an event Ω_0 with $\mathbf{P}(\Omega_0) = 1$ such that for any $\omega \in \Omega_0$, we can integrate over any rectangles $(y_1, y_2) \in [0,a) \times [a,\infty)$ with $a > 0$ to obtain

$$\begin{aligned}
&\int_0^a \mathcal{Z}_{u-\frac{1}{2}}(t,x_1|0,y_1)\exp(W_u(y_1))\mathrm{d}y_1 \int_a^\infty \mathcal{Z}_{u-\frac{1}{2}}(t,x_2|0,y_2)\exp(W_u(y_2))\mathrm{d}y_2 \\
&\geq \int_a^\infty \mathcal{Z}_{u-\frac{1}{2}}(t,x_1|0,y_2)\exp(W_u(y_2))\mathrm{d}y_2 \int_0^a \mathcal{Z}_{u-\frac{1}{2}}(t,x_2|0,y_1)\exp(W_u(y_1))\mathrm{d}y_1,
\end{aligned}$$

and both sides are positive and finite. Add

$$\int_0^a \mathcal{Z}_{u-\frac{1}{2}}(t,x_1|0,y)\exp(W_u(y))\mathrm{d}y \int_0^a \mathcal{Z}_{u-\frac{1}{2}}(t,x_2|0,y)\exp(W_u(y))\mathrm{d}y$$

to both sides and then divide by the product $Z_{u-\frac{1}{2}}(t,x_1)Z_{u-\frac{1}{2}}(t,x_2)$, we obtain (4.5.10).

To prove (4.5.11), by (4.5.10), for any fixed $x > 0$,

$$\begin{aligned}
&\int_0^\infty y\rho_u^R(y|t,0)\mathrm{d}y = \frac{\int_0^\infty y\mathcal{Z}_{u-\frac{1}{2}}(t,0|0,y)\exp(W_u(y))\mathrm{d}y}{\int_0^\infty \mathcal{Z}_{u-\frac{1}{2}}(t,0|0,y')\exp(W_u(y'))\mathrm{d}y'} \\
&\leq \int_0^\infty y\rho_u^R(y|t,x)\mathrm{d}y = \frac{\int_0^\infty y\mathcal{Z}_{u-\frac{1}{2}}(t,x|0,y)\exp(W_u(y))\mathrm{d}y}{\int_0^\infty \mathcal{Z}_{u-\frac{1}{2}}(t,x|0,y')\exp(W_u(y'))\mathrm{d}y'} \quad \mathbf{P}\text{-almost surely.}
\end{aligned}$$

Multiplying the positive denominators on both sides, this is equivalent to the inequality

$$\begin{aligned}
&\int_0^\infty y\mathcal{Z}_{u-\frac{1}{2}}(t,0|0,y)\exp(W_u(y))\mathrm{d}y \int_0^\infty \mathcal{Z}_{u-\frac{1}{2}}(t,x|0,y')\exp(W_u(y'))\mathrm{d}y' \\
&\leq \int_0^\infty \mathcal{Z}_{u-\frac{1}{2}}(t,0|0,y')\exp(W_u(y'))\mathrm{d}y' \int_0^\infty y\mathcal{Z}_{u-\frac{1}{2}}(t,x|0,y)\exp(W_u(y))\mathrm{d}y.
\end{aligned} \tag{4.5.12}$$

By Proposition 4.2.7 (ii), both sides are continuous in $x \in [0,\infty)$ $\mathbf{P}$-almost surely as the integrals are mild solutions to SHE. It follows that (4.5.12) holds for all $x \in [0,\infty)$ simultaneously.

When $u < 0$, by (4.5.4), the left-hand side of (4.5.12) is integrable in x with respect to the density (4.5.2) $\mathbf{P}$-almost surely. Similar integrability holds for the right-hand side. Thus with probability one, we have an inequality

$$\begin{aligned}
&\int_0^\infty y\mathcal{Z}_{u-\frac{1}{2}}(t,0|0,y)\exp(W_u(y))\mathrm{d}y \\
&\int_0^\infty\int_0^\infty \mathcal{Z}_{u-\frac{1}{2}}(t,x|0,y')\exp(W_u(y'))\mathrm{d}y'\exp(\tilde{W}_u(x))\mathrm{d}x \\
\le&\int_0^\infty \mathcal{Z}_{u-\frac{1}{2}}(t,0|0,y')\exp(W_u(y'))\mathrm{d}y' \\
&\int_0^\infty\int_0^\infty y\mathcal{Z}_{u-\frac{1}{2}}(t,x|0,y)\exp(W_u(y))\mathrm{d}y\exp(\tilde{W}_u(x))\mathrm{d}x,
\end{aligned}$$

with all the integrals being positive and finite. This is equivalent to

$$\begin{aligned}
&\frac{\int_0^\infty y\mathcal{Z}_{u-\frac{1}{2}}(t,0|0,y)\exp(W_u(y))\mathrm{d}y}{\int_0^\infty \mathcal{Z}_{u-\frac{1}{2}}(t,0|0,y')\exp(W_u(y'))\mathrm{d}y'} \\
\le&\frac{\int_0^\infty\int_0^\infty y\mathcal{Z}_{u-\frac{1}{2}}(t,x|0,y)\exp(W_u(y))\exp(\tilde{W}_u(x))\mathrm{d}y\mathrm{d}x}{\int_0^\infty\int_0^\infty \mathcal{Z}_{u-\frac{1}{2}}(t,x'|0,y')\exp(W_u(y'))\exp(\tilde{W}_u(x'))\mathrm{d}y'\mathrm{d}x'},
\end{aligned}$$

and the proof is complete. □

Remark 4.5.3. We would like to point out that in [93], the authors proved a strict inequality '>' for (4.5.5) after coupling the full space CDRP. Another proof using a different argument has been conducted in [121]. We do not pursue it here.

Since we only need an inequality with '≤', an alternative approach is to prove the strong Markov property for polymer measures $\mathbb{Q}_x^{u,t}$ and then use the coupling method. For continuum time models, proving the strong Markov property requires both the Markov and the Feller properties of the associated semigroup [122, Theorem 6.17], which we believe can also be done for this model.

To complete the proof of Theorem 4.1.2 and Corollary 4.1.3, it remains to compute the annealed mean of the stationary measure-to-measure polymer's endpoint $\mathbf{E}\mathbb{E}_{\tilde{W}_u}^{\mathbb{Q}^{u,t}}(X_t)$.

Lemma 4.5.4. *For any $u < 0$ and $t > 0$, we have*

$$\mathbf{E}\int_0^\infty y\tilde{\rho}_u^R(y|t)\mathrm{d}y = \psi'(2|u|). \tag{4.5.13}$$

The functions ψ and ψ' are the digamma and trigamma functions defined in (4.1.5)–(4.1.6).

Proof. By definition (4.5.3),

$$\begin{aligned}\mathbf{E}\int_0^\infty y\tilde{\rho}_u^R(y|t)\mathrm{d}y &= \mathbf{E}\frac{\int_0^\infty y\int_0^\infty \mathcal{Z}_{u-\frac{1}{2}}(t,x|0,y)e^{\tilde{W}_u(x)}\mathrm{d}xe^{W_u(y)}\mathrm{d}y}{\int_0^\infty\int_0^\infty \mathcal{Z}_{u-\frac{1}{2}}(t,x'|0,y')e^{\tilde{W}_u(x')}\mathrm{d}x'e^{W_u(y')}\mathrm{d}y'}\\ &= \mathbf{E}\frac{\int_0^\infty y\int_0^\infty \mathcal{Z}_{u-\frac{1}{2}}(t,y|0,x)e^{\tilde{W}_u(x)}\mathrm{d}xe^{W_u(y)}\mathrm{d}y}{\int_0^\infty\int_0^\infty \mathcal{Z}_{u-\frac{1}{2}}(t,y'|0,x')e^{\tilde{W}_u(x')}\mathrm{d}x'e^{W_u(y')}\mathrm{d}y'},\end{aligned}$$

where the second equality follows from the time reversal of the field $\mathcal{Z}$ in (4.2.20). Since W and $\tilde{W}$ are independent and identical, we can interchange them under the expectation, and the above equals to

$$\begin{aligned}\mathbf{E}\frac{\int_0^\infty yZ_{u-\frac{1}{2}}(t,y)e^{\tilde{W}_u(y)}\mathrm{d}y}{\int_0^\infty Z_{u-\frac{1}{2}}(t,y')e^{\tilde{W}_u(y')}\mathrm{d}y'} &= \mathbf{E}\frac{\int_0^\infty yZ_{u-\frac{1}{2}}(t,y)Z_{u-\frac{1}{2}}(t,0)^{-1}e^{\tilde{W}_u(y)}\mathrm{d}y}{\int_0^\infty Z_{u-\frac{1}{2}}(t,y')Z_{u-\frac{1}{2}}(t,0)^{-1}e^{\tilde{W}_u(y')}\mathrm{d}y'}\\ &= \mathbf{E}\frac{\int_0^\infty ye^{W_u(y)}e^{\tilde{W}_u(y)}\mathrm{d}y}{\int_0^\infty e^{W_u(y')}e^{\tilde{W}_u(y')}\mathrm{d}y'},\end{aligned}$$

where we used the invariance (4.3.2) and our assumption that $W, \tilde{W}, \xi$ are all independent. The last expression is time independent and can be computed explicitly. In fact, since $W(y) + \tilde{W}(y)$

has the same distribution as $W(2y)$, the above expectation equals to

$$\mathbf{E}\frac{\int_0^\infty y\exp(W(2y)+2uy)\mathrm{d}y}{\int_0^\infty \exp(W(2y')+2uy')\mathrm{d}y'} = \frac{1}{2}\mathbf{E}\frac{\int_0^\infty y\exp(W(y)+uy)\mathrm{d}y}{\int_0^\infty \exp(W(y')+uy')\mathrm{d}y'} = \frac{1}{2}\mathbf{E}\frac{\mathrm{d}}{\mathrm{d}v}\log\int_0^\infty e^{W(y)+vy}\mathrm{d}y\Big|_{v=u}$$
$$= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}v}\mathbf{E}\log\int_0^\infty e^{W(y)+vy}\mathrm{d}y\Big|_{v=u} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}v}\left(-\psi(-2v)+\log 2\right)\Big|_{v=u} = \psi'(-2u). \tag{4.5.14}$$

In the second from last equality, we have used (4.3.1) together with the fact that

$$\mathbf{E}\left[\log(1/\gamma(\theta))\right] = -\psi(\theta)$$

for any $\theta > 0$. To check the interchange of differentiation and expectation, we first use Fatou's lemma to derive that $\mathbf{E}\frac{\int_0^\infty y\exp(W(y)+uy)\mathrm{d}y}{\int_0^\infty \exp(W(y')+uy')\mathrm{d}y'} \leq 2\psi'(-2u)$, then the interchange can be justified by applying the dominated convergence theorem and the mean value theorem. □

Remark 4.5.5. With the convergence conjecture mentioned in Remark 4.1.9, (4.5.13) should also equal to the annealed mean of the midpoint position of long half-space polymers (i.e. as $t \to \infty$) with both endpoints fixed near the wall at times 0 and $2t$ in the bound phase $u < 0$. For a discussion on such midpoint distributions, see [24, Section IV B].

4.6 Symmetry: proof of Proposition 4.1.5

The goal of this section is to prove the symmetry identity in Proposition 4.1.5. This symmetry would allow us to study the height function statistics and polymer endpoint displacement for all values of $u > 0$, using the results we obtained in the case of $u < 0$. It follows from results in [23] regarding the half-space Macdonald processes. We learned this symmetry from [30, Claim 4.9], where the authors stated a proof of convergence heuristically. This convergence was later rigorously proved in [26], so we now summarize these results to provide a proof.

For this section, we use uppercase letters (e.g. S, T, X, Y) for continuum variables and lowercase letters (e.g. r, s, t, x, y, w) for discrete variables. We use $\mathbf{Q}$ to denote the set of rational numbers, and $\mathbf{Z}$ to denote the set of integers.

We use the inhomogeneous half-space log-gamma (HSLG) polymer models as defined in [26, Definition 2.1]. Following the notations there, let $\alpha_\circ, \alpha_1, \alpha_2, \ldots$ be real parameters such that $\alpha_i + \alpha_\circ > 0$ for all $i \geq 1$ and $\alpha_i + \alpha_j > 0$ for all $i \neq j \geq 1$. Let $(\varpi_{i,j})_{i \geqslant j}$ be a family of independent random variables such that for $i > j, \varpi_{i,j} \sim 1/\gamma(\alpha_i + \alpha_j)$ and $\varpi_{i,i} \sim 1/\gamma(\alpha_\circ + \alpha_i)$. The partition function of the half-space log-gamma polymer is defined as

$$z(n,m) = \sum_{\pi:(1,1)\to(n,m)} \prod_{(i,j)\in\pi} \varpi_{i,j}, \tag{4.6.1}$$

where the sum is over all up-right paths from $(1,1)$ to (n,m) in octant $\{(i,j) \in \mathbf{Z}^2_{>0} : i \geqslant j\}$.

Derived from a symmetry result of the half-space Macdonald process, the partition functions $z(m,m)$ for any $m \geq 1$ as defined in (4.6.1) will not change their laws when the parameters $\alpha_\circ$ and α_1 are interchanged. This is a generalization of [23, Proposition 8.1] for inhomogeneous HSLG polymer.

Proposition 4.6.1. *For any $m \geq 1$, the law of the partition function $z(m,m)$ with parameters $\alpha_\circ, \alpha_1, \alpha_2, ..., \alpha_m$ equals to the law of the partition function $\tilde{z}(m,m)$ with parameters $\alpha_1, \alpha_\circ, \alpha_2, ..., \alpha_m$.*

Proof. The law of the partition function $z(m,m)$ with parameters $\alpha_\circ, \alpha_1, ..., \alpha_m$ equals to the limiting law of $(1-q)^{2m-1}q^{-\lambda_1}$ as $q \to 1$, where λ_1 is distributed according to the q-Whittaker measure [23,123] with parameters $\{(q^{\alpha_1}, ..., q^{\alpha_m}), q^{\alpha_\circ}\}$. One can think of $(q^{\alpha_1}, ..., q^{\alpha_m})$ as the bulk parameters, and $q^{\alpha_\circ}$ as the diagonal parameter. By [23, Proposition 2.6], λ_1 has the same distribution as π_1, where π_1 is distributed to the q-Whittaker measure with parameters $\{(q^{\alpha_1}, ..., q^{\alpha_m}, q^{\alpha_\circ}), 0\}$, where $(q^{\alpha_1}, ..., q^{\alpha_m}, q^{\alpha_\circ})$ is the bulk parameter and 0 is the diagonal parameter. Since the q-Whittaker measure is invariant under permutation of its bulk parameters (see [26, Lemma 2.7] for the proof and further applications of this symmetry), π_1 equals in law to $\tilde{\pi}_1$, which is defined to be distributed as the q-Whittaker measure with parameters $\{(q^{\alpha_\circ}, q^{\alpha_2}, ..., q^{\alpha_m}, q^{\alpha_1}), 0\}$. Apply [23, Proposition 2.6] again backward, we have $\tilde{\pi}_1$ has the same distribution as $\tilde{\lambda}_1$, which is distributed according to the q-Whittaker measure with parameters $\{(q^{\alpha_\circ}, q^{\alpha_2}, ..., q^{\alpha_m}), q^{\alpha_1}\}$. Thus we have

$$(1-q)^{2m-1} q^{-\lambda_1} \overset{\text{law}}{=} (1-q)^{2m-1} q^{-\tilde{\lambda}_1}.$$

Taking the limit $q \to 1$ as in [23, Proposition 8.1], we have the partition function $z(m, m)$ with parameters $\alpha_\circ, \alpha_1, ..., \alpha_m$ equals in law to the partition function $\tilde{z}(m, m)$ with parameters $\alpha_1, \alpha_\circ, \alpha_2, ..., \alpha_m$.□

From now on, we assume $u - v > 0$. We set $\alpha_\circ = u$ and $\alpha_1 = -v$, and for all $i \geq 2$, $\alpha_i = \alpha$ for some $\alpha > \min(0, -u, v)$. We use $z^{\alpha}_{u,-v}(n, m) := z(n, m)$ to emphasize the dependence on parameters u, v and α. We define the ratio

$$z^{\text{stat},\alpha}_{u,-v}(m, m) := \frac{z^{\alpha}_{u,-v}(m, m)}{\varpi_{1,1}}.$$

Corollary 4.6.2. *Assume $u > v$. For any fixed $m \geq 1$,*

$$z^{\textit{stat},\alpha}_{u,-v}(m, m) \overset{\textit{law}}{=} z^{\textit{stat},\alpha}_{-v,u}(m, m). \tag{4.6.2}$$

Proof. Proposition 4.6.1 says that for any $m \geq 1$, $z^{\alpha}_{u,-v}(m,m) \overset{\text{law}}{=} z^{\alpha}_{-v,u}(m,m)$. Since $(\varpi_{i,j})_{i\geqslant j}$ are independent and $\varpi_{1,1}$ appears in the product in all of the terms in (4.6.1), the ratio $z^{\text{stat},\alpha}_{u,-v}(m,m)$ is independent of $\varpi_{1,1}$ and we have $\log z^{\text{stat},\alpha}_{u,-v}(m,m) + \log \varpi_{1,1} \overset{\text{law}}{=} \log z^{\text{stat},\alpha}_{-v,u}(m,m) + \log \varpi_{1,1}$. Since $\alpha_0 + \alpha_1 = u - v > 0$ and the characteristic function of a log-gamma random variable is always nonzero, the independence implies (4.6.2). □

We next take the weak noise scaling limit to obtain the result for the half-space KPZ equation, based on [26, Theorem 5.4] for the above half-space HSLG models.

Proposition 4.6.3. *Assume $u > v$. For any fixed $T, X \geq 0$, $u > v$, with $\alpha = \alpha^{(n)} = \frac{1}{2} + \sqrt{n}$, we have*

$$(\sqrt{n})^{nT+\sqrt{n}X} z^{stat,\alpha^{(n)}}_{u,-v}\left(\frac{nT}{2} + \sqrt{n}X + 1, \frac{nT}{2} + 1\right) \overset{law}{\to} \int_0^\infty \mathcal{Z}_{u-\frac{1}{2}}(T, X|0, Y)\exp(W_v(Y))\mathrm{d}Y, \tag{4.6.3}$$

as $n \to \infty$, where $z^{stat,\alpha^{(n)}}_{u,-v}(\cdot,\cdot)$ are defined by linear interpolation when $nT/2 \notin \mathbf{Z}_{\geq 0}$ or $\sqrt{n}X \notin \mathbf{Z}_{\geq 0}$.

Proof. The proof follows closely the proof of [26, Theorem 1.4], except that we need to verify the convergence of a different initial condition. Following the convention there, we let $\overline{\varpi} := \mathbb{E}[\varpi_{3,2}] = (2\alpha - 1)^{-1}$ be the mean of a generic log-gamma polymer bulk weight. Since we are dealing with "one-row partition functions" (see [26]), we define

$$\mathfrak{z}^{1\mathrm{r}}_{u,-v;\alpha}(t,y) := \left(\frac{1}{2\overline{\varpi}}\right)^{2t+y} z^{\text{stat},\alpha}_{u,-v}(t+y+1, t+1),$$

and

$$\mathfrak{z}^{1\mathrm{r}}_{-v;\alpha}(x) := \mathfrak{z}^{1\mathrm{r}}_{u,-v;\alpha}(0, x+1),$$

where we drop the subscript u as these weights only depend on $-v$ and α. We have that for any $y \in \mathbf{Z}_{\geq 0}$, and $t \in \mathbf{Z}_{\geq 1}$,

$$\mathscr{Z}^{1\mathrm{r}}_{u,-v;\alpha}(t,y) = \sum_{x=0}^{t+y-1} \mathscr{Z}^{1\mathrm{r}}_{-v;\alpha}(x) \left(\frac{1}{2\overline{\varpi}}\right)^{2t+y-x-1} z_{u;\alpha}(x+2,2;t+y+1,t+1), \tag{4.6.4}$$

where $z_{u;\alpha}(a,b;a',b')$ denotes the log-gamma partition function for up-right paths starting at (a,b) and ending at (a',b') as a generalization of the definition (4.6.1). This partition function only depends on parameters u and α. Similar to [26], by assigning proper boundary weights, one can match the log-gamma polymer paths to reflected symmetric simple random walks. Consequently, as a process in $x, y \in \mathbf{Z}_{\geq 0}$ and $t \in \mathbf{Z}_{\geq 1}$,

$$z_{u;\alpha}(x+2,2;t+y+1,t+1) \overset{\text{law}}{=} \frac{1}{2} \cdot \left(\frac{\omega(2t+y-2,y)}{\overline{\varpi} 2^{\mathscr{I}(y)}}\right) \cdot 2^{\mathscr{I}(y)} z_{\boldsymbol{\mathcal{X}},\boldsymbol{\omega},\beta}(x,x;2t+y-2,y), \tag{4.6.5}$$

where we define $\mathscr{I}(x) \coloneqq \mathbb{1}_{x=0}$, and $z_{\boldsymbol{\mathcal{X}},\boldsymbol{\omega},\beta}(s,x;t,y)$ is the "modified polymer partition function" as defined in [26, (5.13)] for reflected simple symmetric random walks, with the boundary random variables $\boldsymbol{\mathcal{X}} = (\mathcal{X}(r))_{r \in \mathbf{Z}_{\geq 0}}$ being i.i.d., the bulk random variables $\boldsymbol{\omega} = (\omega(r,w))_{r \in \mathbf{Z}_{\geq 0}, w \in \mathbf{Z}_{\geq 1}}$ being i.i.d., and the distributions and the parameter $\beta > 0$ are so that

$$\begin{aligned} 1 + \beta\omega(r,w) &\overset{\text{law}}{=} \frac{\varpi_{3,2}}{\overline{\varpi}} \sim (2\alpha-1)\mathrm{Gamma}^{-1}(2\alpha), \\ \mathcal{X}(r) &\overset{\text{law}}{=} \frac{\varpi_{2,2}}{2\overline{\varpi}} \sim \frac{2\alpha-1}{2}\mathrm{Gamma}^{-1}(\alpha+u). \end{aligned}$$

Then (4.6.4) and (4.6.5) imply that as a process in $t \in \mathbf{Z}_{\geq 1}$ and $y \in \mathbf{Z}_{\geq 0}$,

$$\begin{aligned} \mathfrak{z}^{1\mathrm{r}}_{u,-v;\alpha}(t,y) \overset{\text{law}}{=} \frac{2^{\mathscr{I}(y)}}{2} \cdot \left[\mathscr{I}(y)\mathcal{X}(2t+y-2) + (1-\mathscr{I}(y))(1+\beta\omega(2t+y-2,y))\right] \\ \cdot z^{\swarrow}_{\boldsymbol{\mathcal{X}},\boldsymbol{\omega},\beta,z^{\swarrow 1\mathrm{r}}}(2t+y-2,y), \end{aligned} \tag{4.6.6}$$

where $z^{\swarrow}_{\boldsymbol{\mathcal{X}},\boldsymbol{\omega},\beta,z^{\swarrow 1\mathrm{r}}}(\cdot,\cdot)$ is the partition function as defined in [26, (5.19)] with the initial data $z^{\swarrow 1\mathrm{r}}(\cdot) = \mathfrak{z}^{1\mathrm{r}}_{-v;\alpha}(\cdot)$ independent of the boundary and bulk weights $\boldsymbol{\mathcal{X}},\boldsymbol{\omega}$. For any $S \geq 0, Y \geq 0$, define $z^{\swarrow}_{\boldsymbol{\mathcal{X}},\boldsymbol{\omega},\beta,z^{\swarrow 1\mathrm{r}}}(S,Y)$ by linear interpolation when $S \notin \mathbf{Z}_{\geq 0}$ or $Y \notin \mathbf{Z}_{\geq 0}$ or $S+Y$ is not even. Note that the only difference between the field $z^{\swarrow}_{\boldsymbol{\mathcal{X}},\boldsymbol{\omega},\beta,z^{\swarrow 1\mathrm{r}}}(\cdot,\cdot)$ here and the field $z^{\swarrow}_{\boldsymbol{\mathcal{X}},\boldsymbol{\omega},\beta,z^{\swarrow}}(\cdot,\cdot)$ in [26, (6.3)] is that the initial condition differs. The boundary and the bulk random variables $\boldsymbol{\mathcal{X}},\boldsymbol{\omega}$ are the same. Let $\alpha = \alpha^{(n)} = \sqrt{n} + 1/2$ and $\beta = \beta^{(n)} = n^{-1/4}/\sqrt{2}$. In [26, Section 6], the authors have verified the conditions on the same $\boldsymbol{\mathcal{X}},\boldsymbol{\omega}$ to apply [26, Theorem 5.4]. As long as we can verify the conditions on the initial data $z^{\swarrow 1\mathrm{r}}(\cdot)$ here, [26, Theorem 5.4] implies the convergence in law

$$\frac{2^{\mathscr{I}(X)}}{2} z^{\swarrow}_{\boldsymbol{\mathcal{X}},\boldsymbol{\omega},\beta,z^{\swarrow 1\mathrm{r}}}(nT+\sqrt{n}X-2,\sqrt{n}X) \overset{\text{law}}{\to} \int_0^\infty \mathcal{Z}_{u-\frac{1}{2}}(T,X|0,Y)\exp(W_v(Y))\mathrm{d}Y, \tag{4.6.7}$$

as $n \to \infty$, for any fixed $T, X \geq 0$. The convergence (4.6.3) then follows from (4.6.6) and (4.6.7), together with the fact that $\mathcal{X}(r) \to 1$ and $1+\beta\omega(r,w) \to 1$ in probability as $n \to \infty$ (the dependence on n was kept implicit through $\alpha^{(n)}$ and $\beta^{(n)}$).

To verify the conditions on the initial data, with $\alpha = \alpha^{(n)} = \sqrt{n}+1/2$, we use Donsker's theorem to check convergence in law

$$\mathfrak{z}^{1\mathrm{r}}_{-v;\alpha}(\sqrt{n}X) \overset{\text{law}}{\to} \exp(W(X)+vX) = \exp(W_v(X)) \quad \text{as a process in } X \in [0,\infty).$$

Moreover, for any $n \in \mathbf{Z}_{\geq 1}$ and $X \geq 0$ such that $\sqrt{n}X \in \mathbf{Z}_{\geq 0}$, we have that $\mathbb{E}[\widetilde{z}^{1\mathrm{r}}_{-v;\alpha}(\sqrt{n}X)^2] = [nM_2(-v)]^{\sqrt{n}X+1}$ with $M_k(-v) := \mathbb{E}[(\varpi^{(n)})^k]$ where $\varpi^{(n)} \sim \mathrm{Gamma}^{-1}(\alpha^{(n)}-v)$. By [26, (1.1)],

$$nM_2(-v) = 1 + (2+2v)n^{-1/2} + O(n^{-1}),$$

which implies the uniform bound

$$\sup_{n \in \mathbf{Z}_{\geq 1}} \sup_{X \geq 0} e^{-aX}\mathbb{E}[\widetilde{z}^{1\mathrm{r}}_{-v;\alpha}(\sqrt{n}X)^2] < \infty,$$

for some $a > 0$ that only depends on v. The conditions imposed on the initial data in [26, Theorem 5.4] are thus verified for our $z^{\swarrow}_{\boldsymbol{\mathcal{X}},\boldsymbol{\omega},\beta,z^{\swarrow 1\mathrm{r}}}(\cdot,\cdot)$. $\square$

Combining the results (4.6.2) and (4.6.3), we prove the identity in law for the partition function on the boundary in the continuum setting under the condition $u > v$. This identity in law was previously claimed in [30, Claim 4.9] with a slightly different convention on the parameters. For discussion on general $u, v \in \mathbb{R}$, see Remark 4.6.6 below.

Proposition 4.6.4. *For any $T \geq 0$ and $u, v \in \mathbb{R}$ with $u > v$, we have*

$$\int_0^\infty \mathcal{Z}_{u-\frac{1}{2}}(T,0|0,Y)\exp(W_v(Y))\mathrm{d}Y \overset{law}{=} \int_0^\infty \mathcal{Z}_{-v-\frac{1}{2}}(T,0|0,Y)\exp(W_{-u}(Y))\mathrm{d}Y. \quad (4.6.8)$$

Proof. Since Corollary 4.6.2 is not generalized to joint law, we need to be careful to avoid using any linear interpolations for the identity in law. For any fixed $T \in \mathbf{Q}_{\geq 0}$, there exists a series of integers $\{n_k \in \mathbf{Z}_{\geq 0}\}_{k \geq 1}$ such that $n_k \to \infty$ as $k \to \infty$ and $\frac{n_k T}{2} \in \mathbf{Z}_{\geq 0}$. Then Corollary 4.6.2 implies that for each $k \geq 1$,

$$(\sqrt{n_k})^{n_k T} z_{u,-v}^{\text{stat},\alpha^{(n_k)}}\left(\frac{n_k T}{2}+1, \frac{n_k T}{2}+1\right) \overset{\text{law}}{=} (\sqrt{n_k})^{n_k T} z_{-v,u}^{\text{stat},\alpha^{(n_k)}}\left(\frac{n_k T}{2}+1, \frac{n_k T}{2}+1\right).$$

By taking the limit $k \to \infty$, Proposition 4.6.3 implies that for any fixed $T \in \mathbf{Q}_{\geq 0}$, (4.6.8) holds.

Using the continuity of mild solutions, we extend (4.6.8) to any $T \in [0, \infty)$. □

Remark 4.6.5. By sending $v \to -\infty$ in Proposition 4.6.4, one can formally recover [124, Theorem 1.1].

To complete the proof of Proposition 4.1.5, it remains to pass the limit $v \to u$.

Proof of Proposition 4.1.5. By the Minkowsiki and Hölder's inequalities, the $\mathbf{P}$-almost surely continuity of the Green's function $\mathcal{Z}_\cdot(\cdot,\cdot|\cdot,\cdot)$ (Proposition 4.2.7), the uniform moment bounds in (4.2.17) and the dominated convergence theorem, as $v \to u$, we have

$$\left\| \int_0^\infty \mathcal{Z}_{u-\frac{1}{2}}(T,0|0,Y)\exp(W_v(Y))\mathrm{d}Y - \int_0^\infty \mathcal{Z}_{u-\frac{1}{2}}(T,0|0,Y)\exp(W_u(Y))\mathrm{d}Y \right\|_2 \to 0,$$

and

$$\left\| \int_0^\infty \mathcal{Z}_{-v-\frac{1}{2}}(T,0|0,Y)\exp(W_{-u}(Y))\mathrm{d}Y - \int_0^\infty \mathcal{Z}_{-u-\frac{1}{2}}(T,0|0,Y)\exp(W_{-u}(Y))\mathrm{d}Y \right\|_2 \to 0.$$

The identity (4.6.8) can thus be extended to $v = u$, and we have $Z_{u-\frac{1}{2}}(T,0) \overset{\text{law}}{=} Z_{-u-\frac{1}{2}}(T,0)$ for any $T \geq 0$ and $u \in \mathbb{R}$. Taking a logarithm on both sides and applying the continuous mapping theorem then completes the proof. □

Remark 4.6.6. By using analytic continuation, one should be able to extend Corollary 4.6.2 to the more general condition with $u, v \in \mathbb{R}$ and $\alpha > \min(0, -u, v)$. In other words, the identity in law still holds after dropping the assumption $u - v > 0$. Heuristically, this is clear as the ratio $z^{\text{stat},\alpha}_{u,-v}(m,m)$ does not involve $\varpi_{1,1}$ and thus does not depend on the value of $u-v$. This extension has been explained in the proof of [23, Claim 4.9], but a verification for the analytic continuation has not yet been proved.

As a last result, we prove Corollary 4.1.6.

Proof of Corollary 4.1.6. From (4.1.8) in Theorem 4.1.1, we know that for any $u \in \mathbb{R}, T \in [0,\infty)$,

$$\mathbf{E}\mathbb{E}_0^{\mathbb{Q}^{u,T}}[X_T] = \frac{1}{2}\left(\mathbf{Var}[H_u(T,0)] + uT\right). \tag{4.6.9}$$

By Proposition 4.1.5, Theorem 4.1.2 is extended to all $u > 0$ with

$$uT \leq \mathbf{Var}[H_u(T,0)] \leq 2\psi'(2u) + uT.$$

Substituting the above bounds of $\mathbf{Var}[H_u(T,0)]$ into (4.6.9) gives (4.1.14). □

4.7 Supercritical regime: proof of Theorem 4.1.7

In this last section, we prove (4.1.15), which provides an upper bound to the polymer endpoint displacement when the boundary parameter u is in the supercritical regime: $u = ct^{-\alpha}$ with $\alpha > 1/3$ and $c \in \mathbb{R}$. Combining with the variance identity obtained in Theorem 4.1.1, we further derive an upper bound for the fluctuation size of the height function in the supercritical regime, as stated in (4.1.16).

To prove (4.1.15), we use a stochastic monotonicity argument different from Section 4.5.

The main idea is the following.

For any realization of (ξ, W) and any $u_1 < u_2$, one may expect that the endpoint distribution of the half-space continuum directed random polymer $\mathbb{Q}_0^{u_1,t}$ would be stochastically dominated by that of the half-space continuum directed random polymer $\mathbb{Q}_0^{u_2,t}$. Intuitively, this is because that when the parameter u becomes greater, the "wall attraction" at $x = 0$ imposed on the polymer would be weakened, while at the same time, the terminal data at time t would have a greater drift. Both the weakened attraction force and the greater drift would drive the polymer endpoint farther away from the wall. Recall that in (4.2.28) we defined $\rho_u^R(\cdot|t,0)$ as the quenched density of polymer endpoints under $\mathbb{Q}_0^{u,t}$. Let $c \in \mathbb{R}$, $u_1 = ct^{-\alpha}$ for some $\alpha > 1/3$, and $u_2 = (|c| \vee 1)t^{-1/3}$. For any $t \geq 1$, we expect that $\rho_{u_1}^R(\cdot|t,0) \leq_{st} \rho_{u_2}^R(\cdot|t,0)$ $\mathbf{P}$-almost surely.

On the other hand, from the upper bound in (4.1.14) of Corollary 4.1.6, we know that when $u_2 = ct^{-1/3}$ for some $c > 0$, for any $t \geq 1$, the annealed mean of the polymer endpoint is bounded by

$$\mathbf{E}\mathbb{E}_0^{\mathbb{Q}^{u_2,t}}[X_t] \leq Ct^{2/3}$$

for some $C > 0$ depending only on c. Combining the above estimate and the stochastic dominance of $\rho_{u_1}^R(\cdot|t,0) \leq_{st} \rho_{u_2}^R(\cdot|t,0)$ gives (4.1.15).

We now prove the aforementioned results. The Lemma 4.7.1 below is the main ingredient to prove $\rho_{u_1}^R(\cdot|t,0) \leq_{st} \rho_{u_2}^R(\cdot|t,0)$ $\mathbf{P}$-almost surely. One can compare it with Lemma 4.5.1 above. In fact, our approach to the stochastic monotonicity here is partly inspired by Section 4.5. As in Section 4.5, we first analyze the determinants in (4.7.1) formed by four copies of Green's functions, and we will work with a smoothed noise to utilize the Feynman-Kac formula before passing to the limit. The determinant in (4.7.1) being nonnegative further leads to the relation

$\rho^R_{u_1}(\cdot|t,0) \le_{st} \rho^R_{u_2}(\cdot|t,0)$.

For reference, we note that in Section 4.5, we proved $\rho^R_u(\cdot|t,x_1) \le_{st} \rho^R_u(\cdot|t,x_2)$ for any $x_1 < x_2$. Here we are expecting that $\rho^R_{u_1}(\cdot|t,0) \le_{st} \rho^R_{u_2}(\cdot|t,0)$ for any $u_1 < u_2$.

Lemma 4.7.1. *Fix $t > 0$. For any $u_1 < u_2$ and $0 \le y_1 < y_2$, we have*

$$\det[\mathcal{Z}_{u_i-\frac{1}{2}}(t,0|0,y_j)]_{i,j\in\{1,2\}} \ge 0 \quad \mathbf{P}\text{-}\textit{almost surely}. \tag{4.7.1}$$

Before proving Lemma 4.7.1, we first use it to prove Theorem 4.1.7.

Proof of Theorem 4.1.7. By (4.7.1), for any fixed $t > 0$ and $u_1 < u_2$, with probability one,

$$\mathcal{Z}_{u_1-\frac{1}{2}}(t,0|0,y_1)\mathcal{Z}_{u_2-\frac{1}{2}}(t,0|0,y_2) \ge \mathcal{Z}_{u_1-\frac{1}{2}}(t,0|0,y_2)\mathcal{Z}_{u_2-\frac{1}{2}}(t,0|0,y_1), \tag{4.7.2}$$

for all $0 \le y_1 < y_2$ simultaneously. (As in the proof of Corollary 4.5.2, one can first consider rational y_1, y_2 then use the continuity of $\mathcal{Z}$.)

Similar to Proposition 4.2.8, it is straightforward to show that for any fixed $u_1, u_2 \in \mathbb{R}$, we have $\mathbf{P}$-almost surely

$$0 < \int_0^\infty \mathcal{Z}_{u_2-\frac{1}{2}}(t,0|0,y)\exp(W_{u_1}(y))\mathrm{d}y < \infty.$$

Define

$$\mathcal{U}(t,u,\theta) := \frac{\int_0^\infty y\mathcal{Z}_{u-\frac{1}{2}}(t,0|0,y)\exp(W_\theta(y))\mathrm{d}y}{\int_0^\infty \mathcal{Z}_{u-\frac{1}{2}}(t,0|0,y)\exp(W_\theta(y))\mathrm{d}y},$$

which is the quenched mean of the endpoint displacement of a half-space polymer with boundary parameter u and initial parameter θ. As in the proof of Corollary 4.5.2, one can obtain from

(4.7.2) that for any fixed $t > 0$ and $u_1 < u_2$,

$$\mathcal{U}(t, u_1, u_1) \leq \mathcal{U}(t, u_2, u_1) \quad \mathbf{P}\text{-almost surely}.$$

(To compare with the proof of Corollary 4.5.2, one only needs to replace $u_i \mapsto x_i, i = 1, 2$.)

On the other hand, by the Jensen's inequality,

$$\begin{aligned}&\partial_\theta \mathcal{U}(t, u, \theta)\\ &= \frac{\int_0^\infty y^2 \mathcal{Z}_{u-\frac{1}{2}}(t, 0|0, y) \exp(W_\theta(y)) \mathrm{d}y}{\int_0^\infty \mathcal{Z}_{u-\frac{1}{2}}(t, 0|0, y) \exp(W_\theta(y)) \mathrm{d}y} - \frac{[\int_0^\infty y \mathcal{Z}_{u-\frac{1}{2}}(t, 0|0, y) \exp(W_\theta(y)) \mathrm{d}y]^2}{[\int_0^\infty \mathcal{Z}_{u-\frac{1}{2}}(t, 0|0, y) \exp(W_\theta(y)) \mathrm{d}y]^2} \geq 0,\end{aligned}$$

$\mathbf{P}$-almost surely. It thus follows that $\mathcal{U}(t, u_2, u_1) \leq \mathcal{U}(t, u_2, u_2)$ $\mathbf{P}$-almost surely.

Combining the above results, for any $u_1 < u_2$, we have $\mathcal{U}(t, u_1, u_1) \leq \mathcal{U}(t, u_2, u_2)$ $\mathbf{P}$-almost surely, which further implies that

$$\mathbf{E}\mathbb{E}_0^{\mathbb{Q}^{u_1,t}}[X_t] \leq \mathbf{E}\mathbb{E}_0^{\mathbb{Q}^{u_2,t}}[X_t]. \tag{4.7.3}$$

Now for any $u_1 = ct^{-\alpha}$ with $\alpha > 1/3$ and $c \in \mathbb{R}$, let $u_2 = (|c| \vee 1)t^{-1/3}$. For any $t \geq 1$, we have $u_1 \leq u_2$, and by (4.1.14),

$$\mathbf{E}\mathbb{E}_0^{\mathbb{Q}^{u_2,t}}[X_t] \leq u_2 t + \psi'(2u_2) \leq Ct^{2/3}, \tag{4.7.4}$$

for some constant $C > 0$ depending on c only. (4.7.3) and (4.7.4) together prove (4.1.15), and (4.1.16) follows immediately using (4.1.8). □

To finish the proof, it only remains to prove Lemma 4.7.1.

Proof of Lemma 4.7.1. Similar to the proof of Lemma 4.5.1, we first prove an inequality analogous to (4.7.1) for the Green's functions of half-space SHE with a smoothed noise ((4.7.6) below). With smoothed noise, one can prove such an inequality using the Feynman-Kac representation and the strong Markov property of reflected Brownian motions. We then pass to the limit using Lemma 4.8.1 to obtain (4.7.1) for the Green's functions of half-space SHE with the white noise.

As in the proof of Lemma 4.5.1, let $\xi_\delta(t,x) = \int_{\mathbb{R}} p_\delta(x-y)\xi(t,\mathrm{d}y)$ be a Gaussian noise that is white in time and smooth in space, with the covariance function

$$Q_\delta(t_1,t_2,x_1,x_2) = \mathbf{E}\xi_\delta(t_1,x_1)\xi_\delta(t_2,x_2) = \delta_0(t_1-t_2)p_{2\delta}(x_1-x_2).$$

Using the same notations as in the proof of Lemma 4.5.1, for any $\mu \in \mathbb{R}, \delta > 0, x, y \in [0,\infty)$, define

$$\begin{aligned}\mathcal{Z}_\mu^\delta(t,x|0,y) &:= \mathscr{P}^N(t,x|0,y)\mathbb{E}_B^x\left[\exp\left(-\mu L_t^0 + \int_0^t \xi_\delta(t-r,|B_r|)\mathrm{d}r - \frac{t}{2}p_{2\delta}(0)\right)\Big||B_t| = y\right]\\ &= \mathscr{P}^N(t,x|0,y)\mathbb{E}_\Upsilon^{x,y}\left[\exp\left(-\mu L_t^{0,\Upsilon} + \int_0^t \xi_\delta(t-r,\Upsilon_r)\mathrm{d}r - \frac{t}{2}p_{2\delta}(0)\right)\right],\end{aligned}$$

with the expectation $\mathbb{E}_\Upsilon^{x,y}$ being taken with respect to the reflected Brownian bridges starting from $\Upsilon_0 = x$ and ending at $\Upsilon_t = y$, and the local time term $L_t^{0,\Upsilon}$ is the same as defined in (4.5.6). Furthermore, by the same argument following after (4.8.22) in Appendix 4.8.1.4, one can always make a time reversal to the paths of reflected Brownian bridges to obtain

$$\mathcal{Z}_\mu^\delta(t,x|0,y) = \mathscr{P}^N(t,y|0,x)\mathbb{E}_\Upsilon^{y,x}\left[\exp\left(-\mu L_t^{0,\Upsilon} + \int_0^t \xi_\delta(r,\Upsilon_r)\mathrm{d}r - \frac{t}{2}p_{2\delta}(0)\right)\right].$$

We use $L_{[s_1,s_2]}^{0,\Upsilon}$ to denote the local time at zero for reflected Brownian motions restricted to the time interval $[s_1, s_2] \subset [0, t]$:

$$L^{0,\Upsilon}_{[s_1,s_2]} := \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_{s_1}^{s_2} \mathbb{1}_{[0,\varepsilon]}(\Upsilon_s)\, \mathrm{d}s. \tag{4.7.5}$$

One may rewrite $L^{0,\Upsilon}_t = L^{0,\Upsilon}_{[0,t]}$. To ease notations, we also define the following functional on the paths of reflected Brownian motions between any time interval $[s_1, s_2] \subset [0, t]$:

$$F^{\delta,\mu}_{[s_1,s_2]}(\Upsilon) := \exp\left(-\mu L^{0,\Upsilon}_{[s_1,s_2]} + \int_{s_1}^{s_2} \xi_\delta(r, \Upsilon_r)\mathrm{d}r - \frac{s_2 - s_1}{2} p_{2\delta}(0)\right).$$

For any $0 \le s_1 \le s_2 \le t$, $F^{\delta,\mu}_{[s_1,s_2]}(\Upsilon)$ is a $\mathbf{P}$-almost surely continuous, strictly positive, and multiplicative functional of Υ. We also note that the functional $F^{\delta,\mu}_{[s_1,s_2]}(\cdot)$ depends on the boundary parameter μ through the multiplier of the local time term.

Our goal is to prove a δ-level mollified version of (4.7.1), namely, for any $t > 0, \delta > 0$, $u_1 < u_2$ and $0 \le y_1 < y_2$,

$$\det\left[\mathcal{Z}^\delta_{u_i - \frac{1}{2}}(t, 0|0, y_j)\right]_{i,j\in\{1,2\}} = \det\left[\mathscr{P}^N(t, y_j|0, 0)\mathbb{E}^{y_j,0}_\Upsilon\left[F^{\delta, u_i - \frac{1}{2}}_{[0,t]}\right]\right] \ge 0, \tag{4.7.6}$$

$\mathbf{P}$-almost surely.

One can check that (4.7.1) follows from (4.7.6) with Lemma 4.8.1 (i). In fact, it is easy to see that by Lemma 4.8.1 (i) (with a mollification in space only), for any $t > 0, u_1, u_2 \in \mathbb{R}, y_1, y_2 \in [0, \infty)$ and $p \in [1, \infty)$,

$$\det\left[\mathcal{Z}^\delta_{u_i - \frac{1}{2}}(t, 0|0, y_j)\right]_{i,j\in\{1,2\}} \to \det\left[\mathcal{Z}_{u_i - \frac{1}{2}}(t, 0|0, y_j)\right]_{i,j\in\{1,2\}} \quad \text{in } L^p(\Omega) \quad \text{as } \delta \to 0.$$

Thus the proof is complete once we prove (4.7.6).

To prove (4.7.6), we use a path-switching argument. Let $\mathbb{P}^{\mathbf{y},\mathbf{0}}_{\Upsilon^1,\Upsilon^2}$ be the measure of a two-dimensional reflected Brownian bridge (Υ^1, Υ^2) starting from $\mathbf{y} = (y_1, y_2)$ at time 0 and ending at $\mathbf{0} = (0, 0)$ at time t. Since $\mathscr{P}^N(t, y|0, 0)$ is positive for any $y \in [0, \infty)$, it suffices to show that

$$\mathbb{E}^{\mathbf{y},\mathbf{0}}_{\Upsilon^1,\Upsilon^2}\left[F^{\delta,u_1-\frac{1}{2}}_{[0,t]}(\Upsilon^1)F^{\delta,u_2-\frac{1}{2}}_{[0,t]}(\Upsilon^2) - F^{\delta,u_2-\frac{1}{2}}_{[0,t]}(\Upsilon^1)F^{\delta,u_1-\frac{1}{2}}_{[0,t]}(\Upsilon^2)\right] \geq 0 \quad \mathbf{P}\text{-almost surely}, \tag{4.7.7}$$

for any $t > 0, \delta > 0$, $u_1 < u_2$ and $0 \leq y_1 < y_2$.

Define the stopping time

$$\tau = \inf\{s \in [0, t] : \Upsilon^1(s) = \Upsilon^2(s)\}.$$

Since $\Upsilon^1(t) = \Upsilon^2(t) = 0$ and $\Upsilon^1(0) = y_1 < y_2 = \Upsilon^2(0)$, we have $0 < \tau \leq t$ $\mathbb{P}^{\mathbf{y},\mathbf{0}}_{\Upsilon^1,\Upsilon^2}$-almost surely. By the path continuity of reflected Brownian motions, $\Upsilon^1(s) \leq \Upsilon^2(s)$ for all $s \in [0, \tau]$. It thus follows from the definition of the local time through approximation (4.7.5) that

$$L^{0,\Upsilon^1}_{[0,\tau]} \geq L^{0,\Upsilon^2}_{[0,\tau]} \quad \mathbb{P}^{\mathbf{y},\mathbf{0}}_{\Upsilon^1,\Upsilon^2}\text{-almost surely}. \tag{4.7.8}$$

Define

$$\left(\tilde{\Upsilon}^1_r, \tilde{\Upsilon}^2_r\right) = \begin{cases} (\Upsilon^1_r, \Upsilon^2_r) & r \leq \tau \\ (\Upsilon^2_r, \Upsilon^1_r) & r > \tau. \end{cases}$$

As stated in [117, Proposition 1], the reflected Brownian bridges $\Upsilon_\cdot$ under $\mathbb{P}^{x,y}_{\Upsilon}$ for any $x, y \in [0, \infty)$ are strong Markov processes. (The proof follows from the optional stopping theorem and that the reflected Brownian motion is strong Markov.) Hence $\left(\tilde{\Upsilon}^1_r, \tilde{\Upsilon}^2_r\right)$ is also a two-dimensional reflected Brownian bridge starting from $\mathbf{y} = (y_1, y_2)$ at time 0 and ending at $\mathbf{0} = (0,0)$ at time t, and the left-hand-side of (4.7.7) equals to

$$\mathbb{E}^{\mathbf{y},\mathbf{0}}_{\tilde{\Upsilon}^1,\tilde{\Upsilon}^2}\left[F^{\delta,u_1-\frac{1}{2}}_{[0,t]}(\tilde{\Upsilon}^1)F^{\delta,u_2-\frac{1}{2}}_{[0,t]}(\tilde{\Upsilon}^2) - F^{\delta,u_2-\frac{1}{2}}_{[0,t]}(\tilde{\Upsilon}^1)F^{\delta,u_1-\frac{1}{2}}_{[0,t]}(\tilde{\Upsilon}^2)\right],$$

which can be further expanded as

$$\begin{aligned}\mathbb{E}^{\mathbf{y},\mathbf{0}}_{\Upsilon^1,\Upsilon^2}\Bigg[&F^{\delta,u_1-\frac{1}{2}}_{[0,\tau]}(\Upsilon^1)F^{\delta,u_1-\frac{1}{2}}_{[\tau,t]}(\Upsilon^2)F^{\delta,u_2-\frac{1}{2}}_{[0,\tau]}(\Upsilon^2)F^{\delta,u_2-\frac{1}{2}}_{[\tau,t]}(\Upsilon^1)\\ &- F^{\delta,u_2-\frac{1}{2}}_{[0,\tau]}(\Upsilon^1)F^{\delta,u_2-\frac{1}{2}}_{[\tau,t]}(\Upsilon^2)F^{\delta,u_1-\frac{1}{2}}_{[0,\tau]}(\Upsilon^2)F^{\delta,u_1-\frac{1}{2}}_{[\tau,t]}(\Upsilon^1)\Bigg] =: \mathcal{S}_1.\end{aligned}$$

Meanwhile, one can also expand the left-hand-side of (4.7.7) directly so that it equals to

$$\begin{aligned}\mathbb{E}^{\mathbf{y},\mathbf{0}}_{\Upsilon^1,\Upsilon^2}\Bigg[&F^{\delta,u_1-\frac{1}{2}}_{[0,\tau]}(\Upsilon^1)F^{\delta,u_1-\frac{1}{2}}_{[\tau,t]}(\Upsilon^1)F^{\delta,u_2-\frac{1}{2}}_{[0,\tau]}(\Upsilon^2)F^{\delta,u_2-\frac{1}{2}}_{[\tau,t]}(\Upsilon^2)\\ &- F^{\delta,u_2-\frac{1}{2}}_{[0,\tau]}(\Upsilon^1)F^{\delta,u_2-\frac{1}{2}}_{[\tau,t]}(\Upsilon^1)F^{\delta,u_1-\frac{1}{2}}_{[0,\tau]}(\Upsilon^2)F^{\delta,u_1-\frac{1}{2}}_{[\tau,t]}(\Upsilon^2)\Bigg] =: \mathcal{S}_2.\end{aligned}$$

Denote the left-hand-side of (4.7.7) by $\mathcal{S}$, then by a symmetrization we have

$$\begin{aligned}&\mathcal{S} = \mathcal{S}_1 = \mathcal{S}_2 = \frac{1}{2}\mathcal{S}_1 + \frac{1}{2}\mathcal{S}_2\\ &= \frac{1}{2}\mathbb{E}^{\mathbf{y},\mathbf{0}}_{\Upsilon^1,\Upsilon^2}\left[\left(F^{\delta,u_1-\frac{1}{2}}_{[0,\tau]}(\Upsilon^1)F^{\delta,u_2-\frac{1}{2}}_{[0,\tau]}(\Upsilon^2) - F^{\delta,u_1-\frac{1}{2}}_{[0,\tau]}(\Upsilon^2)F^{\delta,u_2-\frac{1}{2}}_{[0,\tau]}(\Upsilon^1)\right)F^{\delta,u_1-\frac{1}{2}}_{[\tau,t]}(\Upsilon^2)F^{\delta,u_2-\frac{1}{2}}_{[\tau,t]}(\Upsilon^1)\right]\\ &+ \frac{1}{2}\mathbb{E}^{\mathbf{y},\mathbf{0}}_{\Upsilon^1,\Upsilon^2}\left[\left(F^{\delta,u_1-\frac{1}{2}}_{[0,\tau]}(\Upsilon^1)F^{\delta,u_2-\frac{1}{2}}_{[0,\tau]}(\Upsilon^2) - F^{\delta,u_1-\frac{1}{2}}_{[0,\tau]}(\Upsilon^2)F^{\delta,u_2-\frac{1}{2}}_{[0,\tau]}(\Upsilon^1)\right)F^{\delta,u_1-\frac{1}{2}}_{[\tau,t]}(\Upsilon^1)F^{\delta,u_2-\frac{1}{2}}_{[\tau,t]}(\Upsilon^2)\right]\\ &= \frac{1}{2}\mathbb{E}^{\mathbf{y},\mathbf{0}}_{\Upsilon^1,\Upsilon^2}\left[\det\left[F^{\delta,u_i-\frac{1}{2}}_{[0,\tau]}(\Upsilon^j)\right]_{i,j\in\{1,2\}}\left(F^{\delta,u_1-\frac{1}{2}}_{[\tau,t]}(\Upsilon^1)F^{\delta,u_2-\frac{1}{2}}_{[\tau,t]}(\Upsilon^2) + F^{\delta,u_1-\frac{1}{2}}_{[\tau,t]}(\Upsilon^2)F^{\delta,u_2-\frac{1}{2}}_{[\tau,t]}(\Upsilon^1)\right)\right].\end{aligned}$$

By (4.7.8) and the assumption that $u_1 < u_2$, we have

$$(u_1 - u_2)\left(L^{0,\Upsilon^1}_{[0,\tau]} - L^{0,\Upsilon^2}_{[0,\tau]}\right) \le 0 \quad \mathbb{P}^{\mathbf{y},\mathbf{0}}_{\Upsilon^1,\Upsilon^2}\text{-almost surely.} \tag{4.7.9}$$

One can directly check that (4.7.9) implies $\det\left[F^{\delta,u_i-\frac{1}{2}}_{[0,\tau]}(\Upsilon^j)\right]_{i,j\in\{1,2\}} \ge 0$, $\mathbb{P}^{\mathbf{y},\mathbf{0}}_{\Upsilon^1,\Upsilon^2}$-almost surely. Since the functionals $F^{\delta,u_i-\frac{1}{2}}_{[\tau,t]}(\cdot)$ are always positive, $\mathcal{S} \ge 0$ is thus proved. □

4.8 Appendix

4.8.1 Auxiliary lemmas

4.8.1.1 Proof of Lemma 4.2.4

The proof follows a standard argument as in [125]. We use $\alpha \lesssim \beta$ to denote $\alpha \le C\beta$ for any constant $C = C(a, b, \mu_0, p, \zeta)$.

By the chaos expansion (4.2.12), for any fixed $\mu \in [\mu_0, \infty), s \in [a, b), t \in (s, b]$ and $x \in [0, \infty)$,

$$Z_\mu(t, x|s, \zeta) = \sum_{n=0}^{\infty} z_n(t, x|s, \zeta, \mu),$$

where

$$z_0(t, x|s, \zeta, \mu) := \int_0^\infty \mathscr{P}^R_\mu(t, x|s, y)\zeta(\mathrm{d}y),$$

$$\text{and} \quad z_{n+1}(t, x|s, \zeta, \mu) := \int_s^t \int_0^\infty \mathscr{P}^R_\mu(t, x|r, w) z_n(r, w|s, \zeta, \mu)\xi(\mathrm{d}w\mathrm{d}r).$$

Define

$$f_n(r) = \sup_{x\in[0,\infty)} \|z_n(r, x|s, \zeta, \mu)\|_p^2 [z_0(r, x|s, \zeta, \mu)]^{-1}.$$

We will show below that this supremum always exists. By Hypothesis 2, for any $\mu \in [\mu_0, \infty), r \in (s, b]$,

$$f_0(r) = \sup_{x\in[0,\infty)} z_0(r, x|s, \zeta, \mu) \lesssim (r-s)^{-1/2}.$$

While $z_{n+1}(t, x|s, \zeta, \mu)$ is not a martingale in t, the process

$$M_{n+1}(v) := \int_s^v \int_0^\infty \mathscr{P}_\mu^R(t, x|r, w) z_n(r, w|s, \zeta, \mu) \xi(\mathrm{d}w\mathrm{d}r)$$

is a martingale in $v \in [s, t]$. By applying the Burkholder-Davis-Gundy (to $M_{n+1}(v)$) and the Minkowski inequalities, (4.2.3) and semigroup property of the Robin heat kernel, we derive

$$\begin{aligned}
&\|z_{n+1}(t, x|s, \zeta, \mu)\|_p^2 = \|M_{n+1}(t)\|_p^2 \lesssim \left\| \int_s^t \int_0^\infty \left[\mathscr{P}_\mu^R(t, x|r, w) z_n(r, w|s, \zeta, \mu)\right]^2 \mathrm{d}r\mathrm{d}w \right\|_{p/2} \\
&\leq \int_s^t \int_0^\infty \mathscr{P}_\mu^R(t, x|r, w)^2 \|z_n(r, w|s, \zeta, \mu)\|_p^2 \mathrm{d}r\mathrm{d}w \\
&\leq \int_s^t \int_0^\infty \mathscr{P}_\mu^R(t, x|r, w)^2 z_0(r, w|s, \zeta, \mu) f_n(r) \mathrm{d}r\mathrm{d}w \\
&\lesssim z_0(t, x|s, \zeta, \mu) \int_r^t f_n(r)(t-r)^{-1/2} \mathrm{d}r.
\end{aligned}$$

where we used inequality (4.2.3) and semigroup property of the Robin heat kernel in the last inequality. Thus

$$f_{n+1}(t) \lesssim \int_s^t f_n(r)(t-r)^{-1/2} \mathrm{d}r.$$

It is straightforward to check that $f_1(r)$ and $f_2(r)$ are both bounded on $[s, t]$. By iteration, for any $n \geq 2$, (with the convention $(n/2)! = \Gamma(\frac{n}{2}+1)$,)

$$f_n(t) \leq C \int_s^t f_{n-2}(r) \mathrm{d}r \leq \frac{[C(t-s)]^{(n-1)/2}}{(n/2)!}$$

for some constant $C = C(a,b,\mu_0,p,\zeta)$. Now by the Minkowski's inequality,

$$\begin{aligned}
&\|Z_\mu(t,x|s,\zeta)\|_p \le \sum_{n=0}^{\infty} \|z_n(t,x|s,\zeta,\mu)\|_p \le \sum_{n=0}^{\infty} [z_0(t,x|s,\zeta,\mu) f_n(t)]^{1/2} \\
&\lesssim z_0(t,x|s,\zeta,\mu)^{1/2}(t-s)^{-1/4} \sum_{k=0}^{\infty} \left[\frac{[C(t-s)]^{n/2}}{(n/2)!}\right]^{1/2} \lesssim z_0(t,x|s,\zeta,\mu)^{1/2}(t-s)^{-1/4}.
\end{aligned}$$

This completes the proof.

4.8.1.2 Feynman-Kac Approximation

We describe how to approximate the half-space SHE solution $Z_\mu(t,x|s,\zeta)$ with Feynman-Kac type representations. This approximation is used throughout this section. The approximation is similar to full-space Feynman-Kac type approximations, except that the Brownian paths are now restricted to the positive half-plane (become the reflected Brownian motion), and we have a local time term on the boundary.

Here we consider a mollification of the space-time white noise $\xi(t,x)$. For its use in the proof of Lemma 4.2.5 below, we will smooth both time and spatial variables. As in [6], the Feynman-Kac type approximation is also valid when we only mollify the spatial variable. We omit the statements and proofs for the "spatial mollification only" approximation, as they are similar.

Let $g \in \mathcal{C}_c^\infty(\mathbb{R},[0,\infty))$ be a smooth symmetric function with compact support such that $\int_{\mathbb{R}} g(x)\mathrm{d}x = 1$. We define $g_\varepsilon(x) := \frac{1}{\varepsilon} g(\frac{x}{\varepsilon})$ and

$$\xi_{\varepsilon,\delta}(t,x) := e^{-\frac{1}{2}\varepsilon|t|^2-\frac{1}{2}\delta|x|^2} \int_{\mathbb{R}} \int_{\mathbb{R}} g_\varepsilon(t-s) p_\delta(x-y) \xi(\mathrm{d}y\mathrm{d}s), \tag{4.8.1}$$

where $p_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/(2t)}$ is the standard heat kernel. In the above mollification the extra factor $e^{-\frac{1}{2}\varepsilon|t|^2 - \frac{1}{2}\delta|x|^2}$ will be used later for the negative moments bound. With "$*$" denoting the convolution, the covariance of $\xi_{\varepsilon,\delta}$ is a nonnegative function

$$
\begin{aligned}
Q_{\varepsilon,\delta}(t_1, t_2, x_1, x_2) &:= \mathbf{E}[\xi_{\varepsilon,\delta}(t_1, x_1)\xi_{\varepsilon,\delta}(t_2, x_2)] \\
&= e^{-\frac{1}{2}\varepsilon(|t_1|^2 + |t_2|^2)}(g_\varepsilon * g_\varepsilon)(t_1 - t_2)e^{-\frac{1}{2}\delta(|x_1|^2 + |x_2|^2)} p_{2\delta}(x_1 - x_2),
\end{aligned} \tag{4.8.2}
$$

and $\xi_{\varepsilon,\delta}(\cdot,\cdot) \in \mathcal{C}^\infty(\mathbb{R}^2, \mathbb{R})$. One can check that for any $\varepsilon, \delta > 0$,

$$
\mathbf{E}\int_{\mathbb{R}}\int_{\mathbb{R}} |\xi_{\varepsilon,\delta}(t,x)|^2 \mathrm{d}t\mathrm{d}x = \int_{\mathbb{R}}\int_{\mathbb{R}} Q_{\varepsilon,\delta}(t,t,x,x)\mathrm{d}t\mathrm{d}x < \infty,
$$

so the noise $\xi_{\varepsilon,\delta}(\cdot,\cdot) \in L^2(\mathbb{R}^2)$ $\mathbf{P}$-almost surely.

For any $\varepsilon, \delta > 0$ and $\zeta(\cdot)$ under Hypothesis 2, define

$$
\begin{aligned}
&Z_\mu^{\varepsilon,\delta}(t, x|s, \zeta) \\
&:= \int_0^\infty \mathscr{P}^N(t, x|s, y)\mathbb{E}_B^x\left[\exp\left(-\mu L_{t-s}^0\right) : \exp : \left\{\int_0^{t-s} \xi_{\varepsilon,\delta}(t - r, |B_r|)\mathrm{d}r\right\} \middle| |B_{t-s}| = y\right]\zeta(\mathrm{d}y),
\end{aligned} \tag{4.8.3}
$$

with

$$
\begin{aligned}
&: \exp : \left\{\int_0^{t-s} \xi_{\varepsilon,\delta}(t - r, |B_r|)\mathrm{d}r\right\} \\
&\quad := \exp\left(\int_0^{t-s} \xi_{\varepsilon,\delta}(t - r, |B_r|)\mathrm{d}r - \frac{1}{2}\int_0^{t-s}\int_0^{t-s} Q_{\varepsilon,\delta}(t - r_1, t - r_2, |B_{r_1}|, |B_{r_2}|)\mathrm{d}r_1\mathrm{d}r_2\right).
\end{aligned}
$$

It may be possible to interpret (4.8.3) as the random field solution to the following equation in the Skorohod sense, with $\diamond$ denoting the Wick product. For equation of Skorohod type on the full-space, we refer to [126]. Here we do not pursue this interpretation for our half-space problem.

$$\begin{aligned}
\partial_t Z_\mu^{\varepsilon,\delta}(t,x|s,\zeta) &= \frac{1}{2}\partial_x^2 Z_\mu^{\varepsilon,\delta}(t,x|s,\zeta) + Z_\mu^{\varepsilon,\delta}(t,x|s,\zeta)\diamond \xi_{\varepsilon,\delta}(t,x),\\
\partial_x Z_\mu^{\varepsilon,\delta}(t,x|s,\zeta)\big|_{x=0} &= \mu Z_\mu^{\varepsilon,\delta}(t,0|s,\zeta),\\
Z_\mu^{\varepsilon,\delta}(s,\cdot|s,\zeta) &= \zeta(\cdot).
\end{aligned}$$

Let $\mathscr{L}_t^a(X)$ denote the (symmetric) local time at level a and at time t for any continuous semimartingale X, which means that, with $\langle X\rangle_\cdot$ denoting the quadratic variation of X, we define

$$\mathscr{L}_t^a(X) \coloneqq \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_0^t \mathbb{1}_{[a-\varepsilon,a+\varepsilon]}(X_s)\mathrm{d}\langle X\rangle_s.$$

We are using the usual local time definition so that it is consistent with the local time of Brownian motions defined after (4.2.4). When X is the difference of two independent standard Brownian motions, this definition of local time is two times the local time used in [6].

The following lemma is on the approximate of the solution $Z_\mu(t,x|s,\zeta)$. The expression in (4.2.14) can now be understood as the $L^p(\Omega)$ limit of $Z_\mu^{\varepsilon,\delta}(t,x|s,\zeta)$ as $\varepsilon,\delta\to 0$.

Lemma 4.8.1 (Feynman-Kac approximation)**.** *Let $\zeta(\cdot)$ be an initial condition satisfying Hypothesis 2. The following holds for any $\mu,s,t\in\mathbb{R}$ with $s<t$, and $x\in[0,\infty)$:*

(i). As $\varepsilon,\delta\to 0$, the approximated solution $Z_\mu^{\varepsilon,\delta}(t,x|s,\zeta)$ in (4.8.3) converges to $Z_\mu(t,x|s,\zeta)$ in $L^p(\Omega,\mathscr{F},\mathbf{P})$ for any $p\in[1,\infty)$, where $Z_\mu(t,x|s,\zeta)$ is the unique mild solution to (4.2.8). The convergence is uniform for $x\in[0,\infty)$, and for t in any compact subset of (s,∞).

(ii). For any integer $p\in[2,\infty)$, let $B_t^1,\dots,B_t^p$ be p independent Brownian motions starting at x and independent of ξ, with $L_t^{0,j}\coloneqq\mathscr{L}_t^0(B^j)$ for any B^j, $j=1,\cdots,p$, $t\geq 0$. For any

$x \in [0, \infty)$,

$$\mathbf{E}[Z_\mu^{\varepsilon,\delta}(t,x|s,\zeta)^p] = \int_{[0,\infty)} \cdots \int_{[0,\infty)} \zeta(\mathrm{d}y_1)\ldots\zeta(\mathrm{d}y_p) \prod_{j=1}^{p} \mathscr{P}^N(t,x|s,y_j)$$
$$\mathbb{E}^x_{B^1,\ldots,B^p}\left[e^{-\mu\sum_{j=1}^p L^{0,j}_{t-s} + \sum_{1\le i<j\le p}\int_0^{t-s}\int_0^{t-s} Q_{\varepsilon,\delta}(t-r_1,t-r_2,|B^i_{r_1}|,|B^j_{r_2}|)\mathrm{d}r_1\mathrm{d}r_2} \middle| |B^j_{t-s}| = y_j \textit{ for all } j \right]. \tag{4.8.4}$$

(iii). For any integer $p \in [2, \infty)$*,*

$$\mathbf{E}[Z_\mu(t,x|s,\zeta)^p] = \int_{[0,\infty)} \cdots \int_{[0,\infty)} \zeta(\mathrm{d}y_1)\ldots\zeta(\mathrm{d}y_p) \prod_{j=1}^{p} \mathscr{P}^N(t,x|s,y_j)$$
$$\mathbb{E}^x_{B^1,\ldots,B^p}\left[\exp(-\mu\sum_{j=1}^p L^{0,j}_{t-s}) \exp\left(\frac{1}{2}\sum_{1\le i<j\le p} \mathscr{L}^0_{t-s}(|B^i|-|B^j|)\right) \middle| |B^j_{t-s}| = y_j \textit{ for all } j \right]. \tag{4.8.5}$$

In particular, for any fixed $p \in [2,\infty)$*,* $a, b \in \mathbb{R}$ *with* $a < b$*,* $\mu_0 \in \mathbb{R}$*, there exists a constant* $C = C(p, \mu_0, a, b)$ *such that for any* $x \in [0,\infty)$*,* $a \le s < t \le b$*, and* $\mu \in [\mu_0, \infty)$*,*

$$\mathbf{E}[Z_\mu(t,x|s,\zeta)^p] \le C\left[\int_0^\infty \mathscr{P}^N(t,x|s,y)\zeta(\mathrm{d}y)\right]^p. \tag{4.8.6}$$

We first provide the next two useful results.

Lemma 4.8.2. *Let* $B.$ *be a standard Brownian motion. For any* $\tau > 0, \Lambda > 0$*, there exists a constant* $C(\tau,\Lambda) > 0$ *such that for each* $t \in [0,\tau], \lambda \in [0,\Lambda]$*,*

$$\sup_{x,y,a\in\mathbb{R}} \mathbb{E}^x_B\left[\exp(\lambda\mathscr{L}^a_t(B)) \mid B_t = y\right] \le C(\tau,\Lambda). \tag{4.8.7}$$

Proof. There are several ways to prove this property of local times. One way is to directly compute from the distribution of local times of a Brownian bridge (see [127]). An alternative is the arguments used in the proof of [6, Lemma 3.2]. By approximating the local time with some positive definite functions, an argument similar to the one used in the last inequality in (4.8.9) below can give another proof. □

Lemma 4.8.3. *For any $\Lambda > 0$ and $a < b$, there exists some uniform constant $C = C(\Lambda, a, b)$ such that for any two independent Brownian motions B^1, B^2 starting at $x \in [0, \infty)$,*

$$\sup_{\varepsilon,\delta\in(0,1)} \mathbb{E}^x_{B^1,B^2}\left[\exp\left(\lambda\int_0^{t-s}\int_0^{t-s} Q_{\varepsilon,\delta}(t-r_1, t-r_2, |B^1_{r_1}|, |B^2_{r_2}|)\mathrm{d}r_1\mathrm{d}r_2\right)\middle| |B^1_{t-s}| = y_1, |B^2_{t-s}| = y_2\right] \leq C, \tag{4.8.8}$$

for any $x, y_1, y_2 \in [0, \infty)$, $a \leq s < t \leq b$, and $\lambda \in [0, \Lambda]$.

Proof. We explain how to relate (4.8.8) to similar estimates for the full-space SHE and use the uniform bounds derived there. We first note that, for each fixed $y_1, y_2 \in [0, \infty)$, the left-hand side of (4.8.8) can be bounded above by a sum over four expectations conditioning on the events $\{B^i_{t-s} = \pm y_i, i = 1, 2\}$ respectively, i.e.,

$$\begin{aligned}
&\mathbb{E}^x_{B^1,B^2}\left[\exp\left(\lambda\int_0^{t-s}\int_0^{t-s} Q_{\varepsilon,\delta}(t-r_1, t-r_2, |B^1_{r_1}|, |B^2_{r_2}|)\mathrm{d}r_1\mathrm{d}r_2\right)\middle| |B^1_{t-s}| = y_1, |B^2_{t-s}| = y_2\right] \\
&\leq \sum_{i_1,i_2=\pm1} \mathbb{E}^x_{B^1,B^2}\Bigg[\exp\left(\lambda\int_0^{t-s}\int_0^{t-s} Q_{\varepsilon,\delta}(t-r_1, t-r_2, |B^1_{r_1}|, |B^2_{r_2}|)\mathrm{d}r_1\mathrm{d}r_2\right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Bigg| B^1_{t-s} = i_1 y_1, B^2_{t-s} = i_2 y_2\Bigg].
\end{aligned}$$

To ease notations, we use $\mathbb{E}^{x_1,x_2,y_1,y_2}_{B^1,B^2}$ as the expectation over independent Brownian bridges B^1, B^2 with $B^1_0 = x_1, B^2_0 = x_2, B^1_{t-s} = y_1$ and $B^2_{t-s} = y_2$ for any $x_1, x_2, y_1, y_2 \in \mathbb{R}$.

A straightforward observation is that, for any realization of B^1, B^2 and any $r_1, r_2 \in [0, t-s]$,

$$Q_{\varepsilon,\delta}(t-r_1, t-r_2, |B^1_{r_1}|, |B^2_{r_2}|) \le Q_{\varepsilon,\delta}(t-r_1, t-r_2, |B^1_{r_1}|, |B^2_{r_2}|) \sum_{i_1, i_2 = \pm 1} \mathbb{1}_{\{i_1 B_{r_1} \ge 0, i_2 B_{r_2} \ge 0\}}(r_1, r_2)$$

$$\le \sum_{i_1, i_2 = \pm 1} Q_{\varepsilon,\delta}(t-r_1, t-r_2, i_1 B^1_{r_1}, i_2 B^2_{r_2}).$$

Thus by Cauchy-Schwarz inequality, for any $\varepsilon, \delta > 0$ and $x_1, x_2, y_1, y_2 \in \mathbb{R}$,

$$\begin{aligned}
&\mathbb{E}^{x,x,y_1,y_2}_{B^1,B^2}\left[\exp\left(\lambda \int_0^{t-s}\int_0^{t-s} Q_{\varepsilon,\delta}(t-r_1, t-r_2, |B^1_{r_1}|, |B^2_{r_2}|)\mathrm{d}r_1\mathrm{d}r_2\right)\right] \\
&\le \prod_{i_1, i_2 = \pm 1}\left\{\mathbb{E}^{x,x,y_1,y_2}_{B^1,B^2}\left[\exp\left(4\lambda \int_0^{t-s}\int_0^{t-s} Q_{\varepsilon,\delta}(t-r_1, t-r_2, i_1 B^1_{r_1}, i_2 B^2_{r_2})\mathrm{d}r_1\mathrm{d}r_2\right)\right]\right\}^{1/4},
\end{aligned}$$

and (4.8.8) will be proved if we can show that there exists $C = C(\Lambda, a, b)$, such that

$$\sup_{\varepsilon,\delta \in (0,1)} \sup_{x_1,x_2,y_1,y_2 \in \mathbb{R}} \mathbb{E}^{x_1,x_2,y_1,y_2}_{B^1,B^2}\left[\exp\left(4\lambda \int_0^{t-s}\int_0^{t-s} Q_{\varepsilon,\delta}(t-r_1, t-r_2, B^1_{r_1}, B^2_{r_2})\mathrm{d}r_1\mathrm{d}r_2\right)\right] \le C,$$

for any $a \le s < t \le b$, and $\lambda \in [0, \Lambda]$.

To give a uniform bound over different starting points and endpoints, we note that for any $\varepsilon, \delta > 0$, $x_1, x_2, y_1, y_2 \in \mathbb{R}$,

$$\begin{aligned}
&\mathbb{E}^{x_1,x_2,y_1,y_2}_{B^1,B^2}\left[\exp\left(4\lambda \int_0^{t-s}\int_0^{t-s} Q_{\varepsilon,\delta}(t-r_1, t-r_2, B^1_{r_1}, B^2_{r_2})\mathrm{d}r_1\mathrm{d}r_2\right)\right] \\
&\le \mathbb{E}^{x_1,x_2,y_1,y_2}_{B^1,B^2}\left[\exp\left(4\lambda \int_0^{t-s}\int_0^{t-s} (g_\varepsilon * g_\varepsilon)(r_1 - r_2) p_{2\delta}(B^1_{r_1} - B^2_{r_2})\mathrm{d}r_1\mathrm{d}r_2\right)\right] \\
&= \mathbb{E}^{0,0,0,0}_{B^1,B^2}\Bigg[\exp\Bigg(4\lambda \int_0^{t-s}\int_0^{t-s} (g_\varepsilon * g_\varepsilon)(r_1 - r_2) \qquad (4.8.9) \\
&p_{2\delta}\left(B^1_{r_1} - B^2_{r_2} + x_1 - x_2 + \frac{r_1(y_1 - x_1) - r_2(y_2 - x_2)}{t-s}\right)\mathrm{d}r_1\mathrm{d}r_2\Bigg)\Bigg] \\
&\le \mathbb{E}^{0,0,0,0}_{B^1,B^2}\left[\exp\left(4\lambda \int_0^{t-s}\int_0^{t-s} (g_\varepsilon * g_\varepsilon)(r_1 - r_2) p_{2\delta}\left(B^1_{r_1} - B^2_{r_2}\right)\mathrm{d}r_1\mathrm{d}r_2\right)\right],
\end{aligned}$$

where the equation follows from rescaling the Brownian bridges, and the last inequality follows from a generalization of [128, Lemma 4.1] (see also [129, (4.5)]). Roughly speaking, the proof of the last inequality can be done by expanding the exponential, applying Fourier transforms to the Gaussians, and using the fact that $p_{2\delta}$ is a positive definite function for any $\delta > 0$. Heuristically, this is to say that an approximated "local time at level 0" of Brownian bridges starting from $x_1 - x_2$ and ending at $y_1 - y_2$ would reach its maximum when $x_1 - x_2 = 0$ and $y_1 - y_2 = 0$.

Combining the results above, we have reduced the problem of showing (4.8.8) to showing that for any $a \leq s < t \leq b$, and $\lambda \in [0, \Lambda]$,

$$\sup_{\varepsilon,\delta\in(0,1)} \mathbb{E}^{0,0,0,0}_{B^1,B^2}\left[\exp\left(4\lambda \int_0^{t-s}\int_0^{t-s} (g_\varepsilon * g_\varepsilon)(r_1 - r_2) p_{2\delta}\left(B^1_{r_1} - B^2_{r_2}\right) \mathrm{d}r_1 \mathrm{d}r_2\right)\right] \leq C(\Lambda, a, b),$$

where the Brownian bridges no longer have reflections. This uniform bound would follow from the same arguments as for a similar result [126, (3.17)]. Compared with [126, (3.17)], a few adaptions are needed for the noise ξ being white in time (instead of colored as in [126, (3.17)]) and B^1, B^2 being Brownian bridges (instead of Brownian motions as in [126, (3.17)]). The adaptation for time independence is straightforward from the computations. For the adaption to Brownian bridges, we refer to [128, Proposition 4.2], where essential estimates have been proved.

□

We now prove Lemma 4.8.1.

Proof of Lemma 4.8.1. A direct Gaussian calculation gives (ii). To prove (i) and (4.8.5), one can follow the same arguments as used in [126, Theorem 3.6] for the analogous full-space SHEs. In particular, the proof mainly needs two estimates: (i) passing to the limit: for each $p \in [1, \infty)$,

$$\lim_{\varepsilon,\delta\to 0} \mathbb{E}^x_{B^1,B^2}\Bigg[\bigg| \int_0^{t-s}\int_0^{t-s} Q_{\varepsilon,\delta}(t-r_1, t-r_2, |B^1_{r_1}|, |B^2_{r_2}|)\mathrm{d}r_1\mathrm{d}r_2 \\ - \frac{1}{2}\mathscr{L}^0_{t-s}\left(|B^1| - |B^2|\right)\bigg|^p \bigg| |B^1_{t-s}| = y_1, |B^2_{t-s}| = y_2\Bigg] = 0, \tag{4.8.10}$$

uniformly for $x, y_1, y_2 \in [0, \infty)$ and t in compact subset of (s, ∞); (ii) a uniform bound: for any $p \in [1, \infty)$,

$$\sup_{\varepsilon,\delta\in(0,1)} \sup_{x,y_1,\dots,y_p\in[0,\infty)} \mathbb{E}^x_{B^1,\dots,B^p}\Bigg[\exp(-\mu\sum_{j=1}^p L^{0,j}_{t-s}) \\ \exp\left(\sum_{1\le i<j\le p}\int_0^{t-s}\int_0^{t-s} Q_{\varepsilon,\delta}(t-r_1, t-r_2, |B^i_{r_1}|, |B^j_{r_2}|)\mathrm{d}r_1\mathrm{d}r_2\right)\bigg| |B^j_{t-s}| = y_j \text{ for all } j\Bigg] \\ \le C(p, \mu, s, t). \tag{4.8.11}$$

The limit (4.8.10) follows from the continuity of the semimartingale $|B^1| - |B^2|$, the density of occupation time formula [122, Corollary 9.7], and that the local time process $\mathscr{L}^a_t(|B^1| - |B^2|)$ is continuous in $a \in \mathbb{R}$ almost surely (which can be verified using e.g. [122, Theorem 9.4]). The uniform bound (4.8.11) follows from Hölder's inequality applied to the conditional expectation and applying Lemma 4.8.2 and Lemma 4.8.3.

The proof also shows that the right-hand side of (4.8.5) is the limit of the right-hand side of (4.8.4) as $\varepsilon, \delta \to 0$. To prove (4.8.6), we apply Hölder's inequality to the conditional expectation in (4.8.4), and use Lemma 4.8.2 and Lemma 4.8.3 with the convergence. From the uniform bounds in Lemma 4.8.2 and Lemma 4.8.3, it is not hard to see that the constant in (4.8.6) is uniform for $x \in [0, \infty)$, $\mu \in [\mu_0, \infty)$ and $a \le s < t \le b$. □

4.8.1.3 Proof of Lemma 4.2.5

In this section, we provide a proof of the negative moments bounds (Lemma 4.2.5) following the argument in [103]. To simplify notations, we only present the case when the initial condition ζ has a continuous density $f(\cdot)$ with respect to Lebesgue measure, but this is not required as the Feynman-Kac approximation Lemma 4.8.1 is valid for general ζ under Hypothesis 2.

Using the notations as in Appendix 4.8.1.2, we define

$$\begin{aligned}\Phi_{\mu}^{\varepsilon,\delta}(t,x|0,\zeta) &:= \mathbb{E}^{x}_{B^1,B^2}\Bigg[f\left(|B_t^1|\right) f\left(|B_t^2|\right) \exp(-\mu L_t^{0,1} - \mu L_t^{0,2}) \\ &\exp\left(\int_0^t \int_0^t Q_{\varepsilon,\delta}(t-s,t-r,|B_r^1|,|B_s^2|)\mathrm{d}s\mathrm{d}r + \frac{1}{2}\mathscr{L}_t^0(|B^1|-|B^2|)\right)\Bigg].\end{aligned} \tag{4.8.12}$$

Recall that $z_\mu(t,x|0,\zeta) := \int_0^\infty \mathscr{P}_\mu^R(t,x|0,y)\zeta(\mathrm{d}y)$. We first prove the following auxiliary result:

Proposition 4.8.4. *Let ζ be the constant initial condition or the Dirac-δ initial condition $\zeta = \delta_y$ for some $y \in [0,\infty)$. For any $\tau > 0$, there exists some positive constant $C = C(\mu,\tau)$ such that*

$$\sup_{x\in[0,\infty)} \sup_{\varepsilon,\delta\in(0,1)} \frac{\mathbf{E}[Z_\mu^{\varepsilon,\delta}(t,x|0,\zeta)^2]}{z_\mu(t,x|0,\zeta)^2} \le C(\mu,\tau), \tag{4.8.13}$$

and

$$\sup_{x\in[0,\infty)} \sup_{\varepsilon,\delta\in(0,1)} \frac{\Phi_\mu^{\varepsilon,\delta}(t,x|0,\zeta)}{z_\mu(t,x|0,\zeta)^2} \le C(\mu,\tau), \tag{4.8.14}$$

for any $t \in (0,\tau]$. The constant $C(\mu,\tau)$ can be chosen uniformly for μ in any compact subset of $\mathbb{R}$.

Proof. The proof is essentially applying Hölder's inequality to the conditional expectations and using the uniform bounds in Lemma 4.8.2 and Lemma 4.8.3. These estimates would give us

(4.8.13) and (4.8.14) with $z_\mu(t,x|0,\zeta)^2$ replaced by $z_0(t,x|0,\zeta)^2$ on the denominators. Note that when $\zeta = \delta_y$, $z_0(t,x|0,\zeta) = \mathscr{P}_N(t,x|0,y)$. Another estimate useful here is that following Lemma 4.8.2, for any $\tau > 0, \mu_0 \in \mathbb{R}$, there exists a constant $C(\mu_0,\tau) > 0$ such that

$$\sup_{t\in(0,\tau]}\sup_{\mu\le\mu_0}\frac{z_0(t,x|0,\delta_y)}{z_\mu(t,x|0,\delta_y)} = \sup_{t\in(0,\tau]}\sup_{\mu\le\mu_0}\frac{1}{\mathbb{E}_B^x[\exp(-\mu L_t)\,|\,|B_t|=y]} \le C(\mu_0,\tau), \tag{4.8.15}$$

for all $x, y \in [0,\infty)$. Similarly, we also have that

$$\sup_{t\in(0,\tau]}\sup_{\mu\le\mu_0}\frac{z_0(t,x|0,\mathbb{1})}{z_\mu(t,x|0,\mathbb{1})} = \sup_{t\in(0,\tau]}\sup_{\mu\le\mu_0}\frac{1}{\mathbb{E}_B^x[\exp(-\mu L_t)]} \le C'(\mu_0,\tau),$$

for all $x \in [0,\infty)$ with some constant $C'(\mu_0,\tau) > 0$. □

Following [103], we now present the proof of Lemma 4.2.5.

For any fixed $\mu \in \mathbb{R}, t > 0$ and any $L^2(\mathbb{R}^2)$-valued sample path $\xi_{\varepsilon,\delta}$ of the smoothed space-time noise (4.8.1), define

$$\Theta(B_t,\xi_{\varepsilon,\delta}) := f(|B_t|)\exp(-\mu L_t^0) : \exp : \left\{\int_0^t \xi_{\varepsilon,\delta}(t-r,|B_r|)\mathrm{d}r\right\}.$$

$\Theta(B_t,\xi_{\varepsilon,\delta})$ is a continuous functional of the Brownian motion B_t starting from $x \in [0,\infty)$. For any bounded functional F of the Brownian motion B, we define its weighted expectation as

$$\begin{aligned}\mathbb{E}^B_{x,\xi_{\varepsilon,\delta}}[F] &= \frac{\mathbb{E}_B^x\left[F(\cdot)\Theta(B_t,\xi_{\varepsilon,\delta})\right]}{\mathbb{E}_B^x\left[\Theta(B_t,\xi_{\varepsilon,\delta})\right]}\\ &= \frac{\mathbb{E}_B^x\left[F(\cdot)f(|B_t|)\exp(-\mu L_t^0) : \exp : \left\{\int_0^t \xi_{\varepsilon,\delta}(t-r,|B_r|)\mathrm{d}r\right\}\right]}{\mathbb{E}_B^x\left[f(|B_t|)\exp(-\mu L_t^0) : \exp : \left\{\int_0^t \xi_{\varepsilon,\delta}(t-r,|B_r|)\mathrm{d}r\right\}\right]}.\end{aligned}$$

For any bounded functional F of two paths B^1, B^2, we define

$$\mathbb{E}_{x,\xi_{\varepsilon,\delta}}^{B^1,B^2}[F] = \frac{\mathbb{E}_{B^1,B^2}^x\left[F(\cdot)\Theta(B_t^1,\xi_{\varepsilon,\delta})\Theta(B_t^2,\xi_{\varepsilon,\delta})\right]}{\mathbb{E}_{B^1,B^2}^x\left[\Theta(B_t^1,\xi_{\varepsilon,\delta})\Theta(B_t^2,\xi_{\varepsilon,\delta})\right]}.$$

To ease notations, we write

$$Z_\mu(t,x,\xi_{\varepsilon,\delta}) := \mathbb{E}_B^x\left[f(|B_t|)\exp\left(-\mu L_t^0\right) : \exp : \left\{\int_0^t \xi_{\varepsilon,\delta}(t-r,|B_r|)\mathrm{d}r\right\}\right]$$

to emphasize the dependence of $Z_\mu^{\varepsilon,\delta}(t,x|0,\zeta)$ on the noise $\xi_{\varepsilon,\delta}$. For any fixed Brownian path B_t,

$$\mathbf{E}\left[: \exp : \left\{\int_0^t \xi_{\varepsilon,\delta}(t-r,|B_r|)\mathrm{d}r\right\}\right] = 1.$$

Tonelli's theorem gives $\mathbf{E}\left[Z_\mu(t,x,\xi_{\varepsilon,\delta})\right] = z_\mu(t,x|0,\zeta)$. As above, we use $\mathscr{L}_t^a(\cdot)$ to denote the local time at level $a \in \mathbb{R}$ up to time $t \geq 0$ for continuous semimartingales.

Now for each $\lambda > 0, (t,x) \in (0,\infty)\times[0,\infty)$, define the event

$$A_\lambda(t,x) = \left\{\xi'_{\varepsilon,\delta} : Z_\mu(t,x,\xi'_{\varepsilon,\delta}) > \frac{1}{2}z_\mu(t,x|0,\zeta), \quad \mathbb{E}_{x,\xi'_{\varepsilon,\delta}}^{B^1,B^2}\left[\frac{1}{2}\mathscr{L}_t^0(|B^1|-|B^2|)\right] \leq \lambda\right\}, \tag{4.8.16}$$

and a distance

$$d(\xi_{\varepsilon,\delta}, A_\lambda(t,x)) := \inf\{\|\xi_{\varepsilon,\delta} - \xi'_{\varepsilon,\delta}\|_{L^2(\mathbb{R}^2)} : \xi'_{\varepsilon,\delta} \in A_\lambda(t,x)\}.$$

We first prove the following lemmas.

Lemma 4.8.5. *For any sample path $\xi_{\varepsilon,\delta}$ and any $\lambda > 0$,*

$$Z_\mu(t,x,\xi_{\varepsilon,\delta}) \geq \frac{1}{2} z_\mu(t,x|0,\zeta) \exp\left[-\sqrt{\lambda} d(\xi_{\varepsilon,\delta}, A_\lambda(t,x))\right].$$

Proof. Fix any $\xi'_{\varepsilon,\delta} \in A_\lambda(t,x)$. By definitions and Jensen's inequality, we have

$$\begin{aligned} Z_\mu(t,x,\xi_{\varepsilon,\delta}) &= Z_\mu(t,x,\xi'_{\varepsilon,\delta}) \mathbb{E}^B_{x,\xi'_{\varepsilon,\delta}} \left[\frac{:\exp:\left\{\int_0^t \xi_{\varepsilon,\delta}(t-r,|B_r|)\mathrm{d}r\right\}}{:\exp:\left\{\int_0^t \xi'_{\varepsilon,\delta}(t-r,|B_r|)\mathrm{d}r\right\}} \right] \\ &= Z_\mu(t,x,\xi'_{\varepsilon,\delta}) \mathbb{E}^B_{x,\xi'_{\varepsilon,\delta}} \left\{ \exp\left[\int_0^t \xi_{\varepsilon,\delta}(t-r,|B_r|)\mathrm{d}r - \int_0^t \xi'_{\varepsilon,\delta}(t-r,|B_r|)\mathrm{d}r \right] \right\} \\ &\geq Z_\mu(t,x,\xi'_{\varepsilon,\delta}) \exp\left\{ \mathbb{E}^B_{x,\xi'_{\varepsilon,\delta}} \left[\int_0^t \xi_{\varepsilon,\delta}(t-r,|B_r|)\mathrm{d}r - \int_0^t \xi'_{\varepsilon,\delta}(t-r,|B_r|)\mathrm{d}r \right] \right\}. \end{aligned} \tag{4.8.17}$$

The exponent can be estimated as

$$\begin{aligned} &\left| \mathbb{E}^B_{x,\xi'_{\varepsilon,\delta}} \left[\int_0^t \xi_{\varepsilon,\delta}(t-r,|B_r|)\mathrm{d}r - \int_0^t \xi'_{\varepsilon,\delta}(t-r,|B_r|)\mathrm{d}r \right] \right| \\ &\leq \mathbb{E}^B_{x,\xi'_{\varepsilon,\delta}} \left[\int_0^t \left| \xi_{\varepsilon,\delta}(t-r,|B_r|) - \xi'_{\varepsilon,\delta}(t-r,|B_r|) \right| \mathrm{d}r \right] \\ &\leq \left[\frac{1}{2} \mathbb{E}^{B^1,B^2}_{x,\xi'_{\varepsilon,\delta}} \left[\mathscr{L}^0_t(|B^1| - |B^2|) \right] \right]^{1/2} \|\xi_{\varepsilon,\delta} - \xi'_{\varepsilon,\delta}\|_{L^2(\mathbb{R}^2)}. \end{aligned} \tag{4.8.18}$$

The last "≤" will be proved at the end. Since $\xi'_{\varepsilon,\delta} \in A_\lambda(t,x)$, by the second inequality in (4.8.16), the above expression is further bounded by $\sqrt{\lambda}\|\xi_{\varepsilon,\delta} - \xi'_{\varepsilon,\delta}\|_{L^2(\mathbb{R}^2)}$. Therefore, we have

$$Z_\mu(t,x,\xi_{\varepsilon,\delta}) \geq Z_\mu(t,x,\xi'_{\varepsilon,\delta}) \exp(-\sqrt{\lambda}\|\xi_{\varepsilon,\delta} - \xi'_{\varepsilon,\delta}\|_{L^2(\mathbb{R}^2)}).$$

The proof will be complete when we combine the above with the first inequality in (4.8.16) and take the supremum over all $\xi'_{\varepsilon,\delta} \in A_\lambda(t,x)$ on the right-hand side.

It remains to prove the last inequality in (4.8.18). Formally, this inequality follows from the density of occupation time formula (see e.g. [122, Corollary 9.7]) and Hölder's inequality on $\mathbb{R}^2$. However, since the local time process $\frac{1}{2}\mathscr{L}_t^a$ is not differentiable in t, we will prove it by approximation.

Set $\Delta\xi(t,x) \coloneqq |\xi'_{\varepsilon,\delta}(t,x) - \xi_{\varepsilon,\delta}(t,x)|$ for $(t,x) \in \mathbb{R}^2$. Since $\Delta\xi$ is deterministic, continuous and in $L^2(\mathbb{R}^2)$, the second line of (4.8.18) can be approximated by a family of convolutions

$$\mathbb{E}^B_{x,\xi'_{\varepsilon,\delta}}\left[\int_0^t \Delta\xi(t-r,|B_r|)\mathrm{d}r\right] = \lim_{\kappa\to 0}\mathbb{E}^B_{x,\xi'_{\varepsilon,\delta}}\left[\int_0^t\int_{\mathbb{R}}\Delta\xi(t-r,y)p_\kappa(|B_r|-y)\mathrm{d}y\mathrm{d}r\right].$$

For each $\kappa > 0$, by Tonelli's theorem and Hölder's inequality,

$$\begin{aligned}
&\mathbb{E}^B_{x,\xi'_{\varepsilon,\delta}}\left[\int_0^t\int_{\mathbb{R}}\Delta\xi(t-r,y)p_\kappa(|B_r|-y)\mathrm{d}y\mathrm{d}r\right] = \int_0^t\int_{\mathbb{R}}\Delta\xi(t-r,y)\mathbb{E}^B_{x,\xi'_{\varepsilon,\delta}}\left[p_\kappa(|B_r|-y)\right]\mathrm{d}y\mathrm{d}r \\
&\leq \left[\int_0^t\int_{\mathbb{R}}\left(\mathbb{E}^B_{x,\xi'_{\varepsilon,\delta}}\left[p_\kappa(|B_r|-y)\right]\right)^2\mathrm{d}y\mathrm{d}r\right]^{1/2}\|\Delta\xi\|_{L^2(\mathbb{R}^2)} \\
&= \left[\int_0^t\int_{\mathbb{R}}\left(\mathbb{E}^{B^1,B^2}_{x,\xi'_{\varepsilon,\delta}}\left[p_\kappa(|B^1_r|-y)p_\kappa(|B^2_r|-y)\right]\right)\mathrm{d}y\mathrm{d}r\right]^{1/2}\|\Delta\xi\|_{L^2(\mathbb{R}^2)} \\
&= \left[\mathbb{E}^{B^1,B^2}_{x,\xi'_{\varepsilon,\delta}}\left(\int_0^t p_{2\kappa}(|B^1_r|-|B^2_r|)\mathrm{d}r\right)\right]^{1/2}\|\Delta\xi\|_{L^2(\mathbb{R}^2)}.
\end{aligned}$$

By the density of occupation time formula, since the quadratic variation of the semimartingale $|B^1_r| - |B^2_r|$ equals to $2r$,

$$\int_0^t p_{2\kappa}(|B^1_r|-|B^2_r|)\mathrm{d}r = \frac{1}{2}\int_{\mathbb{R}}p_{2\kappa}(a)\mathscr{L}^a_t(|B^1|-|B^2|)\mathrm{d}a.$$

The proof is complete since

$$\int_{\mathbb{R}}p_{2\kappa}(a)\mathbb{E}^{B^1,B^2}_{x,\xi'_{\varepsilon,\delta}}\left[\frac{1}{2}\mathscr{L}^a_t(|B^1|-|B^2|)\right]\mathrm{d}a \to \mathbb{E}^{B^1,B^2}_{x,\xi'_{\varepsilon,\delta}}\left[\frac{1}{2}\mathscr{L}^0_t(|B^1|-|B^2|)\right], \quad \text{as } \kappa\to 0.$$

□

We next show that the probability of $\xi'_{\varepsilon,\delta} \in A_\lambda(t,x)$ is always positive.

Lemma 4.8.6. *For any $\mu \in \mathbb{R}$, $\tau > 0$, there exists some constant $\lambda = \lambda(\mu,\tau)$ such that for any $x \in [0,\infty)$, $t \in (0,\tau]$ and ζ addressed in Proposition 4.8.4,*

$$\mathbf{P}(A_\lambda(t,x)) \geq \frac{1}{8}\frac{1}{C(\mu,\tau)} =: b(\mu,\tau),$$

where $C(\mu,\tau)$ is the constant in (4.8.13) *and* (4.8.14).

Proof. From (4.8.16), we see that the set of noise

$$\Big\{\xi'_{\varepsilon,\delta} : Z_\mu(t,x,\xi'_{\varepsilon,\delta}) > \frac{1}{2}z_\mu(t,x|0,\zeta),$$
$$\mathbb{E}^x_{B^1,B^2}\left[\frac{1}{2}\mathscr{L}^0_t(|B^1|-|B^2|)\Theta(B^1_t,\xi'_{\varepsilon,\delta})\Theta(B^2_t,\xi'_{\varepsilon,\delta})\right] \leq \frac{\lambda}{4}z_\mu(t,x|0,\zeta)^2\Big\}$$

is a subset of $A_\lambda(t,x)$. Thus we have that

$$\mathbf{P}(A_\lambda(t,x)) \geq \mathbf{P}\left(Z_\mu(t,x,\xi'_{\varepsilon,\delta}) > \frac{1}{2}z_\mu(t,x|0,\zeta)\right) - \mathbf{P}(B_\lambda(t,x)), \tag{4.8.19}$$

where

$$B_\lambda(t,x) = \left\{\xi'_{\varepsilon,\delta} : \mathbb{E}^x_{B^1,B^2}\left[\frac{1}{2}\mathscr{L}^0_t(|B^1|-|B^2|)\Theta(B^1_t,\xi'_{\varepsilon,\delta})\Theta(B^2_t,\xi'_{\varepsilon,\delta})\right] > \frac{\lambda}{4}z_\mu(t,x|0,\zeta)^2\right\}.$$

By the Paley–Zygmund inequality and (4.8.13), when $\varepsilon, \delta \in (0,1)$, for any $t \in (0,\tau]$, $x \in [0,\infty)$,

$$\mathbf{P}\left(Z_\mu(t,x,\xi'_{\varepsilon,\delta}) > \frac{1}{2} z_\mu(t,x|0,\zeta)\right) \geq \frac{1}{4}\frac{z_\mu(t,x|0,\zeta)^2}{\mathbf{E}\left[Z_\mu(t,x,\xi'_{\varepsilon,\delta})^2\right]} \geq \frac{1}{4}\frac{1}{C(\mu,\tau)}, \tag{4.8.20}$$

where $C(\mu,\tau)$ is the constant in Proposition 4.8.4. To estimate $\mathbf{P}(B_\lambda(t,x))$, by Tonelli's theorem and Lemma 4.8.1 (ii),

$$\begin{aligned}
&\mathbf{E}\mathbb{E}^x_{B^1,B^2}\left[\frac{1}{2}\mathscr{L}^0_t(|B^1|-|B^2|)\Theta(B^1_t,\xi'_{\varepsilon,\delta})\Theta(B^2_t,\xi'_{\varepsilon,\delta})\right] \\
&= \mathbb{E}^x_{B^1,B^2}\left[\frac{1}{2}\mathscr{L}^0_t(|B^1|-|B^2|)\mathbf{E}[\Theta(B^1_t,\xi'_{\varepsilon,\delta})\Theta(B^2_t,\xi'_{\varepsilon,\delta})]\right] \\
&= \mathbb{E}^x_{B^1,B^2}\Bigg[\frac{1}{2}\mathscr{L}^0_t(|B^1|-|B^2|) f\left(|B^1_t|\right) f\left(|B^2_t|\right) \\
&\qquad \exp(-\mu L^{0,1}_t - \mu L^{0,2}_t)\exp\left(\int_0^t\int_0^t Q_{\varepsilon,\delta}(t-s,t-r,|B^1_r|,|B^2_s|)\mathrm{d}s\mathrm{d}r\right)\Bigg].
\end{aligned}$$

Using a trivial bound $a \leq \exp(a)$ for all $a \geq 0$, the last expression is bounded above by the value $\Phi^{\varepsilon,\delta}_\mu(t,x|0,\zeta)$ defined in (4.8.12). By Chebyshev inequality and (4.8.14), when $\varepsilon,\delta \in (0,1)$, for any $t \in (0,\tau]$, $x \in [0,\infty)$,

$$\mathbf{P}(B_\lambda(t,x)) \leq \frac{4\Phi^{\varepsilon,\delta}_\mu(t,x|0,\zeta)}{\lambda z_\mu(t,x|0,\zeta)^2} \leq \frac{4C(\mu,\tau)}{\lambda}.$$

Together with (4.8.19) and (4.8.20), we have that for any $x \in [0,\infty)$, $t \in (0,\tau]$,

$$\mathbf{P}(A_\lambda(t,x)) \geq \frac{1}{4}\frac{1}{C(\mu,\tau)} - \frac{4C(\mu,\tau)}{\lambda}.$$

Let

$$\lambda_0(\mu,\tau) := 32C(\mu,\tau)^2.$$

By Proposition 4.8.4, $\lambda_0(\mu,\tau) < \infty$. Choose some $\lambda = \lambda(\mu,\tau) > \lambda_0(\mu,\tau)$, then for any $x \in [0,\infty)$, $t \in (0,\tau]$,

$$\mathbf{P}(A_\lambda(t,x)) \geq \frac{1}{8}\frac{1}{C(\mu,\tau)} > 0.$$

□

With Lemma 4.8.6, we can further bound the probability of any sample path $\xi_{\varepsilon,\delta}$ deviating from the set $A_\lambda(t,x)$: for any $t \in (0,\tau], x \in [0,\infty)$ and $a > 0$,

$$\mathbf{P}\left(d(\xi_{\varepsilon,\delta}, A_\lambda(t,x)) \geq a + 2\sqrt{\log\frac{2}{b(\mu,\tau)}}\right) \leq 2e^{-a^2/4}. \tag{4.8.21}$$

This is derived from Talagrand's concentration inequality. The proof only depends on estimates of the space-time noise $\xi_{\varepsilon,\delta}$ and the lower bound on the probability of event $A_\lambda(t,x)$ from Lemma 4.8.6. A technical (though not difficult) part here is that one needs to use an $L^2(\mathbb{R}^2)$ approximation $\xi_{\varepsilon,\delta,n}$ for $\xi_{\varepsilon,\delta}$ and prove concentration at each n. Since we are using the same $\xi_{\varepsilon,\delta}$ as in the (1+1) full-space SHEs, we refer to [103, Lemma 4.5] for the proof of (4.8.21).

We are now ready to prove Lemma 4.2.5.

Proof of Lemma 4.2.5. By Lemma 4.8.5, for any fixed $\mu \in \mathbb{R}, t \in (0,\tau], x \in [0,\infty)$, we have

$$Z_\mu(t,x,\xi_{\varepsilon,\delta}) \geq \frac{1}{2} z_\mu(t,x|0,\zeta)\exp\left[-\sqrt{\lambda}\, d(\xi_{\varepsilon,\delta}, A_\lambda(t,x))\right].$$

Applying (4.8.21), for any $\lambda > \lambda_0(\mu,\tau)$, $a > 0$,

$$\mathbf{P}\left(\frac{Z_\mu(t,x,\xi_{\varepsilon,\delta})}{z_\mu(t,x|0,\zeta)} \leq \frac{1}{2}\exp\left(-\sqrt{\lambda}\left[a + 2\sqrt{\log\frac{2}{b(\mu,\tau)}}\right]\right)\right) \leq 2e^{-a^2/4}.$$

As λ, b are independent of ε, δ, by Lemma 4.8.1 (i), by sending $\varepsilon, \delta \to 0$, we have

$$\mathbf{P}\left(F(t,x) \le \exp\left(-\sqrt{\lambda}a\right)\right) \le 2e^{-a^2/4}, \quad \text{for } F(t,x) := \frac{Z_\mu(t,x|0,\zeta)}{z_\mu(t,x|0,\zeta)} 2\exp\left(2\sqrt{\lambda \log \frac{2}{b(\mu,\tau)}}\right).$$

As in [103, Corollary 4.8], for any $p \ge 0$, we compute

$$\begin{aligned}
&\mathbf{E}F^{-p} = p\int_0^\infty r^{-p}\mathbf{P}(F<r)\frac{\mathrm{d}r}{r} \le p\int_0^1 r^{-p}\mathbf{P}(F<r)\frac{\mathrm{d}r}{r} + p\int_1^\infty r^{-p}\frac{\mathrm{d}r}{r} \\
&= p\sqrt{\lambda}\int_0^\infty \exp(p\sqrt{\lambda}a)\mathbf{P}\left(F<\exp(-\sqrt{\lambda}a)\right)\mathrm{d}a + 1 \le p\sqrt{\lambda}\int_0^\infty 2\exp(p\sqrt{\lambda}a - \frac{a^2}{4})\mathrm{d}a + 1 \\
&\le 4\sqrt{\pi p^2\lambda}\exp(p^2\lambda) + 1.
\end{aligned}$$

Thus we have that for any $t \in (0,\tau], x \in [0,\infty)$,

$$\mathbf{E}\left[(Z_\mu(t,x|0,\zeta))^{-p}\right] \le z_\mu(t,x|0,\zeta)^{-p}2^p \exp\left(2p\sqrt{\lambda \log \frac{2}{b(\mu,\tau)}}\right)\left(4\sqrt{\pi p^2\lambda}\exp(p^2\lambda) + 1\right),$$

which implies (4.2.15).

Additionally, Proposition 4.8.4 above has assured that the constant $C(\mu,\tau)$ in (4.8.13) and (4.8.14) can be chosen uniformly over μ in compact subsets of $\mathbb{R}$. It thus implies that $\lambda_0(\mu,\tau)$ and $b(\mu,\tau)^{-1}$ can also be uniform in the corresponding sets of μ. □

4.8.1.4 Proof of Proposition 4.2.7 (vii)

Let $\xi_\delta(t,x) = \int_{\mathbb{R}} p_\delta(x-y)\xi(t,\mathrm{d}y)$, which is a Gaussian noise white in time and smooth in space, with the covariance function

$$Q_\delta(t_1,t_2,x_1,x_2) = \delta_0(t_1-t_2)p_{2\delta}(x_1-x_2).$$

For any $\delta > 0$, $\mu, s, t \in \mathbb{R}, t > s$, $x, y \in [0, \infty)$, define

$$\begin{aligned}&\mathscr{Z}_\mu^\delta(t,x|s,y)\\&:= \mathscr{P}^N(t,x|s,y)\mathbb{E}_B^x\left[\exp\left(-\mu L_{t-s}^0 + \int_0^{t-s}\xi_\delta(t-r,|B_r|)\mathrm{d}r - \frac{1}{2}(t-s)p_{2\delta}(0)\right)\middle||B_{t-s}| = y\right]\end{aligned}$$

Let $\Upsilon_\cdot = |B_\cdot|$ be the reflected Brownian motion starting from $\Upsilon_0 = |B_0| = x$. We use $L_t^{0,\Upsilon}$ to denote the local time

$$L_t^{0,\Upsilon} := \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_0^t \mathbb{1}_{[0,\varepsilon]}(\Upsilon_s)\,\mathrm{d}s.$$

As discussed after (4.2.5), we can rewrite the above as

$$\begin{aligned}&\mathscr{Z}_\mu^\delta(t,x|s,y)\\&= \mathscr{P}^N(t,x|s,y)\mathbb{E}_\Upsilon^x\left[\exp\left(-\mu L_{t-s}^{0,\Upsilon} + \int_0^{t-s}\xi_\delta(t-r,\Upsilon_r)\mathrm{d}r - \frac{1}{2}(t-s)p_{2\delta}(0)\right)\middle|\Upsilon_{t-s} = y\right],\end{aligned}$$

where the conditional expectation is taken over the reflected Brownian bridges started from $\Upsilon_0 = x$ and ending at $\Upsilon_{t-s} = y$. Since

$$\{\xi_\delta(s+r,x)\}_{r\in[0,t-s],x\in[0,\infty)} \overset{\text{law}}{=} \{\xi_\delta(t-r,x)\}_{r\in[0,t-s],x\in[0,\infty)},$$

we have

$$\begin{aligned}&\left\{\mathbb{E}_\Upsilon\left[\exp\left(-\mu L_{t-s}^{0,\Upsilon} + \int_0^{t-s}\xi_\delta(t-r,\Upsilon_r)\mathrm{d}r\right)\middle|\Upsilon_0 = x, \Upsilon_{t-s} = y\right]\right\}_{x,y\in[0,\infty)}\\&\overset{\text{law}}{=}\left\{\mathbb{E}_\Upsilon\left[\exp\left(-\mu L_{t-s}^{0,\Upsilon} + \int_0^{t-s}\xi_\delta(s+r,\Upsilon_r)\mathrm{d}r\right)\middle|\Upsilon_0 = x, \Upsilon_{t-s} = y\right]\right\}_{x,y\in[0,\infty)}.\end{aligned} \tag{4.8.22}$$

Let $r' = t - s - r$, $\tilde{\Upsilon}_{r'} := \Upsilon_{t-s-r'}$ for $0 \le r' \le t - s$ and $L^{0,\tilde{\Upsilon}}_{t-s}$ be the local time at zero associated with $\tilde{\Upsilon}$. The tuples $((\Upsilon_{t-s-r})_{r\in[0,t-s]}, L^{0,\Upsilon}_{t-s})$ and $((\tilde{\Upsilon}_r)_{r\in[0,t-s]}, L^{0,\tilde{\Upsilon}}_{t-s})$ are identical ω-by-ω, and the process $(\tilde{\Upsilon}_{r'})_{0\le r'\le t-s}$ is a reflected Brownian bridge started from $\tilde{\Upsilon}_0 = y$ and ending at $\tilde{\Upsilon}_{t-s} = x$. It then follows that the right-hand side of (4.8.22) equals to

$$\left\{\mathbb{E}_{\tilde{\Upsilon}}\left[\exp\left(-\mu L^{0,\tilde{\Upsilon}}_{t-s} + \int_0^{t-s} \xi_\delta(t - r', \tilde{\Upsilon}_{r'})\mathrm{d}r'\right)\middle|\tilde{\Upsilon}_0 = y, \tilde{\Upsilon}_{t-s} = x\right]\right\}_{x,y\in[0,\infty)}.$$

Since the deterministic function $\mathscr{P}^N(t,x|s,y)$ is also symmetric in (x,y)-variables, we have proved that for any $\delta > 0$,

$$\{\mathcal{Z}^\delta_\mu(t,x|s,y)\}_{x,y\in[0,\infty)} \overset{\text{law}}{=} \{\mathcal{Z}^\delta_\mu(t,y|s,x)\}_{x,y\in[0,\infty)}.$$

By Lemma 4.8.1 (i) (with a mollification in space only), $\mathcal{Z}^\delta_\mu(t,x|s,y)$ converges to $\mathcal{Z}_\mu(t,x|s,y)$ in $L^p(\Omega)$ as $\delta \to 0$. We thus derive (4.2.20).

4.8.1.5 Proof of Proposition 4.2.8

For any $a, u \in \mathbb{R}$, $p \in [1,\infty)$, and $x \in [0,\infty)$, we have identity

$$\|\exp(aW_u(x))\|_p = \exp\left(aux + \frac{1}{2}a^2px\right). \tag{4.8.23}$$

By (4.2.21), Minkowski and Cauchy-Schwarz inequalities, for any $t > 0$,

$$\begin{aligned}\|Z_{u-\frac{1}{2}}(t,x)\|_p &\le \int_0^\infty \|\mathcal{Z}_{u-\frac{1}{2}}(t,x|0,y)\exp(W_u(y))\|_p\mathrm{d}y\\ &\le \int_0^\infty \|\mathcal{Z}_{u-\frac{1}{2}}(t,x|0,y)\|_{2p}\|\exp(W_u(y))\|_{2p}\mathrm{d}y.\end{aligned}$$

Using Lemma 4.8.1 (iii) and (4.8.23) proves (4.2.22).

For the bound of negative moments, we use Jensen's, Minkowski and Cauchy-Schwarz inequalities to obtain

$$\begin{aligned}
&\mathbf{E}\left[Z_{u-\frac{1}{2}}(t,x)^{-p}\right] = \mathbf{E}\left[\left(\int_0^\infty \mathcal{Z}_{u-\frac{1}{2}}(t,x|0,y)\exp(W_u(y))\mathrm{d}y\right)^{-p}\right] \\
&\le \mathbf{E}\left[\left(\int_0^\infty \mathcal{Z}_{u-\frac{1}{2}}(t,x|0,y)\mathrm{d}y\right)^{-p-1}\int_0^\infty \mathcal{Z}_{u-\frac{1}{2}}(t,x|0,y)\exp(-pW_u(y))\mathrm{d}y\right] \\
&\le \left\|\left(\int_0^\infty \mathcal{Z}_{u-\frac{1}{2}}(t,x|0,y)\mathrm{d}y\right)^{-1}\right\|_{2p+2}^{p+1}\left\|\int_0^\infty \mathcal{Z}_{u-\frac{1}{2}}(t,x|0,y)\exp(-pW_u(y))\mathrm{d}y\right\|_2 \\
&\le \left\|\left(\int_0^\infty \mathcal{Z}_{u-\frac{1}{2}}(t,x|0,y)\mathrm{d}y\right)^{-1}\right\|_{2p+2}^{p+1}\int_0^\infty \left\|\mathcal{Z}_{u-\frac{1}{2}}(t,x|0,y)\right\|_4 \|\exp(-pW_u(y))\|_4\,\mathrm{d}y.
\end{aligned}$$

By Lemma 4.2.5, the first term in the last product is uniformly bounded on $x \in [0,\infty)$. The bound of the second term follows from a similar computation.

The constants in Lemma 4.8.1 (iii) or Lemma 4.2.5 do not depend on t when $t \in (0,\tau]$ for any fixed $\tau > 0$ and can be chosen uniformly for u in any compact subset of $\mathbb{R}$. Thus it is not hard to check that the bounds of $\|Z_{u-\frac{1}{2}}(t,x)\|_p$ and $\|Z_{u-\frac{1}{2}}(t,x)^{-1}\|_p$ can also be chosen uniformly for (u,t) in any compact subset of $\mathbb{R}\times[0,\infty)$.

Using the uniform negative moment bounds, by Kolmogorov continuity criteria and the uniqueness of the mild solution, $Z_{u-\frac{1}{2}}(t,x)^{-1}$ is also continuous on $(t,x) \in \mathbb{R}^2_{\ge 0}$, which implies that $Z_{u-\frac{1}{2}}(t,x) > 0$ for all $(t,x) \in \mathbb{R}^2_{\ge 0}$.

To prove (4.2.26), we use an elementary inequality: for any $a \in \mathbb{R}$, $|a| \le e^a + e^{-a}$. By the definition of $H_u(t,x)$, $|H_u(t,x)| \le Z_{u-\frac{1}{2}}(t,x) + Z_{u-\frac{1}{2}}(t,x)^{-1}$, and (4.2.26) follows from the triangle inequality.

4.8.2 Proof of (4.4.3)

Recall that the goal was to show

$$\mathbf{Cov}[\mathcal{H}_u(t,x),\mathcal{H}_u(t,0)] \to 0, \quad \text{as } x \to \infty.$$

The above convergence arises from the spatial decorrelation of the half-space KPZ equation (4.1.1), for fixed $t > 0$. Similar to the analogous result for the full-space KPZ equation (see [90, Proposition 5.2] and [29, Eq. (2.10)] for two different proofs), this decorrelation is independent of the KPZ behavior or the boundary condition.

Using the same notation as above, $\mathscr{W}$ is the spatial white noise associated with the Brownian motion W and $\mathcal{D}$ is the Malliavin derivative operator on the Gaussian space generated by $\mathscr{W}$. We further define $\mathscr{D}$ to be the Malliavin derivative operator on the Gaussian space generated by the noise ξ. For any $t \geq 0, x \geq 0$, $\mathbf{E}H_u(t,x) = \mathbf{E}_W\mathbf{E}_\xi H_u(t,x)$. The law of total covariance gives that

$$\begin{aligned}\mathbf{Cov}[\mathcal{H}_u(t,x),\mathcal{H}_u(t,0)] &= \mathbf{E}_W\mathbf{E}_\xi\left([\mathcal{H}_u(t,x)-\mathbf{E}_\xi\mathcal{H}_u(t,x)]\,[\mathcal{H}_u(t,0)-\mathbf{E}_\xi\mathcal{H}_u(t,0)]\right)\\ &+\mathbf{E}_W\left([\mathbf{E}_\xi\mathcal{H}_u(t,x)-\mathbf{E}\mathcal{H}_u(t,x)]\,[\mathbf{E}_\xi\mathcal{H}_u(t,0)-\mathbf{E}\mathcal{H}_u(t,0)]\right).\end{aligned}$$

For each realization of W, by the Clark-Ocone formula [48, Proposition 6.3], we have

$$\mathcal{H}_u(t,x)-\mathbf{E}_\xi\mathcal{H}_u(t,x) = \int_0^t\int_{\mathbb{R}}\mathbf{E}_\xi\left[\mathscr{D}_{s,z}\mathcal{H}_u(t,x)\mid\mathcal{F}_s\right]\xi(\mathrm{d}s\mathrm{d}z),$$

with $\{\mathcal{F}_s\}_{s\geq 0}$ being the natural filtration in time generated by ξ. Similarly,

$$\mathbf{E}_\xi \mathcal{H}_u(t,x) - \mathbf{E}\mathcal{H}_u(t,x) = \int_0^\infty \mathbf{E}_W[\mathcal{D}_r \mathbf{E}_\xi \mathcal{H}_u(t,x) \mid \tilde{\mathcal{F}}_r] \mathrm{d}W(r),$$

with $\{\tilde{\mathcal{F}}_r\}_{r\geq 0}$ being the natural filtration generated by $\mathscr{W}$. By Itô isometry, Cauchy-Schwarz inequality and Jensen's inequality,

$$\begin{aligned}
&\mathbf{E}\left([\mathcal{H}_u(t,x) - \mathbf{E}_\xi \mathcal{H}_u(t,x)][\mathcal{H}_u(t,0) - \mathbf{E}_\xi \mathcal{H}_u(t,0)]\right) \\
&= \int_0^t \int_{\mathbb{R}} \mathbf{E}\left[\mathbf{E}_\xi\left[\mathscr{D}_{s,z}\mathcal{H}_u(t,x) \mid \mathcal{F}_s\right] \mathbf{E}_\xi\left[\mathscr{D}_{s,z}\mathcal{H}_u(t,0) \mid \mathcal{F}_s\right]\right] \mathrm{d}z\mathrm{d}s \\
&\leq \int_0^t \int_{\mathbb{R}} \|\mathscr{D}_{s,z}\mathcal{H}_u(t,x)\|_2 \|\mathscr{D}_{s,z}\mathcal{H}_u(t,0)\|_2 \mathrm{d}z\mathrm{d}s =: I_1(t,x),
\end{aligned}$$

and

$$\begin{aligned}
&\mathbf{E}_W\left([\mathbf{E}_\xi \mathcal{H}_u(t,x) - \mathbf{E}\mathcal{H}_u(t,x)][\mathbf{E}_\xi \mathcal{H}_u(t,0) - \mathbf{E}\mathcal{H}_u(t,0)]\right) \\
&= \mathbf{E}_W\left[\int_0^\infty \mathbf{E}_W[\mathcal{D}_r \mathbf{E}_\xi \mathcal{H}_u(t,x) \mid \tilde{\mathcal{F}}_r] \mathbf{E}_W[\mathcal{D}_r \mathbf{E}_\xi \mathcal{H}_u(t,0) \mid \tilde{\mathcal{F}}_r] \mathrm{d}r\right] \\
&\leq \int_0^\infty \|\mathcal{D}_r \mathcal{H}_u(t,x)\|_2 \|\mathcal{D}_r \mathcal{H}_u(t,0)\|_2 \mathrm{d}r =: I_2(t,x).
\end{aligned}$$

The proof of (4.4.3) reduces to showing that for any fixed $t > 0$, $I_1(t,x) + I_2(t,x) \to 0$ as $x \to \infty$.

We first analyze I_1. Similar to the result of full-space SHE in [130, Proposition 5.1], for $(s,z) \in [0,t) \times \mathbb{R}$ and $x \geq 0$, we have

$$\begin{aligned}
\mathscr{D}_{s,z} Z_{u-\frac{1}{2}}(t,x) = \mathbb{1}_{[0,\infty)}(z) \mathscr{P}^R_{u-\frac{1}{2}}(t,x|s,z) Z_{u-\frac{1}{2}}(s,z) \\
+ \int_s^t \int_0^\infty \mathscr{P}^R_{u-\frac{1}{2}}(t,x|r,y) \mathscr{D}_{s,z} Z_{u-\frac{1}{2}}(r,y) \xi(r,y) \mathrm{d}y\mathrm{d}r,
\end{aligned}$$

with $\mathscr{D}_{s,z} Z_{u-\frac{1}{2}}(t,x) \equiv 0$ when $z < 0$ or $s \geq t$. When $s < t$, $z \geq 0$, since $Z_{u-\frac{1}{2}}(s,z)$ is strictly positive, we can multiply both sides by $Z_{u-\frac{1}{2}}(s,z)^{-1}$. The uniqueness statement in Proposition 4.2.2

part (2) then implies that

$$\mathscr{D}_{s,z} Z_{u-\frac{1}{2}}(t,x) Z_{u-\frac{1}{2}}(s,z)^{-1} = \mathcal{Z}_{u-\frac{1}{2}}(t,x|s,z),$$

where we recall that $\mathcal{Z}$ is the Green's function. It follows that for any $s \in [0,t), z \in [0,\infty)$,

$$\mathscr{D}_{s,z}\mathcal{H}_u(t,x) = \mathscr{D}_{s,z}\left[\log Z_{u-\frac{1}{2}}(t,x)\right] = \frac{\mathscr{D}_{s,z} Z_{u-\frac{1}{2}}(t,x)}{Z_{u-\frac{1}{2}}(t,x)} = \frac{\mathcal{Z}_{u-\frac{1}{2}}(t,x|s,z) Z_{u-\frac{1}{2}}(s,z)}{Z_{u-\frac{1}{2}}(t,x)}.$$

By Hölder's inequality, (4.2.17), (4.2.22) and (4.2.23), there exists some positive constant $C = C(t,u)$ such that for any $x \in [0,\infty)$,

$$\begin{aligned}\|\mathscr{D}_{s,z}\mathcal{H}_u(t,x)\|_2 &\le \|\mathcal{Z}_{u-\frac{1}{2}}(t,x|s,z)\|_6 \|Z_{u-\frac{1}{2}}(s,z)\|_6 \|Z_{u-\frac{1}{2}}(t,x)^{-1}\|_6 \\ &\le C(t-s)^{-1/2}\exp(-(x-z)^2/4(t-s))\exp(C(x+z)).\end{aligned}$$

Thus we have

$$I_1(t,x) \le C^2 \int_0^t \int_0^\infty (t-s)^{-1} \exp\left(-\frac{(x-z)^2}{4(t-s)} - \frac{z^2}{4(t-s)}\right) \exp(C(x+2z)) \mathrm{d}z \mathrm{d}s.$$

By a straightforward computation, the right-hand side of the above converges to 0 as $x \to \infty$.

For I_2, we note that

$$\begin{aligned}\mathcal{D}_r\mathcal{H}_u(t,x) &= \frac{\int_r^\infty \mathcal{Z}_{u-\frac{1}{2}}(t,x|0,y)\exp(W_u(y))\mathrm{d}y}{\int_0^\infty \mathcal{Z}_{u-\frac{1}{2}}(t,x|0,y)\exp(W_u(y))\mathrm{d}y} - \mathbb{1}_{[0,x]}(r) \\ &= \mathbb{1}_{[x,\infty)}(r)\int_r^\infty \rho_u^R(y|t,x)\mathrm{d}y - \mathbb{1}_{[0,x]}(r)\int_0^r \rho_u^R(y|t,x)\mathrm{d}y,\end{aligned}$$

which further implies

$$\|\mathcal{D}_r\mathcal{H}_u(t,x)\|_2 \le \mathbb{1}_{[0,x]}(r)\int_0^r \|\rho_u^R(y|t,x)\|_2\mathrm{d}y + \mathbb{1}_{[x,\infty)}(r)\int_r^\infty \|\rho_u^R(y|t,x)\|_2\mathrm{d}y.$$

By a similar estimate as in (4.4.5), we can prove that there exists a constant $C = C(t,u)$ such that

$$\|\rho_u^R(y|t,x)\|_2 \le C\exp\left(-(x-y)^2/C + Cx\right), \quad \text{for all } x \in [0,\infty).$$

As a result, we can bound I_2 by

$$\begin{aligned}
&I_2(t,x) \le \exp(Cx)\int_0^x \left(\int_0^r C\exp\left(-(x-y)^2/C\right)\mathrm{d}y\right)\left(\int_r^\infty C\exp\left(-y^2/C\right)\mathrm{d}y\right)\mathrm{d}r\\
&+\exp(Cx)\int_x^\infty \left(\int_r^\infty C\exp\left(-(x-y)^2/C\right)\mathrm{d}y\right)\left(\int_r^\infty C\exp\left(-y^2/C\right)\mathrm{d}y\right)\mathrm{d}r\\
&\le \exp(Cx)\int_0^x \phi(2x-r)\phi(r)\mathrm{d}r + \exp(Cx)\int_x^\infty \phi(r-x)\phi(r)\mathrm{d}r,
\end{aligned}$$

with $\phi(x) := \int_x^\infty C\exp\left(-y^2/C\right)\mathrm{d}y$. By the Gaussian tail of the function ϕ, the last line above converges to 0 as $x \to \infty$.

Chapter 5: Time-dependent averages of a long-range SHE

5.1 Introduction

5.1.1 Main results

We consider the following stochastic heat equation (SHE) in $\mathbb{R}^d$ for $d \geq 3$ starting from flat initial data

$$\partial_t u(t,x) = \frac{1}{2}\Delta u(t,x) + \kappa u(t,x)\dot{F}(t,x), \quad u(0,x) = 1, \tag{5.1.1}$$

with $\kappa > 0$ and $\dot{F}$ being a Gaussian noise that is white in time and has a spatially homogeneous covariance function described by Riesz kernel with index 2. Formally, with $\|\cdot\|$ denoting the Euclidean norm in $\mathbb{R}^d$, the covariance function of the noise $\dot{F}$ is expressed as

$$\mathbb{E}\dot{F}(s,y)\dot{F}(t,x) = \delta(t-s)\|x-y\|^{-2}.$$

The stochastic heat equation (5.1.1) is a borderline critical stochastic PDE. Its solution is previously studied and given a meaning in [31] for $0 < \kappa < \frac{d-2}{2}$. In particular, [31] defines the (martingale problem) solution to (5.1.1), and shows that when $0 < \kappa < \frac{d-2}{2}$, such solutions exist (through Wiener chaos expansion) and are unique in law. We defer the precise definitions of the solution to Section 5.2.1 below. Moreover, [31] shows that any solution to (5.1.1) is a nonnegative singular Radon measure valued process, which we shall denote by $\{u_t(\mathrm{d}x) : t \geq 0\}$.

For any $R > 0$, we use $B_R = \{x \in \mathbb{R}^d : \|x\| < R\}$ to denote the open ball with radius R centered at zero in $\mathbb{R}^d$. The volume of the ball B_R is given by $\omega_d R^d$, with ω_d being the volume of B_1.

Our main result is the following three different limits of the spatial averages of $u_t(\mathrm{d}x)$ under different space-time scales. We write t_R to emphasize that the time scale could depend on the radius. Our results cover the cases of (i) when t is fixed and R goes to infinity, or (ii) when t depends on R and t, R both go to infinity.

Theorem 5.1.1. *For any $0 < \kappa < \frac{d-2}{2}$, let $\{u_t(\mathrm{d}x) : t \geq 0\}$ be a solution to (5.1.1) as defined in [31]. For $t = t_R > 0$, the time-dependent spatial averages $u_{t_R}(B_R)$ have the following limits:*

1. *When $t_R = o(R^2)$, there exists a Gaussian random variable $\mathcal{N}(0, \sigma^2)$, with variance*

$$\sigma^2 = \kappa^2 \int_{B_1} \int_{B_1} \frac{1}{\|x - y\|^2} \mathrm{d}x \mathrm{d}y, \tag{5.1.2}$$

such that

$$\frac{1}{\sqrt{t_R} R^{d-1}} [u_{t_R}(B_R) - \omega_d R^d] \to \mathcal{N}(0, \sigma^2) \quad \textit{in law}, \tag{5.1.3}$$

as $R \to \infty$.

2. *When $t_R = cR^2$ for some $c > 0$, we have*

$$\frac{1}{R^d} u_{t_R}(B_R) = u_c(B_1) \quad \textit{in law}, \tag{5.1.4}$$

for any $R > 0$. Here $u_c(B_1)$ is a non-degenerate random variable and is positive almost *surely.*

3. *When* $t_R^{-1} = o(R^{-2})$, *we have*

$$\frac{1}{R^d} u_{t_R}(B_R) \to 0 \quad \textit{in probability}, \tag{5.1.5}$$

as $R \to \infty$.

We note that the construction of the solution via the chaos expansion (5.2.4) below and the scaling invariance of equation (5.1.1) are the main tools to study the spatial averages. For part (1) under the condition $t_R = o(R^2)$, the first chaos is dominant when we take the limit $R \to \infty$, so the central limit theorem is not chaotic. In part (2) when the time variable grows fast enough to be at the borderline $t_R = cR^2$, the heuristic is that all the chaos now always contribute, and the random variable $\frac{1}{R^d} u_{t_R}(B_R)$ remains non-Gaussian even when $R \to \infty$.

Using the scaling invariance relation of (5.1.1), Theorem 5.1.1 part (1) is actually equivalent to the following corollary.

Corollary 5.1.2. *For any* $0 < \kappa < \frac{d-2}{2}$, *when* $t_R = o(R^2)$,

$$\frac{1}{\sqrt{t_R/R^2}} [u_{t_R/R^2}(B_1) - \omega_d] \to \mathcal{N}(0, \sigma^2) \quad \textit{in law},$$

as $R \to \infty$, *with* σ^2 *as defined in* (5.1.2).

Therefore, Theorem 5.1.1 part (1) can also be understood as the central limit theorem of the random variables $u_t(B_1)$ as $t \to 0$.

Another case that falls into the category of the extinction (Theorem 5.1.1 part (3)) is when

R is fixed and t goes to infinity. In fact, for any fixed radius $R > 0$,

$$\frac{1}{R^d} u_t(B_R) \to 0 \quad \text{in probability,} \quad \text{as } t \to \infty.$$

This result has already been mentioned in [31, Theorem 2], and our Theorem 5.1.1 part (3) is derived from this result together with the scaling invariance of equation (5.1.1).

Remark 5.1.3. The solution theory of the stochastic heat equation (5.1.1) when $\kappa \geq \frac{d-2}{2}$ remains open. When $\kappa > \frac{d-2}{2}$, the chaos series used to construct the solution to (5.1.1) in [31] (see (5.2.4) below) does not converge in L^2, putting spatial averages beyond the scope of this discussion.

For the boundary case $\kappa = \frac{d-2}{2}$, a solution constructed via chaos expansion (5.2.4) does exists, though [31] leaves open the possibility that there are solutions with a different law in this case. One may think of (5.1.1) with $\kappa = \frac{d-2}{2}$ as a critical stochastic heat equation with critical scaling. We believe that Theorem 5.1.1 parts (2) and (3) still hold for the boundary-case solution constructed via chaos expansion (see [31, Remark 3 on page 105]). However, our proof of Theorem 5.1.1 part (1) in Section 5.3.2 relies on the condition $0 < \kappa < \frac{d-2}{2}$, and it remains open whether the central limit theorem (5.1.3) extends to cover solutions via chaos expansion in this boundary case $\kappa = \frac{d-2}{2}$.

Remark 5.1.4. Another intriguing problem is to investigate the possibility of proving a quantitative central limit theorem under the same conditions as in Theorem 5.1.1 part (1). Previous works, such as [131–133], have established various quantitative central limit theorems for spatial averages of stochastic heat equations in one dimension, primarily using Malliavin calculus. Given that the solution to (5.1.1) is not defined point-wise, it is of interest to determine whether this approach remains applicable where the Malliavin derivatives cannot be defined point-wise.

We now introduce the context of our work from the following two perspectives.

5.1.2 Spatial averages of SPDE

The stochastic heat equation (5.1.1) is a critical stochastic partial differential equation (SPDE). A formal computation shows that the noise has the scaling invariance

$$\dot{F}(t,x) = \varepsilon^2 \dot{F}(\varepsilon^2 t, \varepsilon x) \quad \text{in law} \tag{5.1.6}$$

for any $\varepsilon > 0$, which is indicative that the solution will be a borderline distribution. Due to this criticality, the solution theory falls outside the well-posedness theories of regularity structures [10, 134], paracontrolled distributions [11], and more recently flow equation [15]. Similar critical scaling occurs for the stochastic heat equation with space-time white noise in the critical dimension two, although only the scaling limit of the noise-mollified SHE is established in this set-up [18, 44].

Our result considers the spatial average for a critical stochastic PDE with a measure-valued solution. We would like to mention that the critical $2d$ stochastic heat flow (SHF) [44] is also measure-valued, but the asymptotic behaviors of its spatial averages remain open.

Study of the spatial averages of stochastic PDEs started with the work [33] in the context of stochastic heat equation in one dimension with a space-time white noise potential. Since this work there has been an extensive study of spatial averages of solutions in various different set-ups, see for example, [34, 35, 131] study the SHE with colored noise, [135] studies the SHE with fractional in time noise, [136, 137] study the spatial average of stochastic wave equation. Most of these work consider the spatial averages for fixed time, only exception are the works [132, 133] where they study time-dependent averages.

Among all the above results, the limiting distribution of the spatial averages, if obtained, is always Gaussian. However, for our critical SPDE (5.1.1), the scaling invariance relation (5.1.6) enables us to obtain a non-Gaussian spatial average (5.1.4) under a specific space-time scale, which is the first non-Gaussian spatial average of an SPDE observed to the best of our knowledge. Our result also shows that this transition of Gaussian (5.1.3) to non-Gaussian (5.1.4) spatial averages happens when all the chaos begin to contribute in the spatial average limit.

The problem of obtaining non-Gaussian spatial average limit is of its own interest. It has long been conjectured, though not yet proven, that non-Gaussian stable distributions could be the limit of spatial averages of the $1d$ stochastic heat equation with space-time white noise when the average is taken over boxes with a proper time-dependent growth rate. This conjecture is inspired by the work of parabolic Anderson model on lattice [138].

As a final remark, we point out that in [35, Theorem 1.7], the authors proved a central limit theorem for the spatial averages of a stochastic heat equation with a spatial covariance function given by the Riesz kernel $\|\cdot\|^{-p}$ with index $0 < p < 2$ in $d \geq 3$. The SHE in [35] is not a critical SPDE and its point-wise solution is well-defined. While our central limit theorem in Theorem 5.1.1 part (1) (for t being fixed) may look like a result of taking $p \to 2^-$ in [35, Theorem 1.7], it is *not* a simple adaption of [35] as the SHE becomes critical when $p \to 2^-$. Our proof of Theorem 5.1.1 part (1) uses new estimates. In particular, our central limit theorem for $p = 2$ only holds when the coupling constant $0 < \kappa < \frac{d-2}{2}$, while the result from [35] would hold for any positive coupling constant before the noise. We will discuss this difference further in Remark 5.3.4.

5.1.3 Scaling limit

One may also consider (5.1.1) as the macroscopic scaling limit of a long-range SHE. One closely related work in this line is [139], where the authors study the scaling limit of the long-range nonlinear stochastic heat equation in dimension $d \geq 3$. If we only consider [139] for the linear case, we may interpret its result and compare it with ours in the following sense.

Consider the stochastic heat equation in $\mathbb{R}^d$ for $d \geq 3$ starting from flat initial data

$$\partial_t v(t,x) = \frac{1}{2}\Delta v(t,x) + \kappa v(t,x)\dot{V}(t,x), \quad v(0,x) = 1, \tag{5.1.7}$$

with $\kappa > 0$ and $\dot{V}$ being a space-time mean zero Gaussian noise with formal covariance function

$$\mathbb{E}\dot{V}(s,y)\dot{V}(t,x) = \delta(t-s)\left(\|x-y\|^{-p} \wedge 1\right),$$

for some $p \in [2,d)$.

By [31] and our Theorem 5.1.1 part (2), when $p = 2$, $\kappa < \frac{d-2}{2}$, and $t_R = cR^2$ for some $c > 0$, we have

$$\frac{1}{R^d}\int_{B_R} v(t_R,x)\mathrm{d}x \to u_c(B_1) \quad \text{in law}, \tag{5.1.8}$$

as $R \to \infty$. The limit is the same random variable as in (5.1.4) and is non-Gaussian.

On the other hand, by [139, Theorem 1.3] with $\sigma(x) = x$, when $p \in (2,d)$, κ is small enough, and $t_R = cR^2$ for some $c > 0$, we have that for any smooth test function with compact support $g : \mathbb{R}^d \to \mathbb{R}$,

$$\frac{1}{R^{d+1-\frac{p}{2}}} \int_{\mathbb{R}^d} [v(t_R, x) - 1]\, g(x/R)\, \mathrm{d}x \to \int_{\mathbb{R}^d} \mathcal{U}(c, x) g(x) \mathrm{d}x \quad \text{in law}, \tag{5.1.9}$$

as $R \to \infty$, where $\mathcal{U}$ is the solution to the Edwards-Wilkinson equation

$$\partial_t \mathcal{U}(t, x) = \frac{1}{2} \Delta \mathcal{U}(t, x) + \kappa \eta(t, x), \quad \mathcal{U}(0, \cdot) \equiv 0,$$

with noise η being a space-time mean zero Gaussian noise with formal covariance function

$$\mathbb{E}\eta(s, y)\eta(t, x) = \delta(t - s)\|x - y\|^{-p}.$$

In particular, the limit in (5.1.9) is Gaussian.

This comparison shows that the case $p = 2$ distinguishes itself from the cases with $2 < p < d$ as its large-scale fluctuations remain multiplicative, thus non-Gaussian, under diffusive scaling. When $p = 2$, all the chaos contribute to the fluctuation at the space-time scale $t = cR^2$, while when $2 < p < d$, only the first chaos from the Riesz-kernel-correlated noise contributes.

Another related literature is [17], where the authors consider the scaling limit of mollified SHE in $d \geq 3$ with compactly supported covariance function. For this model, the fluctuations of the diffusively scaled solution converge to an Edwards-Wilkinson limit with a nontrivial effective variance.

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5.2 Preliminaries

5.2.1 Solution theory

As discussed above, for the critical stochastic heat equation (5.1.1), the first challenge is to make sense of its solution. In this subsection, we introduce the definition of the solution to (5.1.1) and explain one specific construction of the solution from the Wiener chaos expansion. All of these results follow from [31].

Let $(\Omega, \mathcal{F}, \{\mathcal{F}\}_t, \mathbb{P})$ be a filtered probability space. Let $\mathcal{M}$ denote the nonnegative Radon measures on $\mathbb{R}^d$, $\mathcal{C}_c$ denote the space of continuous functions on $\mathbb{R}^d$ with compact support, and $\mathcal{C}_c^k$ denote the space of functions in $\mathcal{C}_c$ with k continuous derivatives. We define $\mu(f) := \int_{\mathbb{R}^d} f(x)\mu(dx)$ for any $\mu \in \mathcal{M}$ and f being integrable on $\mathbb{R}^d$. We also equip the space $\mathcal{M}$ with the vague topology. We denote the standard d-dimensional heat kernel as

$$G_t(x) := (2\pi t)^{-d/2} \exp\left(-|x|^2/2t\right)$$

for any $x \in \mathbb{R}^d$ and $t > 0$.

In [31], the (martingale problem) solution to (5.1.1) is defined as any adapted continuous $\mathcal{M}$ valued process $\{u_t(\mathrm{d}x) : t \geq 0\}$ such that it satisfies the following three conditions:

1. $\mathbb{P}(u_0(\mathrm{d}x) = \mathrm{d}x) = 1$;

2. $\{u_t(\mathrm{d}x) : t \geq 0\}$ satisfies

$$\mathbb{E}[u_t(f)] = \int_{\mathbb{R}^d} f(x)\mathrm{d}x,$$

and

$$\begin{aligned}&\mathbb{E}\left[(u_t(f))^2\right]\\&\leq C\int_{\mathbb{R}^{4d}} G_t(x-x')\,G_t(y-y')\,f(x')\,f(y')\left(1+\frac{t^\alpha}{|x-y|^\alpha\,|x'-y'|^\alpha}\right)\mathrm{d}x\mathrm{d}y\mathrm{d}x'\mathrm{d}y',\end{aligned} \tag{5.2.1}$$

for some positive constant $C = C(d,\kappa)$ and $\alpha = \frac{d-2}{2} - [(\frac{d-2}{2})^2 - \kappa^2]^{1/2}$;

3. $\{u_t(\mathrm{d}x) : t \geq 0\}$ satisfies the martingale problem, i.e., for all $f \in \mathcal{C}_c^2$,

$$z_t(f) = u_t(f) - u_0(f) - \int_0^t \frac{1}{2}u_s(\Delta f)\mathrm{d}s \tag{5.2.2}$$

is a continuous local $\mathcal{F}_t$ -martingale with finite quadratic variation given by

$$\langle z(f)\rangle_t = \kappa^2\int_0^t\int_{\mathbb{R}^{2d}} \frac{f(x)f(y)}{|x-y|^2}u_s(\mathrm{d}y)u_s(\mathrm{d}x)\mathrm{d}s. \tag{5.2.3}$$

For any $0 < \kappa < \frac{d-2}{2}$, [31] shows that such solution to (5.1.1) exists and is unique in law. In particular, one construction of the solution to (5.1.1) is from the Wiener chaos expansion. We use this construction throughout this section, so we explain it in detail.

For any $f \in \mathcal{C}_c$, define $I_t^{(n)}(f)$ as the integral of f over the n-th order Wiener chaos, given by

$$I_t^{(n)}(f) := \int_{\mathbb{R}^d} f(x) I^{(n)}(t,x) \mathrm{d}x,$$

with $I^{(n)}(t,x)$ being defined as $I^{(0)}(t,x) = 1$, and for any $n \geq 1$, setting $s_{n+1} = t$, $y_{n+1} = x$,

$$I^{(n)}(t,x) := \kappa^n \int_0^{s_{n+1}} \int_0^{s_n} \dots \int_0^{s_2} \int_{\mathbb{R}^{nd}} \prod_{i=1}^{n} G_{s_{i+1}-s_i}(y_{i+1} - y_i) \dot{F}(\mathrm{d}y_i \mathrm{d}s_i),$$

where the stochastic integral can be interpreted as multiple Wiener-Itô integrals (see [47, Section 1.1.2]). Note that one can easily extend the above definition of $I_t^{(n)}(f)$ to f being indicator functions $\mathbf{1}_{B_R}$ for any $R > 0$.

For any $0 < \kappa < \frac{d-2}{2}$, by [31], there exists an adapted $\mathcal{M}$ valued process $\{\tilde{u}_t(\mathrm{d}x) : t \geq 0\}$ such that for any $t \geq 0$,

$$\tilde{u}_t(f) = \sum_{n=0}^{\infty} I_t^{(n)}(f), \quad \text{for all } f \in \mathcal{C}_c, \quad \mathbb{P}\text{-almost surely.} \tag{5.2.4}$$

In fact, it is easy to check that for any $f \in \mathcal{C}_c$, $\mathbb{E}\left[\sum_{n=0}^{\infty} I_t^{(n)}(f)\right] = \int_{\mathbb{R}^d} f(x)\mathrm{d}x$, and by [31, Lemma 3 & Lemma 5], when $0 < \kappa < \frac{d-2}{2}$, $\mathbb{E}\left[\left(\sum_{n=0}^{\infty} I_t^{(n)}(f)\right)^2\right]$ is bounded above by the right-hand-side of (5.2.1). The second moment bound implies that there is a regularization [140, Theorem 2.3.3] so that for any $t > 0$, there exists a random distribution $\tilde{u}_t(\mathrm{d}x)$ satisfying (5.2.4). Moreover, [31] shows that for any $t > 0$, the distribution satisfying (5.2.4) is a nonnegative singular Radon measure, and the $\mathcal{M}$ valued process $\{\tilde{u}_t(\mathrm{d}x) : t \geq 0\}$ has a modification continuous in time. This continuous modification satisfies the martingale problem (5.2.2)–(5.2.3). Therefore, combining all the results above, the continuous modification of the process $\{\tilde{u}_t(\mathrm{d}x) : t \geq 0\}$ with $\tilde{u}_t$ satisfying (5.2.4) is a martingale problem solution to (5.1.1).

We also record the following Feynman-Kac type formula for the chaos expansion, where we refer to [31, Lemma 5] for details. By Urysohn's lemma and monotone convergence theorem, one can extend [31, Lemma 5] to the indicator functions $f = \mathbf{1}_{B_R}$.

Let

$$\beta_t(x) := \int_0^t \frac{\mathrm{d}s}{\left\|x + \sqrt{2}W_s\right\|^2},$$

where $(W_t)_{t\geq 0}$ denotes a standard d-dimensional Brownian motion starting from $W_0 = 0$ independent of ξ. We let E denote the expectation with respect to this Brownian motion. Then for each $n \geq 1$ and any $t > 0, R > 0$, by Itô isometry, with $s_{n+1} = t$ and $y_{n+1} = x_1$,$z_{n+1} = x_2$,

$$\begin{aligned}
\mathbb{E}\left[I_t^{(n)}(\mathbf{1}_{B_R})^2\right] &= \kappa^{2n} \int_{B_R}\int_{B_R}\int_0^{s_{n+1}}\int_0^{s_n}\cdots\int_0^{s_2}\int_{\mathbb{R}^{2nd}} \\
&\qquad \prod_{i=1}^n G_{s_{i+1}-s_i}(y_{i+1}-y_i)\, G_{s_{i+1}-s_i}(z_{i+1}-z_i)\,\frac{1}{\|y_i - z_i\|^2}\mathrm{d}y_i\mathrm{d}z_i\mathrm{d}s_i\mathrm{d}x_1\mathrm{d}x_2 \\
&= \frac{\kappa^{2n}}{n!}\int_{B_R}\int_{B_R}\mathrm{E}\left[\prod_{i=1}^n \int_0^t \frac{1}{\left\|x - y + \sqrt{2}W_{s_i}\right\|^2}\mathrm{d}s_i\right]\mathrm{d}x\mathrm{d}y \\
&= \frac{\kappa^{2n}}{n!}\int_{B_R}\int_{B_R}\mathrm{E}[\beta_t(x-y)^n]\mathrm{d}x\mathrm{d}y,
\end{aligned} \tag{5.2.5}$$

where the second equation is obtained by using the finite dimensional distributions of the Brownian bridges.

By [31], for any $0 < \kappa < \frac{d-2}{2}$, the series sum $\sum_{n=0}^{\infty}\mathbb{E}\left[I_t^{(n)}(\mathbf{1}_{B_R})^2\right]$ is convergent, and we have

$$\mathbb{E}[\tilde{u}_t(B_R)^2] = \sum_{n=0}^{\infty} \mathbb{E}\left[I_t^{(n)}(\mathbf{1}_{B_R})^2\right]$$
$$= \sum_{n=0}^{\infty} \frac{\kappa^{2n}}{n!} \int_{B_R}\int_{B_R} \mathrm{E}[\beta_t(x-y)^n]\mathrm{d}x\mathrm{d}y = \int_{B_R}\int_{B_R} \mathrm{E}\left[\exp\left(\kappa^2\beta_t(x-y)\right)\right]\mathrm{d}x\mathrm{d}y.$$

5.2.2 Technical lemmas

Since we will conduct our main analysis on the Fourier space, we introduce some auxiliary lemmas related to the Fourier transform of the Riesz kernel and the indicator functions.

Throughout this section, we take the Fourier transform of any tempered distribution μ on $\mathbb{R}^d$ in the convention of

$$\mathscr{F}\mu(x) := \int_{\mathbb{R}^d} e^{-\mathbf{i}x\cdot\xi}\mu(\mathrm{d}\xi).$$

Under this convention, for any functions $u, v \in L^2(\mathbb{R}^d)$, the Parseval formula is

$$\int_{\mathbb{R}^d} u(\xi)\overline{v(\xi)}\mathrm{d}\xi = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \mathscr{F}u(x)\overline{\mathscr{F}v(x)}dx.$$

When μ is in the space of tempered distributions, its Fourier transform $\mathscr{F}\mu$ is well-defined through the Parseval formula. By Bochner's theorem, for any continuous positive definite function $\gamma : \mathbb{R}^d \to \mathbb{R}\cup\{\infty\}$, there exists a nonnegative tempered distribution μ on $\mathbb{R}^d$ so that $\gamma(x) = \mathscr{F}\mu(x)$ and μ is known as the spectral measure.

In particular, for the Riesz kernel $\gamma(x) = 1/\|x\|^2$ on $\mathbb{R}^d$ with $d \geq 3$, its spectral measure is

$$\mu(\mathrm{d}\xi) = c_d\|\xi\|^{-d+2}\mathrm{d}\xi, \quad \text{with coefficient} \quad c_d = \frac{\Gamma(\frac{d-2}{2})}{\pi^{d/2}2^2}. \tag{5.2.6}$$

The spectral measure (5.2.6) does not satisfy Dalang's condition [141]:

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1+\|\xi\|^2} < \infty.$$

We now give the composition formula of the Riesz kernels. This formula is well-known and we state it here for the convenience of the readers. We omit the proof, as it follows from an elementary computation with the relation of gamma and beta functions.

Lemma 5.2.1. *For any* $0 < \alpha, \beta, \alpha + \beta < d$ *and* $x \neq y$*,*

$$\int_{\mathbb{R}^d} \|x-z\|^{\alpha-d}\|z-y\|^{\beta-d} dz = k_{\alpha,\beta,d}\|x-y\|^{\alpha+\beta-d},$$

with the coefficient

$$k_{\alpha,\beta,d} = \pi^{d/2}\frac{\Gamma(\frac{\alpha}{2})\Gamma(\frac{\beta}{2})\Gamma(\frac{d-\alpha-\beta}{2})}{\Gamma(\frac{d-\alpha}{2})\Gamma(\frac{d-\beta}{2})\Gamma(\frac{\alpha+\beta}{2})}.$$

Another useful lemma is the Fourier transform of the indicator function over a ball of radius R in $\mathbb{R}^d$. We refer to [35, Lemma 2.1] for its proof.

Lemma 5.2.2 ([35, Lemma 2.1])**.** *For any* $R > 0$*, we have*

$$\mathscr{F}\mathbf{1}_{B_R}(\xi) = \int_{B_R} e^{-\mathbf{i}x\cdot\xi} \mathrm{d}x = (2\pi R)^{d/2}\|\xi\|^{-d/2} J_{d/2}(R\|\xi\|), \quad \textit{for } \xi \in \mathbb{R}^d,$$

where $J_{d/2}(\cdot)$ *is the Bessel function of the first kind with order* $d/2$*:*

$$J_{d/2}(x) = \frac{(x/2)^{d/2}}{\sqrt{\pi}\Gamma\left(\frac{d+1}{2}\right)}\int_0^\pi (\sin\theta)^d \cos(x\cos\theta) d\theta, \quad \forall x \in \mathbb{R}.$$

By [142, page 134], the Bessel function $J_{d/2}(\cdot)$ has the following asymptotic behaviors:

$$\begin{aligned} & J_{d/2}(x) \sim \sqrt{2/(\pi x)} \cos\left(x - \frac{(d+1)\pi}{4}\right) \quad \text{as } x \to \infty, \\ \text{and} \quad & J_{d/2}(x) \sim \frac{x^{d/2}}{2^{d/2}\Gamma(d/2+1)} \quad \text{as } x \to 0. \end{aligned}$$

Consequentially, we have

$$\sup_{x\in\mathbb{R}} \left|J_{d/2}(|x|)\right| < \infty,$$

and there exists a constant $C > 0$ so that

$$|J_{d/2}(|x|)| \le C|x|^{-1/2}, \quad \text{for any } x \in \mathbb{R}\backslash\{0\}.$$

For $0 < \alpha < d+1$, it is easy to verify that the following integral is finite:

$$\int_{\mathbb{R}^d} \|\eta\|^{-2d+\alpha} J_{d/2}(\|\eta\|)^2 d\eta < \infty. \tag{5.2.7}$$

5.3 Proof of Theorem 5.1.1 and Corollary 5.1.2

5.3.1 Scaling

The following lemma is from the scaling invariance relation of equation (5.1.1). Using it together with [31, Theorem 2 & 3] proves Theorem 5.1.1 part (2) and (3).

Lemma 5.3.1. *For any radius $R > 0$, any time $t \ge 0$, and any $\varepsilon > 0$, we have*

$$u_t(B_R) = \varepsilon^{-d/2} u_{\varepsilon t}(B_{\sqrt{\varepsilon}R}) \quad \textit{in law}. \tag{5.3.1}$$

Proof. For any fixed $\varepsilon > 0, t \geq 0$, define the measure $v_t^\varepsilon(\mathrm{d}x) \in \mathcal{M}$ so that for any Borel measurable set $A \subset \mathbb{R}^d$, we have $v_t^\varepsilon(A) = \varepsilon^{-d/2} u_{\varepsilon t}(\sqrt{\varepsilon} A)$, where $\sqrt{\varepsilon} A = \{\sqrt{\varepsilon} x : x \in A\}$. By the scaling invariance relation (5.1.6), $\{v_t^\varepsilon(\mathrm{d}x) : t \geq 0\}$ is also a solution to (5.1.1). The uniqueness in law of the solution to (5.1.1) then implies (5.3.1). The result that$\{v_t^\varepsilon(\mathrm{d}x) : t \geq 0\}$ is a solution to (5.1.1) is straightforward to check, which we omit here. One can also check the invariance chaos by chaos, using (5.1.6). A result more general has also been previously mentioned in [31, Lemma 2]. We note that the prefactor $\varepsilon^{-d/2}$ is chosen here so that the rescaled measure-valued process is the solution to (5.1.1) with the same initial data. □

Proof of Theorem 5.1.1 parts (2) and (3). For any $R > 0$ and $t = t_R > 0$, applying Lemma 5.3.1 with $\varepsilon = R^{-2}$, we have

$$\frac{1}{R^d} u_{t_R}(B_R) = u_{t_R/R^2}(B_1) \quad \text{in law.}$$

When $t = t_R = cR^2$ for some $c > 0$, we have proved (5.1.4). By (5.2.5), $\mathbb{V}ar[u_c(B_1)] > 0$. The positivity follows from [31, Theorem 3 ii].

When $t = t_R$ with $t_R^{-1} = o(R^{-2})$, we have $t_R/R^2 \to \infty$ as $R \to \infty$. By [31, Theorem 2 ii], $u_t(B_1)$ converges to zero in probability as $t \to \infty$. Thus the proof is complete. □

We also note that, assuming Theorem 5.1.1 part (1) holds, Corollary 5.1.2 is another consequence of Lemma 5.3.1 with $\varepsilon = R^{-2}$.

5.3.2 Central limit theorem

In this subsection, we prove Theorem 5.1.1 part (1).

Without loss of generality, we only need to prove Theorem 5.1.1 part (1) for the case of $t_R = 1$ being fixed. This is also due to the scaling invariance relation. In fact, for any $t = t_R > 0$, applying Lemma 5.3.1 with $\varepsilon = {t_R}^{-1}$, we have

$$\begin{aligned}
&\frac{1}{\sqrt{t_R}R^{d-1}}\left[u_{t_R}(B_R) - \omega_d R^d\right] \overset{\text{law}}{=} \frac{1}{\sqrt{t_R}R^{d-1}}\left[t_R^{d/2}u_1\left(B_{t_R^{-1/2}R}\right) - \omega_d R^d\right] \\
&= \frac{t_R^{d/2}}{\sqrt{t_R}R^{d-1}}\left[u_1\left(B_{t_R^{-1/2}R}\right) - \omega_d\left(t_R^{-1/2}R\right)^d\right] = \frac{1}{\left({t_R}^{-1/2}R\right)^{d-1}}\left[u_1\left(B_{t_R^{-1/2}R}\right) - \omega_d\left(t_R^{-1/2}R\right)^d\right] \\
&= \frac{1}{\tilde{R}^{d-1}}\left[u_1\left(B_{\tilde{R}}\right) - \omega_d\tilde{R}^d\right],
\end{aligned}$$

where we define $\tilde{R} := t_R^{-1/2}R$ in the last equation. Note that when $t_R = o(R^2)$, we have $\tilde{R} \to \infty$ as $R \to \infty$, so it is sufficient to prove (5.1.3) for the fixed time case $t_R = 1$.

Additionally, by [31], the (martingale problem) solution to (5.1.1) is unique in law, so we only need to prove (5.1.3) for the process $\{\tilde{u}_t(\mathrm{d}x) : t \geq 0\}$, where $\{\tilde{u}_t(\mathrm{d}x) : t \geq 0\}$ is the adapted $\mathcal{M}$ valued process defined by the chaos expansion (5.2.4).

From now on, we will only consider the series $\tilde{u}_1(B_R) = \sum_{n=0}^{\infty} I_1^{(n)}(\mathbf{1}_{B_R})$. It is immediate that

$$\mathbb{E}[\tilde{u}_1(B_R)] = \mathbb{E}\left[I_1^{(0)}(\mathbf{1}_{B_R})\right] = \omega_d R^d,$$

and Theorem 5.1.1 part (1) would follow from the following two results.

Proposition 5.3.2. *For any $\kappa > 0$, we have*

$$\lim_{R\to\infty}\frac{\mathbb{V}ar\left[I_1^{(1)}(\mathbf{1}_{B_R})\right]}{R^{2d-2}} = \sigma^2,$$

where σ^2 is given by (5.1.2).

Proposition 5.3.3. *For any $0 < \kappa < \frac{d-2}{2}$, we have*

$$\lim_{R\to\infty} \frac{1}{R^{2d-2}} \sum_{n=2}^{\infty} \mathbb{V}ar\left[I_1^{(n)}(\mathbf{1}_{B_R})\right] = 0.$$

Remark 5.3.4. Before proving these two propositions, we point out how our results are similar or not similar to [35, Theorem 1.7]. In [35, Theorem 1.7], the authors proved a central limit theorem for the spatial averages of a stochastic heat equation with a spatial covariance function given by the Riesz kernel $\|\cdot\|^{-p}$ with index $0 < p < 2$ in $d \geq 3$, using the same method as checking the order of the variance of the first order chaos and then the sum of all higher order chaos.

However, although Proposition 5.3.2 and Lemma 5.3.5 below are indeed similar to the analogous results in the proof of [35, Theorem 1.7], our proof of Proposition 5.3.3 is not an adaption of [35].

From the viewpoint of (5.1.1) being a critical SPDE, we only expect to prove Proposition 5.3.3 under the restriction $0 < \kappa < \frac{d-2}{2}$. This is a strong indication that [35] cannot be directly adapted to prove Proposition 5.3.3 here. In fact, if we consider the counterpart of (5.1.1) with Riesz index $0 < p < 2$, the central limit theorem from [35] should hold for all $\kappa > 0$. However, in our case ($p = 2$), for any fixed-radius ball B_R with $R > 0$, the chaos expansion $\sum_{n=0}^{\infty} I_1^{(n)}(\mathbf{1}_{B_R})$ is not bounded in $L^2(\Omega)$ when $\kappa > \frac{d-2}{2}$ (see [31, page 114]). This shows that it would be unreasonable to expect a central limit theorem for all $\kappa > 0$, and it further suggests that the estimates from [35] would not work here.

If one dives into the technical detail, when Dalang's condition holds, one can always choose a sufficiently large N so that the integral $\int_{|\xi|\geq N} \frac{1}{\|\xi\|^2}\mu(\mathrm{d}\xi)$ becomes arbitrarily small. Here μ is the spectral measure of the spatial covariance function as discussed in Section 5.2.2. The estimates from [35] rely on this fact, and it is also the reason why the central limit theorem would hold for any coupling constant $\kappa > 0$ before the noise when the Riesz kernel is of index $0 < p < 2$. When the index of the Riesz kernel becomes $p = 2$, Dalang's condition no longer holds, and the above integral of the tail is not finite for any arbitrary N. The estimates from [35] thus become inapplicable, and the estimates we use to prove Proposition 5.3.3 are new.

5.3.2.1 Notations and auxiliary lemmas

We first give some notations. For any $n \geq 1$, $1 \leq k \leq n-1$, we use $\boldsymbol{\eta}_n = (\eta_1, \ldots, \eta_n) \in \mathbb{R}^{nd}$ and $\boldsymbol{\eta_{k,n}} = (\eta_{k+1}, \ldots, \eta_n) \in \mathbb{R}^{(n-k)d}$ to denote the n-tuple and $(n-k)$-tuple spatial variables respectively. We also use $\boldsymbol{w}_n = (w_1, \ldots, w_n) \in \mathbb{R}^n_{>0}$ for the n-tuple time variables. Since we only need to consider for $t_R = 1$, we set

$$\Delta_n := \{\boldsymbol{s}_n \in \mathbb{R}^n_{>0} : 0 < s_n < \cdots < s_1 < 1\}, \tag{5.3.2}$$

$$S_n := \{\boldsymbol{w}_n \in \mathbb{R}^n_{>0} : w_1 + \cdots + w_n < 1\}, \tag{5.3.3}$$

and define

$$\varphi(\boldsymbol{\eta}_n) := \int_{S_n} e^{-\sum_{i=1}^n w_i \|\eta_i\|^2} \mathrm{d}\boldsymbol{w}. \tag{5.3.4}$$

Now for any $n \geq 1$, we give the formula of the variance of $I_1^{(n)}(\mathbf{1}_{B_R})$ on the Fourier side. This computation is similar to the one in [35, Theorem 1.7] when an SHE with a spatial covariance function given by the Riesz kernel $\|\cdot\|^{-p}$ with index $0 < p < 2$ is considered. For completeness, we give the proof.

Lemma 5.3.5. *For any $n \geq 1$, $R > 0$, setting $\eta_0 = 0$, we have*

$$\mathbb{V}ar[I_1^{(n)}(\mathbf{1}_{B_R})] = \kappa^{2n}(2\pi)^d R^{2d-2n} c_d^n \int_{\mathbb{R}^{nd}} \prod_{j=1}^n \|\eta_j - \eta_{j-1}\|^{-d+2} \|\eta_n\|^{-d} J_{d/2}(\|\eta_n\|)^2 \varphi(\boldsymbol{\eta}_n/R) \mathrm{d}\boldsymbol{\eta}_n,$$

where c_d is defined as in (5.2.6), and the functions φ and $J_{d/2}$ are defined as in (5.3.4) and Lemma 5.2.2.

Proof. By (5.2.5), for any $n \geq 1$,

$$\mathbb{V}ar[I_1^{(n)}(\mathbf{1}_{B_R})] = \mathbb{E}\left[I_1^{(n)}(\mathbf{1}_{B_R})^2\right] = \frac{\kappa^{2n}}{n!} \int_{B_R} \int_{B_R} \mathrm{E}[\beta_1(x-y)^n] \mathrm{d}x \mathrm{d}y. \tag{5.3.5}$$

By the Fourier transform of Gaussian random variables and (5.2.6), we have

$$\begin{aligned}
\mathrm{E}[\beta_1(x-y)^n] &= c_d^n \int_{[0,1]^n} \int_{\mathbb{R}^{nd}} \prod_{j=1}^n \|\xi_j\|^{-d+2} e^{-\mathbf{i}(x-y)\cdot(\xi_1+\cdots+\xi_n)} \mathrm{E}[e^{-\mathbf{i}\sum_{j=1}^n \xi_j \cdot \sqrt{2} W_{s_j}}] \mathrm{d}\boldsymbol{\xi}_n \mathrm{d}\boldsymbol{s}_n \\
&= c_d^n n! \int_{\Delta_n} \int_{\mathbb{R}^{nd}} \prod_{j=1}^n \|\xi_j\|^{-d+2} e^{-\mathbf{i}(x-y)\cdot(\xi_1+\cdots+\xi_n)} e^{-\sum_{j=1}^n (s_j - s_{j+1})\|\xi_1+\cdots+\xi_j\|^2} \mathrm{d}\boldsymbol{\xi}_n \mathrm{d}\boldsymbol{s}_n,
\end{aligned}$$

where we set $s_{n+1} = 0$, c_d is as defined in (5.2.6), and Δ_n is as defined in (5.3.2). Now making the change of variables $\eta_j := \xi_1 + \cdots + \xi_j$ and $w_j := s_j - s_{j+1}$, we can rewrite the above as

$$\begin{aligned}
&c_d^n n! \int_{S_n} \int_{\mathbb{R}^{nd}} \prod_{j=1}^n \|\eta_j - \eta_{j-1}\|^{-d+2} e^{-\mathbf{i}(x-y)\cdot\eta_n} e^{-\sum_{j=1}^n w_j \|\eta_j\|^2} \mathrm{d}\boldsymbol{\eta}_n \mathrm{d}\boldsymbol{w}_n \\
&= c_d^n n! \int_{\mathbb{R}^{nd}} \prod_{j=1}^n \|\eta_j - \eta_{j-1}\|^{-d+2} e^{-\mathbf{i}(x-y)\cdot\eta_n} \varphi(\boldsymbol{\eta}_n) \mathrm{d}\boldsymbol{\eta}_n,
\end{aligned}$$

where S_n is as defined in (5.3.3) and φ is as defined in (5.3.4). Now plugging back into (5.3.5), we have

$$\begin{aligned}\mathbb{V}ar[I_1^{(n)}(\mathbf{1}_{B_R})] &= \kappa^{2n} c_d^n \int_{\mathbb{R}^{nd}} \prod_{j=1}^n \|\eta_j - \eta_{j-1}\|^{-d+2} \left(\int_{B_R} \int_{B_R} e^{-\mathbf{i}(x-y)\cdot\eta_n} \mathrm{d}x \mathrm{d}y \right) \varphi(\boldsymbol{\eta}_n) \mathrm{d}\boldsymbol{\eta}_n \\ &= \kappa^{2n} (2\pi R)^d c_d^n \int_{\mathbb{R}^{nd}} \prod_{j=1}^n \|\eta_j - \eta_{j-1}\|^{-d+2} \|\eta_n\|^{-d} J_{d/2}(R\|\eta_n\|)^2 \varphi(\boldsymbol{\eta}_n) \mathrm{d}\boldsymbol{\eta}_n,\end{aligned}$$

where the last line follows from Lemma 5.2.2. The proof is now complete by applying another change-of-variable $\eta_j \to \eta_j / R$ for all $j = 1, 2, \ldots, n$. $\square$

Another useful lemma is the bounds of $\varphi(\boldsymbol{\eta}_n)$:

Lemma 5.3.6. *For any $n \geq 1$ and $\boldsymbol{\eta}_n \in \mathbb{R}^{nd}$,*

$$\varphi(\boldsymbol{\eta}_n) \leq \frac{1}{n!}. \tag{5.3.6}$$

Thus for any $n \geq 2$ and $\boldsymbol{\eta}_n \in \mathbb{R}^{nd}$,

$$\lim_{R\to\infty} \frac{1}{R^{2n-2}} \varphi(\boldsymbol{\eta}_n / R) = 0. \tag{5.3.7}$$

Moreover, for any $n \geq 2$ and $1 \leq k \leq n-1$, when $\eta_i \neq 0$ for $1 \leq i \leq k$, we have

$$\varphi(\boldsymbol{\eta}_n) \leq \left(1 \wedge \frac{1}{\|\eta_1\|^2 \ldots \|\eta_k\|^2} \right) \varphi(\boldsymbol{\eta}_{k,n}). \tag{5.3.8}$$

Proof. The proof of (5.3.6) and (5.3.7) is trivial. For (5.3.8), we have that for any $n \geq 2$,

$$\begin{aligned}
\varphi(\boldsymbol{\eta}_n) &= \int_{S_n} e^{-w_1\|\eta_1\|^2} e^{-\sum_{i=2}^n w_i\|\eta_i\|^2} \mathrm{d}\boldsymbol{w}_n \\
&= \int_{\mathbb{R}_{>0}^{n-1}} \mathbf{1}_{(\sum_{i=2}^n w_i)<1} \frac{1}{\|\eta_1\|^2} \left(1 - e^{-(1-\sum_{i=2}^n w_i)\|\eta_1\|^2}\right) e^{-\sum_{i=2}^n w_i\|\eta_i\|^2} \mathrm{d}w_2 \ldots \mathrm{d}w_n \\
&\leq \left(1 \wedge \frac{1}{\|\eta_1\|^2}\right) \int_{S_{n-1}} e^{-\sum_{i=2}^n w_{i-1}\|\eta_i\|^2} \mathrm{d}\boldsymbol{w}_{n-1} = \left(1 \wedge \frac{1}{\|\eta_1\|^2}\right) \varphi(\boldsymbol{\eta}_{1,n}),
\end{aligned}$$

and we can conclude by induction. □

We are now ready to prove Proposition 5.3.2 and Proposition 5.3.3.

5.3.2.2 Proof of Proposition 5.3.2

For any $R > 1$, $\eta_1 \in \mathbb{R}^d$, it is easy to see that for any $\eta_1 \neq 0$,

$$\varphi(\eta_1/R) = \int_0^1 e^{-w_1\|\eta_1/R\|^2} \mathrm{d}w_1 = \frac{1 - e^{-\|\eta_1/R\|^2}}{\|\eta_1/R\|^2} \leq 1,$$

and $\lim_{R\to\infty} \varphi(\eta_1/R) = 1$. Applying the dominated convergence theorem with the bound (5.2.7), we have

$$\lim_{R\to\infty} \int_{\mathbb{R}^d} \|\eta_1\|^{-2d+2} J_{d/2}(\|\eta_1\|)^2 \varphi(\eta_1/R) \mathrm{d}\eta_1 = \int_{\mathbb{R}^d} \|\eta\|^{-2d+2} J_{d/2}(\|\eta\|)^2 \mathrm{d}\eta.$$

Using Lemma 5.3.5 with $n = 1$, it remains to verify that the variance σ^2 in (5.1.2) equals to

$$\kappa^2 (2\pi)^d c_d \int_{\mathbb{R}^d} \|\eta\|^{-2d+2} J_{d/2}(\|\eta\|)^2 \mathrm{d}\eta,$$

which is pure computation of Fourier transforms.

5.3.2.3 Proof of Proposition 5.3.3

By Lemma 5.3.5, for any $n \geq 2$, $R > 1$, setting $\eta_0 = 0$, we have

$$\begin{aligned}
&\frac{1}{R^{2d-2}} \mathbb{V}ar\left[I_1^{(n)}(\mathbf{1}_{B_R}) \right] \\
&= \kappa^{2n}(2\pi)^d R^{2-2n} c_d^n \int_{\mathbb{R}^{nd}} \prod_{j=1}^{n} \|\eta_j - \eta_{j-1}\|^{-d+2} \|\eta_n\|^{-d} J_{d/2}(\|\eta_n\|)^2 \varphi(\boldsymbol{\eta}_n/R) \mathrm{d}\boldsymbol{\eta}_n \\
&= \kappa^{2n}(2\pi)^d c_d^n [\mathcal{J}^{(n)}(R) + \mathcal{K}^{(n)}(R)],
\end{aligned}$$

where we define the functions

$$\mathcal{J}^{(n)}(R) := R^{2-2n} \int_{\mathbb{R}^{nd}} \mathbf{1}_{B_1^c}(\eta_1) \prod_{j=1}^{n} \|\eta_j - \eta_{j-1}\|^{-d+2} \|\eta_n\|^{-d} J_{d/2}(\|\eta_n\|)^2 \varphi(\boldsymbol{\eta}_n/R) \mathrm{d}\boldsymbol{\eta}_n,$$

and

$$\mathcal{K}^{(n)}(R) := R^{2-2n} \int_{\mathbb{R}^{nd}} \mathbf{1}_{B_1}(\eta_1) \prod_{j=1}^{n} \|\eta_j - \eta_{j-1}\|^{-d+2} \|\eta_n\|^{-d} J_{d/2}(\|\eta_n\|)^2 \varphi(\boldsymbol{\eta}_n/R) \mathrm{d}\boldsymbol{\eta}_n.$$

The proof then follows from the following two lemmas.

Lemma 5.3.7. *For any $0 < \kappa < \frac{d-2}{2}$, we have*

$$\lim_{R\to\infty} \sum_{n=2}^{\infty} \kappa^{2n} c_d^n \mathcal{J}^{(n)}(R) = 0.$$

Proof. By (5.3.8), we have that for any $R > 1$, $n \geq 2$, $\boldsymbol{\eta}_n \in \mathbb{R}^{nd}$ such that $\eta_i \neq 0$ for $i = 1, \dots, n-1$,

$$R^{2-2n}\varphi(\boldsymbol{\eta}_n/R) \leq R^{2-2n}\prod_{i=1}^{n-1}\|\eta_i/R\|^{-2} = \prod_{i=1}^{n-1}\|\eta_i\|^{-2}. \tag{5.3.9}$$

It follows that for any $R > 1, n \geq 2$,

$$\mathcal{J}^{(n)}(R) \leq \int_{\mathbb{R}^{nd}} \mathbf{1}_{B_1^c}(\eta_1)\prod_{j=1}^{n}\|\eta_j - \eta_{j-1}\|^{-d+2}\|\eta_n\|^{-d}J_{d/2}(\|\eta_n\|)^2\prod_{i=1}^{n-1}\|\eta_i\|^{-2}\mathrm{d}\boldsymbol{\eta}_n. \tag{5.3.10}$$

We first prove that for any $n \geq 2$, there exists some constant $C(n,d) > 0$ such that the right-hand-side of (5.3.10) is bounded from above by $C(n,d)$. To show this, we use the fact that for any $x \in B_1^c$ and any $\gamma > 0$,

$$\|x\|^{-2} \leq \|x\|^{-2+\gamma}.$$

Thus for some $\gamma > 0$ that we will choose later, we have

$$\begin{aligned}
&\int_{\mathbb{R}^{nd}} \mathbf{1}_{B_1^c}(\eta_1)\prod_{j=1}^{n}\|\eta_j - \eta_{j-1}\|^{-d+2}\|\eta_n\|^{-d}J_{d/2}(\|\eta_n\|)^2\prod_{i=1}^{n-1}\|\eta_i\|^{-2}\mathrm{d}\boldsymbol{\eta}_n \\
&\leq \int_{\mathbb{R}^{nd}} \|\eta_1\|^{-2+\gamma}\prod_{j=1}^{n}\|\eta_j - \eta_{j-1}\|^{-d+2}\|\eta_n\|^{-d}J_{d/2}(\|\eta_n\|)^2\prod_{i=2}^{n-1}\|\eta_i\|^{-2}\mathrm{d}\boldsymbol{\eta}_n \\
&= \int_{\mathbb{R}^{nd}} \|\eta_1\|^{-d+\gamma}\prod_{j=2}^{n}\left(\|\eta_j - \eta_{j-1}\|^{-d+2}\|\eta_j\|^{-2}\right)\|\eta_n\|^{-d+2}J_{d/2}(\|\eta_n\|)^2\mathrm{d}\boldsymbol{\eta}_n.
\end{aligned}$$

When $0 < \gamma < d-2$, for any $n \geq 2$, applying Lemma 5.2.1 $(n-1)$ times, the above equals to

$$(k_{\gamma,2,d})^{n-1}\int_{\mathbb{R}^d}\|\eta_n\|^{-2d+\gamma+2}J_{d/2}(\|\eta_n\|)^2\mathrm{d}\eta_n. \tag{5.3.11}$$

Let $\gamma = \frac{d-2}{2}$. The integral in (5.3.11) is finite by (5.2.7), and

$$k_{\gamma,2,d} = \pi^{d/2}\frac{\Gamma(\frac{\gamma}{2})\Gamma(\frac{d-\gamma-2}{2})}{\Gamma(\frac{d-\gamma}{2})\Gamma(\frac{d-2}{2})\Gamma(\frac{2+\gamma}{2})} = \pi^{d/2}\frac{1}{\frac{d-\gamma-2}{2}\Gamma(\frac{d-2}{2})\frac{\gamma}{2}} = \frac{1}{c_d(d-2-\gamma)\gamma} = \frac{1}{c_d\gamma^2}.$$

Thus the right-hand-side of equation (5.3.10) is bounded from above by

$$C(n,d) := \left(\frac{1}{c_d}\right)^{n-1}\left(\frac{2}{d-2}\right)^{2n-2}\int_{\mathbb{R}^d}\|\eta\|^{-2d+\frac{d+2}{2}}J_{d/2}(\|\eta\|)^2\mathrm{d}\eta < \infty,$$

and for any $0 < \kappa < \frac{d-2}{2}$, the following sum is finite

$$\sum_{n=2}^{\infty}\kappa^{2n}c_d^n C(n,d) < \infty. \tag{5.3.12}$$

Now for any $n \geq 2$,

$$\mathcal{J}^{(n)}(R) \to 0 \quad \text{as } R \to \infty. \tag{5.3.13}$$

In fact, using again the bound (5.3.9) and that the right-hand-side of (5.3.10) is bounded, (5.3.13) follows from the dominated convergence theorem and (5.3.7).

By (5.3.10), $\mathcal{J}^{(n)}(R) \leq C(n,d)$ for all $R > 1, n \geq 2$. The proof is complete once we apply the dominated convergence theorem again to the sum of $\kappa^{2n}c_d^n\mathcal{J}^{(n)}(R)$ from $n = 2$ to infinity and use the results (5.3.12) and (5.3.13). □

Lemma 5.3.8. *For any* $0 < \kappa < \frac{d-2}{2}$, *we have*

$$\lim_{R\to\infty}\sum_{n=2}^{\infty}\kappa^{2n}c_d^n\mathcal{K}^{(n)}(R) = 0.$$

Proof. Similar to the proof of Lemma 5.3.7, we first show that, for any $R > 1$, $n \geq 2$, $\mathcal{K}^{(n)}(R) \leq C'(n,d)$ for some constant $C'(n,d) > 0$. In order to have $\kappa^{2n} c_d^n C'(n,d)$ be summable from $n = 2$ to infinity for all $0 < \kappa < \frac{d-2}{2}$, we conduct the following analysis.

Let $R > 1, n \geq 2$, and m be an integer so that $0 \leq m \leq n-2$. By (5.3.8), for any $\boldsymbol{\eta}_n \in \mathbb{R}^{nd}$ such that $\eta_i \neq 0$ for $i = m+2, \dots, n-1$, we have

$$R^{2-2n} \varphi(\boldsymbol{\eta}_n / R) \leq R^{2-2n} \prod_{i=m+2}^{n-1} \|\eta_i / R\|^{-2} = R^{-2m-2} \prod_{i=m+2}^{n-1} \|\eta_i\|^{-2}. \tag{5.3.14}$$

It follows that for any $R > 1, n \geq 2$ and $0 \leq m \leq n-2$,

$$\mathcal{K}^{(n)}(R) \leq R^{-2m-2} \int_{\mathbb{R}^{nd}} \mathbf{1}_{B_1}(\eta_1) \prod_{j=1}^{n} \|\eta_j - \eta_{j-1}\|^{-d+2} \|\eta_n\|^{-d} J_{d/2}(\|\eta_n\|)^2 \prod_{i=m+2}^{n-1} \|\eta_i\|^{-2} \mathrm{d}\boldsymbol{\eta}_n. \tag{5.3.15}$$

Note that for any $x \in B_1$ and any $\gamma' \geq 0$,

$$\|x\|^{-d+2} \leq \|x\|^{-d+2-\gamma'}.$$

Thus for some $\gamma' \geq 0$ and some $0 \leq m \leq n-2$ that we will choose later, we have

$$\begin{aligned} &\int_{\mathbb{R}^{nd}} \mathbf{1}_{B_1}(\eta_1) \prod_{j=1}^{n} \|\eta_j - \eta_{j-1}\|^{-d+2} \|\eta_n\|^{-d} J_{d/2}(\|\eta_n\|)^2 \prod_{i=m+2}^{n-1} \|\eta_i\|^{-2} \mathrm{d}\boldsymbol{\eta}_n \\ &\leq \int_{\mathbb{R}^{nd}} \|\eta_1\|^{-d+2-\gamma'} \prod_{j=2}^{n} \|\eta_j - \eta_{j-1}\|^{-d+2} \|\eta_n\|^{-d} J_{d/2}(\|\eta_n\|)^2 \prod_{i=m+2}^{n-1} \|\eta_i\|^{-2} \mathrm{d}\boldsymbol{\eta}_n. \end{aligned}$$

When $0 < 2-\gamma' < d-2(m+1)$, applying Lemma 5.2.1 $(m+1)$ times, the above equals to

$$\prod_{i=0}^{m} k_{2i+2-\gamma',2,d} \int_{\mathbb{R}^{(n-m-1)d}} \|\eta_{m+2}\|^{-d+2m+4-\gamma'} \prod_{j=m+3}^{n} \|\eta_j-\eta_{j-1}\|^{-d+2} \|\eta_n\|^{-d} J_{d/2}(\|\eta_n\|)^2 \prod_{i=m+2}^{n-1} \|\eta_i\|^{-2} \mathrm{d}\boldsymbol{\eta}_{m+1,n}.$$

Then again applying Lemma 5.2.1 $(n-2-m)$ times, the above further equals to

$$\prod_{i=0}^{m} k_{2i+2-\gamma',2,d} \left(k_{2m+2-\gamma',2,d}\right)^{n-2-m} \int_{\mathbb{R}^d} \|\eta_n\|^{-2d+2m+4-\gamma'} J_{d/2}(\|\eta_n\|)^2 \mathrm{d}\eta_n. \tag{5.3.16}$$

Note that when $0 < 2m+4-\gamma' < d$, the integral in (5.3.16) is finite by (5.2.7), and

$$\begin{aligned} k_{2m+2-\gamma',2,d} &= \pi^{d/2} \frac{\Gamma\left(\frac{2m+2-\gamma'}{2}\right)\Gamma\left(\frac{d-2-(2m+2-\gamma')}{2}\right)}{\Gamma\left(\frac{d-(2m+2-\gamma')}{2}\right)\Gamma\left(\frac{d-2}{2}\right)\Gamma\left(\frac{2+(2m+2-\gamma')}{2}\right)} \\ &= \pi^{d/2} \frac{1}{\left(\frac{d-2-(2m+2-\gamma')}{2}\right)\Gamma\left(\frac{d-2}{2}\right)\left(\frac{2m+2-\gamma'}{2}\right)} \\ &= \frac{1}{c_d(d-2-(2m+2-\gamma'))(2m+2-\gamma')}. \end{aligned}$$

We need to choose m and γ' in some clever way so that the series $\kappa^{2n} c_d^n \left(k_{2m+2-\gamma',2,d}\right)^{n-2-m}$ is summable. The crucial observation here is that for any $d \geq 3$, we can find an integer $m_0 \geq 0$ and some $\gamma_0' \in \{0, 1/2, 1, 3/2\}$ such that

$$2m_0 + 2 - \gamma_0' = \frac{d-2}{2}. \tag{5.3.17}$$

In fact, let m_0 be the largest nonnegative integer such that $4m_0 \leq d-3$ and let $\gamma_0' = \frac{3}{2} + \frac{4m_0-(d-3)}{2}$. It is easy to see that $0 \leq \gamma_0' < 2$, (5.3.17) is satisfied, and $0 < 2m_0 + 4 - \gamma_0' < d$ when $d \geq 3$. The values of m_0 and γ'_0 only depend on the dimension d. With $m = m_0, \gamma' = \gamma'_0$, we have

$$k_{2m+2-\gamma',2,d} = \frac{1}{c_d(\frac{d-2}{2})^2},$$

and $\kappa^2 c_d k_{2m+2-\gamma',2,d} < 1$ for all $0 < \kappa < \frac{d-2}{2}$.

Note that m_0 only depends on the dimension d and could be greater than some small n, so we need to choose m differently according to the values of n. For any $2 \leq n \leq m_0 + 2$, we choose $m = 0$ and $\gamma' = 3/2$; for any $n > m_0 + 2$, we choose $m = m_0$ and $\gamma' = \gamma'_0$. With these choices, the conditions $\gamma' \geq 0$, $0 \leq m \leq n - 2$ and $0 < 2 - \gamma' < d - 2(m + 1)$ are always satisfied, and $\kappa^{2n} c_d^n \left(k_{2m+2-\gamma',2,d}\right)^{n-2-m}$ is summable from $n = 2$ to infinity for all $0 < \kappa < \frac{d-2}{2}$. By (5.3.15), define $C'(n, d)$ as the value of (5.3.16) under such choices of m and γ', we have $\mathcal{K}^{(n)}(R) \leq C'(n, d)$ for any $R > 1$ and $n \geq 2$, and also

$$\sum_{n=2}^{\infty} \kappa^{2n} c_d^n C'(n, d) < \infty,$$

for any $0 < \kappa < \frac{d-2}{2}$.

It also follows from (5.3.15) that for any $n \geq 2$, $\mathcal{K}^{(n)}(R) \leq R^{-2m-2} C'(n, d)$, so $\mathcal{K}^{(n)}(R) \to 0$ as $R \to \infty$. The proof is thus complete once we apply the dominated convergence theorem to the sum of $\kappa^{2n} c_d^n \mathcal{K}^{(n)}(R)$ from $n = 2$ to infinity and use the above results together. □

Bibliography

[1] Ran Tao. Gaussian fluctuations of a nonlinear stochastic heat equation in dimension two. *Stochastics and Partial Differential Equations: Analysis and Computations*, 12(1):220–246, 2024.

[2] Ran Tao. Mesoscopic averaging of the two-dimensional kpz equation. *Journal of Statistical Physics*, 191(6), 2024.

[3] Yu Gu and Ran Tao. Fluctuation exponents of the half-space kpz at stationarity. *arXiv preprint arXiv:2410.01653*, 2025.

[4] Sefika Kuzgun and Ran Tao. Time-dependent averages of a critical long-range stochastic heat equation. *arXiv preprint arXiv:2411.09058*, 2024.

[5] Mehran Kardar, Giorgio Parisi, and Yi-Cheng Zhang. Dynamic scaling of growing interfaces. *Physical Review Letters*, 56(9):889, 1986.

[6] Lorenzo Bertini and Nicoletta Cancrini. The stochastic heat equation: Feynman-kac formula and intermittence. *Journal of statistical Physics*, 78:1377–1401, 1995.

[7] Gideon Amir, Ivan Corwin, and Jeremy Quastel. Probability distribution of the free energy of the continuum directed random polymer in $1 + 1$ dimensions. *Comm. Pure Appl. Math.*, 64(4):466–537, 2011.

[8] Ivan Corwin. The Kardar-Parisi-Zhang equation and universality class. *Random Matrices Theory Appl.*, 1(1):1130001, 76, 2012.

[9] Martin Hairer. Solving the KPZ equation. *Ann. of Math. (2)*, 178(2):559–664, 2013.

[10] Martin Hairer. A theory of regularity structures. *Inventiones mathematicae*, 198(2):269–504, 2014.

[11] Massimiliano Gubinelli, Peter Imkeller, and Nicolas Perkowski. Paracontrolled distributions and singular PDEs. *Forum Math. Pi*, 3:e6, 75, 2015.

[12] Massimiliano Gubinelli and Nicolas Perkowski. KPZ reloaded. *Comm. Math. Phys.*, 349(1):165–269, 2017.

[13] Patrícia Gonçalves and Milton Jara. Nonlinear fluctuations of weakly asymmetric interacting particle systems. *Arch. Ration. Mech. Anal.*, 212(2):597–644, 2014.

[14] Massimiliano Gubinelli and Nicolas Perkowski. Energy solutions of KPZ are unique. *J. Amer. Math. Soc.*, 31(2):427–471, 2018.

[15] Paweł Duch. Flow equation approach to singular stochastic pdes. *arXiv preprint arXiv:2109.11380*, 2021.

[16] Chiranjib Mukherjee, Alexander Shamov, and Ofer Zeitouni. Weak and strong disorder for the stochastic heat equation and continuous directed polymers in $d \geq 3$. *Electronic Communications in Probability*, 21:1–12, 2016.

[17] Yu Gu and Jiawei Li. Fluctuations of a nonlinear stochastic heat equation in dimensions three and higher. *SIAM Journal on Mathematical Analysis*, 52(6):5422–5440, 2020.

[18] Francesco Caravenna, Rongfeng Sun, and Nikos Zygouras. Universality in marginally relevant disordered systems. *The Annals of Applied Probability*, 27(5):3050–3112, 2017.

[19] Alexander Dunlap and Yu Gu. A forward-backward SDE from the 2d nonlinear stochastic heat equation. *The Annals of Probability*, 50(3):1204–1253, 2022.

[20] Francesco Caravenna, Rongfeng Sun, and Nikos Zygouras. The two-dimensional KPZ equation in the entire subcritical regime. *The Annals of Probability*, 48(3):1086–1127, 2020.

[21] Sourav Chatterjee. Weak convergence of directed polymers to deterministic kpz at high temperature. *Annales de l'Institut Henri Poincare (B) Probabilites et statistiques*, 59(2):774–794, 2023.

[22] Mehran Kardar. Depinning by quenched randomness. *Physical review letters*, 55(21):2235, 1985.

[23] Guillaume Barraquand, Alexei Borodin, and Ivan Corwin. Half-space macdonald processes. *Forum of Mathematics, Pi*, 8:e11, 2020.

[24] Guillaume Barraquand and Pierre Le Doussal. Kardar-parisi-zhang equation in a half space with flat initial condition and the unbinding of a directed polymer from an attractive wall. *Physical Review E*, 104(2):024502, 2021.

[25] Guillaume Barraquand, Alexandre Krajenbrink, and Pierre Le Doussal. Half-space stationary kardar–parisi–zhang equation beyond the brownian case. *Journal of Physics A: Mathematical and Theoretical*, 55(27):275004, 2022.

[26] Guillaume Barraquand and Ivan Corwin. Stationary measures for the log-gamma polymer and kpz equation in half-space. *The Annals of Probability*, 51(5):1830–1869, 2023.

[27] Guillaume Barraquand, Ivan Corwin, and Sayan Das. Kpz exponents for the half-space log-gamma polymer. *arXiv preprint arXiv:2310.10019*, 2023.

[28] Takashi Imamura, Matteo Mucciconi, and Tomohiro Sasamoto. Solvable models in the kpz class: approach through periodic and free boundary schur measures. *arXiv preprint arXiv:2204.08420*, 2022.

[29] Yu Gu and Tomasz Komorowski. Another look at the balázs-quastel-seppäläinen theorem. *Transactions of the American Mathematical Society*, 376(04):2947–2962, 2023.

[30] Guillaume Barraquand, Alexandre Krajenbrink, and Pierre Le Doussal. Half-space stationary kardar–parisi–zhang equation. *Journal of Statistical Physics*, 181(4):1149–1203, 2020.

[31] Carl Mueller and Roger Tribe. A singular parabolic Anderson model. *Electron. J. Probab.*, 9:no. 5, 98–144, 2004.

[32] Le Chen, Davar Khoshnevisan, David Nualart, and Fei Pu. Central limit theorems for spatial averages of the stochastic heat equation via malliavin–stein's method. *Stochastics and Partial Differential Equations: Analysis and Computations*, pages 1–55, 2021.

[33] Jingyu Huang, David Nualart, and Lauri Viitasaari. A central limit theorem for the stochastic heat equation. *Stochastic Processes and Their Applications*, 130(12):7170–7184, 2020.

[34] Jingyu Huang, David Nualart, Lauri Viitasaari, and Guangqu Zheng. Gaussian fluctuations for the stochastic heat equation with colored noise. *Stochastics and Partial Differential Equations: Analysis and Computations*, 8:402–421, 2020.

[35] David Nualart and Guangqu Zheng. Averaging Gaussian functionals. *Electron. J. Probab.*, 25:Paper No. 48, 54, 2020.

[36] Yu Gu. Gaussian fluctuations from the 2D KPZ equation. *Stochastics and Partial Differential Equations: Analysis and Computations*, 8(1):150–185, 2020.

[37] Shuta Nakajima and Makoto Nakashima. Fluctuations of two-dimensional stochastic heat equation and kpz equation in subcritical regime for general initial conditions. *Electronic Journal of Probability*, 28:1–38, 2023.

[38] Giuseppe Cannizzaro, Dirk Erhard, and Philipp Schönbauer. 2d anisotropic kpz at stationarity: Scaling, tightness and nontriviality. *The Annals of Probability*, 49(1):122–156, 2021.

[39] Francis Comets, Clément Cosco, and Chiranjib Mukherjee. Space-time fluctuation of the Kardar-Parisi-Zhang equation in $d \geq 3$ and the Gaussian free field. *Ann. Inst. Henri Poincaré Probab. Stat.*, 60(1):82–112, 2024.

[40] Clément Cosco, Shuta Nakajima, and Makoto Nakashima. Law of large numbers and fluctuations in the sub-critical and L^2 regions for SHE and KPZ equation in dimension $d \geq 3$. *Stochastic Processes and their Applications*, 2022.

[41] Alexander Dunlap, Yu Gu, Lenya Ryzhik, and Ofer Zeitouni. Fluctuations of the solutions to the KPZ equation in dimensions three and higher. *Probability Theory and Related Fields*, 176(3):1217–1258, 2020.

[42] Yu Gu, Lenya Ryzhik, and Ofer Zeitouni. The edwards–wilkinson limit of the random heat equation in dimensions three and higher. *Communications in Mathematical Physics*, 363(2):351–388, 2018.

[43] Jacques Magnen and Jeremie Unterberger. The scaling limit of the KPZ equation in space dimension 3 and higher. *Journal of Statistical Physics*, 171(4):543–598, 2018.

[44] Francesco Caravenna, Rongfeng Sun, and Nikos Zygouras. The critical 2d stochastic heat flow. *Inventiones mathematicae*, 233:325–460, 2023.

[45] Yu Gu, Jeremy Quastel, and Li-Cheng Tsai. Moments of the 2d she at criticality. *Probability and Mathematical Physics*, 2(1):179–219, 2021.

[46] Francesco Caravenna, Rongfeng Sun, and Nikos Zygouras. On the moments of the $(2+1)$ -dimensional directed polymer and stochastic heat equation in the critical window. *Communications in Mathematical Physics*, 372(2):385–440, 2019.

[47] David Nualart. *The Malliavin calculus and related topics*. Probability and its Applications (New York). Springer-Verlag, New York, 1995.

[48] Le Chen, Davar Khoshnevisan, David Nualart, and Fei Pu. Spatial ergodicity for spdes via poincaré-type inequalities. *Electronic Journal of Probability*, 26:1–37, 2021.

[49] Dimitris Lygkonis and Nikos Zygouras. Moments of the 2d directed polymer in the subcritical regime and a generalisation of the erdös-taylor theorem, 2021.

[50] Le Chen and Jingyu Huang. Comparison principle for stochastic heat equation on $\mathbb{R}^d$. *The Annals of Probability*, 47(2):989–1035, 2019.

[51] Ivan Nourdin and Giovanni Peccati. *Normal approximations with Malliavin calculus: from Stein's method to universality*. Number 192. Cambridge University Press, 2012.

[52] Ioannis Karatzas and Steven Shreve. *Brownian motion and stochastic calculus*, volume 113. Springer Science & Business Media, 2012.

[53] Sourav Chatterjee and Alexander Dunlap. Constructing a solution of the $(2+1)$-dimensional KPZ equation. *The Annals of Probability*, 48(2):1014–1055, 2020.

[54] Hermine Biermé, Olivier Durieu, and Yizao Wang. Generalized random fields and lévy's continuity theorem on the space of tempered distributions. *Communications on Stochastic Analysis*, 12(4):4, 2018.

[55] Lorenzo Bertini and Giambattista Giacomin. Stochastic burgers and KPZ equations from particle systems. *Communications in mathematical physics*, 183(3):571–607, 1997.

[56] Lorenzo Bertini and Nicoletta Cancrini. The two-dimensional stochastic heat equation: renormalizing a multiplicative noise. *Journal of Physics A: Mathematical and General*, 31(2):615, 1998.

[57] Francesco Caravenna, Rongfeng Sun, and Nikos Zygouras. The dickman subordinator, renewal theorems, and disordered systems. *Electron. J. Probab*, 24(101):1–40, 2019.

[58] Francesco Caravenna, Rongfeng Sun, and Nikos Zygouras. The critical 2d stochastic heat flow is not a gaussian multiplicative chaos. *The Annals of Probability*, 51(6):2265–2300, 2023.

[59] Francesco Caravenna, Francesca Cottini, and Maurizia Rossi. Quasi-critical fluctuations for 2d directed polymers. *arXiv preprint arXiv:2307.02453*, 2023.

[60] Simon Gabriel. Central limit theorems for the (2+ 1)-dimensional directed polymer in the weak disorder limit. *arXiv preprint arXiv:2104.07755v4*, 2021.

[61] Francis Comets, Clément Cosco, and Chiranjib Mukherjee. Renormalizing the kardar–parisi–zhang equation in $d \geq 3$ in weak disorder. *Journal of Statistical Physics*, 179(3):713–728, 2020.

[62] Clément Cosco and Shuta Nakajima. Gaussian fluctuations for the directed polymer partition function in dimension d≥ 3 and in the whole l^2-region. *Annales de l'Institut Henri Poincaré-Probabilités et Statistiques*, 57(2):872–889, 2021.

[63] Dimitris Lygkonis and Nikos Zygouras. Edwards–Wilkinson fluctuations for the directed polymer in the full L^2-regime for dimensions $d \geq 3$. *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*, 58(1):65 – 104, 2022.

[64] Sourav Chatterjee and Panagiotis E Souganidis. Convergence of deterministic growth models. *Archive for Rational Mechanics and Analysis*, 245(2):863–898, 2022.

[65] Sourav Chatterjee. Universality of deterministic kpz. *arXiv preprint arXiv:2102.13131*, 2021.

[66] Yasufumi Ito and Kazumasa A Takeuchi. When fast and slow interfaces grow together: connection to the half-space problem of the kardar-parisi-zhang class. *Physical Review E*, 97(4):040103, 2018.

[67] Kurt Johansson. Shape fluctuations and random matrices. *Communications in mathematical physics*, 209:437–476, 2000.

[68] Jinho Baik and Eric M. Rains. Algebraic aspects of increasing subsequences. *Duke Mathematical Journal*, 109(1):1 – 65, 2001.

[69] Jinho Baik and Eric M. Rains. Symmetrized random permutations. *Random Matrix Models and Their Applications*, 40, 2001.

[70] Jinho Baik and Eric M. Rains. The asymptotics of monotone subsequences of involutions. *Duke Mathematical Journal*, 109(2):205 – 281, 2001.

[71] Dan Betea, Patrik L Ferrari, and Alessandra Occelli. Stationary half-space last passage percolation. *Communications in Mathematical Physics*, 377:421–467, 2020.

[72] Dan Betea, PL Ferrari, and Alessandra Occelli. The half-space airy stat process. *Stochastic Processes and their Applications*, 146:207–263, 2022.

[73] Patrik Ferrari and Alessandra Occelli. Time-time covariance for last passage percolation in half-space. *The Annals of Applied Probability*, 34(1A):627–674, 2024.

[74] Guillaume Barraquand and Shouda Wang. An identity in distribution between full-space and half-space log-gamma polymers. *International Mathematics Research Notices*, 2023(14):11877–11929, 2023.

[75] Xuan Wu. Intermediate disorder regime for half-space directed polymers. *Journal of Statistical Physics*, 181(6):2372–2403, 2020.

[76] Sayan Das and Weitao Zhu. The half-space log-gamma polymer in the bound phase. *arXiv preprint arXiv:2310.10960*, 2023.

[77] Sayan Das and Christian Serio. Convergence to stationary measures for the half-space log-gamma polymer. *arXiv preprint arXiv:2402.16834*, 2024.

[78] Victor Ginsburg. Pinning, diffusive fluctuations, and gaussian limits for half-space directed polymer models. *arXiv preprint arXiv:2312.11439*, 2023.

[79] Jimmy He. Shift invariance of half space integrable models. *arXiv preprint arXiv:2205.13029*, 2022.

[80] Jimmy He. Boundary current fluctuations for the half-space asep and six-vertex model. *Proceedings of the London Mathematical Society*, 128(2):e12585, 2024.

[81] Xincheng Zhang. Tasep in half-space. *arXiv preprint arXiv:2409.09974*, 2024.

[82] Jacopo De Nardis, Alexandre Krajenbrink, Pierre Le Doussal, and Thimothée Thiery. Delta-bose gas on a half-line and the kardar–parisi–zhang equation: boundary bound states and unbinding transitions. *Journal of Statistical Mechanics: Theory and Experiment*, 2020(4):043207, 2020.

[83] Alexandre Krajenbrink and Pierre Le Doussal. Replica bethe ansatz solution to the kardar-parisi-zhang equation on the half-line. *SciPost Physics*, 8(3):035, 2020.

[84] Timo Seppäläinen. Scaling for a one-dimensional directed polymer with boundary conditions. *Ann. Probab.*, 40(1):19–73, 2012.

[85] Marton Balazs, Eric Cator, and Timo Seppalainen. Cube Root Fluctuations for the Corner Growth Model Associated to the Exclusion Process. *Electronic Journal of Probability*, 11:1094 – 1132, 2006.

[86] Timo Seppäläinen and Benedek Valkó. Bounds for scaling exponents for a 1+ 1 dimensional directed polymer in a brownian environment. *Latin American Journal of Probability and Mathematical Statistics*, 7:451–476, 2010.

[87] Márton Balázs and Timo Seppäläinen. Order of current variance and diffusivity in the asymmetric simple exclusion process. *Annals of mathematics*, pages 1237–1265, 2010.

[88] Benjamin Landon, Christian Noack, and Philippe Sosoe. KPZ-type fluctuation exponents for interacting diffusions in equilibrium. *The Annals of Probability*, 51(3):1139 – 1191, 2023.

[89] Leandro PR Pimentel. Integration by parts and the kpz two-point function. *The Annals of Probability*, 50(5):1755–1780, 2022.

[90] Márton Balázs, Jeremy Quastel, and Timo Seppäläinen. Fluctuation exponent of the kpz/stochastic burgers equation. *Journal of the American Mathematical Society*, 24(3):683–708, 2011.

[91] Daniel Dufresne. The distribution of a perpetuity, with applications to risk theory and pension funding. *Scandinavian Actuarial Journal*, 1990(1):39–79, 1990.

[92] Guillaume Barraquand and Pierre Le Doussal. Steady state of the kpz equation on an interval and liouville quantum mechanics. *EPL-Europhysics Letters*, 137, 2022.

[93] Tom Alberts, Christopher Janjigian, Firas Rassoul-Agha, and Timo Seppäläinen. The green's function of the parabolic anderson model and the continuum directed polymer. *arXiv preprint arXiv:2208.11255*, 2022.

[94] Ivan Corwin and Hao Shen. Open asep in the weakly asymmetric regime. *Communications on Pure and Applied Mathematics*, 71(10):2065–2128, 2018.

[95] Mark Iosifovich Freidlin. *Functional integration and partial differential equations*. Number 109. Princeton university press, 1985.

[96] Kiyosi Ito and Henry Jr McKean. *Diffusion processes and their sample paths*. Springer-Verlag., 1965.

[97] Shalin Parekh. The kpz limit of asep with boundary. *Communications in Mathematical Physics*, 365:569–649, 2019.

[98] Le Chen and Robert C Dalang. Hölder-continuity for the nonlinear stochastic heat equation with rough initial conditions. *Stochastic Partial Differential Equations: Analysis and Computations*, 2:316–352, 2014.

[99] Le Chen and Robert C Dalang. Moments and growth indices for the nonlinear stochastic heat equation with rough initial conditions. *The Annals of Probability*, pages 3006–3051, 2015.

[100] Carl Mueller. On the support of solutions to the heat equation with noise. *Stochastics: An International Journal of Probability and Stochastic Processes*, 37(4):225–245, 1991.

[101] Carl Mueller and David Nualart. Regularity of the density for the stochastic heat equation. *Electronic Journal of Probability*, 13(none):2248 – 2258, 2008.

[102] Gregorio R Moreno Flores. On the (strict) positivity of solutions of the stochastic heat equation. *The Annals of Probability*, 42(4):1635–1643, 2014.

[103] Yaozhong Hu and Khoa Lê. Asymptotics of the density of parabolic anderson random fields. *Annales de l'Institut Henri Poincare (B) Probabilites et statistiques*, 58(1):105–133, 2022.

[104] Tom Alberts, Konstantin Khanin, and Jeremy Quastel. The continuum directed random polymer. *Journal of Statistical Physics*, 154(1):305–326, 2014.

[105] Alexander Dunlap, Yu Gu, and Tomasz Komorowski. Fluctuation exponents of the kpz equation on a large torus. *Communications on Pure and Applied Mathematics*, 76(11):3104–3149, 2023.

[106] Fabrice Baudoin and Neil O'Connell. Exponential functionals of brownian motion and class-one whittaker functions. *Annales de l'IHP Probabilités et statistiques*, 47(4):1096–1120, 2011.

[107] Yu Gu and Tomasz Komorowski. Some recent progress on the periodic kpz equation. *arXiv preprint arXiv:2408.14174*, 2024.

[108] Marc Yor. On some exponential functionals of brownian motion. *Advances in applied probability*, 24(3):509–531, 1992.

[109] Guillaume Barraquand. Integral formulas for two-layer schur and whittaker processes. *arXiv preprint arXiv:2409.08927*, 2024.

[110] Francis Comets and Jacques Neveu. The sherrington-kirkpatrick model of spin glasses and stochastic calculus: the high temperature case. *Communications in Mathematical Physics*, 166:549–564, 1995.

[111] Francis Comets. *Directed polymers in random environments*. Springer, 2017.

[112] Hiroyuki Matsumoto and Marc Yor. Exponential functionals of brownian motion, i: Probability laws at fixed time. *Probability Surveys*, 2:312, 2005.

[113] Giuseppe Da Prato and Jerzy Zabczyk. *Stochastic equations in infinite dimensions*. Cambridge university press, 2014.

[114] Tadahisa Funaki and Jeremy Quastel. Kpz equation, its renormalization and invariant measures. *Stochastic Partial Differential Equations: Analysis and Computations*, 3(2):159–220, 2015.

[115] Tom Alberts, Konstantin Khanin, and Jeremy Quastel. The intermediate disorder regime for directed polymers in dimension 1+ 1. *The Annals of Probability*, pages 1212–1256, 2014.

[116] Neil O'Connell and Jon Warren. A multi-layer extension of the stochastic heat equation. *Communications in Mathematical Physics*, 341:1–33, 2016.

[117] Pat Fitzsimmons, Jim Pitman, and Marc Yor. Markovian bridges: construction, Palm interpretation, and splicing. 33:101–134, 1993.

[118] Jim Pitman and Marc Yor. On the distribution of ranked heights of excursions of a brownian bridge. *The Annals of Probability*, 29(1):361–384, 2001.

[119] David Aldous and Jim Pitman. Two recursive decompositions of brownian bridge related to the asymptotics of random mappings. *In Memoriam Paul-André Meyer: Séminaire de Probabilités XXXIX*, pages 269–303, 2006.

[120] Samuel Karlin and James McGregor. Coincidence probabilities. *Pacific Journal of Mathematics*, 9(4):1141 – 1164, 1959.

[121] Chin Hang Lun and Jon Warren. Continuity and strict positivity of the multi-layer extension of the stochastic heat equation. *Electronic Journal of Probability*, 25, 2020.

[122] Jean-François Le Gall. *Brownian motion, martingales, and stochastic calculus*. Springer, 2016.

[123] Neil O'Connell, Timo Seppäläinen, and Nikos Zygouras. Geometric rsk correspondence, whittaker functions and symmetrized random polymers. *Inventiones mathematicae*, 197(2):361–416, 2014.

[124] Shalin Parekh. Positive random walks and an identity for half-space SPDEs. *Electron. J. Probab.*, 27:Paper No. 45, 47, 2022.

[125] John B Walsh. An introduction to stochastic partial differential equations. *Lecture notes in mathematics*, pages 265–439, 1986.

[126] Yaozhong Hu, Jingyu Huang, David Nualart, and Samy Tindel. Stochastic heat equations with general multiplicative gaussian noises: Hölder continuity and intermittency. *Electronic Journal of Probability*, 20:1–50, 2015.

[127] Jim Pitman. The distribution of local times of a brownian bridge. *Séminaire de probabilités de Strasbourg*, 33:388–394, 1999.

[128] Jingyu Huang, Khoa Lê, and David Nualart. Large time asymptotics for the parabolic anderson model driven by spatially correlated noise. *Annales de l'Institut Henri Poincaré-Probabilités et Statistiques*, 53(3):1305–1340, 2017.

[129] Jingyu Huang, Khoa Lê, and David Nualart. Large time asymptotics for the parabolic anderson model driven by space and time correlated noise. *Stochastics and Partial Differential Equations: Analysis and Computations*, 5:614–651, 2017.

[130] Le Chen, Yaozhong Hu, and David Nualart. *Regularity and strict positivity of densities for the nonlinear stochastic heat equations*, volume 273. American Mathematical Society, 2021.

[131] David Nualart, Panqiu Xia, and Guangqu Zheng. Quantitative central limit theorems for the parabolic Anderson model driven by colored noises. *Electron. J. Probab.*, 27:Paper No. 120, 43, 2022.

[132] Le Chen, Davar Khoshnevisan, David Nualart, and Fei Pu. Central limit theorems for spatial averages of the stochastic heat equation via Malliavin-Stein's method. *Stoch. Partial Differ. Equ. Anal. Comput.*, 11(1):122–176, 2023.

[133] Kunwoo Kim and Jaeyun Yi. Limit theorems for time-dependent averages of nonlinear stochastic heat equations. *Bernoulli*, 28(1):214–238, 2022.

[134] Felix Otto, Jonas Sauer, Scott Smith, and Hendrik Weber. A priori bounds for quasi-linear spdes in the full sub-critical regime. *arXiv preprint arXiv:2103.11039*, 2021.

[135] David Nualart, Xiaoming Song, and Guangqu Zheng. Spatial averages for the parabolic Anderson model driven by rough noise. *ALEA Lat. Am. J. Probab. Math. Stat.*, 18(1):907–943, 2021.

[136] Francisco Delgado-Vences, David Nualart, and Guangqu Zheng. A Central Limit Theorem for the stochastic wave equation with fractional noise. *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*, 56(4):3020 – 3042, 2020.

[137] Raul Bolaños Guerrero, David Nualart, and Guangqu Zheng. Averaging 2d stochastic wave equation. *Electronic Journal of Probability*, 26:1–32, 2021.

[138] Jürgen Gärtner and Adrian Schnitzler. Stable limit laws for the parabolic Anderson model between quenched and annealed behaviour. *Ann. Inst. Henri Poincaré Probab. Stat.*, 51(1):194–206, 2015.

[139] Luca Gerolla, Martin Hairer, and Xue-Mei Li. Fluctuations of stochastic pdes with long-range correlations. *arXiv preprint arXiv:2303.09811*, 2023.

[140] Kiyosi Ito. *Foundations of Stochastic Differential Equations in Infinite Dimensional Spaces*, volume 47 of *CBMS-NSF Regional Conference Series in Applied Mathematics*. SIAM, 1984.

[141] Robert C. Dalang. Extending the martingale measure stochastic integral with applications to spatially homogeneous s.p.d.e.'s. *Electron. J. Probab.*, 4:no. 6, 29, 1999.

[142] Nikolai N. Lebedev. *Special Functions and Their Applications*. Dover Publications, New York, revised english edition edition, 1972.