ABSTRACT

Title of Dissertation: Exit Problem and Stochastic Resonance for a Class of Random Perturbations

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The asymptotic exit problems for diffusion processes with small parameter were considered in the classic work of Freidlin and Wentzell. In 2000, a mathematical theory of stochastic resonance for systems with random perturbations was established by Freidlin in the frame of the large deviation theory.

This dissertation concerns exit problems and stochastic resonance for a class of random perturbations approximating white noise. The tools used in the proofs are the large deviation theory and the Markov property of the processes. The first problem considered is the exit problem and stochastic resonance for random perturbations of random walks. It turns out that a specific random walk can be chosen which approximates the large deviation asymptotics of the Wiener process in the best way. Analogous results concerning exit problems and stochastic resonance for this type of random perturbations were obtained under appropriate assumptions and compared with those of white noise type perturbation. The

second problem I consider is the exit problems for random perturbations of a Gaussian process $\eta_t^{\mu,\varepsilon}$ which satisfies the equation $\mu\dot{\eta}_t^{\mu,\varepsilon} = -\eta_t^{\mu,\varepsilon} + \sqrt{\varepsilon}\dot{W}_t$, $\eta_0^{\mu,\varepsilon} = y$, $0 < \mu << 1$, $0 < \varepsilon << 1$. One can check that $\int_0^t \eta_s^{\mu,\varepsilon} ds$ converges to $\sqrt{\varepsilon}W_t$ uniformly on $[0, T]$ in probability as $\mu \downarrow 0$. Results concerning asymptotic exit problems for this type of random perturbation were obtained under appropriate assumptions. Since $\eta_t^{\mu,\varepsilon}$ is not a Markov process, this creates some difficulties for the proof. A new Markov process was constructed and the Markov property of the new process was used in the proof.

Exit Problem and Stochastic Resonance

for a Class of Random Perturbations

by

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Chapter 1

Introduction and a Review of Large Deviations

1.1 Introduction

This thesis is concerned with exit problems and stochastic resonance caused by random perturbations of dynamical systems. If a non-perturbed system has several asymptotically stable equilibrium points (or attracting compacts), the perturbed system could make transitions between the equilibrium points (or attracting compacts) in large time intervals. This could cause stochastic resonance. Two important tools are used in the proofs. First, the asymptotics of probabilities of large deviations allows the analysis of the long time behavior of random processes. In general, a large deviation principle can be described by an action functional. A review of large deviation theory will be given in §1.2. Second, the property of Markov processes plays an important role in the proof of the main results concerning exit problems and stochastic resonance.

Over the last two decades, stochastic resonance has continuously attracted considerable attention. The models with stochastic resonance and its modifications are used in various areas of physics, chemistry, neurophysiology and engineering. We mention here a famous model, initially suggested in [1] and [2], as an example exploiting stochastic resonance (see [4]). Let the time evolution of the "earth temperature", denoted by X_t^ε, be described by the following equation:

$$\dot{X}_t^\varepsilon = -B^{'}(X_t^\varepsilon) + f(t/T) + \sqrt{\varepsilon}\dot{W}_t, \quad X_0^\varepsilon = x \in R^1, \ \ 0 < \varepsilon << 1. \tag{1.1}$$

Here the potential $B(x)$ has two wells, $\lim_{|x|\to\infty} B(x) = \infty$, $f(t)$ is a 1-periodic function, $T = T(\varepsilon)$ is a large parameter for $0 < \varepsilon << 1$ and $\dot{W}_t$ is a standard white noise.

If $\varepsilon = 0$, the solution of the equation cannot be transferred from one well to another because the periodic term $f(t/T)$ has a small amplitude. The trajectory may have small oscillations near the bottom of the well containing the initial point, but it stays inside the well forever. If $\varepsilon > 0$ but $f \equiv 0$, the solution of the equation will make transitions between the wells. The transition times, say τ_{12}^ε and τ_{21}^ε, are random variables and there is no periodicity in the transitions. If both terms $f(t/T)$ and $\sqrt{\varepsilon}\dot{W}_t$ are included in the equation, the trajectory of X_t^ε, under certain relations between ε and $T(\varepsilon)$, will be close in an appropriate topology to a periodic function of large period $T(\varepsilon)$. This explains the phenomenon of large amplitude periodicity of the earth's temperature and this effect is called stochastic resonance.

A survey of applications of stochastic resonance is given by L. Gammaitoni, P.Hanggi and P. Jung in [3]. About 500 papers are cited in [3]. In many papers the main tools used to study stochastic resonance are digital or analog simulations. However, there were no papers where a satisfactory mathematical theory of stochastic resonance was given until the publication of paper [4] by Freidlin. In

[4], a mathematical theory of stochastic resonance is established in the framework of a large deviation theory.

Let us recall some results from [4]. Consider a dynamical system in R^d:

$$\dot{X}_t = b(X_t), \quad X_0 = x \in R^d. \tag{1.2}$$

Here $b(x) = (b^1(x), \ldots, b^d(x))$ is a vector field in R^d and $b(x)$ is Lipschitz continuous. Assume for brevity that the system has a finite number of asymptotic stable equilibrium points $K_1, \ldots, K_l$. Each trajectory of (1.2), besides the trajectories belonging to the separatrix surfaces, is attracted to one of the points K_i as $t \to \infty$. Let $i(x)$ be the index such that the trajectory starting at $x \in R^d$ is attracted to $K_{i(x)}$.

Now consider the system with a small additive white noise type perturbation,

$$\dot{X}^\varepsilon_t = b(X^\varepsilon_t) + \sqrt{\varepsilon}\sigma(X^\varepsilon_t)\dot{W}_t, \quad X^\varepsilon_0 = x \in R^d, 0 < \varepsilon \ll 1, \tag{1.3}$$

where $\dot{W}_t$ is a standard d-dimensional white noise and $\sigma(x)$ is a $d \times d$ matrix. Notice that X^ε_t is a Markov process.

To analyze the qualitative behavior of the perturbed system in large time intervals, the action functional for the family of processes X^ε_t, denoted by $\varepsilon^{-1}S^X_{[0,T]}(\varphi)$, is introduced. The physical meaning of the action functional is that $\exp\{\varepsilon^{-1}S^X_{[0,T]}(\varphi)\}$ is, roughly speaking, the main term of the probability that $X^\varepsilon_t, 0 < t \le T$, belongs to a small neighborhood of a function $\varphi : [0, T] \to R^d$ as $\varepsilon \downarrow 0$.

From the action functional, one can introduce a function $V(x, y)$:

$$V(x, y) = \inf_{\varphi \in C_{0T}(R^r)} \{S^X_{[0,T]}(\varphi) : \varphi_0 = x, \varphi_T = y, T > 0\}.$$

In particular, if $b(x)$ is of potential-type and $\sigma(x)$ is a unit matrix, then $V(x, y)$ can be expressed through the potential.

Define $V_{ij} = V(K_i, K_j)$, where K_i and K_j are the stable equilibrium points of the field $b(x), i, j \in \{1, ..., l\}$. Using the numbers V_{ij}, a hierarchy of cycles can be constructed and it defines the sequence of transition of X_t^ε between the stable equilibrium points as $\varepsilon \downarrow 0$: Cycles of rank 0 are the equilibrium states $L = \{1, \ldots, l\}$ themselves. For each $i \in L$, define "the closest" $j = J(i) \in L$ such that $V_{ij} = \min_{k \in L \setminus \{i\}} V_{ik}$. Such a closest state is unique in the generic system. Starting from any $i \in L$, one can consider the sequence $i, J(i), J^2(i), \ldots, J^n(i), \ldots$ where $J^{n+1}(i) = J(J^n(i))$. Since L is finite, the sequence, starting from some $m \in L$, is periodic: $i, J(i), \ldots, J^{n-1}(i), J^n(i) \to J^{n+1}(i) \to \ldots, J^m(i) = J^n(i)$. This sequence $i, J(i), \ldots, J^{n-1}(i), J^n(i) \to J^{n+1}(i) \to \ldots, J^m(i) = J^n(i)$ is called the cycle of rank 1 (1-cycle) generated by the state $i \in L$. From any 1-cycle C, one can define a 1-cycle which follows C. The 1-cycles form cycles of second rank (2-cycles). The second rank cycles form 3-cycles. Since L is finite, a hierarchy of cycles up to rank m^* can be constructed so that the m^*-cycles contain all the stable equilibrium points of L. The values of V_{ij} together with the hierarchy of cycles define, for each cycle, a rotation rate, an exit rate and a main state. In the generic case, all of these notions are defined in a unique way.

The exit rate of a cycle gives the asymptotics (non-random) of the logarithms of the transition times from this cycle to the next closest cycle. The rotation rate characterizes the rate of convergence to the sub-limiting distribution inside the cycle. The main state $m^* = M(C)$ of a cycle C defines the attracting point such that X_t^ε spends most of its time in the basin of K_{m^*} until it leaves the basin of $\cup_{i \in C} K_i$. Notice that the hierarchy of cycles and the main states are not random although the transitions between the stable points are caused by the random perturbations.

Let $T = T(\varepsilon)$ be a large parameter such that $\lim_{\varepsilon\downarrow 0} \varepsilon \ln T(\varepsilon) = \lambda > 0$. Let $X_0^\varepsilon = x \in R^d$ not belong to a separatrix. For any $\lambda > 0$, except for a finite number of values, there exists a cycle C such that for any $\alpha > 0$, X_t^ε will come into the basin of $\cup_{i\in C} K_i$ before time $\alpha T(\varepsilon)$ with probability close to 1 as $\varepsilon \downarrow 0$. However, X_t^ε does not have enough time to leave that basin before the time $AT(\varepsilon), \alpha < A < \infty$ with probability close to 1 as $\varepsilon \downarrow 0$. Moreover, the rotation time for the cycle C is $o(T(\varepsilon))$ as $\varepsilon \downarrow 0$, so that $X_{tT(\varepsilon)}^\varepsilon, 0 < t \leq A < \infty$, approaches the sub-limiting distribution concentrated at $K_{\mu(x,\lambda)}$ as $\varepsilon \downarrow 0$, where $\mu(x, \lambda)$ is the main state of the cycle C.

The state $K_{\mu(x,\lambda)}$ is called the metastable state. Such a metastable state is unique in the generic system In general, the metastable state depends on λ and x. For any $A > 0$, X_t^ε spends most of its time around the state $K_{\mu(x,\lambda)}$ in the time interval $[0, AT(\varepsilon)]$ with probability close to 1 as $\varepsilon \downarrow 0$.

Let $\Lambda(G)$ be the Lebesgue measure of a set $G \subset R^1$ and let $\rho(.,.)$ be the Euclidean metric in R^r. Under appropriate assumptions, for any $\delta > 0$ and $A > 0$,

$$\Lambda\{t \in [0, A] : \rho(X_{tT(\varepsilon)}^\varepsilon, K_{\mu(x,\lambda)}) > \delta\} \to 0$$

in P_x probability as $\varepsilon \to 0$.

Now consider a system where the characteristics of the system and its perturbations are changing slowly in time:

$$\dot{X}_t^\varepsilon = b(t/T, X_t^\varepsilon) + \sqrt{\varepsilon}\sigma(t/T, X_t^\varepsilon)\dot{W}_t, \quad X_0^\varepsilon = x \in R^d, \quad 0 < \varepsilon << 1. \tag{1.4}$$

Here $T = T(\varepsilon) \asymp e^{\lambda/\varepsilon}, \lambda > 0$, is a large parameter as $\varepsilon \downarrow 0$ so that the coefficients of (1.4) are changing very slowly. Therefore, the positions of the equilibrium points $K_i(t)$ as well as their number now depend on time. The numbers $V_{ij}(t)$

and the function $\mu^t(x,\lambda)$ also depend on time. This implies the trajectory $X^{\varepsilon}_{tT(\varepsilon)}$ first approaches the metastable state for the system with frozen dependence on time, and then evolves together with the metastable state. Therefore, the process $X^{\varepsilon}_{tT(\varepsilon)}, 0 < t < A < \infty$, will be close to a function $\Phi(t) = \Phi(t, x, \lambda) = K^t_{\mu(x,\lambda)}$.

Now, let $b(t, x)$ and $\sigma(t, x)$ be 1-periodic in t. Furthermore, suppose that the unperturbed system has only a finite number of stable equilibrium points. Then $\Phi(t)$ is also periodic. Thus, the trajectory of $X^{\varepsilon}_{tT(\varepsilon)}$ will be close to a periodic function as $\varepsilon \downarrow 0$. This effect is called stochastic resonance.

In this thesis, we are especially interested in the following problems:

1. Let $n \geq 1$ be a fixed integer. We replace W_t in (1.3) by a random walk ξ^{δ}_t and let σ be the unit matrix. In the case $d = 1$, the random walk $\xi^{\delta,1}_t$, $t \in N_\delta = \{0, \delta, ...k\delta, ...\}$, can jump to $0, \pm\sqrt{\delta}, \ldots, \pm n\sqrt{\delta}$ such that $\xi^{\delta,1}_{t+\delta} - \xi^{\delta,1}_t = \pm i\sqrt{\delta}$ with probability $\frac{1}{2}p_i, i = 1, \ldots, n$. The probability that $\xi^{\delta,1}_t$ jumps to 0 is p_0 and $p_0 + p_1 + \ldots + p_n = 1$. It can be shown that, using the same idea as in [7], the random walk $\xi^{\delta,1}_t$ converges when $\delta \downarrow 0$ and $\sum_{i=1}^n i^2 p_i = 1$ to a one-dimensional Wiener process W_t uniformly on $[0, T]$ with probability 1. The random walk $\xi^{\delta}_t = (\xi^{\delta,1}_t, ..., \xi^{\delta,d}_t)$, $t \in N_\delta$, on a d-dimensional lattice $Z^d_{\sqrt{\delta}}$ ($\sqrt{\delta}$ is the step of the lattice), with components independent and identically distributed, converges to a d-dimensional Wiener process as $\delta \downarrow 0$ and $\sum_{i=1}^n i^2 p_i = 1$.

Such a replacement is, roughly speaking, equivalent to replacing the differential equation by an appropriate difference equation. For convenience, one can construct a continuous and time non-homogeneous process $X^{\delta,\varepsilon}_t$ based on the difference equation. It is of interest to study the asymptotic behavior of $X^{\delta,\varepsilon}_t$ as $\delta, \varepsilon \downarrow 0$. In 2002, the case when $n = 1$ was discussed by Freidlin in [6]. A large deviation principle for the family of processes $X_t^{\delta,\varepsilon}$, as $\delta, \varepsilon \downarrow 0$, $\delta/\varepsilon = \mu^2 = O(1)$, was established in an explicit way in [6]. It turns out it is related to but different from those in the case of white noise. Exit problems and stochastic resonance are also discussed briefly in [6].

In this thesis, we construct a continuous, time non-homogeneous process $X_t^{\delta,\varepsilon}, t \in [0,T]$ as in [6]. For $t = k\delta$ where k is an integer and $t \in [0,T]$, $X_t^{\delta,\varepsilon}$ is defined by the following equation:

$$X_{t+\delta}^{\delta,\varepsilon} - X_t^{\delta,\varepsilon} = \int_t^{t+\delta} b(X_s^{\delta,\varepsilon})ds + \sqrt{\varepsilon}(\xi_{t+\delta}^{\delta} - \xi_t^{\delta}), \;\; X_0^{\delta,\varepsilon} = x. \tag{1.5}$$

For $t \in [k\delta, (k+1)\delta]$, $X_t^{\delta,\varepsilon}$ is defined as the linear function connecting the points $X_{k\delta}^{\delta,\varepsilon}$ and $X_{(k+1)\delta}^{\delta,\varepsilon}$. It can be shown that $X_t^{\delta,\varepsilon}$ converges to X_t^{ε} uniformly on $[0,T]$ in probability when $\delta \downarrow 0$ and $\sum_{i=1}^{n} i^2 p_i = 1$.

However, the large deviations for $X_t^{\delta,\varepsilon}$ are different from those in the case of white noise. In §2.1, the action functional for the family of processes $X_t^{\delta,\varepsilon}, 0 \leq t \leq T$, as $\delta, \varepsilon \downarrow 0$, $\delta/\varepsilon = \mu^2 = O(1)$ in the uniform topology is established. The parameters $p_0, \ldots, p_n$ are chosen so that the action functional for this type of perturbation approximates the action functional for the white-noise-type perturbation in the best way. The tools used in the proof are the limit theorem on large deviations for Markov processes ([8]) and the contraction principle ([5]).

In §2.2, we describe the exit problem of the process $X_t^{\delta,\varepsilon}$ from a bounded domain, as $\delta, \varepsilon \downarrow 0$, $\delta/\varepsilon = \mu^2$. The case when $\delta, \varepsilon \downarrow 0, \delta/\varepsilon = o(1)$ is also considered. Results regarding exit problems for $X_t^{\delta,\varepsilon}$ are obtained following the ideas of Theorems 4.2.1 and 4.4.1 of [5] in which exit problems for X_t^{ε} as $\varepsilon \downarrow 0$ are described. In §2.3, we formulate results concerning stochastic resonance for $X_t^{\delta,\varepsilon}$ following the same idea as in [4]. An example for $n = 2$ is given in §2.4.

2. In Chapter 3, we study the large deviation principle and exit problem for another type of random perturbation approximating white noise. Consider a mean-zero Gaussian process $\eta_t^{\mu,\varepsilon}$. Here $\eta_t^{\mu,\varepsilon} = (\eta_t^{\mu,\varepsilon,1}, ..., \eta_t^{\mu,\varepsilon,d})$, $t \in [0, T]$, with each component identically and independently distributed, satisfying

$$\mu \dot{\eta}_t^{\mu,\varepsilon} = -\eta_t^{\mu,\varepsilon} + \sqrt{\varepsilon}\dot{W}_t, \;\; \eta_0^{\mu,\varepsilon} = y \in R^d.$$

Here W_t is a standard d-dimensional Wiener process and μ is a positive constant. This process η_t^{μ} is called the Ornstein-Uhlenbeck process. Now, let us replace $\sqrt{\varepsilon}\dot{W}_t$ in (1.3) with $\eta_t^{\mu,\varepsilon}$ and let σ be a unit matrix. Then (1.3) becomes

$$\dot{X}_t^{\mu,\varepsilon} = b(X_t^{\mu,\varepsilon}) + \eta_t^{\mu,\varepsilon}, \;\; X_0^{\mu,\varepsilon} = x \in R^d. \tag{1.6}$$

It is of interest to consider the exit problem for $X_t^{\mu,\varepsilon}$ as $\mu, \varepsilon \downarrow 0$. We want to formulate the results of exit problems for $X_t^{\mu,\varepsilon}$ following the same ideas as Theorems 4.2.1 and 4.4.1 of [5], where properties of Markov processes plays an important role in the proofs. However, since $X_t^{\mu,\varepsilon}$ is not a Markov process, we should consider the $2d$-dimensional Markov process $(X_t^{\mu,\varepsilon}, \eta_t^{\mu,\varepsilon})$ in the proof, where

$$\begin{cases} \dot{X}_t^{\mu,\varepsilon} = b(X_t^{\mu,\varepsilon}) + \eta_t^{\mu,\varepsilon}, \\ \mu \dot{\eta}_t^{\mu,\varepsilon} = -\eta_t^{\mu,\varepsilon} + \sqrt{\varepsilon}\dot{W}_t, \\ X_0^{\mu,\varepsilon} = x \in R^d, \;\; \eta_0^{\mu,\varepsilon} = y \in R^d, \;\; 0 < \varepsilon \ll 1. \end{cases} \tag{1.7}$$

In §3.1, we establish a large deviation principle for the family of processes $(X_t^{\mu,\varepsilon}, \eta_t^{\mu,\varepsilon})$ as $\varepsilon \downarrow 0$. In §3.2, we describe the exit problem for the processes $X_t^{\mu,\varepsilon}$ from a bounded domain as $\mu, \varepsilon \downarrow 0$ under appropriate assumptions. Although we follow the same ideas of Theorems 4.2.1 and 4.4.1 of [5], we underline that the proof needs some modification. A detailed proof is given in §3.2.

1.2 A Review of Large Deviations

The large deviation principle for stochastic processes is an essential tool used to analyze the long time behavior of stochastic processes. Here, we give a brief review of this principle. For more details, one may consult [5] and [8].

The first results regarding the large deviation principle were obtained by Craḿer in 1937 [9] and Chernoff in 1952 [10], also proved classical limit theorems for sums of independent random variables. Following the ideas of Freidlin and Wentzell ([5],[8]), we consider stochastic processes and families of measures in infinite-dimensional spaces.

Let $\{\xi_t\}_{t\geq 0}$ be a stochastic process on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ taking values in a measurable phase space $(X, \mathcal{B})$, where $\mathcal{B}$ is the σ-field on X. Let $\mathcal{X}$ be a space of functions $\varphi : [0, T] \to X$. Let $\mathcal{C}(\mathcal{X})$ be the σ-field generated by the cylinder sets $\{\varphi \in \mathcal{X} : (\varphi_{t_1}, ...\varphi_{t_n}) \in C\}, t_i \in [0, T], C \in \mathcal{B}^n$. Denote by μ_ξ the measure on the space $(\mathcal{X}, \mathcal{C}(\mathcal{X}))$ generated by the process ξ_t.

In this thesis, $\mathcal{X}$ will usually be $C_{[0,T]}(X)$, the space of continuous functions defined in $[0, T]$ with values on a metric space (X, ρ). Here we define ρ_{0T} to be the uniform metric,

$$\rho_{0T}(\phi, \varphi) = \sup_{0\leq t\leq T} \rho(\phi_t, \varphi_t).$$

It is known that $\mathcal{C}(\mathcal{X}) = \mathcal{B}_{[0T]}(\mathcal{X})$, where $\mathcal{B}_{[0T]}(\mathcal{X})$ is the σ-field of the Borel sets of $(\mathcal{X}, \rho_{0T})$ (see [11]). For more details of stochastic processes and Markov processes, one may consult [17] and [18].

Suppose that we have a family of stochastic processes $\{\xi_t^\varepsilon\}_{t\geq 0}$, $\varepsilon > 0$, such that $\xi^\varepsilon \to \varphi$ as $\varepsilon \to 0$ in probability. This deterministic function φ can be regarded as the "most probable path" for ξ^ε as $\varepsilon \downarrow 0$. For any measurable set $A \subset \mathcal{X}$ which is of positive distance from φ, $P(\xi \in A) \to 0$ as $\varepsilon \downarrow 0$. The large deviation principle for the family $\{\xi_t^\varepsilon\}_{t\geq 0}$, $\varepsilon > 0$, describes the rate of convergence of $P(\xi \in A) \to 0$ as $\varepsilon \downarrow 0$.

In general, a large deviation principle can be described by an action functional. Let $(\mathcal{X}, \rho_{\mathcal{X}})$ be a metric space. Let μ^ε be a family of probability measures depending on $\varepsilon > 0$ defined on the σ-algebra of Borel subsets of $\mathcal{X}$. Let $\lambda(\varepsilon)$ be a positive real-valued function going to $+\infty$ as $\varepsilon \downarrow 0$, and let $S(x)$ be a function on $\mathcal{X}$ assuming values in $[0, \infty]$. We say that $\lambda(\varepsilon)S(x)$ is an action functional for μ^ε as $\varepsilon \downarrow 0$ if the following assertions hold:

(1) The set $\Phi(s) = \{x : S(x) \leq s\}$ is compact for every $s > 0$.

(2) For any $\delta > 0$, $\gamma > 0$ and $x \in \mathcal{X}$ there exists an $\varepsilon_o > 0$ such that

$$\mu^\varepsilon\{y : \rho(x, y) < \delta\} \geq \exp\{-\lambda(\varepsilon)[S(x) + \gamma]\}$$

for all $\varepsilon \leq \varepsilon_o$.

(3) for any $\delta > 0$, $\gamma > 0$, there exists an $\varepsilon_o > 0$ such that

$$\mu^\varepsilon\{y : \rho(y, \Phi(s)) \geq \delta\} \leq \exp\{-\lambda(\varepsilon)(s - \gamma)\}$$

for $\varepsilon \leq \varepsilon_o$.

The function $S(x)$ is called the <u>normalized action function</u> and $\lambda(\varepsilon)$ is the normalizing coefficient. If the above assertions $(1) - (3)$ are satisfied we say that $\{\mu^\varepsilon\}_{\varepsilon>0}$ obeys a large deviation principle with the action function S. If $\mathcal{X}$ is a space of functions, we use the term <u>action functional</u>. If μ_ξ^ε is the measure on $\mathcal{X}$ generated by the process $\{\xi_t^\varepsilon\}$, then the action functional for the family of processes is the action functional for μ_ξ^ε.

The concept of large deviation principle can also be given in a different form (see [5]).

The following Theorem (named "the contraction principle" in [5]), gives the relationship between the action functionals of two families of random processes connected by a continuous operator and plays an important role in the proofs in this thesis.

Theorem 1.1 : Let $\lambda(\varepsilon)S^{\mu}(x)$ be the action function for a family of measures μ^{ε} on a metric space $(\mathcal{X}, \rho_{\mathcal{X}})$ as $\varepsilon \downarrow 0$. Let G be a continuous mapping of $\mathcal{X}$ into $\mathcal{Y}$ with metric $\rho_{\mathcal{Y}}$ and let a measure υ^{ε} on $\mathcal{Y}$ be given by the formula $\upsilon^{\varepsilon}(A) = \mu^{\varepsilon}(G^{-1}(A))$. The asymptotics of the family of measures υ^{ε} as $\varepsilon \downarrow 0$ is given by the action function $\lambda(\varepsilon)S^{\upsilon}(x)$, where $S^{\upsilon}(y) = \min\{S^{\mu}(x) : x \in G^{-1}(y)\}$ (the minimum over the empty set is set equal to $+\infty$).

Now, we review action functionals for some families of processes which will be useful in this thesis.

1. Consider X_t^{ε} defined in (1.3) and let σ be a unit matrix. By [5], the action functional for the family X_t^{ε} as $\varepsilon \downarrow 0$ in the space $C_{[0,T]}(R^d)$ is equal to $\varepsilon^{-1}S_{[0,T]}(\varphi)$, where

$$S_{[0,T]}(\varphi) = \begin{cases} \frac{1}{2}\int_0^T |\dot{\varphi}_s - b(\varphi_s)|^2 ds, & \text{if } \varphi \text{ is absolutely continuous and } \varphi_0 = x, \\ +\infty, & \text{otherwise.} \end{cases}$$

2. Consider in R^1 a family of discrete Markov processes $X_t^h, 0 \le t \le T$ with time jump size $\tau = \tau(h)$. For each $h > 0$, $X_{(k+1)\tau}^h = X_{k\tau}^h + hU$ where the random variable U has distribution $P_{k\tau, X_{k\tau}^h}$. Let us consider two different cases of the asymptotic problems for the processes X_t^h as $\tau, h \downarrow 0$. The first case is when $\tau = h$ (or τ and h are of the same order) and it is called the case of very large deviation; the second case is when $\tau = o(h), h \downarrow 0$ and it is called the case of not very large deviation. In §3.2 and §4.2 − 4.3 of [8], Wentzell describes the Large Deviation Principle for the process X_t^h as $\tau, h \downarrow 0$ in uniform topology for both of these two cases. We formulate the following Theorems without giving proofs. (One may find proofs in Theorems 3.2.3′ and 4.4.1 of [8].)

Define the cumulant $G^{\tau,h}$ of the process X_t^h by

$$G^{\tau,h}(t,x;z) := \tau^{-1} \ln E_{t,x}^{\tau,h} \exp\{z(X_{t+\tau}^h - x)\},$$

Let $k(h)$ be a real valued function tending to $+\infty$ as $h \to 0$. Let the following conditions be satisfied for the cumulant $G^{\tau,h}(t,x;z)$ and some function $G_0(t,x;z)$:

I1: $k(h)^{-1}G^{\tau,h}(t,x;k(h)z) \to G_0(t,x;z)$ as $h \to 0$, uniformly with respect to t, x and every bound set of values of z.

I2: $\nabla_z(k(h)^{-1}G^{\tau,h}(t,x;k(h)z)) \to \nabla_z G_0(t,x;z)$ as $h \to 0$, uniformly with respect to t, x and every bound set of values of z.

I3: For every bounded set K, let

$$\left| \frac{\partial^2}{\partial z_i \partial z_j}(k(h)^{-1}G^{\tau,h}(t,x;k(h)z)) \right| \leq \mathit{constant} < \infty$$

for all sufficiently small h, for all t, x and $z \in K$.

Theorem 1.2 : Let X_t^h be the family of Markov processes described above. Let τ and h be of the same order. Suppose $k(h) \to \infty$ as $h \to 0$. Let the conditions I1-I3 be satisfied for the cumulant $G^{\tau,h}(t,x;z)$ and some function $G_0(t,x;z)$. Let $L_0(t,x;u)$ be the corresponding Legendre transformation of $G_0(t,x;z)$. That is $L_0(t,x;u) = \sup_z(uz - G_0(t,x;z))$. Suppose the functions $G_0(t,x;z)$ and $L_0(t,x;u)$ satisfy the following conditions:

A1: $G_0(t,x;z) \le \bar{G}_0(z)$ for all t,x,z, where $\bar{G}_0$ is a downward convex non-negative function, finite for all z, and such that $G_0(t,x;0) \equiv \bar{G}(0) = 0$.

Let $\underline{L}(u)$ be the corresponding legendre transformation of $\bar{G}(z)$.

A2: $L_0(t,x;u) < \infty$ for any u such that $\underline{L}(u)$ is finite.

A3: $\triangle L_0(h,\delta') = \sup_{|t-s|\le h, |x-y|\le\delta', L_0(t,x;u)<\infty} \frac{L_0(s,y;u)-L_0(t,x;u)}{1+L_0(t,x;u)} \to 0$ for all $\delta', h \downarrow 0$.

A4: The set $\{u : \underline{L}_0(u) < \infty\}$ has at least one interior point u_o and $\sup_{t,x} L_0(t,x;u_o) < \infty$.

A5: The set of points u of the closure $\bar{U}$ of the set $\{u : \underline{L}_0(u) < \infty\}$ for which $\underline{L}_0(u) = \infty$ is closed.

A6: For any compact $U_K \subseteq \{u : \underline{L}_0(u) < \infty\}$, the function $L_0(t,x;u)$ is continuous in u uniformly with respect to t,x and $u \in U_K$.

A7: For any compact U_K consisting entirely of interior points of $\{u : \underline{L}_0(u) < \infty\}$, the first derivative of $L_0^\mu(u)$, $\frac{dL_0(t,x;u)}{du}$, is bounded and continuous with respect to u uniformly with respect to $t,x,u \in U_K$.

Then the action functional for the family of processes X_t^h as $h \downarrow 0$ in the uniform topology is $k(h)S_{[0,T]}(\varphi)$, where

$$S_{[0,T]}(\varphi) = \begin{cases} \int_0^T L_0(s,\varphi(s);\dot{\varphi}(s))ds, & \text{if } \varphi \text{ is absolutely cont. and } \varphi_0 = x, \\ +\infty, & \text{otherwise.} \end{cases}$$

Theorem 1.3 : Let a family of Markov processes X_t^h be as described above. Suppose $\tau h^{-2} \to \infty, \tau h^{-1} \to 0$ as $h \to 0$. Let the conditions I1-I3 be satisfied for the cumulant $G^{\tau,h}(t,x;z)$ and some function $G_0(t,x;z)$. Let the function $G_0(t,x;z)$ be finite and bounded for all t,x and all sufficiently small $|z|$, let $\frac{dG_0(t,x;z)}{dz}|_{z=0} \equiv 0$ and let the matrix

$$(A^{ij}(t,x)) = (\frac{\partial^2 G_0}{\partial z_i \partial z_j}(t,x,0)))$$

be bounded, uniformly positive definite and uniformly continuous with respect to t,x. Put $(A_{ij}(t,x)) = (A^{ij}(t,x))^{-1}$. Put

$$H_0(t,x;u) = \frac{1}{2}\sum_{i,j} A_{ij}(t,x)u^i u^j$$

Then the action functional for the family of processes X_t^h as $h \to 0$ is $\tau h^{-2} S_{[0,T]}(\varphi)$ uniformly with respect to the initial point where

$$S_{[0,T]}(\varphi) = \begin{cases} \int_0^T H_0(s,\varphi(s);\dot{\varphi}(s))ds, & \text{if } \varphi \text{ is absolutely continuous and } \varphi_0 = x, \\ +\infty, & \text{otherwise.} \end{cases}$$

1.3 Main Results

We describe the main results of this dissertation in this section.

1. Consider the family of processes $X_t^{\delta,\varepsilon}$ defined in (1.5).

Theorem 1.4 : Let $n \geq 1$ be an integer. Let $p_0, \ldots, p_n$ satisfy $\sum_{i=1}^n p_i i^{2k} = (2k-1)!!$ for $k = 1, \ldots, n$. The action functional for the family $X_t^{\delta,\varepsilon}$ as $\delta, \varepsilon \downarrow 0, \delta/\varepsilon = \mu^2$, in the space $C_{[0,T]}(R^d)$, is equal to $\varepsilon^{-1} S^{\mu}_{[0,T]}(\varphi)$, where

$$S^{\mu}_{[0,T]}(\varphi) = \begin{cases} \int_0^T \sum_{i=1}^d L_0^{\mu}(\dot{\varphi}_s^i - b^i(\varphi^i))ds, & \text{if } \varphi \text{ is absolutely continuous and } \varphi_0 = x, \\ +\infty, & \text{otherwise.} \end{cases}$$

where

$$L_0^{\mu}(u) = \frac{u^2}{2} + \left[\frac{(2n+1)!! - \sum_{i=1}^n p_i i^{2n+2}}{(2n+2)!}\right] u^{2n+2}\mu^{2n} + O(\mu^{2n+2}), \quad 0 < \mu << 1.$$

Theorem 1.5 : The action functional for the family of processes $X_t^{\delta,\varepsilon}$ as $\delta, \varepsilon \downarrow 0, \delta/\varepsilon \downarrow 0$ in the space $C_{[0,T]}(R^d)$ is $\varepsilon^{-1}S_{[0,T]}(\varphi)$, the same as the action functional for X_t^ε when $\varepsilon \downarrow 0$.

Consider the system

$$\dot{X}_t = b(X_t), \;\; X_0 = x \in R^d,$$

where the vector field $b(x)$ is Lipschitz continuous.

Assumption 1: The vector field $b(x), x \in R^d$, has an asymptotically stable equilibrium at a point $O \in R^d$.

Let $G \subset R^d$ be a bounded domain with boundary ∂G.

Assumption 2: The domain G is attracted to $O \in G$: $\lim_{t\uparrow\infty} X_t = O$ for each trajectory of $\dot{X}_t = b(X_t)$, $X_0 = x \in G$.

Assumption 3: The domain G has a smooth boundary ∂G and $(b(x) \cdot n(x)) < 0, x \in \partial G$ where $n(x)$ is the exterior normal of the boundary of G.

Now, consider the continuous process $X_t^{\delta,\varepsilon}$ defined in equation (1.5) and the process X_t^ε in equation (1.3) with σ as a unit matrix. Let $X_0^\varepsilon = X_0^{\delta,\varepsilon} = x \in G$. Denote by $\tau = \tau^\varepsilon$ ($\tau^\delta = \tau^{\delta,\varepsilon}$) the first exit time from G for the process X_t^ε ($X_t^{\delta,\varepsilon}$): $\tau^\varepsilon = \min\{t : X_t^\varepsilon \in \partial G\}$, $\tau^{\delta,\varepsilon} = \min\{t : X_t^{\delta,\varepsilon} \in \partial G\}$.

Define $V^\mu(x) = \inf_{\varphi \in C_{[0,T]}(R^d)}\{S^\mu_{[0,T]}(\varphi) : \varphi_0 = O, \varphi_T = x, T > 0\}$ and $V_o^\mu = \min_{x\in\partial G} V^\mu(x)$. Define $V(x) = \inf_{\varphi \in C_{[0,T]}(R^d)}\{S_{[0,T]}(\varphi) : \varphi_0 = O, \varphi_T = x, T > 0\}$ and $V_o^\mu = \min_{x\in\partial G} V^\mu(x)$. Here $\varepsilon^{-1}S^\mu_{[0,T]}(\varphi)$ is the action functional for $X_t^{\delta,\varepsilon}$ as $\delta\varepsilon \downarrow 0, \delta/\varepsilon = \mu^2 = O(1)$ in the space $C_{[0,T]}$ and $\varepsilon^{-1}S_{[0,T]}(\varphi)$ is the action functional for X_t^ε when $\varepsilon \downarrow 0$ in the space $C_{[0,T]}$.

Theorem 1.6: Let Assumptions 1-3 be satisfied. Then for any initial point

$x \in G$ and $h > 0$

$$\lim_{\varepsilon,\delta\downarrow 0;\delta\varepsilon^{-1}=\mu^2} \varepsilon \ln E_x \tau^{\delta,\varepsilon} = V_o^\mu,$$

$$\lim_{\varepsilon,\delta\varepsilon^{-1}\downarrow 0} \varepsilon \ln E_x \tau^{\delta,\varepsilon} = \lim_{\varepsilon\downarrow 0} \varepsilon \ln E_x \tau^{\varepsilon} = V_o.$$

$$\lim_{\varepsilon,\delta\downarrow 0;\delta\varepsilon^{-1}=\mu^2} P_x(e^{\frac{V_o^\mu-h}{\varepsilon}} < \tau^{\delta,\varepsilon} < e^{\frac{V_o^\mu+h}{\varepsilon}}) = 1,$$

$$\lim_{\varepsilon,\delta\varepsilon^{-1}\downarrow 0} P_x(e^{\frac{V_o-h}{\varepsilon}} < \tau^{\delta,\varepsilon} < e^{\frac{V_o+h}{\varepsilon}}) = \lim_{\varepsilon\downarrow 0} P_x(e^{\frac{V_o-h}{\varepsilon}} < \tau^{\varepsilon} < e^{\frac{V_o+h}{\varepsilon}}) = 1,$$

If $\min_{x\in\partial G} V^\mu(x)$ ($\min_{x\in\partial G} V(x)$) is achieved just at one point $x_*^\mu \in \partial G$ ($x_* \in \partial G$), then

$$\lim_{\varepsilon,\delta\downarrow 0;\delta\varepsilon^{-1}=\mu^2} P_x(|X^{\delta,\varepsilon}_{\tau^{\delta,\varepsilon}} - x_*^\mu| > h) = 0$$

$$\lim_{\varepsilon,\delta\varepsilon^{-1}\downarrow 0} P_x(|X^{\delta,\varepsilon}_{\tau^{\delta,\varepsilon}} - x_*| > h) = \lim_{\varepsilon\downarrow 0} P_x(|X^{\varepsilon}_{\tau^{\varepsilon}} - x_*| > h) = 0.$$

Theorem 1.7: Let $p_0, \ldots, p_n$ satisfy $\sum_{i=1}^n p_i i^{2k} = (2k-1)!!,\ k = 1, \ldots, n$. Let $b(x)$ be of potential type, that is, there exists $U(x)$ such that $\nabla U(x) = -b(x)$. We assume that $U(x)$ is smooth enough, $U(O) = 0$, and that $U(x) > 0, \nabla U(x) \neq 0$ for $x \neq 0$. Then

$$V^\mu(x) = 2U(x) + V_n(x)\mu^{2n} + O(\mu^{2n+2}),\ 0 < \mu << 1$$

where $V_n(x)$ is given by the equation

$$V_n(x) = 2^{2n+2} B_n \int_{-\infty}^0 |\nabla U(Z_t)|^{2n+2} dt.$$

Here $B_n = \frac{(2n+1)!!-\sum_{i=1}^n p_i i^{2n+2}}{(2n+2)!}$ and Z_t is the solution of the equation $\dot{Z}_t = \nabla U(Z_t), Z_0 = x \in G, t < 0$.

2. Consider the process $X_t^{\mu,\varepsilon}$ defined in (1.6). We are interested in the asymptotic behavior of $\tau^X = \min\{t : X_t^{\mu,\varepsilon} \in \partial G\}$ and the exit point $X^{\mu,\varepsilon}_{\tau^X}$ when $\varepsilon \downarrow 0$. Since $X_t^{\mu,\varepsilon}$ is not a Markov process, this creates some difficulties to obtain results

concerning τ^X and the exit point. The $2d$-dimensional Markov process $(X_t^{\mu,\varepsilon}, \eta_t^{\mu,\varepsilon})$ should be considered in the proof.

Let $b(x)$ be of potential type, that is, there exists $U(x)$ such that $\nabla U(x) = -b(x)$. We assume $U(x)$ is smooth enough and $U(O) = 0$ and $U(x) > 0, \nabla U(x) \neq 0$ for $x \neq 0$. In this case, we notice that the unperturbed system $(X_t^{\mu}, \eta_t^{\mu})$ has an asymptotically stable point at $(O \times O)$.

Theorem 1.8: The action functional for the family of processes $(X_t^{\mu,\varepsilon}, \eta_t^{\mu,\varepsilon})$ in the space $C_{[0,T]}(R^{2d})$ as $\varepsilon \downarrow 0$ is $\varepsilon^{-1} S^{\mu}_{[0,T]}(\varphi, \phi)$, where

$$S^{\mu}_{[0,T]}(\varphi, \phi) = \frac{1}{2} \int_0^T |(\dot{\varphi}_t - b(\varphi_t) + \mu \frac{d}{dt} |\dot{\varphi}_t - b(\varphi_t)|^2 dt,$$

if $\dot{\varphi}_t$ is absolutely continuous, $\phi_t = \dot{\varphi}_t - b(\varphi_t)$ and $\varphi_0 = x, \phi_0 = y$. Otherwise $S^{\mu}_{[0T]}(\varphi, \phi) = \infty$ for the remaining functions in $C_{[0,T]}(R^{2d})$.

Define $V^{\mu}(x) = \inf_{\varphi, \phi \in C_{[0,T]}(R^d)} \{S^{\mu}_{[0,T]}(\varphi, \phi) : \varphi_0 = O, \phi_0 = O, \varphi_T = x, T > 0\}$ and $V_o^{\mu} = \min_{x \in \partial G} V^{\mu}(x)$.

Theorem 1.9: Let assumptions 1-3 be satisfied. Let $b(x)$ be of the potential type and let all the assumptions concerning the potential are satisfied. Let N be any positive constant such that the unperturbed system $(X_t^{\mu}, \eta_t^{\mu})$, $(X_0^{\mu}, \eta_0^{\mu}) = (x, y)$ with $x \in G, |y| < N$, never leaves the domain $G \times R^d$. Then, for any $x \in G$, $|y| \leq N$ and $\alpha > 0$, there exists a μ^* such that for any $\mu < \mu^*$,

$$\lim_{\varepsilon \downarrow 0} \varepsilon \ln E_{x,y} \tau^X < V_o^{\mu} + \alpha, \quad \lim_{\varepsilon \downarrow 0} \varepsilon \ln E_{x,y} \tau^X > V_o^{\mu} - \alpha,$$

$$\lim_{\varepsilon \downarrow 0} P_{x,y}(e^{\frac{V_o^{\mu} - \alpha}{\varepsilon}} < \tau^X < e^{\frac{V_o^{\mu} + \alpha}{\varepsilon}}) = 1.$$

If $\min_{x \in \partial G} V^{\mu}(x)$ $(\min_{x \in \partial G} V(x))$ is achieved at just one point $x_o^{\mu} \in \partial G$ $(x_o \in \partial G)$, then

$$\lim_{\varepsilon \downarrow 0} P_{x,y}(|X^{\mu,\varepsilon}_{\tau^X} - x^{\mu}_{o}| < \alpha) = 1.$$

Chapter 2

Stochastic Resonance for Random-walk Perturbations Approximating White Noise

2.1 Large Deviations

Recall from the introduction that we constructed a continuous d-dimensional stochastic process $X_t^{\delta,\varepsilon}$ in (1.5) when replacing the Wiener process W_t in (1.3) with the random walk ξ_t^δ. Our goal in this section is to establish a large deviation principle for the family of processes $X_t^{\delta,\varepsilon}$ as $\delta, \varepsilon \downarrow 0$. We describe this large deviation principle by means of an action functional in the space $C_{[0,T]}(R^d)$.

Let $\delta, \varepsilon \downarrow 0$ and $\delta/\varepsilon = \mu^2$, where $\mu > 0$ is a fixed constant. First, let us consider the one-dimensional case and assume $b(x) \equiv 0$. Then $X_{m\delta}^{\delta,\varepsilon} - x$ is the sum of m independent random variables, $\eta_k^\delta = \sqrt{\varepsilon}(\xi_{(k+1)\delta}^\delta - \xi_{k\delta}^\delta)$. Here $\eta_k^\delta = \pm i\sqrt{\delta\varepsilon}$, with probability $\frac{1}{2}p_i$ for $i = 1, \ldots, n$, and $\eta_k^\delta = 0$ with probability p_o. Furthermore, we assume $\sum_{i=0}^n p_i = 1, \sum_{i=1}^n i^2 p_i = 1$.

In order to calculate the action functional for $X_t^{\delta,\varepsilon}$ as $\delta, \varepsilon \downarrow 0, \delta/\varepsilon = \mu^2$, which is the case of very large deviation, we apply Theorem 1.2 (see §1.2): One should first calculate the cumulant $G^{\mu,\varepsilon}$ of the process $X_t^{\delta,\varepsilon}$ where

$$G^{\mu,\varepsilon}(z) = G^{\delta,\varepsilon}(t,x;z) = \frac{1}{\delta}\ln E\exp\{z\eta_i^\delta\}.$$

Then one should find $G_0^\mu(z) := \lim_{\varepsilon\downarrow 0}\varepsilon G_0^{\mu,\varepsilon}(z\varepsilon^{-1})$. Let $L_0^\mu(u)$ be the Legendre transform of $G_0^\mu(z)$. If conditions A1–A7 in Lemma 1.2 are satisfied, the action functional for the family of processes $X_t^{\delta,\varepsilon}$, as $\delta,\varepsilon\downarrow 0$, $\delta/\varepsilon = \mu^2$ in the space $C_{[0,T]}(R^1)$, is equal to $\varepsilon^{-1}S_{[0,T]}^{\mu,0}$ where

$$S_{[0,T]}^{\mu,0}(\varphi) = \begin{cases} \int_0^T L_0^\mu(\dot\varphi_s)ds, & \text{if } \varphi \text{ is absolutely continuous and } \varphi_0 = x, \\ +\infty, & \text{otherwise.} \end{cases} \tag{2.1}$$

Before we start the calculations, let us mention, without proof, some properties of convex functions and the Legendre transformation. (see Theorem 2.6.5 of [12] or §1.1.2 [8].)

Lemma 2.1 : Let $G(z)$ be a downward convex, lower semi-continuous function of $z\in R^1$, taking values in $(-\infty,\infty]$. The Legendre transform is the function $L(u)$, $u\in R^1$, defined by the formula $L(u) = \sup_z(zu - G(z))$. This transform is also downward convex and lower semi-continuous. Furthermore,

(1) if the function $L(u)$ is differentiable at a point u, then

$$L(u) = (L^{'}(u))u - G(L^{'}(u));$$

(2) if, in addition, the function G is differentiable at the point $z = L^{'}(u)$, then

$$G^{'}(L^{'}(u)) = u;$$

(3) if the function G is differentiable $k \geq 2$ times at a point z, then the function $L(u)$ is also k times differentiable at the point $G^{'}(z)$ and

$$L^{''}(u)|_{u=G^{'}(z)} = (G^{''}(z))^{-1}.$$

$\blacksquare$

Now, let us calculate the cumulant $G^{\mu,\varepsilon}(z)$ and $G_0^{\mu}(z)$. In our case,

$$G^{\mu,\varepsilon}(z) := \frac{1}{\delta} \ln E \exp\{z\eta_i^{\delta}\} = \frac{1}{\delta} \ln\{p_0 + \sum_{j=1}^{n} \frac{p_i}{2}(e^{jz\sqrt{\delta\varepsilon}} + e^{-jz\sqrt{\delta\varepsilon}})\};$$

$$G_0^{\mu}(z) := \lim_{\varepsilon \downarrow 0} \varepsilon G_0^{\mu,\varepsilon}(z\varepsilon^{-1}) = \frac{1}{\mu^2} \ln\{p_0 + \sum_{j=1}^{n} \frac{p_i}{2}(e^{jz\mu} + e^{-jz\mu})\}.$$

We summarize some properties of $G_0^{\mu}(z)$ in the following:

G1: $G_0^{\mu}(0) = 0$; $G_0^{\mu}(z)$ is an even and non-negative function; $G_0^{\mu}(z)$ is finite for all $z \in R$.

G2: The first derivative of $G_0^{\mu}(z)$ is

$$(G_0^{\mu})^{'}(z) = \frac{\sum_{i=1}^{n} p_i i(e^{iz\mu} - e^{-iz\mu})}{\mu(2p_0 + \sum_{i=1}^{n} p_i(e^{iz\mu} + e^{iz\mu}))}.$$

It is easily checked that $\lim_{z\uparrow\pm\infty}(G_0^{\mu})^{'}(z) = \pm\frac{n}{\mu}$ and $(G_0^{\mu})^{'}(z)$ is an odd function. Furthermore,

$$\begin{cases} (G_0^{\mu})^{'}(z) < 0, & z < 0; \\ (G_0^{\mu})^{'}(z) = 0, & z = 0; \\ (G_0^{\mu})^{'}(z) > 0, & z > 0. \end{cases}$$

G3: The second derivative of $G_0^\mu(z)$ is

$$\begin{aligned}
(G_0^\mu)''(z) &= \frac{4 + 2\sum_{i=1}^n p_0 p_i i^2 (e^{iz\mu} + e^{-iz\mu})}{(2p_0 + \sum_{i=1}^n p_i(e^{iz\mu} + e^{-iz\mu}))^2} \\
&+ \frac{\sum_{1\le i<j\le n} p_i p_j (i-j)^2 (e^{(i+j)z\mu} + e^{-(i+j)z\mu})}{(2p_0 + \sum_{i=1}^n p_i(e^{iz\mu} + e^{-iz\mu}))^2} \\
&+ \frac{\sum_{1\le i<j\le n} p_i p_j (i+j)^2 (e^{(i-j)z\mu} + e^{-(i-j)z\mu})}{(2p_0 + \sum_{i=1}^n p_i(e^{iz\mu} + e^{-iz\mu}))^2}, \quad z \in R \\
&\ge \frac{4}{(2p_0 + \sum_{i=1}^n p_i(e^{iz\mu} + e^{-iz\mu}))^2} \\
&> 0.
\end{aligned}$$

G4: From property G3, $G_0^\mu(z)$ is a downward convex and even function.

G5: From property G3, $(G_0^\mu)'(z)$ is monotone increasing for $z \in R$.

Now, let us summarize some properties of $L_0^\mu(u)$, the Legendre transformation of $G_0^\mu(z)$.

L1: $L_0^\mu(0) = \sup_z(-G_0^\mu(z)) = -\inf_z G_0^\mu(z) = 0$; $L_0^\mu(u)$ is an even function because

$$L_0^\mu(-u) = \sup_z(-uz - G_0^\mu(z)) = \sup_z(-uz - G_0^\mu(-z)) = L_0^\mu(u).$$

L2: By Lemma 2.1, $L_0^\mu(u)$ is downward convex and lower semi-continuous because $G_0^\mu(z)$ is a downward convex and continuous function.

L3: For $u \in (-n/\mu, +n/\mu)$ $L_0^\mu(u)$ is twice differentiable and $(L_0^\mu)''(u) > 0$. The reason is the following:

Since $\lim_{z\uparrow\pm\infty}(G_0^\mu)'(z) = \pm n/\mu$ and $(G_0^\mu)'(z)$ is a continuous and monotone increasing function for $z \in R$ (see Properties G2 and G5), there exists a z_o such that $(G_o^\mu)'(z)|_{z=z_o} = u$ for any $u \in (-n/\mu, +n/\mu)$. Following part (3) of Lemma 2.1, $(L_0^\mu)''(u)$ exists and $(L_0^\mu)''(u) = ((G_0^\mu)''(z_o))^{-1} > 0$ for any $u \in (-n/\mu, +n/\mu)$.

L4: Following part (2) of Lemma 2.1 and Property L3, for any $u \in (-n/\mu, +n/\mu)$ $(L_0^\mu)'(u)$ exists and $(G_0^\mu)'(z)|_{z=(L_0^\mu)'(u)} = u$. Furthermore, it follows from Property G2 that for $u \in (-n/\mu, +n/\mu)$, $(L_0^\mu)'(-u) = -(L_0^\mu)'(-u)$ and

$$\begin{cases} (L_0^\mu)'(u) < 0, & u \in (-n/\mu, 0); \\ (L_0^\mu)'(u) = 0, & u = 0; \\ (L_0^\mu)'(u) > 0, & u \in (0, n/\mu). \end{cases}$$

L5: For $|u| = n/\mu$, $L_0^\mu(u) = \frac{1}{\mu^2}\ln\frac{2}{p_n}$; for $|u| < +n/\mu$, $L_0^\mu(u) \le \frac{1}{\mu^2}\ln\frac{2}{p_n}$; for $|u| > +n/\mu$, $L_0^\mu(u) = \infty$.

Proof: Since $L_0^\mu(0) = 0$ and $L_0^\mu(u)$ is an even function, it is sufficient to prove property L5 for $u > 0$. Let $u > 0$.

$$\begin{aligned} L_0^\mu(u) &= \sup_z \left\{ uz - \frac{1}{\mu^2}\ln\left(p_0 + \sum_{i=1}^n \frac{p_i}{2}(e^{iz\mu} + e^{-iz\mu})\right)\right\} \\ &= \frac{1}{\mu^2}\sup_z \left\{ \ln e^{\mu^2 uz} - \ln\left(p_0 + \sum_{i=1}^n \frac{p_i}{2}(e^{iz\mu} + e^{-iz\mu})\right)\right\} \\ &= \frac{1}{\mu^2}\ln\sup_z f(z). \end{aligned}$$

where

$$f(z) = \frac{e^{\mu^2 uz}}{p_0 + \sum_{i=1}^n (p_i/2)(e^{iz\mu} + e^{-iz\mu})}$$

and the first derivative of $f(z)$ is

$$f'(z) = \frac{\mu e^{n\mu^2 uz}\left[\mu u p_0 + \mu u \sum_{i=1}^n (p_i/2)(e^{iz\mu} + e^{-iz\mu}) - \sum_{i=1}^n (p_i/2) i (e^{iz\mu} + e^{-iz\mu})\right]}{\left[p_0 + \sum_{i=1}^n (p_i/2)(e^{iz\mu} + e^{-iz\mu})\right]^2}.$$

Notice that $f^{'}(z) \geq 0$ for $\mu u \geq n$. Therefore, for $u \geq n/\mu$,

$$\begin{aligned} L_0^\mu(u) &= \frac{1}{\mu^2}\ln \sup_z f(z) \\ &= \frac{1}{\mu^2}\ln \lim_{z\uparrow\infty} f(z) \\ &= \frac{1}{\mu^2}\ln \lim_{z\uparrow\infty} \frac{e^{\mu^2 uz}}{p_0+\sum_{i=1}^n (p_i/2)(e^{iz\mu}+e^{-iz\mu})}. \end{aligned}$$

It can be easily checked that $L_0^\mu(n/\mu) = (1/\mu^2)\ln(2/p_n)$ and $L_0^\mu(u) = \infty$ for $u > +n/\mu$.

To show $L_0^\mu(u) \leq L_0^\mu(n/\mu) = (1/\mu^2)\ln(2/p_n)$ for $0 < u < +n/\mu$, it is sufficient to show $L_0^\mu(u)$ is non-decreasing function for $u > 0$. Notice that for any $0 < u_1 \leq u_2$,

$$\begin{aligned} L_0^\mu(u_2) - L_0^\mu(u_1) &= \sup_{z\in R}(u_2 z - G_0^\mu(z)) - \sup_{z\in R}(u_1 z - G_0^\mu(z)) \\ &= \sup_{z\geq 0}(u_2 z - G_0^\mu(z)) - \sup_{z\geq 0}(u_1 z - G_0^\mu(z)) \geq 0. \end{aligned}$$

This completes the proof.

L6: When $n = 1$, i.e., $p_0 = 0, p_1 = 1$, the Legendre transform was calculated explicitly in [6]:

$$L_0^\mu(u) = \begin{cases} \frac{1}{2\mu^2}[(1+\mu u)\ln(1+\mu u) + (1-\mu u)\ln(1-\mu u)], & \text{if } |u| \leq \frac{1}{\mu}, \\ +\infty, & \text{if } |\text{u}| > \frac{1}{\mu}. \end{cases}$$

Lemma 2.2 : The action functional for $X_t^{\delta,\varepsilon}$ as $\delta, \varepsilon \downarrow 0$, $\delta/\varepsilon = \mu^2$ in $C_{[0,T]}(R^1)$ is $\varepsilon^{-1}S_{[0,T]}^{\mu,0}$ where $S_{[0,T]}^{\mu,0}$ is given by the formula (2.1).

Proof: This is an application of Lemma 1.2. Let us verify that the conditions $A1 - A7$ in Lemma 1.2 are satisfied. Notice that $G_0^\mu(z)$ is independent of t and x. Let the function $\bar{G}_0^\mu(z)$ in the conditions $A1 - A7$ be $\bar{G}_0^\mu(z) \equiv G_0^\mu(z)$, then

$L_0^\mu(u) \equiv \underline{L}_0^\mu(u)$. It is easily checked that conditions $A1 - A5$ are satisfied. In particular, $\{u : \underline{L}_0^\mu(u) < \infty\} = \{u : |u| \leq n/\mu\}$. In condition $A4$, we can take $u_0 = 0$. Condition $A7$ is satisfied because $(L_0^\mu)''(u)$ exists for $|u| < n/\mu$. To show that condition $A6$ is satisfied, it is sufficient to show the function $L_0^\mu(u)$ is left continuous at $u = n/\mu$. Notice that $L_0^\mu(u)$ is a lower semi-continuous and non-decreasing function for $u > 0$ (see the proof of L5), it is easily verified that $\lim_{u \uparrow n/\mu} L_0^\mu(u) = L_0^\mu(n/\mu)$. This completes the proof.

■

Consider now the d-dimensional stochastic process $X_t^{\delta,\varepsilon}$ with $b(x) = 0$. Since the components of this process are independent, the action functional for the family $X_t^{\delta,\varepsilon}$ as $\delta, \varepsilon \downarrow 0, \delta/\varepsilon = \mu^2$ in $C_{[0,T]}(R^d)$ is equal to

$$\varepsilon^{-1} \tilde{S}_{0T}^{\mu,0}(\varphi) = \varepsilon^{-1} \sum_{i=1}^{d} S_{[0,T]}^{\mu,0}(\varphi^i), \;\; \varphi_t = (\varphi_t^1, ..., \varphi_t^d) \in C_{[0,T]}(R^d).$$

To calculate the action functional in the case when $b(x) \neq 0$, note that the map $R : \varphi \mapsto X$ in $C_{[0,T]}(R^d)$ defined by the equation

$$X_t = x + \int_0^t b(X_s)ds + (\varphi_t - \varphi_0)$$

is continuous and $R^{-1}(X) = \varphi$ is defined uniquely, if we assume that $\varphi_0 = 0$. Thus, the following theorem can be derived from the contraction principle:

Theorem 2.1 : The action functional for the family $X_t^{\delta,\varepsilon}$ as $\delta, \varepsilon \downarrow 0, \delta/\varepsilon = \mu^2$, in the space $C_{[0,T]}(R^d)$, is equal to $\varepsilon^{-1} S_{[0,T]}^\mu(\varphi)$, where

$$S_{[0,T]}^\mu(\varphi) = \begin{cases} \int_0^T \sum_{i=1}^d L_0^\mu(\dot{\varphi}_s^i - b^i(\varphi^i))ds, & \text{if} \varphi \text{ is absolutely continuous and } \varphi_0 = x, \\ +\infty, & \text{otherwise.} \end{cases}$$

$\blacksquare$

Now we will look for $L_0^\mu(u)$ in the form: $L_0^\mu(u) = L^0(u) + L^1(u)\mu + L^2(u)\mu^2 + \ldots, 0 < \mu << 1$. We will choose parameters $p_0, \ldots, p_n$ so that the action functional for this type of perturbations approximates the action functional for the white noise type perturbation in the best way.

Recall from property L4 that $(G_0^\mu)'(z)|_{z=(L_0^\mu)'(u)} = u$ for any $u \in (-n/\mu, +n/\mu)$. Together with the expression of $(G_0^\mu)'(u)$(see Property G2), we have

$$u\mu(2p_0 + \sum_{i=1}^{n} p_i(e^{iz^*\mu} + e^{-iz^*\mu})) = \sum_{i=1}^{n} ip_i(e^{iz^*\mu} - e^{-iz^*\mu}). \tag{2.2}$$

where $z^* := (L_0^\mu)'(u)$. Now let us look for $z^* = (L_0^\mu)'(u)$ in the form $z^* = \sum_k c_k\mu^k, 0 < \mu << 1$.

First, we state the following lemma which is useful.

Lemma 2.3 : Let z^* be defined as above. Then, for $\mu > 0$ and $|\mu u| < |(np_n/2)|$,

$$|z^*| < \left|\frac{2u}{n^2 p_n}\right|.$$

Proof: Recall from Property L4 that $(L_0^\mu)'(u)$ is an odd function for $u \in (-n/\mu, n/\mu)$ and $(L_0^\mu)'(u) > 0$ when $u \in (0, n/\mu)$. Therefore, to prove the lemma, it is sufficient to prove $z^* < 2u/(n^2 p_n)$ when $\mu u < (np_n)/2, u > 0, \mu > 0$.

Let $u > 0, \mu > 0$ and $\mu u < \frac{np_n}{2}$. Recall from Property G5 that $(G_0^\mu)'(z)$ is a monotone increasing function for $z \in R$. Hence,

$$z^* < \frac{2u}{n^2 p_n} \Leftrightarrow (G_0^\mu)'(z^*) < (G_0^\mu)'\left(\frac{2u}{n^2 p_n}\right).$$

Recall from Property L4 that $(G_0^\mu)'(z)|_{z=(L_0^\mu)'(u)} = u$ for any $u \in (-n/\mu, +n/\mu)$.

It implies $(G_0^\mu(z^*))' = u$. Hence,

$$(G_0^\mu)'(z^*) < (G_0^\mu)'\left(\frac{2u}{n^2p_n}\right) \Leftrightarrow u < (G_0^\mu)'\left(\frac{2u}{n^2p_n}\right).$$

For convenience, let us denote $\tilde{z} := \frac{2u}{n^2p_n}$. From above argument and the expression of $(G_0^\mu)'(u)$(see Property G2), it is sufficient to prove the following inequality:

$$u\mu \leq \frac{\sum_{i=1}^n ip_i(e^{i\tilde{z}\mu} - e^{-i\tilde{z}\mu})}{(2p_0 + \sum_{i=1}^n p_i(e^{i\tilde{z}\mu} + e^{-i\tilde{z}\mu}))}.$$

Notice that

$$\frac{np_n(e^{n\tilde{z}\mu} - e^{-n\tilde{z}\mu})}{e^{n\tilde{z}\mu} + e^{-n\tilde{z}\mu}} \leq \frac{\sum_{i=1}^n ip_i(e^{i\tilde{z}\mu} - e^{-i\tilde{z}\mu})}{(2p_0 + \sum_{i=1}^n p_i(e^{i\tilde{z}\mu} + e^{-i\tilde{z}\mu}))}.$$

Hence, to prove this lemma it is sufficient to show $u\mu \leq \frac{np_n(e^{n\tilde{z}\mu} - e^{-n\tilde{z}\mu})}{e^{n\tilde{z}\mu} + e^{-n\tilde{z}\mu}}$. Now, let us prove it.

$$\begin{aligned}
& u\mu \leq \frac{np_n(e^{n\tilde{z}\mu} - e^{-n\tilde{z}\mu})}{e^{n\tilde{z}\mu} + e^{-n\tilde{z}\mu}} \\
\Leftrightarrow\quad & u\mu e^{n\tilde{z}\mu} + u\mu e^{-n\tilde{z}\mu} \leq np_n e^{n\tilde{z}\mu} - np_n e^{-n\tilde{z}\mu} \\
\Leftrightarrow\quad & (np_n + u\mu)e^{-n\tilde{z}\mu} \leq (np_n - u\mu)e^{n\tilde{z}\mu} \\
\Leftrightarrow\quad & \frac{1 + \frac{u\mu}{nP_n}}{1 - \frac{u\mu}{nP_n}} \leq e^{2n\tilde{z}\mu} \\
\Leftrightarrow\quad & \frac{1}{2n\mu}\left[\ln\left(1 + \frac{u\mu}{nP_n}) - \ln(1 - \frac{u\mu}{nP_n}\right)\right] \leq \tilde{z}.
\end{aligned}$$

To prove the above inequality, notice that

$$\begin{aligned}
& \frac{1}{2n\mu}\left[\ln(1 + \frac{u\mu}{nP_n}) - \ln(1 - \frac{u\mu}{nP_n})\right] \\
& = \frac{1}{2n\mu}\sum_{k=1}^{\infty}\frac{2(\frac{u\mu}{np_n})^{2k-1}}{2k-1} \\
& = \frac{1}{n}\frac{u}{np_n}\sum_{k=1}^{\infty}\frac{(\frac{u}{np_n})^{2k-2}\mu^{2k-2}}{2k-1} \\
& \leq \frac{1}{n}\frac{u}{np_n}\sum_{k=1}^{\infty}(\frac{u}{np_n})^{2k-2}\mu^{2k-2} \\
& \leq \frac{1}{n}\frac{u}{np_n}\sum_{k=1}^{\infty}(\frac{1}{2})^{2k-2} < 2\frac{u}{n^2p_n} = \tilde{z}.
\end{aligned}$$

The above inequality holds because $|\mu u| < |\frac{np_n}{2}|$. This completes the proof.

$\blacksquare$

Lemma 2.4 : Consider $z^* = (L_o^\mu)'(u)$ as in Lemma 2.3. Let $n \geq 1$ be a fixed integer. Assume $p_0, \dots, p_n$ satisfy $\sum_{i=1}^n p_i i^{2k} = (2k-1)!!,\ k = 1, \dots, n$. Then

$$z^* = u + \left(\frac{(2n+1)!! - \sum_{i=1}^n p_i i^{2n+2}}{(2n+1)!} \right) u^{2n+1}\mu^{2n} + O(\mu^{2n+2}), \quad 0 < \mu << 1.$$

Proof : When $n = 1$ it is easily seen that $p_0 = 0$ and $p_1 = 1$. From Lemma 2.3, for $0 < \mu < |1/u|$,

$$\begin{aligned} z^* = (L_0^\mu)'(u) &= \frac{1}{2\mu}(\ln(1+u\mu) - \ln(1-u\mu)) \\ &= u \sum_{k=1}^{\infty} \frac{(u\mu)^{2k-2}}{2k-1} \\ &= u + \frac{1}{3}u^3\mu^2 + \left[\sum_{k=1}^{\infty} \frac{(u\mu)^{2k-2}}{2k+3} \right] u^5\mu^4. \end{aligned}$$

Since $|u\mu| < 1$, there exists $\alpha < 1$ such that $|u\mu| \leq \alpha < 1$. It can be easily checked that $\sum_{k=1}^{\infty} \frac{(u\mu)^{2k-2}}{2k+3}$ is bounded by some constant $M = M(n)$. Therefore, Lemma 2.4 holds for $n = 1$. It is now sufficient to prove this lemma for $n \geq 2$.

For convenience, we denote

$$c_{2n} := \frac{(2n+1)!! - \sum_{i=1}^n p_i i^{2n+2}}{(2n+1)!}; \qquad b_{2k} := \sum_{i=1}^n p_i i^{2k},\ k = 1, 2, ...$$

Step 1. Applying Taylor's formula, we have

$$e^{iz^*\mu} - e^{-iz^*\mu} = 2\sum_{k=0}^{\infty} \frac{(iz^*)^{2k+1}}{(2k+1)!}\mu^{2k+1}; \quad e^{iz^*\mu} + e^{-iz^*\mu} = 2\sum_{k=0}^{\infty} \frac{(iz^*)^{2k}}{(2k)!}\mu^{2k}.$$

Substituting the above equalities into equation (2.2) and taking into account that $\sum_{i=0}^{i=n} p_i = 1$ and $\sum_{i=0}^{i=n} i^2 p_i = 1$, equality (2.2) becomes

$$u + u\sum_{k=1}^{\infty} \left(\sum_{i=1}^n p_i i^{2k} \right) \frac{(z^*)^{2k}}{(2k)!}\mu^{2k} = z^* + \sum_{k=1}^{\infty} \left(\sum_{i=1}^n p_i i^{2k+2} \right) \frac{(z^*)^{2k+1}}{(2k+1)!}\mu^{2k}. \quad (2.3)$$

Step 2. Assume that $z^* = \sum_{k=0}^{\infty} a_k \mu^k$. We are to find all the coefficients a_k for $k = 0, 1, 2, ...$ by induction.

Let us substitute $z^* = a_0 + a_1\mu + a_2\mu^2 + ...$ into equation (2.3) to determine the coefficients of μ^0 and μ on both sides of the equation:

On the left side of equation (2.3), the coefficient of μ^0 is u while the coefficient of μ^1 is 0.

On the right side of the equation (2.3), the coefficient of μ^0 is $\sum_{i=1}^{n} p_i i^2 a_0$ while the coefficients of μ^1 is $\sum_{i=1}^{n} p_i i^2 a_1$.

Taking into account that $\sum_{i=1}^{n} p_i i^2 = 1$, we have $a_0 = u$ and $a_1 = 0$.

Assume that we already have $a_{2j-1} = 0$, $j = 1, ..., N$ where N is any positive integer. Let us show $a_{2N+1} = 0$. We substitute $z^* = u + \sum_{k=2N+1}^{\infty} a_{2k+1}\mu^{2k+1} + \sum_{k=1}^{\infty} a_{2k}\mu^{2k}$ into equation (2.3) to determine the coefficients of μ^{2N+1} on both sides of the equation:

On the left side of equation (2.3), the coefficient of μ^{2N+1} is 0.

On the right side of equation (2.3), the coefficient of μ^{2N+1} is $\sum_{i=1}^{n} p_i i^2 a_{2N+1}$.

This implies $a_{2N+1} = 0$.

In conclusion, we can have $a_{2N+1} = 0, N = 1, 2, \ldots$.

Substituting $z^* = u + \sum_{k=1}^{\infty} a_{2k}\mu^{2k}$ into equation (2.3) to determine the coefficients of μ^{2N} on both sides of the equation, we find the following:

The coefficient of μ^{2N} on the left side of equation (2.3) is

$$\frac{u}{(2N)!}\left[\sum_{k=1}^{N} b_{2k}\left(\sum_{\substack{\alpha_1,\ldots,\alpha_{2k}\ is\ integer \\ 0\le\alpha_1,\ldots,\alpha_{2k}\le N-k}}^{\alpha_1+\alpha_2+\ldots\alpha_{2k}=N-k}(a_{2\alpha_1}a_{2\alpha_2}\cdots a_{2\alpha_{2k}})\right)\right];$$

The coefficient of μ^{2N} on the right side of equation (2.3) is

$$a_{2N}\mu^{2N} + \frac{1}{(2N+1)!}\left[\sum_{k=1}^{N} b_{2k+2}\left(\sum_{\substack{\alpha_1+\alpha_2+\ldots\alpha_{2k+1}=N-k \\ \alpha_1,\ldots,\alpha_{2k+1} \text{ is integer} \\ 0\le\alpha_1,\ldots\alpha_{2k+1}\le N-k}} (a_{2\alpha_1}\cdots a_{2\alpha_{2k}}a_{2\alpha_{2k+1}})\right)\right].$$

Therefore

$$\begin{aligned} a_{2N} &= \frac{u}{(2N)!}\left[\sum_{k=1}^{N} b_{2k}\left(\sum_{\substack{\alpha_1+\alpha_2+\ldots\alpha_{2k}=N-k \\ \alpha_1,\ldots,\alpha_{2k} \text{ is integer} \\ 0\le\alpha_1,\ldots,\alpha_{2k}\le N-k}} (a_{2\alpha_1}a_{2\alpha_2}\cdots a_{2\alpha_{2k}})\right)\right] \\ &- \frac{1}{(2N+1)!}\left[\sum_{k=1}^{N} b_{2k+2}\left(\sum_{\substack{\alpha_1+\alpha_2+\ldots\alpha_{2k+1}=N-k \\ \alpha_1,\ldots,\alpha_{2k+1} \text{ is integer} \\ 0\le\alpha_1,\ldots\alpha_{2k+1}\le N-k}} (a_{2\alpha_1}\cdots a_{2\alpha_{2k}}a_{2\alpha_{2k+1}})\right)\right]. \end{aligned}$$

Step 3. Let $p_0,\ldots,p_n$ satisfy $\sum_{i=1}^{n} p_i i^{2k} = (2k-1)!!$ for $k=1,\ldots,n$. The coefficients $a_2,\ldots,2_{2n}$ will then have an explicit expression. Let us find a_{2k} for $k=1,2,\ldots n$ by induction.

From the expression for a_{2N} in step 2, we have $a_2 = \frac{b_2}{2!}u^3 - \frac{b_4}{3!}u^3$.

Let n be any integer such that $n \ge 2$. We have $b_2 = 1$ and $b_4 = \sum_{i=1}^{n} p_i i^4 = 3!!$. Hence, $a_2 = \frac{b_2}{2!}u^3 - \frac{b_4}{3!}u^3 = 0$.

Assume that $a_{2k} = 0$ for $k = 1,2,\ldots,K$ where $K < n$. Then

$$\begin{aligned} a_{2(K+1)} &= \left(\frac{b_{2(K+1)}}{2(K+1)!} - \frac{b_{2(K+1)+2}}{(2(K+1)+1)!}\right)u^{2(K+1)+1} \\ &= \left(\frac{(2(K+1)+1)\sum_{i=1}^{n} p_i i^{2(K+1)} - \sum_{i=1}^{n} p_i i^{2(K+1)+2}}{(2(K+1)+1)!}\right)u^{2(K+1)+1} \\ &= \begin{cases} 0, & \text{if } K+1 < n, \\ \left(\frac{(2n+1)!!-\sum_{i=1}^{n} p_i i^{2n+2}}{(2n+1)!}\right)u^{2n+1}, & \text{if } K+1 = n. \end{cases} \end{aligned}$$

By mathematical induction, we conclude that

$$a_{2k} = 0,\ k = 1,\ldots n-1, \text{ and } a_{2n} = \left(\frac{(2n+1)!! - \sum_{i=1}^{n} p_i i^{2n+2}}{(2n+1)!}\right)u^{2n+1}.$$

Step 4. Let $p_0, \ldots, p_n$ satisfy $\sum_{i=1}^{n} p_i i^{2k} = (2k-1)!!$ for $k = 1, \ldots, n$. Assume $z^* = u + c_{2n}u^{2n+1}\mu^{2n} + C(u,\mu)\mu^{2n+2}$. Let us find an expression for $C = C(u,\mu)$.

Let us substitute $z^* = u + c_{2n}u^{2n+1}\mu^{2n} + C\mu^{2n+2}$ into the both sides of equation (2.3). Taking into account that $\sum_{i=1}^{n} p_i i^{2k} = (2k-1)!!$, $k = 1, ..., n$, we find the following: The left side of equation (2.3) is equal to

$$\begin{aligned}
& u + u\sum_{k=1}^{\infty}\Big(\sum_{i=1}^{n} p_i i^{2k}\Big)\frac{(z^*)^{2k}}{(2k)!}\mu^{2k} \\
&= u + u\sum_{k=1}^{n}\frac{(2k-1)!!}{(2k)!}(z^*)^{2k}\mu^{2k} + u\sum_{k=n+1}^{\infty}\Big(\sum_{i=1}^{n} p_i i^{2k}\Big)\frac{(z^*)^{2k}}{(2k)!}\mu^{2k} \\
&= u + u\sum_{k=1}^{n-1}\frac{1}{(2k)!!}(z^*)^{2k}\mu^{2k} + u\frac{1}{(2n)!!}(z^*)^{2n}\mu^{2n} + u\sum_{k=n+1}^{\infty}\frac{b_{2k}}{(2k)!}(z^*)^{2k}\mu^{2k}.
\end{aligned}$$

The right side of equation (2.3) is equal to

$$\begin{aligned}
& z + \sum_{k=1}^{\infty}\Big(\sum_{i=1}^{n} p_i i^{2k+2}\Big)\frac{(z^*)^{2k+1}}{(2k+1)!}\mu^{2k} \\
&= u + c_{2n}u^{2n+1}\mu^{2n} + C\mu^{2n+2} + \sum_{k=1}^{\infty}\frac{b_{2k+2}}{(2k+1)!}(z^*)^{2k+1}\mu^{2k} \\
&= u + c_{2n}u^{2n+1}\mu^{2n} + C\mu^{2n+2} \\
&\quad + \sum_{k=1}^{\infty}\frac{b_{2k+2}}{(2k+1)!}(z^*)^{2k}(u + c_{2n}u^{2n+1}\mu^{2n} + C\mu^{2n+2})\mu^{2k} \\
&= u + c_{2n}u^{2n+1}\mu^{2n} + C\mu^{2n+2} + u\sum_{k=1}^{\infty}\frac{b_{2k+2}}{(2k+1)!}(z^*)^{2k}\mu^{2k} \\
&\quad + \sum_{k=1}^{\infty}\frac{b_{2k+2}}{(2k+1)!}(z^*)^{2k}c_{2n}u^{2n+1}\mu^{2n+2k} + C\sum_{k=1}^{\infty}\frac{b_{2k+2}}{(2k+1)!}(z^*)^{2k}\mu^{2n+2k+2} \\
&= u + c_{2n}u^{2n+1}\mu^{2n} + C\mu^{2n+2} \\
&\quad + u\sum_{k=1}^{n-1}\frac{(2k+1)!!}{(2k+1)!}(z^*)^{2k}\mu^{2k} + u\frac{b_{2n+2}}{(2n+1)!}(z^*)^{2n}\mu^{2n} + u\sum_{k=n+1}^{\infty}\frac{b_{2k+2}}{(2k+1)!}(z^*)^{2k}\mu^{2k} \\
&\quad + c_{2n}\sum_{k=1}^{\infty}\frac{b_{2k+2}}{(2k+1)!}(z^*)^{2k}u^{2n+1}\mu^{2n+2k} + C\sum_{k=1}^{\infty}\frac{b_{2k+2}}{(2k+1)!}(z^*)^{2k}\mu^{2n+2k+2}.
\end{aligned}$$

Now, let us equate the left and right sides. The terms u and $u\sum_{k=1}^{n-1}\frac{(2k+1)!!}{(2k+1)!}(z^*)^{2k}\mu^{2k}$ can be canceled, therefore, the left side of equation (2.3) is equal to

$$\begin{aligned}&u\frac{1}{(2n)!!}(z^*)^{2n}\mu^{2n}+u\sum_{k=n+1}^{\infty}\frac{b_{2k}}{(2k)!}(z^*)^{2k}\mu^{2k}\\&=\frac{1}{(2n)!}(z^*)^{2n}u\mu^{2n}+u\sum_{k=0}^{\infty}\frac{b_{2n+2k+2}}{(2n+2k+2)!}(z^*)^{2n+2k+2}\mu^{2n+2k+2}.\end{aligned}$$

The right side of equation (2.3) is equal to

$$\begin{aligned}&c_{2n}u^{2n+1}\mu^{2n}+C\mu^{2n+2}+u\frac{b_{2n+2}}{(2n+1)!}(z^*)^{2n}\mu^{2n}+u\sum_{k=n+1}^{\infty}\frac{b_{2k+2}}{(2k+1)!}(z^*)^{2k}\mu^{2k}\\&+c_{2n}\sum_{k=1}^{\infty}\frac{b_{2k+2}}{(2k+1)!}(z^*)^{2k}u^{2n+1}\mu^{2n+2k}+C\sum_{k=1}^{\infty}\frac{b_{2k+2}}{(2k+1)!}(z^*)^{2k}\mu^{2n+2k+2}\\&=c_{2n}u^{2n+1}\mu^{2n}+C\mu^{2n+2}+u\frac{b_{2n+2}}{(2n+1)!}(z^*)^{2n}\mu^{2n}\\&\quad+u\sum_{k=0}^{\infty}\frac{b_{2n+2k+4}}{(2n+2k+3)!}(z^*)^{2n+2k+2}\mu^{2n+2k+2}\\&\quad+c_{2n}u^{2n+1}\sum_{k=0}^{\infty}\frac{b_{2k+4}}{(2k+3)!}(z^*)^{2k+2}\mu^{2n+2k+2}+C\sum_{k=1}^{\infty}\frac{b_{2k+2}}{(2k+1)!}(z^*)^{2k}\mu^{2n+2k+2}.\end{aligned}$$

Let us add $(-c_{2n}u^{2n+1}\mu^{2n}-u\frac{b_{2n+2}}{(2n+1)!}(z^*)^{2n}\mu^{2n})$ to both sides. Then the left side is equal to

$$\begin{aligned}&c_{2n}((z^*)^{2n}-u^{2n})u\mu^{2n}+u\sum_{k=0}^{\infty}\frac{b_{2n+2k+2}}{(2n+2k+2)!}(z^*)^{2n+2k+2}\mu^{2n+2k+2}\\&=c_{2n}\frac{(z^*)^{2n}-u^{2n}}{z^*-u}(c_{2n}u^{2n+1}\mu^{2n}+C\mu^{2n+2})u\mu^{2n}\\&\quad+u\sum_{k=0}^{\infty}\frac{b_{2n+2k+2}}{(2n+2k+2)!}(z^*)^{2n+2k+2}\mu^{2n+2k+2}\\&=(c_{2n})^2\frac{(z^*)^{2n}-u^{2n}}{z^*-u}u^{2n+2}\mu^{4n}+u\sum_{k=0}^{\infty}\frac{b_{2n+2k+2}}{(2n+2k+2)!}(z^*)^{2n+2k+2}\mu^{2n+2k+2}\\&\quad+C(c_{2n})\frac{(z^*)^{2n}-u^{2n}}{z^*-u}u\mu^{4n+2}.\end{aligned}$$

The right side is equal to

$$C(\mu^{2n+2}+\sum_{k=1}^{\infty}\frac{b_{2k+2}}{(2k+1)!}(z^*)^{2k}\mu^{2n+2k+2})$$
$$+u\sum_{k=0}^{\infty}\frac{b_{2n+2k+4}}{(2n+2k+3)!}(z^*)^{2n+2k+2}\mu^{2n+2k+2}+c_{2n}u^{2n+1}\sum_{k=0}^{\infty}\frac{b_{2k+4}}{(2k+3)!}(z^*)^{2k+2}\mu^{2n+2k+3}.$$

Now, let us equate the left and right sides. It can be checked that $C = C(u,\mu) = A/B$ where $A = A1 + A2 + A3 + A4$ and

$$\begin{aligned}
A1 &= c_{2n}^2\left(\frac{(z^*)^{2n}-u^{2n}}{z^*-u}\right)u^{2n+2}\mu^{2n-2},\ \ A2 = u\sum_{k=0}^{\infty}\left(\frac{b_{2n+2k+2}}{(2n+2k+2)!}\right)(z^*)^{2n+2k+2}\mu^{2k}\\
A3 &= -u\sum_{k=0}^{\infty}(\frac{b_{2n+2k+4}}{(2n+2k+3)!})(z^*)^{2n+2k+2}\mu^{2k},\ \ A4 = -c_{2n}u^{2n+1}\sum_{k=0}^{\infty}\frac{b_{2k+4}}{(2k+3)!}(z^*)^{2k+2}\mu^{2k},\\
B &= 1+\sum_{k=1}^{\infty}\frac{b_{2k+2}}{(2k+1)!}(z^*)^{2k}\mu^{2k}-c_{2n}\left(\frac{(z^*)^{2n}-u^{2n}}{z^*-u}\right)u\mu^{2n}.
\end{aligned}$$

Step 5. Assume $0<\mu\le|\frac{np_n}{2u}|$. Let us show that $|C|$ is bounded for $0<\mu\le|\frac{np_n}{2u}|$ and $n\ge 2$. Before we start to estimate C, we need to verify some inequalities.

I1. Applying Lemma 2.3 and taking into account that $n^2p_n\le 1$, it can be checked that for $0<\mu\le|\frac{np_n}{2u}|$

$$\begin{aligned}
\left|\frac{(z^*)^{2n}-u^{2n}}{z^*-u}\right| &= \left|(z^*+u)((z^*)^{2n-2}+(z^*)^{2n-4}u^2+...+u^{2n-2})\right|\\
&\le (|z^*|+|u|)(|(z^*)|^{2n-2}+|(z^*)|^{2n-4}u^2+...+u^{2n-2})\\
&\le \left(\frac{2}{n^2p_n}+1\right)|u|\left[\left(\frac{2}{n^2p_n}\right)^{2n-2}+\left(\frac{2}{n^2p_n}\right)^{2n-4}+...+1\right]|u|^{2n-2}\\
&\le \frac{3}{n^2p_n}\left(\frac{(\frac{2}{n^2p_n})^{2n}-1}{\frac{2}{n^2p_n}-1}\right)|u|^{2n-1}\\
&= \frac{3(\frac{2}{n^2p_n})^{2n}}{2-n^2p_n}|u|^{2n-1}\le 3\left(\frac{2}{n^2p_n}\right)^{2n}|u|^{2n-1}.
\end{aligned}$$

I2. Notice that $b_{2n+2} = \sum_{i=1}^{n} i^{2n+2} p_i \leq n^2 \sum_{i=1}^{n} i^{2n} p_i = n^2 b_{2n}$. Then

$$\begin{aligned} |c_{2n}| = |\frac{(2n+1)b_{2n} - b_{2n+2}}{(2n+1)!}| &\leq \frac{((2n+1)+n^2)b_{2n}}{(2n+1)!} \\ &= \frac{1}{(2n)!!}\left(1 + \frac{n^2}{2n+1}\right) \\ &= \frac{1}{(2n)!!}\frac{(n+1)^2}{2n+1} \leq \frac{1}{(2n)!!}2n \leq 1. \end{aligned}$$

I3. Taking into account inequality I1 and I2, it can be checked that for $0 < \mu \leq |np_n/(2u)|$ and $n \geq 2$,

$$\begin{aligned} \left| c_{2n} \frac{(z^*)^{2n} - u^{2n}}{z^* - u} u\mu^{2n} \right| &\leq 3\left(\frac{2}{n^2 p_n}\right)^{2n} |u|^{2n}\mu^{2n} \\ &\leq \left(\frac{2}{n^2 p_n}\right)^{2n} \left(\frac{np_n}{2}\right)^{2n} \\ &\leq 3\frac{1}{n^{2n}} \leq \frac{1}{2}. \end{aligned}$$

I4. Applying Lemma 2.3, it can be checked that for any integer $N \geq 0$ and $0 < \mu \leq |\frac{np_n}{2u}|$,

$$\begin{aligned} \sum_{k=0}^{\infty} \left[\frac{b_{2N+2k}}{(2N+2k)!}\right] (z^*\mu)^{2k} &= \frac{1}{2}\sum_{i=1}^{n} p_i i^{2N} (e^{iz^*\mu} + e^{-iz^*\mu}) \\ &\leq b_{2N}(e^{nz^*\mu} + e^{-nz^*\mu}) \\ &\leq b_{2N}\left[e^{n\frac{2}{n^2 p_n}|u\mu|} + e^{-n\frac{2}{n^2 p_n}|u\mu|}\right] \\ &\leq b_{2N}\left[e^{n\frac{2}{n^2 p_n}|\frac{np_n}{2}|} + e^{-n\frac{2}{n^2 p_n}|\frac{np_n}{2}|}\right] \\ &= b_{2N}(e + e^{-1}). \end{aligned}$$

Now, let us estimate $A = A1 + A2 + A3 + A4$ and B for $0 < \mu \leq |np_n/(2u)|$

and $n \geq 2$. Applying inequalities **I1-I4**, we have

$$\begin{aligned}
|A1| &= \left| c_{2n}^2 \frac{(z^*)^{2n} - u^{2n}}{z^* - u} u^{2n+2} \mu^{2n-2} \right| \\
&\leq 3 \left(\frac{2}{n^2 p_n} \right)^{2n} |u|^{2n-1} u^{2n+2} \mu^{2n-2} \\
&= 3 \left(\frac{2}{n^2 p_n} \right)^{2n} (u\mu)^{2n-2} |u|^{2n+3} \\
&\leq 3 \left(\frac{2}{n^2 p_n} \right)^{2n} \left(\frac{n p_n}{2} \right)^{2n-2} |u|^{2n+3} = \frac{12}{n^{2n+2} p_n^2} |u|^{2n+3}.
\end{aligned}$$

$$\begin{aligned}
|A2| &= \left| u \sum_{k=0}^{\infty} \left(\frac{b_{2n+2k+2}}{(2n+2k+2)!} \right) (z^*)^{2n+2k+2} \mu^{2k} \right| \\
&= |u|(z^*)^{2n+2} \sum_{k=0}^{\infty} \left(\frac{b_{2n+2k+2}}{(2n+2k+2)!} \right) (z^* \mu)^{2k} \\
&\leq |u|(z^*)^{2n+2} \sum_{k=0}^{\infty} \left(\frac{b_{2n+2k+2}}{(2k+2)!} \right) (z^* \mu)^{2k} \\
&\leq |u|(z^*)^{2n+2} (e + e^{-1}) b_{2n+2} \\
&\leq \left[\left(\frac{2}{n^2 p_n} \right)^{2n-2} (e + e^{-1}) b_{2n+2} \right] |u|^{2n+3}.
\end{aligned}$$

$$\begin{aligned}
|A3| &= \left| u \sum_{k=0}^{\infty} \left(\frac{b_{2n+2k+4}}{(2n+2k+3)!} \right) (z^*)^{2n+2k+2} \mu^{2k} \right| \\
&\leq \left[\left(\frac{2}{n^2 p_n} \right)^{2n-2} (e + e^{-1}) b_{2n+4} \right] |u|^{2n+3}.
\end{aligned}$$

$$\begin{aligned}
|A4| &= \left| c_{2n} u^{2n+1} \sum_{k=0}^{\infty} \frac{b_{2k+4}}{(2k+3)!} (z^*)^{2k+2} \mu^{2k} \right| \\
&\leq (z^*)^2 u^{2n+1} \sum_{k=0}^{\infty} \frac{b_{2k+4}}{(2k+3)!} (z^* \mu)^{2k} \\
&\leq \left[\left(\frac{2}{n^2 p_n} \right)^2 (e + e^{-1}) b_4 \right] |u|^{2n+3}.
\end{aligned}$$

$$
\begin{aligned}
|B| &= \left|1+\sum_{k=1}^{\infty}\frac{b_{2k+2}}{(2k+1)!}(z^*)^{2k}\mu^{2k} - c_{2n}\frac{(z^*)^{2n}-u^{2n}}{z^*-u}u\mu^{2n}\right| \\
&\geq \left|1+\sum_{k=1}^{\infty}\frac{b_{2k+2}}{(2k+1)!}(z^*)^{2k}\mu^{2k}\right| - \left|c_{2n}\frac{(z^*)^{2n}-u^{2n}}{z^*-u}u\mu^{2n}\right| \\
&\geq 1-\left|c_{2n}\frac{(z^*)^{2n}-u^{2n}}{z^*-u}u\mu^{2n}\right| \geq \frac{1}{2}.
\end{aligned}
$$

From above inequalities, we conclude that for $\mu \leq \frac{np_n}{2|u|}$ and $n \geq 2$, there exists a function, say $M(n)$, depending on n such that

$$
|C| = \frac{|A|}{|B|} \leq \frac{|A1|+|A2|+|A3|+|A4|}{|B|} < M(n)|u|^{2n+3}.
$$

Therefore, it can be concluded that

$$
z^* = u + \left(\frac{(2n+1)!! - \sum_{i=1}^{n} p_i i^{2n+2}}{(2n+1)!}\right) u^{2n+1}\mu^{2n} + O(\mu^{2n+2}), \ \ 0<\mu<<1.
$$

This completes the proof.

■

Remark: From Lemma 2.4, it can be verified that

$$
(G_0^{\mu})'(z) = z - \left(\frac{(2n+1)!! - \sum_{i=1}^{n} p_i i^{2n+2}}{(2n+1)!}\right) z^{2n+1}\mu^{2n} + O(\mu^{2n+2}), \ \ 0<\mu<<1.
$$

We sketch the proof here. Step 1: For any $z \in R$, there exists a unique $u \in (-n/\mu, n/\mu)$ such that $z = (L_0^{\mu})'(u)$ (see Property L4 and G2). Furthermore, by equation (8), we have the two inequalities $|u\mu| < n(e^{n|z\mu|} - e^{-n|z\mu|})$ and $|u| < 2n^2 z e^{n|z\mu|}$. Step 2: From Lemma 2.4, it is known that

$$
(L_0^{\mu}(u))' = u + \left(\frac{(2n+1)!! - \sum_{i=1}^{n} p_i i^{2n+2}}{(2n+1)!}\right) u^{2n+1}\mu^{2n} + C(u,\mu)\mu^{2n+2},
$$

where $|C(u,\mu)| < M(n)|u|^{2n+3}$ provided $\mu \leq \frac{np_n}{2|u|}$. Here $M(n)$ is some function depending on n. Taking into account the two inequalities in step 1 and

$(G_0^\mu)'(z)|_{z=(L_0^\mu)'(u)} = u$, it can be checked that

$$(G_0^\mu)'(z) = z - \left(\frac{(2n+1)!! - \sum_{i=1}^n p_i i^{2n+2}}{(2n+1)!} \right) z^{2n+1}\mu^{2n} + \tilde{C}(z,\mu)\mu^{2n+2}$$

where $|\tilde{C}(z,\mu)| < \tilde{M}(n)|z|^{2n+3}$ provided $|\mu|$ is small enough. Here $\tilde{M}(n)$ is some function depending on n. This completes the proof.

Lemma 2.5 : Let $n \geq 1$ be an integer. Let $p_0, \ldots, p_n$ satisfy $\sum_{i=1}^n p_i i^{2k} = (2k-1)!!$ for $k = 1, \ldots, n$. Then

$$L_0^\mu(u) = \frac{u^2}{2} + \left(\frac{(2n+1)!! - \sum_{i=1}^n p_i i^{2n+2}}{(2n+2)!} \right) u^{2n+2}\mu^{2n} + O(\mu^{2n+2}), \;\; 0 < \mu << 1. \tag{2.4}$$

Proof: Recall that $z^* = (L_0^\mu)'(u)$. Recall from property L1 that $L_0^\mu(0) = 0$ and $L_0^\mu(u)$ is an even function.

Recall from the proof of Lemma 2.4 that

$$z^* = (L_0^\mu)'(u) = u + \left(\frac{(2n+1)!! - \sum_{i=1}^n p_i i^{2n+2}}{(2n+1)!} \right) u^{2n+1}\mu^{2n} + C(u,\mu)\mu^{2n+2}$$

where $|C(u,\mu)| < M(n)|u|^{2n+3}$ for $\mu \leq np_n/(2|u|)$.

From all the above arguments, this lemma can be easily derived.

■

So far, we have considered the case $\delta, \varepsilon \downarrow 0, \delta\varepsilon^{-1} = \mu^2 = constant > 0$. Consider now the case $\varepsilon, \delta\varepsilon^{-1} \downarrow 0$. Notice that

$$\begin{aligned} G_0^\mu(u) &:= \frac{1}{\mu^2} \ln \left\{ p_0 + \sum_{i=1}^n \frac{p_i}{2}(e^{iu\mu} + e^{-iu\mu}) \right\} \\ &= \frac{u^2}{2} + O(\mu^2), \quad \mu \to 0. \end{aligned}$$

Therefore, when $\mu^2 = \delta/\varepsilon \downarrow 0$, $G_0(u) = u^2/2$. Using Lemma 1.3, we can derive the following result:

Theorem 2.2 : The action functional for the family $X_t^{\delta,\varepsilon}$ as $\delta, \varepsilon \downarrow 0, \delta/\varepsilon \downarrow 0$ in the space $C_{[0,T]}(R^d)$ is equal to $\varepsilon^{-1}S_{[0,T]}(\varphi)$, where

$$S_{[0,T]}(\varphi) = \begin{cases} \int_0^T |\dot{\varphi}_s - b(\varphi_s)|^2 ds, & \text{if } \varphi \text{ is absolutely continuous and } \varphi_0 = x, \\ +\infty, & \text{otherwise.} \end{cases}$$

Notice that $\varepsilon^{-1}S_{[0,T]}(\varphi)$ is the action functional for the family X_t^ε in $C_{[0,T]}(R^d)$ as $\varepsilon \downarrow 0$.

2.2 Exit Problem for $X_t^{\delta,\varepsilon}$

Consider the system:

$$\dot{X}_t = b(X_t), \;\; X_0 = x \in R^d.$$

where the vector field $b(x)$ is Lipschitz continuous.

Assumption 1: The vector field $b(x)$, $x \in R^d$ have an asymptotically stable equilibrium at a point $O \in R^d$.

Let $G \subset R^d$ be a bounded domain with boundary ∂G.

Assumption 2: The domain G is attracted to $O \in G$: $\lim_{t\uparrow\infty} X_t = O$ for each trajectory of $\dot{X}_t = b(X_t)$, $X_0 = x \in G$.

Assumption 3: The domain G has a smooth boundary ∂G and $(b(x) \cdot n(x)) < 0, x \in \partial G$ where $n(x)$ is the exterior normal of the boundary of G.

Now, consider the continuous process $X_t^{\delta,\varepsilon}$ defined in the equation (1.5) and the process X_t^ε in the equation (1.3) with σ as a unit matrix. Let $X_0^\varepsilon = X_0^{\delta,\varepsilon} = x \in G$. Denote by $\tau = \tau^\varepsilon(\tau^\delta = \tau^{\delta,\varepsilon})$ the first exit time from G for the process

$X_t^\varepsilon(X_t^{\delta,\varepsilon})$: $\tau^\varepsilon = \min\{t : X_t^\varepsilon \in \partial G\}$, $\tau^{\delta,\varepsilon} = \min\{t : X_t^{\delta,\varepsilon} \in \partial G\}$. To describe the asymptotic behavior of τ^ε and $\tau^{\delta,\varepsilon}$, let us introduce the functions $V(x,y)$, $V^\mu(x,y)$:

$$V(x,y) = \inf_{\varphi \in C_{0T}(R^d)} \{S_{[0,T]}(\varphi) : \varphi_0 = x, \varphi_T = y, T > 0\},$$

$$V^\mu(x,y) = \inf_{\varphi \in C_{0T}(R^d)} \{S^\mu_{[0,T]}(\varphi) : \varphi_0 = x, \varphi_T = y, T > 0\}.$$

Here $S_{[0,T]}(\varphi)$ is the action functional for X_t^ε in the space $C_{[0,T]}(R^d)$ as $\varepsilon \downarrow 0$ and $S^\mu_{[0,T]}(\varphi)$ is the action functional for $X_t^{\delta,\varepsilon}$ in the space $C_{[0,T]}(R^d)$ as $\delta, \varepsilon \downarrow 0$, $\frac{\delta}{/}\varepsilon = \mu^2$. Define $V(x) := V(O,x)$ and $V^\mu(x) := V^\mu(O,x)$. The functions $V(x)$ and $V^\mu(x)$ are non-negative and equal to zero only at $x = 0$. Furthermore, if the functions $V(x)$ and $V^\mu(x)$ are continuously differentiable, they can be found as the solutions of corresponding Hamilton-Jacobi equations:

$$\frac{1}{2}|\nabla V(x)|^2 + (b(x) \cdot \nabla V(x)) = 0,\ V(O) = 0, V(x) > 0 \ \ x \neq O;$$

$$\frac{1}{\mu^2}\sum_{k=1}^{d} \ln\left[p_0 + \sum_{i=1}^{n} \frac{p_i}{2}\left(e^{i\mu\frac{\partial V^\mu(x)}{\partial x^k}} + e^{-i\mu\frac{\partial V^\mu(x)}{\partial x^k}}\right)\right] + (b(x) \cdot \nabla V^\mu(x)) = 0, \quad (2.5)$$
$$V^\mu(O) = 0,\ V^\mu(x) > 0 \ if \ \ x \neq O.$$

The Hamilton-Jacobi equation for $V(x)$ can be found in §4.2 of [5]. Equation (2.5) for $V^\mu(x)$ can be derived following the same idea as in [4] (see also §4.2 of [5] and [6]). We briefly discuss the one dimensional case here. Define

$$\tilde{V}(t_1, t_2, O, x) = \inf_{\varphi \in C_{[t_1,t_2]}(R^1)} \{S^\mu_{t_1 t_2}(\varphi) : \varphi_{t_1} = O, \varphi_{t_2} = x\}.$$

From §4.23 of [13], we can have the Hamilton-Jacobi equation for $\tilde{V}(t_1, t_2, O, x)$:

$$\frac{\partial \tilde{V}(t_1,t_2,O,x)}{\partial t_2} + G_0^\mu\left(\frac{\partial \tilde{V}(t_1,t_2,O,x)}{\partial x}\right) + (b(x) \cdot \frac{\partial \tilde{V}(t_1,t_2,O,x)}{\partial x}) = 0. \quad (2.6)$$

We have to add the condition $\tilde{V}(t_1,t_2,O,O) = 0$, $\tilde{V}(t_1,t_2,O,x) > 0$ if $x \neq O$. Notice that the infimum for $V^\mu(x)$ is only attained for functions defined on a semiaxis infinite from the left: there exists a function $\varphi(t)$, $-\infty \leq t \leq t_2$ such that $\varphi(-\infty) = O, \varphi(t_2) = x$ and $S^{\mu}_{-\infty,t_2}(\varphi) = V^{\mu}(x)$ (see §4.2 of [5]). Hence, we have $V^{\mu}(x) = \lim_{t_1 \downarrow -\infty} \tilde{V}(t_1, t_2, O, x)$. It can be checked that (see §4.2 of [5] and [4], [6]) if $V^{\mu}(x)$ is smooth enough, then

$$G_0^{\mu}(\frac{dV^{\mu}(x)}{dx}) + (b(x) \cdot \frac{dV^{\mu}(x)}{dx}) = 0.$$

In general, the equation (2.5) can have just generalized solutions. Since $L_0^{\mu}(z) = \infty$ if $||z|| = \max_{1 \leq k \leq d} |z_k| > n/u$, the function $V^{\mu}(x)$ can be equal to ∞. For example, if the point O is separated from ∂G by a smooth surface Γ with exterior normal $\nu(x), x \in \Gamma$, and $b(x) \cdot \nu(x) < -1/\mu$, then ${V_0}^{\mu} = +\infty$ and $X_t^{\delta,\varepsilon}$ never leaves the domain G starting inside the region bounded by Γ if ε is small enough and $\delta\varepsilon^{-1} = \mu^2$.

Define $V_o = \min_{x \in \partial G} V(x)$ and $V_o^{\mu} = \min_{x \in \partial G} V^{\mu}(x)$.

Theorem 2.3: Let $p_0, \ldots, p_n$ satisfy $\sum_{i=1}^{n} p_i i^{2k} = (2k-1)!!,\ k = 1, \ldots, n$. Let Assumptions 1-3 be satisfied. Then for any initial point $x \in G$ and $h > 0$

$$\lim_{\varepsilon,\delta \downarrow 0; \delta\varepsilon^{-1} = \mu^2} \varepsilon \ln E_x \tau^{\delta,\varepsilon} = V_o^{\mu},$$

$$\lim_{\varepsilon,\delta\varepsilon^{-1} \downarrow 0} \varepsilon \ln E_x \tau^{\delta,\varepsilon} = \lim_{\varepsilon \downarrow 0} \varepsilon \ln E_x \tau^{\varepsilon} = V_o,$$

$$\lim_{\varepsilon,\delta \downarrow 0; \delta\varepsilon^{-1} = \mu^2} P_x(e^{\frac{V_o^{\mu} - h}{\varepsilon}} < \tau^{\delta,\varepsilon} < e^{\frac{V_o^{\mu} + h}{\varepsilon}}) = 1,$$

$$\lim_{\varepsilon,\delta\varepsilon^{-1} \downarrow 0} P_x(e^{\frac{V_o - h}{\varepsilon}} < \tau^{\delta,\varepsilon} < e^{\frac{V_o + h}{\varepsilon}}) = \lim_{\varepsilon \downarrow 0} P_x(e^{\frac{V_o - h}{\varepsilon}} < \tau^{\varepsilon} < e^{\frac{V_o + h}{\varepsilon}}) = 1,$$

If $\min_{x \in \partial G} V^{\mu}(x)$ ($\min_{x \in \partial G} V(x)$) is achieved just at one point $x_*^{\mu} \in \partial G$ ($x_* \in \partial G$), then

$$\lim_{\varepsilon,\delta \downarrow 0; \delta\varepsilon^{-1} = \mu^2} P_x(|X^{\delta,\varepsilon}_{\tau^{\delta,\varepsilon}} - x_*^{\mu}| > h) = 0$$

$$\lim_{\varepsilon,\delta\varepsilon^{-1} \downarrow 0} P_x(|X^{\delta,\varepsilon}_{\tau^{\delta,\varepsilon}} - x_*| > h) = \lim_{\varepsilon \downarrow 0} P_x(|X^{\varepsilon}_{\tau^{\varepsilon}} - x_*| > h) = 0.$$

Proof: Taking into account Theorems 2.1 and 2.2, the proof of this theorem is a modification of the proof of theorems 4.4.1 and 4.4.2 from [5].

■

Assumption 4: Assume $b(x) = -\nabla U(x) + l(x)$, where the function $U(x)$ and the vector field $l(x)$ are smooth enough, $\nabla U(x) \cdot l(x) = 0$ for $x \in G$, $U(O) = 0, U(x) > 0$ and $\nabla U(x) \neq 0$ for $x \neq O$.

1. Let Assumptions 1-4 be satisfied. Then $V(x) = 2U(x)$ (see Theorem 4.3.1 of [5]).

2. We look for the solution of $V^\mu(x)$ under the condition $V^\mu(O) = 0$ in the form: $V^\mu(x) = V(x) + \mu^2 V_1(x) + \ldots, 0 < \mu << 1$.

Let $p_0, \ldots, p_n$ satisfy $\sum_{i=1}^n p_i i^{2k} = (2k-1)!!,\ k = 1, \ldots, n$. Let Assumptions 1-4 be satisfied. Then we have

$$V^\mu(x) = 2U(x) + V_n(x)\mu^{2n} + O(\mu^{2n+2}),\ 0 < \mu << 1,$$

where $V_n(x)$ is given by the equation

$$V_n(x) = 2^{2n+2}\left[\frac{(2n+1)!! - \sum_{i=1}^n p_i i^{2n+2}}{(2n+2)!}\right]\int_{-\infty}^0 \sum_{k=1}^d \left[\frac{\partial U}{\partial x^k}(Z_t)\right]^{2n+2} dt. \qquad (2.7)$$

Here Z_t is the solution of the equation

$$\dot{Z}_t = \nabla U(Z_t) + l(Z_t),\ \ Z_0 = x \in G,\ \ t < 0.$$

Let us prove the above statement. For convenience, denote $B_n := \frac{(2n+1)!! - \sum_{i=1}^n p_i i^{2n+2}}{(2n+2)!}$.

Step 1: Recall from the remark of Lemma 2.4 that

$$(G_0^\mu)'(z) = z - (2n+2)B_n z^{2n+1}\mu^{2n} + O(\mu^{2n+2}), 0 < \mu << 1.$$

Hence

$$G_0^\mu(z) = \frac{1}{2}z^2 - B_n z^{2n+2}\mu^{2n} + O(\mu^{2n+2}), 0 < \mu << 1.$$

Applying this equality to equation (2.5), we have

$$(\nabla U(x) \cdot \nabla V^\mu(x)) - (l(x) \cdot \nabla V^\mu(x)) = \frac{1}{2}|\nabla V^\mu(x)|^2 - B_n |\nabla V^\mu(x)|^{2n+2}\mu^{2n} + O(\mu^{2n+2}), \tag{2.8}$$

$$V^\mu(O) = 0,\ V^\mu(x) > 0 \quad x \neq O,\ 0 < \mu << 1.$$

Step 2: We look for the solution of (2.8) under the condition $V^\mu(O) = 0$ in the form: $V^\mu(x) = V_0(x) + \mu^2 V_1(x) + \ldots$, $0 < \mu << 1$. Substituting this series into (2.8), we have

$$(-\nabla U(x) \cdot \nabla V_0(x)) + (l(x) \cdot \nabla V_0(x)) + \frac{1}{2}|\nabla V_0(x)|^2 = 0,\ \ V_0(O) = 0.$$

It can be checked that $V_0(x) = 2U(x)$. Substituting the series $V^\mu(x) = 2U(x) + V_1(x)\mu^2 + ...$ into (2.8), we find that the functions $V_k(x)$ for $1 \le k \le n$ satisfy the following equations:

$$((-\nabla U(x) + l(x)) \cdot \nabla V_k(x)) + \sum_{i+j=k, 0 \le i < j \le k} (\nabla V_i(x) \cdot \nabla V_j(x)) = 0,\ V_k(O) = 0,\ k < n,$$

and

$$((-\nabla U(x) + l(x)) \cdot \nabla V_n(x)) + \sum_{i+j=n, 0 \le i < j \le n} (\nabla V_i(x) \cdot \nabla V_j(x)) = 2^{2n+2} B_n |\nabla U(x)|^{2n+2}$$

$$V_n(O) = 0.$$

Let us solve for $V_1(x)$. When $n = 1$, the function $V_1(x)$ satisfies

$$((\nabla U(x) + l(x)) \cdot \nabla V_1(x)) = \frac{4}{3}|\nabla U(x)|^4,\ V_1(O) = 0. \tag{2.9}$$

Solving this first order partial differential equation can be reduced to solving ordinary differential equations (see [19]): Let Z_t be the solution of the equation

$$\dot{Z}_t = \nabla U(Z_t) + l(Z_t), \;\; Z_0 = x \in G, \;\; t < 0.$$

Notice that

$$\frac{dU(Z_t)}{dt} = \nabla U(Z_t)(\nabla U(Z_t) + l(Z_t)) = |\nabla U(Z_t)|^2 > 0 \;\; Z_0 = x \in G, \; t < 0.$$

Thus, we have $\lim_{t \downarrow -\infty} Z_t = O$. On the other hand, we notice that

$$\frac{dV_1(Z_t)}{dt} = \nabla V_1(Z_t)(\nabla U(Z_t) + l(Z_t)) = 0.$$

Therefore the solution of (2.9) can be written in the form

$$V_1(x) = \frac{4}{3}\int_{-\infty}^{0} |\nabla U(Z_t)|^4 dt.$$

When $n > 1$, the function $V_1(x)$ satisfies the equation

$$((\nabla U(x) + l(x)) \cdot \nabla V_1(x)) = 0, \; V_1(O) = 0.$$

By the same arguments above, we have $V_1(x) \equiv 0$.

Now, let us find the functions $V_k(x)$ for $1 < k \leq n$. Using mathematical induction, it turns out that:

$$((\nabla U(x) + l(x)) \cdot \nabla V_k(x)) = 0, \; V_k(O) = 0, \quad k < n;$$

$$((\nabla U(x) + l(x)) \cdot \nabla V_n(x)) = 2^{2n+2} b_n |\nabla U(x)|^{2n+2}, \; V_n(O) = 0.$$

Solving these two equations, we have $V_k(x) \equiv 0, \; k < n$ and

$$V_n(x) = 2^{2n+2} b_n \int_{-\infty}^{0} |\nabla U(Z_t)|^{2n+2} dt.$$

This completes the proof.

∎

2.3 Stochastic Resonance for $X_t^{\delta,\varepsilon}$

Consider the system:

$$\dot{X}_t = b(X_t), \;\; X_0 = x \in R^d,$$

where the vector field $b(x)$ is Lipschitz continuous.

Assumption 5: Assume that a finite number of asymptotically stable equilibrium points $K_1, \ldots, K_l \in R^d$ exist such that any trajectory of the system $\dot{X}_t = b(X_t)$, $X_0 = x$, except for the trajectories belonging to the separatrix surfaces, is attracted to one of the points K_i as $t \to \infty$.

Now, consider the continuous process $X_t^{\delta,\varepsilon}$ defined in equation (1.5) and the process X_t^{ε} in equation (1.3) with σ as a unit matrix.

Lemma 2.6: For any $h > 0$, we have

$$\lim_{\delta,\varepsilon \downarrow 0, \delta/\varepsilon = \mu^2} P(\max_{0 \le t \le T} |X_t^{\delta,\varepsilon} - X_t| > h) = 0.$$

The proof is a modification of the proof in §2.1 of [5]. We omit it.

Let $i(x)$ be the index of the point such that the trajectory of X_t starting at $x \in R^d$ is attracted to $K_{i(x)}$. Let $x \in R^d$ not belong to a separatrix. From Lemma 2.6, the trajectory of the process $X_t^{\delta,\varepsilon}$ starting from $X_0^{\delta,\varepsilon} = x$ will be attracted to the asymptotically stable point $K_{i(x)}$ as $\delta, \varepsilon \downarrow 0, \delta/\varepsilon = \mu^2$.

However, because of the small perturbations, the process $X_t^{\delta,\varepsilon}, X_0^{\delta,\varepsilon} = x$ will make transitions between the attractor points in large time intervals. To describe the sequence of transitions of $X_t^{\delta,\varepsilon}$ as $\delta, \varepsilon \downarrow 0, \delta/\varepsilon = \mu^2$, a hierarchy of cycles related to the processes $X_t^{\delta,\varepsilon}$ is introduced (see §6.6 of [5] and [4]).

Define $V_{ij} := \inf_{\varphi \in C_{0T}(R^d)}\{S^{\mu}_{[0,T]}(\varphi) : \varphi_0 = K_i, \varphi_T = K_j, T > 0\}$ where $\varepsilon^{-1}S^{\mu}_{[0,T]}(\varphi)$ is the action functional for the family $X_t^{\delta,\varepsilon}$ as $\delta, \varepsilon \downarrow 0, \delta/\varepsilon \downarrow 0$ in the space $C_{[0,T]}(R^d)$. From V_{ij}, a hierachy of cycles can be constructed: Cycles of rank 0 are states of $L = \{1, \ldots, l\}$ themselves. For each $i \in L$, define "the closest" $j = J(i) \in L$ such that $V_{ij} = \min_{k \in L \setminus \{i\}} V_{ik}$. Such a closest state is unique in a generic system. Starting from any $i \in L$, one can consider the sequence $i, J(i), J^2(i), \ldots, J^n(i), \ldots$ where $J^{n+1}(i) = J(J^n(i))$. Since L is finite, the sequence, starting from some $m \in L$, is periodic: $i, J(i), \ldots, J^{n-1}(i), J^n(i) \to J^{n+1}(i) \to \ldots, J^m(i) = J^n(i)$. This sequence $i, J(i), \ldots, J^{n-1}(i), J^n(i) \to J^{n+1}(i) \to \ldots, J^m(i) = J^n(i)$ is called the cycle of rank 1 (1-cycle) generated by the state $i \in L$. Denote by $D^{(1)}$ the set of 1-cycles generated by any $i \in L$.

Now let us define notions of main state, stationary distribution rate, rotation rate and exit rate for 0-cycles and 1-cycles. Later, we define the cycles of rank greater than 1 and introduce the same notions for the higher rank cycles.

For a 0-cycle C which contains one point $i \in L$, we define the main state $M(C) = i$, stationary distribution rate $m_C(i) = 0$, rotation rate $R(C) = 0$ and the exit rate $\mathcal{E}(C) = V_{iJ(i)}$ as follows. For a 1-cycle C, we define the main state $M(C) = k^* \in C$ such that $V_{k^*J(k^*)} = \max_{i \in C} V_{iJ(i)}$. We assume that the maximum is attained just for one point k^*. The stationary distribution rate $m_C(i)$ for any $i \in C$ is defined as $m_C(i) = V_{iJ(i)} - V_{k^*J(k^*)}$. The rotation rate is defined as $R(C) = \max_{i \in C} V_{iJ(i)}$. The exit rate $\mathcal{E}(C)$ for the 1-cycle C is defined as $\mathcal{E}(C) = \min_{i \in C, j \notin C}(m_C(i) + V_{ij})$. Assume that there exist just one $i = i^* \in C$ and just one $j = j^* \notin C$ for which the maximum and the minimum are attained.

Now we define by induction the cycles of higher ranks as well as their main states, stationary distribution rate, rotation rate and exit rate.

Suppose we already introduced the cycles of rank k (k-cycles). Let $C^{(k)}$ be the set of all k-cycles. For any cycle $C \in C^{(k)}$, suppose the main state $M(C)$ and $m_C(i), R(C), \mathcal{E}(C)$ are well defined. Then for any $C_1 \in C^{(k)}$, define the exit point $i^* \in C_1$ and the entrance point $j^* \in L \backslash C_1$ such that $\min_{i \in C_1, j \notin C_1}(m_{C_1}(i) + V_{ij})$ is attained at $i = i^*, j = j^*$. The k-cycle, say $C_2 \in C^{(k)}$, containing the entrance point j^* is called the k-cycle following after C_1. We denote it $J(C_1)$. Such a closest k-cycle is unique in the generic system. Consider the sequence $C_1, J(C_1), J^2(C_1), \ldots, J^n(C_1), \ldots$ where $J^{n+1}(C_1) = J(J^n(C_1)$. Since C^k has finite number of k-cycles, the sequence, starting from some $m \in L$, is periodic: $C_1, J(C_1), \ldots, J^{n-1}(C_1), J^n(C_1) \rightarrow J^{n+1}(C_1) \rightarrow \ldots, J^m(C_1) = J^n(C_1)$. This sequence $C_1, J(C_1), \ldots, J^{n-1}(C_1), J^n(C_1) \rightarrow J^{n+1}(C_1) \rightarrow \ldots, J^m(C_1) = J^n(C_1)$ is called the cycle of rank $k+1$ (1-cycle) generated by the k-cycle C_1. Let us denote by $C^{(k+1)}$ the set of all $(k+1)$-cycles generated by any k-cycles belonging to $C^{(k)}$.

Now, let us define the main state $M(C)$ and $m_C(i), R(C)$ and $\mathcal{E}(C)$ for any $(k+1)$-cycle $C \in C^{(k+1)}$.

First, let us recall the notion of i-graph for a finite set $L = \{1, \ldots, l\}$ (see page 177 of [5]). A system of directed arrows connecting some of the points $j \in L$ is called an i-graph if any point $j \in L \setminus \{i\}$ is the initial point of exactly one arrow and starting from any point $j \in L \setminus \{i\}$ there exists a sequence of arrows leading from it to i.

Denote by $G_i(L), i \in L$ the set of all i graphs for the finite set L. The main state $M(C)$ for the $(k+1)$-cycle C is defined as $M(C) = j^*(C)$ such that $\min_{j \in C} \min_{g \in G_j(C)} \sum_{(m \to n) \in g} V_{mn}$ is attained at $j = j^*$. We assume that such a j^* is unique. Define the rotation rate $R(C)$ for the $k+1$-cycle C as $R(C) =$

$\max_{i:C_i^{(k)}\in C}\mathcal{E}(C_i^{(k)})$ where $C_i^{(k)}$ are the k-cycles which form the $(k+1)$-cycle C and $\mathcal{E}(C_i^{(k)})$ is the exit rate for the k-cycle $C_i^{(k)}$. The stationary distribution rate $m_C(i), i \in C$ for the $(k+1)$-cycle C is defined as

$$m_C(i) = \min_{g\in G_i(C)} \sum_{(m\to n)\in g} V_{mn} - \min_{g\in G_{j*}(C)} \sum_{(m\to n)\in g} V_{mn}$$

where $j^* = M(C)$ is the main state of C. The exit rate for the $(k+1)$-cycle C is defined as $\mathcal{E}(C) = \min_{i\in C, j\notin C}(m_C(i) + V_{ij})$. We assume the minimum is attained at unique points $i = i^*$ and $j = j^*$.

Since L is finite, a hierarchy of cycles up to rank k^* can be constructed so that the k^*-cycle contain all the stable equilibrium points of L.

Assumption 6: The system $X_t^{\delta,\varepsilon}$ is a generic system such that the hierachy, the main state, stationary distribution rate, rotation rate and the exit rate of each cycle are defined in a unique way.

Notice that the hierarchy of cycles and the main states are not random although the transitions between the stable points are caused by random perturbations.

Denote by $D(i) \subset R^d$ the domain attracted to the equilibrium point $i, i \in \{1, \ldots, l\}$. Let $D(C) = \cup_{i\in C} D(i)$ where C is some cycle. Let τ_C^μ be the exit time from $D(C)$: $\tau_C^\mu = \inf\{t : X_t^{\delta,\varepsilon} \notin D(C)\}$. It follows from Theorem 6.6.2 of [5] and Theorem 2.1.1 of [4] that

$$\lim_{\delta,\varepsilon\downarrow 0, \delta/\varepsilon=\mu^2} \varepsilon \ln E\tau_C^\mu = \mathcal{E}(C), \tag{2.10}$$

and for any $\gamma > 0$,

$$\lim_{\delta,\varepsilon\downarrow 0, \delta/\varepsilon=\mu^2} P_x(e^{\varepsilon\mathcal{E}(C)-\gamma)} < \tau_C^\mu < e^{\varepsilon(\mathcal{E}(C)+\gamma)}) = 1 \tag{2.11}$$

uniformly for any $X_0^{\delta,\varepsilon} = x \in F$ where F is a compact subset of $D(C)$.

Now, let us describe the long-time behavior of $X_t^{\delta,\varepsilon}$ as $\delta, \varepsilon \downarrow 0, \delta/\varepsilon = \mu^2$. For any initial point $x \in D(i), i \in \{1, \dots l\}$, with probability 1, the trajectory of $X_t^{\delta,\varepsilon}, 0 < \delta, \varepsilon << 1, \frac{\delta}{/}\varepsilon = \mu^2$, first is attracted to a small neighborhood of the equilibrium point $i(x)$. Put the 0-cycle $C^{(0)}(x) = i$. At time $\tau^{\varepsilon}_{C^{(0)}(x)}$, the trajectory $X_t^{\delta,\varepsilon}$ leaves $D(C^{(0)}(x))$ for a neighborhood of the equilibrium point $J(C^{(0)}(x))$, and then it leaves $D(J(C^{(0)}(x)))$ for a neighborhood of $J^2(C^{(0)}(x))$ and so on. With probability 1, it then rotates in the 1-cycle $C^{(1)}(x)$ generated by the state $C^{(0)}(x)$. For time greater than $\tau^{\varepsilon}_{C^{(1)}(x)} \asymp \mathcal{E}(C^{(1)}(x))$, with probability 1, the trajectory leaves $D(C^{(1)}(x))$ for a small neighborhood of $J(C^{(1)}(x))$, the 1-cycle following after $C^{(1)}(x)$, and then it leaves $D(J(C^{(1)}(x)))$ for a neighborhood of $J^2(C^{(1)}(x))$ and so on. Then it rotates along $C^{(2)}(x)$, the 2-cycle generated by $C^{(1)}(x)$, and then along a 3-cycle $C^{(3)}(x)$ which is generated by $C^{(2)}(x)$, and so on up to the highest rank k^*-cycle $C^{(k^*)}(x)$ which includes all the equilibrium points in L.

The transition times are described in equations (2.10) and (2.11) and they are described in terms of the action functional for the family $X_t^{\delta,\varepsilon}$ as $\delta, \varepsilon \downarrow 0, \delta/\varepsilon = \mu^2$.

Notice that $C^{(j)}(x) \subset C^{(j+1)}(x)$ for $j = 1, \dots, k^* - 1$ where $C^{(j+1)}(x)$ is the $(j+1)$-cycle generated by the j-cycle $C^{(m^*)}(x)$. Furthermore, it can be checked that for any initial point $x \in D(i)$,

$$\mathcal{E}(C^{(0)}(x)) < \mathcal{E}(C^{(1)}(x)) < \mathcal{E}(C^{(2)}(x)) < \dots < \mathcal{E}(C^{(k^*)}(x)) = \infty$$

and the sequence of the rotation rates for each cycle is also an increasing sequence and $R(C^{(j)}(x)) < \mathcal{E}(C^{(j)}(x)), j = 1, \dots, k^*$.

Let $T = T(\varepsilon)$ be a large parameter such that $\lim_{\varepsilon \downarrow 0} \varepsilon \ln T(\varepsilon) = \lambda > 0$. For any

$x \in R^d$, except for the points belonging to a separatrix, and any $\lambda > 0$, except for a finite number of values, there exists a cycle $C^{(m^*)}(x)$ such that $\mathcal{E}(C^{(m^*)}(x)) < \lambda < \mathcal{E}(C^{(m^*+1)}(x))$ where $C^{(m^*+1)}(x)$ is the (m^*+1)-cycle generated by the m^*-cycle $C^{(m^*)}(x)$. This implies that $X_t^{\delta,\varepsilon}$ will leave the m^*-cycle $C^{(m^*)}(x)$ and come into the basin of $D(C^{(m^*+1)}(x))$ before time $T(\varepsilon)$, but $X_t^{\delta,\varepsilon}$ does not have enough time to leave that basin before the time $AT(\varepsilon), 0 < A < \infty$ with probability 1 as $\delta, \varepsilon \downarrow 0, \delta/\varepsilon = \mu^2$.

There also exists a state $K_{h(x,\lambda)}$ in the cycle $C^{(m^*+1)}(x)$. For any $A > 0$, $X_t^{\delta,\varepsilon}$ spends most of its time around the state $K_{h(x,\lambda)}$ in the time interval $[0, AT(\varepsilon)]$ with probability 1 as $\delta, \varepsilon \downarrow 0, \delta/\varepsilon = \mu^2$. The state $K_{h(x,\lambda)}$ is called the metastable state. The following statements precisely define the metastable state $K_{h(x,\lambda)}$, $h \in C^{(m^*+1)}(x)$.

If $\lambda > R(C^{(m^*+1)}(x))$, then $h(x,\lambda) = M(C^{(m^*+1)}(x))$ the main state of the cycle $C^{(m^*+1)}(x)$.

If $\lambda < R(C^{(m^*+1)}(x)) = \max_{C^{(m^*)}(x) \in C^{(m^*+1)}(x)} \mathcal{E}(C^{(m^*)}(x))$, then the trajectory does not have enough time to rotate through all the m^*-cycles in $C^{(m^*+1)}(x)$ before time $AT(\varepsilon), 0 < A < \infty$ with probability 1 as $\varepsilon \downarrow 0$. Let $\hat{C}^{(m^*)}(x)$ be the first m^*-cycle such that $\lambda < \mathcal{E}(\hat{C}^{(m^*)}(x))$. If $\lambda > R(\hat{C}^{(m^*)}(x))$, then $h(x,\lambda) = M(\hat{C}^{(m^*)}(x))$ the main state of the cycle $\hat{C}^{(m^*)}(x)$.

If $\lambda < R(\hat{C}^{(m^*)}(x)) = \max_{C^{(m^*-1)}(x) \in \hat{C}^{(m^*)}(x)} \mathcal{E}(C^{(m^*-1)}(x))$, then the trajectory does not have enough time to rotate through all the m^*-1-cycles in $\hat{C}^{(m^*)}(x)$ before the time $AT(\varepsilon), 0 < A < \infty$ with probability 1 as $\varepsilon \downarrow 0$. Let $\hat{C}^{(m^*-1)}(x)$ be the first m^*-1-cycle such that $\lambda < \mathcal{E}(\hat{C}^{(m^*-1)}(x))$. If $\lambda > R(\hat{C}^{(m^*-1)}(x))$, then $\mu(x,\lambda) = M(\hat{C}^{(m^*-1)}(x))$ the main state of the cycle $\hat{C}^{(m^*-1)}(x)$.

If $\lambda < R(\hat{C}^{(m^*-1)}(x))$, then we consider the (m^*-2)-cycles in $\hat{C}^{(m^*-1)}(x))$ and so on until we come to a (m^*-n)-cycle $\hat{C}^{(m^*-n)}(x))$ such that $R(\hat{C}^{(m^*-n)}(x)) < \lambda < \mathcal{E}(\hat{C}^{(m^*-n)}(x))$. Then $h(x,\lambda) = M(\hat{C}^{(m^*-n)}(x))$, the main state of the cycle $\hat{C}^{(m^*-n)}(x)$. The metastable state $K_{h(x,\lambda)}$ is well defined.

Denote by $\Lambda(G)$ the Lebesgue measure of a set $G \subset R^1$ and $\rho(\cdot,\cdot)$ the Euclidean metric in R^d. The following lemma can be proved following the same idea as Theorem 1 of [4].

Lemma 2.7: Let Assumptions 5-6 be satisfied and let $T = T(\varepsilon)$ be a large parameter such that $\lim_{\varepsilon\downarrow 0} \varepsilon \ln T(\varepsilon) = \lambda > 0$. Let $x \in R^d$ not belong to a separatrix. Then, for any $h > 0$ and $A > 0$,

$$\Lambda\{t \in [0,A] : \rho(X^{\delta,\varepsilon}_{tT(\varepsilon)}, K_{h(x,\lambda)}) > h\} \to 0$$

in probability as $\delta, \varepsilon \downarrow 0, \delta/\varepsilon = \mu^2$.

■

Now consider a system $X_t^{\delta,\varepsilon}$ with time dependent coefficient when replacing equation (1.5)) with

$$X^{\delta,\varepsilon}_{t+\delta} - X^{\delta,\varepsilon}_t = \int_t^{t+\delta} b\left(\frac{s}{T}, X^{\delta,\varepsilon}_s\right) ds + \sqrt{\varepsilon}(\xi^{\delta,n}_{t+\delta} - \xi^{\delta,n}_t), \;\; X^\varepsilon_0 = x. \tag{2.12}$$

Here $T = T(\varepsilon) \asymp e^{\frac{\lambda}{\varepsilon}}, \lambda > 0$, is a large parameter as $\varepsilon \downarrow 0$ so that the coefficients of $b(t,x)$ are changing very slowly. Without loss of generality, we suppose:

Assumption 7: Assume $b(t,x)$ is a step function in t where $0 \le t < 1$ and that points $0 = t_0 < t_1 < \ldots < t_m = 1$ exists such that

$$b(t,x) = b_k(x), \;\; t_{k-1} \le t \le t_k, \quad k \in \{1,\ldots,m\}.$$

Here each $b_k(x)$ is a vector field taking values in R^d and is Lipschitz continuous in x. Suppose each vector field $b_k(x)$ has l_k asymptotically stable equilibrium points $K_1^{(k)}, \ldots, K_{l_k}^{(k)}$.

For any $x \in R^d$, except for the points belonging to a separatrix, and any $\lambda > 0$, except for a finite number of values, the trajectory $X^{\delta,\varepsilon}_{tT(\varepsilon)}$, $0 \le t < 1$, first approaches the metastable state of the system (2.12) with $b(t,x) = b_1(x)$ and then evolves together with the metastable states. Put $\pi^1(x,\lambda) = h^1(x,\lambda), \pi^k(x,\lambda) = h^k(\pi^{k-1}(x,\lambda),\lambda), k \in \{2,\ldots,m\}$ where $h^k(x,\lambda)$ is the metastable state for the system (2.12) with $b(t,x) = b_k(x)$ and initial point $X^{\delta,\varepsilon}_0 = x \in R^d$. Define a step function $\phi(t) = \phi(t,x,\lambda), 0 \le t \le 1$, where $\phi(t,x,\lambda) = K^{(j)}_{\pi^j(x,\lambda)}$ for $t_{j-1} \le t < t_j, j \in \{1,\ldots,m\}$. The following lemma can be proved following the same idea as Theorem 2 of [4]:

Lemma 2.8: Let Assumptions 6-7 be satisfied and let $T = T(\varepsilon)$ be a large parameter such that $\lim_{\varepsilon \downarrow 0} \varepsilon \ln T(\varepsilon) = \lambda > 0$. Let $x \in R^d$ not belong to a separatrix. Then, for any $h > 0$ and $0 < A < 1$,

$$\Lambda\{t \in [0,A] : \rho(X^{\delta,\varepsilon}_{tT(\varepsilon)}, \phi_t) > h\} \to 0$$

in probability as $\delta, \varepsilon \downarrow 0, \delta/\varepsilon = \mu^2$.

$\blacksquare$

Assumption 8: Let $b(t,x)$ be 1-periodic in t: $b(t+1,x) = b(t,x)$. For $0 \le t < 1$, the function $b(t,x)$ satisfies the conditions in Assumption 7.

Let L^* be the set of all the equilibrium stable points: $L^* = \{1,2,\ldots,l^*\}$ where l^* is a finite number.

Theorem 2.5: Let Assumptions 6 and 8 be satisfied and let $T = T(\varepsilon)$ be a large parameter such that $\lim_{\varepsilon \downarrow 0} \varepsilon \ln T(\varepsilon) = \lambda > 0$. Then for any $x \in D(i)$, there exists a periodic function $\phi(t)$ with period $N = N(i,\lambda,l^*) \le l^*$ such that for any

$h > 0, A > 0,$

$$\Lambda\{t \in [0,A] : \rho(X^{\delta,\varepsilon}_{tT(\varepsilon)}, \phi_t) > h\} \to 0$$

in probability as $\varepsilon \to 0$. Here l^* is the number of the equilibrium stable points of the unperturbed system.

The argument for this theorem is in the following (see [4]):

First, consider the process $X^{\delta,\varepsilon}_{tT(\varepsilon)}$, $0 \le t < 1$. Put

$$\pi^1(x,\lambda) = h^1(x,\lambda), \ldots, \pi^k(x,\lambda) = h^k(\pi^{k-1}(x,\lambda),\lambda), \ldots, \pi^n(x,\lambda) = h^n(\pi^{n-1}(x,\lambda),\lambda).$$

Let $\phi(t) = \phi(t,x,\lambda)$, $0 \le t < 1$ where

$$\phi(t,x,\lambda) = K^{(j)}_{\pi^j(x,\lambda)},\;\; t_{j-1} \le t < t_j,\;\; j \in \{1,\ldots,n\}.$$

Then the trajectory of the process $X^{\delta,\varepsilon}_{tT(\varepsilon)}$ for $0 \le t < 1$ will be close to the step function $\phi(t)$, $0 \le t < 1$, as $\delta, \varepsilon \downarrow 0, \delta/\varepsilon = \mu^2$. For $1 \le t < 2$, the trajectory of the process $X^{\delta,\varepsilon}_t$ will evolve together with the metastable state of $X^{\delta,\varepsilon}_t$ for $0 \le t < 1$. Put

$$\pi^{n+1}(x,\lambda) = h^1(\pi^n(x,\lambda),\lambda), \ldots, \pi^{n+k}(x,\lambda) = h^k(\pi^{n+k-1}(x,\lambda),\lambda), \ldots,$$

$$\pi^{2n}(x,\lambda) = h^n(\pi^{2n-1}(x,\lambda),\lambda).$$

Let $\phi(t) = \phi(t,x,\lambda)$, $1 \le t < 2$ where

$$\phi(t,x,\lambda) = K^{(j)}_{\pi^{n+j}(x,\lambda)},\;\; t_{j-1} \le t-1 < t_j,\;\; j \in \{1,\ldots,n\}.$$

Then the trajectory of the process $X^{\delta,\varepsilon}_{tT(\varepsilon)}$ for $1 \le t < 2$ will be close to the function $\phi(t)$, $1 \le t < 2$ as $\delta, \varepsilon \downarrow 0, \delta/\varepsilon = \mu^2$. Keeping this pattern, we conclude that the trajectory of the process $X^{\delta,\varepsilon}_{tT(\varepsilon)}, t > 0$ will be close to a function $\phi(t)$ as $\delta, \varepsilon \downarrow 0, \delta/\varepsilon = \mu^2$.

Let us show that the function $\phi(t)$ is periodic in t with period less than l^*.

Consider the sequence

$$\pi^0(x,\lambda) = i(x), \pi^n(x,\lambda) = h^n(\pi^{n-1}(x,\lambda),\lambda), \pi^{2n}(x,\lambda) = h^n(\pi^{2n-1}(x,\lambda),\lambda), \ldots.$$

Since the unperturbed system has only a finite number l^* of asymptotically stable equilibrium points, the sequence $\pi^0(x,\lambda), \pi^n(x,\lambda), \pi^{2n}(x,\lambda), \ldots$ starting from some $N = N(i,\lambda,l^*) \le l^*$ is periodic. Therefore, $\phi(t)$ is a periodic function with period $N \le l^*$.

In conclusion, for any $X_0^{\delta,\varepsilon} = x \in D(i)$ and $\lambda > 0$, besides a finite number of values, the trajectory of $X_{tT(\varepsilon)}^{\delta,\varepsilon}$ will be close to a periodic function $\phi(t)$ as $\delta, \varepsilon \downarrow 0, \delta/\varepsilon = \mu^2$. This effect is called stochastic resonance.

2.4 Example

Example 1: (see [5]) Consider (1.4) in the one-dimensional case:

$$\dot{X}_t^\varepsilon = b(\frac{t}{T}, X_t^\varepsilon) + \sqrt{\varepsilon}\dot{W}_t, \;\; X_0^\varepsilon = x \in R^1, \;\; 0 < \varepsilon \ll 1.$$

Let $b(t,x)$ be 1-periodic in t and

$$b(t) = \begin{cases} -B_1'(x), & 0 \le t \le t_1 < 1, \\ -B_2'(x), & t_1 \le t < 1. \end{cases}$$

The potentials $B_1(x)$ and $B_2(x)$ are given in Figure 2.1. There are three stable attractors $k_1 = \{x_1\}, k_2 = \{x_2\}, k_3 = \{x_3\}$ for each of the fields $-B_1'(x)$ and $-B_2'(x)$. We define $V(x,y) = \inf_{\phi \in C_{0T}}\{S_{[0,T]}(\phi), \phi_0 = x, \phi_T = y\}$ and $V_{ij} = V(x,y)$ for $x \in K_i$, $y \in K_j$. In this case, $V^i(x,y)$, $i = 1,2$, for each of these fields can be expressed through the potential: for x and y from the same well, we have

$V^i(x,y) = 2(B_i(y) - B_i(x)) \vee 0$ (see §4.3 of [4], [5]). Hence, the values of V^1_{ij} and V^2_{ij} can be calculated:

$$V^1_{12} = 16,\ V^1_{13} = 22,\ V^1_{21} = 8,\ V^1_{23} = 6,\ V^1_{31} = 12,\ V^1_{32} = 4.$$

$$V^2_{12} = 16,\ V^2_{13} = 20,\ V^2_{21} = 6,\ V^2_{23} = 4,\ V^2_{31} = 12,\ V^2_{32} = 6.$$

There are two 1-cycles for each of the field $-B_1^{'}(x)$ and $-B_2^{'}(x)$, $C^1_1 = \{1\}$ and $C^1_2 = \{2 \to 3 \to 2\}$. The 2-cycles contain all the states. But the main states of cycle C^1_2 are different for the fields $-B_1^{'}(x)$ and $-B_2^{'}(x)$. The main state of C^1_2 for $-B_1^{'}(x)$ is $M_1(C^1_2) = 2$ and $M_1(C^1_2 = 3$ for $-B_2^{'}(x)$. Let $\lambda = \lim_{\varepsilon \downarrow 0} \varepsilon \ln T(\varepsilon) > 0$. Let $X^\varepsilon_0 = x$. Denote by $h^i(x,\lambda)$, $i = 1, 2$, the function of main states calculate for each drift $-B_i^{'}(x)$ respectively. If $x \in D(1)$, the basin $D(1)$ of x_1, then $h^i(x,\lambda) = 1$ for $i = 1, 2$. If $x \in D(2)$, then

$$h^1(x,\lambda) = \begin{cases} 2, & \lambda < 8, \\ 1, & \lambda > 8. \end{cases} \qquad h^2(x,\lambda) = \begin{cases} 2, & \lambda < 4, \\ 3, & 4 < \lambda < 8, \\ 1, & \lambda > 8. \end{cases}$$

If $x \in D(3)$, then

$$h^1(x,\lambda) = \begin{cases} 3, & \lambda < 8, \\ 2, & 4 < \lambda < 8. \\ 1, & \lambda > 8. \end{cases} \qquad h^2(x,\lambda) = \begin{cases} 3, & \lambda < 8, \\ 1, & \lambda > 8. \end{cases}$$

This means if $x \in D(1)$, $X^\varepsilon_{tT(\varepsilon)}, 0 \le t < A < \infty$, $X^\varepsilon_0 = x$, will be close to x_1 for any $\lambda > 0$. No periodic oscillations will be observed in this case. If $x \in D(2) \cup D(3)$ and $4 < \lambda < 8$, then $X^\varepsilon_{tT(\varepsilon)}, 0 \le t < A < \infty$, $X^\varepsilon_0 = x$, will be close to a 1-periodic function $\phi(t)$ as $\varepsilon \downarrow 0$ with probability close to 1:

$$\phi(t) = \begin{cases} x_2, & t \in [0, t_1), \\ x_3, & t \in [t_1, 1). \end{cases}$$

If $\lambda < 4$, then $X^{\varepsilon}_{tT(\varepsilon)}, 0 \leq t < A < \infty$ as $\varepsilon \downarrow 0$ will be stay near the attractor of the initial point with probability close to 1. If $\lambda > 8$, then $X^{\varepsilon}_{tT(\varepsilon)}, 0 \leq t < A < \infty$ as $\varepsilon \downarrow 0$ will be stay near the attractor of x_1 with probability close to 1.

Example 2: Let W_t in Example 1 be replaced by a one-dimensional random walk ξ^{δ}_t with $n = 2$. More precisely, the random walk ξ^{δ}_t, $t \in N_{\delta} = \{0, \delta, ...k\delta, ...\}$, can jump to $0, \pm\sqrt{\delta}, \pm 2\sqrt{\delta}$ such that $\xi^{\delta}_{t+\delta} - \xi^{\delta}_t = \pm\sqrt{\delta}$ with probability $\frac{1}{2}p_1 = \frac{1}{6}$, $\xi^{\delta}_{t+\delta} - \xi^{\delta}_t = \pm\sqrt{\delta}$ with probability $\frac{1}{2}p_1 = \frac{1}{12}$. The probability that ξ^{δ}_t jumps to 0 is $p_0 = \frac{1}{2}$. It can be verified that $p_0 = \frac{1}{2}, p_1 = \frac{1}{3}, p_2 = \frac{1}{6}$ satisfy $p_0 + p_1 + p_2 = 1$, $\sum_{i=1}^{2} p_i i^2 = 1$ and $\sum_{i=1}^{2} p_i i^4 = 3$.

Consider the processes $X^{\delta,\varepsilon}_t$ with same construction as in (1.5)). From Theorem 2.1 and Lemma 2.5, the action functional for the family of processes $X^{\delta,\varepsilon}_t$ as $\delta, \varepsilon \downarrow 0, \delta/\varepsilon = \mu^2$ in the space $C_{[0,T]}$ is $\varepsilon^{-1} S^{\mu}_{[0,T]}(\varphi)$, where

$$S^{\mu}_{[0,T]}(\varphi) = \begin{cases} \int_0^T L^{\mu}_0(\dot{\varphi}_s - b(\varphi_s))ds, & \text{if} \varphi \text{ is a.c. and } \varphi_0 = x, \\ +\infty, & \text{otherwise.} \end{cases}$$

where $L^{\mu}_0 = \frac{u^2}{2} + \frac{1}{180}u^6\mu^4 + O(\mu^4)$, $0 < \mu << 1$. Define

$$V^{\mu}(x, y) = \inf_{\phi \in C_{0T}} \{S^{\mu}_{[0,T]}(\phi), \phi_0 = x, \phi_T = y\}.$$

We have (see §4.3 of [4] and [5]),

$$V^{\mu}(x, y) = 2(B(y) - B(x)) \vee 0 + \frac{16}{45}\mu^2 \int_{-\infty}^{0} (B^{'}(\phi_s))^6 ds + O(\mu^4), \; 0 < \mu << 1.$$

where ϕ_s satisfies $\dot{\phi}_s = B^{'}(\phi_s)$, $\phi_0 = y$ and $\lim_{t\downarrow -\infty} \phi_t = x$.

Hence, values for V^{μ}_{ij}, $i, j = 1, 2, 3$ can be calculated and they are different from V_{ij} in example 1. Different $h_{\mu}(x, \lambda)$ will be obtained also. The corresponding conditions providing stochastic resonance can be obtained from $h_{\mu}(x, \lambda)$.

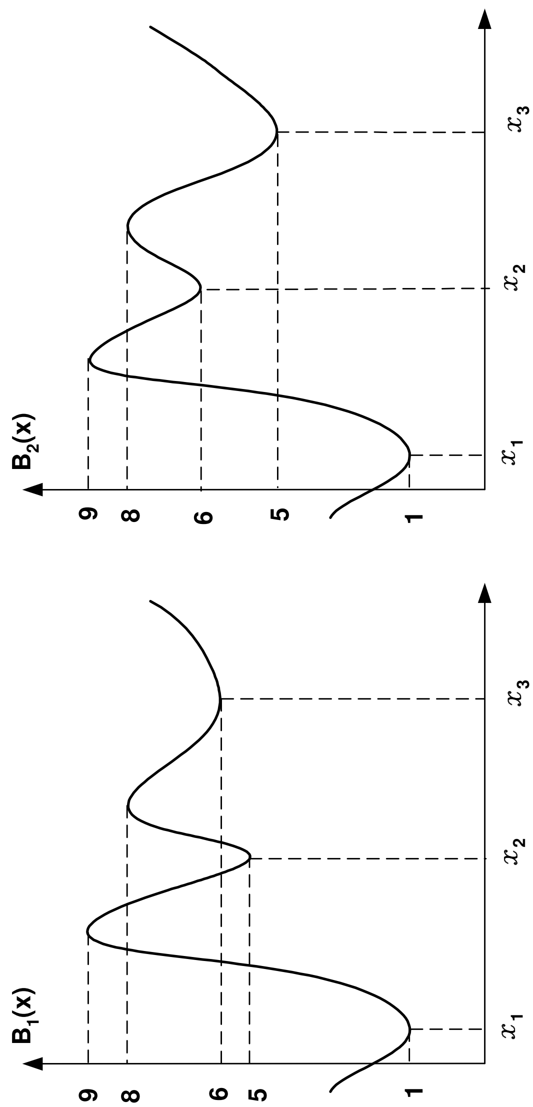

Figure 2.1:

Chapter 3

Exit Problem for Perturbation $\eta_t^{\mu,\varepsilon}$ Approximating White Noise

3.1 Large Deviations for $(X_t^{\mu,\varepsilon}, \eta_t^{\mu,\varepsilon})$

Consider a mean-zero Gaussian process $\eta_t^{\mu,\varepsilon}$. Here $\eta_t^{\mu,\varepsilon} = (\eta_t^{\mu,\varepsilon,1}, ..., \eta_t^{\mu,\varepsilon,d})$, $t \in [0,T]$, with each component identically and independently distributed, satisfies the following equation

$$\mu\dot{\eta}_t^{\mu,\varepsilon} = -\eta_t^{\mu,\varepsilon} + \sqrt{\varepsilon}\dot{W}_t, \;\; \eta_0^{\mu,\varepsilon} = y \in R^d.$$

Here W_t is a standard d-dimensional Wiener process and $\mu \in R^1$ is a positive constant. According to the contraction principle (see Theorem 1.1), a large deviation principle for the family of processes $\eta_t^{\mu,\varepsilon}$ as $\varepsilon \downarrow 0$ can be established:

Theorem 3.1: The action functional for the family of processes $\eta_t^{\mu,\varepsilon}$ in the space $C_{[0,T]}(R^d)$ as $\varepsilon \downarrow 0$ is $(1/\varepsilon)S^{\eta}_{[0,T]}(\varphi, \phi)$ where

$$S^{\eta}_{[0,T]}(\varphi) = \begin{cases} \frac{1}{2}\int_0^T |\mu\dot{\phi}_t + \phi_t|^2 dt, & \text{if } \phi_t \text{ is absolutely continuous and } \phi_0 = y \\ +\infty, & \text{otherwise} \end{cases}$$

■

Let us replace $\sqrt{\varepsilon}\dot{W}_t$ in (1.3) with $\eta_t^{\mu,\varepsilon}$ and let σ be a unit matrix. Then (1.3) becomes

$$\dot{X}_t^{\mu,\varepsilon} = b(X_t^{\mu,\varepsilon}) + \eta_t^{\mu,\varepsilon}, \;\; X_0^{\mu,\varepsilon} = x \in R^d.$$

Since $X_t^{\mu,\varepsilon}$ is not a Markov process, we consider the $2d$-dimensional Markov process $(X_t^{\mu,\varepsilon}, \eta_t^{\mu,\varepsilon})$ which satisfies

$$\begin{cases} \dot{X}_t^{\mu,\varepsilon} = b(X_t^{\mu,\varepsilon}) + \eta_t^{\mu,\varepsilon}, \\ \mu\dot{\eta}_t^{\mu,\varepsilon} = -\eta_t^{\mu,\varepsilon} + \sqrt{\varepsilon}\dot{W}_t, \\ X_0^{\mu,\varepsilon} = x \in R^d, \;\; \eta_0^{\mu,\varepsilon} = y \in R^d, \;\; 0 < \varepsilon \ll 1. \end{cases} \tag{3.1}$$

Our goal in this section is to establish a large deviation principle for the family of processes $(X_t^{\mu,\varepsilon}, \eta_t^{\mu,\varepsilon})$ as $\varepsilon \downarrow 0$.

Let φ_t, ϕ_t be two functions belonging to the space $C_{[0,T]}(R^d)$. Define the functional $S^{\mu}_{[0,T]}(\varphi, \phi)$ on the space $C_{[0,T]}(R^{2d})$ as

$$S^{\mu}_{[0,T]}(\varphi, \phi) = \frac{1}{2}\int_0^T |(\dot{\varphi}_t - b(\varphi_t) + \mu\frac{d}{dt}(\dot{\varphi}_t - b(\varphi_t))|^2 dt. \tag{3.2}$$

If $\dot{\varphi}_t$ is absolutely continuous, $\phi_t = \dot{\varphi}_t - b(\varphi_t)$ and $\varphi_0 = x, \phi_0 = y$. While $S^{\mu}_{0T}(\varphi, \phi) = \infty$ for the remaining functions in $C_{[0,T]}(R^{2d})$.

Theorem 3.2: The functional $S^{\mu}_{[0,T]}(\varphi, \phi)$ is the action functional in the space $C_{[0,T]}(R^{2d})$ for the family of processes $(X_t^{\mu,\varepsilon}, \eta_t^{\mu,\varepsilon})$ as $\varepsilon \downarrow 0$.

Proof: Let ψ_t be a continuous function on $[0, T]$ with values in R^d. In $C_{[0,T]}(R^d)$ we consider the map $B_{x,y} : \psi \to (u, v)$, where $(u, v) = (u_t, v_t)$ is the solution of the equation

$$\begin{cases} u_t = x + \int_0^t b(u_s)ds + \int_0^t v_s ds \\ \mu v_t = \mu y - \int_0^t v_s ds + \psi_t, \;\; t \in [0, T]. \end{cases}$$

It is easy to prove that the solution of the above equation exists and is unique for any continuous function ψ and for any $x, y \in R^d$. And it is also easily checked that $B_{x,y}$ is continuous map from the space $C_{[0,T]}(R^d)$ (with uniform metric) into the space $C_{[0,T]}(R^{2d})$ (with uniform metric). The map $B_{x,y}$ has the inverse

$$(B_{x,y}^{-1}(u,v))_t = \psi_t = \mu(\dot{u}_t - b(u_t)) + u_t - x - \int_0^t b(u_s)ds - \mu y.$$

with $v_t = \dot{u}_t - b(u_t)$ and $u_0 = x$ and $v_0 = y$. According to the contraction principle (see Lemma 1.1), the action functional for the family of processes $(X_t^{\mu,\varepsilon}, \eta_t^{\mu,\varepsilon}) = B_{x,y}(\sqrt{\varepsilon}W_t)$ in the space $C_{[0,T]}(R^{2d})$ has the form $\varepsilon^{-1}S_{[0,T]}^{\mu}(\varphi)$ where

$$\begin{aligned} S_{[0,T]}^{\mu}(\varphi,\phi) &= \frac{1}{2}\int_0^T |\frac{d}{dt}(B_{x,y}^{-1}\varphi)_t|^2 dt \\ &= \frac{1}{2}\int_0^T |(\dot{\varphi}_t - b(\varphi_t) + \mu\frac{d}{dt}(\dot{\varphi}_t - b(\varphi_t)|^2 ds. \end{aligned}$$

if $\phi_t = \dot{\varphi}_t - b(\varphi_t)$ and $\varphi_0 = x, \phi_0 = y$ and the function $(B_{x,y}^{-1}(\varphi,\phi))_t = \mu(\dot{\varphi}_t - b(\varphi_t)) + \varphi_t - x - \int_0^t b(\varphi_s)ds - y$ is absolutely continuous. It is clear that this function is absolutely continuous if and only if $\dot{\varphi}_t$ is absolutely continuous. Also $S_{[0,T]}^{\mu}(\varphi,\phi) = \infty$ for the remaining functions in $C_{[0,T]}(R^{2d})$. This completes the proof.

■

3.2 Exit Problem for $X_t^{\mu,\varepsilon}$

Consider the system:

$$\dot{X}_t = b(X_t), \;\; X_0 = x \in R^d,$$

where the vector field $b(x)$ is Lipschitz continuous.

Assumption 1: The vector field $b(x), x \in R^d$ has an asymptotically stable equilibrium at a point $O \in R^d$.

Assumption 2: Assume $b(x)$ has a potential $U(x)$ such that $\nabla U(x) = -b(x)$ with $U(O) = 0$ and $U(x) > 0, \nabla U(x) \neq 0$ for $x \neq O$.

Let $G \subset R^d$ be a bounded domain with boundary ∂G.

Assumption 3: The domain G is attracted to $O \in G$: $\lim_{t\uparrow\infty} X_t = O$, for each trajectory of $\dot{X}_t = b(X_t)$, $X_0 = x \in G$.

Assumption 4: The domain G has a smooth boundary ∂G and $(b(x)\cdot n(x)) < 0, x \in \partial G$, where $n(x)$ is the exterior normal of the boundary of G.

Now, let us consider the process $X_t^{\mu,\varepsilon}$, $t \in [0,T]$. Let $X_0^{\mu,\varepsilon} = x \in G$. Denote by τ^X the first exit time of the process $X_t^{\mu,\varepsilon}$ from G: $\tau^X = \min\{t : X_t^{\mu,\varepsilon} \in \partial G\}$. Our goal in this section is to consider the asymptotic behavior of τ^X and the exit point of $X_t^{\mu,\varepsilon}$ from G when $\varepsilon \downarrow 0$. We follow the same idea as Theorems 4.2.1, 4.4.1 and 4.4.2 in [5] where properties of Markov processes play an important role in the proof. Since $X_t^{\mu,\varepsilon}$ is not a Markov process, we must consider the $2d$-dimensional Markov process $(X_t^{\mu,\varepsilon}, \eta_t^{\mu,\varepsilon})$ in the proof.

First, consider the system $(X_t^{\mu}, \eta_t^{\mu})$:

$$\begin{cases} \dot{X}_t^{\mu} = b(X_t^{\mu}) + \eta_t^{\mu}, \quad X_0^{\mu} = x \in R^d \\ \mu\dot{\eta}_t^{\mu} = -\eta_t^{\mu}, \quad \eta_0^{\mu} = y \in R^d. \end{cases}$$

It is easy to prove that the system $(X_t^{\mu}, \eta_t^{\mu})$ has an asymptotic stable equilibrium at $(O,O) \in R^{2d}$ under the assumptions concerning $b(x)$ (see [15]). Indeed, since there exists a potential $U(x)$ such that $\nabla U(x) = -b(x)$. Define $V(x,\eta) = \mu\eta^2/2 + B(x)$. It can be checked that $\frac{d}{dt}V(X_t^{\mu}, \eta_t^{\mu}) < 0$ except for the point (O,O). By Theorem 2.3 in [25], $(X_t^{\mu}, \eta_t^{\mu})$ has an asymptotic stable equilibrium at (O,O).

Now, consider the system $(X_t^{\mu,\varepsilon}, \eta_t^{\mu,\varepsilon})$ with $X_0^{\mu,\varepsilon} = x \in G$, $\eta_0^{\mu,\varepsilon} = y \in R^d$. Notice that $\tau^X = \min\{t : X_t^{\mu,\varepsilon} \in \partial G\} = \min\{t : (X_t^{\mu,\varepsilon}, \eta_t^{\mu,\varepsilon}) \in \partial G \times R^d\}$. Let $x, y, z \in R^d$. We introduce the function $V^\mu(x, y, z)$:

$$V^\mu(x,y,z) = \inf_{\varphi,\phi \in C_{[0,T]}(R^d)} \{S^\mu_{[0,T]}(\varphi,\phi) : \varphi_0 = x, \phi_0 = y, \varphi_T = z, T > 0\}.$$

Here $S^\mu_{[0,T]}(\varphi,\phi)$ is the action functional for $(X_t^{\mu,\varepsilon}, \eta_t^{\mu,\varepsilon})$ in the space $C_{[0,T]}(R^{2d})$ as $\varepsilon \downarrow 0$. Define $V^\mu(x) := V^\mu(O, O, x)$. It turns out that

$$\begin{aligned} V^\mu(x) &= \inf_{\varphi,\phi \in C_{0T}(R^d)} \{S^\mu_{[0,T]}(\varphi,\phi) : \varphi_0 = O, \phi_0 = O, \varphi_T = x, T > 0\} \\ &= \inf_{\varphi,\phi \in C_{0T}(R^d)} \{\frac{1}{2}\int_0^T |(\dot\varphi_t - b(\varphi_t)) + \mu\frac{d}{dt}(\dot\varphi_t - b(\varphi_t))|^2 ds : \\ &\qquad \phi_t = \dot\varphi_t - b(\varphi_t),\ \varphi_0 = O, \phi_0 = O, \varphi_T = x,\ T > 0\} \\ &= \inf_{\varphi,\dot\varphi \in C_{0T}(R^d)} \{\frac{1}{2}\int_0^T |(\dot\varphi_t - b(\varphi_t)) + \mu\frac{d}{dt}(\dot\varphi_t - b(\varphi_t))|^2 ds, \\ &\qquad \varphi_0 = O, \dot\varphi_0 = O, \varphi_T = x, T > 0\} \end{aligned}$$

The function $V^\mu(x)$ is non-negative and equal to zero just at $x = O$ under the assumption that O is the only asymptotically stable point of $b(x)$. Let us define $V_o^\mu = \min_{x\in\partial G} V^\mu(x)$. We formulate the following Lemma without proof. For detailed proof, please see [16].

Recall from Chapter 2 that $V(x) = \inf_{\varphi \in C_{0T}(R^d)}\{\frac{1}{2}\int_0^T |\dot\varphi_t - b(\varphi_t)|^2 dt : \varphi_0 = O, \varphi_T = x, T > 0\}$ and $V_o = \min_{x\in\partial G} V(x)$.

Lemma 3.1: Let $V^\mu(x)$ and V_o^μ be defined as above. Assume that $\min_{x\in\partial G} V^\mu(x)$ is achieved just at one point $x_o^\mu \in \partial G$ and $\min_{x\in\partial G} V(x)$ is achieved just at one point $x_o \in \partial G$. Then $\lim_{\mu\downarrow 0} V^\mu(x) = V(x)$, $\lim_{\mu\downarrow 0} V_o^\mu = V_o$ and $\lim_{\mu\downarrow 0} x_o^\mu = x_o$.

Theorem 3.3: Let N be any positive constant such that (X_t^μ, η_t^μ) with $X_0^\mu = x \in G$, $\eta_0^\mu = y$, $|y| < N$ never leaves the domain $G \times [-N, N]$. Let Assumptions

1-4 be satisfied. For any $x \in G$, $|y| \leq N$ and $\alpha > 0$, there exists a μ^* such that for any $\mu < \mu^*$,

(1)

$$\lim_{\varepsilon \downarrow 0} \varepsilon \ln E_{x,y} \tau^X < V_o^\mu + \alpha, \quad \lim_{\varepsilon \downarrow 0} \varepsilon \ln E_{x,y} \tau^X > V_o^\mu - \alpha.$$

(2)

$$\lim_{\varepsilon \downarrow 0} P_{x,y}(e^{\frac{V_o^\mu - \alpha}{\varepsilon}} < \tau^X < e^{\frac{V_o^\mu + \alpha}{\varepsilon}}) = 1.$$

(3) If $\min_{x \in \partial G} V^\mu(x)$ is achieved just at one point $x_o^\mu \in \partial G$ and $\min_{x \in \partial G} V(x)$ is achieved just at one point $x_o \in \partial G$, then

$$\lim_{\varepsilon \downarrow 0} P_{x,y}(|X_{\tau^X}^{\mu,\varepsilon} - x_o^\mu| < \alpha) = 1.$$

For convenience, let us first prove this theorem for the one dimensional case. (Hence G is a one dimensional bounded domain and $b(x)$ is automatically of potential-type). We follow the same idea as Theorems 4.2.1, 4.4.1 and 4.4.2 in [5].

Define

$$V^\eta(x) = \inf_{\phi \in C_{[0,T]}} \{S_{[0,T]}^\eta(\phi) : \phi_0 = O, \phi_T = x\}.$$

where $1/\varepsilon S_{[0,T]}^\eta(\phi)$ is the action functional for the family of processes $\eta_t^{\mu,\varepsilon}$ in the space $C_{[0,T]}$ as $\varepsilon \downarrow 0$. From Theorem 3.4.3 in [5] together with Theorem 3.1, it is easily check that $V^\eta(x) = x^2\mu$.

Let M be some positive constant which will be determined later. Define $V_o^\eta = \min_{x \in \partial[-M,M]} V^\eta(x)$. Hence $V_o^\eta = M^2\mu$.

Define $\Gamma := \{(x,y) \in R^2 : (x,y) \in \mathcal{E}_\delta(O \times O)\}$; $\gamma := \{(x,y) \in R^2 : (x,y) \in \mathcal{E}_{\frac{\delta}{2}}(O \times O)\}$. Here $\mathcal{E}_\delta(O \times O)$ and $\mathcal{E}_{\delta/2}(O \times O)$ are the δ and $\delta/2$ spheres of

$O \times O$, respectively. Let us introduce an increasing sequence of Markov times $\tau_0, \sigma_0, \tau_1, \sigma_1, \tau_2, ...$ and a Markov chain $Z_n = (X^{\mu,\varepsilon}_{\tau_n}, \eta^{\mu,\varepsilon}_{\tau_n})$ in the following way: $\tau_0 = 0$ and

$$\sigma_n := \inf\{t > \tau_n : (X^{\mu,\varepsilon}_t, \eta^{\mu,\varepsilon}_t) \in \Gamma\};$$

$$\tau_n := \inf\{t > \sigma_n, (X^{\mu,\varepsilon}_t : \eta^{\mu,\varepsilon}_t) \in \gamma \cup (\partial G \times (-M, M)) \cup (G \times \partial[-M, M])\}$$

If at a certain step, the process $(X^{\mu,\varepsilon}_t, \eta^{\mu,\varepsilon}_t)$ does not reach the set Γ any more, we set the corresponding Markov time and all subsequent ones equal to $+\infty$. Notice that infinite τ_n and σ_n can be avoinded if we change the field $b(x)$ outside G in an appropriate way. The sequence $Z_n = (X^{\mu,\varepsilon}_{\tau_n}, \eta^{\mu,\varepsilon}_{\tau_n})$ forms a Markov chain on the set $\gamma \cup (\partial G \times (-M, M)) \cup (G \times \partial[-M, M])$. Notice that Z_n is not defined if $\tau_n = \infty$ but this can happen only after exit to $(\partial G \times (-M, M)) \cup (G \times \partial[-M, M])$. Now, let us prove some lemmas which will be useful.

Lemma 3.2: Let F_1 and F_2 be two compact sets in R^1 and let T and α be positive numbers. For each fixed $\mu > 0$, there exist $\varepsilon_o = \varepsilon_o(\mu)$ and $\beta = \beta(\mu) > 0$ such that for any $x \in F_1$, $y \in F_2$ and $\varepsilon < \varepsilon_o$,

$$P_{x,y}\{\rho_{C_{[0,T]}}((X^{\mu,\varepsilon}_t, \eta^{\mu,\varepsilon}_t), (X^{\mu}_t, \eta^{\mu}_t)) > \alpha\} \leq \exp\{-\varepsilon^{-1}\beta\}.$$

Here $(X^{\mu}_t, \eta^{\mu}_t)$ is the trajectory of the dynamic system

$$\begin{cases} \dot{X}^{\mu}_t = b(X^{\mu}_t) + \eta^{\mu}_t, \;\; X_0 = x \in R^d \\ \mu\dot{\eta}^{\mu}_t = -\eta^{\mu}_t, \;\; \eta^{\mu}_0 = y \in R^d. \end{cases}$$

Proof: Put

$$G(x, y) = \{\varphi, \phi \in C_{[0,T]}(R^d) : \varphi_0 = x, \phi_0 = y, \rho_{C_{[0,T]}}((\varphi, \phi), (X^{\mu}_t, \eta^{\mu}_t)) > \alpha\}$$

Then $\cup_{x \in F_1, y \in F_2} G(x, y)$ is a closed set and $S^{\mu}_{[0,T]}$ attains its infimum h on this closed set.

Notice that

$$S_{[0,T]}(\varphi,\phi)=\frac{1}{2}\int_0^T|(\dot{\varphi}_t-b(\varphi_t)+\mu\frac{d}{dt}(\dot{\varphi}_t-b(\varphi_t))|^2dt$$

if $\dot{\varphi}_t$ is absolutely continuous and $\phi_t=\dot{\varphi}_t-b(\varphi_t)$ and $\varphi_0=x,\phi_0=y$, while $S^{\mu}_{[0T]}(\varphi,\phi)=\infty$ for the remaining functions in $C_{[0,T]}$. The functional vanishes only on trajectories of the dynamical system (X^{μ}_t,η^{μ}_t). Therefore, $h>0$.

For any $h^{'}<h$, the sets $\cup_{x\in F_1,y\in F_2}G(x,y)$ and $\cup_{x\in F_1,y\in F_2}\Phi_{x,y}(h^{'})$ are disjoint. Let us denote the distance between them by $\alpha^{'}$ and $\alpha^{'}>0$. Therefore, for any $\gamma>0$, we have

$$\begin{aligned}
&P_{x,y}(\rho_{C_{[0,T]}}((X^{\mu,\varepsilon}_t,\eta^{\mu,\varepsilon}_t),(X^{\mu}_t,\eta^{\mu}_t))>\alpha)\\
&=P_{x,y}((X^{\mu,\varepsilon}_t,\eta^{\mu,\varepsilon}_t)\in G(x,y))\\
&\leq P_{x,y}(\rho_{C_{[0,T]}}((X^{\mu,\varepsilon}_t,\eta^{\mu,\varepsilon}_t),\Phi_{x,y}(h^{'})\geq\alpha^{'})\leq\exp\{-\varepsilon(h^{'}-\gamma)\}
\end{aligned}$$

for sufficiently small ε and for any $x\in F_1,y\in F_2$. Hence this lemma holds for $\beta=h^{'}-\gamma$.

■

Lemma 3.3: Let Assumptions 1-4 be satisfied. Let $M<\infty$ be a positive constant. Let $x\in G$, $|y|<M$. Then, for each $\mu>0$, we have

(1) For any $\alpha>0$, there exist positive constants a and T_o such that for any function (φ_t,ϕ_t) assuming its values in the set $((G\cup\partial G)\times[-M,M])\setminus\mathcal{E}_{\alpha}(O\times O)$ for $t\in[0,T]$ with $\varphi_o=x$, $\phi_0=y$, we have the inequality $S^{\mu}_{[0,T]}(\varphi,\phi)>a(T-T_o)$.

(2) For any $\alpha>0$ there exists positive constant a and T_o such that for all sufficiently small $\varepsilon>0$ and any $(x,y)\in((G\cup\partial G)\times[-M,M])\setminus\mathcal{E}_{\hat{\alpha}}(O\times O)$

we have the inequality

$$P_{x,y}(\zeta_\alpha > T) \leq \exp\{-\varepsilon^{-1}a(T - T_o)\},$$

where $\zeta_\alpha = \inf\{t : (X_t^{\delta,\varepsilon}, \eta_t^{\mu,\varepsilon}) \notin ((G \cup \partial G) \times [-M, M]) \setminus \mathcal{E}_\alpha(O \times O)$.

(3) For any $\alpha > 0$ and any $c > 0$, there exists T large enough such that for all sufficiently small $\varepsilon > 0$ and any $(x, y) \in ((G \cup \partial G) \times [-M, M]) \setminus \mathcal{E}_{\hat{\alpha}}(O \times O)$, we have the inequality

$$P_{x,y}(\zeta_\alpha > T) \leq \exp\{-\varepsilon^{-1}c\}.$$

Proof: (1) Let $\mathcal{E}_{\alpha'}(O, O)$ be a neighborhood of (O, O) such that the trajectories of the dynamical system (X_t^μ, η_t^μ) issuing from $\mathcal{E}_{\alpha'}(O, O)$ never leave $\mathcal{E}_\alpha(O, O)$. We denote by $T(\alpha, x, y)$ the time spent by (X_t^μ, η_t^μ) with $(X_0^\mu, \eta_0^\mu) = (x, y)$ until reaching $\mathcal{E}_{\alpha'}(O, O)$. Since $G \times R$ is attracted to $O \times O$, we have $T(\alpha, x, y) < \infty$ for $x \in G \cup \partial G$ and $y \in R$. The function $T(\alpha, x, y)$ is upper semi-continuous in x and y (because $(X_t^\mu(x), \eta_t^\mu(y))$ depends continuously on x and y). Consequently, it attains its largest value $T_o = \max_{x \in G \cup \partial G, |y| \leq M} T(\alpha, x, y) < \infty$.

The set of functions from $C_{[0,T_o]}$ assuming their values in $((G \cup \partial G) \times [-M, M]) \setminus \mathcal{E}_\alpha(O \times O)$ is closed in $C_{[0T_o]}(R^d)$. The functional $S^\mu_{[0,T_o]}(\varphi, \phi)$ attains its minimum on this closed set and this minimum is different from zero.

Then for all such functions $S^\mu_{[0,T_o]}(\varphi, \phi) \geq A > 0$. By the additivity of S, we have

$$S^\mu_{[0,T]}(\varphi, \phi) \geq A[T/T_o] > A(T/T_o - 1) = a(T - T_o).$$

(2) Since G is attracted to O and $(b(x) \cdot n(x)) < 0$ on the boundary of G, it follows that the same properties will be enjoyed by the β-neighborhood of G for

sufficiently small $\beta > 0$. Let $\beta < \frac{\alpha}{2}$. Then, by assertion (1), there exists constants T_o and A such that $S^{\mu}_{[0,T_o]}(\varphi, \phi) > A$ for functions (φ, ϕ) such that (φ, ϕ) do not get into $\mathcal{E}_{\frac{\alpha}{2}}(O \times O)$, φ do not leave the closed β-neighborhood of G and ϕ does not leave $[-(M+\beta), (M+\beta)]$ during the time $[0, T_o]$. For $(x, y) \in G \times (-M, M)$, we have that either the functions (φ, ϕ) in the set $\Phi_{x,y}(A) = \{(\varphi, \phi) : \varphi_0 = x,\ \phi_0 = y,\ S^{\mu}_{[0,T_o]}(\varphi, \phi) \leq A\}$ get into $\mathcal{E}_{\frac{\alpha}{2}}(O \times O)$ or φ leaves the closed β-neighborhood of G or ϕ leave $[-(M + \beta), (M + \beta)]$ during the time $[0, T_o]$. Then the trajectories of $(X^{\mu,\varepsilon}_t, \eta^{\mu,\varepsilon}_t)$ for which $\zeta_\alpha > T_o$ are at distance not smaller than β from this set $\Phi_{x,y}(A)$. Hence for sufficiently small ε and all $(x, y) \in G \times (-M, M)$, we have

$$P_{x,y}(\zeta_\alpha > T_o) \leq \exp(-\varepsilon^{-1}(A - \gamma))$$

where γ is an arbitrary small number.

Now, we use the Markov property of $(X^{\mu,\varepsilon}_t, \eta^{\mu,\varepsilon}_t)$: For any $(x, y) \in ((G \cup \partial G) \times [-M, M]) \setminus \mathcal{E}_\alpha(O \times O)$,

$$\begin{aligned}
&P_{x,y}(\zeta_\alpha > (n + 1)T_o) \\
&= E_{x,y}[\zeta_\alpha > nT_o; P_{(X^{\mu,\varepsilon}_{nT_o}, \eta^{\mu,\varepsilon}_{nT_o})}(\zeta_\alpha > T_o)] \\
&\leq P_{x,y}(\zeta_\alpha > nT_o) \sup_{x \in G, |y| < M} P_{x,y}(\zeta_\alpha > T_o).
\end{aligned}$$

and we obtain by induction that for any $(x, y) \in ((G \cup \partial G) \times [-M, M]) \setminus \mathcal{E}_\alpha(O \times O)$,

$$\begin{aligned}
P_{x,y}(\zeta_\alpha > T) &\leq P_{x,y}(\zeta_\alpha > (\frac{T}{T_o})T_o) \\
&\leq \left[\sup_{x \in G, |y| \leq M} P_{x,y}(\zeta_\alpha > T_o) \right]^{\frac{T}{T_o}} \\
&\leq \exp\{-\varepsilon^{-1}(\frac{T}{T_o} - 1)(A - \gamma)\} \\
&\leq \exp\{-\varepsilon^{-1} a(T - T_o)\}.
\end{aligned}$$

Hence $a = (A - \gamma)T_o$ where γ is an arbitrary small number.

(3) For any $c > 0$, we can choose T large enough such that $a(T - T_o) > c$ such that for all sufficiently small $\varepsilon > 0$ and any $(x, y) \in ((G \cup \partial G) \times [-M, M]) \setminus \mathcal{E}_{\hat{\alpha}}(O \times O)$, we have the inequality

$$P_{x,y}(\zeta_\alpha > T) \leq \exp\{-\varepsilon^{-1}c\}.$$

This completes the proof.

$\blacksquare$

Lemma 3.4: Let $|x_1|, |x_2|, |y_1|, |y_2| \leq 1$. Then, there exists a smooth function (φ_t, ϕ_t), $t \in [0, T]$, $T = |x_1 - x_2|$ with $\phi_t = \dot{\varphi}_t - b(\varphi_t)$ and $(\varphi_0, \phi_0) = (x_1, y_1)$, $(\varphi_T, \phi_T) = (x_2, y_2)$ such that for μ smaller than $\min\{T, 1\}$, we have

$$S^{\mu}_{[0,T]}(\varphi, \phi) \leq L|x_2 - x_1|.$$

Here L is some constant.

Proof: 1. Since $b(x)$ is Lipschitz continuous, we have that $|b'(x)|$ is bounded for $x \in G \cup \partial G$ (see [20]). That is, there exists a constant, say C_1, such that $|b(x) - b(y)| < C_1|x - y|$. This implies $|b'| < C_1$. Also, we assume $\max_{|x| \leq 1} |b(x)| < C_2$. Define $y_1^* = y_1 + b(x_1)$ and $y_2^* = y_2 + b(x_2)$. Let us put

$$\varphi_t = x_1 + y_1^* t + at^2 + bt^3, \quad \varphi_0 = x_1,\ \varphi_T = x_2,\ \dot{\varphi}_0 = y_1^*,\ \dot{\varphi}_T = y_2^*$$

We can solve for the coefficients to get

$$a = \frac{-(y_2^* + 2y_1^*)}{T} + 3\frac{(x_2 - x_1)}{T^2}, \quad b = \frac{(y_2^* + y_1^*)}{T^2} - \frac{2(x_2 - x_1)}{T^3}.$$

Define $\phi_t = \dot{\varphi}_t - b(\varphi_t)$. It can be verified that $\phi_0 = y_1$ and $\phi_T = y_2$. Furthermore, since $T = |x_1 - x_2|$, it can be checked that $|a| \leq \frac{3C_2+6}{T}$ and $|b| \leq \frac{2C_2+4}{T^2}$ and

$$\begin{aligned} \int_0^T |\dot{\varphi}_t|^2 dt &= \int_0^T |y_1^* + 2at + 3bt^2|^2 dt \\ &\leq 2\int_0^T (|y_1^*|^2 + 4|a|^2t^2 + 9|b|^2t^4) dt \leq L_1 T, \end{aligned}$$

$$
\begin{aligned}
\int_0^T |\ddot{\varphi}|^2 dt &= \int_0^T |2a+6bt|^2 dt \\
&\leq 2\int_0^T (4|a|^2 + 36|b|^2 t^2) dt \leq \frac{L_2}{T},
\end{aligned}
$$

$$
|\varphi_t| \leq |x_1| + |y_1^*|T + |a|T|^2 + |b|T^3 \leq L_3(\max\{T,1\}) = L_4
$$

where L_1, L_2 and L_3 are some constants and $L_4 = L_3(\max\{T,1\})$.

2. Hence, for $\mu \leq \min\{T,1\}$, we have

$$
\begin{aligned}
S^{\mu}_{[0,T]}(\varphi,\phi) &= \frac{1}{2}\int_0^T |(\dot{\varphi}_t - b(\varphi_t) + \mu\frac{d}{dt}(\dot{\varphi}_t - b(\varphi_t)|^2 dt \\
&\leq 2\int_0^T |\dot{\varphi}_t|^2 dt + 2\int_0^T |b(\varphi_t)|^2 + 2\mu^2\int_0^T |\ddot{\varphi}_t|^2 dt \\
&\quad +2\mu^2\int_0^T |b^{'}(\varphi_t)\dot{\varphi}_t|^2 dt \\
&\leq 2\int_0^T |\dot{\varphi}_t|^2 dt + 2(\max_{|x|\leq L_4} |b(x)|)^2 T \\
&\quad +2\mu^2\int_0^T |\ddot{\varphi}_t|^2 dt + 2\mu^2(C_1)^2\int_0^T |\dot{\varphi}_t|^2 dt \\
&\leq 2L_1 T + 2(\max_{|x|\leq L_4} |b(x)|)^2 T + 2\mu^2\frac{(L_2)}{T} + 2\mu^2(C_1)^2(L_1)T \\
&\leq LT = L|x_2 - x_1|.
\end{aligned}
$$

Here L is some constant. This completes the proof.

■

Lemma 3.5: Let C and M be positive constants. Let $|x_1|, |x_2| \leq C$ and $|y_1| \leq M$. Then there exists a smooth function $(\varphi_t, \phi_t), t \in [0,T]$, $T = |x_1 - x_2|$ with $\phi_t = \dot{\varphi} - b(\varphi)$, $(\varphi_0, \phi_0) = (x_1, y_1)$, $\varphi_T = x_2$ such that for $\mu < 1$,

$$
S^{\mu}_{[0,T]}(\varphi,\phi) \leq L|x_2 - x_1|
$$

Here L is some constant depending on M and C.

Proof: 1. Since $b(x)$ is Lipschitz continuous, we assume $|b(x)-b(y)| < C_1|x-y|$. This implies that $|b^{'}| < C_1$. Also, we assume $\max_{|x|\le C}|b(x)| < C_2$. Define $y_1^* = y_1 + b(x_1)$. Put

$$\varphi_t = x_1 + at + b\int_0^t e^{-\frac{s}{\mu}}ds, \quad \varphi_0 = x_1,\ \varphi_T = x_2,\ \dot{\varphi}_0 = y_1^*.$$

We can solve for the coefficients to get

$$a = y_1^* - b,\ \ b = \frac{x_2 - x_1 - y_1^* T}{-(T + \mu e^{-\frac{T}{\mu}})}.$$

Define $\phi_t = \dot{\varphi}_t - b(\varphi_t)$. It can be verified that $\phi_0 = y_1$. Furthermore, since $T = |x_1 - x_2|$, it can be checked that $|b| \le 1 + |y_1^*| \le 1 + M + C_2$ and $|a| \le |y_1^*| + |b| \le 1 + 2M + 2C_2$. Notice that $\dot{\varphi}_t = a + be^{-t/\mu}$ and $\ddot{\varphi}_t = -\frac{1}{\mu}be^{-t/\mu}$. Hence

$$\int_0^T |\dot{\varphi}_t|^2 dt = 2\int_0^T (|a|^2 + |b|^2)dt \le (L_1 M^2 + L_2)T,$$

$$\int_0^T |\dot{\varphi}_t + \mu\ddot{\varphi}_t|^2 dt = \int_0^T |a|^2 dt \le (L_3 M^2 + L_4)T,$$

$$|\varphi_t| = |x_1 + at + b\int_0^t e^{-\frac{1}{\mu}s}ds| \le C + (1 + 2M + 2C_2)2C + (1 + M + C_2) \le C^*$$

where L_1, L_2, L_3, L_4 and C^* are some constants depending on M and C.

2. Hence,

$$\begin{aligned}
S^{\mu}_{[0,T]}(\varphi,\phi) &= \frac{1}{2}\int_0^T |(\dot{\varphi}_t - b(\varphi_t) + \mu\frac{d}{dt}(\dot{\varphi}_t - b(\varphi_t)|^2 dt \\
&\le \int_0^T |\dot{\varphi}_t + \mu\ddot{\varphi}_t|^2 dt + 2\int_0^T |b(\varphi_t)|^2 + 2\mu^2\int_0^T |b^{'}(\varphi_t)\dot{\varphi}_t|^2 dt \\
&\le \int_0^T |\dot{\varphi}_t + \mu\ddot{\varphi}_t|^2 dt + 2(\max_{|x|\le C^*}|b(x)|)^2 T + 2(C_1)^2\int_0^T |\dot{\varphi}_t|^2 dt \\
&\le 2(L_3 M^2 + L_4)T + 2(\max_{|x|\le C^*}|b(x)|)^2 T + 2(C_1)^2(L_1 M^2 + L_4)T \\
&\le LT = L|x_2 - x_1|.
\end{aligned}$$

Here L is some constant depending on M and C. This completes the proof.

■

We now pass to the proof of part (3) of Theorem 3.3. Let $\beta > 0$. We write

$$h^\mu = \min\{V^\mu(x) : x \in \partial G,\ |x - x_o^\mu| \geq \beta\} - V^\mu(x_o^\mu).$$

Since x_o^μ is the only minimum of V^μ, we have $h^\mu > 0$. By Lemma 3.1, we have $\lim_{\mu\downarrow 0} x_o^\mu = x_o$ and $\lim_{\mu\downarrow 0} V_o^\mu = V_o$. Following the same idea in Lemma 3.1, it can be checked that we can pick μ_1 small enough such that $\underline{\text{for any } \mu \text{ smaller than } \mu_1}$, there exists an $h > 0$ independent of μ such that $\underline{h < \min\{h^\mu, 1\}}$ and $\underline{V_o^\mu < V_o + 1}$.

Lemma 3.6: For any $(x, y) \in \gamma$ and for any μ smaller than some $\mu_2(\delta)$, we choose M large enough (say $M^2\mu > V_o + 2$) such that

$$P_{x,y}(Z_1 \in (G \times \partial(-M, M))) \leq \exp\{-\varepsilon^{-1}(V_o + 1.55)\}$$

for ε sufficiently small. (We recall that $\Gamma := \{(x, y) \in R^2 : (x, y) \in \mathcal{E}_\delta(O \times O)\}$; $\gamma := \{(x, y) \in R^2 : (x, y) \in \mathcal{E}_{\frac{\delta}{2}}(O \times O)\}$. Here $\mathcal{E}_\delta(O \times O)$ and $\mathcal{E}_{\frac{\delta}{2}}(O \times O)$ are the δ and $\frac{\delta}{2}$ neighborhood of $O \times O$, respectively. Here δ is a sufficiently small number and $Z_1 = (X_{\tau_1}^{\mu,\varepsilon}, \eta_{\tau_1}^{\mu,\varepsilon})$.)

Proof: 1. The choice of μ and M: To make it clear about the choice of δ, μ and M, let us first prove a statement which will be useful later. We claim that for δ small enough and for any $(\tilde{x}, \tilde{y}) \in \Gamma$ there exists a function $(\varphi_t^{(1)}, \phi_t^{(1)})$, $0 \leq t \leq T_1 = \delta$ with $\phi_t^{(1)} = \dot{\varphi}_t^{(1)} - b(\varphi_t^{(1)})$, $(\varphi_0^{(1)}, \phi_0^{(1)}) = (O \times O)$ and $(\varphi_{T_1}^{(1)}, \phi_{T_1}^{(1)}) = (\tilde{x}, \tilde{y}) \in \Gamma$ such that $S_{[0,T_1]}^\mu(\varphi_t^{(1)}, \phi_t^{(1)}) < 0.2h < 0.2$ provided $\underline{\mu < \mu_2 = \min\{\mu_1, \frac{\delta}{2}\}}$. To prove this statement, let us first assume δ is smaller than 1. We notice that $|\varphi_0^{(1)}|, |\phi_0^{(1)}|, |\varphi_{T_1}^{(1)}|, |\phi_{T_1}^{(1)}| \leq \delta < 1$. By Lemma 3.4, for δ small enough, we can construct such a function $(\varphi_t^{(1)}, \phi_t^{(1)})$ with $S_{[0,T_1]}^\mu(\varphi_t^{(1)}, \phi_t^{(1)}) \leq LT_1 = L\delta < 0.2h < 0.2$.

For any μ smaller than μ_2, let us choose M large enough such that

$$V_o^\eta = \inf_{\phi \in C_{[0,T]}} \{ \int_0^T |\phi_t + \mu \dot{\phi}_t|^2 dt; \phi_0 = O, |\phi_T| = M, T > 0\} = M^2 \mu \geq V_o + 2.$$

2. For any $(x, y) \in \gamma$ and for any μ smaller than μ_2 and large M (say $M^2\mu > V_o + 2$), we have

$$\begin{aligned}
&P_{x,y}(Z_1 \in (G \times \partial[-M, M]) \\
&= E_{x,y} P_{X^{\mu,\varepsilon}_{\sigma_o}, \eta^{\mu,\varepsilon}_{\sigma_o}}(Z_1 \in (G \times \partial[-M, M]) \\
&\leq \sup_{(x,y)\in\Gamma} P_{x,y}(Z_1 \in (G \times \partial[-M, M]) \\
&\leq \sup_{(x,y)\in\Gamma} [P_{x,y}(\tau > T) + P_{x,y}(\tau \leq T, Z_1 \in G \times \partial[-M, M]))]
\end{aligned}$$

By Lemma 3.3, there exists $T > 0$ such that

$$P_{x,y}(\tau > T) \leq \exp\{-\varepsilon^{-1} c\}$$

for any $(x, y) \in \Gamma$ and ε smaller than some ε_o. As c we take, say, $V_o + 1.6$. Now, let us estimate the probability $P_{x,y}(\tau \leq T, Z_1 \in (G \times \partial[-M, M]))$.

Consider the closure of the ϵ-neighborhood of $[-M, M]$: $[-(M - \epsilon), (M - \epsilon)]$. (Here ϵ will be determined later.) For any given T, we claim that no function (φ_t, ϕ_t), $0 \leq t \leq T$ with $\phi_t = \dot{\varphi}_t - b(\varphi_t)$, $(\varphi_0, \phi_0) \in \Gamma$ and $S^\mu_{[0,T]}(\varphi_t, \phi_t) \leq V_o + 1.65$ hits $R \times [-(M - \epsilon), (M - \epsilon)]$. Otherwise, let us assume $|\phi_{t_1}| = M - \epsilon$ for some $t_1 \leq T$. Then $S^\mu_{[0,t_1]}(\varphi_t, \phi_t) \leq S^\mu_{[0,T]}(\varphi_t, \phi_t) \leq V_o + 1.65$.

As we have proved in part 1, for δ smaller than μ_2, there exists a function $(\varphi^{(1)}_t, \phi^{(1)}_t)$, $0 \leq t \leq T_1 = \delta$ with $\phi^{(1)}_t = \dot{\varphi}^{(1)}_t - b(\varphi^{(1)}_t)$, $(\varphi^{(1)}_0, \phi^{(1)}_0) = (O \times O)$ and $(\varphi^{(1)}_{T_1}, \phi^{(1)}_{T_1}) = (\varphi_0, \phi_0) \in \Gamma$ such that $S^\mu_{[0,T_1]}(\varphi^{(1)}_t, \phi^{(1)}_t) \leq 0.2$;

For ϵ small enough, we claim that there exists a function $(\varphi^{(2)}_t, \phi^{(2)}_t)$, $0 \leq t \leq T_2 = \epsilon$ with $\phi^{(2)}_t = \dot{\varphi}^{(2)}_t - b(\varphi^{(2)}_t)$, $(\varphi^{(2)}_0, \phi^{(2)}_0) = (\varphi_{t_1}, \phi_{t_1})$, $|\phi^{(2)}_{T_2} - \phi^{(2)}_0| = \epsilon$,

$|\phi^{(2)}_{T_2}| = M$ such that $S^{\mu}_{[0,T_2]}(\varphi^{(2)}_t, \phi^{(2)}_t) \leq 0.1$. The proof is in the following: Let us define $(\varphi^{(2)}_t, \phi^{(2)}_t)$ as the solution of the following equations:

$$\phi^{(2)}_t = \phi^{(2)}_0 + \frac{\phi^{(2)}_T - \phi^{(2)}_0}{\epsilon} t, \quad 0 \leq t \leq T_2 = \epsilon$$

$$\dot{\varphi}^{(2)}_t = \phi^{(2)}_t + b(\varphi^{(2)}_t), \ \varphi^{(2)}_0 = \varphi_{t_1}$$

Then

$$\begin{aligned} S^{\mu}_{[0,T_2]}(\varphi^{(2)}, \phi^{(2)}) &= \frac{1}{2}\int_0^{T_2} |\dot{\varphi}^{(2)} - b(\varphi^{(2)}) + \mu \frac{d}{dt}(\dot{\varphi}^{(2)} - b(\varphi^{(2)}))|^2 dt \\ &= \frac{1}{2}\int_0^{T_2} |\phi^{(2)}_t + \mu\dot{\phi}^{(2)}_t|^2 dt \leq \int_0^{T_2} (|\phi^{(2)}_0|^2 + t^2 + 1) dt \\ &\leq |M|^2 T_2 + (T_2)^3 + T_2 \leq 0.1 \end{aligned}$$

provided $T_2 = \epsilon$ is small enough. This completes the proof of the statement.

Out of pieces $\varphi^{(1)}, \varphi, \varphi^{(2)}$ we build a new function: $(\hat{\varphi}_t, \hat{\phi}_t) = (\varphi^{(1)}_t, \phi^{(1)}_t)$ for $0 \leq t \leq T_1$; $= (\varphi_{t-T_1}, \phi_{t-T_1})$ for $T_1 \leq t \leq T_1 + t_1$; $= (\varphi^{(2)}_{t-t_1-T_1}, \phi^{(2)}_{t-t_1-T_1})$ for $T_1 + t_1 \leq t \leq T_1 + t_1 + T_2$; Then $(\hat{\varphi}_0, \hat{\phi}_0) = (O, O)$, $|\hat{\phi}_{T_1+t_1+T_2}| = M$ and $S^{\mu}_{[0,T_1+t_1+T_2]}(\hat{\varphi}, \hat{\phi}) \leq V_o + 1.65 + 0.2 + 0.1 < V_o + 1.95$. Notice that $S^{\eta}_{[0,T_1+t_1+T_2]}(\hat{\phi}) = S^{\mu}_{[0,T_1+t_1+T_2]}(\hat{\varphi}, \hat{\phi})$, Contradicting with $V^{\eta}_o \geq M^2\mu > V_o + 2$!

This implies that all functions from $\cup_{(x,y)\in\Gamma}\Phi_{x,y}(V^{\mu}_o + 1.65)$ pass at a distance not smaller than ϵ from $R \times \partial[-M, M]$. Then we obtain that for μ smaller than μ_2 and M large enough (say $M^2\mu > V_o + 2$)) and all $(x, y) \in \Gamma$

$$\begin{aligned} &P_{x,y}(\tau \leq T, Z_1 \in G \times \partial[-M, M])) \\ &\leq P_{x,y}(\rho_{C_{[0,T]}}((X^{\mu,\varepsilon}_t, \eta^{\mu,\varepsilon}_t), \Phi_{x,y}(V_o + 1.65)) \geq \epsilon) \\ &\leq \exp\{-\varepsilon^{-1}(V_o + 1.65 - 0.05)\}. \end{aligned}$$

for sufficiently small ε.

Consequently, for any $(x,y) \in \gamma$,

$$\begin{aligned} & P_{x,y}(Z_1 \in (G \times \partial[-M,M]) \\ & \leq \sup_{(x,y)\in\Gamma} [P_{x,y}(\tau > T) + P_{x,y}(\tau \leq T, Z_1 \in G \times \partial[-M,M]))] \\ & \leq \exp\{-\varepsilon^{-1}(V_o + 0.6) + \exp\{-\varepsilon^{-1}(V_o + 1.65 - 0.05)\} \\ & \leq \exp\{-\varepsilon^{-1}(V_o + 1.55)\} \end{aligned}$$

for ε sufficiently small. This completes the proof.

■

Lemma 3.7: Let us choose the same small μ and large M as in Lemma 3.6. For any $(x,y) \in \gamma$, we have

$$P_{x,y}(Z_1 \in (\partial G \times (-M,M)) \cup (G \times \partial[-M,M])) \geq \exp\{-\varepsilon^{-1}(V_o^\mu + 0.45h)\}$$

for ε sufficiently small.

Proof: Choose a point x_1 outside of $G \cup \partial G$ at a small distance ϵ from x_o^μ (ϵ will be determined later). We claim that, for any point $(x,y) \in \gamma$ and for any small μ we choose (for the choice of μ, please see the Part 1 of the proof of Lemma 3.6), there exists $T > 0$ and a function (φ_t, ϕ_t), $0 \leq t \leq T$ with $\phi_t = \dot{\varphi}_t + b(\varphi_t)$, $(\varphi_0, \phi_0) = (x,y)$, $\varphi_T = x_1$ such that $S^\mu_{[0,T]}(\varphi,\phi) \leq V_o^\mu + 0.4h$.

To prove this statement, let us first construct a function $(\varphi^{(1)}, \phi^{(1)})$ in the following way: Choose a function $(\varphi_t^{(1)}, \phi_t^{(1)})$, $0 \leq t \leq T_1$ with $\phi_t^{(1)} = \dot{\varphi}^{(1)} - b(\varphi^{(1)})$, $(\varphi_0^{(1)}, \phi_0^{(1)}) = (O,O)$, $\varphi_{T_1}^{(1)} = x_o^\mu \in \partial G$ such that $S^\mu_{[0,T_1]}(\varphi^{(1)}, \phi^{(1)}) \leq V_o^\mu + 0.1h$. This function (φ,ϕ) always exists because of the definition of V_o^μ.

We cut off its first portion up to the point $z_1 = (\varphi_{t_1}^{(1)}, \phi_{t_1}^{(1)})$ of the last intersection of $(\varphi_{t_1}^{(1)}, \phi_{t_1}^{(1)})$ with Γ. That is, we introduce the new function $(\varphi_t^{(2)}, \phi_t^{(2)}) =$

$(\varphi^{(1)}_{t_1+t}, \phi^{(1)}_{t_1+t})$, $0 \le t \le T_2 = T_1 - t_1$. We have $\phi^{(2)} := \dot{\varphi}^{(2)}_t - b(\varphi^{(2)}_t)$, $(\varphi^{(2)}_0, \phi^{(2)}_0) = z_1 \in \Gamma$, $\varphi^{(2)}_{T_2} = x^\mu_o$ and $S^\mu_{[0,T_2]}(\varphi^{(2)}, \phi^{(2)}) \le V^\mu_o + 0.1h$.

Moreover, by Lemma 3.4, for the small μ we chose in Lemma 3.6, there exists a function $(\varphi^{(3)}_t, \phi^{(3)}_t)$, $0 \le t \le T_3 = \delta$ with $\phi^{(3)} := \dot{\varphi}^{(3)}_t + b(\varphi^{(3)}_t)$, $(\varphi^{(3)}_0, \phi^{(3)}_0) = (O, O)$, $(\varphi^{(3)}_{T_3}, \phi^{(3)}_{T_3}) = z_1 \in \Gamma$ and $S^\mu_{[0,T_3]}(\varphi^{(3)}, \phi^{(3)}) \le LT_3 = L\delta \le 0.2h$ (see Part 1 of the proof of Lemma 3.6).

With the same idea, by Lemma 3.4, for the choice of small μ in Lemma 3.6 and for any $(x, y) \in \gamma$, there exists a function $(\varphi^{(5)}_t, \phi^{(5)}_t)$, $0 \le t \le T_5 = \frac{\delta}{2}$ with $\phi^{(5)}_t := \dot{\varphi}^{(5)}_t + b(\varphi^{(5)}_t)$ $(\varphi^{(5)}_0, \phi^{(5)}_0) = (x, y) \in \gamma$, $(\varphi^{(5)}_{T_5}, \phi^{(5)}_{T_5}) = (O, O)$ such that $S^\mu_{[0,T_5]}(\varphi^{(5)}, \phi^{(5)}) \le LT_5 = L\frac{\delta}{2} \le 0.1h$.

Recall that M is chosen large enough such that $M^2\mu \ge V_o + 2$. We claim that for such large M, $\|\phi^{(1)}_t\|_{C_{[0,T_1]}} < M$. Otherwise, we assume there exists a $t_o = \min\{t : |\phi^{(1)}_t| = M\}$ with $t_o \le T_1$. Then

$$\begin{aligned}
S^\mu_{[0,T_1]}(\varphi^{(1)}, \phi^{(1)}) &= \frac{1}{2}\int_0^{T_1} |\dot{\varphi}^{(1)} - b(\varphi^{(1)}) + \mu\frac{d}{dt}(\dot{\varphi}^{(1)} - b(\varphi^{(1)}))|^2 dt \\
&= \int_0^{T_1} |\phi^{(1)}_t + \mu\dot{\phi}^{(1)}_t|^2 dt \ge \int_0^{t_o} |\phi^{(1)}_t + \mu\dot{\phi}^{(1)}_t|^2 dt \\
&\ge M^2\mu \ge V_o + 2.
\end{aligned}$$

This contradicts with $S^\mu_{[0,T_1]}(\varphi^{(1)}, \phi^{(1)}) \le V^\mu_o + 0.1h < V_o + 1.1!$ Hence $|\phi^{(1)}_{T_1}| < M$ and $|\phi^{(2)}_{T_2}| = |\phi^{(1)}_{T_1}| < M$.

Now, we construct another function $(\varphi^{(4)}_t, \phi^{(4)}_t)$. We claim that for ϵ small enough, there exists a function $(\varphi^{(4)}_t, \phi^{(4)}_t)$, $0 \le t \le T_4 = \epsilon$ with $\phi^{(4)} := \dot{\varphi}^{(4)}_t + b(\varphi^{(4)}_t)$, $(\varphi^{(4)}_0, \phi^{(4)}_0) = (x^\mu_o, \phi^{(2)}_{T_2})$, $\varphi^{(4)}_{T_4} = x_1$ and $S^\mu_{[0,T_4]}(\varphi^{(4)}, \phi^{(4)}) \le 0.1h$. Since G is a bounded domain, there exists a constant C such that $|\varphi^{(4)}_0|, |\varphi^{(4)}_{T_4}| < C$. Also, notice that $|\phi_0^{(4)}| < M$. This statement can be proved by applying Lemma 3.5.

We construct the function (φ_t, ϕ_t) out of the pieces $(\varphi^{(1)}, \phi^{(5)}), (\varphi^{(3)}, \phi^{(3)})$, $(\varphi^{(2)}, \phi^{(2)})$ and $(\varphi^{(4)}, \phi^{(4)})$: $(\varphi_t, \phi_t) = (\varphi_t^{(5)}, \phi_t^{(5)})$ for $0 \le t < T_5$; $= (\varphi_{t-T_5}^{(3)}, \phi_{t-T_5}^{(3)})$ for $T_5 \le t < T_t + T_3$; $= (\varphi_{t-T_5-T_3}^{(2)}, \phi_{t-T_5-T_3}^{(2)})$ for $T_5 + T_3 \le t < T_5 + T_3 + T_2$; $= (\varphi_{t-T_5-T_3-T_2}^{(4)}, \phi_{t-T_5-T_3-T_2}^{(4)})$ for $T_5 + T_3 + T_2 \le t \le T := T_5 + T_3 + T_2 + T_4$. Then for μ smaller than μ_2 we have constructed a function (φ_t, ϕ_t), $0 \le t \le T$ with $(\varphi_0, \phi_0) = (x, y)$, $\varphi_T = x_1$ and

$$S_{[0,T]}^{\mu}(\varphi, \phi) < V_o^{\mu} + 0.4h.$$

Now, we choose positive $\alpha > 0$ smaller than $\frac{\delta}{4}$ and ϵ. Then for $\mu < \mu_2$ and for all $(x, y) \in \gamma$, we obtain

$$P_{x,y}(\rho_{C_{[0,T]}}((X_t^{\mu,\varepsilon}, \eta_t^{\mu,\varepsilon}), (\varphi_t, \phi_t)) < \alpha) \ge \exp\{-\varepsilon^{-1}(V_o^{\mu} + 0.4h + 0.05h)\}$$

for ε sufficiently small.

On the other hand, if the trajectory of $X_t^{\mu,\varepsilon}$ passes at a distance smaller than α from the curve φ_t it hits the α-neighborhood of $\varphi_T = x_1$. Hence $X_t^{\mu,\varepsilon}$ intersects ∂G on the way, not hitting γ after reaching Γ. If $\phi_t^{\mu,\varepsilon}$ hits $\partial[-M, M]$ before $X_t^{\mu,\varepsilon}$ hit ∂G, then $Z_1 \in (G \times \partial[-M, M])$. If not, then $Z_1 \in (\partial G \times (-M, M))$. Consequently, for any $(x, y) \in \gamma$,

$$P_{x,y}(Z_1 \in (\partial G \times (-M, M)) \cup (G \times \partial[-M, M])) \ge \exp\{-\varepsilon^{-1}(V_o^{\mu} + 0.45h)\}$$

for ε sufficiently small.

$\blacksquare$

Remark: For any $\alpha > 0$ and any $(x, y) \in \Gamma$, there exists a $T_{x,y} > 0$ and a function (φ_t, ϕ_t) such that $\phi_t = \dot{\varphi}_t + b(\varphi_t)$, $(\varphi_0, \phi_0) = (x, y)$ and φ_t reaches the exterior of the ϵ-neighborhood of G at time $T(x,y)$ such that $S^{\mu}_{[0,T(x,y)]}(\varphi,\phi) \leq V^{\mu}_o + \alpha$ provided μ is smaller than some $\mu_3(\alpha)$. Such a function can be constructed in the same way as in Lemma 3.7. Let us choose a small $\mu < \mu_3$ and large M (say, $M^2\mu > V_0 + 2 + \alpha$). Then, by the same arguments as in the proof of Lemma 3.7, we have $\max_{0\leq t\leq T(x,y)} |\phi_t| < M$.

Lemma 3.8: Let us choose the same μ and M as in Lemma 3.6. Then for any $(x,y) \in \gamma$, we have

$$P_{x,y}(Z_1 \in (\partial G \setminus \mathcal{E}_\beta(x^{\mu}_o)) \times (-M,M)) \leq \exp\{-\varepsilon^{-1}(V^{\mu}_o + 0.55h)\}$$

for ε sufficiently small.

Proof: We recall that $Z_1 = (X^{\mu,\varepsilon}_{\tau_1}, \eta^{\mu,\varepsilon}_{\tau_1})$, where

$$\tau_1 = \inf\{t > \sigma_0 : (X^{\mu,\varepsilon}_t, \eta^{\mu,\varepsilon}_t) \in \gamma \cup (\partial G \times (-M,M)) \cup (G \times \partial[-M,M])\}$$

We introduce the notation

$$\tau := \inf\{t > 0 : X^{\mu,\varepsilon}_t \in \gamma \cup (\partial G \times (-M,M) \cup (G \times \partial[-M,M]))\}.$$

Let us use the strong Markov property of $(X^{\mu,\varepsilon}_t, \eta^{\mu,\varepsilon}_t)$ with respect to the Markov time σ_0. Since $(X^{\delta,\varepsilon}_{\sigma_0}, \eta^{\delta,\varepsilon}_{\sigma_0}) \in \Gamma$, we obtain that for any $(x,y) \in \gamma$,

$$\begin{aligned}
&P_{x,y}(Z_1 \in (\partial G \setminus \mathcal{E}_\beta(x^{\mu}_o) \times (-M,M))) \\
&= E_{x,y} P_{X^{\mu,\varepsilon}_{\sigma_o}, \eta^{\mu,\varepsilon}_{\sigma_o}}(Z_1 \in (\partial G \setminus \mathcal{E}_\beta(x^{\mu}_o) \times (-M,M))) \\
&\leq \sup_{(x,y)\in\Gamma} P_{x,y}(Z_1 \in (\partial G \setminus \mathcal{E}_\beta(x^{\mu}_o) \times (-M,M))).
\end{aligned}$$

We estimate the latter probability.

By Lemma 3.3, for any $c > 0$ there exists $T > 0$ such that

$$P_{x,y}(\tau > T) \leq \exp\{-\varepsilon^{-1}c\}$$

for any $(x, y) \in \Gamma$ and ε smaller than some ε_o. As c we take, say, $V_o + 1.6$. To obtain the estimate needed to prove Lemma 3.8, it remains to estimate the probability $P_{x,y}(\tau \leq T, Z_1 \in ((\partial G \setminus \mathcal{E}_\beta(x_o^\mu)) \times (-M, M)))$.

Consider the closure of the ϵ-neighborhood of $\partial G \setminus \mathcal{E}_\beta(x_o^\mu)$. We denote it by K. We claim that no function (φ_t, ϕ_t), $0 \leq t \leq T$ hits $K \times R$ where $\phi_t = \dot{\varphi}_t - b(\varphi_t)$, $(\varphi_0, \phi_0) \in \Gamma$ and $S^\mu_{[0,T]}(\varphi_t, \phi_t) \leq V_o^\mu + 0.65h$. Otherwise, let us assume $\varphi_{t_1} \in K$ for some $t_1 \leq T$. Then $S^\mu_{[0,t_1]}(\varphi_t, \phi_t) \leq S^\mu_{[0,T]}(\varphi_t, \phi_t) \leq V_o^\mu + 0.65h$.

Moreover, by Lemma 3.4, for the small μ we chose in Lemma 3.6, there exists a function $(\varphi_t^{(1)}, \phi_t^{(1)})$, $0 \leq t \leq T_1$ with $\phi_t^{(1)} = \dot{\varphi}_t^{(1)} - b(\varphi_t^{(1)})$, $(\varphi_0^{(1)}, \phi_0^{(1)}) = (O \times O)$ and $(\varphi_{T_1}^{(1)}, \phi_{T_1}^{(1)}) = (\varphi_0, \phi_0) \in \Gamma$ such that $S^\mu_{[0,T_1]}(\varphi_t^{(1)}, \phi_t^{(1)}) \leq 0.2h$ (see part 1 of the proof of Lemma 3.6).

For the small enough μ and large enough M we chose in Lemma 3.6, we claim that $|\phi_{t_1}| < M$. To prove it, we build a new function out of the pieces $(\varphi^{(1)}, \phi^{(1)})$, (φ, ϕ), : $(\tilde{\varphi}_t, \tilde{\phi}_t) = (\varphi_t^{(1)}, \phi_t^{(1)})$ for $0 \leq t \leq T_1$; $= (\varphi_{t-T_1}, \phi_{t-T_1})$ for $T_1 \leq t \leq T_1 + t_1$. Then $\tilde{\phi}_t = \dot{\tilde{\varphi}}_t - b(\tilde{\varphi})$, $(\tilde{\varphi}_0, \tilde{\phi}_0) = (O, O)$ and $S_{[0,T_1+t_1]}(\tilde{\varphi}_t, \tilde{\phi}_t) \leq V_o^\mu + 0.85h < V_o + 1.85$. By the same argument we use in the proof in Lemma 3.7, it can be proved that $|\phi_{t_1}| = |\tilde{\phi}_{T_1+t_1}| < M$.

For ϵ small enough, we claim that there exists a function $(\varphi_t^{(2)}, \phi_t^{(2)})$, $0 \leq t \leq T_2 = \epsilon$ with $\phi_t^{(2)} = \dot{\varphi}_t^{(2)} - b(\varphi_t^{(2)})$, $(\varphi_0^{(2)}, \phi_0^{(2)}) = (\varphi_{t_1}, \phi_{t_1})$, $\varphi_{T_2}^{(2)} \in \partial G \setminus \mathcal{E}_\beta(x_o^\mu)$ such that $S^\mu_{[0,T_2]}(\varphi_t^{(2)}, \phi_t^{(2)}) \leq 0.1h$. Notice that $|\phi_0^{(2)}| < M$. This statement can be proved by applying Lemma 3.5.

Out of pieces $(\varphi^{(1)}, \phi^{(1)})$, (φ, ϕ), $(\varphi^{(2)}, \phi^{(2)})$ we build a new function: $(\hat{\varphi}_t, \hat{\phi}_t) = (\varphi_t^{(1)}, \phi_t^{(1)})$ for $0 \leq t \leq T_1$; $= (\varphi_{t-T_1}, \phi_{t-T_1})$ for $T_1 \leq t \leq T_1 + t_1$; $= (\varphi^{(2)}_{t-t_1-T_1}, \phi^{(2)}_{t-t_1-T_1})$

for $T_1 + t_1 \le t \le T_1 + t_1 + T_2$. Then $(\hat{\varphi}_0, \hat{\phi}_0) = (O, O)$, $\hat{\varphi}_{T_1+t_1+T_2} \in (\partial G \setminus \mathcal{E}_\beta(x_o^\mu))$ and $S^\mu_{[0,T_1+t_1+T_2]}(\hat{\varphi}, \hat{\phi}) \le V_o^\mu + 0.65h + 0.2h + 0.1h$. Contradiction!

This implies that all functions from $\cup_{(x,y)\in\Gamma}\Phi_{x,y}(V_o^\mu + 0.65h)$ pass at a distance not smaller than ϵ from $(\partial G \setminus \mathcal{E}_\beta(x_o^\mu)) \times R$. Then we obtain that for all $(x,y) \in \Gamma$ that

$$
\begin{aligned}
&P_{x,y}(\tau \le T, Z_1 \in (\partial G \setminus \mathcal{E}_\beta(x_o^\mu) \times (-M, M))) \\
&\le P_{x,y}(\rho_{C_{[0,T]}}((X_t^{\mu,\varepsilon}, \eta_t^{\mu,\varepsilon}), \Phi_{x,y}(V_o^\mu + 0.65h) \ge \epsilon) \\
&\le \exp\{-\varepsilon^{-1}(V_o^\mu + 0.65h - 0.05h)\}.
\end{aligned}
$$

for sufficiently small ε. Hence, for any $(x,y) \in \gamma$,

$$
\begin{aligned}
&P_{x,y}(Z_1 \in (\partial G \setminus \mathcal{E}_\beta(x_o^\mu)) \times (-M, M)) \\
&\le \sup_{(x,y)\in\Gamma} P_{x,y}(Z_1 \in (\partial G \setminus \mathcal{E}_\beta(x_o^\mu)) \times (-M, M)) \\
&\le \sup_{(x,y)\in\Gamma} [P_{x,y}(\tau > T) + P_{x,y}(\tau \le T, Z_1 \in (\partial G \setminus \mathcal{E}_\beta(x_o^\mu) \times (-M, M)))] \\
&\le \exp\{-\varepsilon^{-1}(V_o + 1.6) + \exp\{-\varepsilon^{-1}(V_o^\mu + 0.65h - 0.05h)\} \\
&\le \exp\{-\varepsilon^{-1}(V_o^\mu + 0.55h)\}
\end{aligned}
$$

for ε sufficiently small. The last inequality is because $V_o^\mu + 0.6h < V_o + 1.6$. This completes the proof.

■

Now, let us prove part (3) of Theorem 3.3. Recall that

$$
\begin{aligned}
\tau^X &= \min\{t : X_t^{\mu,\varepsilon} \in \partial G\} = \min\{t : (X_t^{\mu,\varepsilon}, \eta_t^{\mu,\varepsilon}) \in \partial G \times R\}, \\
\tau &:= \inf\{t > 0 : X_t^{\mu,\varepsilon} \in \gamma \cup (\partial G \times (-M, M) \cup (G \times \partial[-M, M]))\}.
\end{aligned}
$$

Let us define

$$
\tau^{X,\eta} = \inf\{t > 0 : (X_t^{\mu,\varepsilon}, \eta_t^{\mu,\varepsilon}) \in (\partial G \times (-M, M)) \cup (G \times \partial[-M, M])\}.
$$

Then, for any $x \in G$ and $|y| \le N$

$$\begin{aligned}
P_{x,y}(|X^{\mu,\varepsilon}_{\tau^X} - x^\mu_o| > \beta) &= P_{x,y}(|X^{\mu,\varepsilon}_{\tau^X} - x^\mu_o| > \beta, (X^{\mu,\varepsilon}_{\tau^{X,\eta}}, \eta^{\mu,\varepsilon}_{\tau^{X,\eta}}) \in \partial G \times (-M, M)) \\
&\quad + P_{x,y}(|X^{\mu,\varepsilon}_{\tau^X} - x^\mu_o| > \beta, (X^{\mu,\varepsilon}_{\tau^{X,\eta}}, \eta^{\mu,\varepsilon}_{\tau^{X,\eta}}) \in G \times \partial[-M, M]) \\
&\le P_{x,y}(|X^{\mu,\varepsilon}_{\tau^{X,\eta}} - x^\mu_o| > \beta, (X^{\mu,\varepsilon}_{\tau^{X,\eta}}, \eta^{\mu,\varepsilon}_{\tau^{X,\eta}}) \in \partial G \times (-M, M)) \\
&\quad + P_{x,y}((X^{\mu,\varepsilon}_{\tau^{X,\eta}}, \eta^{\mu,\varepsilon}_{\tau^{X,\eta}}) \in G \times \partial[-M, M])
\end{aligned}$$

1. Let us first estimate the probability $P_{x,y}(|X^{\mu,\varepsilon}_{\tau^{X,\eta}} - x^\mu_o| > \beta, (X^{\mu,\varepsilon}_{\tau^{X,\eta}}, \eta^{\mu,\varepsilon}_{\tau^{X,\eta}}) \in \partial G \times (-M, M))$.

It follows from Lemma 3.7 and 3.8 that, for any small $\underline{\mu < \mu_2}$, large enough M $\underline{\text{(say } M^2\mu > V_o + 2)}$ and any $(x, y) \in \gamma$

$$\begin{aligned}
&P_{x,y}(Z_1 \in (\partial G \setminus \mathcal{E}_\beta(x^\mu_o)) \times (-M, M)) \\
&\le P_{x,y}(Z_1 \in (\partial G \times (-M, M)) \cup (G \times \partial[-M, M])) \exp(-\varepsilon^{-1} 0.1h).
\end{aligned}$$

for ε sufficiently small. We denote by ν the smallest n for which $Z_n \in (\partial G \times (-M, M)) \cup (G \times \partial[-M, M])$. Using the strong Markov property, for any $(x, y) \in \gamma$, we find that

$$\begin{aligned}
&P_{x,y}(|X^{\mu,\varepsilon}_{\tau^{X,\eta}} - x^\mu_o| > \beta, (X^{\mu,\varepsilon}_{\tau^{X,\eta}}, \eta^{\mu,\varepsilon}_{\tau^{X,\eta}}) \in \partial G \times (-M, M))) \\
= &\sum_{n=1}^\infty P_{x,y}(\nu = n, Z_n \in (\partial G \setminus \mathcal{E}_\beta(x^\mu_o)) \times (-M, M)) \\
= &\sum_{n=1}^\infty E_{x,y}(Z_1 \in \gamma, ..., Z_{n-1} \in \gamma) P_{Z_{n-1}}(Z_1 \in (\partial G \setminus \mathcal{E}_\beta(x^\mu_o)) \times (-M, M)) \\
\le &\sum_{n=1}^\infty E_{x,y}(Z_1 \in \gamma, ..., Z_{n-1} \in \gamma) \\
&\times P_{Z_{n-1}}(Z_1 \in (\partial G \times (-M, M)) \cup (G \times \partial[-M, M])) e^{-\varepsilon^{-1} 0.1h} \\
= &\sum_{n=1}^\infty P_{x,y}(\nu = n) \exp(-\varepsilon^{-1} 0.1h) \\
= &\exp(-\varepsilon^{-1} 0.1h).
\end{aligned}$$

For any $x \in G$ and $|y| \le N$, we have

$$
\begin{aligned}
&P_{x,y}(|X^{\mu,\varepsilon}_{\tau^{X,\eta}} - x^\mu_o| > \beta, (X^{\mu,\varepsilon}_{\tau^{X,\eta}}, \eta^{\mu,\varepsilon}_{\tau^{X,\eta}}) \in \partial G \times (-M, M)) \\
&\le P_{x,y}((X^{\mu,\varepsilon}_\tau, \eta^{\mu,\varepsilon}_\tau) \in (\partial G \times (-M, M)) \cup (G \times \partial[-M, M])) \\
&\quad + P_{x,y}((X^{\mu,\varepsilon}_\tau, \eta^{\mu,\varepsilon}_\tau) \in \gamma, |X^{\mu,\varepsilon}_{\tau^{X,\eta}} - x^\mu_o| > \beta, (X^{\mu,\varepsilon}_{\tau^{X,\eta}}, \eta^{\mu,\varepsilon}_{\tau^{X,\eta}}) \in \partial G \times (-M, M))
\end{aligned}
$$

The first probability converges to zero because of Lemma 3.2 and the assumption that (X^μ_t, η^μ_t) never leaves the domain $G \times [-N, N]$ with $x \in G$ and $|y| < N$. Using the strong Markov property, it turns out that the second term

$$
\begin{aligned}
&\le \sup_{(x,y)\in\gamma} P_{x,y}(|X^{\mu,\varepsilon}_{\tau^{\mu,\varepsilon}} - x^\mu_o| > \beta, , (X^{\mu,\varepsilon}_{\tau^{X,\eta}}, \eta^{\mu,\varepsilon}_{\tau^{X,\eta}}) \\
&\le \exp(-\varepsilon^{-1} 0.1h) \downarrow 0
\end{aligned}
$$

as ε tends to 0.

2. Let us now estimate the probability $P_{x,y}((X^{\mu,\varepsilon}_{\tau^{X,\eta}}, \eta^{\mu,\varepsilon}_{\tau^{X,\eta}}) \in G \times \partial[-M, M])$.

Since $V_o + 1.55 > V^\mu_o + 0.55h$, it follows from Lemma 3.6 that, for any small $\underline{\mu < \mu_2}$, large enough M $\underline{\text{(say } M^2\mu > V_o + 2)}$ and any $(x, y) \in \gamma$,

$$
P_{x,y}(Z_1 \in G \times \partial[-M, M]) \le \exp\{-\varepsilon^{-1}(V_o + 1.55)\} \le \exp\{-\varepsilon^{-1}(V^\mu_o + 0.55h)\}.
$$

Consequently, it follows from Lemma 3.6 and 3.7 that

$$
\begin{aligned}
&P_{x,y}(Z_1 \in G \times \partial[-M, M]) \\
&\le P_{x,y}(Z_1 \in (\partial G \times (-M, M)) \cup (G \times \partial[-M, M])) \exp(-\varepsilon^{-1} 0.1h).
\end{aligned}
$$

for ε sufficiently small. By the same idea as in part 1, it can be obtained that for any $x \in G$ and $|y| \le N$

$$
P_{x,y}((X^{\mu,\varepsilon}_{\tau^{X,\eta}}, \eta^{\mu,\varepsilon}_{\tau^{X,\eta}}) \in G \times \partial[-M, M]) \to 0. \tag{3.3}
$$

as ε tends to 0. This complete the proof of part (3) of theorem 3.3 provided $\mu < \mu_2$.

■

Lemma 3.9: Let the assumptions 1-3 be satisfied. Then for any $x \in G$, $|y| \leq N$ and $\alpha > 0$, there exists a $\mu_4 > 0$ such that for any μ smaller than μ_4, we have

$$\lim_{\varepsilon \downarrow 0} \varepsilon \ln E_{x,y} \tau^X \leq V_o^\mu + \alpha.$$

Proof: 1. We choose positive numbers δ, ϵ, T_1, T_2 and M^* such that the following three conditions are satisfied:

First, for any (x, y) lying in the ball Γ (recall that Γ is the δ-neighborhood of $O \times O$ and δ small enough), there exists $T_2 > 0$ and a function (φ_t, ϕ_t) with $\phi_t = \dot{\varphi}_t - b(\varphi_t)$, $(\varphi_0, \phi_0) = (x, y)$ where φ_t reaches the exterior of the ϵ-neighborhood of G at time $T(x, y) \leq T_2$ and (φ_t, ϕ_t) does not hit the $\frac{\delta}{2}$-neighborhood of $O \times O$ after exit from $G \times R$ and $S_{[0,T(x,y)]}(\varphi, \phi) \leq V_o^\mu + \frac{\alpha}{2}$ provided $\underline{\mu \text{ is smaller than some } \mu_3 = \mu_3(\alpha)}$. For the construction of this function (φ_t, ϕ_t), please refer to the remark of lemma 3.7.

Second, we choose M^* large enough such that for any $x \in G$, $|y| > M^*$, the trajectory of X_t^μ reaches the exterior of the ϵ-neighborhood of G before time $T = 1$ $\underline{\text{provided } \mu < \frac{1}{2}}$, where (X_t^μ, η_t^μ) is the unperturbed system starting at points (x, y). To prove this, we notice that since G is a bounded domain, there exists a constant C such that $G \subset [-C, C]$. From the equation

$$\begin{cases} \dot{X}_t^\mu = b(X_t^\mu) + \eta_t^\mu, \;\; X_0^\mu = x \in R^d \\ \mu \dot{\eta}_t^\mu = -\eta_t^\mu, \;\; \eta_0^\mu = y \in R^d. \end{cases}$$

we can solve for (X^μ_t, η^μ_t), finding that $\eta^\mu_t = ye^{-\frac{1}{\mu}t}$ and

$$\begin{aligned}
|X^\mu_t| = & \left| x + \int_0^t b(X^\mu_s)ds + \int_0^t \eta^\mu_s ds \right| \\
= & \ x + \int_0^t b(X^\mu_s)ds + y(1 - \mu e^{-\frac{1}{\mu}t}) \\
\geq & \ |y|(1 - \mu e^{-\frac{1}{\mu}t}) - |x| - \int_0^t |b(X^\mu_s)|ds \\
\geq & \ M^*(1-\mu) - C - \int_0^t |b(X^\mu_s)|ds \\
\geq & \ \frac{M^*}{2} - C - \int_0^t |b(X^\mu_s)|ds
\end{aligned}$$

provided $\mu < \frac{1}{2}$. We claim that for M^* large enough we have $X^\mu_1 > C + \epsilon$. Otherwise, we assume that $|X^\mu_t|_{C_{[0,1]}} \leq C + \epsilon$ and we assume $\max_{|x| \leq C+2\epsilon} |b(x)| < C_1$. Then $|X^\mu_1| \geq \frac{M^*}{2} - C - C_1 > C + \epsilon$ for M^* large enough, contradiction.

Third, all trajectories of (X^μ_t, η^μ_t), the unperturbed system, starting at points $(x, y) \in (G \cup \partial G) \times [-(N + M^*), (N + M^*)]$ hit the $\delta/2$-neighborhood of $O \times O$ before time T_1 (let us take T_1 larger than 1) and after T_1, they don't leave the neighborhood. This assumption is true because $O \times O$ is an asymptotically stable equilibrium position, $G \times (-(N+M), (N+M))$ is attracted to $O \times O$ and $(b(x) \cdot n(x))|_{x \in \partial G} < 0$.

2. From the construction of the (φ_t, ϕ_t) and the definition of an action functional, we obtain that for any $(x, y) \in \Gamma$ and any μ smaller than $\mu_4 = \min\{\mu_3, \frac{1}{2}\}$

$$\begin{aligned}
P_{x,y}(\rho_{[0,T(x,y)]}(X^{\mu,\varepsilon}_t, \eta^{\mu,\varepsilon}_t), (\varphi_t, \phi_t)) < \epsilon) \ \geq & \ \exp\{-\varepsilon^{-1}(S_{[0,T(x,y)]}(\varphi_t, \phi_t) + \frac{\alpha}{2})\} \\
\geq & \ \exp\{-\varepsilon^{-1}(V^\mu_o + \alpha)\}
\end{aligned}$$

for ε sufficiently small. Since $\varphi_{T(x,y)}$ does not belong to the ϵ-neighborhood of G, then for any $(x, y) \in \Gamma$ and for μ smaller than some μ_4,

$$P_{x,y}(\tau^X < T_2) \geq P_{x,y}(\tau^X < T(x,y)) \geq \exp\{-\varepsilon^{-1}(V^\mu_o + \alpha)\}.$$

for ε sufficiently small.

Denote by σ the first entrance time for $(X_t^{\mu,\varepsilon}, \eta_t^{\mu,\varepsilon})$ of Γ: $\sigma := \min\{t : (X_t^{\mu,\varepsilon}, \eta_t^{\mu,\varepsilon}) \in \Gamma\}$. Using the strong Markov property of $(X_t^{\mu,\varepsilon}, \eta_t^{\mu,\varepsilon})$, for any $(x,y) \in G \times [-(N+M^*), (N+M^*)]$ and for any μ smaller than μ_4, we obtain

$$
\begin{aligned}
P_{x,y}(\tau^X < T_1 + T_2) &\geq P_{x,y}(\sigma < T_1, \tau^X < T_1 + T_2) \\
&= P_{x,y}(\sigma < T_1) P_{(X_\sigma^{\mu,\varepsilon}, \eta_\sigma^{\mu,\varepsilon})}(\tau^X < T_2) \\
&\geq P_{x,y}(\sigma < T_1) \inf_{(x,y)\in\Gamma} P_{x,y}(\tau^X < T_2) \\
&\geq P_{x,y}(\sigma < T_1) \exp\{-\varepsilon^{-1}(V_o^\mu + \alpha)\} \\
&\geq \frac{1}{2} \exp\{-\varepsilon^{-1}(V_o^\mu + \alpha)\}
\end{aligned}
$$

for ε sufficiently small. Here we use the fact that the trajectories of $(X_t^{\mu,\varepsilon}, \eta_t^{\mu,\varepsilon})$ converge in probability to (X_t^μ, η_t^μ) uniformly on $[0, T_1]$ as $\varepsilon \downarrow 0$ for $(x,y) \in G \times [-(N+M^*), (N+M^*)]$.

For any $x \in G$ and $|y| > (N+M^*)$, and for any μ smaller than μ_4, we obtain that for any $c > 0$

$$
\begin{aligned}
P_{x,y}(\tau^X < T_1 + T_2) &\geq P_{x,y}(\tau^X < 1) \\
&\geq P_{x,y}(\rho_{[0,1]}(X_t^{\mu,\varepsilon}, \eta_t^{\mu,\varepsilon}), (X_t^\mu, \eta_t^\mu)) < \frac{\epsilon}{2}) \\
&\geq \exp\{-\varepsilon^{-1} c\}
\end{aligned}
$$

for ε sufficiently small. Let us take $c = \frac{\alpha}{2}$.

Consequently, for any $(x,y) \in G \times R$ and for any μ smaller than μ_4,

$$
\begin{aligned}
P_{x,y}(\tau^X < T_1 + T_2) &\geq \min\{\frac{1}{2} \exp\{-\varepsilon^{-1}(V_o^\mu + \alpha)\}, \exp\{-\varepsilon^{-1} \frac{\alpha}{2}\}\} \\
&\geq \frac{1}{2} \exp\{-\varepsilon^{-1}(V_o^\mu + \alpha)\}
\end{aligned}
$$

provided ε small enough.

3. For any $(x,y) \in G \times [-N,N]$ and for any μ smaller than μ_4, using the Markov property of $(X_t^{\mu,\varepsilon}, \eta_t^{\mu,\varepsilon})$, we obtain,

$$\begin{aligned}
E_{x,y}\tau^{X,\eta} &\leq \sum_{n=0}^{\infty}(n+1)(T_1+T_2)P_{x,y}(n(T_1+T_2) < \tau^X < (n+1)(T_1+T_2)) \\
&= (T_1+T_2)\sum_{n=0}^{\infty} P_{x,y}(\tau^X > n(T_1+T_2)) \\
&\leq (T_1+T_2)\sum_{n=0}^{\infty}(\sup_{(x,y)\in G\times R} P_{x,y}(\tau^X > T_1+T_2))^n \\
&= (T_1+T_2)\sum_{n=0}^{\infty}(1 - \inf_{(x,y)\in G\times R} P_{x,y}(\tau^X \leq T_1+T_2))^n \\
&= (T_1+T_2)(\inf_{(x,y)\in G\times R} P_{x,y}(\tau^X \leq T_1+T_2))^{-1} \\
&\leq 2(T_1+T_2)\exp\{\frac{1}{\varepsilon}(V_o^\mu + \alpha)\}
\end{aligned}$$

for ε small enough. This completes the proof.

■

Lemma 3.10: Let the assumptions 1-3 be satisfied. We choose the same μ_4 as in Lemma 3.9. Then for any $(x,y) \in G \times [-N,N]$, $\alpha > 0$ and for any $\mu < \mu_4$, we have

$$\lim_{\varepsilon \downarrow 0} \varepsilon \ln E_{x,y}\tau^X > V_o^\mu - \alpha.$$

for ε small enough.

Proof: Recall that

$$\tau^X = \min\{t : X_t^{\mu,\varepsilon} \in \partial G\} = \min\{t : (X_t^{\mu,\varepsilon}, \eta_t^{\mu,\varepsilon}) \in \partial G \times R\}$$

$$\tau^{X,\eta} = \inf\{t > 0 : (X_t^{\mu,\varepsilon}, \eta_t^{\mu,\varepsilon}) \in (\partial G \times (-M,M)) \cup (G \times \partial[-M,M])\}$$

Here M is a positive constant large enough (say $M^2\mu > V_o+2$). Since $\tau^X > \tau^{X,\eta}$, it is sufficient to show $\lim_{\varepsilon\downarrow 0}\varepsilon \ln E_{x,y}\tau^{X,\eta} > V_o^\mu - \alpha$

Recall that we introduced the Markov times τ_k and σ_k and Z_n is a Markov chain in the phase space $\gamma \cup (\partial G \times (-M, M)) \cup (G \times \partial[-M, M])$ where γ is the $\delta/2$-neighborhood of $O \times O$. Then

$$\begin{aligned}
&P_{x,y}(Z_1 \in (\partial G \times (-M, M)) \cup (G \times \partial[-M, M])) \\
&\leq \sup_{(x,y)\in\Gamma} P_{x,y}(\tau_1 = \tau^{X,\eta}) \\
&= \sup_{(x,y)\in\Gamma} P_{x,y}(\tau_1 = \tau^{X,\eta} < T) + P_{x,y}(\tau_1 = \tau^{X,\eta} \geq T).
\end{aligned}$$

In order to estimate the first probability, we note that for large M(say $M^2\mu > V_o + 2$), the trajectories of $(X_t^{\mu,\varepsilon}, \eta_t^{\mu,\varepsilon})$ starting from $(x, y) \in \Gamma$, $t \leq T$, for which $\tau_1 = \tau^{X,\eta} < T$, are at a positive distance from the set $\{(\varphi, \phi) \in C_{[0,T]} : (\varphi_0, \phi_0) = (x, y) \in \Gamma, S_{[0,T]}(\varphi, \phi) < V_o^\mu - \frac{\theta}{2}\}$ provided $\theta > 0$ is arbitrary and δ sufficiently small enough. Then,

$$P_{x,y}(\tau_1 = \tau^{X,\eta} < T) \leq \exp\{-\varepsilon^{-1}(V_o^\mu - \theta)\} \tag{3.4}$$

for ε sufficiently small.

Following Lemma 3.3, T can be chosen large enough such that

$$\sup_{(x,y)\in\Gamma} P_{x,y}(\tau_1 = \tau^{X,\eta} > T) \leq \frac{1}{2}\exp\{-\varepsilon(V_o^\mu - \theta)\}. \tag{3.5}$$

With the two above equations (3.4) and (3.5), we obtain that for any $(x, y) \in \Gamma$ and for small μ and large M, we have

$$P_{x,y}(Z_1 \in (\partial G \times (-M, M) \cup (G \times \partial[-M, M])) \leq \exp\{-\varepsilon^{-1}(V_o^\mu - \theta)\}.$$

Denote by ν the smallest n for which $Z_n \in (\partial G \times (-M, M) \cup (G \times \partial[-M, M]))$. Then for any $(x, y) \in \Gamma$,

$$P_{x,y}(\nu > n) \geq (1 - \exp(-\varepsilon^{-1}(V_o^\mu - \theta)\})^{n-1}.$$

It is obvious that $\tau^{X,\eta} = (\tau_1 - \tau_0) + (\tau_2 - \tau_1) + ... + (\tau_\nu - \tau_{\nu-1})$. Therefore,

$$\begin{aligned} E_{x,y}\tau^{X,\eta} &= \sum_{n=1}^{\infty} E_{x,y}(\nu \geq n, \tau_n - \tau_{n-1}) \\ &\geq \sum_{n=1}^{\infty} E_{x,y}(\nu \geq n, \tau_n - \sigma_{n-1}) \\ &\geq \sum_{n=1}^{\infty} P_{x,y}(\nu \geq n) \inf_{(x,y)\in\Gamma} E_{x,y}\tau_1 \end{aligned}$$

The last infimum is greater than some positive constant t_1 independent of ε because the trajectory of the system will spend some time going from Γ to γ.

Then for any $(x, y) \in \gamma$ and for sufficiently small δ, and for the small μ and the large M, we have

$$\begin{aligned} E_{x,y}\tau^{X,\eta} &\geq t_1 \sum_{n=1}^{\infty} \min_{(x,y)\in\gamma} P_{x,y}(\nu \geq n) \\ &\geq t_1 \sum_{n=1}^{\infty} (1 - \exp(-\varepsilon^{-1}(V_o^\mu - \theta)))^{n-1} \\ &= t_1 \exp\{\varepsilon^{-1}(V_o^\mu - \theta)\} \end{aligned}$$

For any $x \in G$, $|y| < N$, taking into account that $P_{x,y}(\tau^{X,\eta} > \tau_1) \to 1$ as $\varepsilon \downarrow 0$, we have

$$\begin{aligned} E_{x,y}\tau^{X,\eta} &= E_x(\tau^{X,\eta} \leq \tau_1, \tau^{X,\eta}) + E_{x,y}(\tau^{X,\eta} > \tau_1, \tau^{X,\eta}) \\ &\geq P_{x,y}(\tau^{X,\eta} > \tau_1) E_{(X_{\tau_1}^{\mu,\varepsilon}, \eta_{\tau_1}^{\mu,\varepsilon})}(\tau^{X,\eta}) \\ &\geq \frac{t_1}{2} \exp\{\varepsilon^{-1}(V_o^\mu - \theta)\} \end{aligned}$$

This completes the proof.

$\blacksquare$

Following from Lemma 3.9 and Lemma 3.10, the first part of Theorem 3.3 can be proved provided $\mu < \mu_4$.

Let us prove the second part of Theorem 3.3 for any small $\mu < \mu_4$. Notice that for any $\alpha > 0$ and $x \in G, |y| < N$ and any $\mu < \mu_4$, if

$$\overline{\lim}_{\varepsilon\downarrow 0} P_{x,y}(\tau^X > e^{\varepsilon^{-1}(V_o^\mu + \alpha)}) > 0$$

then we will have

$$\lim_{\varepsilon\downarrow 0} \varepsilon \ln E_{x,y}\tau^X \geq V_o^\mu + \alpha.$$

which contradicts Lemma 3.10. Therefore, for any $\alpha > 0$ and $x \in G, |y| < N$ we have

$$\lim_{\varepsilon\downarrow 0} P_{x,y}(\tau^X < \exp\{\varepsilon^{-1}(V_o^\mu + \alpha)\}) = 1. \tag{3.6}$$

Now, let us show that for any $\alpha > 0$ and $x \in G, |y| < N$, $\lim_{\varepsilon\downarrow 0} P_{x,y}(\tau^X < e^{\varepsilon^{-1}(V_o^\mu - \alpha)}) = 0$ provided $\mu < \mu_4$. Let us choose the same large M(say $M^2\mu > V_o + 2$). Since $\tau^{X,\eta} \leq \tau^X$, then $P_{x,y}(\tau^X < e^{\varepsilon^{-1}(V_o^\mu - \alpha)}) \leq P_{x,y}(\tau^{X,\eta} < e^{\varepsilon^{-1}(V_o^\mu - \alpha)})$. Hence, it is sufficient to show $\lim_{\varepsilon\downarrow 0} P_{x,y}(\tau^{X,\eta} < e^{\varepsilon^{-1}(V_o^\mu - \alpha)}) = 0$.

For any $\alpha > 0$ and $x \in G, |y| < N$, using the same notation we introduced in the proof of Lemma 3.10, we notice that

$$\begin{aligned} P_{x,y}(\tau^{X,\eta} < e^{\varepsilon^{-1}(V_o^\mu - \alpha)}) \;\; &\leq \;\; E_{x,y}(\tau_1 < \tau^{X,\eta}, \sum_{n=1}^{\infty} P_{(X_{\tau_1}^{\mu,\varepsilon}, \eta_{\tau_1}^{\mu,\varepsilon})}(\nu = n, \tau^{X,\eta} < e^{\varepsilon^{-1}(V_o^\mu - \alpha)})) \\ &\quad + P_{x,y}(\tau^{X,\eta} = \tau_1) \end{aligned}$$

The last probability $P_{x,y}(\tau^{X,\eta} = \tau_1)$ converges to zero as $\varepsilon \downarrow 0$ according to Lemma 3.3. Let us estimate the remaining term. Let $m_{\varepsilon,\mu} = [C \exp\{\varepsilon^{-1}(V_o^\mu - \alpha)\}]$. We choose the constant C later. For $(x,y)\in\gamma$, we have

$$\sum_{n=1}^{\infty} P_{(X^{\mu,\varepsilon}_{\tau_1},\eta^{\mu,\varepsilon}_{\tau_1})}(\nu=n,\tau^{X,\eta}<e^{\varepsilon^{-1}(V^{\mu}_o-\alpha)})$$

$$\leq P_{x,y}(\nu<m_{\varepsilon,\mu})+\sum_{n=m_{\varepsilon,\,u}}^{\infty} P_{(X^{\mu,\varepsilon}_{\tau_1},\eta^{\mu,\varepsilon}_{\tau_1})}(\nu=n,\tau_n<e^{\varepsilon^{-1}(V^{\mu}_o-\alpha)})$$

$$\leq P_{x,y}(\nu<m_{\varepsilon,\mu})+P_{x,y}(\tau_{m_{\varepsilon,\,u}}<e^{\varepsilon^{-1}(V^{\mu}_o-\alpha)}). \tag{3.7}$$

Using the equality $P_{x,y}(\nu=1)<\exp\{-\varepsilon^{-1}(V^{\mu}_o-\theta)\}$, which holds for any $(x,y)\in\gamma$, $\theta>0$ and sufficiently small ε, then

$$P_{x,y}(\nu<m_{\varepsilon,\mu})\leq 1-(1-\exp\{-\varepsilon^{-1}(V^{\mu}_o-\theta)\})^{m_{\varepsilon,\mu}}\to 0$$

as $\varepsilon\to 0$, for any $C,\alpha>0$ and θ sufficiently small.

Now, let us estimate the second probability of (3.7), $P_{x,y}(\tau_{m_{\varepsilon,\,u}}<e^{\varepsilon^{-1}(V^{\mu}_o-\alpha)})$. For the fixed μ we chose, there exists a $\lambda>0$ such that $P_{x,y}(\tau_1>\lambda)\geq\frac{1}{2}$ for all $(x,y)\in\gamma$ and $\varepsilon>0$. For the number S_m of successes in m Bernoulli trials with probability of success $\frac{1}{2}$, we have the inequality

$$P_{x,y}(S_m>\frac{m}{3})>1-\epsilon$$

for m larger than some m_o. Since $\tau_m=(\tau_1-\tau_0)+(\tau_2-\tau_1)+...+(\tau_m-\tau_{m-1})$, using the strong Markov property of the process, we obtain that

$$P_{x,y}(\tau_{m_{\varepsilon,\mu}}<e^{\varepsilon^{-1}(V^{\mu}_o-\alpha)})=P_{x,y}\left(\frac{\tau_{m_{\varepsilon,\mu}}}{m_{\varepsilon,\mu}}<\frac{1}{C}\right)<\epsilon$$

for $\lambda/3>1/C$ and ε sufficiently small, $m_{\varepsilon,\mu}$ is sufficiently large. Consequently, we have proved $\lim_{\varepsilon\downarrow 0}P_{x,y}(\tau_{m_{\varepsilon,\,u}}<e^{\varepsilon^{-1}(V^{\mu}_o-\alpha)})=0$ for any $x\in G,|y|<N$. This completes the proof of part (2) of Theorem 3.3.

■

In conclusion, let us choose $\mu^* = \min\{\mu_2, \mu_4\}$. For any $\mu < \mu^*$, Theorem 3.3 is proved. For the multi-dimensional case, Theorem 3.3 can be proved in the same way.

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