

ABSTRACT

Title of Dissertation: *K*-THEORY OF TWO-DIMENSIONAL
SUBSTITUTION TILING SPACES
FROM *AF*-ALGEBRAS

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Given a two-dimensional substitution tiling space, we show that, under some reasonable assumptions, the K -theory of the groupoid C^* -algebra of its unstable groupoid can be explicitly reconstructed from the K -theory of the AF -algebras of the substitution rule and its analogue on the 1-skeleton. We prove this by generalizing the calculations done for the chair tiling in [JS16] using relative K -theory and excision, and packaging the result into an exact sequence purely in topology. From this exact sequence, it appears that one cannot use only ordinary K -theory to compute using the dimension-filtration on the unstable groupoid. Several examples are computed using Sage and the results are compiled in a table.

K-THEORY OF TWO-DIMENSIONAL
SUBSTITUTION TILING SPACES FROM AF -ALGEBRAS

by

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Preface

Ian Putnam visited Maryland in spring, 2016, and gave a talk at the dynamics conference held in honor of Mike Boyle's retirement. I remember him starting the talk by apologizing for the non-dynamical nature of what was to follow, then proceeding to give an awe-inspiring lecture on Elliott's classification program, so much so that it was the only talk I took any pictures of at the conference, and out of everything I was interested in, made me conclude, "that is what I want to work on!" Alas, graduate school course and exam requirements over the next year or two gradually wore down that inspiration, very much not due to the tedium (which they were not), but due to the lack of kindling.

I was the last person in my cohort to pick an advisor. I had hemmed and hawed for nearly a year after coursework, because nothing particularly leapt at me. When my indecision finally concluded and I asked Giovanni Forni, he had too many students then, and instead suggested Rodrigo Treviño. At the time, Rodrigo was a *very* new hire. I was introduced to him by a fellow graduate student at the dynamics seminar, and, in an act of spontaneity that is very much unlike myself, during the brief period of solace between the two halves of the seminar known as "cookie time", I asked him to be my advisor, without any knowledge (even superficial) of his subject of expertise.

After taking some time to understand the basics of tiling spaces, he introduced me to this problem of reconstructing the K -theory of tiling spaces from that of AF -algebras. During one

of our discussions, he mentioned that it was at the 2016 Maryland dynamics conference that he initially met Ian. Jogging my memory for the first time in a few years, I realized that this problem I had been working on is a part of Elliott’s classification program.

This is one of several coincidences that made this project into what it is today, another significant one being Niranjana Ramachandran teaching a topics course on étale cohomology and Rodrigo on Cantor minimal systems in the same semester, with the term “partial homeomorphism” coinciding between the two courses.

Acknowledgments

It took considerable time and effort to assemble the complete picture depicted in this project, neither of which I would have been able to afford without the generous assistance of my advisor, Rodrigo Treviño, and the supportive environment of the math department.

I am deeply grateful for the patience and generosity Rodrigo showed me. He was always enthusiastic about discussing anything that caught my fancy at the moment, despite me frequently dropping by his office unannounced. Although he occasionally alleges that he has no feedback whatsoever due to our discussion being too far afield from his subject of expertise, he will (sometimes off-handedly) mention a thought or resource or two that nudges me (and this project) further along.

I am also very thankful of Professors Bill Goldman, Jonathan Rosenberg, and Christian Zickert for lending me their expertise in their numerous topics classes I have audited, particularly their abilities to reduce multiple layers of abstraction down to more basic concepts to build up my own intuition.

Lastly, I cannot express enough gratitude towards my office mates, David Bekkerman, Charles Daly, S Gilles, Zack Greenberg, and Dani Kaufman, for putting up with my antics all these years.

Table of Contents

Preface	ii
Acknowledgements	iv
Table of Contents	v
Chapter 1: Introduction	1
Chapter 2: Substitution tiling spaces and inverse limits	7
2.1 Substitution tiling spaces	7
2.2 The Anderson–Putnam inverse limit	10
Chapter 3: Groupoid C^* -algebras	18
3.1 Continuous groupoids	18
3.2 Discrete groupoids	30
Chapter 4: K -theory	40
4.1 Ordinary K -theory	40
4.2 Relative K -theory	42
4.3 Excision	43
Chapter 5: Čech cohomology and K -theory	45
5.1 Thickening cochains	46
5.2 Isomorphism for $d \leq 2$	57
5.3 Interpretation	76
Chapter 6: Computations	78
6.1 $d = 1$	78
6.2 $d = 2$	93
Bibliography	132

Chapter 1: Introduction

A tiling T (such as the Penrose tiling) is a partition of \mathbb{R}^d into compact sets. Our goal is to study and classify tilings using (topological or C^* -algebraic) invariants. It is a classic mathematical idea that rather than studying a single tiling, one instead studies a space of tilings, then deduces characteristics of the tiling from the properties of the space. To make this more precise, we impose a notion of distance between two tilings that measures the radius to which two tilings agree. The tiling space, Ω_T , is closure of all translations of T by elements of \mathbb{R}^d under this metric. Due to this construction of the tiling space, Ω_T inherits an action by \mathbb{R}^d translations of tilings.

There are generally two methods of creating tiling spaces, one via substitution rules, the other via cut-and-project schemes. We will focus on the former, tiling spaces that arise from substitution rules.

Under suitable conditions, a d -dimensional substitution tiling space Ω_T has its substitution rule as a self-homeomorphism. With this self-homeomorphism, the tiling space carries a Smale space structure, and an associated Smale bracket returns an unstable and a stable coordinate. The unstable coordinate turns out to correspond to the \mathbb{R}^d direction that is translations, and the stable coordinate the Cantor set direction that is substitutions, giving us the classical picture that locally, the tiling space is a product of the two. From the orbits returned of each of the two types of actions, one obtains a groupoid and a groupoid C^* -algebra ([Put96]). Due to its characterization,

it was shown in [Gon11] and [GRS17] that, for $d \leq 2$, one can directly calculate the K -theory of the C^* -algebra of the stable groupoid using the substitution rule. Can one do the same for the unstable groupoid?

To be more explicit, by identifying each prototile as a copy of \mathbb{C} , the substitution rule induces a natural inclusion, up to unitary equivalence, of a direct sum of matrix algebras into another, each of the summands being a matrix algebra whose diagonal represents the tiles inside a given supertile. Its limit completes to an AF -algebra that includes into the groupoid C^* -algebra of the unstable groupoid, and this inclusion induces a map between their K -theories. We would like to see if this induced map (and perhaps its lower-dimensional analogues) is sufficient in reconstructing the K -theory of the unstable groupoid.

One motivation for studying the K -theory of the unstable groupoid from this perspective is due to Elliott's classification of AF -algebras ([Eli76]), where they are completely classified by their ordered K_0 -groups (K_1 being trivial). For $d = 1$, by using the order inherent on \mathbb{R} , one can form an associated Bratteli–Vershik system whose underlying Bratteli diagram is generated from the substitution rule, giving us that the C^* -algebra of the unstable groupoid associated to the Bratteli–Vershik system is a nontrivial extension of the AF -algebra whose presence in the unstable groupoid is as an orbit-breaking subalgebra. It was computed in [Put89] that the K_0 -group is generally a quotient of the K_0 -group of the AF -algebra, and K_1 is \mathbb{Z} , realized as a generator of the translation action. In fact, by [DHS99], one can reencode the substitution so that it becomes proper, i.e. every supertile begins with the same prototile and ends with the same prototile. This reencoding process leaves the tiling space unmodified, but the K_0 -group of the unstable groupoid becomes order isomorphic to the K_0 -group of this new AF -algebra. In higher dimensions, it was shown that there exists an order structure ([Kel97] and [Put00] via traces; [ORS02] via the

Perron–Frobenius eigenvector).

There is also a physical reason for analyzing this induced inclusion map, at least for the K_0 -group of the unstable groupoid. With the discovery of quasicrystals in [SBGC84], it was realized that punctured tiling spaces can be used as models in their study, with one of the two main constructions of tiling spaces being those arising from substitution rules. One then attempts to characterize the spectra of the associated Schrödinger operators through gap labelling, which was discovered to be intricately related to the K_0 -group, specifically the image of the K_0 -group of the AF -algebra under the induced inclusion map ([Bel86], [Bel92], [Kel95], [Kel97]). In fact, it was shown in [vEI94] that for $d = 2$, a quotient of the K_0 -group of the AF -algebra recovers the K_0 -group of the unstable groupoid, up to a direct summand of \mathbb{Z} .

In [JS16], it was demonstrated for the chair tiling that the K_0 -group of the AF -algebra arising from the induced substitution rule on the boundaries of prototiles is required in reconstructing the K_1 -group of the unstable groupoid. However, the authors there relied on splitting each of the supertiles into three squares and reencoding the substitution based on these squares, thus obtaining two induced substitution rules, one on the horizontal boundaries, the other on the vertical, both of which are stationary and one-dimensional. They then use a more general form of [Put89] in [Put97] and [Put98] to compute the K_1 -group from these one-dimensional substitutions. It is unclear that if one attempts the same procedure on a general substitution rule, the induced substitution rules on the boundaries of the newly-split supertiles would remain stationary.

For $d \leq 2$, using the six-term sequence in relative K -theory ([Has21]) and an observation that, frequently, the K_0 -groups of the AF -algebra obtained from inducing the substitution rule on lower-dimensional boundaries coincide exactly with the cochain group of the same dimension, we show that there is an isomorphism of exact sequences from the six-term sequence arising

from the inclusion map of the AF -algebra into the C^* -algebra of the unstable groupoid to an exact sequence in topology. Using the observation, we recover the fact that a quotient (with a direct sum of \mathbb{Z} for $d = 2$) of the K_0 -group of the AF -algebra gives the K_0 -group of the unstable groupoid, and the K_1 -group is a subgroup of a quotient of the K_0 -group of the AF -algebra of the one-dimensional substitution rule.

Theorem 1.1. *For $d = 1$, we have an isomorphism of exact sequences*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \check{H}^0(\Omega_T) & \hookrightarrow & C^0 & \xrightarrow{\delta^0} & C^1 & \twoheadrightarrow & \check{H}^1(\Omega_T) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow \text{th} & & \downarrow \text{CF} & & \downarrow & & \\
0 & \longrightarrow & K_{1,u} & \hookrightarrow & K_{0,AF;u} & \xrightarrow{\text{ev}} & K_{0,AF} & \xrightarrow{\iota_*} & K_{0,u} & \longrightarrow & 0
\end{array}$$

where $K_{i,u} = K_i(C_r^*(\dot{G}_u))$, $K_{i,AF} = K_i(C_r^*(\dot{G}_{AF}))$, and $K_{i,AF;u} = K_i(C_r^*(\dot{G}_{AF}); C_r^*(\dot{G}_u))$.

Thus

$$K_0(C_r^*(\dot{G}_u)) \cong K_0(C_r^*(\dot{G}_{AF}))/\text{im ev}$$

$$K_1(C_r^*(\dot{G}_u)) \cong \mathbb{Z}.$$

Theorem 1.2. *For $d = 2$, we have an isomorphism of exact sequences*

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \check{H}^1(\Omega_T) & \hookrightarrow & C^1/\text{im } \delta^0 & \xrightarrow{\delta^1} & C^2 & \twoheadrightarrow & \check{H}^2(\Omega_T) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow \text{th} & & \oplus & & \oplus & & \\
& & & & & & 0 & \longrightarrow & \check{H}^0(\Omega_T) & \hookrightarrow & \check{H}^0(\Omega_T) \longrightarrow 0 \\
& & & & & & \downarrow \text{CF} & & \downarrow & & \downarrow \\
0 & \longrightarrow & K_{1,u} & \hookrightarrow & K_{0,AF;u} & \xrightarrow{\text{ev}} & K_{0,AF} & \xrightarrow{\iota_*} & K_{0,u} & \twoheadrightarrow & K_{1,AF;u} \longrightarrow 0.
\end{array}$$

where $K_{i,u} = K_i(C_r^*(\dot{G}_u))$, $K_{i,AF} = K_i(C_r^*(\dot{G}_{AF}))$, and $K_{i,AF;u} = K_i(C_r^*(\dot{G}_{AF}); C_r^*(\dot{G}_u))$.

Thus, if T satisfies the boundary hyperplane condition,

$$K_0(C_r^*(\dot{G}_u)) \cong K_0(C_r^*(\dot{G}_{AF})) / \text{im ev} \oplus \mathbb{Z}$$

$$K_1(C_r^*(\dot{G}_u)) \cong \ker[K_0(C_r^*(\dot{G}_{AF}^{(1)})) / \text{th}(K_0(C_r^*(\dot{G}_{AF}^{(0)}))) \xrightarrow{\text{ev}} K_0(C_r^*(\dot{G}_{AF}))]$$

where $\text{th} : K_0(C_r^*(\dot{G}_{AF}^{(0)})) \rightarrow K_0(C_r^*(\dot{G}_{AF}^{(1)}))$ returns an alternating sum of elements of $K_0(C_r^*(\dot{G}_{AF}^{(1)}))$ that coincides with δ^0 .

We then use this to compute the K -theory (in fact the six-term sequence in relative K -theory) for several examples, the two-dimensional ones being done via Sage and summarized in Table 6.1. Many of these computations are not in the literature.

While the statement and the proof of this result can be given very succinctly by starting from punctures, we have elected to take a more scenic route by starting from the unstable groupoid as defined on the tiling space, carrying it over to the Anderson–Putnam inverse limit by the Robinson map, constructing various subgroupoids, then restricting to the punctures. This allows us to very easily observe the existence AF -algebras for the boundaries of each of the intermediate dimensions, and how one can inductively construct the unstable groupoid from these AF -algebras, thereby recovering weaker forms of [BJS10, Theorem 3.22] and [JS10, Theorem 4.9] that are only as groupoid equivalences rather than homeomorphisms.

Furthermore, this isomorphism of exact sequences relies on the classical fact that in low dimensions, K -theory is isomorphic to direct sums of Čech cohomology groups of the same parity, with a dimension shift. While there are a variety of proofs of this fact (e.g. the Chern and Connes–Thom isomorphisms, and the Pimsner–Voiculescu exact sequence), using any of them requires unwrapping said isomorphisms in order to provide the explicit correspondences between

the generators and relations of each group. Since our isomorphism of exact sequences uses relative K -theory, we believe it to be in the spirit of the paper to reprove this fact purely using the six-term sequence in relative K -theory (and excision, [Put97], [Put21], [Put98]) by adapting the computations performed in [JS16] for the chair tiling.

In Chapter 2, we recall the basic definitions of substitution tiling spaces and sketch the topological conjugacy to the Anderson–Putnam inverse limit. In Chapter 3, we construct various groupoids on the inverse limit and define their (reduced) C^* -algebras. In Chapter 4, we briefly recall K -theory and state the facts we need from relative K -theory and excision. In Chapter 5, we first state and prove the existence of a map of exact sequences between topology and the six-term sequence in relative K -theory, then show that it is an isomorphism for $d = 1$. For $d = 2$, we reprove the isomorphism between K -theory and Čech cohomology, then derive our desired isomorphism as a corollary by realizing that this isomorphism coincides with the maps introduced at the beginning of this chapter. In Chapter 6, for $d = 1$, we manually churn through the computations for each of the K -groups (including relative K_0) for the Fibonacci and the Silver Mean substitutions. We then illustrate the difference in computations for the two-dimensional dyadic solenoid, and leave the computations for the less trivial substitution tiling spaces to be done using a script in Sage, whose results are presented in a table. Other than the latter half of Chapter 5 and Chapter 6, our statements hold for tilings of arbitrary dimensions.

Chapter 2: Substitution tiling spaces and inverse limits

In this chapter, we recall the necessary definitions and theorems of substitution tiling spaces and their inverse limit structure.

2.1 Substitution tiling spaces

A *tiling* is a partition of \mathbb{R}^d into compact sets with disjoint interiors (possibly with labels), each of which is homeomorphic to the closed ball in \mathbb{R}^d . Each element in this partition is a *tile*. The set of *prototiles* is the quotient of the set of tiles by translations in \mathbb{R}^d . A *patch* is a finite collection of tiles whose union is homeomorphic to the closed ball. If P is a patch, denote $\text{supp } P$, the *support* of the patch, the union of the tiles in P .

Given a tiling T and an $x \in \mathbb{R}^d$, denote $T - x$ the tiling where each tile is translated by $-x$. Alternatively, this is the tiling with the origin shifted to x . Two tilings T_1 and T_2 are less than ϵ -apart if there exist $x_1, x_2 \in B_\epsilon(0)$ such that the tilings $T_1 - x_1$ and $T_2 - x_2$ agree on $B_{1/\epsilon}(0)$. This is the *tiling metric*. The *tiling space* associated to T , Ω_T , is the closure of all of its translations under the tiling metric, that is

$$\Omega_T = \overline{\{T - x : x \in \mathbb{R}^d\}}^{\text{tiling metric}}.$$

This metric yields a basis of *cylinder sets* for the topology, defined as

$$C(P, U) = \{T' \in \Omega_T : \exists x \in U \text{ such that } P - x \subseteq T'\},$$

where P is a patch and $U \subseteq \text{supp } P$ is an open ball in \mathbb{R}^d . If T arises from a substitution rule, we can refine this basis further to only use (unions of) supertiles (of the same level), since there is a mutual refining of topologies generated by both types of cylinder sets.

We shall assume that

- T is (strongly) *aperiodic*, or Ω_T contains no periodic components, and
- T has *finite local complexity*, or given any $r > 0$, there exist finitely many patches, up to translation, of radius r .

There is an important recurring property called *extension* that we first state in terms of cylinder sets that allows us to extend (or reduce) the patch of a cylinder set.

Proposition 2.1 (Extension). *Let P be a patch. If Q is a patch that contains P so that each occurrence of P in T induces the same inclusion of P in Q up to translation, then for any open $U \subseteq \text{supp } P$, $C(P, U) = C(Q, U)$. More generally, if $\{Q_i\}_i$ is a collection of patches, each containing P so that each occurrence P in T is contained in some Q_i , and each occurrence of Q_i in T contains P in the same relative location, then $C(P, U) = \bigcup_i C(Q_i, U)$.*

Proof. The inclusion $C(Q, U) \hookrightarrow C(P, U)$ and the restriction $C(P, U)|_{C(Q, U)}$ maps are well-defined, and are inverses to each other. The second statement is a straightforward generalization by partitioning the occurrences of P correctly. □

Since translations are used to form the tiling space, they induce an action $\mathbb{R}^d \curvearrowright \Omega_T$ by translations as well, by

$$x \cdot T' = T' - x.$$

There is a notion of a *punctured prototile*, consisting of a prototile with a choice of a point in its interior. One can then form a *punctured tiling* by replacing each tile with a punctured tile and requiring the origin to be situated on a puncture, and a *punctured tiling space*, where the only allowed translations are those moving between punctured tilings.

A *substitution rule*, ς , is a map from the set of tiles to the set of patches, formed from the composition of two operations

1. *Inflation*, where a tile (with its position relative to the origin) is expanded by a scalar greater than 1, and
2. *Subdivision*, where the inflated tile is subdivided into a set of tiles.

Substitution rules extend to patches. The patch that is the result of applying the substitution rule n -times to a single tile is a *level- n supertile*, denoted $\varsigma^n(t)$ with t a tile. A *supertile* is a level-1 supertile, and a level-0 supertile is a tile. We may similarly consider patches and supertiles up to translation, which (unfortunately) are again called patches and supertiles. We will let context dictate which type we are referring to. A tiling *arises from a substitution rule* if every patch is contained in a level- n supertile, for some sufficiently large n .

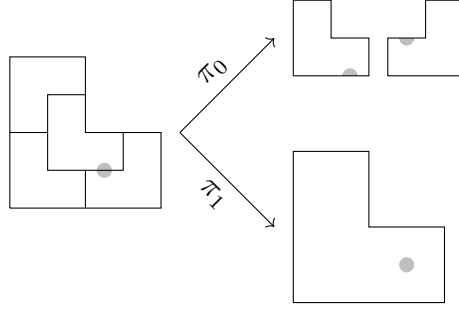


Figure 2.1: A cylinder set in a substitution tiling space where we have fixed the (level-1) supertile and a small ball of origins (light gray), as projected onto the first two coordinates of the inverse system.

2.2 The Anderson–Putnam inverse limit

If T arises from a substitution rule, then one can define an inverse limit space using supertiles of all levels, each viewed as the support of the corresponding patch. More precisely, given any $n \in \mathbb{N}$, let the *level- n Anderson–Putnam complex*, denoted AP_n , be formed from the quotient of the set of (supports of) level- n supertiles (up to translation) by the identifications given in the tiling T . In other words, we identify the boundary of the supertiles to yield a continuous projection map $T \twoheadrightarrow AP_n$ that extends to

$$\pi_n : \Omega_T \twoheadrightarrow AP_n$$

where a tiling is sent to the point in AP_n corresponding to the placement of the origin relative to the level- n supertiles it is situated in (Fig. 2.1). The substitution rule induces a map

$$\sigma : AP_n \rightarrow AP_{n-1}$$

by sending each d -cell that is a level- n supertile to a collection of d -cells obtained by subdividing the level- n supertile into level- $n - 1$ supertiles. This gives us an inverse system of CW -complexes, more particularly of branched manifolds, whose inverse limit is denoted $\varprojlim_n (AP_n, \sigma)$, and, since each π_n is compatible with σ , a canonical projection map

$$\pi : \Omega_T \twoheadrightarrow \varprojlim_n (AP_n, \sigma),$$

called the *Robinson map*. Letting $AP_n^{(k)}$ denote the k -skeleton of the corresponding CW -complex, we similarly get an inverse limit $\varprojlim_n (AP_n^{(k)}, \sigma)$ for each $0 \leq k \leq d$, that we call the k -skeleton of $\varprojlim_n (AP_n, \sigma)$.

Hereafter, we shall assume that our tiling space Ω_T arises from a substitution rule σ . Furthermore, we will assume that

- σ is *primitive*, or there exists an $n \in \mathbb{N}$ such that each level- n supertile contains all of the prototiles, and
- T , and therefore each of the AP -complexes, is *collared*, or the supertiles in T are further labelled by their incident same-level supertiles.

The substitution map induces a map on collared AP -complexes, and one can form a corresponding inverse limit and obtain an analogous Robinson map.

We have [AP98, Theorem 4.3].

Theorem 2.2 (Anderson–Putnam). *The Robinson map is a homeomorphism.*

We will only provide explanation and intuition to convince the reader of why the Robinson map is injective after collaring. From collaring, we will then see that there exists an action by translations

by \mathbb{R}^d on $\varprojlim_n (AP_n, \sigma)$, even though an action does not appear to exist on a finite level if AP_n is branched. Under this action, the Robinson map turns into a topological conjugacy.

Suppose that we are given a tile whose boundary contains the origin. Substituting infinitely many times may only yield a partial tiling of \mathbb{R}^d , particularly if the tile has its support on one side of a hyperplane intersecting the origin. There may be multiple tilings that contain this partial tiling. Applying the Robinson map gives that all such tilings map to the same point, and thus the map cannot be injective. Phrased differently, the inverse limit is one of branched manifolds, and it may be that the limit still results in at least one branching point. This branching point gives multiple tilings that project to it.

One way to resolve this is with collaring, since placing the origin on the boundary of a collared tile still results in the origin being completely surrounded by tiles coming from the boundary tile labels, and substituting infinitely always results in a complete tiling, even though, formally, we lack the knowledge of the pattern on all of \mathbb{R}^d . Therefore, the moral is that *under the collaring process, branches eventually disappear*. Let us phrase this more precisely with a definition and a proposition.

Definition 2.3. For $k \geq 1$, a set is

- *Contractible in $AP_n^{(k)}$* if its restriction and quotient to each $AP_n^{(\ell)}/AP_n^{(\ell-1)}$ is contractible for all $1 \leq \ell \leq k$, and
- *Unbranched in $AP_n^{(k)}$* if it is the homeomorphic image of an open ball in \mathbb{R}^k .

Part of the goal of this definition of contractibility is to prevent an unbranched set from intersecting too many supertiles of a certain level, because such sets should be thought of as belonging to a higher-level AP -complex.

Proposition 2.4 (Border-forcing). *There exists an $N \in \mathbb{N}$ such that for any open ball $U \subseteq AP_N$ with $\sigma^N(U) \subseteq AP_0$ contractible, $\sigma^N(U)$ is unbranched. For such an N and for all $n \geq N$, this holds between AP_n and AP_{n-N} .*

Proof. Let $c(t)$ denote the patch formed from t and its collars. Let N be the level at which for all collared prototiles t , for all $t' \in \varsigma^N(c(t) \setminus t)$ so that t' intersects the boundary of $\text{supp } \varsigma^N(t)$, $c(t') \subseteq \varsigma^N(c(t))$. In other words, $\varsigma^N(c(t))$ contains the neighbors of neighbors of $\varsigma^N(t)$. This N exists since the substitution rule is expansive.

With this choice of N , the boundary collared tiles of $\varsigma^N(t)$ are explicitly determined. More precisely, given a collection $\{t_i\}_i$ of all collared tiles so that each t_i intersects t at the same (branched) boundary and each forms a patch with t along that boundary in T , the collared tiles in $\varsigma^N(t_i)$ that intersect the boundary of $\text{supp } \varsigma^N(t)$ are identical among all i . Picking a t_i then repeating the same argument inductively until all supertiles intersecting this boundary has been exhausted gives that the projection π_N to AP_0 of the small neighborhood of this boundary in AP_N is unbranched.

For the first statement, $U \subseteq AP_N$ being contractible in AP_0 means that for an appropriate choice of boundary, it is contained inside this small neighborhood.

The second statement immediately follows from self-similarity of the substitution rule. \square

The minimum such N is called the level at which the substitution rule *forces the border*, and the boundary tiles are the *forced border*. For $n \geq N$, we take the forced border to be the patch obtained from applying ς^{n-N} to the forced border at level- N ¹. Hereafter, when we refer to a (super)tile, we will always assume that it is collared.

¹Strictly speaking, the forced border can be taken to be larger, since the border itself may yield additional forced border.

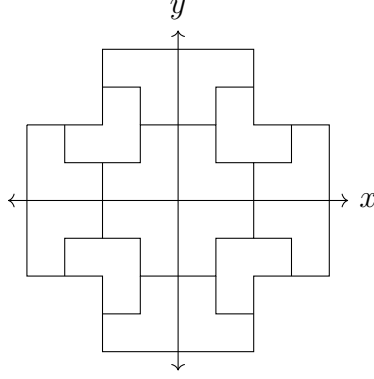


Figure 2.2: The substitution rule sends this particular chair tiling to itself. Observing that the 1-skeleton contains the x - and y -axes and is sent to itself under the induced substitution rule, the origin remains branched in $\varprojlim_n (AP_n^{(1)}, \sigma)$.

We have the following corollary as an immediate consequence to the proof.

Corollary 2.5 (Extension). *Let N be the level at which the substitution rule forces the border, and let $n \geq N$. If $\varsigma^n(t)$ is a supertile and P is the patch formed from the supertile together with its forced border, then $C(\varsigma^n(t), U) = C(P, U)$.*

Proof. The forced border is unique and appears with every occurrence of $\varsigma^n(t)$. An application of extension gives equality between the two cylinder sets. \square

Warning 2.6. The same cannot be said about the lower-dimensional skeleta! For example, one can observe this failure in the chair tiling on the 1-skeleton (Fig. 2.2). More generally, this exists in the 1-skeleton for all two-dimensional substitution tiling spaces.

Even after collaring, not all contractible, unbranched $U \subseteq AP_n$ can be realized as the Robinson map applied to some patch. In the proof of the above proposition, this is the reason for referring back to patches in the tiling and inducing on the pairs of branches that are images of patches under the Robinson map. See the Silver Mean substitution, for example (Fig. 2.3). This forces us make the following definition intrinsic to the inverse system that we will show is the

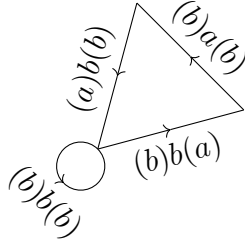


Figure 2.3: The AP -complex of the collared Silver Mean substitution, $a \mapsto b$, $b \mapsto bab$. The prototile $(b)b(b)$ has its endpoints attached, but $(b)b.b(b)$ is not a patch in the tiling.

analogue of a patch in the tiling.

Definition 2.7. By an *allowed* $U \subseteq AP_n$, we mean one so that there exists an $m \geq n$ and an open ball $V \subseteq AP_m$ in the subspace topology so that $\sigma^{m-n}(V) = U$.

Let us abbreviate “allowed, contractible, and unbranched” by *acu*.

Proposition 2.8. *Acu sets correspond to patches in the tiling.*

Proof. Given an acu $U \subseteq AP_n$ with a preimage $V \subseteq AP_m$ that is an open ball, since the Robinson map is a homeomorphism, there exist collections of level- m supertiles each of whose union forms a patch under the inverse of the Robinson map and whose intersection with V projects to U via σ^{m-n} . This makes $\pi^{-1}(U)$ subsets of each of the patches, and $\pi^{-1}(U)$ itself is therefore a collection of patches. In fact, existence of a single such preimage implies that for sufficiently large m , each of the preimages $\sigma^{-(m-n)}(U)$ will be an open ball, since a patch forms a cylinder set, and its image under the Robinson map is open.

Conversely, given any patch P , we consider the level n at which $\pi_n(P)$ is contractible and a collection of open balls. This exists since the substitution rule is expansive. Each open ball can be further written as a union of acu sets. □

These two propositions tell us that acu sets and the standard basis for the subspace topology in

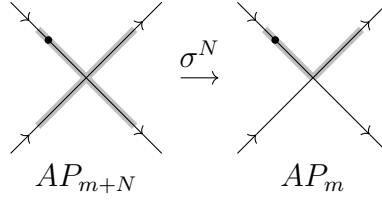


Figure 2.4: We want to define a translation on AP_m on a point that crosses over a branching point (right). By collaring (and border-forcing), any open ball in the branched manifold topology that contains both the point and the branching point (left, light gray) has its image under σ^N unbranched (right, light gray). The choice of the branch associated to the translation is given by this unbranched image.

the AP -complexes mutually refine each other in $\varprojlim_n (AP_n, \sigma)$. In light of this, let us restrict the cylinder sets in Ω_T to those that have the second parameter satisfying the contractibility condition under π_0 .

We can now give a nice description of the cylinder sets in $\varprojlim_n (AP_n, \sigma)$, which have the form

$$C(U_m) = \{(p_0, \dots) \in \varprojlim_n (AP_n, \sigma) : p_m \in U_m\}$$

where $U_m \subseteq AP_m$ is acu. While it appears to not be open in the subspace topology of the AP -complexes, border-forcing says that one can look at sufficiently-high preimages to obtain (unions of) open balls in the correct topology.

Notice that, on taking the inverse system at face value, it appears impossible for there to be a translation action, since it may be that the translation vector is large enough to cross from one supertile into another, thus cross a branch, where it becomes ill-defined. In fact, this is possible had we not collared. Border-forcing resolves this issue of crossing a boundary.

Suppose that we are given a $(p_0, \dots) \in \varprojlim_n (AP_n, \sigma)$ and an $x \in \mathbb{R}^d$ to act on the point. Let N be the level at which the substitution forces the border, and let n be the level at which π_n on every patch of radius $x + \epsilon$ is contractible. Then, for all $m \geq n$, patches of radius $x + \epsilon$ also

have their images under π_m contractible. For each such m , let $U \subseteq AP_{m+N}$ be the preimage of the set of d -cells incident to p_m that contains p_{m+N} . By border-forcing, its image under σ^N is unbranched. By construction, this image contains the well-defined translation $p_m + x$ (Fig. 2.4). Repeating this for all $m \geq n$ picks a branch in each of the AP_m -complexes and a point in it compatible with σ . Applying σ sufficiently many times yields the rest of the coordinates between 0 and n .

We denote this action by $(p_0, \dots) + x$, as the coordinates themselves represent the location of the origin in the tiling relative to each of the supertiles.

We have proven the following corollary to Theorem 2.2.

Corollary 2.9. *Under the action $\mathbb{R}^d \curvearrowright \varprojlim_n (AP_n, \sigma)$, where*

$$x \cdot (p_0, \dots) = (p_0, \dots) + x,$$

the Robinson map is a topological conjugacy.

Summarized succinctly, border-forcing allows us to construct, starting at a sufficiently-high index, a well-defined (up to extension) sequence of acu sets compatible with substitution (up to extension), each of which containing the corresponding coordinate p_n and its translation $p_n + x$.

Chapter 3: Groupoid C^* -algebras

In this chapter, we define two main types of groupoids, one associated to the action, the other to the substitution rule, then discuss their associated C^* -algebras. While the reader with sufficient background may choose to skim this chapter, we will not follow the more standard perspective of defining the groupoids from punctured tiling spaces using the so-called doubly-pointed patterns introduced in [Kel95]. Rather, we apply the Robinson map on the entire tiling space and obtain the analogue of the unstable and AF -groupoids, then restrict to the punctures to obtain a groupoid equivalence between the unstable groupoid on the punctured tiling space and the groupoid arising from a quotient of the AF -equivalence relation present on the associated Bratteli diagram constructed in [BJS10] and [JS10]. We hope that this approach provides more intuition to the relationship between the unstable and the AF -groupoids.

3.1 Continuous groupoids

We start with a proposition on sets of the form $C(P, U) + B_r(x)$ that we call *partial translations* (Fig. 3.1). Fig. 3.2 shows why sets of this form are given such a name. These are not the same as partial translations as defined in [KP00].

Proposition 3.1. *The set of partial translations and the set of cylinder sets of the form $C(P, U + B_r(x))$ where $U + B_r(x) \subseteq \text{supp } P$ are mutual refinements.*

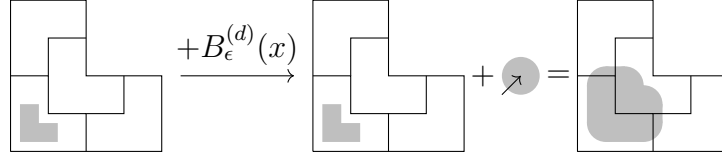


Figure 3.1: A partial translation taking a cylinder set (choice of origin in light gray) to the cylinder set plus a ball.

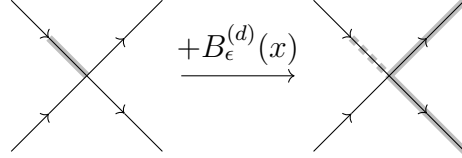


Figure 3.2: A partial translation under the Robinson map. We have a cylinder set that does not intersect a branching point (left), translated by a sufficiently small ball so that the resulting set passes through a branching point (right). The dotted segment indicates that while origin placement is no longer allowed there, we still retain knowledge of its “pattern”, and thus should include it as part of the set of partial translations. This set does not witness all of the branches, hence our usage of the term “partial”.

Proof. Certainly cylinder sets of the latter form are partial translations. For the direction that the latter set is finer than the former, due to border-forcing and extension, the only case we have to check is that addition by sufficiently large x still returns unions of sets of the desired form.

Let n be sufficiently large so that the border that is forced contains balls of radius $R = \text{diam } U + x + r + \epsilon$ centered around the boundaries of each of the supertiles. Consider the set of all inclusions of P into patches of the form of unions of level- $2n$ supertiles, so that the border that is forced is twice as large. For each such inclusion into $\bigcup_i \zeta^{2n}(t_i)$, for addition to make sense, let us consider the patch where we include the R -bordering tiles into the union. Then addition by $B_r(x)$ never moves the resulting set outside of half of the forced border. However, two patches differing by the forced border return the same cylinder sets by extension, provided we pick the sets of origins to be outside of the forced border, thus while the patch of union of supertiles has its border grown by R to $2R$, it is the same cylinder set as if its border stayed at R . Repeating for each such inclusion gives that it can be written as a union of sets of the desired form. \square

For notational purposes, by $C(P, U) + B_r(x)$ we will assume that we work with sets of the latter form.

We recall a few basic definitions on groupoids.

Definition 3.2. A *groupoid* G is a set with a subset $G^2 \subseteq G \times G$ of *composable elements* of G , a multiplication $G^2 \rightarrow G$ defined via $(g_1, g_2) \mapsto g_2 g_1$, and an inversion $G \rightarrow G$ defined via $g \mapsto g^{-1}$ so that

- For all $g \in G$, $(g^{-1})^{-1} = g$,
- For all $(g_1, g_2), (g_2, g_3) \in G^2$, $(g_2 g_1, g_3) \in G^2$, and $(g_3 g_2) g_1 = g_3 (g_2 g_1)$, and
- For all $g \in G$, $(g, g^{-1}) \in G^2$, and for all $(g_1, g_2) \in G^2$, $(g_1, g_2) g_1^{-1} = g_2$ and $g_2^{-1} (g_1, g_2) = g_1$.

$G^0 = \{g^{-1}g : g \in G\}$ is called the *unit space*, and its elements are *units*. The *source* is the map $s(g) = g^{-1}g$, and the *range* is the map $r(g) = gg^{-1}$.

G is a *topological groupoid* if it is a topological space with the source, the range, and the inversion maps continuous, and the multiplication map continuous with respect to the subspace topology of the product topology on $G \times G$.

G is *étale* if it is a topological groupoid and the range map is a *local homeomorphism*, or every $g \in G$ has an open neighborhood U so that $r(U)$ is open, and $r : U \rightarrow r(U)$ is a homeomorphism.

Note that if a groupoid is defined by an action on a topological space, then the unit space can be identified to the topological space itself.

There is an important notion of equivalence from [MRW87, Example 2.7] that relates one groupoid to another.

Definition 3.3. Let G be a locally compact Hausdorff groupoid. A closed subset $N \subseteq G^0$ is an *abstract transversal* if for all $g_1 \in G$, there exists $(g_1, g_2) \in G^2$ so that $r(g_2 g_1) \in N$. $H = s^{-1}(N) \cap r^{-1}(N)$ is the *restriction of G to N* , and G and H are *equivalent (via $s^{-1}(N)$ or just N)*.

It turns out that K -theory is invariant under equivalence of groupoids, so we will consider groupoids up to restriction to abstract transversals.

We now define the groupoid on Ω_T that we wish to study.

Definition 3.4. The *unstable groupoid*, G_u , on Ω_T is the topological groupoid consisting of pairs of tilings that are translations of each other, i.e.

$$G_u = \{(T', T'') \in \Omega_T^2 : \exists x \in \mathbb{R}^d \text{ such that } T'' = T' - x\}.$$

Its topology has the basis consisting of sets of the form

$$(C(P, U) + B_{r_1}(x_1), C(P, U) + B_{r_2}(x_2))$$

where $r_1, r_2 > 0$ and $x_1, x_2 \in \mathbb{R}^d$.

We can replace this basis with the collection of partial homeomorphisms.

Definition 3.5. Given a homeomorphism of cylinder sets $\phi : C(P, U_1) \rightarrow C(P, U_2)$, the *source* is the map $s(\phi) = C(P, U_1)$ (also denoted $\phi^{-1}\phi$), and the *range* is the map $r(\phi) = C(P, U_2)$

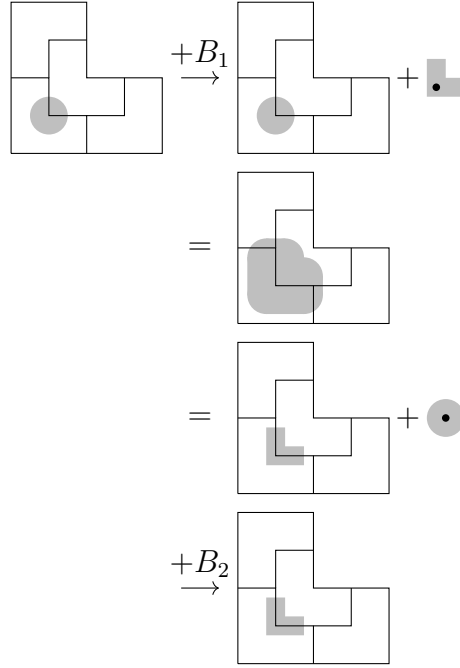


Figure 3.3: A construction of a basic open set in the unstable groupoid (modulo moving down sufficiently many levels) from a partial homeomorphism between the first and the last cylinder sets.

(also denoted $\phi\phi^{-1}$).

A *partial homeomorphism* on Ω_T is an equivalence class of homeomorphisms of cylinder sets, where we quotient by the source and the range.

Proposition 3.6. *The basis of the unstable groupoid and the set of partial homeomorphisms (up to sources and ranges) are mutual refinements.*

Proof. $U + B_{r_1}(x_1)$ and $U + B_{r_2}(x_2)$ are both homeomorphic to open balls, thus there exists a homeomorphism taking one to the other, giving us a partial homeomorphism between the cylinder sets.

Conversely, given a partial homeomorphism $\phi : C(P, U_1) \rightarrow C(P, U_2)$, let $\{Q_i\}_i$ be an extension of P so that for each i , $\text{supp } Q_i$ contains balls of radius $\text{diam } P$ centered around the boundary of $\text{supp } P$. Pick any $x_1 \in U_1$ and $x_2 \in U_2$, then for each i , it is not too hard to see

(Fig. 3.3) that

$$C(Q_i, U_1) + U_2 - (x_1 + x_2)/2 = C(Q_i, U_2) + U_1 - (x_1 + x_2)/2$$

and $U_1 + U_2 - (x_1 + x_2)/2 \subseteq \text{supp } Q_i$, forming a basic open set in the unstable groupoid. \square

Using the topological conjugacy that is the Robinson map, we bring this over to $\varprojlim_n (AP_n, \sigma)$ and obtain the unstable groupoid, also denoted G_u , whose topology is generated by partial homeomorphisms. However, due to the possible existence of branches on each of the AP -complexes, our definitions look slightly different from what is expected if we applied the Robinson map directly. Applying the description given towards the end of the last chapter gives that the two are equivalent.

Definition 3.7. The *unstable groupoid* on $\varprojlim_n (AP_n, \sigma)$ is the set pairs of points $(p_0, \dots), (q_0, \dots)$ such that there exists a set $\sigma^n(U_n) \times \dots \times \sigma(U_n) \times U_n \times U_{n+1} \times \dots$ such that for all $m \geq n$, $U_m \subseteq AP_m$ is acu, compatible with σ , and $p_m, q_m \in U_m$, i.e.

$$G_u = \left\{ ((p_0, \dots), (q_0, \dots)) \in \varprojlim_n (AP_n, \sigma)^2 : \begin{array}{l} \exists U_n \times U_{n+1} \times \dots \text{ such that} \\ \forall m \geq n, U_m \subseteq AP_m \text{ is acu,} \\ \sigma(U_{m+1}) = U_m, \text{ and } p_m, q_m \in U_m \end{array} \right\}.$$

A *partial homeomorphism* on $\varprojlim_n (AP_n, \sigma)$ is an equivalence class of homeomorphisms $\phi : C(U_{1,n}) \rightarrow C(U_{2,n})$ together with some acu $U_n \subseteq AP_n$ such that $U_{1,n}, U_{2,n} \subseteq U_n$ that is its *support*, denoted $\text{supp } \phi$, modulo the source, the range, and the support.

When we refer to sets of the form $\sigma^n(U_n) \times \dots \times \sigma(U_n) \times U_n \times U_{n+1} \times \dots$, we will always assume

acu starting at the level where σ is not used. An analogous proposition for $\varprojlim_n (AP_n, \sigma)$ under the Robinson map shows that the set of partial homeomorphism forms a basis of this unstable groupoid.

A concrete description of a partial homeomorphism can be given as follows. Let $\phi : U_{1,n} \rightarrow U_{2,n}$ denote the underlying partial homeomorphism on AP_n . Then for any $(p_0, \dots) \in \varprojlim_n (AP_n, \sigma)$, $\phi(p_0, \dots) = (q_0, \dots)$ has its coordinates $\phi(p_n) = q_n$, and for all $m \geq 0$, $\sigma^{n-m}(q_n) = q_m$. This is possibly not well-defined for $m > n$, but our choice of support has a single preimage under σ^{n-m} that contains p_m . We pick q_m to be the unique point in this preimage. That is to say, if $\phi : U_{1,n} \rightarrow U_{2,n}$ is a homeomorphism, then for all $m > n$, each of the preimages of the support under σ^{n-m} determines a homeomorphism between the pair of corresponding preimages of $U_{1,n}$ and $U_{2,n}$ under σ^{n-m} .

Before proceeding further, since we will utilize it later, this description lets us extend partial homeomorphisms.

Proposition 3.8 (Extension). *Suppose that $\phi : C(U_{1,n}) \rightarrow C(U_{2,n})$ is a partial homeomorphism.*

- *If $m \geq n$ and $\psi : C(V_{1,m}) \rightarrow C(V_{2,m})$ is a partial homeomorphism so that $\sigma^{n-m}(\text{supp } \phi) \subseteq \text{supp } \psi$ and $V_{i,m} = \sigma^{n-m}(U_{i,n})$ for $i = 1, 2$, then $\phi = \psi$.*
- *More generally, if $m \geq n$ and $\{\psi_i\}$ is a collection of partial homeomorphisms in AP_m so that $\sigma^{n-m}(\text{supp } \phi) \subseteq \bigcup_i \text{supp } \psi_i$ with each $\text{supp } \psi_i$ containing exactly one preimage in $\sigma^{n-m}(\text{supp } \phi)$, and the sources and ranges are exactly $\sigma^{n-m}(U_{i,n})$, respectively, then $\phi = \bigcup_i \psi_i$.*

Proof. Extending the cylinder set $C(\text{supp } \phi)$ to some level $m \geq n$ then restricting the preimages of $C(U_{1,n})$ and $C(U_{2,n})$ to those that belong to $\sigma^{n-m}(\text{supp } \phi)$ and are compatible under σ^{m-n}

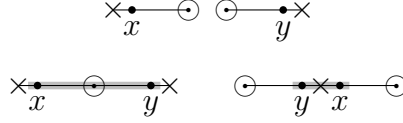


Figure 3.4: An AP_0 -complex with two prototiles with the \times 's and \odot 's identified and two distinct points (top) whose translation is not well-defined until we choose a support that contains them (bottom).

gives our desired collection of partial homeomorphisms. \square

Remark 3.9. One may notice that this is a restricted version of a similar proposition that exists on the unstable groupoid on Ω_T . This proposition can be generalized to allow mixing of different levels of $\text{supp } \psi_i$ in $\varprojlim_n (AP_n, \sigma)$, but it is harder to state. We will also not require this generality.

Remark 3.10. The support is vital in the definition of a partial homeomorphism. If one does not choose the support, then being a compact branched manifold without boundary, AP_n allows multiple ways to translate one point in $U_{1,n}$ to another in $U_{2,n}$ (Fig. 3.4). Choosing the support fixes the translation vector and lets the action to be well-defined. Alternatively, one can use the correspondence between acu sets in AP_n and patches in Ω_T .

Noting that if $\text{supp } \phi$ resides in the interior of a d -cell, its preimages under σ necessarily also only reside in the interiors of d -cells, there is a dichotomy on the set of partial homeomorphisms, based on the support. To describe it, we first define a new topological groupoid.

Definition 3.11. The *tail_n groupoid*, denoted $G_{\text{tail},n}$, is the subgroupoid of the unstable groupoid with its topology generated by partial homeomorphisms whose supports reside within a single d -cell in AP_n . This groupoid consists of pairs of points $(p_0, \dots), (q_0, \dots) \in \varprojlim_n (AP_n, \sigma)$ so that there exists a $\sigma^n(U_n) \times \dots \times \sigma(U_n) \times U_n \times U_{n+1} \times \dots$ containing both points and for all $m \geq n$,

$U_m \subseteq AP_m$ in the interior of a single d -cell, i.e.

$$G_{\text{tail},n} = \left\{ \begin{array}{l} \exists U_n \times U_{n+1} \times \dots \text{ such that} \\ ((p_0, \dots), (q_0, \dots)) \in G_u : \quad \forall m \geq n, p_m, q_m \in U_m \text{ and} \\ \exists d\text{-cells } e_{d,m} \subseteq AP_m \text{ such that } U_m \subseteq e_{d,m} \end{array} \right\}.$$

$U_n \times U_{n+1} \times \dots$ is the *tail*.

There exists a bonding map $\sigma^\top : G_{\text{tail},n} \rightarrow G_{\text{tail},n+1}$. Given any two points that belong to $G_{\text{tail},n}$ via some $\sigma^n(U_n) \times \dots \times \sigma(U_n) \times U_n \times U_{n+1} \times \dots$, since $\sigma(U_{n+1}) = U_n$, one can instead write it as $\sigma^{n+1}(U_{n+1}) \times \dots \times \sigma^2(U_{n+1}) \times \sigma(U_{n+1}) \times U_{n+1} \times \dots$, so the two points belong to $G_{\text{tail},n+1}$ as well, and σ^\top is the inclusion map. While this notation for the map may be misleading, as it is not the dual to the substitution map, on inducing it on the level of K -theory it becomes the dual.

Definition 3.12. We can form the *tail groupoid*, denoted G_{tail} , by taking the inductive limit

$$G_{\text{tail}} = \varinjlim_n (G_{\text{tail},n}, \sigma^\top) = \bigcup_n G_{\text{tail},n}.$$

The *residual groupoid*, denoted G_{res} , is generated, as a groupoid, by $G_u \setminus G_{\text{tail}}$. More concretely, this groupoid is generated from pairs of points such that for all $\sigma^n(U_n) \times \dots \times \sigma(U_n) \times U_n \times U_{n+1} \times \dots$ containing them, each set in each coordinate intersects the interiors of two d -cells, i.e.

$$G_{\text{res}} = \left\langle \left\{ \begin{array}{l} \forall U_n \times U_{n+1} \times \dots \text{ such that} \\ ((p_0, \dots), (q_0, \dots)) \in G_u : \quad \forall m \geq n, p_m, q_m \in U_m, \\ U_m \cap AP_m^{(d-1)} \neq \emptyset \end{array} \right\} \right\rangle.$$

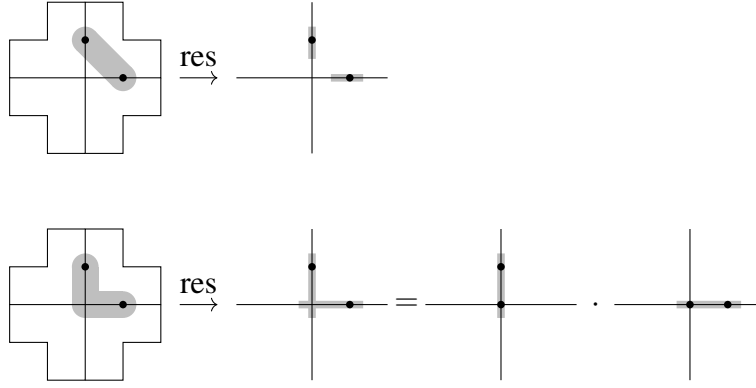


Figure 3.5: If we do not impose contractibility on the supports of partial homeomorphisms, then a partial homeomorphism on G_u may not yield a partial homeomorphism on $G_u^{(1)}$ from restriction to the 1-skeleton (top). Assuming contractibility gives a sequence of partial homeomorphisms in $G_u^{(1)}$ that composes to the desired partial homeomorphism (bottom).

Warning 3.13. The tail_n and tail groupoids on $\varprojlim_n (AP_n, \sigma)$ are very much not finite- or approximately finite-dimensional! We use this term since on restricting to punctures, they become so, and the terminologies tail and AF coincide there.

This dichotomy allows us to put a filtration by dimension on the unstable groupoid. Consider the unstable groupoid on the $d - 1$ -skeleton, $G_u^{(d-1)}$, that is the restriction of G_u to $\varprojlim_n (AP_n^{(d-1)}, \sigma)$. By unrestricting to a small neighborhood of the $d - 1$ -skeleton, we can view $G_u^{(d-1)}$ as a subgroupoid of G_u . Then the groupoid generated by G_{tail} and $G_u^{(d-1)}$ is exactly G_u , and we can replace G_{res} by $G_u^{(d-1)}$ since they present the same topological information in G_u .

Proceeding in the same way G_{tail} and G_{res} are constructed, we obtain $G_{\text{tail}}^{(d-1)}$ and $G_{\text{res}}^{(d-1)}$. To obtain the filtration by dimension, we repeat this construction for each dimension $0 \leq k < d$ inductively in descending order. $G_{\text{res}}^{(0)}$ is empty, since $G_u^{(0)}$ has its diagonal discrete points and there is no nontrivial action among them, so $G_{\text{tail}}^{(0)}$ sees the entire groupoid $G_u^{(0)}$. We will use $G_{\text{tail}}^{(0)}$ instead of $G_u^{(0)}$.

Notice that due to the contractibility condition, if we are given two points in the $d - 1$ -

skeleton that belong in G_u via some $\sigma^n(U_n) \times \cdots \times \sigma(U_n) \times U_n \times U_{n+1} \times \cdots$, restriction of this set to the skeleton gives a similar collection that is connected. Had we not imposed contractibility, the resulting restriction might be disconnected (Fig. 3.5).

Unfortunately, in the same vein as the warning in the previous chapter regarding lower-dimensional skeleta, this restriction presents an issue when we want to put a topology on $G_u^{(k)}$ for $0 \leq k < d$. If we have a partial homeomorphism whose source and range cylinder sets restrict to interiors of different k -cells, thus are open balls in \mathbb{R}^k , then its support crosses a $k - 1$ -cell. If the k -skeleton is branched at the $k - 1$ -cell, then the support, coming from restriction, necessarily sees all of the branches.

As we want the partial homeomorphism to still arise from a k -dimensional translation action, one could fix the branching by deformation retracting the “unused” branches until we obtain an open ball in \mathbb{R}^k . This is then an actual partial homeomorphism. However, this allows for sources or ranges, and therefore cylinder sets, to be centered about branch points. At the branching $k - 1$ -cell, intersecting two cylinder sets that share a single branch yields a set that is not open.

Using a trick adapted from [GMPS08] and [GMPS09], one easy way to resolve this issue is to, rather than restricting to skeleta, instead restrict to their ϵ -neighborhoods that become acyclic upon border-forcing, where contractibility and unbranching are conditions per cell, not on the overall skeleton. Thus, while we morally want to work with groupoids restricted to skeleta, we will instead consider the subgroupoids of G_u with topology generated by partial homeomorphisms with supports in ϵ -neighborhoods of the skeleta. Each k -skeleton forms an abstract transversal of G_u restricted to the ϵ -neighborhood of the skeleton, and the new groupoids that we form by restricting to the ϵ -neighborhood, still called $G_u^{(k)}$, $G_{\text{tail}}^{(k)}$, and $G_{\text{res}}^{(k)}$, respectively, are therefore groupoid

equivalent to the ones defined on the k -skeleton.

We now state these definitions precisely.

Definition 3.14. Let $\epsilon > 0$.

Two open balls in the k -skeleton are *partially homeomorphic* if there exists a partial homeomorphism in G_u , supported on an ϵ -neighborhood of the k -skeleton so that its source and range, restricted to the skeleton, are the two open balls.

- The *unstable groupoid of the k -skeleton*, $G_u^{(k)}$, is the topological space with a basis of such partial homeomorphisms.
- The *tail groupoid of the k -skeleton*, $G_{\text{tail}}^{(k)}$, is the subgroupoid of $G_u^{(k)}$ generated by partial homeomorphisms (that live in G_u) whose support does not intersect the $k - 1$ -skeleton.
- The *residual groupoid of the k -skeleton*, $G_{\text{res}}^{(k)}$, is generated by the difference of the two previous groupoids.

One can also define the tail_n groupoid of the k -skeleton, $G_{\text{tail},n}^{(k)}$, similarly, by restricting to supports that do not intersect $AP_n^{(k)}$. The same process as before allows us to replace the residual groupoid $G_{\text{res}}^{(k)}$, for $1 \leq k \leq d$ by $G_u^{(k-1)}$.

It is crucial that we define each of the groupoids starting from G_u , followed by restriction, since otherwise one can form compositions of partial homeomorphisms on $G_u^{(k)}$ whose sources and ranges belong to different k -cells but avoid the $k - 1$ -skeleton. This is in fact the key reason our groupoids are stated as restrictions from G_u . This is guaranteed to not occur since the supports of partial homeomorphisms in G_u are contractible.

Remark 3.15. The term “partial homeomorphism” is no longer accurate, since there exist partial

homeomorphisms in G_u taking open balls intersecting $k - 1$ -cells to neighboring k -cells. If a $k - 1$ -cell is branched, the restriction of the source is branched, which is not homeomorphic to the restriction of the range, which is unbranched.

If one forgoes restriction, then this is not an issue. However, the fact that partial homeomorphisms can “bypass” lower-dimensional skeleta makes the groupoid equivalence to the punctured tiling space harder to state.

Lastly, while a sequence of partial homeomorphisms can still avoid the $k - 2$ -skeleton (or lower), our filtration on the unstable groupoid allows us to build skeleta dimension-by-dimension. More precisely, inductively, since the attaching maps are given in terms of consecutive-dimensional cells, this avoidance is a nonissue.

We now state an obvious but more useful version of extension for each $G_u^{(k)}$ and $G_{\text{tail}}^{(k)}$.

Proposition 3.16 (Extension). *Any partial homeomorphism in $G_u^{(k)}$ can have its support extended to the entire interiors of (the ϵ -neighborhood of) the k -cells it intersects. For partial homeomorphisms in $G_{\text{tail}}^{(k)}$, we may assume that the support is the interior of (the ϵ -neighborhood of) a single k -cell.*

From now on, we will assume that the supports of partial homeomorphisms in the two groupoids have these forms.

3.2 Discrete groupoids

Let us now restrict to punctures to obtain the more familiar discrete groupoids. Since we will want to work with lower-dimensional skeleta, let us puncture each of the lower-dimensional boundaries of the tiles in the tiling as well, called k -punctures for dimension k , and pass them

through the Robinson map (or π_n if we want to work with the AP_n -complex), so as to mimic our filtration by dimension. On AP_n , there are as many punctures on each k -cell as there are k -cells on applying σ^n , counting each occurrence of the same k -cells in AP_0 separately. On the tiling space, this would be the number of k -dimensional boundaries of tiles that are in a k -dimensional boundary of a level- n supertile.

The set of k -punctures forms an abstract transversal of the k -skeleton, thus restriction gives a groupoid equivalence. We will, however, give a slightly different presentation of the partial homeomorphisms that form a basis of the topology. To properly restrict, we will slightly amend the partial homeomorphisms that generate the topology on G_u to be those whose sources and ranges are sufficiently small, so that they contain one puncture each at the most. We also pick ϵ (in the definitions of lower-dimensional groupoids) sufficiently small so that they only contain punctures from their respective skeleton. Then, on restriction, cylinder sets that contain punctures have their support turn into a collection of punctures and their open ball of origins turn into punctures themselves.

Definition 3.17. The *discrete unstable groupoid of the k -skeleton*, $\dot{G}_u^{(k)}$, is the restriction of $G_u^{(k)}$ to the k -punctures. A *partial homeomorphism* on the k -skeleton is a triple $[P, t', t]$, where P is a set of punctures inside an acu set, or a *punctured acu set*, in some $AP_n^{(k)}$, and t and t' are elements of P . Composition of partial homeomorphisms is given by

$$[P, t'', t'] \cdot [P, t', t] = [P, t'', t].$$

Let $s([P, t', t]) = t$ be the *source*, and $r([P, t', t]) = t'$ be the *range*. We omit the superscript if $k = d$.

It is well-known that \dot{G}_u is étale, although the unstable groupoids of the skeleta may not be.

Remark 3.18. A partial homeomorphism defined this way is also called a *doubly-pointed pattern*.

In order to maintain the set of doubly-pointed patterns as a left action on the set of punctures, we have elected to use the second puncture as the source and the first as the range, rather than the reverse as in [KP00].

It is important to bear in mind that a punctured acu set comes from an underlying acu set, since there are numerous sets that restrict down to the same set of punctures, some of which may not be allowed, contractible, or unbranched. To prevent ambiguity, we will specify the support using the underlying acu set.

We can similarly define the restricted tail_n , tail , and residual groupoids, but we switch to different terminology for the first two that is more suggestive of their defining features.

Definition 3.19.

- The *finite-dimensional groupoid at level- n of the k -skeleton*, or AF_n -groupoid, denoted $\dot{G}_{AF,n}^{(k)}$, is the restriction of $G_{\text{tail},n}^{(k)}$ to the k -punctures.
- The *approximately finite-dimensional groupoid of the k -skeleton*, or AF -groupoid, denoted $\dot{G}_{AF}^{(k)}$, is the restriction of $G_{\text{tail}}^{(k)}$ to the k -punctures.
- The *discrete residual groupoid of the k -skeleton*, denoted $\dot{G}_{\text{res}}^{(k)}$, is the restriction of $G_{\text{res}}^{(k)}$ to the k -punctures.

Each is groupoid equivalent to their unrestricted analogues. Whenever it is unclear, to distinguish from the discrete groupoid, we will refer to the unrestricted groupoids as *continuous*.

We now look at their associated (reduced) groupoid C^* -algebras. Recall that given a étale

groupoid G , we can form the continuous compactly-supported functions on G with values in \mathbb{C} , denoted $C_c(G)$, and give it the convolution product

$$fg(a) = \sum_{bc=a} f(b)g(c)$$

where the sum is finite due to local compactness, and involution

$$f^*(a) = \overline{f(a^{-1})}.$$

From [Sim17, Proposition 3.3.1], for each $x \in G^0$, we have *the regular representation associated to x* , $\pi_x : C_c(G) \rightarrow \mathcal{B}(\ell^2(s^{-1}(x)))$, such that

$$\pi_x(f)\delta_a = \sum_{s(b)=r(a)} f(b)\delta_{ba}.$$

Forming the direct sum over all elements of G^0 gives us the (*reduced*) *groupoid C^* -algebra* of G , $C_r^*(G)$, as the completion of

$$\bigoplus_{x \in G^0} \pi_x(C_c(G)) \subseteq \bigoplus_{x \in G^0} \mathcal{B}(\ell^2(s^{-1}(x)))$$

under the supremum of the operator norm.

Our goal will be to obtain K -theoretic information of $C_r^*(\dot{G}_u)$ from that of $C_r^*(\dot{G}_{AF}^{(k)})$. For most of the remainder of this section, we establish some elementary facts and consequences. To do so, let us denote the characteristic function on partial homeomorphisms by $e[P, t', t]$. Note that these are elements of $C_r^*(\dot{G}_u)$.

There are three types of elements that are going to be crucial in our discussion,

- *Projections*, i.e. elements p so that $p^2 = p$,
- *Partial isometries*, i.e. elements v so that the *source*, v^*v , and the *range*, vv^* , are projections, and
- *Unitaries*, i.e. elements u so that $u^*u = uu^* = 1$.

Let $\mathcal{P}(\cdot)$ denote the set of projections of a C^* -algebra, and let $\mathcal{U}(\cdot)$ denote the group of unitaries.

We have several properties inherited from partial homeomorphisms that we will not prove.

Proposition 3.20.

1. $e[P, t'', t']e[P, t', t] = e[P, t'', t]$.
2. $e[P, t, t]$ is a projection, denoted $e[P, t]$, and $e[P, t', t]$ is a partial isometry.
3. $e[P, t', t]^* = e[P, t, t']$.

Furthermore, the extension properties also hold here. We skip their statements since they are the same, but restricted to punctures.

These allow us to show that $C_r^*(\dot{G}_{AF}^{(k)})$ is a familiar C^* -algebra.

Proposition 3.21. $C_r^*(\dot{G}_{AF}^{(k)})$ is the closure of the direct limit of direct sums of finite-dimensional algebras, where the bonding maps are specified by the transpose of the induced substitution map on k -cells.

Proof. Given any partial homeomorphism in $\dot{G}_{AF,n}^{(k)}$ where the support is some k -cell in $AP_n^{(k)}$, suppose that we are given an order on the punctures, say $\{t^i\}_{i=1}^m$. Consider the matrix algebra

$M_m(\mathbb{C})$ where the diagonal entries represent the ordered set of punctures. Then the elementary matrix e_{ji} is exactly the element $e[P, t^j, t^i]$. Repeating this argument for each k -cell and noting that there are no partial homeomorphisms between different k -cells shows that $C_r^*(\dot{G}_{AF,n}^{(k)})$ is the direct sum of matrix algebras consisting of the punctures inside each k -cell, with one summand for each k -cell. As we will see, the punctures cannot be arbitrarily ordered.

There are as many punctures in a single k -cell in $AP_{n+1}^{(k)}$ as there are punctures in each of the k -cells that it subdivides to in $AP_n^{(k)}$. Given any partial homeomorphism in $\dot{G}_{AF,n}^{(k)}$ with support P an entire k -cell, by extension, we can write it as a sum $\sum_i e[P_i, t'_i, t_i]$, with each P_i k -cells in $AP_{n+1}^{(k)}$, possibly duplicated. Applying this to any pair of punctures in P , we see that each P_i contains copies of the matrix algebra coming from P , specified by σ^\top . The order on the punctures is thus contingent on the order on the previous level, and we are only allowed a choice of order between sets of punctures of consecutive levels. By stationarity of the substitution rule, we maintain the same order between any two consecutive levels, and the order on the punctures on $AP_1^{(k)}$ completely determines the rest. \square

Closures of direct limits of direct sums of finite-dimensional algebras are called *AF-algebras*.

The above proof motivates a type of graphs associated to *AF-algebras*.

Definition 3.22. A *Bratteli diagram* is a graph (V, E) consisting of a vertex set $V = \{V_n\}_{n=0}^\infty$, each of the V_n nonempty and consisting of finitely many vertices, and an edge set $E = \{E_n\}_{n=0}^\infty$, each of the E_n nonempty and consisting of finitely many edges between V_n and V_{n+1} , so that for all $n \geq 0$, each vertex in each V_n is incident to an edge in E_n , and for all $n \geq 1$, each vertex in each V_n is incident to an edge in E_{n-1} .

The *path space* X_E is the subset of $\prod_{n=0}^\infty E_n$ consisting of paths in the graph. The AF_n -

equivalence relation on the path space is the set of pairs of paths in the path space whose edges coincide after level- n , i.e.

$$\{((p_0, \dots), (q_0, \dots)) \in X_E^2 : \forall m \geq n, p_m = q_m\}.$$

The *AF-equivalence relation* is the union of all AF_n -equivalence relations.

For each AF -algebra, we can form an associated Bratteli diagram by taking each vertex at level- n to be each direct summand in the same level of the direct system, with each edge representing a single inclusion of a direct summand in one level into a direct summand in the next.

Fixing a $0 \leq k \leq d$, let us form a Bratteli diagram for the direct system for $C_r^*(\dot{G}_{AF}^{(k)})$, where at each level there are exactly as many vertices as there are k -cells, and the incidence matrix between consecutive levels is the transpose of the substitution matrix σ^\top . It follows from the proof of Proposition 3.21 that each finite path in $\prod_{n=0}^{N-1} E_n$ between V_0 and V_N can be identified to a k -puncture at level- N , and each vertex in the Bratteli diagram at level- N is a matrix algebra whose elementary matrices represent the characteristic functions on partial homeomorphisms among each pair of k -punctures. That is, the AF -equivalence relation on the Bratteli diagram formed from $C_r^*(\dot{G}_{AF}^{(k)})$ is exactly the AF -groupoid of the k -skeleton.

Remark 3.23. For $d = 1$, there is an order one should choose on the d -punctures in AP_1 that is the order of the punctures under π^{-1} . This gives a well-defined (up to a sign) *ordered* Bratteli diagram where the order of the punctures yields a *Bratteli–Vershik map*.

As an immediate consequence, together with the description of the unstable and residual groupoids, we have a weaker form (groupoid equivalence rather than homeomorphism) of

[BJS10, Theorem 3.22] and [JS10, Theorem 4.9].

Theorem 3.24 (Bellissard–Julien–Savinien, Julien–Savinien).

- *The unstable groupoid on the punctured tiling space is groupoid equivalent to a quotient of the AF-equivalence relation on the top-dimensional Bratteli diagram.*
- *The unstable groupoid on the punctured tiling space is groupoid equivalent to an inductive quotient of the AF-equivalence relation on the k -dimensional Bratteli diagram by a quotient of the AF-equivalence relation on the $k - 1$ -dimensional Bratteli diagram.*

Proof. We start from the continuous groupoid on $\varprojlim_n (AP_n, \sigma)$, $G_u^{(k)} = G_{\text{tail}}^{(k)} \cup G_{\text{res}}^{(k)}$. Let us consider $G_{u,\epsilon}^{(k)} \leq G_u^{(k)}$ whose diagonal contains exactly the k -cells neighboring the $k - 1$ -skeleton. Restricting this groupoid to the k - and $k - 1$ -punctures, we see that the set of $k - 1$ -punctures in the $k - 1$ -skeleton forms an abstract transversal for $\dot{G}_{u,\epsilon}^{(k)}$, thus it is groupoid equivalent to $\dot{G}_u^{(k-1)}$. Furthermore, any element in $\dot{G}_{\text{res}}^{(k)}$ can be written as a composition of elements from $\dot{G}_{AF}^{(k)}$ and $\dot{G}_{u,\epsilon}^{(k)}$. Since $\dot{G}_{AF}^{(k)}$ is the AF-equivalence relation on the k -dimensional Bratteli diagram, $\dot{G}_u^{(k)}$ is obtained by having certain AF-equivalence classes identified by $\dot{G}_{u,\epsilon}^{(k)}$, and hence by $\dot{G}_u^{(k-1)}$ by groupoid equivalence.

To obtain the overall groupoid equivalence in the theorem, we restrict the unstable groupoid on Ω_T to the d -punctures, and compose this with the Robinson map applied to the above explanation. □

In the interest of giving an interpretation that will be useful in the next chapter, we will provide an explicit description purely in terms of doubly-pointed patterns. Given a partial homeomorphism $[P, t', t] \in \dot{G}_u^{(k)} \setminus \dot{G}_{AF,n}^{(k)}$ for $k \geq 1$, we can decompose it into a sequence of partial

homeomorphisms $\{[P, t'_i, t_i]\}_{i=1}^m$ so that t_i and t'_i are k -punctures neighboring a $k - 1$ -cell on projecting to AP_0 , $t_{i+1} = t'_i$ for $1 \leq i < m$, and $t_1 = t$ and $t'_m = t$, so that

$$[P, t'_m, t_m] \cdots [P, t'_2, t_2] \cdot [P, t'_1, t_1] = [P, t', t].$$

Since the partial homeomorphism $[P, t', t]$ does not belong to the AF_n -groupoid of the k -skeleton, t and t' belong to different k -cells in AP_n , and therefore there exists at least one index $1 \leq i \leq m$ so that t_i and t'_i belong to different k -cells. In particular, they neighbor a $k - 1$ -puncture in a $k - 1$ -cell in AP_n . We can then think of such a $[P, t'_i, t_i]$ as flowing from t_i to this $k - 1$ -puncture, then to t'_i . Thus this sequence of partial homeomorphisms forms a path of neighboring k - and $k - 1$ -punctures, with segments that stay within a single k -cell, and to move between different k -cells, we move first to a $k - 1$ -puncture, then to the other k -cell. In other words, $k - 1$ -punctures in AP_n can be thought of as the nontrivial partial homeomorphisms in $\dot{G}_u^{(k)}$ relative to $\dot{G}_{AF,n}^{(k)}$.

Remark 3.25. An arbitrary sequence is unlikely to be efficient in terms of crossing as few $k - 1$ -boundaries as possible. However, one can “subtract” by loops until it is.

Unfortunately, this interpretation highlights the insufficiency of this theorem towards a naïve calculation of the K -theory of \dot{G}_u using the dimension filtration. That is, to compute the K -theory of \dot{G}_u , one needs to inductively compute the K -theory of $\dot{G}_u^{(k)}$ using $\dot{G}_{AF}^{(k)}$ and $\dot{G}_u^{(k-1)}$, where one thinks of $\dot{G}_u^{(k-1)}$ as “gluing” elements of $\dot{G}_{AF}^{(k)}$ together. If $AP_n^{(k)}$ is branched at some $k - 1$ -cell with m -branches, then that corresponding puncture simultaneously represents, up to orientation, $m - 1$ partial homeomorphisms (all others can be generated by combining). Furthermore, this m is not consistent between different $k - 1$ -cells, which presents difficulties when there are cylinder sets in the k -skeleton that witness different $k - 1$ -cells when extending.

One way to remedy this is to use a more general K -theoretic object called *relative K -theory*. We will then see a more precise topological explanation for why the naïve computation does not work.

Remark 3.26. There is a way around where one employs a process similar to *state-splitting* that *does not preserve the K -theory of the AF -groupoid* (but does preserve the K -theory of the unstable groupoid), to the substitution rule to ensure the existence of a subgroupoid of $\dot{G}_u^{(k)}$ closed under a k -parameter subgroup of \mathbb{R}^d , e.g. [JS16] where one takes the chair tiling and splits each supertile into three squares. It is unclear if this process always returns a stationary substitution rule.

Chapter 4: K -theory

In this chapter we recall ordinary K -theory, then briefly outline the key facts from relative K -theory and excision, leaving the details for the reader to fill in from [Has21], [Put97], and [Put21]. We will only sketch the relative K_0 -group, and leave the relative K_1 -group to be deduced from topology and exactness. We will also only state excision as it appears in [Put98] since we do not need its full power.

4.1 Ordinary K -theory

Given a unital C^* -algebra A , let $M_n(A)$ denote the algebra of $n \times n$ matrices with entries in A . Let

$$M_\infty(A) = \bigcup_{n=1}^{\infty} M_n(A)$$

under the embedding $a \mapsto \text{diag}(a, 0)$. Let the binary operation on $\mathcal{P}(M_\infty(A))$ be direct sum, i.e.

$$p \oplus q = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}.$$

Two projections are *Murray–von Neumann (MvN) equivalent* if there exists a partial isometry whose source and range are the two projections. The K_0 -group of A , $K_0(A)$, is the Grothendieck

group of the abelian semigroup formed from $\mathcal{P}(M_\infty(A))$ with direct sum, under Murray–von Neumann equivalence in $M_\infty(A)$.

We now consider

$$M_\infty(A) = \bigcup_{n=1}^{\infty} M_n(A)$$

under the embedding $a \mapsto \text{diag}(a, 1)$. Let the binary operation on $\mathcal{U}(M_\infty(A))$ still be direct sum. Two unitaries $u, v \in \mathcal{U}(M_\infty(A))$ are *stably homotopic* if there exists a sufficiently large n so that $\text{diag}(u, 1_k), \text{diag}(v, 1_\ell) \in \mathcal{U}(M_n(A))$ are homotopic within $\mathcal{U}(M_n(A))$. The K_1 -group of A , $K_1(A)$, is the group of equivalence classes of stably homotopic unitaries in $\mathcal{U}(M_\infty(A))$ with direct sum.

We have the following standard properties. See, for example, [RLL00].

Proposition 4.1. *For $i = 1, 2$, K_i is a continuous functor. That is, if $\varinjlim_n (A_n, \phi_n)$ is the closure of a direct limit of $*$ -homomorphisms of C^* -algebras $\phi_n : A_n \rightarrow A_{n+1}$, then $K_i(\varinjlim_n (A_n, \phi_n)) \cong \varinjlim_n (K_i(A_n), \phi_{n*})$.*

Proposition 4.2. $K_0(\mathbb{C}) = K_0(M_n(\mathbb{C})) = \mathbb{Z}$, and $K_1(\mathbb{C}) = K_1(M_n(\mathbb{C})) = 0$.

Corollary 4.3. *If an AF-algebra A arises from a Bratteli diagram (V, E) with incidence matrices $\{h_{E_n}\}$, then $K_0(A) = \varinjlim_n (\mathbb{Z}V_n, h_{E_n})$ and $K_1(A) = 0$.*

The following is [MRW87, Theorem 2.8] adapted for abstract transversals.

Theorem 4.4 (Muhly–Renault–Williams). *If H is a subgroupoid of an étale groupoid G , and the two are equivalent via an abstract transversal N , then for $i = 1, 2$, $K_i(C_r^*(G)) \cong K_i(C_r^*(H))$.*

4.2 Relative K -theory

We now move on to relative K -theory. Given a map of unital C^* -algebras $\phi : A \rightarrow B$, consider triples of the form (p, q, v) where $p, q \in M_\infty(A)$ are projections and $v \in M_\infty(B)$ is a partial isometry so that $s(v) = \phi(p)$ and $r(v) = \phi(q)$. That is, p and q are Murray–von Neumann equivalent under ϕ . Two triples (p_1, q_1, v_1) and (p_2, q_2, v_2) are *isomorphic* if there exist partial isometries $w_1, w_2 \in M_\infty(A)$ so that $s(w_1) = p_1$, $r(w_1) = p_2$, $s(w_2) = q_1$, $r(w_2) = q_2$, and $\phi(w_2)v_1 = v_2\phi(w_1)$. A triple (p, q, v) is *elementary* if $p = q$ and there is a path of partial isometries $\{v_t\}_{0 \leq t \leq 1} \subseteq B$ so that $v_0 = \phi(p)$, $v_1 = v$, and $s(v_t) = r(v_t) = \phi(p)$. Letting the binary operation be component-wise direct sum, we define the *relative K_0 -group of ϕ* , denoted $K_0(A; B)$, to be the equivalence classes of triples under isomorphism and stabilization with respect to taking direct sums with elementary triples. Note that the map ϕ affects the group, but we leave it implicit in the notation.

We have several key properties that we will use as black boxes.

Proposition 4.5.

1. $[p_1, q_1, v_1] + [p_2, q_2, v_2] = [p_1, q_2, v_2 v_1]$ if $q_1 = p_2$.
2. $-[p, q, v] = [q, p, v^*]$.
3. *Relative K -theory is functorial. That is, if the diagram*

$$\begin{array}{ccc} A_1 & \xrightarrow{\phi_1} & B_1 \\ \downarrow \psi_1 & & \downarrow \psi_2 \\ A_2 & \xrightarrow{\phi_2} & B_2 \end{array}$$

of $*$ -homomorphisms between C^* -algebras commutes, then there exists a ψ_* :

$K_i(A_1; B_1) \rightarrow K_i(A_2; B_2)$ for $i = 0, 1$ such that $\psi_*([p, q, v]) = [\psi_1(p), \psi_1(q), \psi_2(v)]$.

The main tool is the *six-term sequence in relative K -theory* ([Has21, Theorem 2.1]).

Theorem 4.6 (Haslehurst). *There is an exact sequence*

$$\begin{array}{ccccc} K_0(A; B) & \xrightarrow{\text{ev}} & K_0(A) & \xrightarrow{\phi_*} & K_0(B) \\ \uparrow & & & & \downarrow \\ K_1(B) & \longleftarrow & K_1(A) & \longleftarrow & K_1(A; B) \end{array}$$

where $\text{ev}([p, q, v]) = [p] - [q]$ is the evaluation map.

4.3 Excision

We now proceed with excision in the context of open subgroupoids in (reduced) groupoid C^* -algebras in [Put98]. Let G be a groupoid that is an equivalence relation on a topological space, and let $L \subseteq G$ as a subspace (*not as a subgroupoid!*) be such that

- L is closed,
- $r(L) \cap s(L) = \emptyset$,
- $G' = G \setminus (L \cup L^{-1})$ is such that $G'G' \leq G$, and
- $LG', G'L \subseteq L$.

Let the topology on $s(L) = L^{-1}L$ be given by the convergence of sequences, $\{x_n\}_n$ converges to x if and only if there exist sequences $\{y_n\}_n$ and $\{z_n\}_n$ in L converging in G so that $x_n = y_n^{-1}z_n$,

and similarly for $r(L) = LL^{-1}$. Let $H' = s(L) \cup r(L)$ and $H = H' \cup L \cup L^{-1}$ be given the disjoint union topology. Then we have the following *excision* property from [Put21].

Theorem 4.7 (Putnam). *For $i = 0, 1$,*

$$K_i(C_r^*(G'); C_r^*(G)) \cong K_i(C_r^*(H'); C_r^*(H)).$$

The following corollary from [Put98] will be very useful. We will provide a sketch of the proof as we will need the map explicitly.

Theorem 4.8 (Putnam). *For $i = 0, 1$,*

$$K_i(C_r^*(G'); C_r^*(G)) \cong K_i(C_r^*(H)).$$

Proof. H' has two components, and under the new topology, each forms an abstract transversal to H . Thus $K_i(C_r^*(H')) \cong K_i(C_r^*(H)) \oplus K_i(C_r^*(H))$. Each of the rows of the six-term sequence in relative K -theory then reads

$$K_i(C_r^*(H'); C_r^*(H)) \xrightarrow{\Delta^\top} K_i(C_r^*(H)) \oplus K_i(C_r^*(H)) \xrightarrow{+} K_i(C_r^*(H))$$

where first map $\Delta^\top = (1, -1)$ is the skew-diagonal map and the second is summation. Thus $K_i(C_r^*(H'); C_r^*(H)) \cong K_i(C_r^*(H))$, and excision gives the statement. \square

The way we will use this theorem is that there will be an obvious abstract transversal to H that comes from our choice of L (rather, $L \cup L^{-1}$) that we then restrict to, where the restriction is continuous with respect to the new topology on H , resulting in a groupoid equivalence.

Chapter 5: Čech cohomology and K -theory

Since $\dot{G}_{AF} \leq \dot{G}_u$ is an inclusion of an open subgroupoid, a continuous compactly-supported function on \dot{G}_{AF} is also one on \dot{G}_u , giving us the inclusion $C_c(\dot{G}_{AF}) \hookrightarrow C_c(\dot{G}_u)$ and therefore the inclusion $\iota : C_r^*(\dot{G}_{AF}) \hookrightarrow C_r^*(\dot{G}_u)$. Applying the six-term sequence in relative K -theory to ι gives us

$$\begin{array}{ccccc} K_0(C_r^*(\dot{G}_{AF}); C_r^*(\dot{G}_u)) & \xrightarrow{\text{ev}} & K_0(C_r^*(\dot{G}_{AF})) & \xrightarrow{\iota_*} & K_0(C_r^*(\dot{G}_u)) \\ \uparrow & & & & \downarrow \\ K_1(C_r^*(\dot{G}_u)) & \longleftarrow & 0 & \longleftarrow & K_1(C_r^*(\dot{G}_{AF}); C_r^*(\dot{G}_u)). \end{array}$$

The goal of this chapter is to construct an isomorphic exact sequence in Čech cohomology for $d = 1, 2$, then use it to explicitly illustrate the roles of each $K_0(C_r^*(\dot{G}_{AF}^{(k)}))$. We endeavor to describe as many things in generality as possible before focusing on $d = 1, 2$, and begin by establishing some notation and convention. We will occasionally make use of the boundary hyperplane condition, and will state explicitly where it is used. Although it factors in to some of the proofs, the only place it is required is the reconstruction of $K_1(C_r^*(\dot{G}_u))$ from $K_0(C_r^*(\dot{G}_{AF}^{(1)}))$.

If P_1 and P_2 are punctured acu sets in AP_n , denote $P = P_1 \cup P_2$ the union of the two punctured acu sets. *This is not the union of cylinder sets!* There may be many such P , and we will let context dictate the relevant one. Furthermore, noting that in AP_0 , there is exactly one k -puncture for each k -cell, allowing us to identify the two, we will make use of the fact that we

have a substitution rule and write $\varsigma^n(t)$, t a k -cell in AP_0 , to denote the punctured acu set in $AP_n^{(k)}$ corresponding to subdividing t n -times. By extension, we may replace any punctured acu set with $\bigcup_i \varsigma^n(t_i)$, for some appropriate $\{t_i\}_i$.

5.1 Thickening cochains

Cohomology, and therefore the cellular structure on the AP -complexes, will be integral in this chapter. Let us assume that our k -cells are formed from the intersection of multiple d -cells. There is a type of k -cell that behaves particularly nicely with respect to substitutions.

Definition 5.1. A k -cell satisfies the *boundary hyperplane condition* if it belongs to a hyperplane. T satisfies the *boundary hyperplane condition* if, for all $0 \leq k < d$, all k -cells satisfy the boundary hyperplane condition.

Note that d -cells automatically satisfy the boundary hyperplane condition. With this definition, we will assign orientations to k -cells, for $k < d$, one of two ways.

- If a k -cell does not satisfy the boundary hyperplane condition, then its orientation is assigned arbitrarily.
- If a k -cell satisfies the boundary hyperplane condition, then its orientation is assigned so that if it and another k -cell have preimages under some σ^n belonging in the same k -cell, then they, and each of their preimages, carry the same orientation. For example, assigning the same orientation for parallel k -cells guarantees this. In other words, we want orientations to stay the same within “primitive components”.

For the sake of signs working out, we will pick the right-handed orientation for the d -cells.

Given the cochain complexes for the AP -complexes, one easily verifies that, from the conditions we placed on the orientations of the cells, the substitution map induces a map between consecutive levels that can be identified with σ^\top , letting us form a cochain complex of direct limits

$$\begin{array}{ccccccc}
0 & \longrightarrow & C_0^0 & \xrightarrow{\delta_0^0} & C_0^1 & \xrightarrow{\delta_0^1} & \dots \xrightarrow{\delta_0^{d-1}} C_0^d \longrightarrow 0 \\
& & \downarrow \sigma^\top & & \downarrow \sigma^\top & & \downarrow \sigma^\top \\
0 & \longrightarrow & C_1^0 & \xrightarrow{\delta_1^0} & C_1^1 & \xrightarrow{\delta_1^1} & \dots \xrightarrow{\delta_1^{d-1}} C_1^d \longrightarrow 0 \\
& & \downarrow \sigma^\top & & \downarrow \sigma^\top & & \downarrow \sigma^\top \\
& & \vdots & & \vdots & & \vdots \\
& & \downarrow \sigma^\top & & \downarrow \sigma^\top & & \downarrow \sigma^\top \\
0 & \longrightarrow & C_n^0 & \xrightarrow{\delta_n^0} & C_n^1 & \xrightarrow{\delta_n^1} & \dots \xrightarrow{\delta_n^{d-1}} C_n^d \longrightarrow 0 \\
& & \downarrow \sigma^\top & & \downarrow \sigma^\top & & \downarrow \sigma^\top \\
& & \vdots & & \vdots & & \vdots \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \varinjlim_n (C_n^0, \sigma^\top) & \xrightarrow{\delta^0} & \varinjlim_n (C_n^1, \sigma^\top) & \xrightarrow{\delta^1} & \dots \xrightarrow{\delta^{d-1}} \varinjlim_n (C_n^d, \sigma^\top) \longrightarrow 0.
\end{array}$$

To simplify notation, let us call $C^k = \varinjlim_n (C_n^k, \sigma^\top)$ for each $0 \leq k \leq d$.

We have a key observation that is the motivation behind most things we do.

Lemma 5.2 (The Cochain–Finite-dimensional Algebra Isomorphism (CF)). $K_0(C_r^*(\dot{G}_{AF})) \cong C^d$. For $k < d$, $\varinjlim_n (K_0(C_r^*(\dot{G}_{AF,n}^{(k)})), \sigma^\top) \cong C^k$, and if the boundary hyperplane condition is satisfied, then $K_0(C_r^*(\dot{G}_{AF}^{(k)})) \cong C^k$.

Proof. We picked the bonding maps $K_0(C_r^*(\dot{G}_{AF,n}^{(k)})) \rightarrow K_0(C_r^*(\dot{G}_{AF,n+1}^{(k)}))$ to be the one given by $\sigma^\top : C_n^k \rightarrow C_{n+1}^k$, so we only need the identification $K_0(C_r^*(\dot{G}_{AF,n}^{(k)})) \cong C_n^k$.

Given a k -cochain corresponding to a k -cell AP_n , we pick a projection in $C_r^*(\dot{G}_{AF,n}^{(k)})$ whose

support is the entire k -cell, which is a k -punctured acu set, together with a puncture in it. K_0 asserts that all punctures in this set are the same, making this identification well-defined.

If the boundary hyperplane condition is satisfied, then each of the entries in $\sigma^\top : C_n^k \rightarrow C_{n+1}^k$ is nonnegative, and the map $K_0(C_r^*(\dot{G}_{AF,n}^{(k)})) \rightarrow K_0(C_r^*(\dot{G}_{AF,n+1}^{(k)}))$ is induced by the algebra map $C_r^*(\dot{G}_{AF,n}^{(k)}) \rightarrow C_r^*(\dot{G}_{AF,n+1}^{(k)})$.

Finally, we chose each of the d -cells to have the same orientations, therefore σ^\top always has nonnegative entries. \square

In other words, if we wish to show that $K_i(C_r^*(\dot{G}_u))$ can be constructed out of $K_0(C_r^*(\dot{G}_{AF}^{(k)}))$ for $0 \leq k \leq d$ using direct sums, quotients, and subgroups, it suffices to demonstrate this for Čech cohomology and the cochains groups, then pass back to K -theory using an isomorphism between K -theory and Čech cohomology. In order to make use of this observation, we need an identification between C^{d-1} and $K_0(C_r^*(\dot{G}_{AF}); C_r^*(\dot{G}_u))$ that arises from extension.

Let us provide a simple picture on certain types of elements of $K_0(C_r^*(\dot{G}_{AF}); C_r^*(\dot{G}_u))$. Observe that a partial isometry $e[P, t', t] \in C_r^*(\dot{G}_u)$ is a triple (p, q, v) by taking $p = e[P, t]$, $q = e[P, t']$, and v to be the partial isometry itself. Partial isometries with support single punctured d -cells are elementary triples, thus are trivial in $K_0(C_r^*(\dot{G}_{AF}); C_r^*(\dot{G}_u))$. However, it is not evident that if $e[P, t', t] \notin C_r^*(\dot{G}_{AF})$, we still have that $e[P, t], e[P, t'] \in C_r^*(\dot{G}_{AF})$, given that the support P may intersect multiple d -cells. This is resolved in the next few definitions and propositions.

Definition 5.3. Given a $d - 1$ -cell in the AP_n -complex, let $\{\zeta^n(t_i)\}_i$ be the set of punctured d -cells, counting multiplicity, incident to and with orientations agreeing with it, and let $\{\zeta^n(t_j)\}_j$ be the set of punctured d -cells, counting multiplicity and with orientations disagreeing. We call

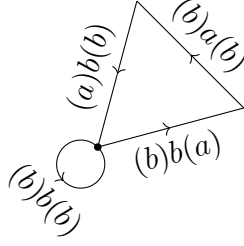


Figure 5.1: The AP -complex of the collared Silver Mean substitution, $a \mapsto b, b \mapsto bab$. The basic open ball around the marked point that is a 0-cell is formed from the three 1-cells $(a)b(b)$, $(b)b(a)$, and $(b)b(b)$. The preimages in the tiling space under the Robinson map are $(a)b.b(a)$, $(a)b.b(b)$, $(b)b.b(a)$, so $(b)b(b)$ is counted twice. The 0-cell partitions this set into $\{(a)b(b), (b)b(b)\}$ (source) and $\{(b)b(b), (b)b(a)\}$ (range), which is a full translation.

such a pair a *full translation*. The first collection satisfies the *right-hand rule*, and the second satisfies the *left-hand rule*.

A single d -cell may have multiple parts of its boundary attached to a single $d - 1$ -cell (Fig. 5.1). By *counting multiplicity*, we mean that we count each occurrence separately. Taking union of a full translation yields an open ball in AP_n . In other words, given the set of d -cells incident to a $d - 1$ -cell forming an open ball, counting multiplicity, we are forming a partition based on handedness.

Remark 5.4. Rather than counting multiplicities, one can instead apply a version of state-splitting. This amounts to decorating each d -cell in AP_n with its image d -cells in AP_{n-m} , where m is sufficiently large so that none of the decorations is incident to the $d - 1$ -cell in more than one way. That is, for this particular m , we pass to the induced substitution applied to sets of finite coordinates of length $n - m$ in $\varprojlim_n (AP_n, \sigma)$ that do not necessarily start at level-0.

Proposition 5.5 (Extension). *Given a partial isometry $e[P, t', t] \in C_r^*(\dot{G}_u)$, if t and t' belong to the same punctured d -cell for some AP_n , then $e[P, t', t] \in C_r^*(\dot{G}_{AF})$, up to shifting by a level given by border-forcing. More generally, if t and t' belong to different punctured d -cells, it can*

be written as a sum of partial isometries

$$\sum_{i=1}^{m-1} e[\varsigma^n(t_i) \cup \varsigma^n(t_{i+1}), t'_i, t'_i]$$

where for $1 \leq i \leq m$, $t'_i \in \varsigma^n(t_i)$ and $t''_i \in \varsigma^n(t_{i+1})$, and for $1 \leq i < m$, each pair $\varsigma^n(t_i)$ and $\varsigma^n(t_{i+1})$ intersects at a $d - 1$ -cell and $t'_{i+1} = t''_i$.

Proof. We will prove the second statement, since it implies the first.

Let N be the level at which the substitution forces the border. By extension, we may assume that we are given a partial isometry of the form $e[\bigcup_i \varsigma^n(t_i), t', t]$. Form a sequence $\{i_j\}_{j=1}^m \subseteq \{i\}_i$ so that the corresponding punctured d -cells pairwise neighbor a $d - 1$ -cell, with $t \in \varsigma^n(t_{i_1})$ and $t' \in \varsigma^n(t_{i_m})$. Consider a single preimage U of $\sigma^{-N}(\bigcup_i \varsigma^n(t_i))$. By border-forcing, U is an open ball in the subspace topology of AP_{n+N} . Furthermore, since $\bigcup_i \varsigma^n(t_i)$ contains each of the $d - 1$ -cells that are pairwise intersections of our sequence of punctured d -cells, $\sigma^{-N}(\varsigma^n(t_{i_j}) \cup \varsigma^n(t_{i_{j+1}})) \cap U$ are also open balls for each $1 \leq j < m$. By extension, we may reduce the support from U to $\bigcup_{j=1}^{m-1} \sigma^{-N}(\varsigma^n(t_{i_j}) \cup \varsigma^n(t_{i_{j+1}})) \cap U$, then for each partial isometry between adjacent d -cells, we may further reduce support to the corresponding $\sigma^{-N}(\varsigma^n(t_{i_j}) \cup \varsigma^n(t_{i_{j+1}})) \cap U$.

These are full translations around $d - 1$ -cells. To obtain partial isometries, we convert each full translation into punctured acu sets around each of the $d - 1$ -cells. \square

Proposition 5.6. *Given any $d - 1$ -cell in the AP_n -complex, let $\{\varsigma^n(t_i)\}_i$ and $\{\varsigma^n(t_j)\}_j$ be a full translation. Then $\sum_i e[\varsigma^n(t_i), t'_i]$ is Murray–von Neumann equivalent to $\sum_j e[\varsigma^n(t_j), t''_j]$ in $C_r^*(\dot{G}_u)$, for any punctures t'_i in $\varsigma^n(t_i)$ and t''_j in $\varsigma^n(t_j)$.*

Proof. Let I denote the pairs of indices (i, j) such that $\varsigma^n(t_i) \cup \varsigma^n(t_j)$ forms a punctured acu set,

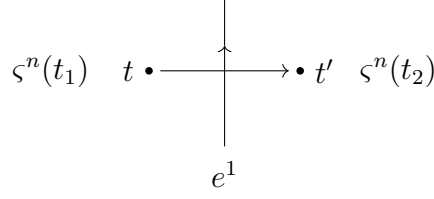


Figure 5.2: Assuming no branches occur at e^1 , th_n maps it to relative K_0 -equivalence class of the partial isometry $e[\varsigma^n(t_1) \cup \varsigma^n(t_2), t', t]$. In other words, $\varsigma^n(t_1)$ satisfies the right-hand rule with the direction of e^1 , and t is defined to be the source, and $\varsigma^n(t_2)$ satisfies the left-hand rule, and t' is defined to be the range. Inverting the partial isometry results in a sign on e^1 .

and let I^\top be the set with index order swapped. Noting that a sum in I and one in I^\top differ in the way individual projections and partial isometries are grouped, namely using the right-handed and the left-handed cells, we have that

$$\begin{aligned}
\sum_i e[\varsigma^n(t_i), t'_i] &= \sum_I e[\varsigma^n(t_i) \cup \varsigma^n(t_j), t'_i] && \text{(Extension)} \\
&= \sum_I e[\varsigma^n(t_i) \cup \varsigma^n(t_j), t'_i, t''_j] e[\varsigma^n(t_i) \cup \varsigma^n(t_j), t''_j, t'_i] \\
&\sim_{\text{MvN}} \sum_{I^\top} e[\varsigma^n(t_i) \cup \varsigma^n(t_j), t''_j, t'_i] e[\varsigma^n(t_i) \cup \varsigma^n(t_j), t'_i, t''_j] \\
&&& \text{(Reordering summands)} \\
&= \sum_{I^\top} e[\varsigma^n(t_i) \cup \varsigma^n(t_j), t''_j] \\
&= \sum_j e[\varsigma^n(t_j), t''_j]. && \text{(Extension)}
\end{aligned}$$

□

Together, these say that partial isometries given by doubly-pointed patterns that are nontrivial in $K_0(C_r^*(\dot{G}_{AF}); C_r^*(\dot{G}_u))$ are exactly those that arise from full translations, whose source and range projections land back in $K_0(C_r^*(\dot{G}_{AF}))$.

We next define the key map between the $d - 1$ -cochain group and the relative K_0 -group that puts these propositions into topological context. We begin by defining it at a finite level, then extend it using functoriality of relative K -theory.

Definition 5.7. The *thickening map*¹, $\text{th}_n : C_n^{d-1} \rightarrow K_0(C_r^*(\dot{G}_{AF,n}); C_r^*(\dot{G}_u))$, sends a $d - 1$ -cell to the relative K_0 -equivalence class of partial isometries formed from the full translation surrounding it, with sources in each of the right-handed cells and ranges in the left-handed ones (Fig. 5.2), i.e.

$$\text{th}_n(e^{d-1}) = \sum_I e[\varsigma^n(t_i) \cup \varsigma^n(t_j), t'_j, t'_i]$$

where the right sum is under the relative K_0 -equivalence, $\{\varsigma^n(t_i)\}_i$ and $\{\varsigma^n(t_j)\}_j$ is a full translation around the $d - 1$ -cochain $e^{d-1} \in C_n^{d-1}$, I is the set of pairs of indices (i, j) such that $\varsigma^n(t_i) \cup \varsigma^n(t_j)$ forms a punctured acu set, and t'_i and t'_j are punctures in $\varsigma^n(t_i)$ and $\varsigma^n(t_j)$ for each i and j , respectively.

Remark 5.8. If one reverses the order of the sources and ranges, then the map th_n will have the opposite sign. More generally, the map will carry signs on the $d - 1$ -cells whose orientations are reversed.

Proposition 5.9. $K_0(C_r^*(\dot{G}_{AF}); C_r^*(\dot{G}_u)) = \varinjlim_n (K_0(C_r^*(\dot{G}_{AF,n}); C_r^*(\dot{G}_u)), \sigma^\top)$.

¹This terminology is borrowed from [JS16], where one thickens a codimension-1 cell into an open ball in the AP -complex.

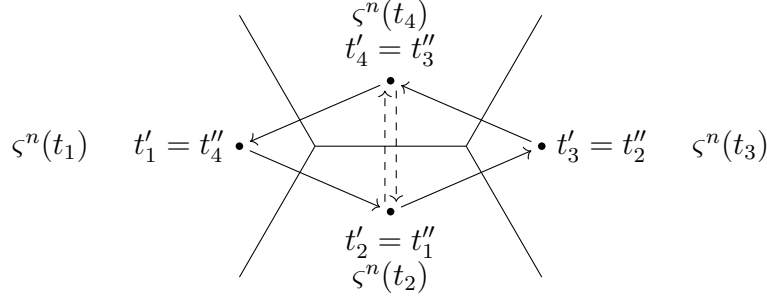


Figure 5.3: A relation in $K_0(C_r^*(\dot{G}_{AF}); C_r^*(\dot{G}_u))$ (solid arrows forming a quadrilateral) that surrounds two 0-cells decomposing into a sum of two relations (solid arrows and dashed arrows forming two triangles), each of which surrounds only one 0-cell. The vertical partial isometries (dashed arrows) are negatives of each other, which cancel when we add the two smaller relations together to obtain the larger relation.

Proof. Since the diagram

$$\begin{array}{ccc}
 C_r^*(\dot{G}_{AF,n}) & \longrightarrow & C_r^*(\dot{G}_u) \\
 \downarrow & & \parallel \\
 C_r^*(\dot{G}_{AF,n+1}) & \longrightarrow & C_r^*(\dot{G}_u) \\
 \downarrow & & \parallel \\
 \vdots & & \vdots \\
 \downarrow & & \parallel \\
 C_r^*(\dot{G}_{AF}) & \longrightarrow & C_r^*(\dot{G}_u)
 \end{array}$$

commutes, functoriality of relative K -theory gives the existence of a map $\sigma^\top : K_0(C_r^*(\dot{G}_{AF,n}); C_r^*(\dot{G}_u)) \rightarrow K_0(C_r^*(\dot{G}_{AF,n+1}); C_r^*(\dot{G}_u))$, and together with the Universal Property of Direct Limits and the Five Lemma, we get the result. \square

Before stating the thickening map as extended to direct limits, we notice that there are relations among full translations that one can fashion by forming a “loop” so that the source of the first full translation is the range of the last (Fig. 5.3). Any such loop can always be decomposed to elementary ones that surround exactly one $d-2$ -cell, thus we will only work with relations of this type. The thickening map identifies such loops with $\text{im } \delta_n^{d-2}$. We package this with compatibility

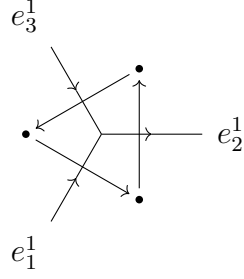


Figure 5.4: The relation given by the sum of the partial isometries identifies with $e_1^1 - e_2^1 + e_3^1 = 0$ under th_n , which coincides with δ_n^0 applied to the center 0-cell. Here we arbitrarily prescribed some orientations to the 1-cells.

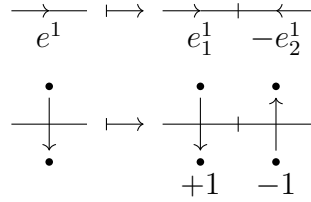


Figure 5.5: The substitution map is $e^1 \mapsto e_1^1 - e_2^1$ with the orientation of e_2^1 reversed in comparison to e^1 (top). Under thickening, the partial isometry $\text{th}(e_2^1)$ carries a sign and therefore has its direction reversed to agree with the substitution map applied to the partial isometry $\text{th}(e^1)$ (bottom).

with σ^\top into the following proposition.

Proposition 5.10. *We have the commutative diagram*

$$\begin{array}{ccccc}
 C_n^{d-1} & \xrightarrow{\sigma^\top} & C_{n+1}^{d-1} & & \\
 \searrow & & \searrow & & \\
 & C_n^{d-1} / \text{im } \delta_n^{d-2} & \xrightarrow{\sigma^\top} & C_{n+1}^{d-1} / \text{im } \delta_{n+1}^{d-2} & \\
 \searrow \text{th}_n & \downarrow \text{th}_n & \searrow \text{th}_{n+1} & \downarrow \text{th}_{n+1} & \\
 & K_0(C_r^*(\dot{G}_{AF,n}); C_r^*(\dot{G}_u)) & \longrightarrow & K_0(C_r^*(\dot{G}_{AF,n+1}); C_r^*(\dot{G}_u)) &
 \end{array}$$

where the quotients exist for $d \geq 2$.

Proof. We first show, for $d \geq 2$, the existence of the map $\text{th}_n : C_n^{d-1} / \text{im } \delta_n^{d-2} \rightarrow K_0(C_r^*(\dot{G}_{AF,n}); C_r^*(\dot{G}_u))$ by showing that $\text{th}_n(\text{im } \delta_n^{d-2}) = 0$. Given a $d-2$ -cell whose image

under δ_n^{d-2} is a signed sum $\sum_i (-1)^{m_i} e_i^{d-1}$, we can apply the thickening map summand-wise to each e_i^{d-1} to obtain an associated full translation $\text{th}_n(e_i^{d-1})$. We then obtain the sum

$$\sum_i (-1)^{m_i} \text{th}_n(e_i^{d-1}) = \sum_i \text{th}_n(e_i^{d-1})^{*m_i}$$

with the exponent on the right denoting that we take adjoint m_i -times. Noticing that taking the union of all such full translations forms an open ball around the $d - 2$ -cell, we see that if two full translations intersect nontrivially at either their sources or ranges (possibly source-to-source or range-to-range), then they must share the entire source or range. Consider any two full translations with associated indices i and $i + 1$ so that they share either the source or the range. We wish to show that with $*m_i$, the sharing must always be source-to-range (Fig. 5.4). Suppose that δ_n^{d-2} assigns the same sign to both $d - 1$ -cells. Since we chose the orientations of each of the d -cells to be the same, the attaching maps to the two $d - 1$ -cells assign opposite signs to the shared source or range of the two full translations, thus one must be the source and the other the range. If the $d - 1$ -cells have opposite signs, then the shared source or range are either both the source or both the range. However, on applying $*m$, the one with the sign has the source and range flipped, giving that the sharing is source-to-range. Procedurally applying adjoints as m_i requires gives us a loop, which, by principality of our groupoid, is trivial. Thus, by the Universal Property of Quotients, thickening descends to the quotient.

For $d \geq 2$, the top square commutes because while the map $\sigma : AP_{n+1} \rightarrow AP_n$ may not be cellular, one can easily make it cellular by introducing k -cells in AP_{n+1} as they exist under σ .

We next show that the bottom slanted square also commutes, that is that thickening is compatible with $\sigma^\top : C_n^{d-1} \rightarrow C_{n+1}^{d-1}$. Traversing the bottom composition, by continuity, σ^\top sends

the underlying open ball of the full translation that is the thickened $d - 1$ -cell to a collection of open balls. Such an open ball either belongs to the interior of a single d -cell, or contains a $d - 1$ -cell that is the image of the $d - 1$ -cell under σ^\top . In the former case, it becomes a partial isometry whose source and range belongs in a single d -cell, and is thus trivial. In the latter, the partitioning of the full translation as inherited from its preimage agrees with the top composition since if σ^\top reverses orientation on a $d - 1$ -cell, the cell will carry a sign, which undoes the reversed orientation of the corresponding partial isometry (Fig. 5.5).

Finally, a diagram chase shows the front square commutes. \square

Thus by functoriality and the Universal Property of Direct Limits, we obtain a map $\text{th} : C^{d-1}/\text{im } \delta^{d-2} \rightarrow K_0(C_r^*(\dot{G}_{AF}); C_r^*(\dot{G}_u))$, allowing us to state the theorem that relates the top two Čech cohomology groups to K -theory.

Theorem 5.11. *We have the commutative diagram*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \check{H}^{d-1}(\Omega_T) & \hookrightarrow & C^{d-1}/\text{im } \delta^{d-2} & \xrightarrow{\delta^{d-1}} & C^d & \twoheadrightarrow & \check{H}^d(\Omega_T) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow \text{th} & & \downarrow \text{CF} & & \downarrow & & \\
0 & \longrightarrow & K_{1,u} & \hookrightarrow & K_{0,AF;u} & \xrightarrow{\text{ev}} & K_{0,AF} & \xrightarrow{\iota_*} & K_{0,u} & \twoheadrightarrow & K_{1,AF;u} \longrightarrow 0
\end{array}$$

where $K_{i,u} = K_i(C_r^*(\dot{G}_u))$, $K_{i,AF} = K_i(C_r^*(\dot{G}_{AF}))$, and $K_{i,AF;u} = K_i(C_r^*(\dot{G}_{AF}); C_r^*(\dot{G}_u))$.

Proof. Since $\delta^{d-1} \circ \delta^{d-2} = 0$, we obtain the existence of the map $\delta^{d-1} : C^{d-1}/\text{im } \delta^{d-2} \rightarrow C^d$.

Then the top exact sequence is none other than the exact sequence

$$0 \longrightarrow \ker \phi \hookrightarrow A \xrightarrow{\phi} B \twoheadrightarrow \text{coker } \phi \longrightarrow 0$$

applied to this map. It suffices to check that the middle square commutes at a finite level, which

it does since thickening counts multiplicities, so $\text{ev} \circ \text{th}_n([e^{d-1}]) = [\sum_i c_i e_i^d] - [\sum_j c_j e_j^d]$ where c_i is the number of occurrences of e^{d-1} in e_i^d with the orientations of the attaching maps agreeing, and c_j is disagreeing, thus this difference is $\delta_n^{d-1}([e^{d-1}])$. \square

5.2 Isomorphism for $d \leq 2$

Over \mathbb{Z} , it is known that for $d \leq 3$ and for $i = 0, 1$, $K_i(C_r^*(\dot{G}_u)) \cong \bigoplus_{k=0}^{\lfloor (d-i)/2 \rfloor} \check{H}^{d-i-2k}(\Omega_T)$. For example, by [SW03], tiling spaces are fiber bundles over \mathbb{Z}^d , thus are decorations of \mathbb{Z}^d , and one can then apply [vEI94]² and observe that the generators and relations given are in terms of cohomology, or one can use the Chern and Connes–Thom isomorphisms. For an incomplete list, see [Put89], [Kel97], and [AP98]. However, in the interest of explicitly computing the map between generators and relations of the two, and of promoting relative K -theory, we will reprove this fact for $d \leq 2$.

The argument follows the same basic structure demonstrated in [JS16] for the chair tiling, and is similar in spirit to [vEI94], where one inductively “peels off” dimensions by appealing to the techniques demonstrated in the following theorem whose conclusion has been known since [Put89].

Theorem 5.12 (Theorem 1.1). *For $d = 1$, we have an isomorphism of exact sequences*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \check{H}^0(\Omega_T) & \hookrightarrow & C^0 & \xrightarrow{\delta^0} & C^1 & \twoheadrightarrow & \check{H}^1(\Omega_T) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \text{th} & & \downarrow \text{CF} & & \downarrow & & \\ 0 & \longrightarrow & K_{1,u} & \hookrightarrow & K_{0,AF;u} & \xrightarrow{\text{ev}} & K_{0,AF} & \xrightarrow{\iota_*} & K_{0,u} & \longrightarrow & 0 \end{array}$$

where $K_{i,u} = K_i(C_r^*(\dot{G}_u))$, $K_{i,AF} = K_i(C_r^*(\dot{G}_{AF}))$, and $K_{i,AF;u} = K_i(C_r^*(\dot{G}_{AF}); C_r^*(\dot{G}_u))$.

²Modulo mistakes for $d = 3$.

Thus

$$K_0(C_r^*(\dot{G}_u)) \cong K_0(C_r^*(\dot{G}_{AF}))/\text{im ev}$$

$$K_1(C_r^*(\dot{G}_u)) \cong \mathbb{Z}.$$

Proof. Noting that the nontrivial elements in $K_0(C_r^*(\dot{G}_{AF}); C_r^*(\dot{G}_u))$ are partial isometries that intersect $\varprojlim_n (AP_n^{(0)}, \sigma)$, and thus $\dot{G}_{AF}^{(0)}$ forms an abstract transversal, we apply Theorem 4.8 followed by restriction to $\varprojlim_n (AP_n^{(0)}, \sigma)$ that is a groupoid equivalence. The thickening map is the inverse map.

More precisely, let L be the set of elements in G_u whose source belongs to the left and range belongs to the right of $\varprojlim_n (AP_n^{(0)}, \sigma)$. Then, forming the corresponding subgroupoids H and H' , $H \leq \dot{G}_u$ is the subgroupoid supported on points in $\varprojlim_n (AP_n, \sigma)$ that are translations of $\varprojlim_n (AP_n^{(0)}, \sigma)$, and $H' \leq \dot{G}_{AF}$ is similar, with appropriate topologies that is measured with respect to $\varprojlim_n (AP_n^{(0)}, \sigma)$ so that $\varprojlim_n (AP_n^{(0)}, \sigma)$ forms an abstract transversal to H and gives a groupoid equivalence between H and $\dot{G}_{AF}^{(0)}$.

By Theorem 4.8, we have the sequence of isomorphisms

$$K_i(C_r^*(\dot{G}_{AF}); C_r^*(\dot{G}_u)) \cong K_i(C_r^*(H'); C_r^*(H)) \cong K_i(C_r^*(H)) \cong K_i(C_r^*(\dot{G}_{AF}^{(0)})).$$

Applying Lemma 5.2 and reversing the sequence of isomorphisms gives the thickening map. \square

Remark 5.13. This theorem allows us to prove, for $d = 1$, facts regarding proper substitutions and substitutions that do not force the border.

- If a substitution is proper³, i.e. every supertile begins with the same prototile and ends with the same prototile, then $K_0(C_r^*(\dot{G}_{AF})) \cong K_0(C_r^*(\dot{G}_u))$. Properness holds for all higher-level supertiles as well, thus there is only one nontrivial partial isometry. $K_1(C_r^*(\dot{G}_u)) \cong \mathbb{Z} \leq K_0(C_r^*(\dot{G}_{AF}); C_r^*(\dot{G}_u))$, and $K_0(C_r^*(\dot{G}_{AF}))$ is torsion-free, therefore $\text{ev} = 0$.
- If a substitution does not force the border, then $K_0(C_r^*(\dot{G}_u)) \cong K_0(C_r^*(\dot{G}_{AF}))/\text{im ev}$ is a nontrivial quotient. There are multiple partial isometries that persist in the limit, thus $K_1(C_r^*(\dot{G}_u)) \not\leq K_0(C_r^*(\dot{G}_{AF}); C_r^*(\dot{G}_u))$, and $\text{ev} \neq 0$.

For $d = 2$, we now pass to \mathbb{Z}^2 -decorations using [SW03]. Summarizing, the set of vectors forming the boundaries of prototiles in T forms loops, thus they correspond to an element in the kernel of a linear system of equations in \mathbb{R}^2 solved over \mathbb{R} . There exists an element in the kernel over \mathbb{Q} that is arbitrarily close to the original element, and deforming⁴ the prototiles to this new element gives a homeomorphism between Ω_T and the new tiling space formed from a tiling whose 0-cells of tiles belong to \mathbb{Q}^2 . One can then uniformly scale the coordinates of the 0-cells until they become integral, there being finitely many, up to translation, due to finite local complexity, then replace each 1-cell with a collection of piecewise-constant 1-cells where at least one of the coordinates of each piece is integral. Finally, one splits each of the new prototiles into unit squares decorated by the prototile they reside in. Each step is at least a homeomorphism of tiling spaces, and so the resulting tiling space is homeomorphic to the original. Let us denote this new tiling by T_\square , and the homeomorphism by $\square : \Omega_T \rightarrow \Omega_{T_\square}$.

Rather than attempting to bring the substitution structure from T through \square , we construct a new inverse system that is a hybrid of the Anderson–Putnam and Gähler ([Gäh02]) inverse

³The resulting Bratteli–Vershik system is proper.

⁴Such deformations can be viewed as elements of $\check{H}^1(\Omega_T; \mathbb{R}^2)$.

systems. Let Γ_n be the CW -complex where the 2-cells are patches of squares tiles residing in $[i \cdot 2^n, (i+1) \cdot 2^n] \times [j \cdot 2^n, (j+1) \cdot 2^n]$ collared by their neighboring 2-cells of the same size, for $i, j \in \mathbb{Z}$, up to translation, and we quotient by identifications as they exist in T_\square . The substitution map $\sigma_\square : \Gamma_n \rightarrow \Gamma_{n-1}$ is subdividing each 2-cell into Γ_n into their respective 2-cells in Γ_{n-1} . One easily checks, following the arguments given in [AP98, Theorem 4.3] and Corollary 2.9, that, under the translation action, $\varprojlim_n (\Gamma_n, \sigma_\square)$ is topologically conjugate to Ω_{T_\square} . Note that T_\square satisfies the boundary hyperplane condition, thus the cells in each Γ_n are assigned orientations as aforescribed.

Let us denote the unstable and tail groupoids of each dimension from this inverse limit $\varprojlim_n (\Gamma_n, \sigma_\square)$ by the same notation as before, and introduce two new types of objects that are refinements of these groupoids by observing that in Γ_n , the 1-cells are either horizontal or vertical. Let us denote $\Gamma_{n,h}^{(1)}$ and $\Gamma_{n,v}^{(1)}$ the horizontal and vertical parts of the 1-skeleton of Γ_n , respectively. Let \bullet be either h or v , and let $h^\top = v$ and $v^\top = h$.

- Let $G_{u,\bullet}^{(1)}$ be the subgroupoid of $G_u^{(1)}$ restricted to $\varprojlim_n (\Gamma_{n,\bullet}^{(1)}, \sigma_\square)$, i.e., with the source and range maps taken to be on $G_u^{(1)}$,

$$G_{u,\bullet}^{(1)} = s^{-1}(\varprojlim_n (\Gamma_{n,\bullet}^{(1)}, \sigma_\square)) \cap r^{-1}(\varprojlim_n (\Gamma_{n,\bullet}^{(1)}, \sigma_\square)),$$

where we have identified the unit space with the underlying topological space.

- Let $G_{\text{tail},\bullet}^{(1)}$ be the tail subgroupoid of $G_{u,\bullet}^{(1)}$ induced by σ , i.e.

$$G_{\text{tail},\bullet}^{(1)} = \left\{ \begin{array}{l} \exists U_n \times U_{n+1} \times \dots \text{ such that} \\ ((p_0, \dots), (q_0, \dots)) \in G_{u,\bullet}^{(1)} : \quad \forall m \geq n, p_m, q_m \in U_m \text{ and} \\ \exists 1\text{-cells } e_{1,m} \subseteq \Gamma_{m,\bullet}^{(1)} \text{ such that } U_m \subseteq e_{1,m} \end{array} \right\}.$$

- Let $G_{u,\bullet}$ be the subgroupoid of G_u with the elements crossing $\varprojlim_n (\Gamma_{n,\bullet}^{(1)}, \sigma_\square)$ removed, e.g.

for $\bullet = h$,

$$G_{u,h} = \left\{ \begin{array}{l} \exists y_1, y_2 \text{ such that } \text{sign}(y_1) = \text{sign}(y_2) \text{ and} \\ ((p_0, \dots), (q_0, \dots)) \in G_u : \quad \begin{array}{l} (p_0, \dots) - (0, y_1), \\ (q_0, \dots) - (0, y_2) \end{array} \in \varprojlim_n (\Gamma_{n,h}^{(1)}, \sigma_\square) \end{array} \right\}.$$

- Let C_\bullet^1 be the subgroup of C^1 restricted to $\varprojlim_n (\Gamma_{n,\bullet}^{(1)}, \sigma_\square)$, and $\delta_\bullet^0 : C^0 \rightarrow C_\bullet^1$ and $\delta_\bullet^1 : C_\bullet^1 \rightarrow C^2$ be the induced coboundary maps on the corresponding cochains. Note that $\delta_\bullet^1 \circ \delta_\bullet^0$ is not necessarily zero.

As before, we introduce punctures to each of the cells, the natural choice being the exact middle of each of the cells, and denote the restricted groupoids with a dot.

Let us consider the three inclusions of groupoids

$$\dot{G}_{u,\bullet} \hookrightarrow \dot{G}_u \tag{5.1}$$

$$\dot{G}_{AF} \hookrightarrow \dot{G}_{u,\bullet}^\top \tag{5.2}$$

$$\dot{G}_{AF,\bullet}^{(1)} \hookrightarrow \dot{G}_{u,\bullet}^{(1)} \tag{5.3}$$

and their associated six-term sequences in relative K -theory.

We have the following proposition that simplifies the relative K -groups.

Proposition 5.14. *For $i = 0, 1$,*

1. $K_i(C_r^*(\dot{G}_{u,\bullet}); C_r^*(\dot{G}_u)) \cong K_i(C_r^*(\dot{G}_{u,\bullet}^{(1)})),$
2. $K_i(C_r^*(\dot{G}_{AF}); C_r^*(\dot{G}_{u,\bullet^\top})) \cong K_i(C_r^*(\dot{G}_{AF,\bullet}^{(1)})),$ and
3. $K_i(C_r^*(\dot{G}_{AF,\bullet}^{(1)}); C_r^*(\dot{G}_{u,\bullet}^{(1)})) \cong K_i(C_r^*(\dot{G}_{AF}^{(0)})).$

Proof. These all follow from Theorem 4.8 with appropriate choices of L , so that $L \cup L^{-1}$ is

1. The subset of the groupoid \dot{G}_u that cross $\varprojlim_n (\Gamma_{n,\bullet}^{(1)}, \sigma_\square),$
2. The subset of the groupoid \dot{G}_{u,\bullet^\top} that cross $\varprojlim_n (\Gamma_{n,\bullet}^{(1)}, \sigma_\square),$ and
3. The subset of the groupoid $\dot{G}_{u,\bullet}^{(1)}$ that cross $\varprojlim_n (\Gamma_n^{(0)}, \sigma_\square).$

Let us explain the first one with $\bullet = h$. The others are similar. Let L be the subset of \dot{G}_u consisting of elements of the form $((p_0, \dots) + (0, y_1), (q_0, \dots) + (0, y_2)),$ with $(p_0, \dots), (q_0, \dots) \in \varprojlim_n (\Gamma_{n,h}^{(1)}, \sigma_\square)$ and $y_1 < 0$ and $y_2 > 0$. In the notation of Theorem 4.8, $G = \dot{G}_u,$ $G' = \dot{G}_{u,h},$ and $H = \dot{G}_{u,h}^{(1)},$ and the isomorphism is as desired. \square

Composing these isomorphisms (in reverse) with the evaluation maps (and the counterparts from the relative K_1 -groups) for each of the three six-term sequences, we obtain the three maps

$$\begin{aligned} \delta_i &: K_i(C_r^*(\dot{G}_{u,\bullet}^{(1)})) \rightarrow K_i(C_r^*(\dot{G}_{u,\bullet})) \\ \delta_\bullet^1 &: K_i(C_r^*(\dot{G}_{AF,\bullet}^{(1)})) \rightarrow K_i(C_r^*(\dot{G}_{AF})) \\ \delta_\bullet^0 &: K_i(C_r^*(\dot{G}_{AF}^{(0)})) \rightarrow K_i(C_r^*(\dot{G}_{AF,\bullet}^{(1)})) \end{aligned}$$

where the latter two names are as given because they are the coboundary maps restricted to either the horizontal or the vertical 1-cells.

The three six-term sequences in relative K -theory then read

$$\begin{array}{ccccc}
K_0(C_r^*(\dot{G}_{u,h}^{(1)})) & \xrightarrow{\delta_0} & K_0(C_r^*(\dot{G}_{u,h})) & \xrightarrow{\iota_*} & K_0(C_r^*(\dot{G}_u)) \\
\uparrow & & & & \downarrow \\
K_1(C_r^*(\dot{G}_u)) & \xleftarrow{\iota_*} & K_1(C_r^*(\dot{G}_{u,h})) & \xleftarrow{\delta_1} & K_1(C_r^*(\dot{G}_{u,h}^{(1)}))
\end{array} \tag{5.4}$$

$$\begin{array}{ccccc}
K_0(C_r^*(\dot{G}_{AF,v}^{(1)})) & \xrightarrow{\delta_v^1} & K_0(C_r^*(\dot{G}_{AF})) & \xrightarrow{\iota_*} & K_0(C_r^*(\dot{G}_{u,h})) \\
\uparrow & & & & \downarrow \\
K_1(C_r^*(\dot{G}_{u,h})) & \xleftarrow{\quad} & 0 & \xleftarrow{\quad} & 0
\end{array} \tag{5.5}$$

$$\begin{array}{ccccc}
K_0(C_r^*(\dot{G}_{AF}^{(0)})) & \xrightarrow{\delta_h^0} & K_0(C_r^*(\dot{G}_{AF,h}^{(1)})) & \xrightarrow{\iota_*} & K_0(C_r^*(\dot{G}_{u,h}^{(1)})) \\
\uparrow & & & & \downarrow \\
K_1(C_r^*(\dot{G}_{u,h}^{(1)})) & \xleftarrow{\quad} & 0 & \xleftarrow{\quad} & 0.
\end{array} \tag{5.6}$$

From Eq. (5.6),

$$K_0(C_r^*(\dot{G}_{u,h}^{(1)})) \cong \text{coker } \delta_h^0$$

$$K_1(C_r^*(\dot{G}_{u,h}^{(1)})) \cong \ker \delta_h^0,$$

and from Eq. (5.5),

$$K_0(C_r^*(\dot{G}_{u,h})) \cong \text{coker } \delta_v^1$$

$$K_1(C_r^*(\dot{G}_{u,h})) \cong \ker \delta_v^1.$$

Using the identification from Lemma 5.2, let us show that δ_0 can be identified with $\tilde{\delta}_h^1$ in the diagram

$$\begin{array}{ccc}
C_h^1 & \xrightarrow{\delta_h^1} & C^2 \\
\downarrow & & \downarrow \\
C_h^1 / \text{im } \delta_h^0 & \xrightarrow{\tilde{\delta}_h^1} & C^2 / \text{im } \delta_v^1 \\
\downarrow \text{CF} & & \downarrow \text{CF} \\
K_0(C_r^*(\dot{G}_{AF,h}^{(1)})) / \text{im } \delta_h^0 & \xrightarrow{\quad} & K_0(C_r^*(\dot{G}_{AF})) / \text{im } \delta_v^1 \\
\downarrow \sim & & \downarrow \sim \\
K_0(C_r^*(\dot{G}_{u,h}^{(1)})) & \xrightarrow{\sim} K_0(C_r^*(\dot{G}_{u,h}); C_r^*(\dot{G}_u)) \xrightarrow{\quad} & K_0(C_r^*(\dot{G}_{u,h})). \\
& \searrow \delta_0 &
\end{array}$$

The map δ_0 is thickening a projection supported on a horizontal 1-cell, then evaluating by taking formal differences of projections supported on neighboring 2-cells, while noting that this is modulo horizontal translations. Interpreted topologically, this is given by taking a horizontal 1-cell c_h^1 , taking its coboundary under δ_h^1 , then taking equivalence classes modulo $\text{im } \delta_v^1$. Formally, this is given by the diagram

$$\begin{array}{ccccc}
C^1 & \xrightarrow{\delta^1} & C^2 & \xrightarrow{\quad} & C^2 / \text{im } \delta_v^1 \\
\downarrow & (c_h^1, 0) \mapsto & \delta^1((c_h^1, 0)) \mapsto & \nearrow & [\delta_h^1(c_h^1)] \\
C_h^1 & \downarrow & C_h^1 / \text{im } \delta_h^0 & \xrightarrow{\quad} & \\
& c_h^1 \mapsto & [c_h^1] & \nwarrow &
\end{array}$$

$\tilde{\delta}_h^1$

where this is independent of the representative chosen in $[c_h^1]$ (Fig. 5.6). In words, given $[c_h^1] \in C_h^1 / \text{im } \delta_h^0$, we lift it to an element $c_h^1 \in C_h^1$. This is $(c_h^1, 0) \in C^1$. Then $\delta^1((c_h^1, 0)) = \delta_h^1(c_h^1)$.

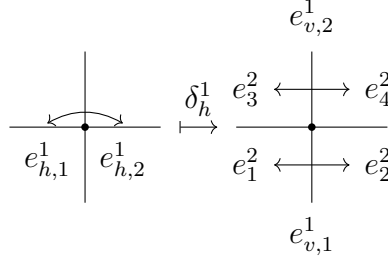


Figure 5.6: The identification of two 1-cells in C_h^1 across $\text{im } \delta_h^0$ (left) is mapped, under δ_h^1 , to the identification of the corresponding 2-cells in C^2 across $\text{im } \delta_v^1$ (right).

Finally, we take its quotient $[\delta_h^1(c_h^1)] \in C^2 / \text{im } \delta_v^1$. Thus the map is $\tilde{\delta}_h^1([c_h^1]) = [\delta_h^1(c_h^1)]$.

Similarly, again using Lemma 5.2, δ_1 can be identified with $\tilde{\delta}_v^0$ from the diagram

$$\begin{array}{ccccc}
 \ker \delta_h^0 & \xrightarrow{\tilde{\delta}_v^0} & & & \ker \delta_v^1 \\
 \downarrow \sim & & \delta_1 & & \downarrow \sim \\
 K_1(C_r^*(\dot{G}_{u,h}^{(1)})) & \xrightarrow{\sim} & K_1(C_r^*(\dot{G}_{u,h}); C_r^*(\dot{G}_u)) & \longrightarrow & K_1(C_r^*(\dot{G}_{u,h})) \\
 \downarrow & & \downarrow & & \downarrow \\
 K_0(C_r^*(\dot{G}_{AF}^{(0)})) & \xrightarrow{\sim} & K_0(C_r^*(\dot{G}_{AF,v}^{(1)}); C_r^*(\dot{G}_{u,v}^{(1)})) & \longrightarrow & K_0(C_r^*(\dot{G}_{AF,v}^{(1)})) \\
 \downarrow \text{CF}^{-1} & & & & \downarrow \text{CF}^{-1} \\
 C^0 & \xrightarrow{\delta_v^0} & & & C_v^1
 \end{array}$$

with $\tilde{\delta}_v^0$ given by the diagram

$$\begin{array}{ccccccc}
 \ker \delta_h^0 & \hookrightarrow & C^0 & \xrightarrow{\delta^0} & C^1 & \xrightarrow{\delta^1} & C^2 \\
 & & c^0 & \xrightarrow{\quad} & c^0 & \xrightarrow{\quad} & (0, c_v^1) \xrightarrow{\quad} 0 \\
 & \searrow \tilde{\delta}_v^0 & & & \downarrow & & \downarrow \\
 & & & & \ker \delta_v^1 & \hookrightarrow & C_v^1 \\
 & & & & c_v^1 & \xrightarrow{\quad} & c_v^1
 \end{array}$$

In words, given $c^0 \in \ker \delta_h^0$, let $\delta^0(c^0) = (0, c_v^1)$. $\delta^1 \circ \delta^0(c^0) = \delta^1((0, c_v^1)) = \delta_v^1(c_v^1) = 0$, so

$c_v^1 \in \ker \delta_v^1$. The map is $\tilde{\delta}_v^0(c^0) = c_v^1$.

Remark 5.15. The maps $\tilde{\delta}_h^1$ and $\tilde{\delta}_v^0$ essentially arise from this diagram that *does not commute*

$$\begin{array}{ccc}
\ker \delta_{\bullet}^0 & \xrightarrow{\tilde{\delta}_{\bullet\tau}^0} & \ker \delta_{\bullet\tau}^1 \\
\downarrow & & \downarrow \\
C_{\bullet}^0 & \xrightarrow{\delta_{\bullet\tau}^0} & C_{\bullet\tau}^1 \\
\downarrow \delta_{\bullet}^0 & \text{\textcircled{X}} & \downarrow \delta_{\bullet\tau}^1 \\
C_{\bullet}^1 & \xrightarrow{\delta_{\bullet}^1} & C^2 \\
\downarrow & & \downarrow \\
C_{\bullet}^1 / \text{im } \delta_{\bullet}^0 & \xrightarrow{\tilde{\delta}_{\bullet}^1} & C^2 / \text{im } \delta_{\bullet\tau}^1.
\end{array}$$

Finally, phrased purely in terms of topological objects, the six-term sequence in Eq. (5.4)

becomes

$$\begin{array}{ccccc}
\text{coker } \delta_h^0 & \xrightarrow{\tilde{\delta}_h^1} & \text{coker } \delta_v^1 & \longrightarrow & K_0(C_r^*(\dot{G}_u)) \\
\uparrow & & & & \downarrow \\
K_1(C_r^*(\dot{G}_u)) & \longleftarrow & \ker \delta_v^1 & \xleftarrow{\tilde{\delta}_v^0} & \ker \delta_h^0.
\end{array}$$

From the definition of the maps,

- $\text{coker } \tilde{\delta}_h^1 \cong C^2 / (\text{im } \tilde{\delta}_h^1 + \text{im } \delta_v^1) = C^2 / (\text{im } \delta_h^1 + \text{im } \delta_v^1)$, and
- $\ker \tilde{\delta}_v^0 \cong \ker \delta_h^0 \cap \ker \delta_v^0 = \ker \delta^0$.

This allows us to break the six-term sequence into two short exact sequences

$$0 \longrightarrow C^2 / (\text{im } \delta_h^1 + \text{im } \delta_v^1) \hookrightarrow K_0(C_r^*(\dot{G}_u)) \twoheadrightarrow \ker \delta^0 \longrightarrow 0 \quad (5.7)$$

$$0 \longrightarrow \ker \delta_v^1 / \text{im } \tilde{\delta}_v^0 \hookrightarrow K_1(C_r^*(\dot{G}_u)) \twoheadrightarrow \ker \tilde{\delta}_h^1 \longrightarrow 0. \quad (5.8)$$

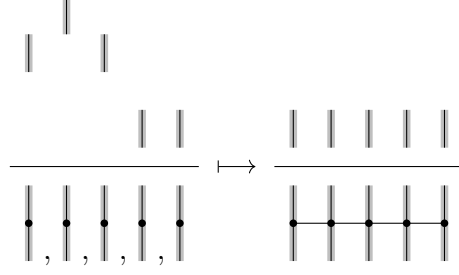


Figure 5.7: An isomorphism $\ker(\pi_h^2 \circ \delta_v^1) / \text{im } \delta_v^0 \xrightarrow{\sim} \ker \delta_v^1 / \text{im } \widetilde{\delta}_v^0$. The 1-cochains in the numerators and denominators are highlighted in light gray. On the left, we are allowed identification across $\text{im } \delta_v^0$ for each individual 1-cochain. On the right, we identify only if they are joined by $\delta_v^0(\ker \delta_h^0)$ (unhighlighted horizontal line).

Eq. (5.7) splits because $\ker \delta^0 \cong \mathbb{Z}$, giving us

$$K_0(C_r^*(\dot{G}_u)) \cong C^2 / (\text{im } \delta_h^1 + \text{im } \delta_v^1) \oplus \ker \delta^0.$$

Towards Eq. (5.8), we observe that by reversing the roles of h and v , we obtain the analogous induced maps $\widetilde{\delta}_h^0$ and $\widetilde{\delta}_v^1$, giving us the following proposition.

Proposition 5.16. *Denote $\pi_\bullet^2 : C^2 \twoheadrightarrow C^2 / \text{im } \delta_\bullet^1$. Then $\ker \widetilde{\delta}_\bullet^1 = \ker(\pi_\bullet^2 \circ \delta_\bullet^1) / \text{im } \delta_\bullet^0 \cong \ker \delta_\bullet^1 / \text{im } \widetilde{\delta}_\bullet^0$.*

Proof. We will prove this for $\bullet = v$. If $c_v^1 \in \ker \delta_v^1$, then c_v^1 is a collection of 1-cochains in C_v^1 that are vertical boundaries of 2-cochains closed under horizontal translations.

Suppose that $c_v^1 \in \ker(\pi_h^2 \circ \delta_v^1)$. The difference from $\ker \delta_v^1$ is that its image in C^2 under δ_v^1 is allowed identification across $\text{im } \delta_h^1$. In other words, the 1-cochains in c_v^1 are allowed to be shifted across $\text{im } \delta_v^0$. $\ker \widetilde{\delta}_v^1$ then quotients by identifying across $\text{im } \delta_v^0$.

On the other hand, $\text{im } \widetilde{\delta}_v^0 = \delta_v^0(\ker \delta_h^0)$, and $\ker \delta_h^0$ is the collection of 0-cochains in C^0 that are boundaries of horizontal 1-cochains closed under horizontal translations. That is, $\text{im } \widetilde{\delta}_v^0$ shifts each 1-cochain in $c_v^1 \in \ker \delta_v^1$ vertically by the same amount.

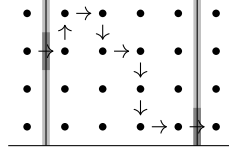


Figure 5.8: A unitary in $K_1(C_r^*(\dot{G}_{u,h}))$ (sequence of arrows on punctures) formed from parallel 1-cochains on $\Gamma_{2,v}^{(1)}$ (light gray). We can freely move the “exit points” (arrows crossing the 1-cochains), and therefore the corresponding 1-cochains on $\Gamma_{0,v}^{(1)}$ (gray), up and down, provided we do not move to a different 1-cochain on $\Gamma_{2,v}^{(1)}$. This is the quotient in $\ker(\pi_h^2 \circ \delta_v^1) / \text{im } \delta_v^0$.

The isomorphism between the two quotients is as follows: given an element in $\ker(\pi_v^2 \circ \delta_h^1) / \text{im } \delta_h^0$ and a 1-cochain in it, form a new sum consisting of all vertical 1-cochains that are horizontal translations of it, then take its equivalence class under $\text{im } \tilde{\delta}_v^0$ (Fig. 5.7). This is independent of representatives by visual inspection. \square

Recall that we have an inductive limit structure on $K_1(C_r^*(\dot{G}_{u,h}))$ using σ_\square , by picking a cochain on $\Gamma_{n,v}^{(1)}$, considering the set of all cochains in it that are horizontal translations of each other, forming the (equivalence class of) unitaries in $K_1(C_r^*(\dot{G}_{u,h}))$ that cross each of these cochains (Fig. 5.8), then taking the union over all n . In other words, the more appropriate topological object to use in place of the domain of the map $K_1(C_r^*(\dot{G}_{u,h})) / \text{im } \delta_1 \rightarrow K_1(C_r^*(\dot{G}_u))$ induced by the map $K_1(C_r^*(\dot{G}_{u,h})) \rightarrow K_1(C_r^*(\dot{G}_u))$ in Eq. (5.4) is $\ker \tilde{\delta}_v^1$. Eq. (5.8) then becomes

$$0 \longrightarrow \ker \tilde{\delta}_v^1 \hookrightarrow K_1(C_r^*(\dot{G}_u)) \twoheadrightarrow \ker \tilde{\delta}_h^1 \longrightarrow 0.$$

Reversing the roles of h and v then gives us the splitting of this short exact sequence, and

$$K_1(C_r^*(\dot{G}_u)) \cong \ker \tilde{\delta}_v^1 \oplus \ker \tilde{\delta}_h^1.$$

This gives us the following theorem.

$$\begin{array}{|c|c|} \hline 1 & -1 \\ \hline -1 & 1 \\ \hline \end{array} + \begin{array}{|c|c|} \hline -1 & 1 \\ \hline 1 & -1 \\ \hline \end{array} = \begin{array}{|c|c|} \hline 0 & 0 \\ \hline 0 & 0 \\ \hline \end{array}$$

Figure 5.9: δ_v^2 applied to an element of $\ker(\pi_h^2 \circ \delta_v^1)$ that does not belong to $\ker \delta_v^1$ (first summand) whose triviality under $\pi_h^2 \circ \delta_v^1$ is by adding by elements of $\text{im } \delta_h^1$ (second summand) that turn out to belong to $\text{im } \delta_h^0$. The 1-cochains are highlighted in light gray, and the signs are indicated by the arrows.

$$\begin{array}{c} \uparrow \\ \downarrow \end{array} \bullet + \begin{array}{c} \leftarrow \bullet \rightarrow \\ \downarrow \end{array} = \begin{array}{c} \uparrow \\ \downarrow \end{array} \bullet + \begin{array}{c} \leftarrow \bullet \rightarrow \\ \downarrow \end{array}$$

Figure 5.10: $\ker(\pi_h^2 \circ \delta_v^1) / \text{im } \delta_v^0$ (left, first summand) added by $\text{im } \delta_h^0$ (left, second summand) gives a sum of an element of $\ker \delta_v^1$ (right, first summand) and an element of $\text{im } \delta^0$ (right, second summand), and vice versa (up to elements in $\text{im } \delta_v^0$ and $\text{im } \delta_h^0$). The signs on the 1-cochains are indicated by the arrows.

Theorem 5.17. For $d = 2$, $K_0(C_r^*(\dot{G}_u)) \cong \check{H}^2(\Omega_{T_\square}) \oplus \check{H}^0(\Omega_{T_\square})$ and $K_1(C_r^*(\dot{G}_u)) \cong \check{H}^1(\Omega_{T_\square})$.

Proof. It remains to show that $\check{H}^1(\Omega_{T_\square}) = \ker \tilde{\delta}_v^1 \oplus \ker \tilde{\delta}_h^1$. We will show this for $\bullet = v$.

Let us begin with the observation that there is an interaction between elements that go to zero only at π_h^2 in $\ker(\pi_h^2 \circ \delta_v^1)$ and elements of $\text{im } \delta_h^0$. Suppose that $\sum_i c_{v,i}^1 \in \ker \delta_v^1$ and for some j and some 0-cochain c^0 , $\delta_v^0(c^0) = c_{v,j}^1 - c_{v,j'}^1$. Then $\sum_{i \neq j} c_v^1 + c_{v,j}^1 \in \ker(\pi_h^2 \circ \delta_v^1)$ but $\sum_{i \neq j} c_v^1 + c_{v,j'}^1 \notin \ker \delta_v^1$. In particular, the quotient from π_h^2 is by two pairs of neighboring 2-cochains that arise from some $\delta_h^1(c_{h,j_1}^1)$ and $\delta_h^1(c_{h,j_2}^1)$, and the difference of the two horizontal 1-cochains is $\delta_h^0(c^0)$. In other words, changing representatives of elements in $\ker(\pi_h^2 \circ \delta_v^1) / \text{im } \delta_v^0$ implicitly includes elements of $\text{im } \delta_h^0$ (Fig. 5.9).

Furthermore, we notice that

$$\delta^0(c^0) = c_{v,j}^1 - c_{v,j'}^1 + c_{h,j_1}^1 - c_{h,j_2}^1,$$

i.e., this changing of representatives can be further realized as adding elements of $\text{im } \delta^0$ (Fig. 5.10).

From these observations, $\ker \tilde{\delta}_v^1 \oplus \ker \tilde{\delta}_h^1 \leq \check{H}^1(\Omega_{T_\square})$.

For the converse, let us consider the inductive limit structure on $\check{H}^1(\Omega_{T_\square})$ afforded by σ_\square , and recall that elements of $\ker \delta^1$ correspond to collections of closed curves intersecting transversely with the 1-cochains, with the oriented intersection numbers coinciding with the values assigned by the 1-cochains ([Hat02, pp. 188–189]). Given a closed curve in Γ_n as described, let $c_n^1 = (c_{n,v}^1, c_{n,h}^1)$ denote the collection of 1-cochains it intersects. Being a closed curve, $c_{n,v}^1$ is formed from the vertical boundaries of a collection of 2-cells that are neighboring, up to vertical translations. That is, $c_{n,v}^1 \in \ker \tilde{\delta}_{n,v}^1$. Same holds for $c_{n,h}^1$. Thus $\check{H}^1(\Omega_{T_\square}) \leq \ker \tilde{\delta}_v^1 \oplus \ker \tilde{\delta}_h^1$. \square

We can then compose these isomorphisms with the map \square^{-1} induced on cohomology to get the generators and relations in terms of the cochains and coboundary maps in Ω_T , from which we see that the isomorphism $K_1(C_r^*(\dot{G}_u)) \cong \check{H}^1(\Omega_T)$ is induced by thickening.

We now revert to groupoids on Ω_T . Using this theorem and Theorem 5.11, we can concisely package the relevant topological objects in the context of relative K -theory into the following corollary that is a generalization of Theorem 1.1 to $d = 2$.

Corollary 5.18 (Theorem 1.2). *For $d = 2$, we have an isomorphism of exact sequences*

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \check{H}^1(\Omega_T) & \hookrightarrow & C^1 / \text{im } \delta^0 & \xrightarrow{\delta^1} & C^2 & \twoheadrightarrow & \check{H}^2(\Omega_T) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow \text{th} & & \oplus & & \oplus & & \oplus \\
& & & & & & 0 & \longrightarrow & \check{H}^0(\Omega_T) & \hookrightarrow & \check{H}^0(\Omega_T) \longrightarrow 0 \\
& & & & & & \downarrow \text{CF} & & \downarrow & & \downarrow \\
0 & \longrightarrow & K_{1,u} & \hookrightarrow & K_{0,AF;u} & \xrightarrow{\text{ev}} & K_{0,AF} & \xrightarrow{\iota_*} & K_{0,u} & \twoheadrightarrow & K_{1,AF;u} \longrightarrow 0.
\end{array}$$

where $K_{i,u} = K_i(C_r^*(\dot{G}_u))$, $K_{i,AF} = K_i(C_r^*(\dot{G}_{AF}))$, and $K_{i,AF;u} = K_i(C_r^*(\dot{G}_{AF}); C_r^*(\dot{G}_u))$.

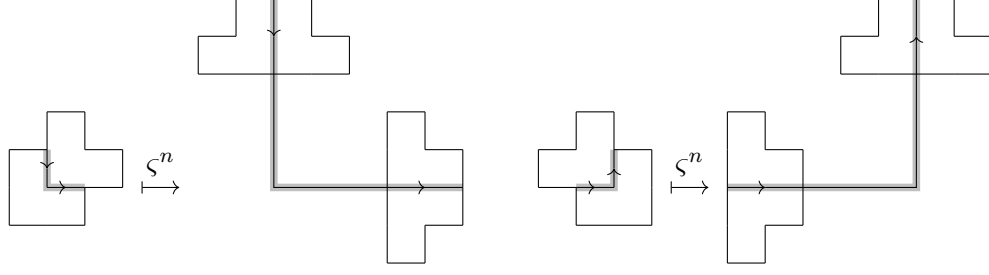


Figure 5.11: In the chair tiling, for sufficiently large n , the two 1-cells (light gray) formed from the two intersections of pairs of chairs in the same orientations substitute to contain the same two types of boundaries (light gray with neighboring tiles drawn). Regardless of the orientations assigned to the unsubstituted 1-cells, one of the substituted 1-cells will always have its orientation reversed, resulting in a sign on σ^\top .

Thus, if T satisfies the boundary hyperplane condition,

$$K_0(C_r^*(\dot{G}_u)) \cong K_0(C_r^*(\dot{G}_{AF})) / \text{im ev} \oplus \mathbb{Z}$$

$$K_1(C_r^*(\dot{G}_u)) \cong \ker[K_0(C_r^*(\dot{G}_{AF}^{(1)})) / \text{th}(K_0(C_r^*(\dot{G}_{AF}^{(0)})))] \xrightarrow{\text{ev}} K_0(C_r^*(\dot{G}_{AF}))]$$

where $\text{th} : K_0(C_r^*(\dot{G}_{AF}^{(0)})) \rightarrow K_0(C_r^*(\dot{G}_{AF}^{(1)}))$ returns an alternating sum of elements of $K_0(C_r^*(\dot{G}_{AF}^{(1)}))$ that coincides with δ^0 .

Therefore for $d = 2$, the K -theory of the unstable groupoid can be reconstructed from the K -theory of the AF -groupoids if T satisfies the boundary hyperplane condition. This includes all triangular substitutions where the tiles are aligned edge-to-edge.

Remark 5.19. The right-hand side of the isomorphism of exact sequences recovers the well-known fact from [Kel97] that the contribution of $K_0(C_r^*(\dot{G}_{AF}))$ towards $K_0(C_r^*(\dot{G}_u))$ is as the top-dimensional cohomology, also called the *integer group of coinvariants*.

Warning 5.20. If T does not satisfy the boundary hyperplane condition, $\sigma^\top : C_n^1 \rightarrow C_{n+1}^1$ may carry signs that *cannot be made nonnegative* through reorienting 1-cells (Fig. 5.11)! This is

the reason we cared little for the orientations on the 1-cells not satisfying the boundary hyperplane condition. Thus we may no longer have a map $K_0(C_r^*(\dot{G}_{AF,n}^{(1)})) \rightarrow K_0(C_r^*(\dot{G}_{AF,n+1}^{(1)}))$ that is both compatible with the map on cochains and induced by an algebra map $C_r^*(\dot{G}_{AF,n}^{(1)}) \hookrightarrow C_r^*(\dot{G}_{AF,n+1}^{(1)})$.

Despite this, $K_0(C_r^*(\dot{G}_{AF}^{(1)}))$ and C^1 may still be isomorphic⁵! Let us call a direct limit under a matrix all of whose eigenvalues are integral *completely split* if it is isomorphic to $\bigoplus_{\lambda} \mathbb{Z}[1/\lambda]$ with λ the eigenvalues of the matrix.

Proposition 5.21. *Let H be the set of 1-cells satisfying the boundary hyperplane condition, and let $\sigma^\top|_H$ be the induced substitution matrix restricted to H . If $\sigma^\top|_H$ has all of its eigenvalues integral and its direct limit splits completely, then $K_0(C_r^*(\dot{G}_{AF}^{(1)})) \cong C^1$.*

Proof. Let H^c be the complement of H in the set of 1-cells. Elements of H cannot subdivide to elements of H^c , so σ^\top , written as a 2×2 block matrix, has the form

$$\begin{pmatrix} \sigma^\top|_{H^c} & A \\ 0 & \sigma^\top|_H \end{pmatrix}.$$

The substitution map, restricted to H^c , is necessarily nonexpanding. Therefore, after removing the eventual kernel, $\sigma^\top|_{H^c}$ is a permutation matrix. By applying a sufficiently-high power (and removing the eventual kernel), we will assume $\sigma^\top|_{H^c} = I$. Letting $B = \sigma^\top|_H$, our block matrix has the form

$$\begin{pmatrix} I & A \\ 0 & B \end{pmatrix}.$$

⁵This isomorphism may not arise from shift equivalence.

Noting that eigenvectors of unit eigenvalues contribute copies of \mathbb{Z} in the limit, suppose that all eigenvalues of B are nonzero and nonunit. Let v be an eigenvector for an eigenvalue λ . Setting $w = Av/(\lambda - 1)$,

$$\begin{aligned} \begin{pmatrix} I & A \\ 0 & B \end{pmatrix} \begin{pmatrix} w \\ v \end{pmatrix} &= \begin{pmatrix} w + Av \\ \lambda v \end{pmatrix} \\ &= \begin{pmatrix} w + (\lambda - 1)w \\ \lambda v \end{pmatrix} \\ &= \lambda \begin{pmatrix} w \\ v \end{pmatrix}, \end{aligned}$$

and we see that each such eigenvector v of B can be used to construct an eigenvector for the block matrix.

To compute the direct limit under this block matrix, we are looking for the set of vectors $c \in \mathbb{Z}^{|H^c|+|H|}$ so that

$$\sum_{i=1}^{|H^c|} c_i e_i + \sum_{j=|H^c|+1}^{|H^c|+|H|} c_j \lambda_j^k \begin{pmatrix} w_j \\ v_j \end{pmatrix} \in \mathbb{Z}^{|H^c|+|H|}.$$

By assumption, since the direct limit under B splits completely, each $c_j \lambda_j^k w_j$ is integral. Thus, each of the c_i is integral (up to a fraction), and the direct limit of the block matrix also splits completely.

Repeating the same argument with A replaced to represent the matrix induced by

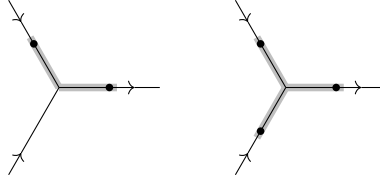


Figure 5.12: A partial homeomorphism yielding a partial isometry, as given by restriction (left), compared to $\text{im } \delta^0$ (right). We expect to quotient the set of all projections on the skeleton by the former, but the latter is what we actually quotient by. This partial isometry cannot be realized as an element of $\text{im } \delta^0$.

$C_r^*(\dot{G}_{AF,n}^{(1)}) \hookrightarrow C_r^*(\dot{G}_{AF,n+1}^{(1)})$, denoted A' , yields our desired isomorphism, given by the map

$$\begin{pmatrix} w_j \\ v_j \end{pmatrix} \mapsto \begin{pmatrix} w'_j \\ v_j \end{pmatrix}$$

where $w' = A'v/(\lambda - 1)$. □

Warning 5.22. In general, it does not appear that $K_0(C_r^*(\dot{G}_{AF}); C_r^*(\dot{G}_u))$ arises as the K_0 -group of the $d-1$ -skeleton! This is true for $d = 1$, as we saw in the proof of Theorem 1.1, where it arises from an AF -algebra. However, for $d = 2$, we possibly have AF -equivalence classes that arise as the intersections of three or more infinite 1-boundaries. If $K_0(C_r^*(\dot{G}_{AF}); C_r^*(\dot{G}_u)) \cong C^1/\text{im } \delta^0$ arises as the K_0 -group of $\varprojlim_n (AP_n^{(1)}, \sigma)$, then we expect it to come from restriction.

On the one hand, there is a partial isometry containing only two of the 1-boundaries. On the other, δ^0 applied to this intersection gives a single signed sum of involving all three infinite 1-boundaries. In particular, $\text{im } \delta^0$ must assign nonzero values to all incident 1-cells, whereas (generating) partial isometries select two 1-cells to assign nonzero values. Under Murray–von Neumann equivalence, we expect the latter to be the relation we quotient by in the K_0 -group of the 1-skeleton, but the relation we have is the former, and we cannot obtain all of the relations of the latter form from the former (Fig. 5.12).

However, in certain situations (e.g. [JS16]), there exist substitution tiling spaces with stationary substitution rules within the same MLD-equivalence class that have a subset of its $d - 1$ -skeleton that intersects all other infinite boundaries and is closed under the induced group action by translation. This subset is a $d - 1$ -dimensional tiling space, and so the relative K -theory relative to this subgroupoid is isomorphic to the K -theory of this new groupoid.

Remark 5.23. Since the Chern and Connes–Thom isomorphisms hold over \mathbb{Z} for $d \leq 3$, we expect that the analogous theorems hold for $d = 3$ as well, at least abstractly,

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \check{H}^2(\Omega_T) & \hookrightarrow & C^2/\text{im } \delta^1 & \xrightarrow{\delta^2} & C^3 & \twoheadrightarrow & \check{H}^3(\Omega_T) & \longrightarrow & 0 \\
& & \oplus & & \oplus & & \oplus & & \oplus & & \oplus \\
0 & \longrightarrow & \check{H}^0(\Omega_T) & \hookrightarrow & \check{H}^0(\Omega_T) & \longrightarrow & 0 & \longrightarrow & \check{H}^1(\Omega_T) & \hookrightarrow & \check{H}^1(\Omega_T) \longrightarrow 0 \\
& & \downarrow & & \downarrow \text{th} \oplus \phi & & \downarrow \text{CF} & & \downarrow & & \downarrow \\
0 & \longrightarrow & K_{1,u} & \hookrightarrow & K_{0,AF;u} & \xrightarrow{\text{ev}} & K_{0,AF} & \xrightarrow{\iota_*} & K_{0,u} & \twoheadrightarrow & K_{1,AF;u} \longrightarrow 0,
\end{array}$$

where assuming the boundary hyperplane condition and using Lemma 5.2 gives the contributions of each of the AF -groupoids towards the K -theory of the unstable groupoid after undoing the isomorphisms.

Unfortunately, as it is for $d = 2$, the component of $\check{H}^0(\Omega_T)$ in $K_1(C_r^*(\dot{G}_u))$, denoted by ϕ , is harder to identify. By writing $C_r^*(\dot{G}_u)$ as a crossed product via [Kel97, Lemma 11], peeling off one dimension, then applying the Connes–Thom isomorphism to this single dimension gives the K_0 -group of a \mathbb{Z}^2 -crossed product. The reasoning provided in [KP00] results in a rank-1 element that cannot be homotopic to the trivial rank-1 projection. This should be the element seen by ϕ .

5.3 Interpretation

Theorem 1.1 and Theorem 1.2 allow us to give a very concise description of the six-term sequence in relative K -theory of $\iota : C_r^*(\dot{G}_{AF}) \hookrightarrow C_r^*(\dot{G}_u)$ for $d \leq 2$. For motivation, let us first recall an alternate way to record a substitution rule with an example.

Example 5.24 (Fibonacci). The square of the collared Fibonacci substitution is

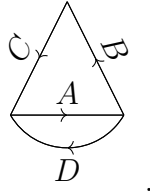
$$A \mapsto (AD)AB(C)$$

$$B \mapsto (B)CAB(C)$$

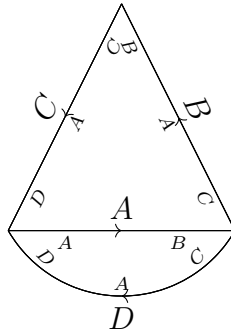
$$C \mapsto (AB)CAD(A)$$

$$D \mapsto (B)CAD(A)$$

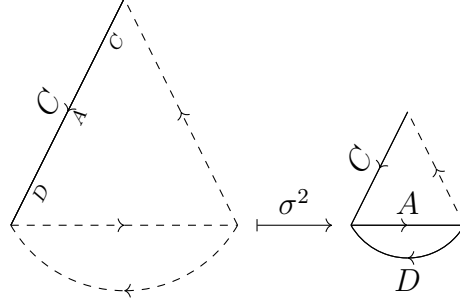
with the AP -complex



We can condense these two pieces of information into a single picture



where the smaller letters are labels we further attach to each of the top-dimensional cells to indicate where they are sent under σ . E.g.



where the first third of the level-2 supertile $\varsigma^2(C)$ is subdivided to C , the second third to A , and the last third to D , so that the path traced out in AP_0 has orientations agreeing with the underlying orientations of the 1-cells.

When we add these labels on top of the supertiles, the substitution map turns into an evaluation map where one evaluates the labels. *The thickening map is exactly labelling the $d - 1$ -cells with their images under δ^{d-1}* , and performing this process in low dimensions turns the exact sequence

$$0 \longrightarrow \check{H}^{d-1}(\Omega_T) \hookrightarrow C^{d-1} / \text{im } \delta^{d-2} \xrightarrow{\delta^{d-1}} C^d \twoheadrightarrow \check{H}^d(\Omega_T) \longrightarrow 0$$

into the nontrivial part of the six-term sequence in relative K -theory that is witnessed by $\text{ev} :$

$$K_0(C_r^*(\dot{G}_{AF}); C_r^*(\dot{G}_u)) \rightarrow K_0(C_r^*(\dot{G}_{AF}))!$$

Chapter 6: Computations

In this chapter, we first perform the computations by hand for $d = 1$ to illustrate the details, then resort to a script in Sage written by the author that automatically collars and computes all of the induced substitution matrices for $d = 2$, whose results are compiled in Table 6.1. Note that the computations are performed over *exact rings*, and therefore carry no floating point errors (aside from the eigenspace computations for the induced substitution matrices in Danzer 7-fold and the GKM examples). Furthermore, it appears that the script works for certain *pseudosubstitutions*, and we perform the computations for [CS06, Figure 4].

6.1 $d = 1$

Example 6.1 (Fibonacci). The Fibonacci substitution has collared prototiles

$$\{A = (b)a(b), D = (a)b(a), B = (a)b(b), C = (b)b(a)\},$$

with the substitution rule

$$A \mapsto (B)C(A)$$

$$D \mapsto AB$$

$$B \mapsto AD(A)$$

$$C \mapsto (D)AB$$

whose square forces the border

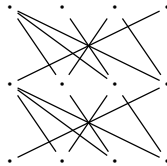
$$A \mapsto (AD)AB(C)$$

$$D \mapsto (B)CAD(A)$$

$$B \mapsto (B)CAB(C)$$

$$C \mapsto (AB)CAD(A).$$

The transpose of the induced substitution on them is given by the stationary ordered Bratteli diagram



where the order of the vertices is the same as the one in the set and the corresponding Bratteli–Vershik system has the order of the edges from left to right as presented in the diagram.

The corresponding matrix is

$$\sigma^\top = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

which has eigenspaces

$$\Lambda_{AF,\phi} = \text{span}\{(\phi^{-1}, 1, 1, 1)\}$$

$$\Lambda_{AF,-1} = \text{span}\{(-1, 1, 0, 1)\}$$

$$\Lambda_{AF,-\phi^{-1}} = \text{span}\{(-\phi, 1, 1, 1)\}$$

$$\Lambda_{AF,0} = \text{span}\{-1, 1, 1, 0\}$$

where $\phi = (1 + \sqrt{5})/2$. The product of the nontrivial eigenvalues is 1, so the induced map

$\sigma^\top : K_0(C_r^*(\dot{G}_{AF,n})) \rightarrow K_0(C_r^*(\dot{G}_{AF,n+1}))$ is invertible after removing the kernel, and

$$K_0(C_r^*(\dot{G}_{AF})) \cong \mathbb{Z}^3 = \Lambda_{AF,\phi} \oplus \Lambda_{AF,-1} \oplus \Lambda_{AF,-\phi^{-1}}.$$

We now look for potentially nontrivial partial isometries that are formed from neighboring prototiles, which are

$$\{(A, D), (A, B), (D, A), (B, C), (C, A)\}$$

giving us that the set of full translations is

$$\{(A, D + B), (D + C, A), (B, C)\}.$$

Since we only deal with basic projections, these can be represented by pairs of sets of matrices where we only keep track of the diagonal. Under $(\sigma^\top)^2$, reading off from the Bratteli diagram, the full translations in level-0 are mapped by

$$\begin{aligned} (1, 0, 0, 0), (0, 1, 1, 0) &\mapsto ((1, 0), (0, 1, 0), (0, 1, 0), (0, 1, 0)), ((0, 1), (0, 0, 1), (0, 0, 1), (0, 0, 1)) \\ &= ((0, 0), (0, 0, 0), (0, 0, 0), (0, 0, 0)), ((0, 0), (0, 0, 0), (0, 0, 0), (0, 0, 0)) \\ (0, 1, 0, 1), (1, 0, 0, 0) &\mapsto ((0, 0), (1, 0, 1), (1, 0, 0), (0, 0, 1)), ((1, 0), (0, 1, 0), (0, 1, 0), (0, 1, 0)) \\ &= ((0, 0), (0, 0, 1), (0, 0, 0), (0, 0, 1)), ((1, 0), (0, 0, 0), (0, 0, 0), (0, 0, 0)) \\ (0, 0, 1, 0), (0, 0, 0, 1) &\mapsto ((0, 1), (0, 0, 0), (0, 0, 1), (0, 0, 0)), ((0, 0), (1, 0, 0), (1, 0, 0), (1, 0, 0)) \\ &= ((0, 1), (0, 0, 0), (0, 0, 0), (0, 0, 0)), ((0, 0), (1, 0, 0), (1, 0, 0), (0, 0, 0)) \\ &\quad + ((0, 0), (0, 0, 0), (0, 0, 1), (0, 0, 0)), ((0, 0), (0, 0, 0), (0, 0, 0), (1, 0, 0)) \end{aligned}$$

where we eliminated the pairs of entries whose sources and ranges belong to $C_r^*(\dot{G}_{AF,2})$, thus are trivial in $K_0(C_r^*(\dot{G}_{AF,2}); C_r^*(\dot{G}_u))$.

Let us explicitly write out the images in terms of tiles in supertiles. For the respective level-2 supertiles, let us denote the position corresponding to the basic projections by underscores and the absense of projections by zeroes. Then, for the second matrix algebra pair, for the second

summand of the source, we have the projection

$$\sigma^2(D) = (B)C\overline{A}\underline{D}(A)$$

which has range

$$\sigma^2(A) = (AD)\underline{A}B(C)$$

which is the first summand of the range. However, (D, A) is not a full translation, and we need to include

$$\sigma^2(C) = (AB)C\overline{A}\underline{D}(A)$$

in the source as well, which is the fourth summand, giving the triple

$$[0 + (B)C\overline{A}\underline{D}(A) + 0 + (AB)C\overline{A}\underline{D}(A), (AD)\underline{A}B(C) + 0 + 0 + 0, 1]$$

as the image, where the last entry indicates that the partial isometry moves left to right by 1 unit.

We can then denote this by

$$[\sigma^2(D) + \sigma^2(C), \sigma^2(A), 1].$$

For the first term in the third matrix algebra pair, we have, for the first summand of the source,

$$\sigma^2(A) = (AD)\underline{A}\underline{B}(C),$$

which has

$$\sigma^2(D) = (B)\underline{C}AD(A)$$

$$\sigma^2(B) = (B)\underline{C}AB(C),$$

the second and third summands, both as its range. This is a full translation, and the triple is

$$[(AD)A\underline{B}(C) + 0 + 0 + 0, 0 + (B)\underline{C}AD(A) + (B)\underline{C}AB(C) + 0, 1]$$

which we can denote as

$$[\sigma^2(A), \sigma^2(D) + \sigma^2(B), 1].$$

For the second term in the third pair, we have, for the third summand,

$$\sigma^2(B) = (B)C\underline{A}\underline{B}(C)$$

which has range

$$\sigma^2(C) = (AB)\underline{C}AD(A).$$

This is a full translation, so we obtain the triple

$$[0 + 0 + (B)C\underline{A}\underline{B}(C) + 0, 0 + 0 + 0 + (AB)\underline{C}AD(A), 1]$$

which we denote as

$$[\sigma^2(B), \sigma^2(C), 1].$$

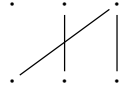
In conclusion, the map is

$$[A, D + B, 1] \mapsto 0$$

$$[D + C, A, 1] \mapsto [\sigma^2(D) + \sigma^2(C), \sigma^2(A), 1]$$

$$[B, C, 1] \mapsto [\sigma^2(A), \sigma^2(D) + \sigma^2(B), 1] + [\sigma^2(B), \sigma^2(C), 1].$$

Since the codomain is a pair of direct sums of matrix algebras, by realizing this as a direct sum of twice the size and repeating the computation for the other two elements, we obtain a map of matrix algebras, which we can represent by a Bratteli diagram



where the order of vertices is the same order we computed. Thus

$$(\sigma^\top)^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

and one computes that the eigenspaces are

$$\Lambda_{AF;u,1} = \text{span}\{(0, 1, 0), (1, 0, 1)\}$$

$$\Lambda_{AF;u,0} = \text{span}\{(1, 0, 0)\}.$$

Evidently there are no nontrivial relations among the two eigenvectors of $\Lambda_{AF;u,1}$, so

$$K_0(C_r^*(\dot{G}_{AF}); C_r^*(\dot{G}_u)) \cong \mathbb{Z}^2 = \Lambda_{AF;u,1}.$$

For each of the two eigenvectors, we have that

$$\text{ev}((0, 1, 0)) = (-1, 1, 0, 1)$$

$$\text{ev}((1, 0, 1)) = (1, -1, 0, -1)$$

where the domain and codomain are both expressed in standard basis in full translations and prototiles, respectively. Thus, expressed in eigenbasis,

$$\text{ev} = \begin{pmatrix} 0 & 0 \\ 1 & -1 \\ 0 & 0 \end{pmatrix}.$$

Finally,

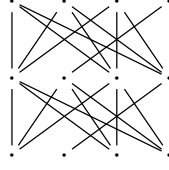
$$K_0(C_r^*(\dot{G}_u)) \cong \mathbb{Z}^2 = \Lambda_{AF,\phi} \oplus \Lambda_{AF,-\phi^{-1}}$$

$$K_1(C_r^*(\dot{G}_u)) \cong \mathbb{Z} = \text{span}\{(1, 1, 1)\} = \Delta(\Lambda_{AF;u,1}).$$

Example 6.2 (Silver Mean). The Silver Mean substitution has collared prototiles

$$\{A = (b)b(a), B = (b)a(b), C = (a)b(b), D = (b)b(b)\}.$$

The transpose of the induced substitution on them is given by the stationary ordered Bratteli diagram



where the order of the vertices is the same as the one in the set, the order of the edges is from left to right, and we provided two levels because the substitution rule forces the border at level-2.

The corresponding matrix is

$$\sigma^\top = \begin{pmatrix} 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 0 \\ 2 & 2 & 2 & 1 \\ 2 & 2 & 2 & 1 \end{pmatrix}.$$

It has eigenspaces

$$\Lambda_{AF, \delta_S^2} = \text{span}\{(1, \delta_S^{-1}, 1, 1)\}$$

$$\Lambda_{AF, \delta_S^{-2}} = \text{span}\{(1, -\delta_S, 1, 1)\}$$

$$\Lambda_{AF, 0} = \text{span}\{(-1, 0, 1, 0), (-1, 1, 0, 0)\}$$

where $\delta_S = 1 + \sqrt{2}$. The product of the nontrivial eigenvalues is again 1, so

$$K_0(C_r^*(\dot{G}_{AF})) \cong \mathbb{Z}^2 = \Lambda_{AF, \delta_S^2} \oplus \Lambda_{AF, \delta_S^{-2}}.$$

We look for partial isometries formed from neighboring prototiles, which are

$$\{(A, B), (B, C), (C, A), (C, D), (D, A)\}$$

giving us that the set of triples in $K_0(C_r^*(\dot{G}_{AF}); C_r^*(\dot{G}_u))$ that we compute the direct limit with is

$$\{[A, B, 1], [B, C, 1], [C + D, A + D, 1]\}.$$

Representing by matrices where we only keep track of the diagonal, under $(\sigma^\top)^2$,

$$\begin{aligned}
(1, 0, 0, 0), (0, 1, 0, 0) &\mapsto ((1, 0, 0, 0, 1, 0, 0), (0, 0, 0), (1, 0, 0, 0, 1, 0, 0), (1, 0, 0, 0, 1, 0, 0)), \\
&((0, 1, 0, 0, 0, 1, 0), (0, 0, 0), (0, 1, 0, 0, 0, 1, 0), (0, 1, 0, 0, 0, 1, 0)) \\
&= ((0, 0, 0, 0, 0, 0, 0), (0, 0, 0), (0, 0, 0, 0, 0, 0, 0), (0, 0, 0, 0, 0, 0, 0)), \\
&((0, 0, 0, 0, 0, 0, 0), (0, 0, 0), (0, 0, 0, 0, 0, 0, 0), (0, 0, 0, 0, 0, 0, 0)) \\
(0, 1, 0, 0), (0, 0, 1, 0) &\mapsto ((0, 1, 0, 0, 0, 1, 0), (0, 0, 0), (0, 1, 0, 0, 0, 1, 0), (0, 1, 0, 0, 0, 1, 0)), \\
&((0, 0, 1, 0, 0, 0, 1), (0, 0, 0), (0, 0, 1, 0, 0, 0, 1), (0, 0, 1, 0, 0, 0, 1)) \\
&= ((0, 0, 0, 0, 0, 0, 0), (0, 0, 0), (0, 0, 0, 0, 0, 0, 0), (0, 0, 0, 0, 0, 0, 0)), \\
&((0, 0, 0, 0, 0, 0, 0), (0, 0, 0), (0, 0, 0, 0, 0, 0, 0), (0, 0, 0, 0, 0, 0, 0)) \\
(0, 0, 1, 1), (1, 0, 0, 1) &\mapsto ((0, 0, 1, 1, 0, 0, 1), (0, 0, 1), (0, 0, 1, 1, 0, 0, 1), (0, 0, 1, 1, 0, 0, 1)), \\
&((1, 0, 0, 1, 1, 0, 0), (1, 0, 0), (1, 0, 0, 1, 1, 0, 0), (1, 0, 0, 1, 1, 0, 0)) \\
&= ((0, 0, 0, 0, 0, 0, 1), (0, 0, 1), (0, 0, 0, 0, 0, 0, 1), (0, 0, 0, 0, 0, 0, 1)), \\
&((1, 0, 0, 0, 0, 0, 0), (1, 0, 0), (1, 0, 0, 0, 0, 0, 0), (1, 0, 0, 0, 0, 0, 0)) \\
&= ((0, 0, 0, 0, 0, 0, 1), (0, 0, 0), (0, 0, 0, 0, 0, 0, 0), (0, 0, 0, 0, 0, 0, 0)), \\
&((0, 0, 0, 0, 0, 0, 0), (1, 0, 0), (0, 0, 0, 0, 0, 0, 0), (0, 0, 0, 0, 0, 0, 0)) \\
&+ ((0, 0, 0, 0, 0, 0, 0), (0, 0, 1), (0, 0, 0, 0, 0, 0, 0), (0, 0, 0, 0, 0, 0, 0)), \\
&((0, 0, 0, 0, 0, 0, 0), (0, 0, 0), (1, 0, 0, 0, 0, 0, 0), (0, 0, 0, 0, 0, 0, 0)) \\
&+ ((0, 0, 0, 0, 0, 0, 0), (0, 0, 0), (0, 0, 0, 0, 0, 0, 1), (0, 0, 0, 0, 0, 0, 1)), \\
&((1, 0, 0, 0, 0, 0, 0), (0, 0, 0), (0, 0, 0, 0, 0, 0, 0), (1, 0, 0, 0, 0, 0, 0))
\end{aligned}$$

where we eliminated the pairs of entries that are in $C_r^*(\dot{G}_{AF,2})$. For clarity, let us illustrate the

process again for computing the images in terms of full translations.

Following the same convention as before, the projection from the first summand of the first term of the third matrix algebra pair is

$$\sigma^2(A) = (BCDABC)ABCDABC\underline{C}(AB)$$

which, under the partial isometry, is sent to

$$\sigma^2(B) = (BCDABC)\underline{A}BC(ABCDAB).$$

Since (A, B) is a full translation, the triple is

$$[(BCDABC)ABCDABC\underline{C}(AB) + 0 + 0 + 0, 0 + (BCDABC)\underline{A}BC(ABCDAB) + 0 + 0, 1],$$

denoted

$$[\sigma^2(A), \sigma^2(B), 1].$$

For the second summand of the second term, we have

$$\sigma^2(B) = (BCDABC)ABC\underline{C}(ABCDAB),$$

and

$$\sigma^2(C) = (BC)\underline{A}BCDABC(ABCDAB)$$

is the range, and since (B, C) is a full translation, the triple is

$$[0 + (BCDABC)ABC\underline{C}(ABCDAB) + 0 + 0, 0 + 0 + (BC)\underline{A}BCDABC(ABCDAB) + 0, 1],$$

denoted

$$[\sigma^2(B), \sigma^2(C), 1].$$

Finally, the third summand of the third term is

$$\sigma^2(C) = (BC)ABCDABC\underline{C}(ABCDAB)$$

and has both

$$\sigma^2(A) = (BCDABC)\underline{A}BCDABC(AB)$$

$$\sigma^2(D) = (BCDABC)\underline{A}BCDABC(ABCDAB)$$

as the range. However, $(C, A + D)$ is not a full translation. We need to include D , therefore

$$\sigma^2(D) = (BCDABC)ABCDABC\underline{C}(ABCDAB),$$

in the source as well. Thus the corresponding triple is

$$[0 + 0 + (BC)\underline{A}BCDABC(ABCDAB) + (BCDABC)\underline{A}BCDABC(ABCDAB), \\ (BCDABC)\underline{A}BCDABC(AB) + 0 + 0 + (BCDABC)\underline{A}BCDABC(ABCDAB), 1],$$

denoted

$$[\sigma^2(C) + \sigma^2(D), \sigma^2(A) + \sigma^2(D), 1].$$

Thus the map is

$$[A, B, 1] \mapsto 0$$

$$[B, C, 1] \mapsto 0$$

$$\begin{aligned} [C + D, A + D, 1] &\mapsto [\sigma^2(A), \sigma^2(B), 1] + [\sigma^2(B), \sigma^2(C), 1] \\ &\quad + [\sigma^2(C) + \sigma^2(D), \sigma^2(A) + \sigma^2(D), 1]. \end{aligned}$$

Since the codomain is a pair of direct sums of matrix algebras, by realizing this as a direct sum of twice the size, we obtain a map of matrix algebras, which we can represent by a Bratteli diagram



where the order of vertices is the same order we computed. Thus

$$(\sigma^\top)^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

and the eigenspaces are

$$\Lambda_{AF;u,1} = \text{span}\{(1, 1, 1)\}$$

$$\Lambda_{AF;u,0} = \text{span}\{(1, 0, 0), (0, 1, 0)\}.$$

There are no nontrivial relations among the eigenvectors, so

$$K_0(C_r^*(\dot{G}_{AF}); C_r^*(\dot{G}_u)) \cong \mathbb{Z} = \Lambda_{AF;u,1}.$$

For the single vector, we have that

$$\text{ev}((1, 1, 1)) = (0, 0, 0, 0)$$

where the domain and codomain are both expressed in standard basis in full translations and prototiles, respectively.

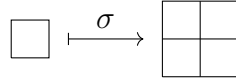
Finally,

$$K_0(C_r^*(\dot{G}_u)) \cong \mathbb{Z}^2 = \Lambda_{AF, \delta_S^2} \oplus \Lambda_{AF, \delta_S^{-2}}$$

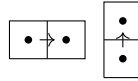
$$K_1(C_r^*(\dot{G}_u)) \cong \mathbb{Z} = \Lambda_{AF;u,1}.$$

6.2 $d = 2$

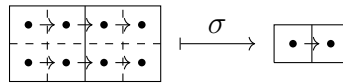
Example 6.3 (Dyadic solenoid). The only “example” we can do by hand¹ is the two-dimensional dyadic solenoid, given by the substitution



where we call the puncture in the single square t . The substitution forces the border, and it is not too hard to see that the AP -complex is a torus, giving us the two generating partial isometries that are simultaneously full translations



which we denote $[t, t, (1, 0)]$ and $[t, t, (0, 1)]$. Notice that as a substitution, there are multiple preimages that subdivide to each, for example for $[t, t, (1, 0)]$,



but if the image is a generator in $K_0(C_r^*(\dot{G}_{AF,n}); C_r^*(\dot{G}_u))$, four of the partial isometries in the domain have sources and ranges belonging to $C_r^*(\dot{G}_{AF,n+1})$, thus are trivial in

¹Rather, the only two-dimensional computation the author is willing to do by hand, particularly since the matrix representations of the translation action that identifies neighboring tiles in a supertile is not as readily transparent as it is for $d = 1$. See [Kel97] for the computation for the Robinson triangle tiling.

$K_0(C_r^*(\dot{G}_{AF,n+1}); C_r^*(\dot{G}_u))$, and the transpose map becomes

$$\begin{array}{|c|c|} \hline \bullet & \bullet \\ \hline \end{array} \xrightarrow{\sigma^\top} \begin{array}{|c|c|c|c|} \hline \bullet & \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet & \bullet \\ \hline \end{array}.$$

There is a single generating relation, given by

$$\begin{array}{|c|c|} \hline \bullet & \bullet \\ \hline \bullet & \bullet \\ \hline \end{array}$$

where the top full translation is $[t, t, -(1, 0)]$ and the left is $[t, t, -(0, 1)]$, giving us

$$\begin{aligned} & [t, t, (1, 0)] + [t, t, (0, 1)] + [t, t, -(1, 0)] + [t, t, -(0, 1)] \\ &= [t, t, (1, 0)] + [t, t, (0, 1)] - [t, t, (1, 0)] - [t, t, (0, 1)] \\ &= 0. \end{aligned}$$

This sum of full translations is always trivial since addition in K -theory is commutative, thus we do not need to find the induced substitution map. The evaluation map is trivial, since $[t, t, (1, 0)] \mapsto [t] - [t] = 0$ and $[t, t, (0, 1)] \mapsto [t] - [t] = 0$. Thus, taking direct limits, the six-term sequence is

$$\begin{array}{ccccc} \mathbb{Z}[1/2]^2 & \xrightarrow{0} & \mathbb{Z}[1/4] & \xrightarrow{\iota_*} & K_0(C_r^*(\dot{G}_u)) \\ \uparrow & & & & \downarrow \\ K_1(C_r^*(\dot{G}_u)) & \longleftarrow & 0 & \longleftarrow & \mathbb{Z}. \end{array}$$

Therefore $K_0(C_r^*(\dot{G}_u)) \cong \mathbb{Z}[1/4] \oplus \mathbb{Z}$ and $K_1(C_r^*(\dot{G}_u)) \cong \mathbb{Z}[1/2]^2$, and the roles $K_0(C_r^*(\dot{G}_{AF}))$ and $K_0(C_r^*(\dot{G}_{AF}^{(1)}))$ are as

$$K_0(C_r^*(\dot{G}_u)) \cong K_0(C_r^*(\dot{G}_{AF})) \oplus \mathbb{Z}$$

and

$$K_1(C_r^*(\dot{G}_u)) \cong K_0(C_r^*(\dot{G}_{AF}^{(1)})).$$

Example 6.4 (Half-hex). Even though the half-hex substitution forces the border, the script collars everything, yielding 24 collared prototiles, 60 partial isometries, and 38 “loops” of partial isometries. Out of the partial isometries, there are only 42 full translations, meaning that there are at least $60 - 42 = 18$ partial isometries that do not come from subdividing full translations. Similarly, out of the loops, there are only 20 that persist. Note that *this does not mean that they do not belong to the kernel of σ^\top !*

In C^2 , there is one eigenvector of eigenvalue 4, one of eigenvalue 2, and four of 1. In C^1 , there are three each of eigenvalues 2 and 1. $\text{im } \delta^0 \cong \mathbb{Z}$ as one of the eigenvectors of eigenvalue 1 in C^1 . One easily checks that the direct limits of the induced substitution matrices split completely, giving us the six-term sequence

$$\begin{array}{ccccc} \mathbb{Z}[1/2]^3 \oplus \mathbb{Z}^2 & \xrightarrow{\text{ev}} & \mathbb{Z}[1/4] \oplus \mathbb{Z}[1/2] \oplus \mathbb{Z}^4 & \xrightarrow{\iota_*} & K_0(C_r^*(\dot{G}_u)) \\ \uparrow & & & & \downarrow \\ K_1(C_r^*(\dot{G}_u)) & \xleftarrow{\quad} & 0 & \xleftarrow{\quad} & \mathbb{Z} \end{array}$$

where the evaluation map maps an eigenvector of eigenvalue 2 and both of the eigenvectors of eigenvalue 1 isomorphically onto their images, with the rest mapped to 0, giving us $K_0(C_r^*(\dot{G}_u)) \cong \mathbb{Z}[1/4] \oplus \mathbb{Z}^3$ and $K_1(C_r^*(\dot{G}_u)) \cong \mathbb{Z}[1/2]^2$. The component of $K_0(C_r^*(\dot{G}_{AF}))$

in $K_0(C_r^*(\dot{G}_u))$ is as

$$\begin{aligned} K_0(C_r^*(\dot{G}_u)) &\cong (\mathbb{Z}[1/4] \oplus \mathbb{Z}[1/2] \oplus \mathbb{Z}^4) / (\mathbb{Z}[1/2] \oplus \mathbb{Z}^2) \oplus \mathbb{Z} \\ &\cong K_0(C_r^*(\dot{G}_{AF})) / (\mathbb{Z}[1/2] \oplus \mathbb{Z}^2) \oplus \mathbb{Z}. \end{aligned}$$

Since the substitution rule satisfies the boundary hyperplane condition, we can read off the role of $K_0(C_r^*(\dot{G}_{AF}^{(1)}))$ in $K_1(C_r^*(\dot{G}_u))$ easily as

$$\begin{aligned} K_1(C_r^*(\dot{G}_u)) &\cong (\mathbb{Z}[1/2]^2 \oplus \mathbb{Z}) / \mathbb{Z} \\ &\leq (\mathbb{Z}[1/2]^3 \oplus \mathbb{Z}^3) / \mathbb{Z} \\ &\cong K_0(C_r^*(\dot{G}_{AF}^{(1)})) / \mathbb{Z}. \end{aligned}$$

Example 6.5 (Chair). Its K -theory has already been computed in [JS16] using an MLD-equivalent substitution with squares, but here we compute directly with the original “chair” prototiles. Each of the induced substitution matrices have integral eigenvalues. One easily checks that the direct limits of the induced substitution matrices split completely. It turns out that $\text{im } \delta^0 = 0$, giving us $K_0(C_r^*(\dot{G}_{AF}); C_r^*(\dot{G}_u)) \cong K_0(C_r^*(\dot{G}_{AF}^{(1)}))$. The six-term sequence is

$$\begin{array}{ccccc} \mathbb{Z}[1/2]^2 \oplus \mathbb{Z}^4 & \xrightarrow{\text{ev}} & \mathbb{Z}[1/4] \oplus \mathbb{Z}[1/2]^2 \oplus \mathbb{Z}^4 & \xrightarrow{\iota_*} & \mathbb{Z}[1/4] \oplus \mathbb{Z}[1/2]^2 \oplus \mathbb{Z} \\ \uparrow & & & & \downarrow \\ \mathbb{Z}[1/2]^2 & \longleftarrow & 0 & \longleftarrow & \mathbb{Z} \end{array}$$

where the evaluation map sends $\mathbb{Z}^4 \leq K_0(C_r^*(\dot{G}_{AF}); C_r^*(\dot{G}_u))$ isomorphically onto its image and

the rest to 0. We have that

$$\begin{aligned} K_0(C_r^*(\dot{G}_u)) &\cong (\mathbb{Z}[1/4] \oplus \mathbb{Z}[1/2]^2 \oplus \mathbb{Z}^4)/\mathbb{Z}^4 \oplus \mathbb{Z} \\ &\cong K_0(C_r^*(\dot{G}_{AF}))/\mathbb{Z}^4 \oplus \mathbb{Z} \end{aligned}$$

and

$$\begin{aligned} K_1(C_r^*(\dot{G}_u)) &\cong \mathbb{Z}[1/2]^2 \\ &\leq \mathbb{Z}[1/2]^2 \oplus \mathbb{Z}^4 \\ &\cong C^1 \\ &\cong K_0(C_r^*(\dot{G}_{AF}^{(1)})) \end{aligned} \tag{Proposition 5.21}$$

where we have to apply a slightly nontrivial isomorphism in the last step since this substitution does not satisfy the boundary hyperplane condition. The proposition applies since the component in C^1 contributed by the 1-cells satisfying the boundary hyperplane condition splits completely, being $\mathbb{Z}[1/2]^2$.

Example 6.6 (Chair 3×3^2). The substitution rule is a variant of the chair tiling with expansion factor 3, and is shown in Fig. 6.1. Unlike the standard chair tiling, rotational symmetry has been eliminated. Each of the induced substitution matrices have integral eigenvalues. Due to the sizes of the induced matrices, we assume each of the direct limits split completely. The six-term

²Supplied by Rodrigo Treviño.

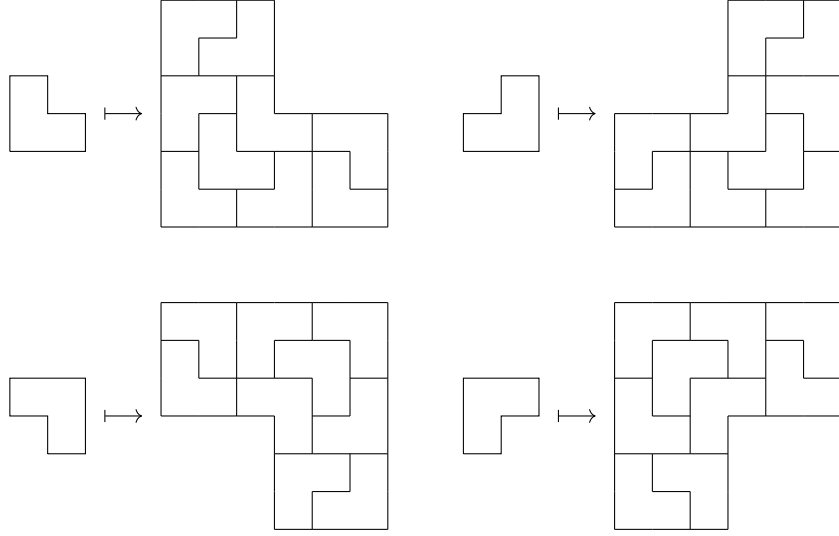


Figure 6.1: Substitution rule for a variant of the chair tiling with expansion factor 3.

sequence is

$$\begin{array}{ccccc}
 \mathbb{Z}[1/3]^4 \oplus \mathbb{Z}[1/2]^2 \oplus \mathbb{Z}^{76} & \xrightarrow{\text{ev}} & \mathbb{Z}[1/9] \oplus \mathbb{Z}[1/3]^6 & \xrightarrow{\iota_*} & \mathbb{Z}[1/9] \oplus \mathbb{Z}[1/3]^4 \\
 & & \oplus \mathbb{Z}[1/2]^4 \oplus \mathbb{Z}^{101} & & \oplus \mathbb{Z}[1/2]^2 \oplus \mathbb{Z}^{26} \\
 \uparrow & & & & \downarrow \\
 \mathbb{Z}[1/3]^2 & \longleftarrow & 0 & \longleftarrow & \mathbb{Z}
 \end{array}$$

where the evaluation map sends $\mathbb{Z}[1/3]^2 \oplus \mathbb{Z}[1/2]^2 \oplus \mathbb{Z}^{76} \leq K_0(C_r^*(\dot{G}_{AF}); C_r^*(\dot{G}_u))$ isomorphically onto its image and the rest to 0. We have that

$$\begin{aligned}
 K_0(C_r^*(\dot{G}_u)) &\cong (\mathbb{Z}[1/9] \oplus \mathbb{Z}[1/3]^6 \oplus \mathbb{Z}[1/2]^4 \oplus \mathbb{Z}^{101}) / (\mathbb{Z}[1/3]^2 \oplus \mathbb{Z}[1/2]^2 \oplus \mathbb{Z}^{76}) \oplus \mathbb{Z} \\
 &\cong K_0(C_r^*(\dot{G}_{AF})) / (\mathbb{Z}[1/3]^2 \oplus \mathbb{Z}[1/2]^2 \oplus \mathbb{Z}^{76}) \oplus \mathbb{Z}
 \end{aligned}$$

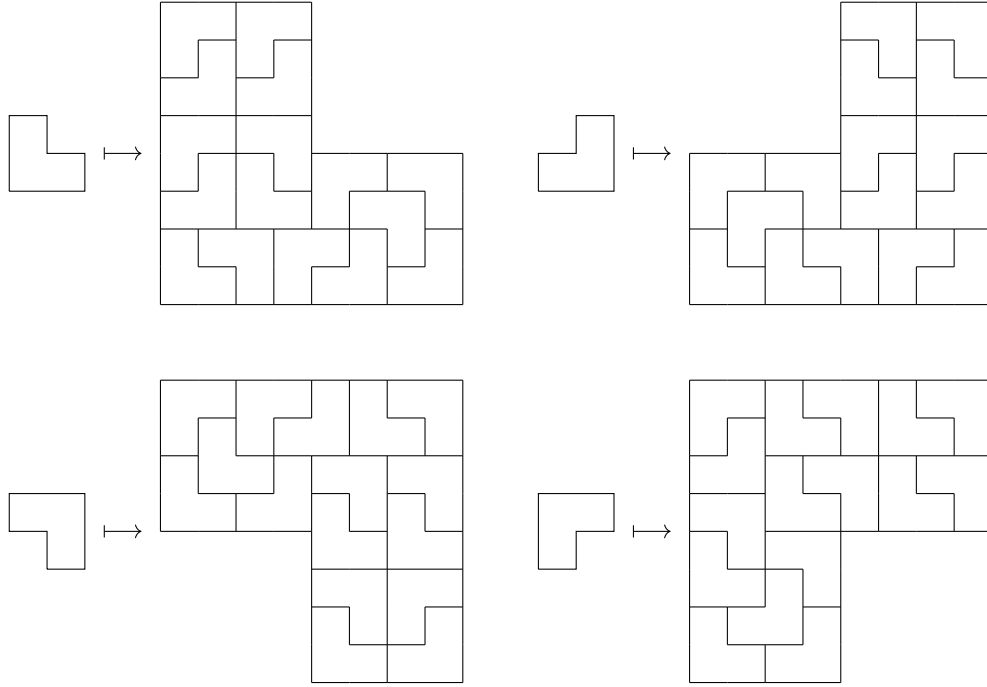


Figure 6.2: Substitution rule for a variant of the chair tiling with expansion factor 4.

and

$$\begin{aligned}
K_1(C_r^*(\dot{G}_u)) &\cong (\mathbb{Z}[1/3]^2 \oplus \mathbb{Z}^{34})/\mathbb{Z}^{34} \\
&\leq (\mathbb{Z}[1/3]^4 \oplus \mathbb{Z}[1/2]^2 \oplus \mathbb{Z}^{110})/\mathbb{Z}^{34} \\
&\cong C^1/\mathbb{Z}^{34} \\
&\cong K_0(C_r^*(\dot{G}_{AF}^{(1)}))/\mathbb{Z}^{34} \quad (\text{Proposition 5.21})
\end{aligned}$$

where we have to apply a slightly nontrivial isomorphism in the last step since this substitution does not satisfy the boundary hyperplane condition. The proposition applies since we assumed that C^1 splits completely.

Example 6.7 (Chair 4×4 ³). The substitution rule is a variant of the chair tiling with expansion

³Supplied by Rodrigo Treviño.

factor 4, and is shown in Fig. 6.2. Unlike the standard chair tiling, rotational symmetry has been eliminated. Each of the induced substitution matrices have integral eigenvalues. Due to the sizes of the induced matrices, we assume each of the direct limits split completely. On a finite level of the direct limit, $\text{ev} : K_0(C_r^*(\dot{G}_{AF,n}); C_r^*(\dot{G}_u)) \rightarrow K_0(C_r^*(\dot{G}_{AF,n}))$ contains a vector that is multiplied by 2, hence $\text{im } \iota_*$ has a torsion term. (Un)fortunately, this torsion term belongs to the kernel of σ^\top . It also turns out that $\text{im } \delta^0 = 0$, giving us $K_0(C_r^*(\dot{G}_{AF}); C_r^*(\dot{G}_u)) \cong K_0(C_r^*(\dot{G}_{AF}^{(1)}))$.

The six-term sequence is

$$\begin{array}{ccccc}
\mathbb{Z}[1/4]^2 \oplus \mathbb{Z}[1/2] \oplus \mathbb{Z}^{19} & \xrightarrow{\text{ev}} & \mathbb{Z}[1/16] \oplus \mathbb{Z}[1/2]^4 \oplus \mathbb{Z}^{30} & \xrightarrow{\iota_*} & \mathbb{Z}[1/16] \oplus \mathbb{Z}[1/2]^3 \oplus \mathbb{Z}^{12} \\
\uparrow & & & & \downarrow \\
\mathbb{Z}[1/4]^2 & \longleftarrow & 0 & \longleftarrow & \mathbb{Z}
\end{array}$$

where the evaluation map sends $\mathbb{Z}[1/2] \oplus \mathbb{Z}^{19} \leq K_0(C_r^*(\dot{G}_{AF}); C_r^*(\dot{G}_u))$ isomorphically onto its image and the rest to 0. We have that

$$\begin{aligned}
K_0(C_r^*(\dot{G}_u)) &\cong (\mathbb{Z}[1/16] \oplus \mathbb{Z}[1/2]^4 \oplus \mathbb{Z}^{30}) / (\mathbb{Z}[1/2] \oplus \mathbb{Z}^{19}) \oplus \mathbb{Z} \\
&\cong K_0(C_r^*(\dot{G}_{AF})) / (\mathbb{Z}[1/2] \oplus \mathbb{Z}^{19}) \oplus \mathbb{Z}
\end{aligned}$$

and

$$\begin{aligned}
K_1(C_r^*(\dot{G}_u)) &\cong \mathbb{Z}[1/4]^2 \\
&\leq \mathbb{Z}[1/4]^2 \oplus \mathbb{Z}[1/2]^2 \oplus \mathbb{Z}^{19} \\
&\cong C^1 \\
&\cong K_0(C_r^*(\dot{G}_{AF}^{(1)})) \quad \quad \quad (\text{Proposition 5.21})
\end{aligned}$$

where we have to apply a slightly nontrivial isomorphism in the last step since this substitution does not satisfy the boundary hyperplane condition. The proposition applies since we assumed that C^1 splits completely.

Example 6.8 (Robinson triangle). This is MLD-equivalent to the Penrose tiling. This substitution rule is known to be border-forcing, but we still collar everything, resulting in a large number of collared prototiles. The induced substitution matrices, after removing the eventual kernels, are all of determinant 1. The six-term sequence is

$$\begin{array}{ccccc} \mathbb{Z}^{37} & \xrightarrow{\text{ev}} & \mathbb{Z}^{40} & \xrightarrow{\iota_*} & \mathbb{Z}^9 \\ \uparrow & & & & \downarrow \\ \mathbb{Z}^5 & \longleftarrow & 0 & \longleftarrow & \mathbb{Z} \end{array}$$

where the evaluation map sends $\mathbb{Z}^{32} \leq K_0(C_r^*(\dot{G}_{AF}); C_r^*(\dot{G}_u))$ isomorphically onto its image and the rest to 0. We have that

$$\begin{aligned} K_0(C_r^*(\dot{G}_u)) &\cong \mathbb{Z}^{40}/\mathbb{Z}^{32} \oplus \mathbb{Z} \\ &\cong K_0(C_r^*(\dot{G}_{AF}))/\mathbb{Z}^{32} \oplus \mathbb{Z} \end{aligned}$$

and, since the substitution rule satisfies the boundary hyperplane condition,

$$\begin{aligned} K_1(C_r^*(\dot{G}_u)) &\cong \mathbb{Z}^8/\mathbb{Z}^3 \\ &\leq \mathbb{Z}^{40}/\mathbb{Z}^3 \\ &\cong K_0(C_r^*(\dot{G}_{AF}^{(1)}))/\mathbb{Z}^3. \end{aligned}$$

Example 6.9 (Tübingen triangle). On a finite level of the direct limit, $\text{ev} :$

$K_0(C_r^*(\dot{G}_{AF,n}); C_r^*(\dot{G}_u)) \rightarrow K_0(C_r^*(\dot{G}_{AF,n}))$ contains 2 vectors that are each multiplied by 5, hence $\text{im } \iota_*$ has 2 torsion terms. These torsion terms both have the same eigenvalue of 4 under σ^\top , which is invertible. Aside from this term, the induced substitution matrices, after removing the eventual kernels, are all of determinant 1. The six-term sequence is

$$\begin{array}{ccccc} \mathbb{Z}^{101} & \xrightarrow{\text{ev}} & \mathbb{Z}^{120} & \xrightarrow{\iota_*} & \mathbb{Z}^{25} \oplus \mathbb{Z}_5^2 \\ \uparrow & & & & \downarrow \\ \mathbb{Z}^5 & \longleftarrow & 0 & \longleftarrow & \mathbb{Z} \end{array}$$

where the evaluation map sends $\mathbb{Z}^{94} \leq K_0(C_r^*(\dot{G}_{AF}); C_r^*(\dot{G}_u))$ isomorphically onto its image, multiplies $\mathbb{Z}^2 \leq K_0(C_r^*(\dot{G}_{AF}); C_r^*(\dot{G}_u))$ by 5, and sends the rest to 0. We have that

$$\begin{aligned} K_0(C_r^*(\dot{G}_u)) &\cong \mathbb{Z}^{120} / (\mathbb{Z}^{94} \oplus (5\mathbb{Z})^2) \oplus \mathbb{Z} \\ &\cong K_0(C_r^*(\dot{G}_{AF})) / (\mathbb{Z}^{94} \oplus (5\mathbb{Z}^2)) \oplus \mathbb{Z} \end{aligned}$$

and, since the substitution rule satisfies the boundary hyperplane condition,

$$\begin{aligned} K_1(C_r^*(\dot{G}_u)) &\cong \mathbb{Z}^{14} / \mathbb{Z}^9 \\ &\leq \mathbb{Z}^{110} / \mathbb{Z}^9 \\ &\cong K_0(C_r^*(\dot{G}_{AF}^{(1)})) / \mathbb{Z}^9. \end{aligned}$$

Example 6.10 (Danzer 7-fold). We are using the original version of the Danzer 7-fold substitution for simplicity, as it is a substitution of triangles, rather than of (degenerate) quadrilaterals. The expansion factor is $\lambda = 1 + 2 \cos \pi/7$, which is *not a PV number*, thus is weak mixing. Due to the sizes of the induced substitution matrices on $K_0(C_r^*(\dot{G}_{AF,n}))$ and $K_0(C_r^*(\dot{G}_{AF,n}); C_r^*(\dot{G}_u))$,

their eigenvalues had to be computed numerically. We assume that any eigenvalue with order $\leq 10^{-7}$ is nonzero due to floating point errors.

The determinant of the induced substitution matrix on $K_0(C_r^*(\dot{G}_{AF}))$, upon removing the eventual kernel, is 32761. Fewer than 18 of the nonintegral eigenvalues multiply to it, with the remaining multiplying to 1. Restricting the direct limit to the associated eigenvectors of the 18 eigenvalues that multiply to 32761 and computing it over \mathbb{Q} gives that $K_0(C_r^*(\dot{G}_{AF})) \cong \mathbb{Z}^{\geq 2213} \oplus \mathbb{Q}^{\leq 18}$. These eigenvectors of the 18 eigenvalues are mapped isomorphically onto its image in $\text{im } \iota_*$. The induced substitution matrix on $K_0(C_r^*(\dot{G}_{AF}); C_r^*(\dot{G}_u))$, after removing the eventual kernel, has determinant 1.

The six-term sequence is

$$\begin{array}{ccccc}
 \mathbb{Z}^{2017} & \xrightarrow{\text{ev}} & \mathbb{Z}^{\geq 2222} \oplus \mathbb{Q}^{\leq 18} & \xrightarrow{\iota_*} & \mathbb{Z}^{\geq 212} \oplus \mathbb{Q}^{\leq 18} \\
 \uparrow & & & & \downarrow \\
 \mathbb{Z}^6 & \longleftarrow & 0 & \longleftarrow & \mathbb{Z}
 \end{array}$$

where the evaluation map sends $\mathbb{Z}^{2011} \leq K_0(C_r^*(\dot{G}_{AF}); C_r^*(\dot{G}_u))$ isomorphically onto its image and the rest to 0. We have that

$$\begin{aligned}
 K_0(C_r^*(\dot{G}_u)) &\cong (\mathbb{Z}^{\geq 2222} \oplus \mathbb{Q}^{\leq 18}) / \mathbb{Z}^{2011} \oplus \mathbb{Z} \\
 &\cong K_0(C_r^*(\dot{G}_{AF})) / \mathbb{Z}^{2011} \oplus \mathbb{Z}
 \end{aligned}$$

and, since the substitution rule satisfies the boundary hyperplane condition,

$$\begin{aligned}
K_1(C_r^*(\dot{G}_u)) &\cong \mathbb{Z}^{215}/\mathbb{Z}^{209} \\
&\leq \mathbb{Z}^{2226}/\mathbb{Z}^{209} \\
&\cong K_0(C_r^*(\dot{G}_{AF}^{(1)}))/\mathbb{Z}^{209}.
\end{aligned}$$

Among the 229 nontrivial eigenvalues contributing to $\check{H}^2(\Omega_T) \cong K_0(C_r^*(\dot{G}_{AF})) / \text{im ev} \leq K_0(C_r^*(\dot{G}_u))$, all but 21 belong to $\mathbb{Z}[\lambda]$. The contribution to the determinant of 32761 is purely from them. There are three that multiply to the integral value of 1 which we include into the integral part of $K_0(C_r^*(\dot{G}_u))$,

$$\begin{aligned}
&0.2864 \dots, 2.9122 \dots, \\
&\text{roots of } x^3 - 2x^2 - 3x + 1, \\
&1.1986 \dots, \\
&\text{root of } x^3 + 2x^2 - 3x - 1,
\end{aligned}$$

with the rest, each with multiplicity 2,

$$\begin{aligned}
&0.4826 \dots, 1.1310 \dots, 2.0456 \dots, 2.4534 \dots, \\
&\text{roots of } x^9 + 6x^8 - 4x^7 - 73x^6 - 49x^5 + 277x^4 + 271x^3 - 364x^2 - 289x + 181, \\
&1.7331 \dots, 1.8534 \dots, 1.8872 \dots, 2.9696 \dots, 3.6693 \dots, \\
&\text{roots of } x^9 - 6x^8 - 4x^7 + 73x^6 - 49x^5 - 277x^4 + 271x^3 + 364x^2 - 289x - 181.
\end{aligned}$$

There are four eigenvalues, $\lambda^2 - \lambda - 1$ with multiplicity 4, $3.6694 \dots$ with multiplicity 2, $2.9696 \dots$ with multiplicity 2, and $2.9122 \dots$, that are strictly between λ and λ^2 .

Among the six nontrivial eigenvalues contributing to $K_1(C_r^*(\dot{G}_u)) \cong \check{H}^1(\Omega_T)$, they multiply to 1 with four expanding and two contracting.

Example 6.11 (Ammann A2). Per [AP98, Section 10.3], we use the square of the substitution rule and the 8 prototiles it is primitive on. This substitution rule is known to be border-forcing, but we still collar everything, resulting in a large number of collared prototiles. The induced substitution matrices, after removing the eventual kernels, are all of determinant 1. The six-term sequence is

$$\begin{array}{ccccc} \mathbb{Z}^6 & \xrightarrow{\text{ev}} & \mathbb{Z}^8 & \xrightarrow{\iota_*} & \mathbb{Z}^7 \\ \uparrow & & & & \downarrow \\ \mathbb{Z}^4 & \longleftarrow & 0 & \longleftarrow & \mathbb{Z} \end{array}$$

where the evaluation map sends $\mathbb{Z}^2 \leq K_0(C_r^*(\dot{G}_{AF}); C_r^*(\dot{G}_u))$ isomorphically onto its image and the rest to 0. We have that

$$\begin{aligned} K_0(C_r^*(\dot{G}_u)) &\cong \mathbb{Z}^8 / \mathbb{Z}^2 \oplus \mathbb{Z} \\ &\cong K_0(C_r^*(\dot{G}_{AF})) / \mathbb{Z}^2 \oplus \mathbb{Z} \end{aligned}$$

and

$$\begin{aligned}
K_1(C_r^*(\dot{G}_u)) &\cong \mathbb{Z}^6/\mathbb{Z}^2 \\
&\leq \mathbb{Z}^8/\mathbb{Z}^2 \\
&\cong C^1/\mathbb{Z}^2 \\
&\cong K_0(C_r^*(\dot{G}_{AF}^{(1)}))/\mathbb{Z}^2
\end{aligned}$$

where we have to apply a slightly nontrivial isomorphism in the last step since this substitution does not satisfy the boundary hyperplane condition. Proposition 5.21 *does not apply*, since the eigenvalues are not integral. However, the induced substitution matrix on C^1 has determinant 1 (up to the eventual kernel), thus as does the induced substitution matrix restricted to the 1-cells that satisfy the boundary hyperplane condition. A proof similar to the one in the proposition gives an isomorphism $K_0(C_r^*(\dot{G}_{AF}^{(1)})) \cong C^1$.

Example 6.12 (Ammann A5 decorated). We are using the rhombus-triangle substitution. This is MLD-equivalent to the Ammann–Beenker tiling. The induced substitution matrices, after removing the eventual kernels, are all of determinant 1. The six-term sequence is

$$\begin{array}{ccccc}
\mathbb{Z}^{65} & \xrightarrow{\text{ev}} & \mathbb{Z}^{80} & \xrightarrow{\iota_*} & \mathbb{Z}^{24} \\
\uparrow & & & & \downarrow \\
\mathbb{Z}^8 & \longleftarrow & 0 & \longleftarrow & \mathbb{Z}
\end{array}$$

where the evaluation map sends $\mathbb{Z}^{57} \leq K_0(C_r^*(\dot{G}_{AF}); C_r^*(\dot{G}_u))$ isomorphically onto its image

and the rest to 0. We have that

$$\begin{aligned} K_0(C_r^*(\dot{G}_u)) &\cong \mathbb{Z}^{80}/\mathbb{Z}^{57} \oplus \mathbb{Z} \\ &\cong K_0(C_r^*(\dot{G}_{AF}))/\mathbb{Z}^{57} \oplus \mathbb{Z} \end{aligned}$$

and, since the substitution rule satisfies the boundary hyperplane condition,

$$\begin{aligned} K_1(C_r^*(\dot{G}_u)) &\cong \mathbb{Z}^{15}/\mathbb{Z}^7 \\ &\leq \mathbb{Z}^{72}/\mathbb{Z}^7 \\ &\cong K_0(C_r^*(\dot{G}_{AF}^{(1)}))/\mathbb{Z}^7. \end{aligned}$$

Example 6.13 (Ammann A5 undecorated). We are using the rhombus-triangle substitution. This is MLD-equivalent to the Ammann–Beenker tiling *with the tile labels removed*. The induced substitution matrices, after removing the eventual kernels, are all of determinant 1. It turns out that $\text{im } \delta^0 = 0$, giving us $K_0(C_r^*(\dot{G}_{AF}); C_r^*(\dot{G}_u)) \cong K_0(C_r^*(\dot{G}_{AF}^{(1)}))$. The six-term sequence is

$$\begin{array}{ccccc} \mathbb{Z}^{16} & \xrightarrow{\text{ev}} & \mathbb{Z}^{20} & \xrightarrow{\iota_*} & \mathbb{Z}^{10} \\ \uparrow & & & & \downarrow \\ \mathbb{Z}^5 & \longleftarrow & 0 & \longleftarrow & \mathbb{Z} \end{array}$$

where the evaluation map sends $\mathbb{Z}^{11} \leq K_0(C_r^*(\dot{G}_{AF}); C_r^*(\dot{G}_u))$ isomorphically onto its image

and the rest to 0. We have that

$$\begin{aligned} K_0(C_r^*(\dot{G}_u)) &\cong \mathbb{Z}^{20}/\mathbb{Z}^{11} \oplus \mathbb{Z} \\ &\cong K_0(C_r^*(\dot{G}_{AF}))/\mathbb{Z}^{11} \oplus \mathbb{Z} \end{aligned}$$

and, since the substitution rule satisfies the boundary hyperplane condition,

$$\begin{aligned} K_1(C_r^*(\dot{G}_u)) &\cong \mathbb{Z}^8 \\ &\leq \mathbb{Z}^{16} \\ &\cong K_0(C_r^*(\dot{G}_{AF}^{(1)})). \end{aligned}$$

Example 6.14 (GKM 9.1.1.1). This is the substitution rule in [GKM15, Figure 9], where the enumeration is, from left to right, the first triangle, followed by the first triangle connected by an arc, followed by the first triangle connected by an arc. The expansion factor is $\lambda = 1 + 2 \cos \pi/7$, which is *not a PV number*, thus is weak mixing. Due to the sizes of the induced substitution matrices on $K_0(C_r^*(\dot{G}_{AF,n}))$ and $K_0(C_r^*(\dot{G}_{AF,n}); C_r^*(\dot{G}_u))$, their eigenvalues had to be computed numerically.

The induced substitution matrices, after removing the eventual kernels, are all of determinant 1. The six-term sequence is

$$\begin{array}{ccccc} \mathbb{Z}^{602} & \xrightarrow{\text{ev}} & \mathbb{Z}^{672} & \xrightarrow{\iota_*} & \mathbb{Z}^{84} \\ \uparrow & & & & \downarrow \\ \mathbb{Z}^{12} & \longleftarrow & 0 & \longleftarrow & \mathbb{Z}^2 \end{array}$$

where the evaluation map sends $\mathbb{Z}^{590} \leq K_0(C_r^*(\dot{G}_{AF}); C_r^*(\dot{G}_u))$ isomorphically onto its image

and the rest to 0. Note that the substitution *appears to have two components, and is thus not primitive!* We have that

$$\begin{aligned} K_0(C_r^*(\dot{G}_u)) &\cong \mathbb{Z}^{672}/\mathbb{Z}^{590} \oplus \mathbb{Z}^2 \\ &\cong K_0(C_r^*(\dot{G}_{AF}))/\mathbb{Z}^{590} \oplus \mathbb{Z}^2 \end{aligned}$$

and, since the substitution rule satisfies the boundary hyperplane condition,

$$\begin{aligned} K_1(C_r^*(\dot{G}_u)) &\cong \mathbb{Z}^{96}/\mathbb{Z}^{84} \\ &\leq \mathbb{Z}^{686}/\mathbb{Z}^{84} \\ &\cong K_0(C_r^*(\dot{G}_{AF}^{(1)}))/\mathbb{Z}^{84}. \end{aligned}$$

Among the 84 nontrivial eigenvalues contributing to $\check{H}^2(\Omega_T) \cong K_0(C_r^*(\dot{G}_{AF}))/\text{im ev} \leq K_0(C_r^*(\dot{G}_u))$, they all belong to $\mathbb{Z}[\lambda]$, and there is an eigenvalue $\lambda^2 - \lambda - 1$ with multiplicity 8 that is strictly between λ and λ^2 .

Among the twelve nontrivial eigenvalues contributing to $K_1(C_r^*(\dot{G}_u)) \cong \check{H}^1(\Omega_T)$, they multiply to 1 with eight expanding and four contracting.

Example 6.15 (GKM 9.2.1.1). This is the substitution rule in [GKM15, Figure 9], where the enumeration is, from left to right, the second triangle, followed by the first triangle connected by an arc, followed by the first triangle connected by an arc. The expansion factor is $\lambda = 1 + 2 \cos \pi/7$, which is *not a PV number*, thus is weak mixing. Due to the sizes of the induced substitution matrices on $K_0(C_r^*(\dot{G}_{AF,n}))$ and $K_0(C_r^*(\dot{G}_{AF,n}); C_r^*(\dot{G}_u))$, their eigenvalues had to be computed numerically.

The induced substitution matrices, after removing the eventual kernels, are all of determinant 1. The six-term sequence is

$$\begin{array}{ccccc}
 \mathbb{Z}^{1022} & \xrightarrow{\text{ev}} & \mathbb{Z}^{1148} & \xrightarrow{\iota_*} & \mathbb{Z}^{140} \\
 \uparrow & & & & \downarrow \\
 \mathbb{Z}^{12} & \longleftarrow & 0 & \longleftarrow & \mathbb{Z}^2
 \end{array}$$

where the evaluation map sends $\mathbb{Z}^{1010} \leq K_0(C_r^*(\dot{G}_{AF}); C_r^*(\dot{G}_u))$ isomorphically onto its image and the rest to 0. Note that the substitution *appears to have two components, and is thus not primitive!* We have that

$$\begin{aligned}
 K_0(C_r^*(\dot{G}_u)) &\cong \mathbb{Z}^{1148} / \mathbb{Z}^{1010} \oplus \mathbb{Z}^2 \\
 &\cong K_0(C_r^*(\dot{G}_{AF})) / \mathbb{Z}^{1010} \oplus \mathbb{Z}^2
 \end{aligned}$$

and, since the substitution rule satisfies the boundary hyperplane condition,

$$\begin{aligned}
 K_1(C_r^*(\dot{G}_u)) &\cong \mathbb{Z}^{124} / \mathbb{Z}^{112} \\
 &\leq \mathbb{Z}^{1134} / \mathbb{Z}^{112} \\
 &\cong K_0(C_r^*(\dot{G}_{AF}^{(1)})) / \mathbb{Z}^{112}.
 \end{aligned}$$

Among the 84 nontrivial eigenvalues contributing to $\check{H}^2(\Omega_T) \cong K_0(C_r^*(\dot{G}_{AF})) / \text{im ev} \leq K_0(C_r^*(\dot{G}_u))$, they all belong to $\mathbb{Z}[\lambda]$, and there is an eigenvalue $\lambda^2 - \lambda - 1$ with multiplicity 8 that is strictly between λ and λ^2 .

Among the twelve nontrivial eigenvalues contributing to $K_1(C_r^*(\dot{G}_u)) \cong \check{H}^1(\Omega_T)$, they multiply to 1 with eight expanding and four contracting.

Example 6.16 (GKM 10.1.1.1). This is the substitution rule in [GKM15, Figure 10], where the enumeration is, from left to right, the first triangle, followed by the first triangle connected by an arc, followed by the first triangle connected by an arc. The expansion factor is $\lambda = 1 + 2 \cos \pi/7$, which is *not a PV number*, thus is weak mixing. Due to the sizes of the induced substitution matrices on $K_0(C_r^*(\dot{G}_{AF,n}))$ and $K_0(C_r^*(\dot{G}_{AF,n}); C_r^*(\dot{G}_u))$, their eigenvalues had to be computed numerically.

The determinant of the induced substitution matrix on $K_0(C_r^*(\dot{G}_{AF}))$, upon removing the eventual kernel, is 5547. Fewer than 27 of the nonintegral eigenvalues multiply to it, with the remaining multiplying to 1. Restricting the direct limit to the associated eigenvectors of the 27 eigenvalues that multiply to 5547 and computing it over \mathbb{Q} gives that $K_0(C_r^*(\dot{G}_{AF})) \cong \mathbb{Z}^{\geq 1457} \oplus \mathbb{Q}^{\leq 27}$. These eigenvectors of the 27 eigenvalues are mapped isomorphically onto its image in $\text{im } \iota_*$. The induced substitution matrix on $K_0(C_r^*(\dot{G}_{AF}); C_r^*(\dot{G}_u))$, after removing the eventual kernel, has determinant 1.

The six-term sequence is

$$\begin{array}{ccccc}
 \mathbb{Z}^{1331} & \xrightarrow{\text{ev}} & \mathbb{Z}^{\geq 1457} \oplus \mathbb{Q}^{\leq 27} & \xrightarrow{\iota_*} & \mathbb{Z}^{\geq 133} \oplus \mathbb{Q}^{\leq 27} \\
 \uparrow & & & & \downarrow \\
 \mathbb{Z}^6 & \longleftarrow & 0 & \longleftarrow & \mathbb{Z}
 \end{array}$$

where the evaluation map sends $\mathbb{Z}^{1325} \leq K_0(C_r^*(\dot{G}_{AF}); C_r^*(\dot{G}_u))$ isomorphically onto its image and the rest to 0. We have that

$$\begin{aligned}
 K_0(C_r^*(\dot{G}_u)) &\cong (\mathbb{Z}^{\geq 1457} \oplus \mathbb{Q}^{\leq 27}) / \mathbb{Z}^{1325} \oplus \mathbb{Z} \\
 &\cong K_0(C_r^*(\dot{G}_{AF})) / \mathbb{Z}^{1325} \oplus \mathbb{Z}
 \end{aligned}$$

and, since the substitution rule satisfies the boundary hyperplane condition,

$$\begin{aligned}
K_1(C_r^*(\dot{G}_u)) &\cong \mathbb{Z}^{145}/\mathbb{Z}^{139} \\
&\leq \mathbb{Z}^{1470}/\mathbb{Z}^{139} \\
&\cong K_0(C_r^*(\dot{G}_{AF}^{(1)}))/\mathbb{Z}^{139}.
\end{aligned}$$

Among the 159 nontrivial eigenvalues contributing to $\check{H}^2(\Omega_T) \cong K_0(C_r^*(\dot{G}_{AF}))/\text{im ev} \leq K_0(C_r^*(\dot{G}_u))$, all but 27 belong to $\mathbb{Z}[\lambda]$. The contribution to the determinant of 5547 is purely from them. There are three that multiply to the integral value of 3,

$$0.7608 \dots, 2.6996 \dots,$$

$$\text{roots of } x^3 - 2x^2 - 3x + 3,$$

$$1.4605 \dots,$$

$$\text{root of } x^3 + 2x^2 - 3x - 3,$$

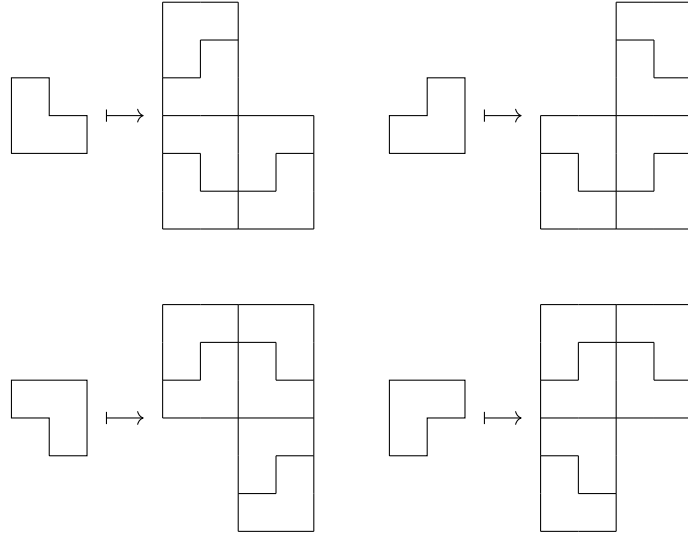


Figure 6.3: Substitution rule for a variant of the chair tiling with affine expansion $\text{diag}(2, 3)$.

with the rest, each with multiplicity 2,

$$0.3856 \dots, 0.5451 \dots, 1.1032 \dots, 1.5015 \dots, 2.6495 \dots, 2.7518 \dots, 2.7699 \dots,$$

$$\begin{aligned} &\text{roots of } x^{12} - 3x^{11} - 19x^{10} + 64x^9 + 90x^8 - 387x^7 - 66x^6 + 797x^5 - 161x^4 \\ &\quad - 573x^3 + 166x^2 + 126x - 43, \end{aligned}$$

$$0.6932 \dots, 0.7677 \dots, 1.4062 \dots, 2.3209 \dots, 3.5188 \dots,$$

$$\begin{aligned} &\text{roots of } x^{12} + 3x^{11} - 19x^{10} - 64x^9 + 90x^8 + 387x^7 - 66x^6 - 797x^5 - 161x^4 \\ &\quad + 573x^3 + 166x^2 - 126x - 43. \end{aligned}$$

There are two eigenvalues, $\lambda^2 - \lambda - 1$ with multiplicity 4 and $3.5188 \dots$ with multiplicity 2, that are strictly between λ and λ^2 .

Among the six nontrivial eigenvalues contributing to $K_1(C_r^*(\dot{G}_u)) \cong \check{H}^1(\Omega_T)$, they multiply to 1 with four expanding and two contracting.

Example 6.17 (Chair 2×3 (affine)⁴). The substitution rule is a variant of the chair tiling with *affine* expansion $\text{diag}(2, 3)$, and is shown in Fig. 6.3. Note that by appropriately grouping pairs of prototiles into rectangles, one can reencode the substitution rule as an affine substitution on rectangles. Each of the induced substitution matrices have integral eigenvalues. One easily checks that the direct limits of the induced substitution matrices split completely. The six-term sequence is

$$\begin{array}{ccccccc}
 \mathbb{Z}[1/3] \oplus \mathbb{Z}[1/2] \oplus \mathbb{Z}^3 & \xrightarrow{\text{ev}} & \mathbb{Z}[1/6] \oplus \mathbb{Z}[1/2] \oplus \mathbb{Z}^4 & \xrightarrow{\iota_*} & \mathbb{Z}[1/6] \oplus \mathbb{Z}[1/2] \oplus \mathbb{Z}^2 & & \\
 \uparrow & & & & \downarrow & & \\
 \mathbb{Z}[1/3] \oplus \mathbb{Z}[1/2] & \longleftarrow & 0 & \longleftarrow & \mathbb{Z} & &
 \end{array}$$

where the evaluation map sends $\mathbb{Z}^3 \leq K_0(C_r^*(\dot{G}_{AF}); C_r^*(\dot{G}_u))$ isomorphically onto its image and the rest to 0. We have that

$$\begin{aligned}
 K_0(C_r^*(\dot{G}_u)) &\cong (\mathbb{Z}[1/6] \oplus \mathbb{Z}[1/2] \oplus \mathbb{Z}^4) / \mathbb{Z}^3 \oplus \mathbb{Z} \\
 &\cong K_0(C_r^*(\dot{G}_{AF})) / \mathbb{Z}^3 \oplus \mathbb{Z}
 \end{aligned}$$

and

$$\begin{aligned}
 K_1(C_r^*(\dot{G}_u)) &\cong \mathbb{Z}[1/3] \oplus \mathbb{Z}[1/2] \\
 &\leq \mathbb{Z}[1/3] \oplus \mathbb{Z}[1/2] \oplus \mathbb{Z}^3 \\
 &\cong C^1 \\
 &\cong K_0(C_r^*(\dot{G}_{AF}^{(1)})) \quad \quad \quad (\text{Proposition 5.21})
 \end{aligned}$$

where we have to apply a slightly nontrivial isomorphism in the last step since this substitution

⁴Supplied by Rodrigo Treviño.

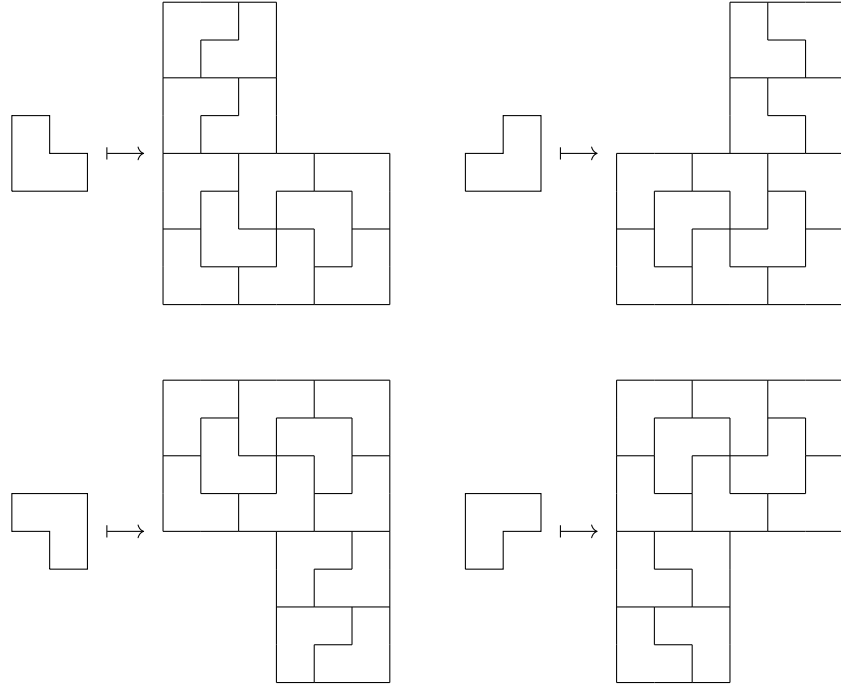


Figure 6.4: Substitution rule for a variant of the chair tiling with affine expansion $\text{diag}(3, 4)$.

does not satisfy the boundary hyperplane condition. The proposition applies since the component in C^1 contributed by the 1-cells satisfying the boundary hyperplane condition splits completely, being $\mathbb{Z}[1/3] \oplus \mathbb{Z}[1/2]$.

Example 6.18 (Chair 3×4 (affine)). The substitution rule is a variant of the chair tiling with *affine* expansion $\text{diag}(3, 4)$, and is shown in Fig. 6.4. Each of the induced substitution matrices have integral eigenvalues. Due to the sizes of the induced matrices, we assume each of the direct limits split completely. The six-term sequence is

$$\begin{array}{ccccc}
 \mathbb{Z}[1/4] \oplus \mathbb{Z}[1/3]^2 & \xrightarrow{\text{ev}} & \mathbb{Z}[1/12] \oplus \mathbb{Z}[1/4] & \xrightarrow{\iota_*} & \mathbb{Z}[1/12] \oplus \mathbb{Z}[1/4] \\
 \oplus \mathbb{Z}[1/2]^3 \oplus \mathbb{Z}^{16} & & \oplus \mathbb{Z}[1/3]^2 \oplus \mathbb{Z}[1/2]^4 \oplus \mathbb{Z}^{22} & & \oplus \mathbb{Z}[1/3] \oplus \mathbb{Z}[1/2] \oplus \mathbb{Z}^7 \\
 \uparrow & & & & \downarrow \\
 \mathbb{Z}[1/4] \oplus \mathbb{Z}[1/3] & \longleftarrow & 0 & \longleftarrow & \mathbb{Z}
 \end{array}$$

where the evaluation map sends $\mathbb{Z}[1/3] \oplus \mathbb{Z}[1/2]^3 \oplus \mathbb{Z}^{16} \leq K_0(C_r^*(\dot{G}_{AF}); C_r^*(\dot{G}_u))$ isomorphically onto its image and the rest to 0. We have that

$$\begin{aligned} K_0(C_r^*(\dot{G}_u)) &\cong \left(\begin{array}{c} \mathbb{Z}[1/12] \oplus \mathbb{Z}[1/4] \\ \oplus \mathbb{Z}[1/3]^2 \oplus \mathbb{Z}[1/2]^4 \oplus \mathbb{Z}^{22} \end{array} \right) / (\mathbb{Z}[1/3] \oplus \mathbb{Z}[1/2]^3 \oplus \mathbb{Z}^{16}) \oplus \mathbb{Z} \\ &\cong K_0(C_r^*(\dot{G}_{AF})) / (\mathbb{Z}[1/3] \oplus \mathbb{Z}[1/2]^3 \oplus \mathbb{Z}^{16}) \oplus \mathbb{Z} \end{aligned}$$

and

$$\begin{aligned} K_1(C_r^*(\dot{G}_u)) &\cong (\mathbb{Z}[1/4] \oplus \mathbb{Z}[1/3] \oplus \mathbb{Z}^4) / \mathbb{Z}^4 \\ &\leq (\mathbb{Z}[1/4] \oplus \mathbb{Z}[1/3]^2 \oplus \mathbb{Z}[1/2]^3 \oplus \mathbb{Z}^{20}) / \mathbb{Z}^4 \\ &\cong C^1 / \mathbb{Z}^4 \\ &\cong K_0(C_r^*(\dot{G}_{AF}^{(1)})) / \mathbb{Z}^4 \quad (\text{Proposition 5.21}) \end{aligned}$$

where we have to apply a slightly nontrivial isomorphism in the last step since this substitution does not satisfy the boundary hyperplane condition. The proposition applies since we assumed that C^1 splits completely.

Example 6.19 (Chair 3×5 variant 1 (affine)). The substitution rule is a variant of the chair tiling with *affine* expansion $\text{diag}(3, 5)$, and is shown in Fig. 6.5. Each of the induced substitution matrices have integral eigenvalues. Due to the sizes of the induced matrices, we assume each of

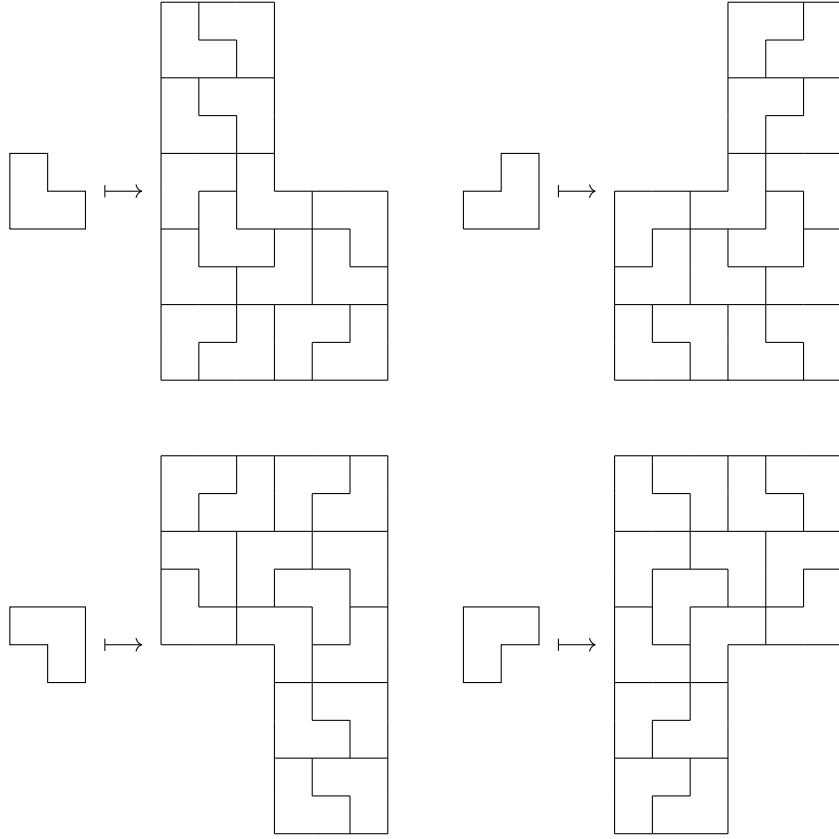


Figure 6.5: Substitution rule for a variant of the chair tiling with affine expansion $\text{diag}(3, 5)$.

the direct limits split completely. The six-term sequence is

$$\begin{array}{ccccc}
 \mathbb{Z}[1/5]^2 \oplus \mathbb{Z}[1/3]^2 \oplus \mathbb{Z}^{32} & \xrightarrow{\text{ev}} & \mathbb{Z}[1/15] \oplus \mathbb{Z}[1/5]^2 & \xrightarrow{\iota_*} & \mathbb{Z}[1/15] \oplus \mathbb{Z}[1/5] \\
 & & \oplus \mathbb{Z}[1/3]^5 \oplus \mathbb{Z}^{48} & & \oplus \mathbb{Z}[1/3]^4 \oplus \mathbb{Z}^{17} \\
 \uparrow & & & & \downarrow \\
 \mathbb{Z}[1/5] \oplus \mathbb{Z}[1/3] & \longleftarrow & 0 & \longleftarrow & \mathbb{Z}
 \end{array}$$

where the evaluation map sends $\mathbb{Z}[1/5] \oplus \mathbb{Z}[1/3] \oplus \mathbb{Z}^{32} \leq K_0(C_r^*(\dot{G}_{AF}); C_r^*(\dot{G}_u))$ isomorphically

onto its image and the rest to 0. We have that

$$\begin{aligned} K_0(C_r^*(\dot{G}_u)) &\cong (\mathbb{Z}[1/15] \oplus \mathbb{Z}[1/5]^2 \oplus \mathbb{Z}[1/3]^5 \oplus \mathbb{Z}^{48}) / (\mathbb{Z}[1/5] \oplus \mathbb{Z}[1/3] \oplus \mathbb{Z}^{32}) \oplus \mathbb{Z} \\ &\cong K_0(C_r^*(\dot{G}_{AF})) / (\mathbb{Z}[1/5] \oplus \mathbb{Z}[1/3] \oplus \mathbb{Z}^{32}) \oplus \mathbb{Z} \end{aligned}$$

and

$$\begin{aligned} K_1(C_r^*(\dot{G}_u)) &\cong (\mathbb{Z}[1/5] \oplus \mathbb{Z}[1/3] \oplus \mathbb{Z}^4) / \mathbb{Z}^4 \\ &\leq (\mathbb{Z}[1/5]^2 \oplus \mathbb{Z}[1/3]^2 \oplus \mathbb{Z}^{36}) / \mathbb{Z}^4 \\ &\cong C^1 / \mathbb{Z}^4 \\ &\cong K_0(C_r^*(\dot{G}_{AF}^{(1)})) / \mathbb{Z}^4 \quad \text{(Proposition 5.21)} \end{aligned}$$

where we have to apply a slightly nontrivial isomorphism in the last step since this substitution does not satisfy the boundary hyperplane condition. The proposition applies since we assumed that C^1 splits completely.

Example 6.20 (Chair 3×5 variant 2 (affine)). The substitution rule is a variant of the chair tiling with *affine* expansion $\text{diag}(3, 5)$, and is shown in Fig. 6.6. Each of the induced substitution matrices have integral eigenvalues. Due to the sizes of the induced matrices, we assume each of

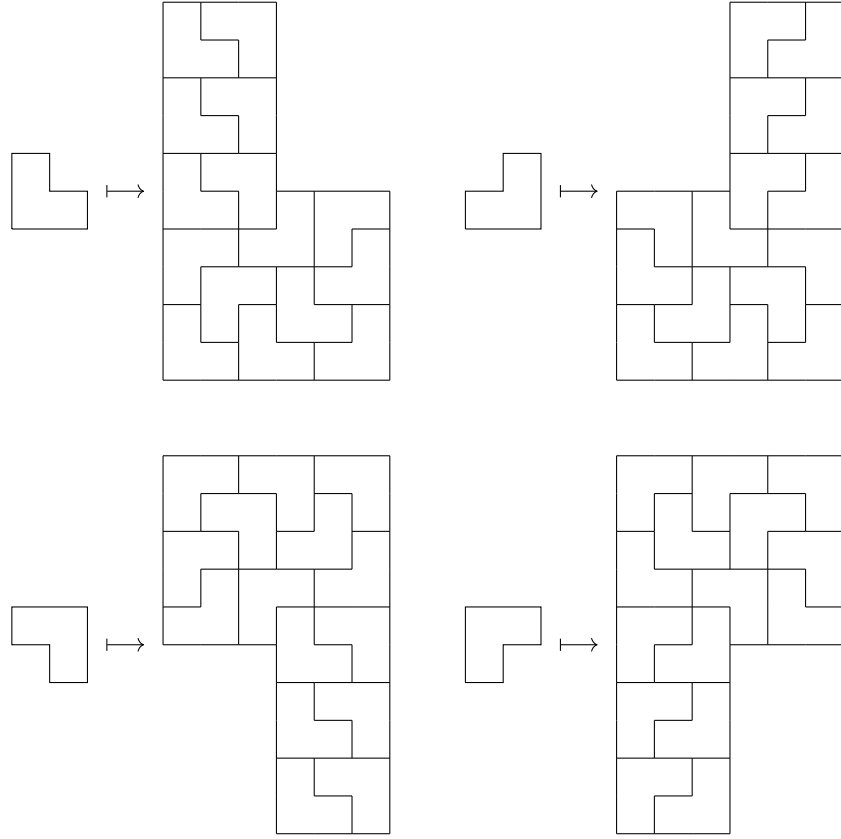


Figure 6.6: Substitution rule for a variant of the chair tiling with affine expansion $\text{diag}(3, 5)$.

the direct limits split completely. The six-term sequence is

$$\begin{array}{ccccc}
 \mathbb{Z}[1/5]^2 \oplus \mathbb{Z}[1/3]^7 & \xrightarrow{\text{ev}} & \mathbb{Z}[1/15] \oplus \mathbb{Z}[1/5]^2 & \xrightarrow{\iota_*} & \mathbb{Z}[1/15] \oplus \mathbb{Z}[1/5] \\
 \oplus \mathbb{Z}[1/2]^2 \oplus \mathbb{Z}^{94} & & \oplus \mathbb{Z}[1/3]^{13} \oplus \mathbb{Z}[1/2]^4 \oplus \mathbb{Z}^{128} & & \oplus \mathbb{Z}[1/3]^7 \oplus \mathbb{Z}[1/2]^2 \oplus \mathbb{Z}^{35} \\
 \uparrow & & & & \downarrow \\
 \mathbb{Z}[1/5] \oplus \mathbb{Z}[1/3] & \longleftarrow & 0 & \longleftarrow & \mathbb{Z}
 \end{array}$$

where the evaluation map sends $\mathbb{Z}[1/5] \oplus \mathbb{Z}[1/3]^6 \oplus \mathbb{Z}[1/2]^2 \oplus \mathbb{Z}^{94} \leq K_0(C_r^*(\dot{G}_{AF}); C_r^*(\dot{G}_u))$

isomorphically onto its image and the rest to 0. We have that

$$\begin{aligned} K_0(C_r^*(\dot{G}_u)) &\cong \left(\begin{array}{c} \mathbb{Z}[1/15] \oplus \mathbb{Z}[1/5]^2 \\ \oplus \mathbb{Z}[1/3]^{13} \oplus \mathbb{Z}[1/2]^4 \oplus \mathbb{Z}^{128} \end{array} \right) / (\mathbb{Z}[1/5] \oplus \mathbb{Z}[1/3]^6 \oplus \mathbb{Z}[1/2]^2 \oplus \mathbb{Z}^{94}) \oplus \mathbb{Z} \\ &\cong K_0(C_r^*(\dot{G}_{AF})) / (\mathbb{Z}[1/5] \oplus \mathbb{Z}[1/3]^6 \oplus \mathbb{Z}[1/2]^2 \oplus \mathbb{Z}^{94}) \oplus \mathbb{Z} \end{aligned}$$

and

$$\begin{aligned} K_1(C_r^*(\dot{G}_u)) &\cong (\mathbb{Z}[1/5] \oplus \mathbb{Z}[1/3] \oplus \mathbb{Z}^{41}) / \mathbb{Z}^{41} \\ &\leq (\mathbb{Z}[1/5]^2 \oplus \mathbb{Z}[1/3]^7 \oplus \mathbb{Z}[1/2]^2 \oplus \mathbb{Z}^{135}) / \mathbb{Z}^{41} \\ &\cong C^1 / \mathbb{Z}^{41} \\ &\cong K_0(C_r^*(\dot{G}_{AF}^{(1)})) / \mathbb{Z}^{41} \quad (\text{Proposition 5.21}) \end{aligned}$$

where we have to apply a slightly nontrivial isomorphism in the last step since this substitution does not satisfy the boundary hyperplane condition. The proposition applies since we assumed that C^1 splits completely.

Example 6.21 (Hexagon (ps)). This is the *pseudosubstitution* rule in [CS06, Figure 4]. Each of the induced substitution matrices have integral eigenvalues. One easily checks that the direct limits of the induced substitution matrices split completely. The six-term sequence is

$$\begin{array}{ccccc} \mathbb{Z}[1/2]^2 \oplus \mathbb{Z}^6 & \xrightarrow{\text{ev}} & \mathbb{Z}[1/4] \oplus \mathbb{Z}^8 & \xrightarrow{\iota_*} & \mathbb{Z}[1/4] \oplus \mathbb{Z}^3 \\ \uparrow & & & & \downarrow \\ \mathbb{Z}[1/2]^2 & \longleftarrow & 0 & \longleftarrow & \mathbb{Z} \end{array}$$

where the evaluation map sends $\mathbb{Z}^6 \leq K_0(C_r^*(\dot{G}_{AF}); C_r^*(\dot{G}_u))$ isomorphically onto its image and the rest to 0. We have that

$$\begin{aligned} K_0(C_r^*(\dot{G}_u)) &\cong (\mathbb{Z}[1/4] \oplus \mathbb{Z}^8)/\mathbb{Z}^6 \oplus \mathbb{Z} \\ &\cong K_0(C_r^*(\dot{G}_{AF}))/\mathbb{Z}^6 \oplus \mathbb{Z} \end{aligned}$$

and, since the substitution rule satisfies the boundary hyperplane condition,

$$\begin{aligned} K_1(C_r^*(\dot{G}_u)) &\cong (\mathbb{Z}[1/2]^2 \oplus \mathbb{Z}^5)/\mathbb{Z}^5 \\ &\leq (\mathbb{Z}[1/2]^2 \oplus \mathbb{Z}^{11})/\mathbb{Z}^5 \\ &\cong K_0(C_r^*(\dot{G}_{AF}^{(1)}))/\mathbb{Z}^5. \end{aligned}$$

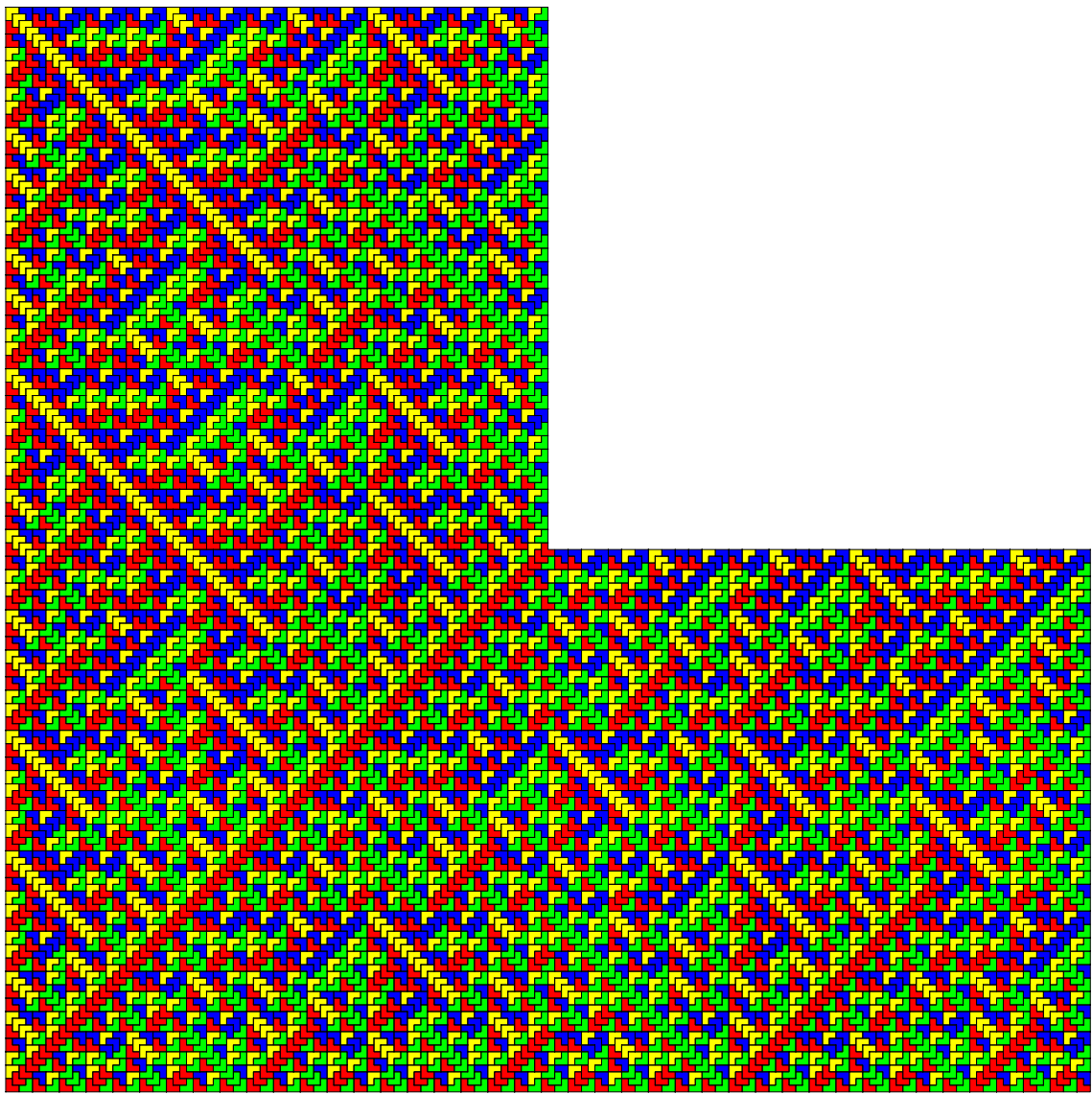


Figure 6.7: A level-4 supertile of the chair 3×3 tiling.

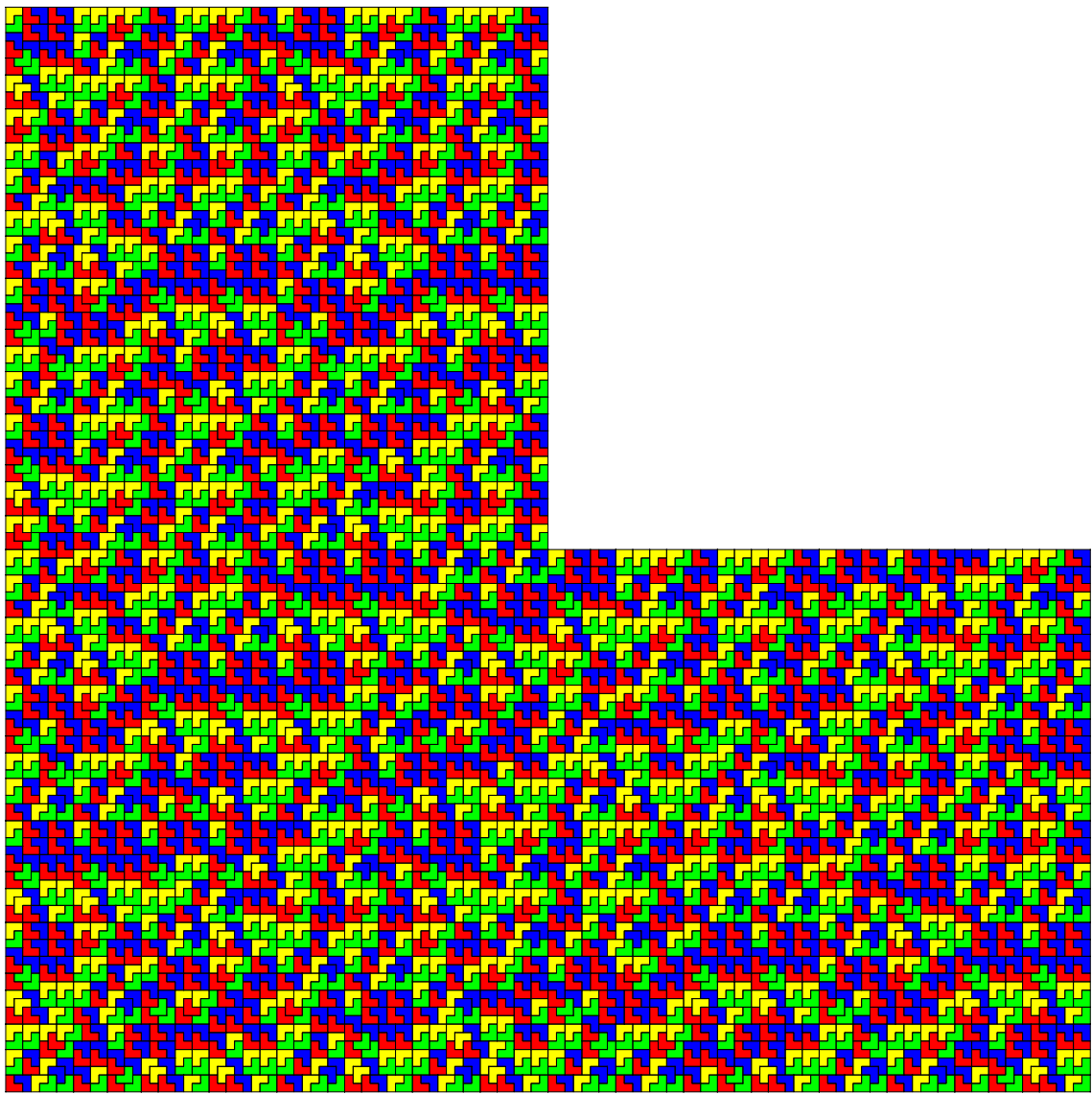


Figure 6.8: A level-3 supertile of the chair 4×4 tiling.

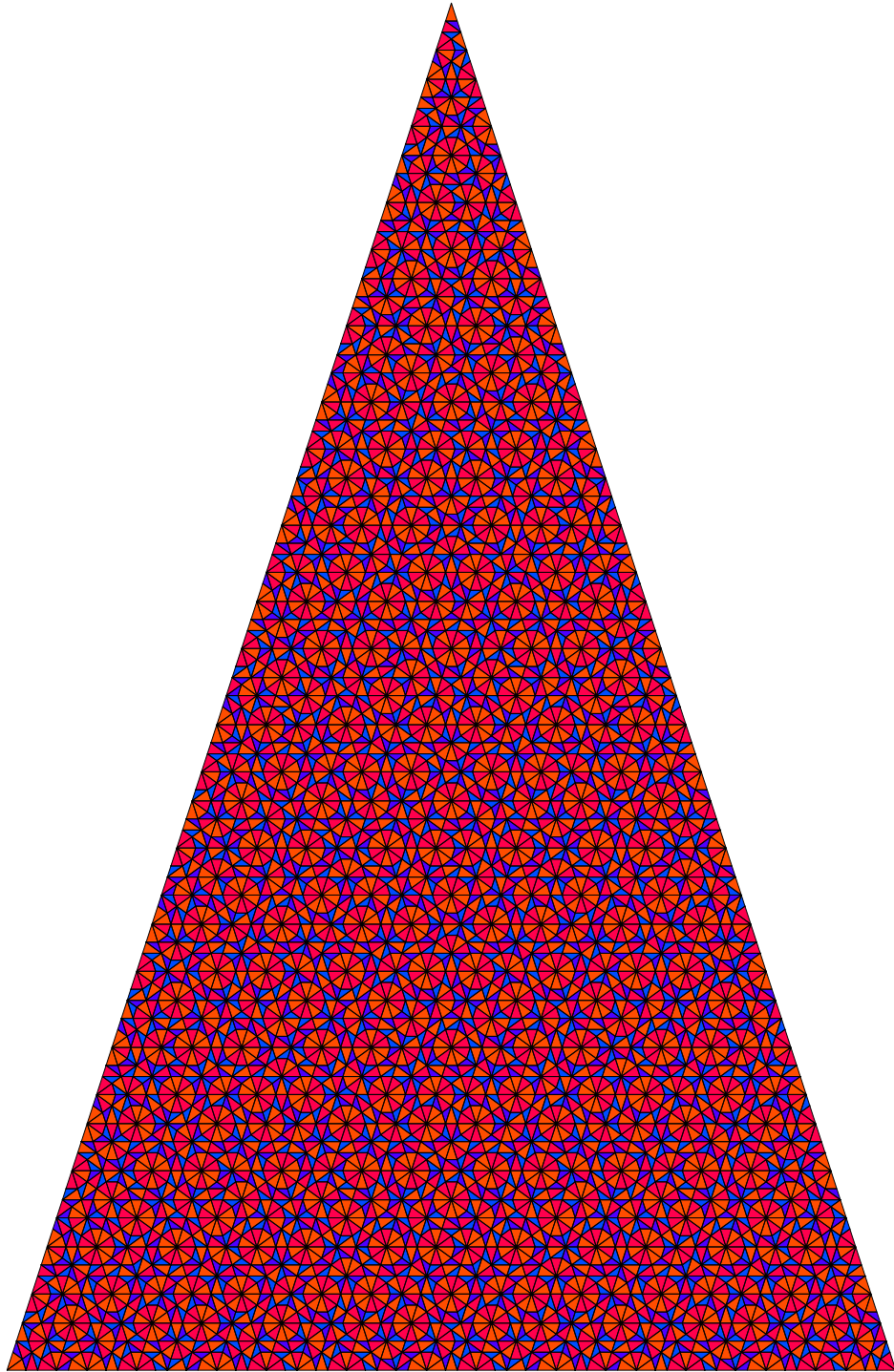


Figure 6.9: A level-9 supertile of the Tübingen triangle tiling.

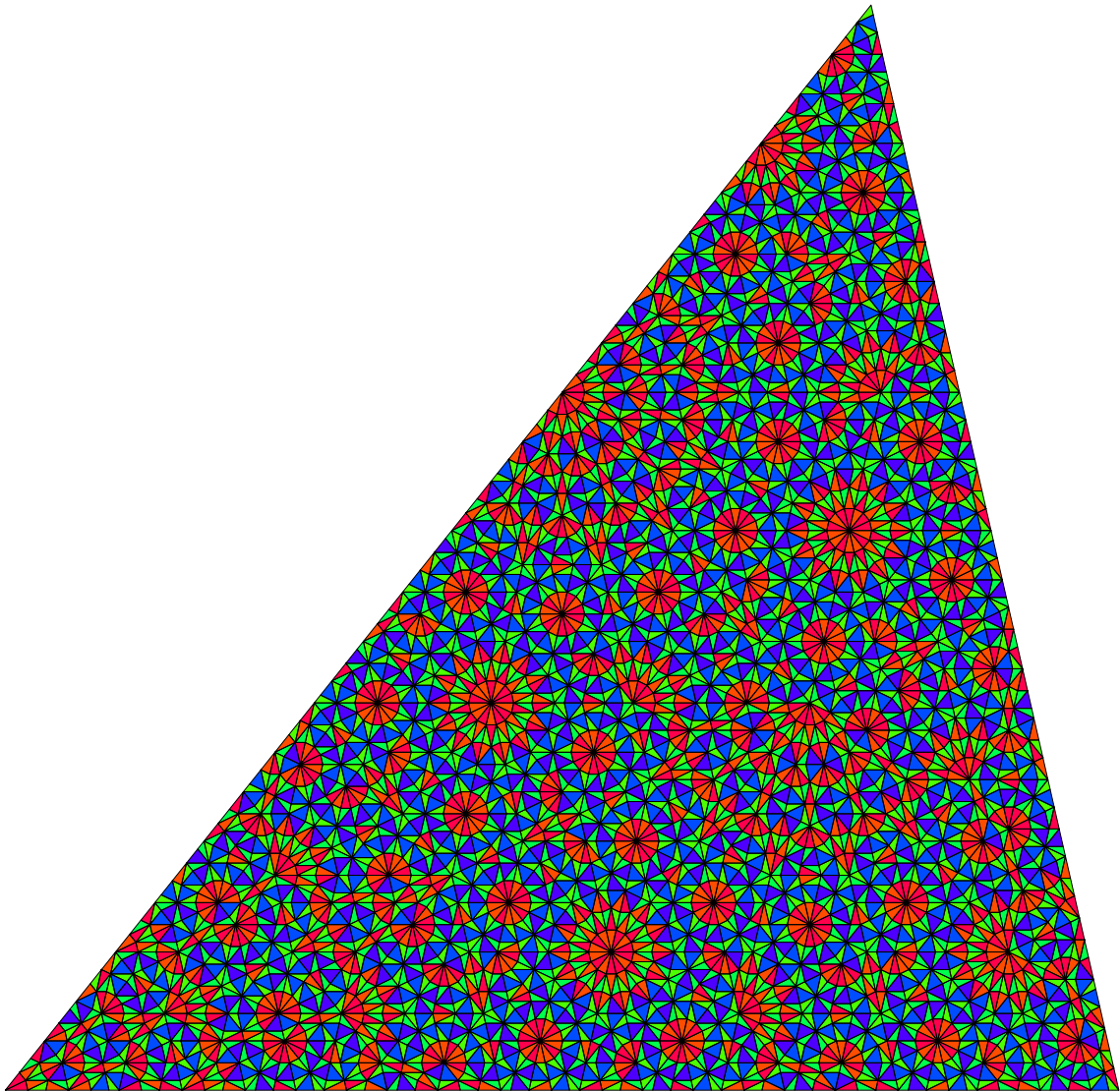


Figure 6.10: A level-4 supertile of the Danzer 7-fold tiling.

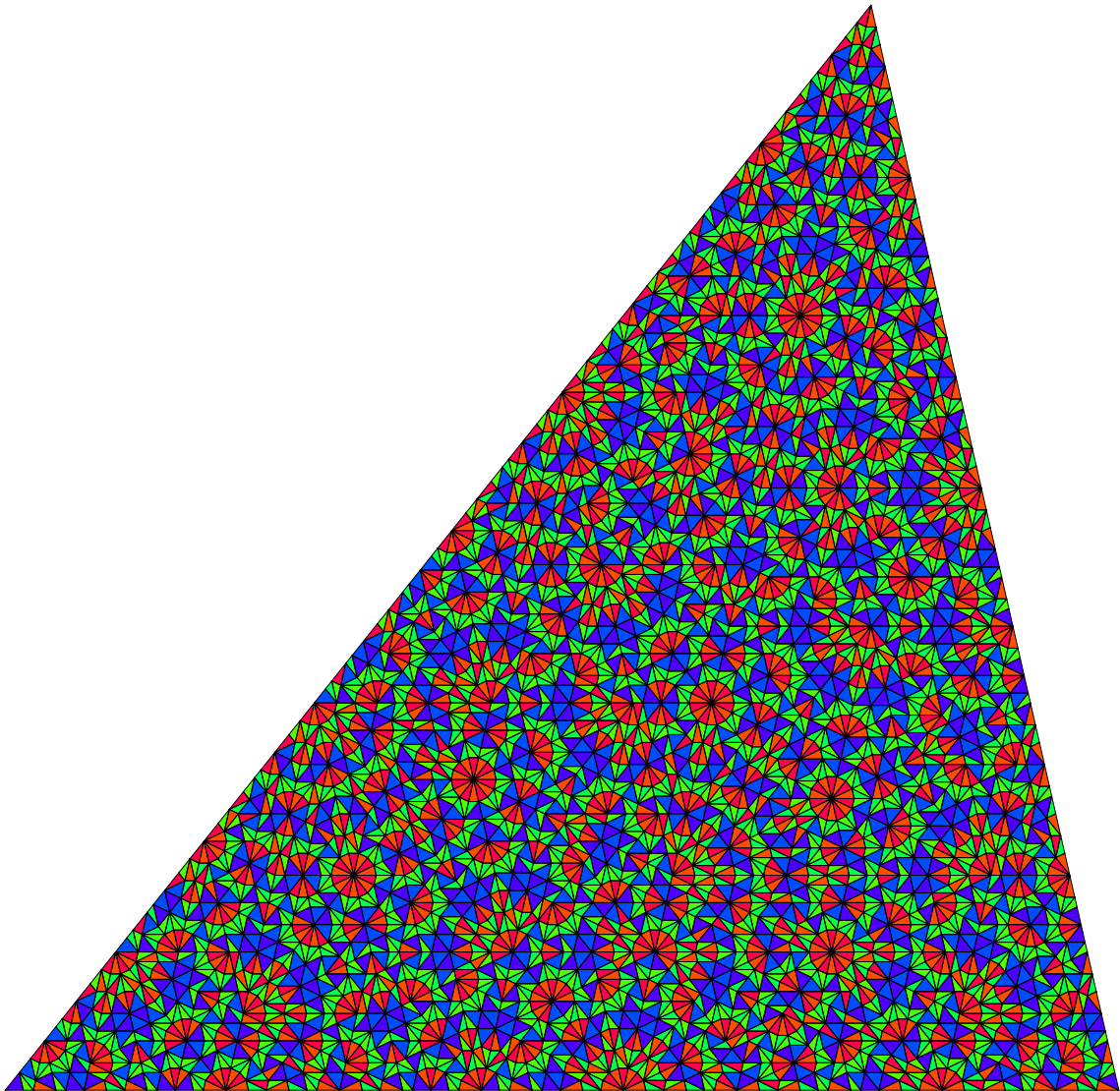


Figure 6.11: A level-4 supertile of the GKM 9.1.1.1 tiling.

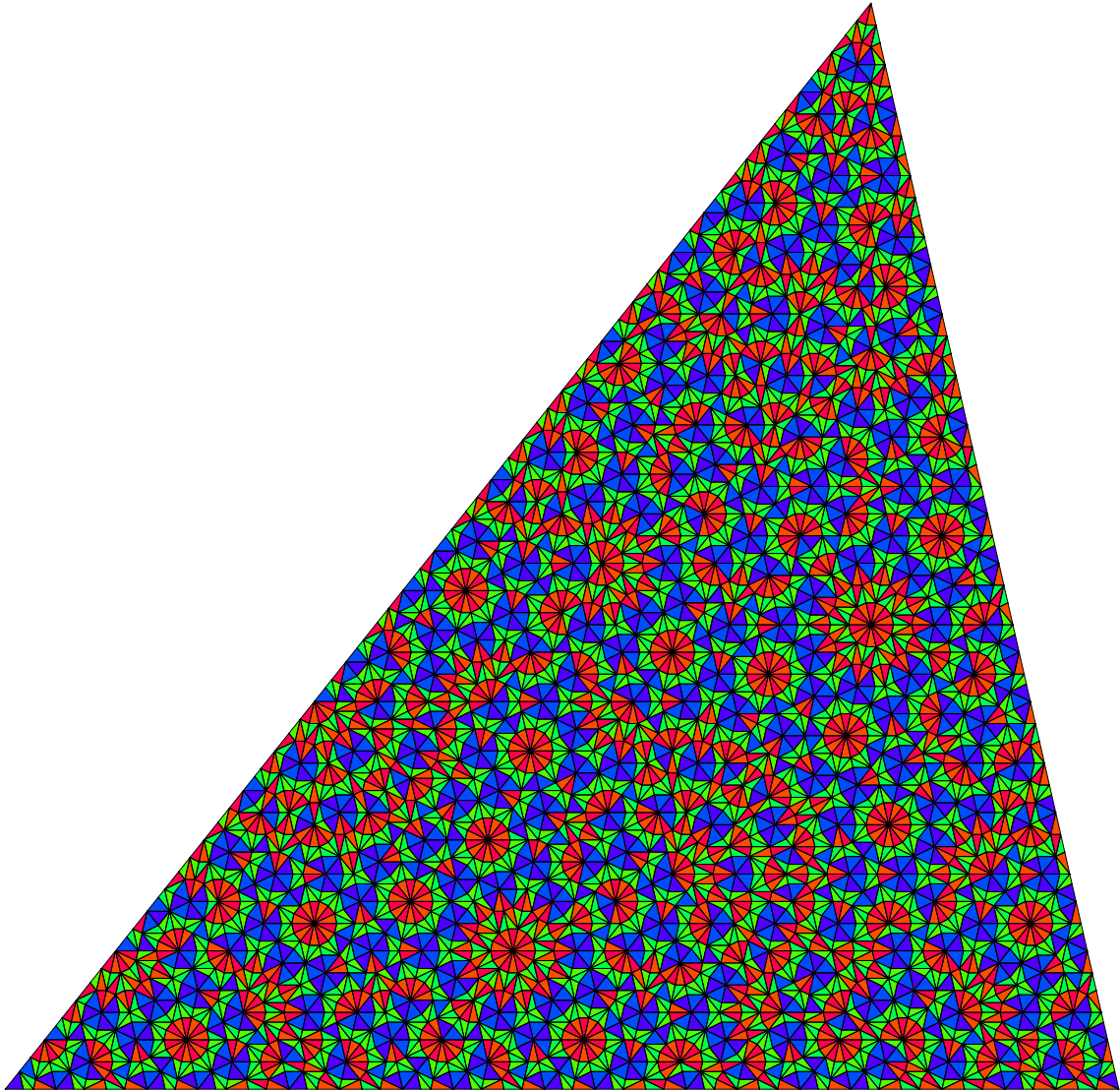


Figure 6.12: A level-4 supertile of the GKM 9.2.1.1 tiling.

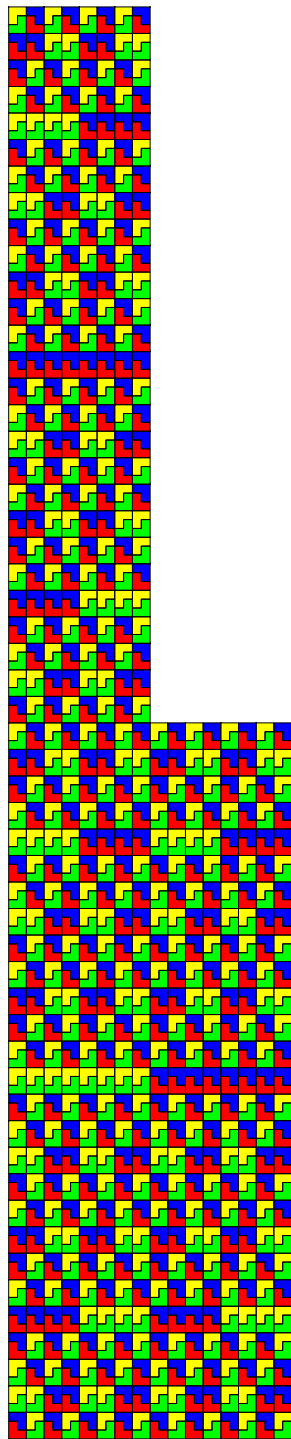


Figure 6.13: A level-4 supertile of the chair 2×3 tiling.

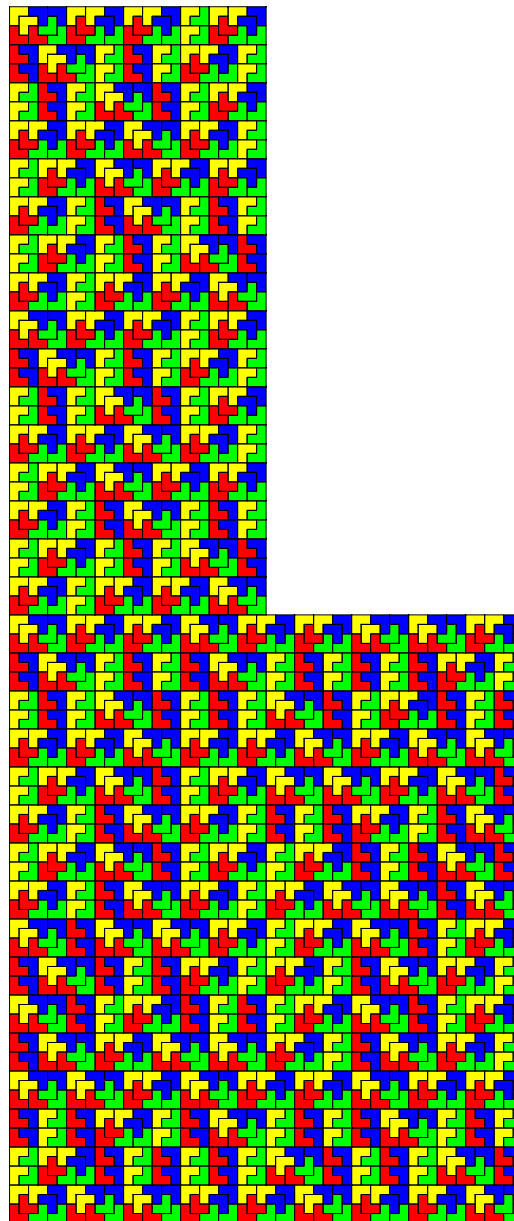


Figure 6.14: A level-3 supertile of the chair 3×4 tiling.

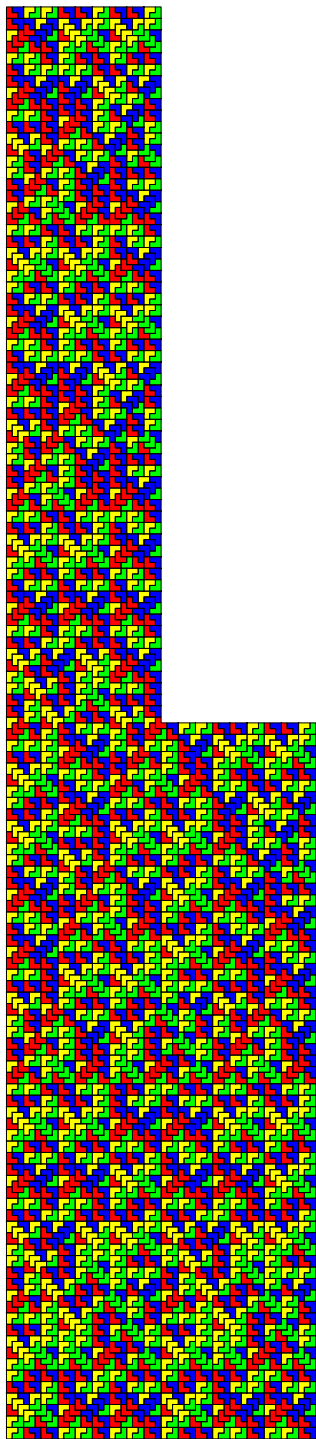


Figure 6.15: A level-3 supertile of the chair 3×5 variant 1 tiling.

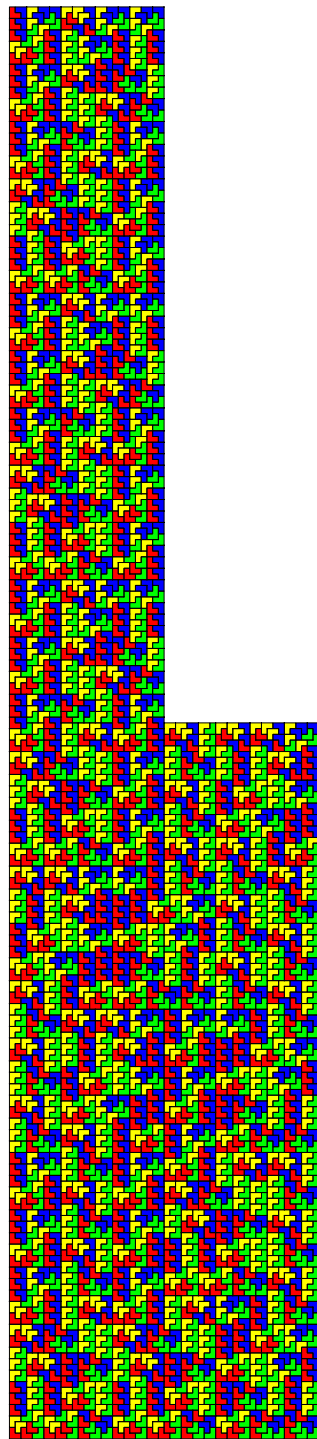


Figure 6.16: A level-3 supertile of the chair 3×5 variant 2 tiling.

	CP	PI	L(PI)	FT	L(FT)	Exact sequence	$K_0(C_r^*(\dot{G}_{AF}^*); C_r^*(\dot{G}_u^*))$ $\cong K_0(C_r^*(\dot{G}_{AF}^{(1)})) / \sim$
Dyadic solenoid	1	2	1	2	1	$\begin{array}{c} \mathbb{Z}[1/2]^2 \xrightarrow{\text{ev}} \mathbb{Z}[1/4] \xrightarrow{\iota_*} \mathbb{Z}[1/4] \oplus \mathbb{Z} \\ \uparrow \\ \mathbb{Z}[1/2]^2 \longleftarrow 0 \longleftarrow \mathbb{Z} \end{array}$	$\mathbb{Z}[1/2]^2$
Half-hex	24	60	38	42	20	$\begin{array}{c} \mathbb{Z}[1/2]^3 \oplus \mathbb{Z}^2 \xrightarrow{\text{ev}} \mathbb{Z}[1/4] \oplus \mathbb{Z}[1/2] \oplus \mathbb{Z}^4 \xrightarrow{\iota_*} \mathbb{Z}[1/4] \oplus \mathbb{Z}^3 \\ \uparrow \\ \mathbb{Z}[1/2]^2 \longleftarrow \mathbb{Z} \end{array}$	$(\mathbb{Z}[1/2]^3 \oplus \mathbb{Z}^3) / \mathbb{Z}$
Chair	56	224	175	84	34	$\begin{array}{c} \mathbb{Z}[1/2]^2 \oplus \mathbb{Z}^4 \xrightarrow{\text{ev}} \mathbb{Z}[1/4] \oplus \mathbb{Z}[1/2]^2 \oplus \mathbb{Z}^4 \xrightarrow{\iota_*} \mathbb{Z}[1/4] \oplus \mathbb{Z}[1/2]^2 \oplus \mathbb{Z} \\ \uparrow \\ \mathbb{Z}[1/2]^2 \longleftarrow 0 \longleftarrow \mathbb{Z} \end{array}$	$\mathbb{Z}[1/2]^2 \oplus \mathbb{Z}^4$
Chair 3×3	364	1616	1333	479	174	$\begin{array}{c} \mathbb{Z}[1/3]^4 \oplus \mathbb{Z}[1/2]^2 \xrightarrow{\text{ev}} \mathbb{Z}[1/9] \oplus \mathbb{Z}[1/3]^6 \xrightarrow{\iota_*} \mathbb{Z}[1/9] \oplus \mathbb{Z}[1/3]^4 \\ \oplus \mathbb{Z}^{76} \oplus \mathbb{Z}^{101} \oplus \mathbb{Z}^{26} \\ \uparrow \\ \mathbb{Z}[1/3]^2 \longleftarrow \mathbb{Z} \end{array}$	$\left(\mathbb{Z}[1/3]^4 \oplus \mathbb{Z}[1/2]^2 \right) / \mathbb{Z}^{34}$
Chair 4×4	435	1782	1464	569	197	$\begin{array}{c} \mathbb{Z}[1/4]^2 \oplus \mathbb{Z}[1/2] \xrightarrow{\text{ev}} \mathbb{Z}[1/16] \oplus \mathbb{Z}[1/2]^4 \xrightarrow{\iota_*} \mathbb{Z}[1/16] \oplus \mathbb{Z}[1/2]^3 \\ \oplus \mathbb{Z}^{19} \oplus \mathbb{Z}^{30} \\ \uparrow \\ \mathbb{Z}[1/4]^2 \longleftarrow 0 \longleftarrow \mathbb{Z} \end{array}$	$\mathbb{Z}[1/4]^2 \oplus \mathbb{Z}[1/2]^2 \oplus \mathbb{Z}^{19}$
Robinson triangle	300	540	264	360	64	$\begin{array}{c} \mathbb{Z}^{37} \xrightarrow{\text{ev}} \mathbb{Z}^{40} \xrightarrow{\iota_*} \mathbb{Z}^9 \\ \uparrow \\ \mathbb{Z}^5 \longleftarrow 0 \longleftarrow \mathbb{Z} \end{array}$	$\mathbb{Z}^{40} / \mathbb{Z}^3$
Tübingen triangle	860	1710	880	950	130	$\begin{array}{c} \mathbb{Z}^{101} \xrightarrow{\text{ev}} \mathbb{Z}^{120} \xrightarrow{\iota_*} \mathbb{Z}^{25} \oplus \mathbb{Z}_5^2 \\ \uparrow \\ \mathbb{Z}^5 \longleftarrow 0 \longleftarrow \mathbb{Z} \end{array}$	$\mathbb{Z}^{110} / \mathbb{Z}^9$
Danzer 7-fold	5628	11270	5922	6118	812	$\begin{array}{c} \mathbb{Z}^{2017} \xrightarrow{\text{ev}} \mathbb{Z}^{\geq 2222} \oplus \mathbb{Q}^{\leq 18} \xrightarrow{\iota_*} \mathbb{Z}^{\geq 212} \oplus \mathbb{Q}^{\leq 18} \\ \uparrow \\ \mathbb{Z}^6 \longleftarrow 0 \longleftarrow \mathbb{Z} \end{array}$	$\mathbb{Z}^{2226} / \mathbb{Z}^{209}$
Ammann A2	68	228	163	110	45	$\begin{array}{c} \mathbb{Z}^6 \xrightarrow{\text{ev}} \mathbb{Z}^8 \xrightarrow{\iota_*} \mathbb{Z}^7 \\ \uparrow \\ \mathbb{Z}^4 \longleftarrow 0 \longleftarrow \mathbb{Z} \end{array}$	$\mathbb{Z}^8 / \mathbb{Z}^2$
Ammann A5 decorated	672	1608	960	824	184	$\begin{array}{c} \mathbb{Z}^{65} \xrightarrow{\text{ev}} \mathbb{Z}^{80} \xrightarrow{\iota_*} \mathbb{Z}^{24} \\ \uparrow \\ \mathbb{Z}^8 \longleftarrow 0 \longleftarrow \mathbb{Z} \end{array}$	$\mathbb{Z}^{72} / \mathbb{Z}^7$
Ammann A5 undecorated	220	496	281	272	65	$\begin{array}{c} \mathbb{Z}^{16} \xrightarrow{\text{ev}} \mathbb{Z}^{20} \xrightarrow{\iota_*} \mathbb{Z}^{10} \\ \uparrow \\ \mathbb{Z}^5 \longleftarrow 0 \longleftarrow \mathbb{Z} \end{array}$	\mathbb{Z}^{16}

GKM 9.1.1.1	4704	9604	5154	4984	660	$\begin{array}{c} \mathbb{Z}^{602} \xrightarrow{\text{ev}} \mathbb{Z}^{672} \xrightarrow{\iota_*} \mathbb{Z}^{84} \\ \uparrow \\ \mathbb{Z}^{12} \longleftarrow 0 \longleftarrow \mathbb{Z}^2 \end{array}$	$\mathbb{Z}^{686} / \mathbb{Z}^{84}$
GKM 9.2.1.1	3864	8008	4230	4060	522	$\begin{array}{c} \mathbb{Z}^{1022} \xrightarrow{\text{ev}} \mathbb{Z}^{148} \xrightarrow{\iota_*} \mathbb{Z}^{140} \\ \uparrow \\ \mathbb{Z}^{12} \longleftarrow 0 \longleftarrow \mathbb{Z}^2 \end{array}$	$\mathbb{Z}^{1134} / \mathbb{Z}^{112}$
GKM 10.1.1.1	4872	9772	5096	5362	742	$\begin{array}{c} \mathbb{Z}^{1331} \xrightarrow{\text{ev}} \mathbb{Z}^{\geq 1457} \oplus \mathbb{Q}^{\leq 27} \xrightarrow{\iota_*} \mathbb{Z}^{\geq 133} \oplus \mathbb{Q}^{\leq 27} \\ \uparrow \\ \mathbb{Z}^6 \longleftarrow 0 \longleftarrow \mathbb{Z} \end{array}$	$\mathbb{Z}^{1470} / \mathbb{Z}^{139}$
Chair 2×3 (affine)	36	104	72	62	28	$\begin{array}{c} \mathbb{Z}[1/3] \oplus \mathbb{Z}[1/2] \oplus \mathbb{Z}^3 \xrightarrow{\text{ev}} \mathbb{Z}[1/6] \oplus \mathbb{Z}[1/2] \oplus \mathbb{Z}^4 \xrightarrow{\iota_*} \mathbb{Z}[1/6] \oplus \mathbb{Z}[1/2] \oplus \mathbb{Z}^2 \\ \uparrow \\ \mathbb{Z}[1/3] \oplus \mathbb{Z}[1/2] \longleftarrow 0 \longleftarrow \mathbb{Z} \end{array}$	$\mathbb{Z}[1/3] \oplus \mathbb{Z}[1/2] \oplus \mathbb{Z}^3$
Chair 3×4 (affine)	160	714	587	194	70	$\begin{array}{c} \mathbb{Z}[1/4] \oplus \mathbb{Z}[1/3]^2 \xrightarrow{\text{ev}} \mathbb{Z}[1/12] \oplus \mathbb{Z}[1/4] \xrightarrow{\iota_*} \mathbb{Z}[1/12] \oplus \mathbb{Z}[1/4] \\ \oplus \mathbb{Z}[1/2]^3 \oplus \mathbb{Z}^{16} \xrightarrow{\text{ev}} \oplus \mathbb{Z}[1/3]^2 \oplus \mathbb{Z}[1/2]^4 \xrightarrow{\iota_*} \oplus \mathbb{Z}[1/3] \oplus \mathbb{Z}[1/2] \\ \oplus \mathbb{Z}^{22} \oplus \mathbb{Z}^7 \\ \uparrow \\ \mathbb{Z}[1/4] \oplus \mathbb{Z}[1/3] \longleftarrow 0 \longleftarrow \mathbb{Z} \end{array}$	$\left(\mathbb{Z}[1/4] \oplus \mathbb{Z}[1/3]^2 \right) / \mathbb{Z}^4$
Chair 3×5 variant 1 (affine)	404	1706	1368	548	192	$\begin{array}{c} \mathbb{Z}[1/5]^2 \oplus \mathbb{Z}[1/3]^2 \xrightarrow{\text{ev}} \mathbb{Z}[1/15] \oplus \mathbb{Z}[1/5]^2 \xrightarrow{\iota_*} \mathbb{Z}[1/15] \oplus \mathbb{Z}[1/5] \\ \oplus \mathbb{Z}^{32} \oplus \mathbb{Z}^{16} \xrightarrow{\text{ev}} \oplus \mathbb{Z}[1/3]^5 \oplus \mathbb{Z}^{48} \xrightarrow{\iota_*} \oplus \mathbb{Z}[1/3]^4 \oplus \mathbb{Z}^{17} \\ \uparrow \\ \mathbb{Z}[1/5] \oplus \mathbb{Z}[1/3] \longleftarrow 0 \longleftarrow \mathbb{Z} \end{array}$	$\left(\mathbb{Z}[1/5]^2 \oplus \mathbb{Z}[1/3]^2 \right) / \mathbb{Z}^4$
Chair 3×5 variant 2 (affine)	620	2958	2497	760	254	$\begin{array}{c} \mathbb{Z}[1/5]^2 \oplus \mathbb{Z}[1/3]^7 \xrightarrow{\text{ev}} \mathbb{Z}[1/15] \oplus \mathbb{Z}[1/5]^2 \xrightarrow{\iota_*} \mathbb{Z}[1/15] \oplus \mathbb{Z}[1/5] \\ \oplus \mathbb{Z}[1/2]^2 \oplus \mathbb{Z}^{94} \xrightarrow{\text{ev}} \oplus \mathbb{Z}[1/3]^{13} \oplus \mathbb{Z}[1/2]^4 \xrightarrow{\iota_*} \oplus \mathbb{Z}[1/3]^7 \oplus \mathbb{Z}[1/2]^2 \\ \oplus \mathbb{Z}^{128} \oplus \mathbb{Z}^{35} \\ \uparrow \\ \mathbb{Z}[1/5] \oplus \mathbb{Z}[1/3] \longleftarrow 0 \longleftarrow \mathbb{Z} \end{array}$	$\left(\mathbb{Z}[1/5]^2 \oplus \mathbb{Z}[1/3]^7 \right) / \mathbb{Z}^{41}$
Hexagon (ps)	24	102	80	42	22	$\begin{array}{c} \mathbb{Z}[1/2]^2 \oplus \mathbb{Z}^6 \xrightarrow{\text{ev}} \mathbb{Z}[1/4] \oplus \mathbb{Z}^8 \xrightarrow{\iota_*} \mathbb{Z}[1/4] \oplus \mathbb{Z}^3 \\ \uparrow \\ \mathbb{Z}[1/2]^2 \longleftarrow 0 \longleftarrow \mathbb{Z} \end{array}$	$(\mathbb{Z}[1/2]^2 \oplus \mathbb{Z}^{11}) / \mathbb{Z}^5$

Table 6.1: CP is the number of partial isometries, PI is the number of full translations, L(\cdot) is the number of loops, the second to last column is the corresponding six-term sequence in relative K -theory, and the last column is the contribution of $K_0(C_r^*(\dot{G}_{AF}^{(1)}))$ towards $K_1(C_r^*(\dot{G}_u))$. If an entry in the last column is not written as a quotient, then $K_1(C_r^*(\dot{G}_u)) \leq K_0(C_r^*(\dot{G}_{AF}^{(1)}))$.

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