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When is the Multiaffine Image of a Cube a Polygon?

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When is the Multiaffine Image of a Cube a Polygon?†

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Abstract

We give two simple sufficient conditions under which the multiaffine image on the complex plane of an m-dimensional cube is a convex polygon. A third condition which, in some generic sense, is necessary and sufficient is then obtained. The conditions involve checking the locations of the image of the vertices of the cube. These results help determine whether a family of parametrized polynomials is stable, and provide a tool for robust control analysis.

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1. Introduction and Preliminary Results

Many problems in robust control (e.g., see [3,4,5,8,10,12]) can be formulated in terms of stability of a family of parametrized polynomials

$$p(s, \gamma) = s^n + a_1(\gamma)s^{n-1} + \cdots + a_n(\gamma), \tag{1}$$

where the parameter γ runs through the set

$$K = \{(\gamma_1, \ldots, \gamma_m) : \underline{\gamma}_i \leq \gamma_i \leq \overline{\gamma}_i,\ i = 1, \ldots, m\}$$

with given $-\infty < \underline{\gamma}_i < \overline{\gamma}_i < \infty$, and where $a_1, \ldots, a_n$ are real valued multiaffine functions of γ. Here a function is *multiaffine* if and only if it is affine (i.e., is a sum of a linear function and a constant function) in each coordinate of its variable γ with the remaining coordinates fixed. By properly scaling and shifting the parameters $\gamma_1, \ldots, \gamma_m$, and absorbing the scaling factors and shifting offsets into the functions a_i, one may assume without loss of generality that K is the m-dimensional hypercube $[0,1]^m$.

Suppose it is known that one member of the polynomial family (1) is (Hurwitz) stable, i.e., all its zeros have negative real parts. Then a simple continuity argument shows that the whole family (1) is stable if and only if

$$0 \notin \{p(j\omega, \gamma) : \gamma \in K\} \quad \forall \omega \in [0, \infty) \tag{2}$$

(zero exclusion principle; e.g., see [2]). For each $\omega \in [0, \infty)$, define $f_\omega : \mathbb{R}^m \to \mathbb{C}$ by $f_\omega(\gamma) = p(j\omega, \gamma)$. Then (2) is equivalent to

$$0 \notin f_\omega(K) \quad \forall \omega \in [0, \infty).$$

Notice that f_ω is multiaffine. It is then of interest to know when the image of K under a multiaffine function can be easily characterized, so that a simple method can be devised to determine if it contains the origin.

For notational convenience, we will drop the symbol ω and simply write f_ω as f, which denotes a general multiaffine function from $\mathbb{R}^m$ to $\mathbb{C}$.

Let us denote the set of *vertices* of K by $V = \{0,1\}^m$. For any $\gamma = (\gamma_1, \ldots, \gamma_m) \in \mathbb{R}^m$, since f is multiaffine, we have (see [11], pg 474)

$$f(\gamma_1, \ldots, \gamma_m) = (1 - \gamma_i)f(\gamma_1, \ldots, \gamma_{i-1}, 0, \gamma_{i+1}, \ldots, \gamma_m) + \gamma_i f(\gamma_1, \ldots, \gamma_{i-1}, 1, \gamma_{i+1}, \ldots, \gamma_m)$$

for $i = 1, \ldots, m$. From this, one deduces that

$$f(\gamma) = \sum_{v \in V} \alpha(\gamma, v) f(v), \tag{3}$$

where

$$\alpha(\gamma, v) = \prod_{i=1}^{m} \beta_i(\gamma, v), \quad \beta_i(\gamma, v) = \begin{cases} 1 - \gamma_i & \text{if } v_i = 0; \\ \gamma_i & \text{if } v_i = 1. \end{cases} \tag{4}$$

The representation of f by (3) implies that any multiaffine $f : \mathbb{R}^m \to \mathbb{C}$ is uniquely determined by the family $\{f(v)\}_{v \in V}$ of 2^m elements in $\mathbb{C}$ that are the image of the vertices of K under f. Moreover, by observing that $\sum\limits_{v \in V} \alpha(\gamma, v) = 1$, and $\alpha(\gamma, v) \geq 0$ if $\gamma \in K$, one sees from (3) that $f(\gamma)$ is actually a convex combination of elements in $f(V)$ if $\gamma \in K$. This gives the following Mapping Theorem of Zadeh and Desoer ([11], pg 476):

Mapping Theorem. $\operatorname{conv} f(K) = \operatorname{conv} f(V)$.

As a result, if $f(K)$ is convex then it must be the polygon $\operatorname{conv} f(V)$.

For any x, y in a vector space X, the *line segment* joining x and y is defined by $[x, y] = \{(1-t)x + ty : t \in [0,1]\}$. An *edge* of K is a line segment $[v^1, v^2]$ where $v^1, v^2 \in V$ are vertices of K that differ from each other by exactly one coordinate. As f is multiaffine, it follows that the image of an edge $[v^1, v^2]$ of K is the line segment $[f(v^1), f(v^2)]$. The Mapping Theorem imply the following [1]:

Suppose the edges of the convex polygon $\operatorname{conv} f(V)$ *are covered by the image of the edges of K, and that $f(K)$ is simply connected. Then $f(K)$ is the convex polygon* $\operatorname{conv} f(V)$.

Notice that $f(K)$ may fail to be simply connected, and a counter-example is given in [1]. This disproves a conjecture of Hollot and Xu [6] that $f(K)$ is a convex polygon if and only if all the edges of $\operatorname{conv} f(V)$ are mapped from edges of K.

The following result is proved in [1] with induction argument and detailed analysis of the $m = 2$ case. For the sake of completeness, we give here a simple proof based on the formula (3).

Proposition 1. *Let $x \in \mathbb{C}$ be on an edge of $\operatorname{conv} f(V)$. If $x \in f(K)$, then x is covered by the image of edges of K.*

Proof. Let $\gamma = (\gamma_1, \ldots, \gamma_m) \in K$, and let $x = f(\gamma)$ be on an edge E of $\operatorname{conv} f(V)$. Define a nonempty subset $V(\gamma)$ of V by

$$V(\gamma) = \{(v_1, \ldots, v_m) \in V : v_i = \gamma_i \text{ if } \gamma_i = 1 \text{ or } 0\},$$

i.e., $V(\gamma)$ is the set of vertices of the "smallest face" of K that contains γ. One can check directly that for $v \in V$ and the function α defined in (3) and (4), $\alpha(\gamma, v) > 0$ if and only if $v \in V(\gamma)$. If $v \in V$ is such that $f(v)$ is not on the edge E, then the coefficient $\alpha(\gamma, v)$ in the convex combination formula (3) must be zero (otherwise $f(\gamma)$ could not be on E). Hence $f\big(V(\gamma)\big) \subset E$, and we may rewrite (3) as the convex combination formula $x = f(\gamma) = \sum_{v \in V(\gamma)} \alpha(\gamma, v) f(v)$. Let $v, w \in V(\gamma)$ be such that $f(v)$ and $f(w)$ be farthest apart, so that $x \in [f(v), f(w)]$. Let k (≥ 1) be the number of coordinates in which v differs from w. It is clear from the definition of $V(\gamma)$ that there exist $v^0 (= v), v^1, \ldots, v^{k-1}, v^k (= w) \in V(\gamma)$ such that $[v^{i-1}, v^i]$ ($i = 1, \ldots, k$) are edges of K. As $f\big(V(\gamma)\big) \subset E$, one concludes that the image $\bigcup_{i=1}^{m-1} [f(v^{i-1}), f(v^i)]$ of these edges of K covers $[f(v), f(w)]$ and hence x. $\square$

As $f(K)$ may not be simply connected, one may consider the subset $\widehat{f(K)}$ which is the union of $f(K)$ and the "holes" of it. More precisely, $\widehat{f(K)} = \{x \in \mathbb{C} : x \text{ is either in } f(K) \text{ or there is a simple closed curve in } f(K) \text{ that encircles } x\}$. The following result, which is a variation of the Hollot and Xu conjecture, follows from the Mapping Theorem and Proposition 1 (see also [1]).

Proposition 2. *$\widehat{f(K)}$ is convex (and is equal to the polygon $\operatorname{conv} f(V)$) if and only if the edges of $\operatorname{conv} f(V)$ are covered by the image of edges of K.*

In [9] and [1], sufficient conditions for $f(K)$ to be a convex polygon are given. These conditions involve the computation of the partial derivatives $\frac{\partial f}{\partial \gamma_i}$ on either K or V. As we have pointed out in connection with (3), f depends only on the the family $\{f(v)\}_{v \in V}$. This suggests that the condition for $f(K)$ to be a polygon may solely rely on the position of those $f(v)$ $(v \in V)$. Two such sufficient conditions will be given in Section 3. They are based on a general result on closed loops which will be presented in Section 2. It turns out that our sufficient conditions are implied by the sufficient conditions given in [9] and [1] (and hence our results imply those in [9] and [1]) and are easier to check. Our sufficient conditions also lead to a third simple condition (see Section 3) which is, in some generic sense, necessary and sufficient for $f(K)$ to be a convex polygon.

2. Closed Loops and Their Interior

We now establish a general result on closed loops in manifolds. This will provide the tool for obtaining our main results in Section 3.

Let D be the unit disk $\{re^{j\theta} : r \in [0,1],\, \theta \in [0, 2\pi]\}$ and ∂D the unit circle $\{e^{j\theta} : \theta \in [0, 2\pi]\}$. For any subset M of either $\mathbb{R}^m$ or $\mathbb{C}$, a continuous function $c : \partial D \to M$ is called a *closed loop* in M. It is clear that if M is simply connected then any closed loop c in M can be extended to a continuous function $C : D \to M$ with $C\big|_{\partial D} = c$. In addition, if $M = \mathbb{C}$ then we define the *interior* of c by

$$\operatorname{int} c = \bigcap_{C \in \mathcal{C}(c)} C(D)$$

where $\mathcal{C}(c)$ is the class of all continuous functions $C : D \to \mathbb{C}$ that satisfy $C\big|_{\partial D} = c$. Accordingly, $x \in \mathbb{C}$ is *not* in $\operatorname{int} c$ if and only if there exists a continuous function $C : D \to \mathbb{C}$ such that $C\big|_{\partial D} = c$ and $x \notin C(D)$. For this continuous C, if we define $c_t : \partial D \to \mathbb{C}$ by $c_t(e^{j\theta}) = C(te^{j\theta})$ for $t \in [0,1]$, then c_t is a continuous deformation of c into a constant function when t decreases from 1 to 0. Hence $x \in \mathbb{C}$ is not in $\operatorname{int} c$ if and only if the closed loop c can be deformed in $\mathbb{C}$ continuously into a single point without hitting x. Note that, in the above, M could be any topological space in the definition of the closed loop, and $\mathbb{C}$ could be replaced by any simply connected 2-dimensional manifold in the definition of the interior.

Lemma 1. *Let M be a simply connected subset of $\mathbb{R}^m$ or $\mathbb{C}$. Suppose c is a closed loop in M, and $g : M \to \mathbb{C}$ is continuous. Then*

$$\operatorname{int}(g \circ c) \subset g(M).$$

Proof. It is clear that $g \circ c$ is a closed loop in $\mathbb{C}$. Since M is simply connected, c can be extended to a continuous $C : D \to M$ with $C\big|_{\partial D} = c$. Now that $g \circ C : D \to \mathbb{C}$ is continuous and $g \circ C\big|_{\partial D} = g \circ c$, we have $\operatorname{int}(g \circ c) \subset g \circ C(D) = g\big(C(D)\big) \subset g(M)$. $\square$

Thus for any continuous map g, the interior of the image under g of a closed loop in M is contained in the image of M. We remark that Lemma 1 still holds if M is any simply connected topological space and g is any continuous map from M to a simply connected 2-dimensional manifold.

3. Main Results

Given $x^0, \ldots, x^{k-1} \in X$, where $k \geq 2$ and X is a topological vector space (e.g., $X = \mathbb{R}^m$ or $\mathbb{C}$), let $x^k = x^0$, call *closed loop with k nodes* $x^0, \ldots, x^{k-1}$ the closed loop $c : \partial D \to X$ defined by

$$c\big(\exp(j\frac{2\pi(i+t)}{k})\big) = (1-t)x^{i-1} + tx^i \quad \forall i = 1, \ldots, k, \ t \in [0,1],$$

and denote it by $c_{[x^0,\ldots,x^{k-1}]}$. The ordering of the nodes is important in determining the interior of $c_{[x^0,\ldots,x^{k-1}]}$, in case X has real dimension 2 (and thus X is a simply connected 2-dimensional manifold). For example, in Figure 1, x is in the interior of $c_{[x_0,x_1,x_2,x_3,x_4,x_5,x_3]}$ but not in the interior of $c_{[x_0,x_1,x_2,x_3,x_5,x_4,x_3]}$.

Theorem 1. *Suppose $v^0, \ldots, v^{k-1} \in V$ are $k \geq 2$ vertices of K such that $[v^{i-1}, v^i]$ ($i = 1, \ldots, k-1$) and $[v^{k-1}, v^0]$ are edges of K. Then for any multiaffine $f : \mathbb{R}^m \to \mathbb{C}$,*

$$\operatorname{int} c_{[f(v^0),\ldots,f(v^{k-1})]} \subset f(K).$$

Proof. Let $v^0, \ldots, v^{k-1} \in V$ satisfy the hypotheses of the theorem. Since K is simply connected and f is continuous, Lemma 1 ensures that $\operatorname{int}(f \circ c_{[v^0,\ldots,v^{k-1}]}) \subset f(K)$. As f is multiaffine and $[v^{i-1}, v^i]$ ($i = 1, \ldots, k-1$) and $[v^{k-1}, v^0]$ are edges of K, we have $f \circ c_{[v^0,\ldots,v^{k-1}]} = c_{[f(v^0),\ldots,f(v^{k-1})]}$. The result then follows. □

The significance of Theorem 1 is that it gives certain subsets of $f(K)$ in terms of the image of the vertices of K. This should be compared with the Mapping Theorem of Zadeh and Desoer, which gives a superset of $f(K)$ (namely, $\operatorname{conv} f(V)$) which is also determined by the image of the vertices of K. These two results give the following sufficient condition for $f(K)$ to be a convex polygon.

Corollary 1. *Suppose*

(I) there exist vertices $v^0, \ldots, v^{k-1}$ ($k \geq 2$) of K such that $[v^{i-1}, v^i]$ ($i = 1, \ldots, k-1$) and $[v^{k-1}, v^0]$ are edges of K, and that $\operatorname{conv} f(V) \subset \operatorname{int} c_{[f(v^0),\ldots,f(v^{k-1})]}$.

Then $f(K)$ is the convex polygon $\operatorname{conv} f(V)$.

Proof. Suppose condition (I) holds. Then by Theorem 1 and the Mapping Theorem, $\operatorname{conv} f(V) \subset \operatorname{int} c_{[f(v^0),\ldots,f(v^{k-1})]} \subset f(K) \subset \operatorname{conv} f(V)$ and the result follows. □

If the closed loop $c = c_{[f(v^0),\dots,f(v^{k-1})]}$ crosses itself many times (i.e., there are many pairs of distinct $x, y \in \partial D$ that satisfy $c(x) = c(y)$), then it may not be easy to check whether $\operatorname{conv} f(V)$ is a subset of $\operatorname{int} c_{[f(v^0),\dots,f(v^{k-1})]}$ or not. However, if c traverses the boundary of $\operatorname{conv} f(V)$ exactly once and never crosses itself, then clearly we have $\operatorname{conv} f(V) \subset \operatorname{int} c_{[f(v^0),\dots,f(v^{k-1})]}$. This results in the following weakened version of Corollary 1.

Corollary 2. *Suppose*

> *(II) there exist vertices $v^0, \dots, v^{k-1}$ $(k \geq 2)$ of K such that $[v^{i-1}, v^i]$ $(i = 1, \dots, k-1)$ and $[v^{k-1}, v^0]$ are edges of K, and $f(v^0), \dots, f(v^{k-1})$ are all the successive vertices of* $\operatorname{conv} f(V)$ *in order.*

Then $f(K)$ is the convex polygon $\operatorname{conv} f(V)$.

Suppose now K' is a simply connected subset of K containing the vertices $v^0, \dots, v^{k-1}$ in the statement of Theorem 1 or in conditions (I) or (II). Then, by Lemma 1, $\operatorname{int} c_{[f(v^0),\dots,f(v^{k-1})]} \subset f(K')$. In particular, if K' is a union of 2-dimensional faces of K and is simply connected and contains the vertices $v^0, \dots, v^{k-1}$ in conditions (I) or (II), then $\operatorname{conv} f(V) = f(K) = f(K')$, i.e., the image of K' fills up the polygon $\operatorname{conv} f(V)$.

In [9], it is given that if

> (III) $\frac{\partial f}{\partial \gamma_i}(\gamma) \neq 0$ and $\arg\big(\pm \frac{\partial f}{\partial \gamma_k}(\gamma)\big) \neq \arg\big(\frac{\partial f}{\partial \gamma_\ell}(\gamma)\big)$ for all $i = 1, \dots, m$, $1 \leq k < \ell \leq m$, $\gamma \in K$,

then $f(K)$ is a convex polygon. In [1] it is shown (in an equivalent form) that if

> (IV) $\frac{\partial f}{\partial \gamma_i}(\gamma) \neq 0$ and the imaginary parts of $\frac{\partial f}{\partial \gamma_k}(\gamma) / \frac{\partial f}{\partial \gamma_\ell}(\gamma)$ are either all positive or all negative for all $i = 1, \dots, m$, $1 \leq k < \ell \leq m$, $\gamma \in V$,

then $f(K)$ is a convex polygon. Simple continuity argument shows that condition (III) implies condition (IV), and hence the result in [1] implies that in [9]. Also, it is proved in [1] that if condition (IV) holds then, after possible reordering of the coordinates in γ, the points $f(u^0), \dots, f(u^m)$, $f(w^1), \dots, f(w^{m-1})$ are the successive vertices of $\operatorname{conv} f(V)$ traversed in counter-clockwise direction. Here u^i (resp., w^i) $\in V$ $(i = 0, \dots, m)$ is such that all but the last i of its coordinates are 0 (resp., 1). Observe that $[u^{i-1}, u^i]$ $(i = 1, \dots, m)$, $[u^m, w^1]$, $[w^{i-1}, w^i]$ $(i = 2, \dots, m-1)$, and $[w^{m-1}, u^0]$ are edges of K. Hence condition (IV) implies condition (II). Therefore the sufficient conditions given in [9] and [1] for $f(K)$ to be a convex polygon follow from our Corollary 2.

The following example shows that conditions (I) and (II) are not necessary for $f(K)$ to be a convex polygon.

Example. Let $f : \mathbb{R}^4 \to \mathbb{C}$ be the unique multiaffine function defined by

$$f(0,0,0,0) = f(0,1,1,1) = 0;$$

$$f(0,0,0,1) = f(0,0,1,0) = f(0,1,0,0) = f(1,0,0,0) = f(1,1,1,1) = 2j;$$

$$f(0,0,1,1) = f(0,1,0,1) = f(0,1,1,0) = 2;$$

$$f(1,0,0,1) = f(1,0,1,0) = f(1,1,0,0) = 1 + j;$$

$$f(1,0,1,1) = f(1,1,0,1) = f(1,1,1,0) = j.$$

Then one can show that $f(K)$ is the convex polygon $\operatorname{conv} f(V) = \operatorname{conv}\{0, 2, 2j\}$ by plotting the image of line segments in (some 2-dimensional faces of) K that are parallel to the coordinate axes (Figure 2). However, direct checking shows that conditions (I) and (II) are not satisfied. Thus the sufficient conditions given in Corollary 1 and Corollary 2 are not necessary for $f(K)$ to be a convex polygon.

While conditions (I) and (II) are in general not necessary for $f(K)$ to be convex, under a certain assumption on f ((*) below), we have the following simple necessary and sufficient condition.

Theorem 2. *Suppose*

> $(*)$ *each vertex of* $\operatorname{conv} f(V)$ *is the image of a unique vertex of* K*, and no other vertex of* K *has its image on an edge of* $\operatorname{conv} f(V)$.

Then $f(K)$ *is a convex polygon (and equals* $\operatorname{conv} f(V)$*) if and only if the edges of* $\operatorname{conv} f(V)$ *are mapped from edges of* K.

Proof. Suppose $(*)$ holds. Then there exist unique vertices $v^0, \ldots, v^{k-1}$ of K such that $f(v^0), \ldots, f(v^{k-1})$ are successively the vertices of $\operatorname{conv} f(V)$ in order. For simplicity of argument and notation we write $v^k = v^0$. If $f(K)$ is convex then, by Proposition 1 and $(*)$, each x on an edge $[f(v^{i-1}), f(v^i)]$ $(i = 1, \ldots, k)$ must be covered by the image of $[v^{i-1}, v^i]$. Hence each edge of $\operatorname{conv} f(V)$ is mapped from an edge of K. Conversely, if each edge $[f(v^{i-1}), f(v^i)]$ $(i = 1, \ldots, k)$ of $\operatorname{conv} f(V)$ is an image of an edge of K then, by $(*)$, that edge of K must be $[v^{i-1}, v^i]$. Now that $[v^{i-1}, v^i]$ $(i = 1 \ldots, m)$ are edges of K, it follows from Corollary 2 that $f(K)$ is the convex polygon $\operatorname{conv} f(V)$. $\square$

Notice that the necessary and sufficient condition in Theorem 2 is exactly that conjectured by Hollot and Xu [6] (without assumption $(*)$), and is similar to that in Proposition 2. Accordingly, what Theorem 2 says is that if f satisfies $(*)$ and if the edges of $\operatorname{conv} f(V)$ are mapped from edges of K, then $f(K)$ is simply connected.

The complexity in checking condition (IV) directly is of order $O(m^2 2^m)$. Checking condition (III) is difficult because the set K is infinite. On the other hand, there are efficient algorithms (e.g., see [7], Section 2.3) of complexity order $O(m2^m)$ for finding the vertices of the polygon $\operatorname{conv} f(V)$. Thus condition (II) in Corollary 2 and the necessary and sufficient condition in Theorem 2 (i.e., that the edges of $\operatorname{conv} f(V)$ are mapped from edges of K) can be checked in $O(m2^m)$ time (or even in $O\big(m(1+\frac{2^m}{p})\big)$ time if p processors are available; see [7], Section 1.5 and Theorem 6.2).

We conclude by showing that condition $(*)$ is generically satisfied when the set of multiaffine functions is equipped with some natural topologies. Recall that a multiaffine $f : \mathbb{R}^m \to \mathbb{C}$ is uniquely determined by the family $\{f(v)\}_{v\in V}$ via (3). Another representation of f is by writing $f(\gamma) = \sum_{v\in V} \delta(v)\gamma_v$, where $\delta(v) \in \mathbb{C}$ is the coefficient of the term $\gamma_v \equiv \prod_{i=1}^{m} \gamma_i^{v_i}$ for $v = (v_1, \ldots, v_m) \in V$, so that f is uniquely determined by the family $\{\delta(v)\}_{v\in V}$.

Suppose we order the vertices of K as $v^0, \ldots, v^{2^m-1}$ in such a way that the coordinates of $v^i = (v_1^i, \ldots, v_m^i)$ give the binary representation of i (i.e., $i = \sum_{k=1}^{m} v_k^i 2^{m-k}$), and define a partial ordering $\prec$ on V by $v = (v_1, \ldots, v_m) \prec w = (w_1, \ldots, w_m)$ if and only if $v_i \le w_i$ for all $i = 1, \ldots, m$. Then for all $v \in V$,

$$f(v) = \sum_{w \prec v} \delta(w). \tag{5}$$

Now identify the class of all multiaffine $f : \mathbb{R}^m \to \mathbb{C}$ with $\mathbb{C}^{2^m}$ (with usual topology) by either

$$f \longleftrightarrow \big(f(v^0), \ldots, f(v^{2^m-1})\big) \tag{6}$$

or

$$f \longleftrightarrow \big(\delta(v^0), \ldots, \delta(v^{2^m-1})\big) \tag{7}.$$

From (5), one sees that (6) and (7) are related through the homeomorphism ϕ on $\mathbb{C}^{2^m}$ defined by

$$\phi\big(\delta(v^0), \ldots, \delta(v^{2^m-1})\big) = \Big(\sum_{w \prec v^0} \delta(w), \ldots, \sum_{w \prec v^{2^m-1}} \delta(w)\Big).$$

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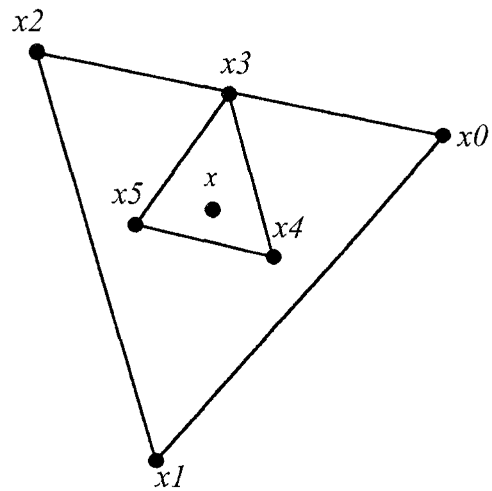

Figure 1

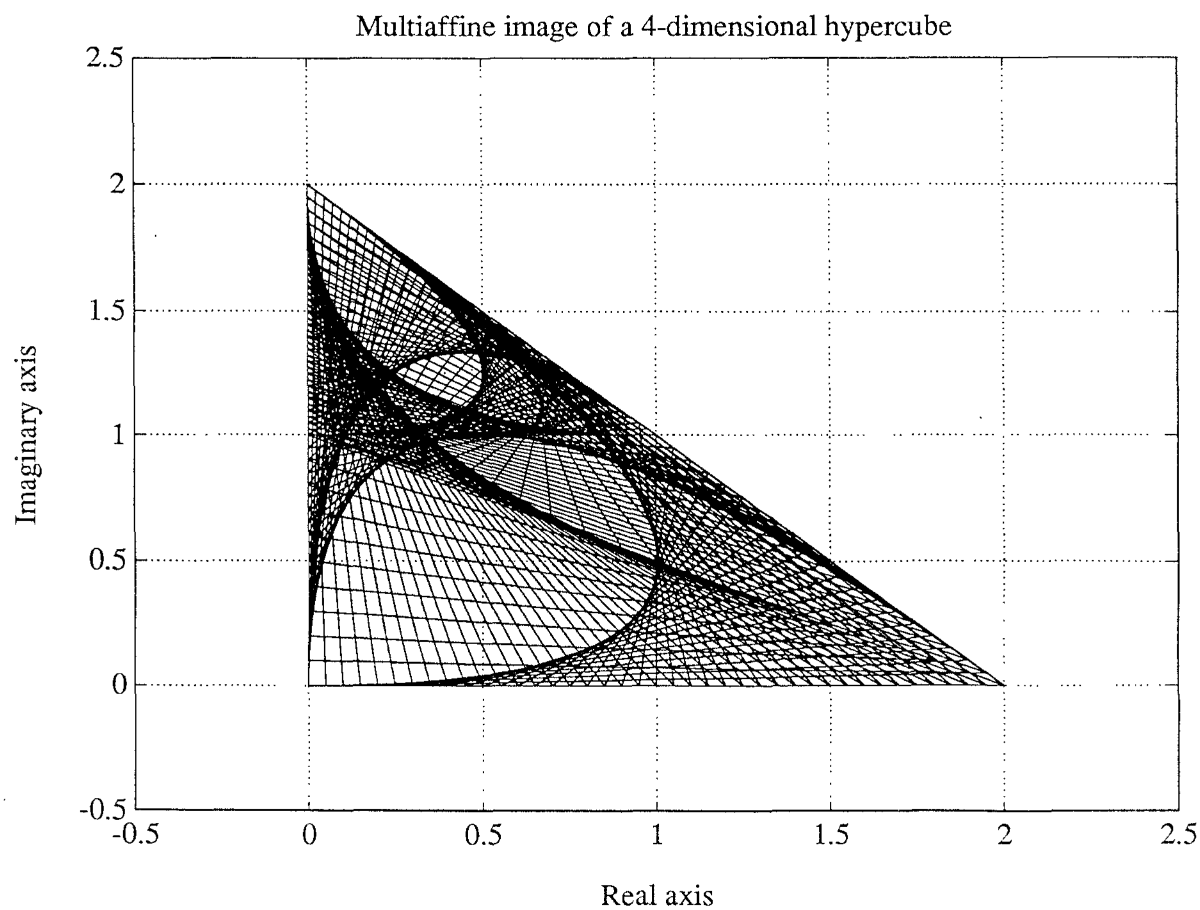

Figure 2