ABSTRACT

Title of Dissertation: FOCK–GONCHAROV COORDINATES FOR SEMISIMPLE LIE GROUPS

S. Gilles
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Dissertation Directed by: Professor Christian Zickert
Department of Mathematics

Fock and Goncharov [FG06b] introduced cluster ensembles, providing a framework for coordinates on varieties of surface representations into Lie groups, as well as a complete construction for groups of type A_n. Later, Zickert [Zic19], Le [Le16b], [Le16a], and Ip [Ip18] described, using differing methods, how to apply this framework for other Lie group types. Zickert also showed that this framework applies to triangulated 3-manifolds. We present a complete, general construction, based on work of Fomin and Zelevinsky. In particular, we complete the picture for the remaining cases: Lie groups of types F_4, E_6, E_7, and E_8.

FOCK–GONCHAROV COORDINATES FOR SEMISIMPLE LIE GROUPS

by

S. Gilles

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Advisory Committee:
Professor Christian Zickert, Chair/Advisor
Professor Jeffrey Adams
Professor William Gasarch
Professor William Goldman
Professor Richard Wentworth

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Finally, I would like to thank S. Kane, who taught me the meaning of work, and whom I am unable to repay.

Soli Deo Gloria.

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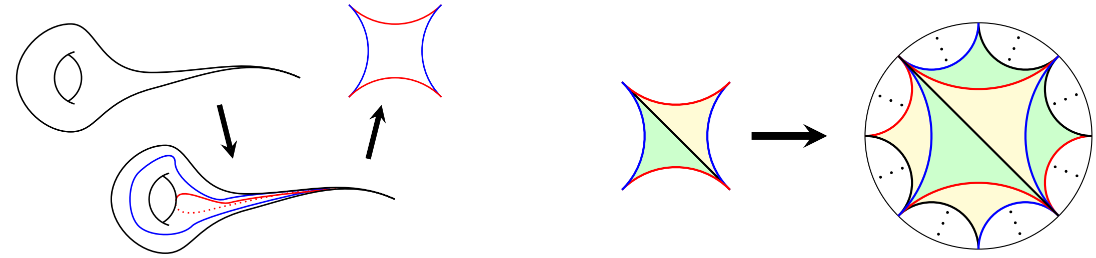
Chapter 1: Introduction

The program of Fock–Goncharov, starting with [FG06b], aims to describe representation spaces of hyperbolic surfaces into Lie groups by moduli spaces defined by polynomial equations. These moduli spaces carry a positive structure (see Definition 4.3.4), and in the case of the Lie group $\mathrm{PSL}_2(\mathbb{R})$, the associated positive spaces can be identified with Teichmüller space and decorated Teichmüller space. For more complicated Lie groups, these positive spaces give (decorated) higher Teichmüller spaces.

Our result regards an additional structure, a cluster ensemble structure (see Section 2.7), which allows manipulating these moduli spaces efficiently via quivers.



(A) Cutting Σ into a fundamental domain.

(B) Embedding a fundamental domain for Σ into the disk model for hyperbolic space.

Figure 1.1: Putting a hyperbolic structure on Σ.

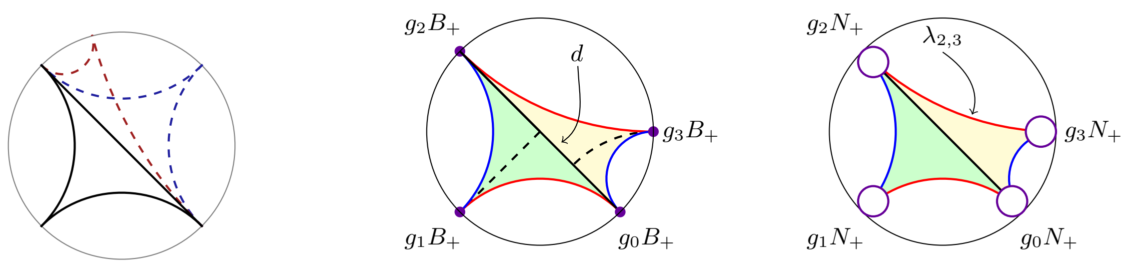
1.1 Classical Teichmüller space

We start by describing the Fock–Goncharov program for the case of Lie groups of type A_1. Let Σ be a surface that admits ideal triangulation, for example the once-punctured torus $\Sigma = \Sigma_{1,1}$. The Fock–Goncharov moduli spaces will describe (decorated) Teichmüller space, so fix a hyperbolic structure on Σ. This can be described by an embedding of Σ's fundamental domain in $\mathbb{H}^2$, as in Figure 1.1.

We now describe the two moduli spaces with seven key points.

1. The structure is determined by ideal points.

This follows from our ideal triangulation: all vertices are on $\partial\mathbb{H}^2$, and all edges between them are unique geodesics. See Figure 1.2A.



(A) Two different hyperbolic structures on Σ.

(B) Defining a representation by B_+-cosets.

(C) Defining a representation by N_+-cosets.

Figure 1.2: Representations of Σ by cosets.

2. We can identify ideal points with cosets of isometries in $\mathrm{Isom}^+(\mathbb{H}^2)$ *that fix them.*

The specific type of isometry determines which moduli space we create.

- We can consider B_+, a maximal borel subgroup, which fixes a point on $\partial\mathbb{H}^2$. Thus different cosets gB_+ fix, and can be identified with, different points on $\partial\mathbb{H}^2$. See Figure 1.2B.

- Replacing points with horocycles, we can consider N_+, a maximal unipotent subgroup, which fixes a horocycle in $\mathbb{H}^2$. See Figure 1.2C.

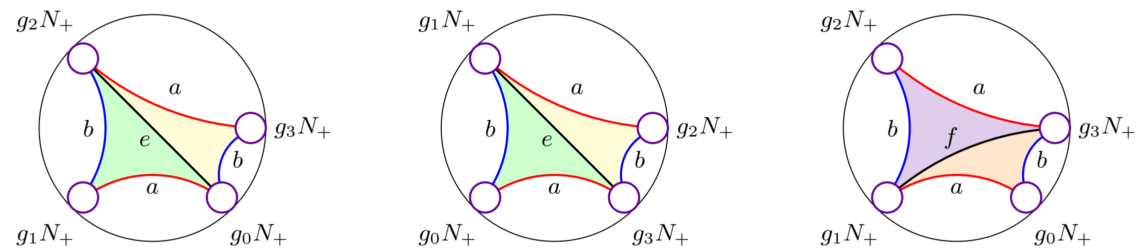
3. Choosing an ordered triangulation attaches a non-degenerate ordered triple of cosets to each triangle.

The non-degeneracy condition requires that the cosets be distinct, so that the triangle has three well-defined edges. Describing how the coordinates of Fock–Goncharov's moduli spaces change under alternate choices for the triangulation and the ordering is one of our primary concerns.

4. Coordinates can be assigned to each triple (or pair of triples) of cosets.

- If we consider B_+-cosets, then the coordinates we attach to each edge resemble Thurston's shear coordinates (as described in e.g. [Bon96]) along that identified edge, as the distance d in Figure 1.2B. Some of these coordinates need two triples to define. We will not focus on these in this introduction.

- If we consider N_+-cosets, then the portion of each geodesic between horocycles has finite length. These are equivalent to Penner's λ-lengths in his parametrization of decorated Teichmüller space of [Pen87]. We associate to each triangle's edge the λ-length of its truncated geodesic as a coordinate. See Figure 1.2C.



(A) Coordinate assignments for one ordered triangulation.

(B) Figure 1.3A under reordering.

(C) Figure 1.3A under retriangulation.

Figure 1.3: Coordinates by N_+-cosets under reordering and retriangulation.

In our example, we label the edges a, b, and e as in Figure 1.3A (identified edges necessarily have the same λ-length). Using the $(0 \to 1, 1 \to 2, 2 \to 0)$-ordering on a triangle's edges, the coordinates we obtain are (a, b, e) on the left and (e, a, b) on the right.

5. We must describe how the coordinates change under (oriented) reordering and retriangulation.

In this A_1 case, reordering has a trivial effect on the coordinates. If we rename g_0, g_1, g_2, g_3, but do not change their values, we merely rearrange the coordinates following the new cyclic ordering on the triangle's edges. The coordinates defined by Figure 1.3B are (b, e, a) on the left and (a, b, e) on the right.

The retriangulation has a more interesting effect, as shown in Figure 1.3C. The coordinates we obtain are (b, a, f) for the top and (a, f, b) for the bottom, but we need to describe the relationship of the new λ-length f to the λ-lengths a, b, e. This is given by the classic Ptolemy's Theorem regarding lengths and diagonals of circumscribed quadrilaterals. In Euclidean space this is given by Figure 1.4A, and in hyperbolic space by Figure 1.4B. For our example, taking identifications into account,

$$ef = a^2 + b^2.$$

(A) Ptolemy's Theorem for lengths in Euclidean space.

(B) Ptolemy's Theorem for λ-lengths in hyperbolic space.

Figure 1.4: Ptolemy's theorem: $ac + bd = ef$

6. Hyperbolic structures are those for which all coordinates are positive.

The coordinates a, b, e, f with the relation $ef = a^2 + b^2$ describe an algebraic variety, but not every point corresponds to a hyperbolic structure. Those which do are exactly the points with positive coordinates, and thus positive geodesic lengths. This positivity need only be checked for one collection of coordinates: if a, b, and e are all positive, then f will be as well.

7. The representation $\rho : \pi_1(\Sigma) \to G$ can be reconstructed.

Via the coordinates we have mentioned, the holonomy $\rho(\gamma)$ corresponding to this hyperbolic structure for $\gamma \in \pi_1(\Sigma)$ can be computed. See Section 4.3 for greater detail.

The moduli space we have described using N_+-cosets is $\mathcal{A}_{\mathrm{SL}_2(\mathbb{R}),\Sigma}$, and points with positive coordinates correspond to points in Penner's decorated Teichmüller space for Σ. The moduli space derived from considering B_+-cosets is $\mathcal{X}_{\mathrm{PGL}_2(\mathbb{R}),\Sigma}$, and positive points correspond to points in Teichmüller space for Σ.

1.2 Cluster ensemble structures

In the above program, the retriangulation identity was given by Ptolemy's Theorem. Another structure that encodes the identity is a **cluster ensemble**. We defer a detailed description to Section 2.7, but for now we only need that a cluster ensemble consists of a **quiver** (a directed graph with a skew-symmetrizable adjacency matrix) and a pair of coordinate structures ($\mathcal{A}$- and $\mathcal{X}$-coordinates), and that the shape of the quiver dictates how the coordinates change under **mutation** (a certain local alteration of the quiver).

The quiver of the cluster ensemble which realizes $ac + bd = ef$ for the $\mathcal{A}$-coordinates is given (along with the effect of the only relevant mutation) in Figure 1.5. The quivers are inscribed in two quadrilaterals with different diagonals.

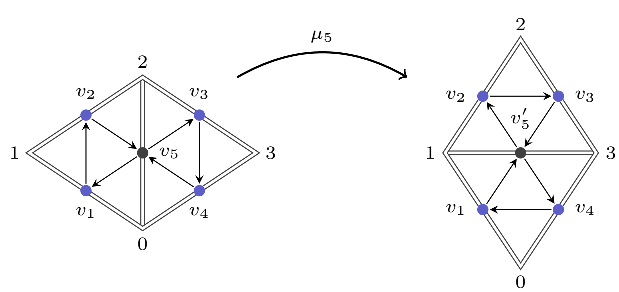


Figure 1.5: Quiver for type A_1. The $\mathcal{A}$-coordinate at v_k is a_k. The relation on $\mathcal{A}$-coordinates for mutating at v_5 is $a_1a_3 + a_2a_4 = a_5a_5'$.

For other surfaces Σ, larger collections of triangles can be glued together. Retriangulations (and reorderings) can be computed as sequences of mutations, and the coordinate changes are given by collections of low-degree polynomial relations.

1.3 Higher Teichmüller spaces

We view higher Teichmüller spaces as collections of representations

$$\{\, \rho : \pi_1(\Sigma) \to G \text{ discrete and faithful} \,\} / G,$$

where the standard Teichmüller space case is given by $G = \mathrm{Isom}^+(\mathbb{H}^2) = \mathrm{PSL}_2(\mathbb{R})$. In order to apply the Fock–Goncharov program, we need the following conditions.

- The surface Σ must admit an ideal triangulation, so we demand that it be hyperbolic with $n \geq 1$ punctures.
- The group G must be a split semisimple algebraic group over $\mathbb{Q}$; for technical reasons the moduli space $\mathcal{A}_{G,\Sigma}$ is defined when G is simply-connected, and $\mathcal{X}_{G,\Sigma}$ is defined when G is centerless. However, the field underlying G is not critical.

As another technical point, when G is simply connected we will parametrize boundary-unipotent representations, and when G is centerless the representations will be boundary-borel (see Definition 4.1.2).

We now replay the program to create moduli spaces $\mathcal{A}_{G,\Sigma}$ and $\mathcal{X}_{G,\Sigma}$ for higher rank G.

1. The structure is determined by ideal points.

By restrictions on Σ, the representation is still defined by ideal points of a fundamental domain.

2. We can identify ideal points with cosets of isometries in $\mathrm{Isom}^+(\mathbb{H}^2)$ *that define them*

We replace $\mathrm{Isom}^+(\mathbb{H}^2)$ with general G. By our restrictions on G, subgroups N_+ and B_+ are still available.

3. Choosing an ordered triangulation attaches a non-degenerate ordered triple of cosets to each triangle.

Non-degenerate cosets are replaced by **sufficiently generic configuration spaces** (see Definition 4.1.1). These ensure that a well-defined element of H, the maximal torus of G, corresponding to translations along geodesics in the $\mathrm{PSL}_2(\mathbb{R})$ case, can be attached to each edge of each triangle.

4. Coordinates can be assigned to each triple (or pairs of triples) of cosets.

To assign coordinates to configuration spaces, Fock–Goncharov use **generalized minors** (see Definition 2.4.6). In the case of GL_n, these correspond to shuffling rows and columns according to two permutations, then taking the upper $i \times i$ minor. In the more general case, two words in the Weyl group of G are invoked and a coordinate associated to the i^{th} fundamental weight is used.

By work of Lusztig, configuration spaces carry a positive structure via **factorization coordinates** (see Definition 2.5.1). By work of Fomin–Zelevinsky, generalized minors inherit this positive structure.

5. We must describe how the coordinates change under (oriented) reordering and retriangulation

A major result of the Fock–Goncharov program is that reordering of triangles (we restrict to orientation-preserving reorderings, which we refer to as **rotation**) and retriangulation of quadrilaterals (which we refer to as a **flip** of the diagonal) preserve this positive structure. All general retriangulations and reorderings can be obtained this way.

6. Hyperbolic structures are those for which all coordinates are positive.

Instead of hyperbolic structures, we are interested in points in higher Teichmüller spaces. Since the general moduli spaces carry positive structures, the sets of positive points $\mathcal{A}^+_{G,\Sigma}$ and $\mathcal{X}^+_{G,\Sigma}$ are well-defined when G is over $\mathbb{R}$. These do, in fact, correspond to higher Teichmüller spaces or decorations of them (see Section 4.3).

7. The representation $\rho : \pi_1(\Sigma) \to G$ can be reconstructed.

The representation $\rho(\gamma)$ can still be computed by $\mathcal{A}$- or $\mathcal{X}$-coordinates (again, see Section 4.3). Here our restrictions on G being simply connected or centerless are necessary.

So the Fock–Goncharov program holds beyond $\mathrm{PSL}_2(\mathbb{R})$. We now address the following question:

The positive structure on coordinates is preserved under retriangulation and reorientation, but how are those two operations realized on coordinates?

In the A_1 case, the cluster ensemble of Figure 1.5 provides the answer. In [FG06b], Fock–Goncharov produced cluster ensembles for all types A_n. The quivers appear as triangular lattices, see Chapter 6 for an overview. Fock–Goncharov also provided descriptions for coordinate changes under the two key operations as sequences of mutations. The **rotation** describes reordering of a triangle, and the **flip** describes retriangulation of a quadrilateral.

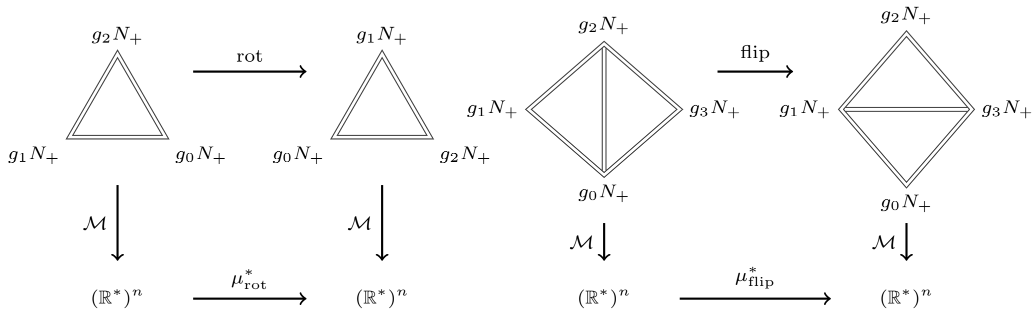
Fock–Goncharov predicted that $\mathcal{A}_{G,\Sigma}$ and $\mathcal{X}_{G,\Sigma}$ would carry cluster ensemble structures as well. To realize those structures, we need, we need

- A way to assign coordinates of $\mathrm{Conf}_4^*(G/K)$ to a triangle. These coordinates will be encoded in a cluster ensemble. We will call the coordinate assignment map $\mathcal{M}$. When the group G is over $\mathbb{R}$, collections of coordinates will be $(\mathbb{R}^*)^n$. Seeds with all coordinates positive will be the **positive points**.

- A way to realize (orientable) symmetries of a triangle on those coordinates (the **rotation**, see Figure 1.6a). The bottom map will be realized by a quiver mutation which we call μ_{rot}.

- A way to realize changes of triangulation on those coordinates (the **flip**, see Figure 1.6b). The bottom map will be realized by a quiver mutation which we call μ_{flip}.

- The maps μ_{rot} and μ_{flip} should preserve the positive structure. That is, if a seed has positive coordinates, applying the rotation or the flip should not change that. Using generalized minors for coordinates is a way to preserve positivity.



(A) The rotation. (B) The flip.

Figure 1.6: Retriangulation and reordering (for a group over $\mathbb{R}$).

1.4 Results

Our main result is an explicit construction for cluster ensembles described above for all semisimple G, including the rotation and the flip. In other words, we present a constructive result of the following theorem.

Theorem 1.4.1. *For a split semisimple simply-connected (or centerless) algebraic group G over $\mathbb{Q}$, there exists a quiver Q, quiver mutations μ_{rot} and μ_{flip}, and a coordinate map $\mathcal{M}$.*

The map $\mathcal{M}$ associates coordinates of a flag configuration to the coordinates on the cluster ensemble for Q, the quiver mutation μ_{rot} describes how the coordinates change under rotation of an ordered triangle and μ_{flip} describes how the coordinates change under retriangulation.

We call this collection $(Q, \mu_{\text{rot}}, \mu_{\text{flip}}, \mathcal{M})$ a **Fock–Goncharov coordinate structure**. The construction algorithm is presented in Chapter 5. First, we use the algorithm of [FZ99, Section 2], using a specific choice of w_0 (the longest word in the Weyl group) by Fact 2.3.2. This produces a quiver whose $\mathcal{A}$-coordinates generate the coordinate ring for $B_- = H \times N_-$. We call this quiver Q_0.

The quiver Q_0 has the same mutable portion that we need for Q. However, the non-mutable portion isn't complete, so Q_0 does not have **triangulation-compatible symmetry** (see Definition 4.2.8). We therefore can apply the rotation μ_{rot} to Q_0, and define Q to be the smallest quiver with triangulation-compatible symmetry containing Q_0.

The quiver Q and the map $\mathcal{M}$ allow constructing representation varieties easily given a triangulation of a surface: a copy of Q is inscribed on each triangle and vertices along edges are identified. The maps μ_{rot} and μ_{flip} describe how the coordinate functions of the variety respond to retriangulation, ensuring that the variety itself is independent of triangulation choices.

Remark 1.4.2. The existence of cluster ensemble structures was shown non-constructively, and by different methods, in [GS18].

This theorem can also be used to compute representation varieties for 3-manifolds, following work of Zickert. In this case, quivers are drawn on each face of each ideal tetrahedron in the triangulation. The map μ_{flip} describes the relation between coordinates on the four triangular faces of each tetrahedron. The variety constructed is not, however, independent of triangulation. If the triangulation is sufficiently fine, however, the variety detects all representations.

1.5 Historical context

Fock and Goncharov introduced cluster ensembles in [FG06b] and [FG09b], employing the cluster algebra work of Fomin–Zelevinsky [FZ02], [FZ03], [BFZ05] as well as Lusztig's study of positivity [Lus94], [Lus97]. The application was laid out for a split semisimple simply-connected Lie group over $\mathbb{R}$, but explicit constructions were only known for type A_n. Since then, there has been ongoing work.

- In [Zic19], Zickert produced examples of cluster ensembles and mutations for types A_2, B_2, C_2, and G_2, directly employing Lusztig's positive maps to explicitly construct varieties. Our work addresses Conjecture 2.7, which predicts the existence of quivers, rotation and flip mutations, and coordinate maps for semisimple G.

  This work also expanded the use of Fock–Goncharov coordinates to representations of ideally-triangulated hyperbolic 3-manifolds, building on [GTZ15].

- With [Le16b] and [Le16a], Le described constructions for quivers of types A_n, B_n, C_n, and D_n using tensor invariants and webs. Our work addresses Conjecture 3.12 (that the cluster algebra for $\mathrm{Conf}^*_m(G/N_+)$ is invariant under retriangulation and reordering) for the orientation-preserving case, for our choice of presentation of w_0.

- Using representations of quivers, Fei constructed quivers and mutations for many types in [Fei16], though associated to a slightly different flag variety.

- In [Ip18], Ip produced similar “basic quivers”, though not presenting mutation sequences in the general case.

- Goncharov–Shen continued work on invariants of cluster ensembles with [GS18], providing existence proofs.

There were several obstacles to extending these results to our proof of Theorem 1.4.1. First, the cluster ensembles for type A_n have trivial triangular symmetry. However, direct dimension counting of the coordinate ring shows that this is not possible for general G.

Second, the algorithm of [BFZ05, Section 2] to produce a quiver carrying the coordinate structure of B_- (which is a significant step) is well-known. However, this algorithm depends on a particular choice of presentation for w_0 in the Weyl group of G. It is not immediately obvious how to choose w_0 for each group, or if this choice should matter.

Third, the same non-triviality of triangular symmetry for Q extends to non-triviality of the coordinate assignment map $\mathcal{M}$. In other words, fully three sides of each square in Figure 1.6 are trivial in the A_n case, and therefore give few clues as to the general case.

Finally, the action of w_0 on a simple root α_i is not necessarily $w_0(\alpha_i) = -\alpha_i$. This creates various complications.

The following observations are the key to our result.

- A_n is the exception, not the base case. Specifically, for all G except type A_{2n}, there exists a very regular presentation of w_0 in terms of Coxeter elements.

- This regularity causes the quiver with the coordinate ring of B_- to be be laid out so that [FZ99, Theorem 1.17] and [YZ08, Theorem 1.5] apply, and these identities hint at certain sequences of mutations.

Chapter 2: Ingredients

Here we review some key components of our construction. These include generalized minors (see Section 2.4), quivers (see Section 2.6), and cluster ensembles (see Section 2.7).

2.1 Root spaces and Weyl groups

We begin with some fundamentals of Lie groups, referring to e.g. [Kna96, Chapter II] or [Bou02] for more detail. We use the language of Lie groups over $\mathbb{C}$, with straightforward generalization to algebraic groups over other fields.

Definition 2.1.1. For $\mathfrak{g}$ a semisimple Lie algebra over $\mathbb{C}$, fix a Cartan subalgebra $\mathfrak{h}$. A **root** is some simultaneous eigenvalue $\alpha \in \mathfrak{h}^*$ of all $\mathrm{ad}_H : X \mapsto [H, X]$ for $H \in \mathfrak{h}$. That is, there is some $X \in \mathfrak{h}$, and for all H we have $[H, X] = \alpha(H)X$. X is the simultaneous eigenvector, and α is the simultaneous eigenvalue, the root. Δ is the set of all roots, the **root system**.

Every root $\alpha \in \mathfrak{h}^*$ corresponds to some $H_\alpha \in \mathfrak{h}$ such that for all $H \in \mathfrak{h}$, $B(H, H_\alpha) = \alpha(H)$ (with B the Killing form). Let $\mathfrak{h}_0$ be the $\mathbb{R}$-linear span of all H_α, and $\mathfrak{h}_0^* \subset \mathfrak{h}^*$ the dual.

There is also a corresponding **weight space** $\mathfrak{g}_\alpha$, defined as

$$\mathfrak{g}_\alpha = \{\, X \in \mathfrak{g} : \text{for all } H \in \mathfrak{h}, \quad [H, X] = \alpha(H)X \,\} \,.$$

With this notation, $\mathfrak{h} = \mathfrak{g}_0$, and $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$.

Definition 2.1.2. For a root system Δ relative to a Cartan subalgebra $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ over $\mathbb{C}$, the **Weyl group** is the subgroup of $\mathrm{GL}(\mathfrak{h}_0^*)$ generated by (where $\langle \cdot, \cdot \rangle$ is the usual inner product, viewing $\mathfrak{g}$ as a complex vector space)

$$\left\langle s_\alpha : \varphi \mapsto \varphi - 2\frac{\langle \alpha, \varphi \rangle}{\langle \alpha, \alpha \rangle}\alpha : \alpha \text{ a root for } \Delta. \right\rangle$$

Each s_α may be thought of as a reflection through the hyperplane perpendicular to α.

A sequence $(i_1, i_2, \ldots, i_m)$ such that $w = s_{\alpha_{i_1}} s_{\alpha_{i_2}} \cdots s_{\alpha_{i_r}}$ is a **presentation** for $w \in W$. The elements $\{\, s_{\alpha_i} \,\}$ follow the braid relations, so presentations are often not unique. If a presentation for w has the shortest possible length, it is **reduced**.

Definition 2.1.3. For a root system Δ and $\mathfrak{h}_0^*$ as above, arbitrarily choose some maximal subset Δ_+ closed under addition and scalar multiplication by positive reals. Such a choice gives **positive roots** for Δ.

A positive root $\alpha \in h^*$ is **simple** if it is positive and cannot be expressed as a sum of other positive roots with positive coefficients. The span $\langle s_\alpha : \alpha \text{ a simple root} \rangle$ generates the Weyl group.

Definition 2.1.4. Each simple root α_i has an associated **fundamental weight** ω_i in $\mathfrak{h}_0^*$, defined by

$$2\frac{\langle \omega_i, \alpha_j \rangle}{\langle \alpha, \alpha \rangle} = \delta_{ij}.$$

Fact 2.1.5. *The Weyl group is finite. In particular, there is a **longest word** w_0. That is, w_0 is an element of W such that for any s_α, the word $w_0 s_\alpha$ has a shorter presentation than the shortest presentation of w_0.*

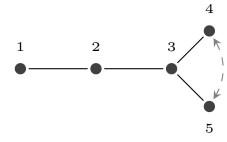
The element w_0 induces an action $\alpha \mapsto -w_0(\alpha)$ on the simple roots.

Definition 2.1.6. Let σ_G be the permutation such that $-w_0(\alpha_i) = \alpha_{\sigma_G(i)}$. This permutation is of order 1 or 2, and is often trivial. Where convenient, we will label $\sigma_G(i)$ as i^*.

In an abuse of notation, we will view σ_G as acting on H as the unique automorphism defined by (looking ahead, χ_i^* is from Definition 2.2.1)

$$\sigma_G(\chi_i^*(t)) = \chi_{\sigma_G(i)}^*(t).$$

Remark 2.1.7. The permutation σ_G may be realized as a graph automorphism of the Dynkin diagram associated to G. For example, for G of type D_5, the involution σ_G acts on the simple roots as shown:



2.2 Unipotent subgroups

Definition 2.2.1. For a Lie group G (the Lie algebra of which is $\mathfrak{g}$), a Cartan subalgebra $\mathfrak{h}$ and a root system Δ with a choice of positive roots Δ_+, there is a **root space decomposition** defining $\mathfrak{n}_\pm$ by

$$\mathfrak{g} = \overbrace{\bigoplus_{\alpha\in\Delta_+} \mathfrak{g}_{-\alpha}}^{\mathfrak{n}_-} \oplus \mathfrak{h} \oplus \overbrace{\bigoplus_{\alpha\in\Delta_+} \mathfrak{g}_{\alpha}}^{\mathfrak{n}_+}.$$

The Lie subgroups of G with these Lie algebras are, respectively, N_-, H, and N_+. The $N_\pm$ subgroups are **maximal unipotent** subgroups of G. H is a **maximal torus**. We also have the **borel subgroups** $B_\pm = HN_\pm$.

We also fix standard generators $e_i \in \mathfrak{g}_{\alpha_i}$, $f_i \in \mathfrak{g}_{-\alpha_i}$, and $h_i \in \mathfrak{h}$ (with $i \in \{1, 2, \ldots, \operatorname{rank} G\}$, so that we may write

$$x_i(t) = \exp(te_i) \in N_+, \qquad y_i(t) = \exp(tf_i) \in N_-, \qquad \chi_i^*(t) = \exp(th_i) \in H.$$

2.3 Coxeter elements

Definition 2.3.1. For a root system generated by simple roots $\Delta = \langle\{\alpha_1, \alpha_2, \ldots, \alpha_r\}\rangle$, any element $c = s_{\alpha_1}s_{\alpha_2}\cdots s_{\alpha_r}$ is a **Coxeter element**. The ordering of the roots is irrelevant: any such product of all simple roots is a Coxeter element. All Coxeter elements have the same order, which is the **Coxeter number**, denoted h.

Fact 2.3.2. *For a Weyl group W with Coxeter element c and even Coxeter number*

h, $c^{h/2} = w_0$. That is, if $c = s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_r}$, a presentation for w_0 is

$$\mathbf{i} = \{ \overbrace{1, 2, \ldots, r}^{1}, \quad \overbrace{1, 2, \ldots, r}^{2}, \quad \ldots, \quad \overbrace{1, 2, \ldots, r}^{h/2} \}.$$

This is standard, see e.g. [Bou02, Chapter VI, § 1.11].

2.4 Generalized minors

We review generalized minors. These are the extension of flag minors on GL_n of [BFZ96], [BZ97] to arbitrary semisimple algebraic groups. This generalization is described fully in [FZ99], to which we refer for more details.

Definition 2.4.1. For G a Lie group over $\mathbb{C}$ admiting H and $N_\pm$, $G_0 = N_- H N_+$. For $x \in G_0$, the **Gaussian decomposition** of x into these components is $x = [x]_-[x]_0[x]_+$, with $[x]_0 \in H$ and $[x]_\pm \in N_\pm$. This decomposition is necessarily unique.

Definition 2.4.2. For G a Lie group over $\mathbb{C}$ admiting H as above, we may identify the Weyl group W with $N_G(H)/H$, the quotient of the normalizer of H in G by H. We denote

$$\overline{s_i} = x_i(-1)y_i(1)x_i(-1) \qquad \overline{\overline{s_i}} = x_i(1)y_i(-1)x_i(1).$$

These satisfy the braid relations, and whenever length $uv =$ length $u +$ length v, we have

$$\overline{uv} = \overline{u} \cdot \overline{v}, \qquad \overline{\overline{uv}} = \overline{\overline{u}} \cdot \overline{\overline{v}}.$$

Remark 2.4.3. If G is semisimple, then the standard choice of bases $\{ e_i \}$, $\{ h_i \}$, and $\{ f_i \}$ for $\mathfrak{n}_-$, $\mathfrak{h}$, and $\mathfrak{n}_+$ give rise, via exp, to one-parameter subgroups in N_-, H, N_+ respectively, and the identification of W with $N_G(H)/H$ identifies w_0 with $\overline{w_0}$. Acting by conjugation on G, this element switches each pair of subgroups $\exp(\mathbb{R}e_i)$, $\exp(\mathbb{R}f_i)$. Thus $\overline{w_0}$ switches N_- and N_+.

Remark 2.4.4. The element $\overline{w_0}^2$ arises frequently. It shall be denoted s_G, and is in the center of G, having order either 1 or 2.

Definition 2.4.5. For G a Lie group over $\mathbb{C}$, and any fundamental weight ω_i, define

$$\widetilde{\Delta}^{\omega_i} : H \mapsto \mathbb{C} \qquad \text{by} \qquad \widetilde{\Delta}^{\omega_i} : h \mapsto \exp(\omega_i \exp^{-1}(h))).$$

Definition 2.4.6. For G a Lie group over $\mathbb{C}$, admitting decomposition as above, v, w elements of the Weyl group W, and ω_i a fundamental weight, the **generalized minor** for the words and fundamental weight $v\omega_i, w\omega_i$ is the regular function $\Delta_{v\omega_i, w\omega_i} : G \mapsto \mathbb{C}$, defined by its restriction on $\overline{\overline{v}}G_0\overline{w^{-1}}$ by

$$\Delta_{v\omega_i, w\omega_i} : g \mapsto \widetilde{\Delta}^{\omega_i}\left(\left[\overline{\overline{v^{-1}}}g\overline{w}\right]_0\right).$$

In the case w is trivial, as it often is in our construction, we denote

$$\Delta^{v\omega_i}(g) = \Delta_{v\omega_i, \omega_i}(g).$$

Example 2.4.7. If $G = \mathrm{SL}(n, \mathbb{C})$ (of type A_{n-1}), we may take H as diagonal matrices, with N_+ and N_- as strictly upper and lower triangular matrices.

The fundamental weights are $\omega_i = (e_1 + e_2 + \cdots + e_i)^*$, and the function $\widetilde{\Delta}^{\omega_i}$ corresponds to taking the product $h_{11}h_{22}\cdots h_{ii}$: the $i \times i$ minor of the first i rows and columns. The reflections s_{α_i} lift to

$$\overline{\overline{s_i}} = \mathrm{Id}_{i-1} \boxplus \begin{bmatrix} & 1 \\ -1 & \end{bmatrix} \boxplus \mathrm{Id}_{n-i-1} \qquad \overline{s_i} = \mathrm{Id}_{i-1} \boxplus \begin{bmatrix} & -1 \\ 1 & \end{bmatrix} \boxplus \mathrm{Id}_{n-i-1}\,.$$

(Here $\boxplus$ is diagonal matrix concatenation.) Thus $\overline{\overline{s_i}}$ acts on $\mathrm{SL}(n, \mathbb{C})$ on the left by permuting rows, $\overline{s_i}$ acts on the right by permuting columns, and the generalized minor for $v\omega_i, w\omega_i$ on $\mathrm{SL}(n, \mathbb{C})$ is exactly the $i \times i$ minor of the top left rows and columns after permutation.

Definition 2.4.8. For a fixed presentation $\mathbf{i} = (i_1, i_2, \ldots, i_m)$ of w_0, define

$$w_k = s_{i_m} \cdots s_{i_{k+1}} s_{i_k},$$

and define the **i-chamber weights**

$$\gamma_k = w_k \omega_{i_k}.$$

We also count ω_{i_k} as **i**-chamber weights.

Remark 2.4.9. The w_k are pairwise distinct and fill the involution set of w_0, see [Bou02, Chapter VI, §1.6].

Remark 2.4.10. Any $\Delta^{w_k \omega_j}$ is some Δ^{γ_k} or Δ^{ω_j}. This follows from the fact that

$\widetilde{\Delta}^i([\overline{\overline{s_{j\neq i}}}g]_0) = \widetilde{\Delta}^i([g]_0)$, so for use with generalized minors, any $w_k\omega_j$ may be reduced to $w_{k+1}\omega_j$ whenever w_k is not trivial and $i_k \neq j$.

Definition 2.4.11. When restricted to H, we may express $\Delta^{v\omega_i}$ in another fashion. For G of rank r, let P denote the **weight lattice**, integral combinations of $\{\,\omega_i\,\}$. For $\beta \in P \cong \mathbb{Z}^r$, and for $A = \prod_{k=1}^{r} \chi_k^*(t_k) \in H$, Definition 2.4.5 is equivalent to

$$\widetilde{\Delta}^{\beta}(A) = \exp(\langle \beta, t \rangle),$$

where $\langle \cdot, \cdot \rangle$ is the normal Euclidean inner product.

Now for $v \in W$, we may consider v acting on $P \subset \mathfrak{h}_0$ by $v\beta = (v\beta^*)^*$, using the action of W on $\mathfrak{h}_0^*$. This allows expanding the above definition to

$$\Delta_{v\beta, v\beta}(A) = \exp(\langle v\beta, t \rangle).$$

Now, define the **non-negative generalized minor**, $\Delta_{\oplus}^{v\beta} : H \to \mathbb{C}$ as

$$\Delta_{\oplus}^{v\beta}(A) = \exp(\langle \max(v\beta, (0, 0, \ldots, 0)), t \rangle),$$

where max operates componentwise.

The $\oplus$ is intended to suggest tropical addition.

Remark 2.4.12. For $A \cong (t_1, t_2, \ldots, t_r)$, the map $\Delta_{\oplus}^{v\omega_i}(A)$ can be computed by computing $\Delta_{v\omega_i, v\omega_i}((s_1, s_2, \ldots, s_r))$ abstractly, taking the numerator of the result, then substituting t_i for s_i.

2.5 Factorization coordinates

We also review factorization coordinates, referring again to [FZ99] for details. In brief, they will give coordinates on $\mathrm{Conf}_3^*(G/N_+)$ (of Definition 4.1.1) which are governed by a word $\mathbf{i}$ in the Weyl group of G.

Definition 2.5.1. For G a Lie group of rank r over $\mathbb{C}$ admitting H, $N_\pm$, x_i, and y_i, with Weyl group W, $\mathbf{i} = (i_1, \ldots, i_k)$ a reduced word in W, and $t \in (\mathbb{C}^*)^k$, we define

$$x_{\mathbf{i}}(t) = x_{i_1}(t_1) \cdots x_{i_k}(t_k) \in N_+, \qquad y_{\mathbf{i}}(t) = y_{i_1}(t_1) \cdots y_{i_k}(t_k) \in N_-.$$

Let $\mathbf{i} = (i_1, i_2, \ldots, i_m)$ be a fixed representation for w_0. Then

$$x_{\mathbf{i}}(\{\, t_k \,\}_{k=1}^m) = x_{i_1}(t_1) x_{i_2}(t_2) \cdots x_{i_m}(t_m) \in N_+ \cap G_0 \overline{w_0}$$

$$y_{\mathbf{i}}(\{\, t_k \,\}_{k=1}^m) = y_{i_m}(t_m) \cdots y_{i_2}(t_2) y_{i_1}(t_1) \in N_- \cap \overline{w_0} G_0.$$

By [FZ99, Theorem 1.3] these are isomorphisms.

Definition 2.5.2. For $u \in N_- \cap \overline{w_0} G_0$, the **factorization coordinates** of u are $(t_1, \ldots, t_m) \in \mathbb{C}^m$ such that $u = x_{\mathbf{i}}(t_1, \ldots, t_m)$.

We also recall two utility functions from [FZ99] and [FG06b].

Definition 2.5.3. The biregular anti-automorphism $\Psi : G \to G$ and the biregular automorphism $\Phi : N_+ \cap G_0 \overline{w_0} \to N_- \cap \overline{w_0} G_0$ are the unique maps such that, for

$t \in \mathbb{C}$, $h \in H$, $x \in N_+$, and $y \in N_-$,

$$\Psi(x_i(t)) = y_i(t) \qquad \Psi(y_i(t)) = x_i(t) \qquad \Psi(h) = h$$

and

$$\Phi(x) = [x\overline{w_0}]_-, \qquad \Phi^{-1}(y) = \overline{w_0}[\overline{w_0}^{-1}y]_-\overline{w_0}^{-1}.$$

2.6 Quivers

We review quivers. These are graphical encodings of cluster algebras (specifically, those of geometric type), studied in [FZ02], [FZ03], and [BFZ05]. The quiver interpretation, first discussed in [MRZ03], is given a complete introduction in [Mar13], which we follow. The only departure we need from Marsh's definition is to allow some edge weights to be in $\frac{1}{2}\mathbb{Z}$ instead of $\mathbb{Z}$. We also give a name to the symmetrized edge weights $\sigma(v, w)$, and encode the symmetrizability of edge weights into vertex weights d_v.

Definition 2.6.1. A **quiver** is a directed graph with no 1- or 2-cycles or multiple edges, with weights on edges and vertices. Vertex weights are in $\mathbb{N}_{>0}$, and edge weights are in $\frac{1}{2}\mathbb{Z}$. The edge weights are denoted $\sigma(v, w)$, and the vertex weights d_v. Also defined is the auxiliary quantity $\epsilon(v, w)$, by

$$\epsilon(v, w) = \frac{d_w}{\gcd(d_v, d_w)}\sigma(v, w).$$

Vertices are either **frozen** or **non-frozen**, and edge weights are integral unless the edge is between two frozen vertices, in which case they are half-integral.

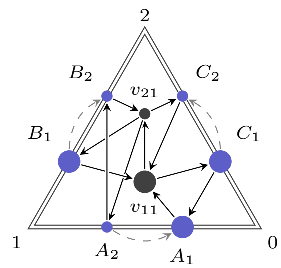


Figure 2.1: A quiver. Larger vertices are of weight 2, and dotted edges correspond to $\sigma(i,j) = \frac{1}{2}$.

Remark 2.6.2. Almost all of the time, $\sigma(i,j)$ will lie in $\{-1, 0, +1\}$. In this case, the edge weights $\sigma(i,j)$ are just the incidence matrix of the graph underlying the quiver.

Remark 2.6.3. The term **quiver** occurs in other areas of literature as a multi-digraph. We use it here as a edge-weighted digraph with skew-symmetrizable adjacency matrix, encoding the exchange matrix of Fomin–Zelevinsky's cluster algebras.

Definition 2.6.4. For each non-frozen v, a quiver $\mu_v(Q)$ is defined. This is the **mutation** of Q at v. Edges in $\mu_v(Q)$ are defined by $\epsilon'(\cdot,\cdot)$, with

$$\epsilon'(x,y) = \begin{cases} \epsilon(x,y) & \text{if } \epsilon(x,v)\epsilon(v,y) \le 0, v \notin \{x,y\} \\ -\epsilon(x,y) & \text{if } v \in \{x,y\} \\ \epsilon(x,y) + |\epsilon(x,v)|\epsilon(v,y) & \text{if } \epsilon(x,v)\epsilon(v,y) > 0, v \notin \{x,y\}. \end{cases}$$

Customarily, the vertex v is renamed in $\mu_v(Q)$, e.g. to v'. This is because μ_v induces a map on the seed torus of Q which changes the coordinate associated to v

(see Definition 2.7.1). For very long sequences, we will usually ignore the renamings for readability and think of v as a vertex in a graph, not as a coordinate function.

Notation 2.6.5. Sequences of mutations are performed left to right: that is $\mu_{v,w} = \mu_w \circ \mu_v$. When subscripts would be awkward, we will also write mutation sequences as $\mu\{v_1, v_2, \ldots\}$. We shall also employ $\prod_{v \in I} \mu_v$ to mean mutating at all vertices in I, in the order given by I.

Remark 2.6.6. When depicting quivers, we shall assume that all black edges have weight $\sigma = 1$, and gray, dashed edges have weight $\sigma = \frac{1}{2}$.

Vertex weights will be depicted by the relative size of circles; we will only use weights 1, 2, and (only in the case of G_2) 3. Frozen vertices will be colored blue, but this will also be described in the text if at all relevant.

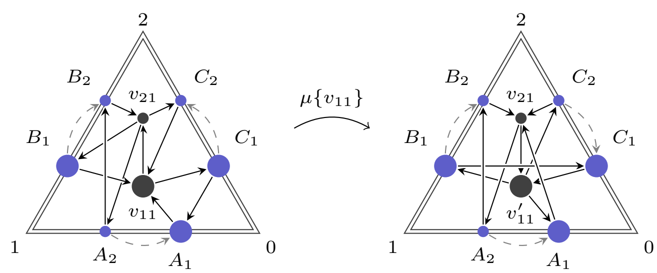


Figure 2.2: Mutating the quiver of Figure 2.1 at v_{11}.

2.7 Cluster ensembles

We now define cluster ensemble structures following [FG09b]. Also useful is [Ish19].

Definition 2.7.1. For a quiver Q with vertices V, let Λ_V be the free abelian group generated by V, with dual $\Lambda_V^* = \mathrm{Hom}(\Lambda_V, \mathbb{Z})$. Let $\{e_v : v \in V\}$ be a basis for Λ_V, and $\{ f_v = d_v^{-1} e_v^* \}$ a basis for Λ_V^*. Let Λ° be the $\mathbb{Q}$-span of $\{ f_v \}$ inside $\Lambda^* \otimes \mathbb{Q}$.

The **seed $\mathcal{X}$-torus** is $T_Q^{\mathcal{X}} = \mathrm{Hom}(\Lambda, \mathbb{C}^*)$. The **seed $\mathcal{A}$-torus** is $T_Q^{\mathcal{A}} = \mathrm{Hom}(\Lambda^\circ, \mathbb{C}^*)$. The coordinates for these tori are denoted $\{ x_v \}_{v \in V}$ and $\{ a_v \}_{v \in V}$ and called **$\mathcal{X}$-coordinates** and **$\mathcal{A}$-coordinates** respectively. A quiver together with these coordinate tori is a **seed**.

There is a map $p : T_Q^{\mathcal{A}} \to T_Q^{\mathcal{X}}$. This map is characterized by the pullback $p^*(x_v)$ of an $\mathcal{X}$-coordinate to $T_Q^{\mathcal{A}}$, which is defined as

$$p^*(x_v) = \prod_{w \in V} a_w^{\epsilon(v,w)}.$$

As these objects are defined in terms of quivers, we must define how quiver mutation affects the coordinates. As with p, we shall define μ_v by pullbacks–in this case, the pullback of the coordinates x'_w and a'_w for vertices w of $\mu_v(Q)$.

$$\mu_v^* x'_w = \begin{cases} x_v^{-1} & \text{if } v' = w \\ x_w(1 + x_v^{\operatorname{sgn} \epsilon(v,w)})^{\epsilon(v,w)} & \text{if } v' \neq w \end{cases}$$

$$\mu_v^* a'_w = \begin{cases} \dfrac{1}{a_v} \left(\displaystyle\prod_{\epsilon(v,z)>0} a_z^{\epsilon(v,z)} + \prod_{\epsilon(v,z)<0} a_z^{-\epsilon(v,z)} \right) & \text{if } v' = w \\ a_w & \text{if } v' \neq w \end{cases}$$

Finally, define a **cluster ensemble** as the orbit of a single seed under all quiver mutations.

Example 2.7.2. In the mutation of Figure 2.2, we have

$$a_{v'_{11}} = \frac{1}{a_{v_{11}}}\left(a_{C_1}a_{v_{21}} + a_{A_1}a_{C_2}a_{B_1}\right)$$

The effect on $\mathcal{X}$-coordinates is

$$\begin{aligned}
&x_{v_{11}} \mapsto \frac{1}{x_{v_{11}}} && x_{v_{21}} \mapsto x_{v_{21}}(1+x_{v_{11}})^2 = x_{v_{21}} + 2x_{v_{11}}x_{v_{21}} + x_{v_{11}}^2 x_{v_{21}} \\
&x_{A_1} \mapsto x_{A_1}(1+x_{v_{11}}^{-1})^{-1} = \frac{x_{A_1}x_{v_{11}}}{1+x_{v_{11}}} && x_{A_2} \mapsto x_{A_2} \\
&x_{B_1} \mapsto x_{B_1}(1+x_{v_{11}}^{-1})^{-1} = \frac{x_{B_1}x_{v_{11}}}{1+x_{v_{11}}} && x_{B_2} \mapsto x_{B_2} \\
&x_{C_1} \mapsto x_{C_1}(1+x_{v_{11}}) = x_{C_1} + x_{C_1}x_{v_{11}} && x_{C_2} \mapsto x_{C_2}(1+x_{v_{11}}^{-1})^{-2} = \frac{x_{C_2}x_{v_{11}}^2}{1+2x_{v_{11}}+x_{v_{11}}^2}
\end{aligned}$$

Remark 2.7.3. The mutation relation for the $\mathcal{A}$- and $\mathcal{X}$-coordinates follows [FZ07, Equation 2.3]. The $\mathcal{A}$-coordinates especially form the **cluster algebra** of Fomin–Zelevinsky which define the quiver. For consistency, we will refer to the cluster variables as $\mathcal{A}$-coordinates.

A cluster ensemble, therefore, is a pair of coordinate structures on spaces, each defined by a quiver. Quiver mutation acts on each coordinate structure by replacing some coordinate functions, and p is always a map (generally non-surjective, non-injective) between the coordinate structures.

Remark 2.7.4. The map p commutes with quiver mutation, as by [FG09b, Section 1.2] and [FZ07, Proposition 3.9].

Chapter 3: Key identities

Here we reproduce two external results of particular significance to our constructions and proofs, from [YZ08] and [FZ99]. We refer to their respective origins for more details. These identities allow us to perform a program we call "adjusting a minor coordinate". Depending on the surrounding conditions, these will allow us to conclude that the action of mutation on $T_Q^{\mathcal{A}}$ is to replace

$$\Delta_{u\omega_i, v\omega_i}(g) \qquad \text{with} \qquad \Delta_{u'\omega_i, v'\omega_i}(q)$$

for some u, v, u', v'.

3.1 Actions of σ_G

We give an overview of the identity used to justify mutations which apply σ_G to the Dynkin diagram-like graph underlying a quiver, as in Remark 2.1.7. We use this in the proofs of Lemmas 7.2.1 and 7.2.2. This identity is collected from several results in [YZ08], so we import some notation.

Definition 3.1.1. Let the **reduced double Bruhat cell** $L^{s,t} = N_+\overline{s}N_+ \cap B_-\overline{t}B_-$. (We shall not need many facts about this object.) Recall that $k^\star = \sigma_G(k)$. The relation $a \prec_c b$ means that a precedes b in the Coxeter element c, and that a and b are connected by an edge in the Dynkin diagram associated to c.

We shall mostly be interested in minors of the form $\Delta_{c^m\omega_k, c^m\omega_k}$. Yang–Zelevinsky denote these as $x_{c^m\omega_k;c}$, but we will avoid this for notational consistency.

The main result is that for a cluster algebra $\mathcal{A}(c)$ defined by an initial seed with quiver of Dynkin type associated to Coxeter element c, there exists a $g \in L^{c,c^{-1}}$ such that all cluster coordinates and coefficients of $\mathcal{A}(c)$ are given by certain generalized minors of g. This allows combinatorially defining the exchange relations of all **source** or **sink mutations** (at vertices where all edges point out or in). These always exchange variables of the form $\Delta_{c^m\omega_i, c^m\omega_i}$ with $\Delta_{c^{m\pm1}\omega_i, c^{m\pm1}\omega_i}$. Yang–Zelevinsky refer to these as **primitive exchange relations**.

Finally, a number $h(i;c)$ is defined such that $\Delta_{c^{h(i;c)}\omega_i, c^{h(i;c)}\omega_i} = \Delta_{\omega_{i^*}, \omega_{i^*}}$. This allows convenient analysis of periodicity of source/sink mutation sequences.

Proposition 3.1.2 ([YZ08, Equation 2.13]). *In some cases, $h(i;c)$ is easily calculated.*

- *When $i^* = i$ (for example, if $\sigma_G = e$) we trivially have $h(i;c) = 0$.*
- *Otherwise, when h is even, $c^{h/2} = w_0$, and by action of w_0 on simple roots, $h(i;c) = \frac{h}{2}$.*

When G is of type A_ℓ, however, the calculation is more delicate. Let

$$t_+ = s_1 s_3 \cdots s_{2\lfloor(\ell+1)/2\rfloor - 1}, \qquad t_- = s_2 s_4 \ldots s_{2\lfloor(\ell+1)/2\rfloor}, \qquad w_0 = \overbrace{t_+ t_- \cdots t_\pm}^{h=\ell+1 \text{ factors}}, \qquad c = t_+ t_-.$$

Then by [YZ08, Equation 2.13], as $h = \ell + 1$ *be the Coxeter number for* A_ℓ*,*

$$h(i;c) = \begin{cases} \lfloor \frac{h}{2} \rfloor = \lfloor \frac{\ell+1}{2} \rfloor & i \text{ even} \\ \lceil \frac{h}{2} \rceil = \lceil \frac{\ell+1}{2} \rceil & i \text{ odd.} \end{cases}$$

Note that

$$\Delta_{c^{m+h(k;c)+1}\omega_k, c^{m+h(k;c)+1}\omega_k} = \Delta_{c^m\omega_{k^*}, c^m\omega_{k^*}}.$$

We now combine several results of [YZ08], mainly Theorem 1.5.

Proposition 3.1.3. *The cluster variables in* $\mathcal{A}(c)$ *satisfy the following primitive exchange relation (letting* A *be the Cartan matrix):*

$$\Delta_{c^{m-1}\omega_k, c^{m-1}\omega_k}\Delta_{c^m\omega_k, c^m\omega_k} = \prod_{i \prec_c k}(\Delta_{c^m\omega_i, c^m\omega_i})^{-A_{i,k}} \prod_{k \prec_c i}(\Delta_{c^{m-1}\omega_i, c^{m-1}\omega_i})^{-A_{i,k}} + 1.$$

Remark 3.1.4. This does not give us much information about $g \in L^{c,c^{-1}}$, but we will restrict our attention to periodic mutations. The only effect will be to change which root the vertices of the quiver are associated to.

3.2 Grid exchange relations

Here we give an overview of the identity [FZ99, Theorem 1.17] used to justify the exchange relations of μ_{rotTw} and $\widetilde{\mu}_{\text{Flipcore}}$, and to prove Lemmas 7.3.1 and 7.4.1:

Theorem 3.2.1 ([FZ99, Theorem 1.17])**.** *For* u, v *two words in the Weyl group* W *such that* $\text{length}(us_i) = \text{length}(u) + 1$ *and* $\text{length}(vs_i) = \text{length}(v) + 1$*, and* A *the*

Cartan matrix,

$$\Delta_{u\omega_i,v\omega_i}\Delta_{us_i\omega_i,vs_i\omega_i} = \Delta_{us_i\omega_i,v\omega_i}\Delta_{u\omega_i,vs_i\omega_i} + \prod_{j\neq i}(\Delta_{u\omega_j,v\omega_j})^{-A_{j,i}}.$$

Remark 3.2.2. It appears that this theorem only allows us to adjust the minor coordinates by one letter at a time: s_i. By the same logic as Remark 2.4.10, however, we actually are able to adjust u and v by longer sequences of letters, as long as they do not contain multiple s_i.

We use this theorem in situations where, mutating at $v_{a,b}$,

$$\begin{aligned}
\Delta_{us_i\omega_i,vs_i\omega_i} &= \text{Coordinate of } v_{a,b} \text{ before mutation}\\
\Delta_{u\omega_i,v\omega_i} &= \text{Coordinate of } v_{a,b} \text{ after mutation}\\
\Delta_{u\omega_i,vs_i\omega_i} &= \text{Coordinate to left of } v_{a,b}\\
\Delta_{us_i\omega_i,v\omega_i} &= \text{Coordinate to right of } v_{a,b}\\
\Delta_{u\omega_j,v\omega_j} &= \text{Coordinates in same column as } v_{a,b}
\end{aligned}$$

Example 3.2.3. We look ahead to the results of Chapter 5 and assume all edge coordinates are 1 for simplicity. Figure 3.1 depicts Q_{C_4} partway through the rotation mutation. The minors associated to relevant coordinates (see the proof of

Lemma 7.3.1 for details) are at this point

$$v_{22} : \Delta_{w_1\omega_2, w_{10}\omega_2} = \Delta_{w_3 s_2 \omega_2, w_{11} s_2 \omega_2}$$

$$v_{21} : \Delta_{w_5\omega_2, w_{10}\omega_2} = \Delta_{w_3\omega_2, w_{11} s_2 \omega_2}$$

$$v_{23} : \Delta_{w_1\omega_2, w_{14}\omega_2} = \Delta_{w_3 s_2 \omega_2, w_{11}\omega_2}$$

$$v_{12} : \Delta_{w_5\omega_1, w_{13}\omega_1} = \Delta_{w_3\omega_1, w_{11}\omega_1}$$

$$v_{32} : \Delta_{w_1\omega_3, w_{11}\omega_3} = \Delta_{w_3\omega_3, w_{11}\omega_3}$$

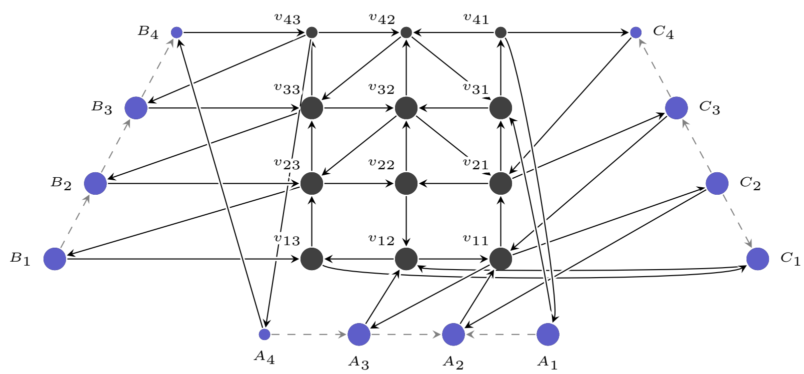


Figure 3.1: After applying $\mu\{\, v_{11}v_{21}v_{31}v_{41}v_{12} \,\}$ to Q_{C_4}. The next mutation in μ_{rot} is at v_{22}.

The equality follows from Remark 2.4.10. Therefore, by applying the identity we see that mutating at v_{22} will change the coordinate to $\Delta_{w_3\omega_2, w_{11}\omega_2} = \Delta_{w_6\omega_2, w_{14}\omega_2}$.

Finally, we note that the requirements on u and v may be loosened, which we need to prove Lemma 7.4.1.

Lemma 3.2.4. *Even if* $\text{length}(us_i) \neq \text{length}(u) + 1$ *or* $\text{length}(vs_i) \neq \text{length}(v) + 1$, *the result of Theorem 3.2.1 still holds as long as u and v are subwords of a repeated*

Coxeter element c.

Proof. From the proof in [FZ99], the only reason for the length condition is to ensure that $\overline{vs_i} = \overline{v}\overline{s_i}$ (and similar for u), since the lifting is not quite a homomorphism: for example $\overline{s_i}\overline{s_i} \neq \overline{e}$.

However, the only situations in which multiplication is not preserved is when the multiplication by s_i induces a length-shortening identity in W. But if u and v are subwords of a repeated Coxeter element, their suffixes will always be of the form w_k. These admit no length-shortening braid relations.

So the only possibility for $\overline{vs_i} \neq \overline{v}\overline{s_i}$ is if vs_i contains a copy of $w_0 = c^{h/2}$. But $\overline{c^h} = s_G$ is an element of H. By prepending copies of c^h to either u or v, we may ensure that the difference in length between u and v is never more than $\frac{h}{2} - 1$. Then, using

$$\Delta_{wu\omega_i, wv\omega_i}(g) = \Delta_{u\omega_i, v\omega_i}(w^{-1}gw)$$

we reduce to the case where u and v are each some w_k shorter than w_0. By replacing g with $wgw^{-1}s_G^{\epsilon}$ (where $\epsilon \in 0, 1$ depending on the exact difference between u and v), the desired identity follows from the regular theorem. □

Chapter 4: Coordinates on generically-decorated representations

Here we review **decorations** of representations and define **Fock–Goncharov coordinate structures**. For a representation ρ and a chosen triangulation, a decoration of ρ is a collection of flags for each simplex in the triangulation. These flags can recreate ρ, and have a canonical form that admits generalized minor coordinates. Fock–Goncharov coordinate structures will describe coordinates on these decorations.

These decorations apply to surfaces, but also to 3-manifolds with fixed ideal triangulations. We follow [FG06b] for the surface case and [Zic19] for the 3-manifold case.

4.1 Configurations and gluings

Definition 4.1.1. Let G be a Lie group over $\mathbb{C}$, with sufficient choices to define G_0 as in Definition 2.4.1. Let K be a subgroup of G (we will ultimately use N_+). A tuple of cosets $(g_0K, g_1K, \ldots, g_mK)$ is **sufficiently generic** if each $g_i^{-1}g_j \in \overline{w_0}G_0$.

Such a tuple corresponds to a labeling of edges in an oriented m-simplex by elements of $\overline{w_0}G_0$, as in Figure 4.1, with the edge from i to j (assuming $j > i$) labeled by $g_i^{-1}g_j$.

The variety of such sufficiently generic $(m+1)$-tuples is the **configuration space** $\mathrm{Conf}_{m+1}(G/K)$. Identifying elements which differ by left-multiplication of G yields the variety $\mathrm{Conf}^*_{m+1}(G/K)$.

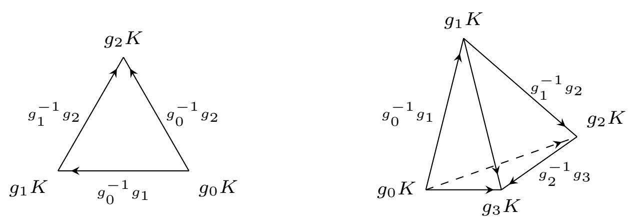


Figure 4.1: Oriented 2- and 3-simplices labeled by elements of $\mathrm{Conf}^*_3(G/K)$ and $\mathrm{Conf}^*_4(G/K)$.

Definition 4.1.2. Let M be a compact manifold (possibly with boundary) and G a Lie group over $\mathbb{C}$, with K a subgroup of G. A subgroup L of $\pi_1(M)$ is **peripheral** if there is a boundary component D of M such that L is induced by the inclusion of D; that is $\iota^*(\pi_1(D)) = L$.

A representation $\rho : \pi_1(M) \to G$ is a **(G,K)-representation** if, for every peripheral subgroup L, the image $\rho(L)$ is a conjugate of K. In the case $K = N_+$, the term **boundary-unipotent** is used. If $K = B_+$, then ρ is **boundary-borel**.

Lemma 4.1.3 ([Zic19, Lemma 5.8, Proposition 5.9])**.** *There is an isomorphism of varieties*

$$\mathrm{Conf}^*_3(G/N_+) \cong H^3 \times_H (N_- \cap \overline{w_0}G_0).$$

The fiber product $\times_H$ means that for $(h_1, h_2, h_3, u) \in \mathrm{Conf}^*_3(G/N_+))$, we have

$$[\overline{w_0}^{-1}u]_0 = (w_0(h_3h_1)h_2)^{-1}.$$

Thus we will often write (h_1, h_2, h_3, u) for α, which has one representative $(N_+, \overline{w_0}h_1N_+, uw_0(h_1)h_2s_GN_+)$. We refer to these as **canonical forms** for $\mathrm{Conf}_3^*(G/N_+)$.

Definition 4.1.4. By expanding N_+ to B_+ above, we obtain that $\alpha \in \mathrm{Conf}_3^*(G/B_+)$ can be given as $(B_+, \overline{w_0}B_+, uB_+)$. Choose the unique u_x such that, for all k, the minor $\Delta_{w_k\omega_1,\omega_1}(u_x) = 1$. This element u_x is the **canonical form** for α.

For an arbitrary $(B_+, \overline{w_0}B_+, uB_+)$, by expanding Definition 2.4.6 there is a unique (up to the center of G) element h such that huh^{-1} is the canonical form u_x. When G is centerless, we denote this element by $n(u)$.

Proposition 4.1.5. *This summarizes [Zic19, Proposition 5.10]. Let* $\alpha \in \mathrm{Conf}_3^*(G/N_+)$ *be given by* $\alpha = (h_1, h_2, h_3, u)$. *Recall* Φ *and* Ψ *of Definition 2.5.3. The map*

$$\mathrm{rot} : \mathrm{Conf}_3^*(G/N_+) \to \mathrm{Conf}_3^*(G/N_+), \qquad (g_0N_+, g_1N_+, g_2N_+) \mapsto (g_2N_+, g_0N_+, g_1N_+)$$

is given, in this form, via

$$\mathrm{rot} : (h_1, h_2, h_3, u) \mapsto (h_3, h_1, h_2, h_2^{-1}(w_0(h_1))^{-1}(\Phi\Psi\Phi\Psi)(u)(w_0(h_1))h_2).$$

Definition 4.1.6. This summarizes [Zic19, Section 2.2.1]. We define and $\mathrm{Conf}_3^*(G/N_+) \times_{13}^{s_G} \mathrm{Conf}_3^*(G/N_+)$ by gluing copies of $\mathrm{Conf}_3^*(G/N_+)$ along matching copies of $\mathrm{Conf}_2^*(G/N_+)$. Specifically,

$$\mathrm{Conf}_3^*(G/N_+) \times_{02}^{s_G} \mathrm{Conf}_3^*(G/N_+) = \left\{ \begin{smallmatrix} (g_0s_GN_+, g_1N_+, g_2N_+), \\ (g_0N_+, g_2N_+, g_3N_+) \end{smallmatrix} : (g_0N_+, g_1N_+, g_2N_+, g_3N_+) \in \mathrm{Conf}_4^*(G/N_+) \right\}$$

$$\mathrm{Conf}_3^*(G/N_+) \times_{13}^{s_G} \mathrm{Conf}_3^*(G/N_+) = \left\{ \begin{smallmatrix} (g_1N_+, g_2N_+, g_3N_+), \\ (g_0N_+, g_1s_GN_+, g_3N_+) \end{smallmatrix} : (g_0N_+, g_1N_+, g_2N_+, g_3N_+) \in \mathrm{Conf}_4^*(G/N_+) \right\}$$

We also define the following maps from $\mathrm{Conf}_4^*(G/N_+)$ to $\mathrm{Conf}_3^*(G/N_+) \times_{jk}^{s_G} \mathrm{Conf}_3^*(G/N_+)$ for $jk = 02, 13$:

$$\Psi_{02} : (g_0N_+, g_1N_+, g_2N_+, g_3N_+) \mapsto \substack{(g_0 s_G N_+, g_1N_+, g_2N_+),\\ (g_0N_+, g_2N_+, g_3N_+)}$$

$$\Psi_{13} : (g_0N_+, g_1N_+, g_2N_+, g_3N_+) \mapsto \substack{(g_1N_+, g_2N_+, g_3N_+),\\ (g_0N_+, g_1 s_G N_+, g_3N_+)}$$

See part of Figure 4.6 for a graphical depiction.

In the surface case, these correspond to the two different ways of triangulating a quadrilateral. The act of retriangulating a single quadrilateral this way is called a "flip". In the 3-manifold case, these correspond to decomposing a tetrahedron's vertices into the two triangles' worth.

Remark 4.1.7. The element s_G is in the center of G, so is only relevant for N_+-cosets. Its purpose is to allow the quiver amalgamation (see Definition 4.2.7 ahead) to agree with the identification of elements of $\mathrm{Conf}_3^*(G/N_+)$ along a copy of $\mathrm{Conf}_2^*(G/N_+)$.

To see this, note that Figure 4.4a shows identifying a 0–2 edge with a 0–1 edge, and Figure 4.4b identifes a 0–2 edge with a 1–2 edge. Since the associated elements of $\mathrm{Conf}_2^*(G/N_+)$ are oriented via the cyclic ordering on $(0, 1, 2)$, this quiver amalgamation corresponds to identifying elements of $\mathrm{Conf}_2^*(G/N_+)$ "backwards".

This use of s_G is why $\mathcal{A}_{G,\Sigma}$ is the moduli space of *twisted*, decorated representations from $\pi_1(\Sigma)$ into G. See [FG06b, Section 8.6] and [Zic19, Section 2.3] for more details.

4.2 Triangular quivers and Fock–Goncharov coordinate structures

We want a variety that is independent of the choice of triangulation. Therefore, we will need coordinate change maps corresponding to changes in triangulation (the rotation of individual triangles, or the quadrilateral flip).

Here we define the outputs of the algorithm of Chapter 5. These are **triangular quivers** (see Definition 4.2.6), which have vertices on edges and in the interior. If, moreover, these quivers admit a rotation and a flip as quiver mutations, they have **triangulation-compatible symmetry** (see Definition 4.2.8). And if we can then associate the $\mathcal{A}$- and $\mathcal{X}$-coordinates to coordinates on $\mathrm{Conf}_3^*(G/N_+)$ and $\mathrm{Conf}_3^*(G/B_+)$, we have a **Fock–Goncharov coordinate structure** (see Definition 4.2.9).

Definition 4.2.1. Let D be a Dynkin diagram, with associated Cartan matrix $\begin{bmatrix} a_{ij} \end{bmatrix}$. A quiver Q is of **Dynkin type** D if $|\epsilon(i,j)| = -a_{ji}$ (for $i \neq j$) and $\sigma(i,j)$ takes values only in $\{-1, 0, +1\}$. The undirected graph of Q carries the adjacency information of D, and the vertex weights carry the edge details.

Q is of **half-Dynkin type** D if $2Q$, the quiver obtained by multiplying all edge weights by 2, is of Dynkin type D.

Definition 4.2.2. Let D be a quiver of Dynkin type (with no oriented cycles). We can obtain a particular Coxeter element c by requiring that s_{α_i} appear before s_{α_j} in c if $\sigma(i,j) > 0$ in D. Such a c is **induced** by the quiver D.

Definition 4.2.3. A quiver of Dynkin type is **tree-like** if, as an unoriented graph, it is a tree, and the directed edges always point away from the root. In this case, there is a partition of vertices $\{T_0, T_1, \ldots, T_m\}$ such that the vertices in T_i are of distance i (in the unoriented graph) from the root vertex.

Note that Coxeter elements induced by tree-like Dynkin quivers always take the form

$$c = \left(\prod_{\alpha \in T_0} s_\alpha\right)\left(\prod_{\alpha \in T_1} s_\alpha\right)\cdots\left(\prod_{\alpha \in T_m} s_\alpha\right).$$

Definition 4.2.4. The involution σ_G acts on quivers of Dynkin type in the same way that it would act on the Dynkin diagram as in Remark 2.1.7. A tree-like quiver Q of Dynkin type D is **well-rooted** if this action of σ_G preserves all T_i partitions. See Figure 4.2.

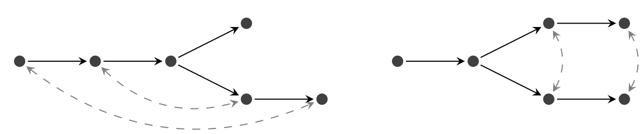


Figure 4.2: A non-well-rooted quiver (left) and a well-rooted quiver (right) for E_6.

Remark 4.2.5. In the most cases, the quiver obtained by applying a naïve "left-to-right" ordering to the common presentation of the Dynkin diagram is well-rooted. The pathological case is E_6.

Definition 4.2.6. A quiver Q is **triangular** if, for a standard 2-simplex σ with vertices $0, 1, 2$, each vertex in Q is associated to the interior of some sub-simplex σ' of σ. $\mathrm{d}_{\sigma'}(Q)$ is defined to be the sub-quiver of Q obtained by deleting all vertices except those that lie on the interior of σ. See Figure 4.3.

In other words, a triangular quiver Q is one that can be split into sub-quivers:

- $d_{(0,1)}(Q)$, $d_{(1,2)}(Q)$, and $d_{(0,2)}(Q)$: the vertices on the edges of the triangle.

- $d_{(0,1,2)}(Q)$: the vertices in the interior of the triangle.

We label the vertices on the $d_{(0,1)}$ edge's vertices by A_i, those on the $d_{(1,2)}$ edge by B_i, and those on the $d_{(0,2)}$ edge by C_i.

An isomorphism φ of triangular quivers must preserve these classifications: $d_\Delta(\varphi(Q)) = \varphi(d_\Delta(Q))$.

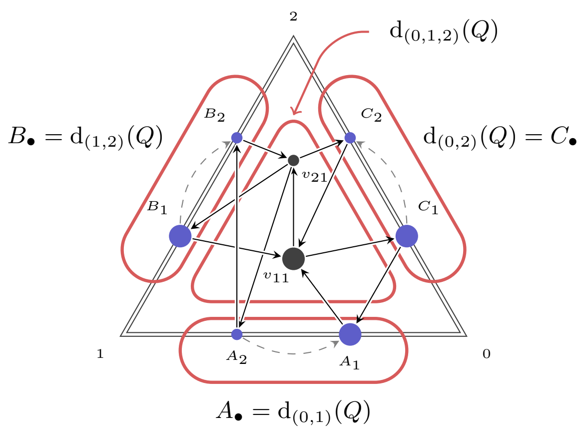


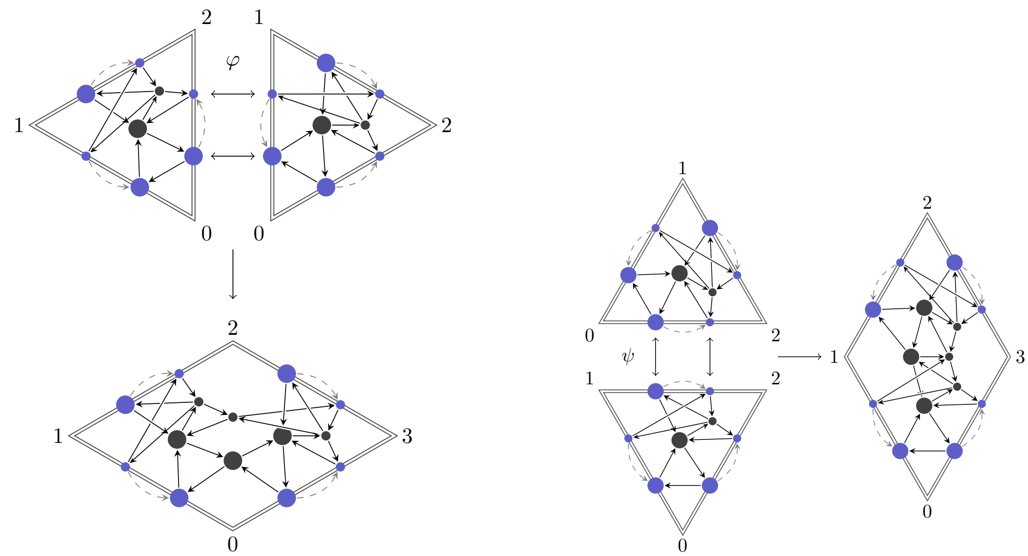
Figure 4.3: A triangular quiver (this one is associated to a Lie group of type C_2).

Definition 4.2.7. Let Q be a triangular quiver such that the edge sub-quivers are isomorphic as sets of vertices. That is, fix isomorphisms $\varphi : d_{(0,1)}(Q) \cong d_{(0,2)}(Q)$ and $\psi : d_{(0,2)}(Q) \cong d_{(1,2)}(Q)$. Then we define

- $Q^{0\text{–}2}$ as the quiver obtained by identifying two copies of Q along φ, and

- $Q^{1\text{–}3}$ as the quiver obtained by identifying two copies of Q along ψ.

In performing the identification of vertices, edge weights are added. They need not agree, and in particular might cancel; see Figures 4.4a and 4.4b.



(A) Construction of $Q^{0\text{–}2}$ (for G of type C_2).

(B) Construction of $Q^{1\text{–}3}$ (for G of type C_2).

Figure 4.4: Amalgamating triangular quivers.

This is a special case of **amalgamation** as defined in [FG06a, §2.2].

Definition 4.2.8. A quiver **has triangulation-compatible symmetry** if

1. It is triangular by Definition 4.2.6,
2. There is a quiver mutation μ_{rot} which is an isomorphism of triangular quivers between Q and Q', where Q' is obtained from Q by permuting simplex indices from $(0, 1, 2)$ to $(2, 0, 1)$.
3. There is a quiver mutation μ_{flip} which transforms $Q^{0\text{–}2}$ to $Q^{1\text{–}3}$.

If $\mathcal{T}$ is an oriented triangulation of a surface, and $\mathcal{T}'$ is an orientation-preserving retriangulation of $\mathcal{T}$, then inscribing Q in each simplex of $\mathcal{T}$ and $\mathcal{T}'$ and amalgamating along edges results in two quivers that differ by some sequence of μ_{rot} and μ_{flip}. Therefore, if the cluster variables of Q are coordinates for the representation variety, the existence of μ_{rot} and μ_{flip} will (eventually) show the variety to be independent of the specific choice of triangulation.

Definition 4.2.9. Fix a semisimple Lie group G over $\mathbb{C}$ with a fixed maximal torus H and a maximal unipotent subgroup N_+, with $B_+ = HN_+$. A quiver Q carries a **Fock–Goncharov coordinate structure for** G if

1. Q has triangulation-compatible symmetry as in Definition 4.2.8.

2. Each of the edge sub-quivers of Q are of half-Dynkin type for G, as by Definition 4.2.1, and the isomorphisms φ and ψ of Definition 4.2.7 identify vertices that come from the same nodes in the Dynkin diagram.

3. There exists a map $\mathcal{M} : \text{Conf}_3^*(G/N_+) \to T_Q^{\mathcal{A}}$, which is a birational equivalence.

4. Moreover, $\mathcal{M}$ respects the rotation. That is, the following diagrams commute:
$$\begin{array}{ccc} \text{Conf}_3^*(G/N_+) & \xrightarrow{\mathcal{M}} & T_Q^{\mathcal{A}} \\ \downarrow{\scriptstyle \text{rot}} & & \downarrow{\scriptstyle \mu_{\text{rot}}^*} \\ \text{Conf}_3^*(G/N_+) & \xrightarrow{\mathcal{M}} & T_{\mu_{\text{rot}}(Q)}^{\mathcal{A}} \end{array} \qquad \begin{array}{ccc} \text{Conf}_3^*(G/B_+) & \xrightarrow{\mathcal{M}} & T_Q^{\mathcal{X}} \\ \downarrow{\scriptstyle \text{rot}} & & \downarrow{\scriptstyle \mu_{\text{rot}}^*} \\ \text{Conf}_3^*(G/B_+) & \xrightarrow{\mathcal{M}} & T_{\mu_{\text{rot}}(Q)}^{\mathcal{X}} \end{array}$$
with $\text{rot} : (g_0K, g_1K, g_2K) \mapsto (g_2K, g_0K, g_1K)$ (see also Figure 4.5).

5. Moreover, $\mathcal{M}$ respects the flip. That is, the following diagrams commute:
$$\begin{array}{ccc} \text{Conf}_3^*(G/N_+) \times_{02}^{s_G} \text{Conf}_3^*(G/N_+) & \xrightarrow{\mathcal{M}} & T_{Q^{0-2}}^{\mathcal{A}} \\ \downarrow{\scriptstyle \Psi_{13} \circ \Psi_{02}^{-1}} & & \downarrow{\scriptstyle \mu_{\text{flip}}^*} \\ \text{Conf}_3^*(G/N_+) \times_{13}^{s_G} \text{Conf}_3^*(G/N_+) & \xrightarrow{\mathcal{M}} & T_{Q^{1-3}}^{\mathcal{A}} \end{array} \qquad \begin{array}{ccc} \text{Conf}_3^*(G/B_+) \times_{02}^{s_G} \text{Conf}_3^*(G/B_+) & \xrightarrow{\mathcal{M}} & T_{Q^{0-2}}^{\mathcal{X}} \\ \downarrow{\scriptstyle \Psi_{13} \circ \Psi_{02}^{-1}} & & \downarrow{\scriptstyle \mu_{\text{flip}}^*} \\ \text{Conf}_3^*(G/B_+) \times_{13}^{s_G} \text{Conf}_3^*(G/B_+) & \xrightarrow{\mathcal{M}} & T_{Q^{1-3}}^{\mathcal{X}} \end{array}$$
where Ψ_{jk} are defined as in Definition 4.1.6 (see also Figure 4.6).

Remark 4.2.10. By the map p of Definition 2.7.1, $\mathcal{M}$ provides a map to $T_Q^{\mathcal{X}}$. In fact, this map factors through $\text{Conf}_3^*(G/B_+)$. By abuse of notation, we also refer to this map as $\mathcal{M}$ in Items 4 and 5.

Example 4.2.11. Figures 4.5 and 4.6 illustrate the last two demands: to move back and forth between $\mathrm{Conf}_3^*(G/N_+)$ and $T_Q^{\mathcal{A}}$, or $\mathrm{Conf}_4^*(G/N_+)$ and $T_{Q^{0-2}}^{\mathcal{A}}$ in terms of coordinates and quiver mutations.

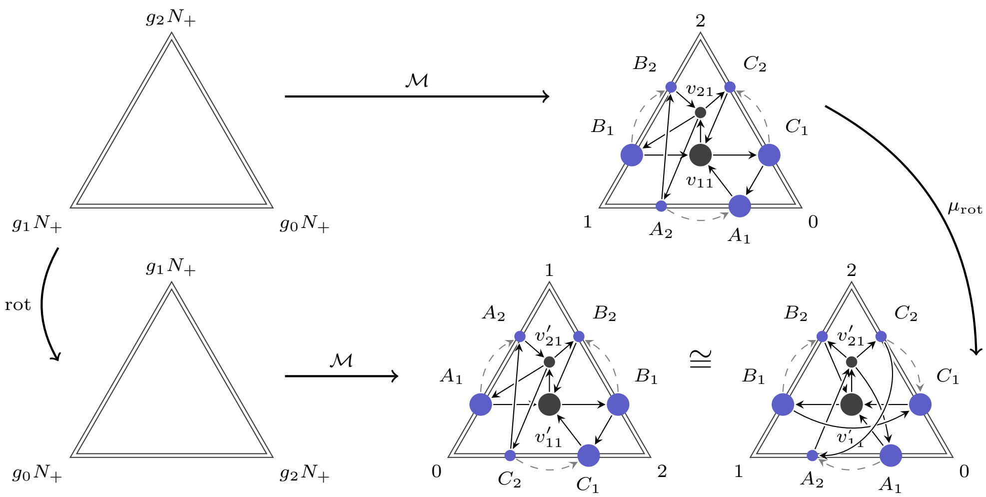


Figure 4.5: Quiver for type C_2, illustrating $\mu_{\mathrm{rot}}^* \circ \mathcal{M} = \mathcal{M} \circ \mathrm{rot}$, with $T_Q^{\mathcal{A}}$ depicted by a graph for Q.

We can now restate Theorem 1.4.1 as the following

Theorem 4.2.12. *For a split semisimple simply-connected (or centerless) algebraic group G over $\mathbb{Q}$, a quiver with a Fock–Goncharov structure exists.*

We will mostly focus on the simple case. The semisimple case follows quickly as described in Chapter 8. The construction for Theorem 4.2.12 is given in Chapter 5.

4.3 From coordinates to representations

The Fock–Goncharov program for defining $\mathcal{A}_{G,\Sigma}$ and $\mathcal{X}_{G,\Sigma}$ works for any compact, oriented Σ with boundary components. In the case that the boundary consists

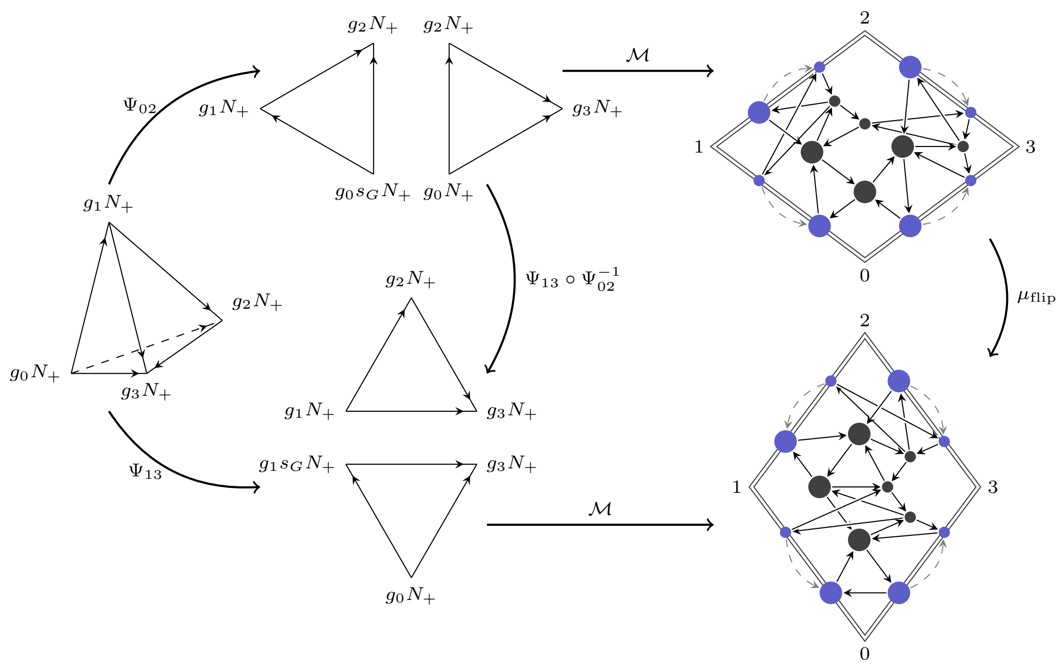


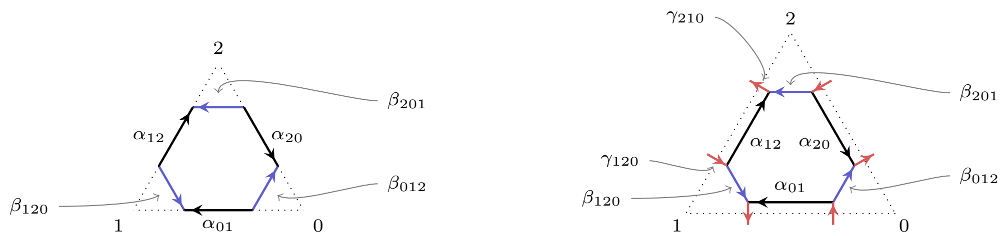
Figure 4.6: Quiver for type C_2, illustrating $\mu^*_{\mathrm{flip}} \circ \mathcal{M} = \mathcal{M} \circ \Psi_{13} \circ \Psi_{02}^{-1}$. The tetrahedron shows the use of the flip in the 3-manifold context.

of punctures, we can describe reconstructing a representation from a point in the moduli space. We start with the $\mathcal{A}$-coordinate version due to complications in technicalities, referring to [FG06b, Section 8], [Zic19, Section 6] for more details.

We start with an ideal triangulation $\mathcal{T}$ of Σ, together with a quiver Q and seed torus $T_Q^{\mathcal{A}}$ for each triangle in $\mathcal{T}$, amalgamated together.

First, we truncate the triangulation, as in Figure 4.7a. We will associate an element of G to each directed edge: elements on the long edges will be called α_{ij}, and elements on short edges will be called β_{ijk}. The local labeling of vertices by 0, 1, and 2 is determined by triangularity (see Definition 4.2.6) of the quiver Q.

To compute $\rho(\gamma)$, we homotope γ to follow these edges, then multiply together the elements in the order given by γ. When the orientation of γ disagrees with the orientation of the edge, we use the inverse of the attached element.



(A) A truncated simplex. "Long" edges are in black, "short" edges are in blue.

(B) A doubly-truncated simplex. "Long" edges are in black, "middle" edges are in blue, "short" edges are in red.

Figure 4.7: Truncations of $\mathcal{T}$. The left is used for $\mathcal{A}$-coordinates, the right for $\mathcal{X}$-coordinates.

All that remains is to give a formula for the α_{ij} and β_{ijk}.

- We start with an element in $T_Q^{\mathcal{A}}$; the $\mathcal{A}$-coordinates of Q.
- Since we have a Fock–Goncharov coordinate structure, we may apply $(\mathcal{M})^{-1}$. This gives an explicit element of $\mathrm{Conf}_3^*(G/N_+)$. It is most conveniently expressed as (h_1, h_2, h_3, u).
- Set

$$\begin{aligned}
\alpha_{01} &= \overline{w_0} h_1 & u_0 &= u & \beta_{201} &= (w_0(h_1)h_2)^{-1}\,(\Psi\Phi\Psi)(u_0)\,(w_0(h_1)h_2) \\
\alpha_{12} &= \overline{w_0} h_2 & u_1 &= (w_0(h_1)h_2)^{-1}\,(\Phi\Psi)^2(u_0)\,(w_0(h_1)h_2) & \beta_{120} &= (w_0(h_2)h_3)^{-1}\,(\Psi\Phi\Psi)(u_1)\,(w_0(h_2)h_3) \\
\alpha_{20} &= \overline{w_0} h_3 & u_2 &= (w_0(h_1)h_2)^{-1}\,(\Phi\Psi)^2(u_1)\,(w_0(h_1)h_2) & \beta_{012} &= (w_0(h_3)h_1)^{-1}\,(\Psi\Phi\Psi)(u_2)\,(w_0(h_3)h_1)\,.
\end{aligned}$$

Remark 4.3.1. Since all β_{ijk} are assigned elements of N_+, and any peripheral γ can be homotoped to follow only short edges, any ρ constructed this way is boundary-unipotent.

Also, given a point in $\mathcal{X}^+_{G,\Sigma}$, we can reconstruct a boundary-borel representation ρ (up to conjugation) as follows. We refer to [FG06b, Section 6], [GGZ15, Section 9] for more details.

Again, we start with an ideal triangulation $\mathcal{T}$ of Σ, together with a quiver Q and seed torus $T_Q^{\mathcal{X}}$ for each triangle in $\mathcal{T}$. Instead of triangulating, however, we doubly triangulate, as in Figure 4.7B. To compute $\rho(\gamma)$, we homotope γ to follow the edges, then multiply α_{ij}, β_{ijk}, or γ_{ijk} (or their inverses) as appropriate.

- We start with elements of $T_Q^{\mathcal{X}}$.
- Taking $(\mathcal{M})^{-1}$ gives elements of $\mathrm{Conf}_3^*(G/B_+)$ for each triangle. To compute edges at a particular triangle, assume its flags are $(g_0B_+, g_1B_+, g_2B_+) = (B_+, \overline{w_0}B_+, u_xB_+)$, and that the neighboring triangles have vertices $(0,1,3)$, $(1,2,4)$, and $(2,0,5)$ (with cosets g_3B_+, g_4B_+, and g_5B_+).
- Recall $n(u)$ of Definition 4.1.4. The element assignments are

$$\begin{aligned}
\alpha_{01} &= \overline{w_0} & \alpha_{12} &= \overline{w_0} & \alpha_{20} &= \overline{w_0} \\
\beta_{012} &= \Phi^{-1}(u_x) & \beta_{120} &= \overline{w_0}^{-1}u_x^{-1}\overline{w_0} & \beta_{201} &= \left(\Phi^{-1}(u_x^{-1})\right)^{-1} \\
\gamma_{012} &= n(g_3) & \gamma_{120} &= n([\overline{w_0}u_x^{-1}g_4]_-) & \gamma_{201} &= n([\overline{w_0}[\overline{w_0}^{-1}u_x^{-1}]_-^{-1}\overline{w_0}^{-1}u_x^{-1}g_5]_-]_-) \\
\gamma_{102} &= (n([\overline{w_0}g_3]_-))^{-1} & \gamma_{210} &= \left(n(u_x^{-1}g_4)\right)^{-1} & \gamma_{021} &= \left(n([\overline{w_0}[\overline{w_0}^{-1}u_x]_-\overline{w_0}^{-1}g_5]_-)\right)^{-1}
\end{aligned}$$

Remark 4.3.2. All middle edges are assigned elements of N_+, and all short edges are assigned elements of H. Therefore any peripheral loop lies in B_+ and ρ is boundary-borel.

Remark 4.3.3. Since $n(u)$ may require taking square roots, the field over which the $\mathcal{X}$-coordinates are defined may need to be extended in order to define ρ.

Since we can now explicitly interpret points in these moduli spaces as representations, we describe the positivity conditions of higher Teichmüller spaces.

Definition 4.3.4. Let N_+ be a maximal unipotent subgroup of G. Then Definition 2.5.1 gives coordinate charts on N_+ for each presentation of w_0. These charts are the **positive structure** on N_+. The elements of N_+ which have coordinates entirely in $\mathbb{R}_{>0}$ for all these charts are the **positive part** of N_+.

The maps χ_{ω_k} of Definition 2.2.1 are also a **positive structure** on the maximal torus H of G. Again, the **positive part** consists of the elements which have coordinates in $\mathbb{R}_{>0}$ by the positive structure.

By [FG06b, Section 8.1], together with the canonical form of Lemma 4.1.3, these provide a **positive structure**, and a **positive part**, of $\mathrm{Conf}_3^*(G/N_+)$. There is also a completely analogous positive structure on $\mathrm{Conf}_3^*(G/B_+)$ following [FG06b, Section 5.5].

Remark 4.3.5. Generalized minors are compatible with these positive structures, as from [FG06b, Theorem 5.1]. Therefore, positive points in the flag varieties $\mathrm{Conf}_3^*(G/K)$ are exactly those points for which all $\mathcal{A}$- and $\mathcal{X}$-coordinates in the cluster ensemble are in $\mathbb{R}_{>0}$.

Definition 4.3.6. The space $\mathcal{T}^+(\Sigma)$ is Teichmüller space, together with choices of orientation for non-cuspidal boundary components. For a surface Σ with $n \geq 0$ boundary circles $b_1, \ldots, b_n$, define

$$\mathcal{T}^+(\Sigma) = \{\, (p, \epsilon_1, \ldots, \epsilon_n) : p \in \mathcal{T}(S), \epsilon_i = \pm 1 \,\},$$

where ϵ_i is positive to denote that the chosen orientation of b_i agrees with that naturally induced by Σ.

Remark 4.3.7. When Σ has only cuspidal boundary components, $\mathcal{T}^+(\Sigma) = \mathcal{T}(\Sigma)$.

Definition 4.3.8. When G is centerless, we may repeat the above construction, but take the $\mathcal{X}$-coordinates of the cluster ensemble. This gives the moduli space $\mathcal{X}_{G,\Sigma}$ of **framed, G-local systems** on Σ by [FG06b, Section 2.1]. The $\mathbb{R}_{>0}$ points also give $\mathcal{X}^+_{G,\Sigma}$. This space is identified with $\mathcal{T}^+(\Sigma)$.

That $\mathcal{T}^+(\Sigma)$ appears instead of $\mathcal{T}(\Sigma)$ is rather a technicality. Restricting our attention to orderings of ideal triangulations that agree with the surface's natural orientation restrictrs $\mathcal{T}^+(\Sigma)$ to a set we can canonically identify with $\mathcal{T}(\Sigma)$.

Definition 4.3.9. When G is simply connected, choose an ordered, oriented triangulation of Σ. Associate a copy of Q to each triangle, and amalgamate all the quivers together by identifying shared edge vertices between triangles. The $\mathcal{A}$-coordinates of the cluster ensemble form the moduli space $\mathcal{A}_{G,\Sigma}$.

By [FG06b, Section 8.6] the $\mathbb{C}$-points of this moduli space parameterize **twisted, decorated representations** into G. The $\mathbb{R}_{>0}$-points give $\mathcal{A}^+_{G,\Sigma}$, a **higher Teichmüller space**. These correspond to flag varieties such that all generalized minors are strictly positive.

These moduli spaces have further interpretations and properties, explored in [FG06b], [FG06a], [FG09a], [GS18], etc.

4.4 Regarding 3-manifolds

The process described can be applied to 3-manifolds as well as surfaces. We omit all details. An element of $\mathrm{Conf}^*_4(G/K)$ is attached to each tetrahedron, and the coordinates for these flag varieties are encoded on a quiver for each face of the tetrahedron.

As in Definition 4.1.6, only two copies of $\mathrm{Conf}_3^*(G/K)$ are necessary to define an element in $\mathrm{Conf}_4^*(G/K)$, so any two copies of Q should determine all the coordinates of the tetrahedron. Therefore, the coordinates on any pair of faces determine the coordinates on the other pair. This relation is given by the mutation μ_{flip}.

Finally, as coordinates are identified along glued edges in the surface case, in the 3-manifold case they are glued along faces, as in Figure 4.8. When reconstructing ρ, the path γ is homotoped as before and the same elements on each segment are used.

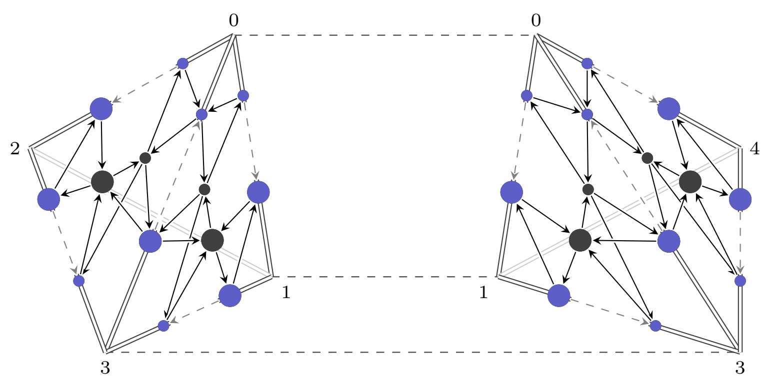


Figure 4.8: Two tetrahedra with a face identification. The quivers Q_{C_2} on the front two faces of each tetrahedron are shown.

The $\mathcal{A}$-coordinate variety constructed this way is called the **Ptolemy variety**, see [GGZ15], [GTZ15], [Zic19] for details. The variety constructed by the $\mathcal{X}$-coordinates will be the analogue of the **shape coordinates** of [GGZ15], and the defining equations will generalize Thurston's gluing equations.

Defining varieties by quivers in this fashion allows efficient computation, and databases have been constructed of Ptolemy varieties for large numbers of triangulations, See [Fal+].

Chapter 5: Main Result: Fock–Goncharov coordinate structures for non-A_n

We now present an algorithm for constructing Fock–Goncharov coordinate structures for a simple Lie group G over $\mathbb{C}$, thus satisfying most of Theorem 4.2.12. We delay all proofs until Chapter 7.

We make one slight demand: the ability to present w_0 via Fact 2.3.2. Luckily, this demand is satisfied as long as the Coxeter number h is even, equivalently $G \neq A_{2n}$. This is acceptable, since the A_n case has a particularly nice form which has been the subject of considerable study, as in [FG06b], [GTZ15]. See Chapter 6 for a review of the results in language consistent with this section.

5.1 Overview

Here we loosely describe the algorithm for constructing Q, μ_{rot}, μ_{flip}, and $\mathcal{M}$.

The quiver Q will be divided into an interior and three edges. Coordinates at each vertex will be assigned generalized minors of elements in the canonical form (h_1, h_2, h_3, u) of an element of Conf_3^*. Each edge should contain information for some h_i, and the interior should contain information for u.

One minor for each simple root determines an element of H; the coordinates on an edge will be of the form $\Delta^{\omega_j}(h_i)$. Accordingly, the edges of the triangle will be quivers of Dynkin type. The interior vertices will be given by generalized minors of the form $\Delta^{w_k \omega_{\mathbf{i}_k}}(u)$. Laying these out to satisfy [FZ99, Theorem 1.17] follows the algorithm of [BFZ05, Section 2]. This, together with our choice of presentation for w_0, means the interior will be a rectangular grid of vertices. Recall that these take the names $A_\bullet$, $B_\bullet$, $C_\bullet$, and $v_{i,j}$ by Definition 4.2.6.

The algorithm of Fomin–Zelevinsky describes a rectangle. Two of the rectangle's edges will be edges in the triangle. We call this Q_0. The chief difficulty of constructing Q is introducing the third edge to Q_0. To do this, we look ahead to μ_{rot}. Since that quiver mutation must rotate Q, and Q and Q_0 share a mutable portion, we can rotate Q_0. And since Q_0 already has two of Q's three edges, the third edge of Q can be deduced from the action of μ_{rot} on Q_0.

The flip mutation μ_{flip} is built in the same way that μ_{rot} is: repeated application of [FZ99, Theorem 1.17]. We need a bit of compensation before and after because the amalgamation does not quite line up the quivers as necessary.

Remark 5.1.1. Our construction will produce μ_{rot} and μ_{flip} as compositions of smaller mutations. They contain smaller mutations (μ_{rotTw} and μ_{flipTw}) which may be more useful for certain applications. See Section 5.7.

Finally, the map $\mathcal{M}$ is the generalized minors, as described above, together with a monomial compensation. At each step of μ_{rot} and μ_{flip}, [FZ99, Theorem 1.17] takes the form described in Section 3.2

$$(\text{initial}) \cdot (\text{final}) = (\text{left}) \cdot (\text{right}) + \prod_{j \neq \mathbf{i}_k} (\text{in same column}) \, .$$

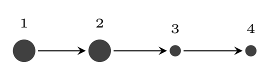
However, at the edges of Q, when j is very low or very high, the left or right elements might be trivially 1 according to the theorem. In our quivers, however, these elements are non-trivial, given by frozen vertices.

Therefore, we have to balance the equation of the theorem. To do so, we treat the edge coordinates in the mutation relation as “Extra” information, and include this extra information in our definition of interior coordinates. Then the quiver mutations take forms similar to the following:

$$\text{Extra} \cdot (\text{initial}) \cdot (\text{final}) = \overbrace{(\text{Extra} \cdot 1)}^{\text{left}} \cdot (\text{right}) + \text{Extra} \cdot \prod_{j \neq \mathbf{i}_k} (\text{in same column}) \, .$$

Since the extraneous factors appear in every term, the desired result still holds by the theorem. This is the effect of the monomial map m in $\mathcal{M}$.

Example 5.1.2. Based on the Dynkin diagram ●-●⇐●-● for F_4, the following is a well-rooted tree-like Dynkin quiver (following Definition 4.2.4) for F_4:



Remark 5.1.3. By Definition 4.2.1, the nodes with higher weight in the quiver correspond to the nodes which are “smaller” in the Dynkin diagram.

With this, we are ready to begin the construction. Let D be a well-rooted tree-like quiver of Dykin type for G. Let $c = \{ c_1, c_2, \ldots, c_n \}$ be the induced Coxeter element as in Definition 4.2.2, and let w_0 be the longest word, with presentation via Fact 2.3.2.

Remark 5.1.4. We will abuse notation slightly by writing $s_{i_1} s_{i_2} \cdots s_{i_n}$ as $\{ i_1, i_2, \ldots, i_n \}$.

5.2 Building the rectangle Q_0

We desire to construct a quiver that holds coordinates for most of $H^3 \times_H N_-$. Conveniently, $H \times N_- = B_-$ is the **double Bruhat cell** $G^{w_0,e} = B_+ w_0 B_+ \cap B_- e B_-$. We will not need any more information about double Bruhat cells, except to note that [BFZ05, Section 2] describes an algorithm which accepts u, v and creates a quiver whose cluster coordinates are coordinates on $G^{u,v}$. Therefore, we will follow this algorithm (with slight modifications) in the special case $u = w_0$, $v = e$. Since this simplifies the algorithm greatly, we can completely reproduce it here.

The result will be a rectangle of $(h/2) + 1$ copies of D, with the first and last copies being frozen, and having half-weight edges. The first copy of D will have vertices labeled $\{ B_i \}_{i \in D}$, the last will be labeled with $\{ C_i \}_{i \in D}$, The middle vertices will be labeled v_{jk}, where j is the corresponding entry in D, and k is the distance along the path from $B_\bullet$ to $C_\bullet$.

Begin by setting Q_0 to be D. Label the vertices $\{ v_{i0} \}_{i \in D}$. Let f_i be the "frontier vertices" of Q_0, with initially $f_i = v_{i0}$ for each $i \in D$. For convenience, let $j : v_{ij} \mapsto j$, so that we may refer to $j(f_i)$. Also for convenience, let $\sigma(f_j, f_k)$ be the weight of the edge between f_j and f_k (taking values in $\{ -1, 0, 1 \}$).

Now, proceed in order through the letters of w_0. For each letter k, let $j' =$

$j(f_k) + 1$.

- Add $v_{kj'}$, a vertex of the same weight as f_k, to the left of f_k.
- For each $\ell \in D$, add an edge of weight $-\sigma(f_k, f_\ell)$ from $v_{kj'}$ to f_ℓ.
- Add an edge of weight 1 from $v_{kj'}$ to f_k.
- Set f_k to $v_{kj'}$.

When finished, all f_k will be $v_{k(h/2)}$. Rename each v_{k0} to C_k, each $v_{k(h/2)}$ to B_k, and halve the weights of any edges if they connect two $B_\bullet$ vertices or two $C_\bullet$ vertices.

Remark 5.2.1. At each step, the frontier vertices form a sub-quiver of Dynkin type. Further, replacing v_{ij} with $v_{i(j+1)}$ operates, graphically, as a quiver mutation on the frontier.

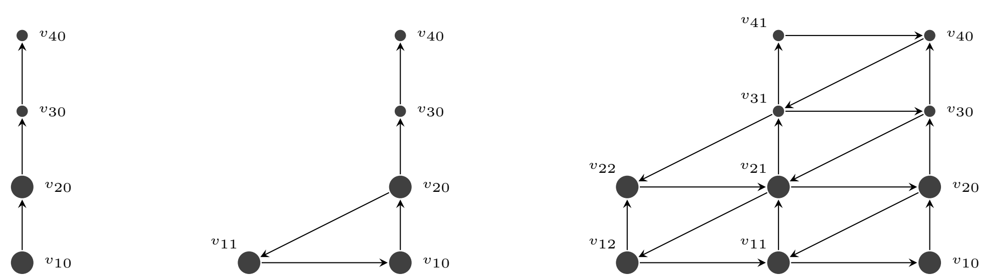


Figure 5.1: In-progress Q_0 for F_4 after 0, 1, and 6 letters of w_0.

Example 5.2.2. The Dynkin-type quiver for $G = F_4$ in Example 5.1.2, with $c = \{1, 2, 3, 4\}$ and $\ell = 6$ gives

$$w_0 = \{1, 2, 3, 4, 1, 2, 3, 4, 1, 2, 3, 4, 1, 2, 3, 4, 1, 2, 3, 4, 1, 2, 3, 4\}.$$

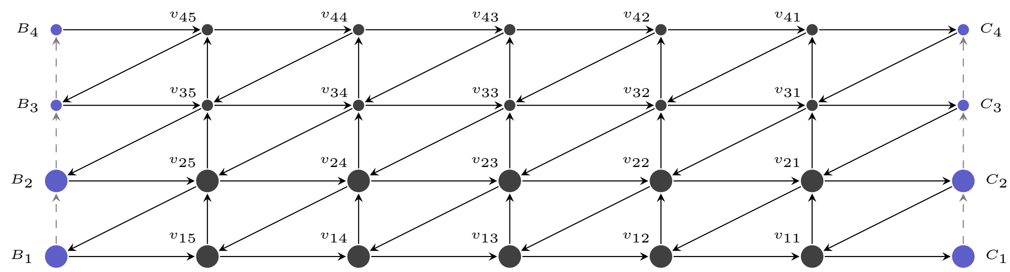


Figure 5.2: Completed Q_0 for F_4.

Carrying out the construction produces the rectangle shown in Figures 5.1 and 5.2.

Using [Gil19], the initial step of Q_0 can be generated by

```
clav-bfzIII -U -c F4 -v "1,2,3,4,1,2,3,4,1,2,3,4,1,2,3,4,1,2,3,4,1,2,3,4"
> Q0.clav.
```

5.3 Construction of μ_{rot}

The most difficult step is modifying Q_0 to become triangular by adding a third group of vertices, $A_\bullet$, also of half-Dynkin type. However, since the $A_\bullet$, $B_\bullet$, and $C_\bullet$ vertices will be frozen, the Q_0 alone is enough to describe the mutation sequence μ_{rot}. It will be composed of three main pieces.

- The mutation μ_{rotTw} transforms coordinates of α to coordinates of $\text{rot}(\alpha)$, in some order. The component mutation $\widetilde{\mu}_{\text{Col}}(j)$ performs a mutation on column j that adjusts the minor coordinates by concatenating the Coxeter element to each word.

- The mutation $\widetilde{\mu}_{\text{O1}}$ ("ordering-1") performs a σ_{A_ℓ} action on each row (a Dynkin diagram of A_ℓ) using $\widetilde{\mu}_{\text{RS}}$ ("row σ_{A_ℓ}")

- The mutation $\widetilde{\mu}_{\text{O2}}$ ("ordering-2") performs a σ_G action on each column (a

Dynkin diagram of G) using $\widetilde{\mu}_{\text{CS}}$ ("column σ_G"), as in Remark 2.1.7. When σ_G is trivial, this can be ignored.

Definition 5.3.1. For any quiver[1] with the same mutable portion as Q_0, fix an induced Coxeter element $c = \{ c_1, c_2, \ldots, c_n \}$, which admits a tree-like partition $\{ T_1, T_2, \ldots, T_m \}$. Let $\ell = \frac{h}{2} - 1$. The sequences μ_{rot} and μ_{rotTw} ("twisting rotation") perform almost the same function, though they permute the vertices of the quiver differently.

$$\widetilde{\mu}_{\text{T}}(i,j) = \prod_{z \in T_i} \mu \{ v_{zj} \}$$

$$\widetilde{\mu}_{\text{Col}}(j) = \prod_{i=1}^{m} \widetilde{\mu}_{\text{T}}(i,j)$$

$$\widetilde{\mu}_{\text{RS}}(i) = \left[\prod_{k=1}^{\ell} \prod_{j=0}^{\ell-k} \widetilde{\mu}_{\text{T}}(i, 1+j) \right] \left[\prod_{j=1}^{\ell} \widetilde{\mu}_{\text{T}}(i, \ell - j) \right]$$

$$\widetilde{\mu}_{\text{CS}}(j) = \left[\prod_{i=0}^{\ell+1} \widetilde{\mu}_{\text{Col}}(j) \right]$$

$$\mu_{\text{rotTw}} = \prod_{x=1}^{\ell} \prod_{y=1}^{\ell+1-x} \widetilde{\mu}_{\text{Col}}(y)$$

$$\widetilde{\mu}_{\text{O1}} = \prod_{i=1}^{m} \widetilde{\mu}_{\text{RS}}(i), \quad \widetilde{\mu}_{\text{O2}} = \begin{cases} \prod_{j=1}^{\ell} \widetilde{\mu}_{\text{CS}}(j) & \sigma_G \text{ non-trivial} \\ \varnothing & \sigma_G \text{ trivial} \end{cases}$$

$$\mu_{\text{rot}} = [\mu_{\text{rotTw}}] \, [\widetilde{\mu}_{\text{O1}}] \, [\widetilde{\mu}_{\text{O2}}]$$

(Recall that, by Notation 2.6.5, multiplied mutations are applied left-to-right, and

[1] We are vague because we will use these mutations on Q_0 to find Q, and then on Q itself as μ_{rot}.

that products carry an ordering.)

Remark 5.3.2. An alternate presentation of μ_{rot} which has the same effect on quivers may be given as roughly "μ_{rotTw} four times", though we will focus on the above definition.

$$\left(\left[\prod_{x=1}^{\ell}\prod_{y=1}^{\ell+1-x}\widetilde{\mu}_{\text{Col}}(y)\right]\left[\prod_{x=1}^{\ell}\prod_{y=1}^{\ell+1-x}\widetilde{\mu}_{\text{Col}}(\ell+1-y)\right]\right)^2.$$

Example 5.3.3. Continuing from Example 5.2.2 gives the following[2]:

$$\mu_{\text{rotTw}} = \mu\left\{\begin{array}{ccccc}
 & & & & v_{11},v_{21},v_{31},v_{41}, \\
 & & & v_{12},v_{22},v_{32},v_{42}, & v_{11},v_{21},v_{31},v_{41}, \\
 & & v_{13},v_{23},v_{33},v_{43}, & v_{12},v_{22},v_{32},v_{42}, & v_{11},v_{21},v_{31},v_{41}, \\
 & v_{14},v_{24},v_{34},v_{44}, & v_{13},v_{23},v_{33},v_{43}, & v_{12},v_{22},v_{32},v_{42}, & v_{11},v_{21},v_{31},v_{41}, \\
v_{15},v_{25},v_{35},v_{45}, & v_{14},v_{24},v_{34},v_{44}, & v_{13},v_{23},v_{33},v_{43}, & v_{12},v_{22},v_{32},v_{42}, & v_{11},v_{21},v_{31},v_{41}
\end{array}\right\}$$

$$\widetilde{\mu}_{\text{O1}} = \mu\left\{\begin{array}{lllll}
w_{41},w_{42},w_{43},w_{44},w_{45}, & w_{41},w_{42},w_{43},w_{44}, & w_{41},w_{42},w_{43}, & w_{41},w_{42}, & w_{41}, \\
w_{45},w_{44},w_{43},w_{42},w_{41}, & & & & \\
w_{31},w_{32},w_{33},w_{34},w_{35}, & w_{31},w_{32},w_{33},w_{34}, & w_{31},w_{32},w_{33}, & w_{31},w_{32}, & w_{31}, \\
w_{35},w_{34},w_{33},w_{32},w_{31}, & & & & \\
w_{21},w_{22},w_{23},w_{24},w_{25}, & w_{21},w_{22},w_{23},w_{24}, & w_{21},w_{22},w_{23}, & w_{21},w_{22}, & w_{21}, \\
w_{25},w_{24},w_{23},w_{22},w_{21}, & & & & \\
w_{11},w_{12},w_{13},w_{14},w_{15}, & w_{11},w_{12},w_{13},w_{14}, & w_{11},w_{12},w_{13}, & w_{11},w_{12}, & w_{11}, \\
w_{15},w_{14},w_{13},w_{12},w_{11} & & & &
\end{array}\right\}$$

$$\widetilde{\mu}_{\text{O2}} = \mu\{\ \}$$

Lemma 5.3.4. *$\mu_{rot}\circ\mu_{rot}\circ\mu_{rot}$ induces the identity on Q_0. Furthermore, μ_{rot} induces a graph isomorphism on the mutable portion of the Q_0 (the sub-quiver containing only the $\{v_{ij}\}$ vertices).*

5.4 The $A_\bullet$ edge

With μ_{rotTw} in hand, we are ready to construct $A_\bullet$. Let Q_1 be Q_0 together with a dummy $A_\bullet$, which is of Dynkin type D with edges removed. Denote $Q_i' = \mu_{\text{rotTw}}(Q_i)$ and $Q_i'' = (\mu_{\text{rotTw}})^{-1}(Q_i)$.

[2]The elements of the sequence are presented in usual reading order. The spacing is to emphasize decomposition into $\widetilde{\mu}_{\text{Col}}$ terms.

We would like Q_1 to be isomorphic to Q_1' and Q_1'' via φ and φ^{-1}, with

$$\varphi : \{\, v_{ij} \mapsto v_{ij}, \qquad A_i \mapsto C_i, \qquad B_i \mapsto A_i, \qquad C_i \mapsto B_i \,\},$$

but unfortunately these are not isomorphisms. However, we can correct for this. Let Q_2 be the quiver containing the vertices of Q_1, and with edges defined by

$$\sigma_{Q_2}(v,w) = \sum \{\, \sigma_{Q_1}(v,w), \sigma_{\varphi^{-1}(Q_1')}(v,w), \sigma_{\varphi(Q_1'')}(v,w) \,\}.$$

That is, we repeatedly rotate Q_1, taking the inclusion of all frozen vertices necessary to ensure that μ_{rot} is of order 3 on the entire quiver.

Remark 5.4.1. It is not yet obvious that μ_{rot} actually does rotate by a third. The proof of Lemma 7.3.1, however, will show that μ_{rot} acts with order 3 on the seed torus for Q_0, and therefore on the quiver. There is no circular dependency here, as that lemma does not depend on the existence of $A_\bullet$, $B_\bullet$, or $C_\bullet$.

We therefore define Q to be Q_2, and we have constructed Q and μ_{rot}.

Remark 5.4.2. To impose Definition 4.2.6, we will customarily put $A_\bullet$ on the $(0,1)$ edge, $B_\bullet$ on the $(1,2)$ edge, and $C_\bullet$ on the $(2,0)$ edge, with the v_{ij} as face vertices.

Example 5.4.3. Continuing from Example 5.3.3, we obtain the quivers of Figure 5.3. Merging them, we obtain $Q = Q_2$ as in Figure 5.4.

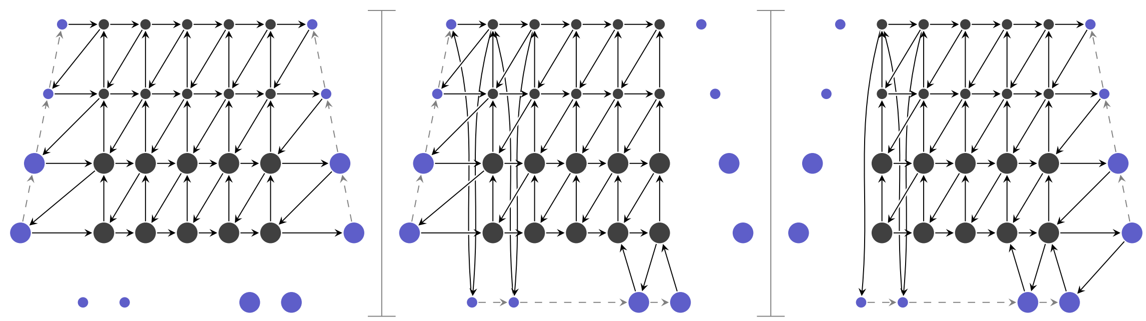


Figure 5.3: Q_1, (Q_1'), and (Q_1'') for F_4 (rotated to agree with id, φ^{-1}, and φ respectively).

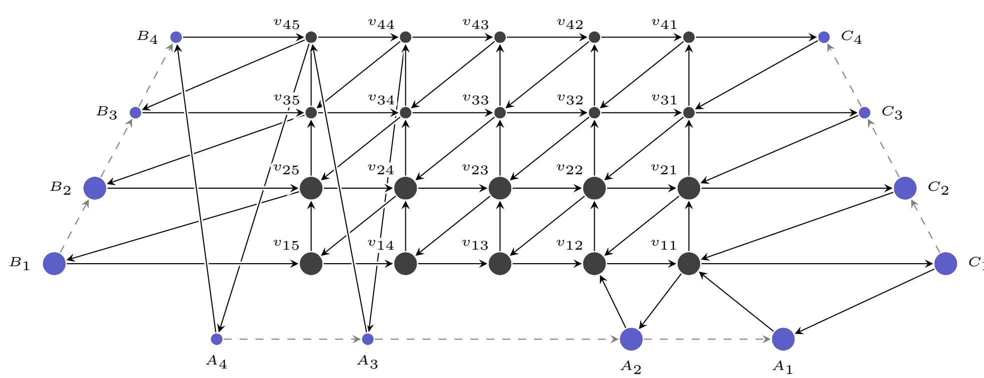


Figure 5.4: Q for F_4.

5.5 Construction of μ_{flip}

As in the case of μ_{rot}, the flip mutation is given by a tedious, repetitive sequence which looks like intertwined mutations on each row and column. It is again composed of pieces.

- The mutation $\widetilde{\mu}_{\text{P}}$ ("Pre-mutation") rotates the sub-quivers on the left and right into a position where all the $w_{i,j}$ vertices forma rectangle. See Figure 5.5.
- The mutation $\widetilde{\mu}_{\text{Flipcore}}$ is the core of the flip. It performs an analogous function to μ_{rotTw}: sequences of $\widetilde{\mu}_{\text{Col}}(j)$ mutations which adjusts minor coordinates (in the sense of Chapter 3) in column j.
- The mutation $\widetilde{\mu}_{\text{O3}}$ ("ordering-3") mirrors the rectangle of $w_{i,j}$ coordinates horizontally, again by Remark 2.1.7. The mutations $\widetilde{\mu}_{\text{O4}}$ and $\widetilde{\mu}_{\text{O5}}$ ("ordering-4" and "ordering-5") do the same, but restricted to the left and right triangles. So the product $\widetilde{\mu}_{\text{O3}}\widetilde{\mu}_{\text{O4}}\widetilde{\mu}_{\text{O5}}$ switches positions of the left and right triangle interiors by translation as a composition of reflection.

To distinguish notation from Q, we shall label the edges of Q^{0-2} by $D_\bullet$, $E_\bullet$, $F_\bullet$, $G_\bullet$, and the interior, mutable vertices by w_{ij}. See Figure 5.5.

For μ a mutation defined on Q, let μ_*^R (resp. μ_*^L) be the mutation defined to act on the right (left) part of the double quiver. Technically, replace each v_{ij} with $w_{i(\ell+1-j)}$ (with $w_{i(L+1-j)}$) in $\mu_\bullet$ to obtain $\mu_\bullet^R$ ($\mu_\bullet^L$).

Definition 5.5.1. Construct Q^{0-2} by Definition 4.2.7 and labeled as in Figure 5.5.

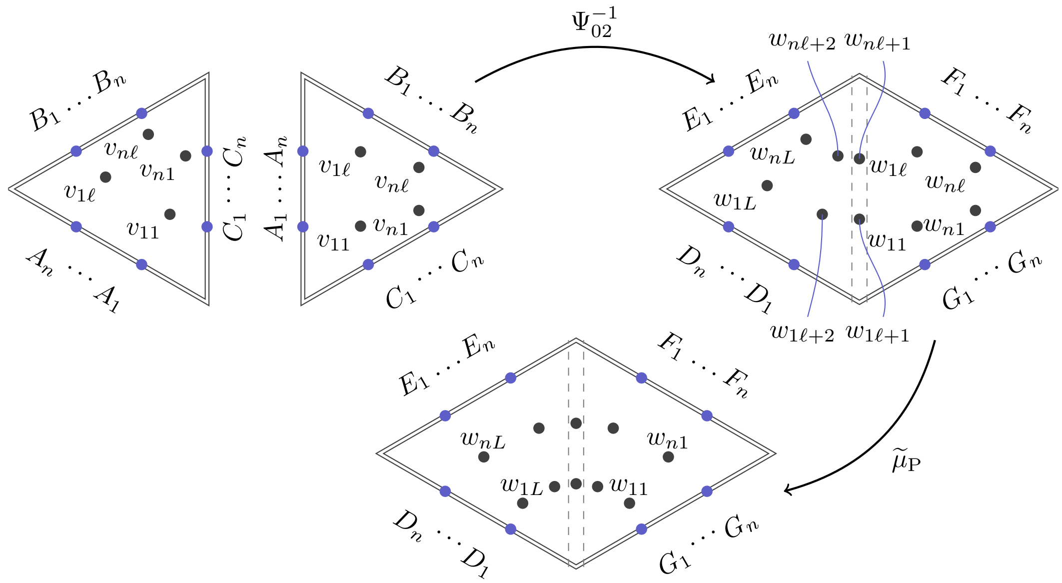


Figure 5.5: Vertex naming for Q^{0-2} (with $L = 2\ell + 1$), agreeing with Figure 4.4A and labeled to induce a mutable rectangle in $\widetilde{\mu}_{\mathrm{P}}(Q^{0-2})$.

Recall that m is the number of partitions T_i of c from Definition 4.2.3. Some names such as $\widetilde{\mu}_{\mathrm{T}}$ are re-used from Definition 5.3.1 with slightly different meanings.

$$\widetilde{\mu}_{\mathrm{P}} = \left(\mu^{L}_{\mathrm{rotTw}}\right)^{-1} \left(\mu^{R}_{\mathrm{rotTw}}\right)^{-1} \qquad \text{(See Definition 5.3.1)}$$

$$\widetilde{\mu}_{\mathrm{T}}(i,j) = \prod_{k \in T_i} \mu \{ w_{kj} \} \quad \text{(possibly empty)}$$

$$\widetilde{\mu}_{\mathrm{Col}}(j) = \prod_{i=1}^{m} \widetilde{\mu}_{\mathrm{T}}(i,j)$$

$$\widetilde{\mu}_{\mathrm{Flipcore}} = \prod_{i=1}^{L} \prod_{j=i}^{L} \widetilde{\mu}_{\mathrm{Col}}(L+i-j)$$

$$\widetilde{\mu}_{\mathrm{RS}}(i,a,b) = \left[\prod_{k=0}^{b-a} \prod_{j=0}^{(b-a)-k} \widetilde{\mu}_{\mathrm{T}}(i,a+j) \right] \left[\prod_{j=0}^{b-a} \widetilde{\mu}_{\mathrm{T}}(i,b-j) \right]$$

$$\widetilde{\mu}_{\mathrm{O3}} = \prod_{i=1}^{m} \widetilde{\mu}_{\mathrm{RS}}(i,1,L), \quad \widetilde{\mu}_{\mathrm{O4}} = \prod_{i=1}^{m} \widetilde{\mu}_{\mathrm{RS}}(i,\ell+2,L), \quad \widetilde{\mu}_{\mathrm{O5}} = \prod_{i=1}^{m} \widetilde{\mu}_{\mathrm{RS}}(i,1,\ell)$$

$$\mu_{\mathrm{flipTw}} = [\widetilde{\mu}_{\mathrm{P}}] \, [\widetilde{\mu}_{\mathrm{Flipcore}}]$$

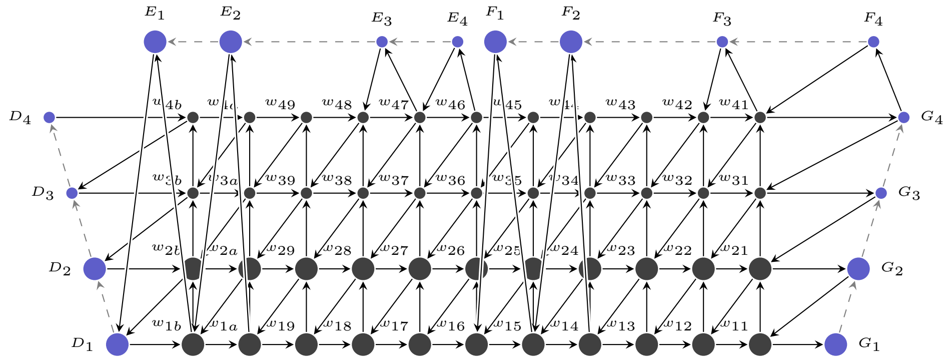


Figure 5.6: $\widetilde{\mu}_{\mathrm{P}}(Q^{0\text{–}2})$ for F_4. We apologize for the hexadecimal notation; rows have > 9 vertices.

$$\mu_{\text{flip}} = [\widetilde{\mu}_{\mathrm{P}}]\,[\widetilde{\mu}_{\text{Flipcore}}]\,[\widetilde{\mu}_{\mathrm{O3}}]\,[\widetilde{\mu}_{\mathrm{O4}}]\,[\widetilde{\mu}_{\mathrm{O5}}]$$

Remark 5.5.2. Because $\widetilde{\mu}_{\mathrm{P}}$ is composed entirely of rotation mutations, we are morally justified in focusing on $\widetilde{\mu}_{\mathrm{P}}(Q^{0\text{–}2})$ instead of $Q^{0\text{–}2}$: it makes the isomorphism with $Q^{1\text{–}3}$ more evident.

The mutation $\widetilde{\mu}_{\mathrm{RS}}(i, a, b)$ permutes the $\mathcal{A}$-coordinates at those vertices, and is purely used for rearranging.

Example 5.5.3. Continuing from Example 5.4.3 constructs $Q^{0\text{–}2}$ such that $\widetilde{\mu}_{\mathrm{P}}(Q^{0\text{–}2})$ is as in Figure 5.6. The sequence of mutations defining the flip is given in Figure 5.7.

Remark 5.5.4. At intermediate stages of μ_{rot} and μ_{flip}, the quiver may contain edges with weights higher than 1. This is one of several phenomena which do not appear in the A_n case. It is possible that alternate presentations of mutations exist which do not exhibit this.

5.6 The map $\mathcal{M}$

We now have Q and the mutations, at least graphically. What remains is to finalize the association to the Lie group G. More specifically, the vertices of Q should be associated to coordinates on $\mathrm{Conf}_3^*(G/N_+)$.

Recall that $\mathrm{Conf}_3^*(G/N_+) \cong H^3 \times_H (N_- \cap \overline{w_0} G_0)$ by Lemma 4.1.3. So let $\alpha \in \mathrm{Conf}_3^*(G/N_+)$ be given by $\alpha = (h_1, h_2, h_3, u)$. By abuse of notation, we denote the coordinate in $T_Q^{\mathcal{A}}$ by the name of the vertex to which it is associated, and regard

$$T_Q^{\mathcal{A}} = \overbrace{\{\, A_i \,\}}^{\widetilde{T}_A^{\mathcal{A}}} \times \overbrace{\{\, B_i \,\}}^{\widetilde{T}_B^{\mathcal{A}}} \times \overbrace{\{\, C_i \,\}}^{\widetilde{T}_C^{\mathcal{A}}} \times \overbrace{\{\, v_{ij} \,\}}^{\widetilde{T}_V^{\mathcal{A}}}.$$

We will define $Y : \mathrm{Conf}_3^*(G/N_+) \to T_Q^{\mathcal{A}}$ and a let $\mathcal{M}$ be in terms of Y.

Definition 5.6.1. Let $k(i,j) = i + rj$. (This has the property that v_{ij} is the $k(i,j)^{\text{th}}$ vertex added to Q_0 in the algorithm of Section 5.2.) Alternately, the prefix of w_0^{-1} used to place vertex v_{ij} in Q_0 via the algorithm of [BFZ05, Section 2] has length $(m+1) - k$, and is equal to $w_{k(i,j)}$ by Definition 2.4.8. Using Definition 2.4.6,

$$
\begin{aligned}
Y_A &: \alpha \mapsto \{\, A_i = \Delta^{\omega_i}(h_1) \,\} \\
Y_B &: \alpha \mapsto \{\, B_i = \Delta^{\omega_i}(h_2) \,\} \\
Y_C &: \alpha \mapsto \{\, C_i = \Delta^{\omega_i}(h_3) \,\} \\
Y_V &: \alpha \mapsto \{\, v_{ij} = \Delta^{w_{k(i,j)}\omega_i}(u) \,\} \\
Y &: \alpha \mapsto Y_A(\alpha) \oplus Y_B(\alpha) \oplus Y_C(\alpha) \oplus Y_V(\alpha)
\end{aligned}
$$

We then define a monomial map $m : T_Q^{\mathcal{A}} \to T_Q^{\mathcal{A}}$ (using Definitions 2.1.6 and 2.4.11) as

$$
\begin{aligned}
m &: A_i \mapsto A_i \\
m &: B_i \mapsto B_i \\
m &: C_i \mapsto C_i \\
m &: v_{ij} \mapsto v_{ij} \cdot \Delta^{\omega_i}(\sigma_G(h_1^{-1})h_2) \cdot \Delta_{\oplus}^{w_{k(i,j)}\omega_i}(\sigma_G(h_1))
\end{aligned}
$$

We define $\mathcal{M} = m \circ Y$.

Remark 5.6.2. The map $\mathcal{M}$ is a birational equivalence, following from [FZ99, Section 2.7].

Example 5.6.3. Continuing from Example 5.5.3, let $\alpha = (h_1, h_2, h_3, u)$. Let $h_{ij} =$

$\Delta^{\omega_j}(h_i)$. Then Y_V produces the following:

$$A_1 = h_{11}, \quad A_2 = h_{12}, \quad \ldots, \quad C_4 = h_{34}, \quad v_{11} = \Delta^{w_5\omega_1}(u), \quad v_{12} = \Delta^{w_6\omega_2}(u), \quad \ldots, \quad v_{45} = \Delta^{w_{24}\omega_4}(u).$$

To compute the monomial map, refer to figure Figure 5.8. For example, to calculate $m(v_{23})$, we have $i = 2$, so $\Delta^{\omega_i}(\sigma_G(h_1^{-1})h_2) = \frac{h_{22}}{h_{12}}$. For the second factor, $k = k(2,3) = 2 + 4 \cdot 3 = 14$. Taking the numerator of the $i = 2$, $k = 14$ entry gives $t_1^2 t_2$, so $\Delta_\oplus^{w_{14}\omega_i}(\sigma_G(h_1)) = h_{11}^2 h_{12}$. This gives $m(v_{23}) = v_{23} \cdot \frac{h_{22}}{h_{12}} \cdot h_{11}^2 h_{12}$, and $\mathcal{M}(\alpha)$ has $v_{23} = \Delta^{w_{14}\omega_2}(u) \cdot h_{11}^2 h_{22}$.

Repeating this for all others, $\mathcal{M}(\alpha)$ has coordinates as given in Figure 5.9.

Remark 5.6.4. There is good reason to suspect that this construction is not unique up to mutation equivalency of the quivers. It is certainly not unique if one relaxes the construction of $\mathcal{M}$ as a monomial map applied to generalized minors. For example, during our investigation $\mathcal{M}$ was considered as $\mathrm{Conf}_3^*(G/N_+) \times \langle \sigma_G \rangle \to T_Q^{\mathcal{A}} \times \langle \sigma_G \rangle$, where ${\mu_{\mathrm{rot}}}^*$ acted by $+1$ on the second factor. An infinite family of $\mathcal{M}$ maps were found, depending on this factor of $[\tau] \in \langle \sigma_G \rangle$ to varying extent.

5.7 The significance of the "twisting" mutations

The quiver mutations μ_{rot} and μ_{flip} are intended to compose easily without requiring renaming the vertices of the quiver. For example, ${\mu_{\mathrm{rot}}}^2(Q)$ should be identical to ${\mu_{\mathrm{rot}}}^{-1}(Q)$.

Unfortunately, quiver mutations that yield equivalent seeds do not necessarily preserve the positions of those seeds. For example, consider the classic "pentagon recurrence", which happens to be equivalent to $\widetilde{\mu}_{\mathrm{RS}}$ on a row which is of Dynkin type A_2. The switching of positions is exactly the the action of σ_{A_2} on the Dynkin diagram.

The $\widetilde{\mu}_{\mathrm{Oi}}$ parts of μ_{rot} and μ_{flip}, distinguishing them from μ_{rotTw} and μ_{flipTw}, are exactly to address these actions of σ_G. If desired, shorter mutations may be used at the expense of slightly more complicated identifications between variables.

Chapter 6: Construction for A_n

The A_n case was studied in detail in [GTZ15]. We merely restate the conclusions in language consistent with the above.

Remark 6.0.1. That the algorithm of Chapter 5 does not work for A_{2n} can be seen in a few different ways, which are interconnected.

- There is no general formula for a Coxeter element c that yields a longest-word presentation $w_0 = cc \cdots c$.
- The Coxeter number for A_{2n} is odd.
- The action of σ_G for A_{2n} preserves no simple roots, therefore there can be no tree partitioning with a unique root node.

Proposition 6.0.2. *For G of type A_n, Theorem 4.2.12 holds.*

Proof. For the construction of Q, take the quiver consisting of mutable vertices $\{ v_{ij} : 1 \leq j \leq n, 1 \leq i \leq n-j \}$ and frozen vertices $\{ A_i, B_i, C_i : 1 \leq i \leq n \}$, all of weight 1. We consider

$$A_i = v_{0,i}, \quad B_i = v_{i,n+1-j}, \quad C_i = v_{i,0}.$$

The edges are given by

$$\sigma(a,b) = \begin{cases} c(a,b) & a = v_{i,j}, b = v_{i+1,j} \text{ or } a = v_{i,j}, b = v_{i-1,j+1} \text{ or } a = v_{i,j}, b = v_{i,j-1} \\ -c(a,b) & a = v_{i+1,j}, b = v_{i,j} \text{ or } a = v_{i-1,j+1}, b = v_{i,j} \text{ or } a = v_{i,j-1}, b = v_{i,j} \\ 0 & \text{else,} \end{cases}$$

and $c(a,b)$ is $\frac{1}{2}$ if a and b are both A_is, both B_is, or both C_is, and is 1 otherwise.

The mutation $\mu_{\text{rot}} = \mu\{ \ \}$ is trivial.

Since the mutable portion of Q_{A_n} is not rectangular, we need another convention for describing coordinates of $Q^{0\text{–}2}$ in order to describe μ_{flip}. We present this by example in Figure 6.1. □

Mechanically, the inclusion of the left Q_{A_n} into $Q^{0\text{–}2}$ is the following:

$$A_i \mapsto D_i, \qquad B_i \mapsto E_i, \qquad C_i \mapsto w_{i,n}, \qquad v_{i,j} \mapsto w_{i,j+n},$$

and for the right Q_{A_n} into $Q^{0\text{–}2}$, the following:

$$A_i \mapsto w_{i,n}, \qquad B_i \mapsto F_i, \qquad C_i \mapsto G_i, \qquad v_{i,j} \mapsto w_{j,n+1-i},$$

The mutation μ_{flip} is then the "diamond sequence":

$$\begin{aligned} t(\ell, j) &= \frac{\ell+1}{2} - \left| j - \frac{\ell+1}{2} \right| \\ \widetilde{\mu}_{\text{Rect}}(\ell) &= \prod_{k=1}^{\ell} \prod_{j=0}^{n-\ell} \mu \left\{ w_{t(\ell,k)+j,2k+n-(\ell+1)} \right\} \\ \mu_{\text{flip}} &= \prod_{\ell=1}^{n} \widetilde{\mu}_{\text{Rect}}(\ell) \end{aligned}$$

The map $\mathcal{M}$ is constructed as in Definition 5.6.1. Where needed, the presentation of the longest word is given by

$$\mathbf{i} = (1, 2, \ldots, n, \quad 1, 2, \ldots, n-1, \quad \ldots, \quad 1, 2, \quad 1).$$

Chapter 7: Proof of Theorem 4.2.12 for simple G

7.1 Overview

Our goal is to prove Theorem 4.2.12. We will only consider the case where G is simple, as products are handled by Chapter 8. We need to show that, for a fixed

G, the results Q, μ_{rot}, μ_{flip}, and $\mathcal{M}$ satisfy all the requirements of Definitions 4.2.8 and 4.2.9.

Proof of Theorem 4.2.12. If G is not simple, then we may appeal to Lemma 8.0.1 and recurse, so assume G is simple. For G of type A_n, Proposition 6.0.2 is sufficient, so assume G is not of type A_n.

The algorithm of Chapter 5 produces Q, μ_{rot}, μ_{flip}, and $\mathcal{M}$. We must show these satisfy Definition 4.2.9.

- Item 1 (triangularity) is handled last. As described in Section 5.4, we need to know that μ_{rot} is of order 3 on the $\mathcal{A}$-coordinates before concluding that the quiver is triangular.

- Item 2 (that the edges are half-Dynkin) is evident by construction.

- Item 3 (that $\mathcal{M}$ is a birational equivalence) is given by Remark 5.6.2.

- We handle Item 4 (the rotation) with Lemmas 7.3.1 and 7.3.2.

- We handle Item 5 (the flip) with Lemma 7.4.1.

- Now, recalling Remark 5.4.1, since μ_{rot}^* is of order 3 acting on $T_Q^{\mathcal{A}}$ and the seed torus of a cluster determines the quiver, μ_{rot} is of order 3 on Q. Likewise, μ_{flip} must transform $Q^{0\text{–}2}$ to $Q^{1\text{–}3}$. Therefore, Q has triangulation-compatible symmetry and Item 1 is satisfied.

□

In what follows, we assume G a simple Lie group over $\mathbb{C}$ of type other than A_n, and that the Q, μ_{rot}, μ_{flip}, $\mathcal{M}$ are given by Chapter 5 for G.

7.2 Ordering mutations

The ends of both μ_{rot} and μ_{flip} are sequences of $\widetilde{\mu}_{\mathrm{RS}}$ and $\widetilde{\mu}_{\mathrm{CS}}$. We must show that these permute Dynkin sub-quivers of Q by switching certain vertices without changing the cluster seed, as in Remark 2.1.7. These are used in Lemmas 7.3.1, 7.3.2 and 7.4.1 to justify the $\widetilde{\mu}_{\mathrm{Oi}}$ parts.

Lemma 7.2.1. *The permutation $\widetilde{\mu}_{CS}$ of Definitions 5.3.1 and 5.5.1 acts on quivers (and cluster ensembles) by permuting the vertices of the appropriate column according to σ_G.*

Lemma 7.2.2. *The permutation $\widetilde{\mu}_{RS}$ of Definitions 5.3.1 and 5.5.1 acts on quivers (and cluster ensembles) by permuting the vertices of the mutated sub-quiver according to σ_{A_ℓ}.*

These proofs rely heavily on results of [YZ08] described in Section 3.1. In short, we shall restrict the quivers on which $\widetilde{\mu}_{\mathrm{CS}}$ and $\widetilde{\mu}_{\mathrm{RS}}$ act to those which Yang–Zelevinsky can strongly analyze. We then use counting arguments to establish that the mutation acts by permutation, and use classic results to apply these results to larger quivers.

Proof of Lemma 7.2.1. In our construction, whenever $\widetilde{\mu}_{\mathrm{CS}}$ is applied, the vertices at which it mutates form a subquiver of Dynkin type. Therefore, first, we shall show that the result holds when $\widetilde{\mu}_{\mathrm{CS}}$ is applied to a quiver which is of Dynkin type. Then we shall show that the result holds for larger quivers which contain a sub-quiver of Dynkin type.

First, suppose $\widetilde{\mu}_{\mathrm{CS}}$ acts on quiver of Dynkin type (non A_{2n}). There exists an element g of $L^{c,c^{-1}}$ such that the initial cluster coordinate at v_k is $\Delta_{\omega_k,\omega_k}(g)$.

Now we note that $\widetilde{\mu}_{\mathrm{CS}}$ is exactly $\ell+2 = \frac{h}{2}+1$ iterations of a mutation following the Coxeter element c, with each mutation replacing $\Delta_{c^m\omega_k,c^m\omega_k}$ with $\Delta_{c^{m+1}\omega_k,c^{m+1}\omega_k}$. Since the number $h(i;c)$ is always $\frac{h}{2}$ by our construction of c, from Proposition 3.1.3 we obtain that the final cluster coordinate at v_k is

$$\Delta_{c^{h(k;c)+1}\omega_k,c^{h(k;c)+1}\omega_k}(g) = \Delta_{\omega_{k^*},\omega_{k^*}}(g),$$

which was the initial coordinate at v_{k^*}.

Thus $\widetilde{\mu}_{\mathrm{CS}}$ acts on the cluster by permuting the associations between vertices and cluster variables according to σ_G. In this way we obtain the desired result without ever having to rely directly facts of $g \in L^{c,c^{-1}}$.

Now it must be shown that this holds when $\widetilde{\mu}_{\mathrm{CS}}$ acts on a larger quiver. Since the action of $\widetilde{\mu}_{\mathrm{CS}}$ is solely a permutation on the mutable portion of the quiver, it preserves the set of cluster variables. By [FZ03, Theorem 1.12] and the finiteness of Dynkin-type quivers, the cluster variables determine the exchange matrix and therefore the quiver. Therefore the action of $\widetilde{\mu}_{\mathrm{CS}}$ on the graph must be trivial up to renaming, and therefore must be exactly the renaming that we have created. □

Proof of Lemma 7.2.2. We would like to apply exactly the same argument as in the proof of Lemma 7.2.1. However, since rows are of type A_ℓ and our construction algorithm for Q does not allow $\widetilde{\mu}_{\mathrm{RS}}$ to be repeated iterations of a single Coxeter-style mutation, we must be a bit more careful.

For a type A_ℓ, define $Q_{\mathrm{Lin}} = Q_{A_\ell,\mathrm{Lin}}$ as the Dynkin quiver where the graph describes a linear ordering from left to right. Also, let $Q_{\mathrm{Alt}} = Q_{A_\ell,\mathrm{Alt}}$ be the Dynkin quiver in source-sink position, see Figure 7.1.

We also introduce one more piece of notation.

Fact 7.2.3. *Let $Y(x)$ be defined as $\frac{1}{2}$ when x is odd and 0 when x is even. Then*

$$Y(a) - Y(b) = (-1)^{\text{parity } a} Y(a-b).$$

As above, we will interpret the cluster coordinates of the mutable Dynkin-type quiver as minor coordinates on some $g \in L^{c,c^{-1}}$, but we cannot interpret them as the initial minors.

The mutation $\widetilde{\mu}_{\mathrm{RS}}$ acts on $\ell = b - a$ vertices, always arranged as in Q_{Lin} of Figure 7.1. Recalling notation of Proposition 3.1.2, by [YZ08] the quiver Q_{Alt} (together with cluster variables $\Delta_{\omega_i,\omega_i}$ at v_i) is an initial seed, governed by an element of $L^{t_+t_-,(t_+t_-)^{-1}}$.

It is straightforward that (up to Langlands dualizing, which has no effect on coordinates) the mutation

$$\widetilde{\mu}_{\mathrm{Lin}} = \begin{cases} \mu\,\{\, v_1, v_3, \ldots, v_{\ell-0}, \quad v_2, v_4, \ldots, v_{\ell-1}, \quad \ldots, \quad v_1, v_3, \quad v_2, \quad v_1 \,\} & \ell \text{ odd} \\ \mu\,\{\, v_1, v_3, \ldots, v_{\ell-1}, \quad v_2, v_4, \ldots, v_{\ell-2}, \quad \ldots, \quad v_1, v_3, \quad v_2, \quad v_1 \,\} & \ell \text{ even} \end{cases}$$

$$= \prod_{j=2\lfloor \ell/2 \rfloor-(\ell-1)}^{\ell-1} \begin{cases} \prod_{k=1}^{\lceil(\ell-j)/2\rceil} \mu\,\{\, v_{2k-1} \,\} & \ell-j \text{ odd} \\ \prod_{k=1}^{(\ell-j)/2} \mu\,\{\, v_{2k} \,\} & \ell-j \text{ even} \end{cases}$$

transforms Q_{Alt} to Q_{Lin}, using only source or sink mutations (thus governed by primitive exchange relations). By considering these relations, and counting mutations, it is also straightforward that the cluster coordinates of Q_{Lin} are given by the following.

$$\text{coordinate at } v_k \text{ is } \Delta_{c^{(\ell-k)/2+Y(\ell)+Y(k)}\omega_k,\, c^{(\ell-k)/2+Y(\ell)+Y(k)}\omega_k}$$

For example, coordinates of $Q_{A_4,\mathrm{Lin}}$ are $\{\, \Delta_{c^2\omega_1,c^2\omega_1}, \Delta_{c^1\omega_2,c^1\omega_2}, \Delta_{c^1\omega_3,c^1\omega_3}, \Delta_{c^0\omega_4,c^0\omega_4} \,\}$, and the coordinates for $Q_{A_3,\mathrm{Lin}}$ are $\{\, \Delta_{c^2\omega_1,c^2\omega_1}, \Delta_{c^1\omega_2,c^1\omega_2}, \Delta_{c^1\omega_3,c^1\omega_3} \,\}$. To prove the lemma, we must show that $\widetilde{\mu}_{\mathrm{RS}}$ permutes these by σ_G, which in the case of A_ℓ replaces each ω_k with $\omega_{\ell+1-k}$.

Since $\widetilde{\mu}_{\mathrm{RS}}$ also consists entirely of source-or-sink mutations, it is also governed by primitive exchange mutations. Let

$$\widetilde{\mu}_{\mathrm{A}} = \prod_{k=1}^{\ell} \prod_{j=0}^{\ell-k} \widetilde{\mu}_{\mathrm{T}}(i, 1+j), \qquad \widetilde{\mu}_{\mathrm{B}} = \prod_{j=1}^{\ell} \widetilde{\mu}_{\mathrm{T}}(i, \ell-j),$$

so that $\widetilde{\mu}_{\mathrm{RS}} = \widetilde{\mu}_{\mathrm{A}}\widetilde{\mu}_{\mathrm{B}}$. We make the following observations, which are all easily checked by induction:

- During $\widetilde{\mu}_{\mathrm{A}}$, whenever a vertex v_k is mutated at with k odd, v_k has coordinate $\Delta_{c^m\omega_k, c^m\omega_k}$ while its neighbors have coordinates $\Delta_{c^{m-1}\omega_{k-1}, c^{m-1}\omega_{k-1}}$ and $\Delta_{c^{m-1}\omega_{k+1}, c^{m-1}\omega_{k+1}}$. Since $k \prec_c k\pm1$, the coordinate at v_k becomes $\Delta_{c^{m-1}\omega_{k+1}, c^{m-1}\omega_{k+1}}$.

- During $\widetilde{\mu}_{\mathrm{A}}$, whenever a vertex v_k is mutated at with k even, v_k has coordinate $\Delta_{c^m\omega_k, c^m\omega_k}$ and its neighbors have coordinates $\Delta_{c^m\omega_{k-1}, c^m\omega_{k-1}}$ and $\Delta_{c^m\omega_{k+1}, c^m\omega_{k+1}}$. Since $k\pm1 \prec_c k$, the coordinate at v_k becomes $\Delta_{c^{m-1}\omega_{k+1}, c^{m-1}\omega_{k+1}}$.

- By the same logic as the above statements every mutation in $\widetilde{\mu}_{\mathrm{B}}$ takes the coordinate at v_k from $\Delta_{c^m\omega_k, c^m\omega_k}$ to $\Delta_{c^{m+1}\omega_k, c^{m+1}\omega_k}$.

Therefore, since $\widetilde{\mu}_{\mathrm{A}}$ touches vertex k a total of $\ell - k + 1$ times and $\widetilde{\mu}_{\mathrm{B}}$ touches each vertex once, we may conclude via Proposition 3.1.3 that after $\widetilde{\mu}_{\mathrm{A}}\widetilde{\mu}_{\mathrm{B}}$ the coordinates are given by

$$\text{coordinate at } v_k \text{ is } \Delta_{c^{(\ell-k)/2+Y(\ell)+Y(k)-(\ell-k)}\omega_k, c^{(\ell-k)/2+Y(\ell)+Y(k)-(\ell-k)}\omega_k}.$$

Now, let us consider the difference between the exponents of c in the final coordinate of v_k and in the initial coordinate of v_{k^*}. If the difference is $-h(k^*; c) - 1$, then Proposition 3.1.3 will show that $\widetilde{\mu}_{\mathrm{RS}}$ acts by permuting vertices according to σ_G.

$$\begin{aligned}
\text{final at } k - \text{initial at } k^* &= \left[\frac{\ell-k}{2} + Y(\ell) + Y(k) - (\ell-k)\right] - \left[\frac{\ell-k^*}{2} + Y(\ell) + Y(k^*)\right] \\
&= \frac{k^*-k}{2} - (\ell-k) + Y(k) - Y(k^*) \\
&= \frac{(\ell+1-k)-k}{2} - (\ell-k) + Y(k) - Y(\ell+1-k) \\
&= -\left[\frac{\ell+1}{2} - Y(k) + Y(\ell+1-k)\right] - 1 \\
&= \begin{cases} -\left[\frac{\ell+1}{2} + Y(\ell+1)\right] - 1 & \ell+1-k \text{ odd} \\ -\left[\frac{\ell+1}{2} - Y(\ell+1)\right] - 1 & \ell+1-k \text{ even} \end{cases} && \text{(Fact 7.2.3)} \\
&= \begin{cases} -\lceil\frac{\ell+1}{2}\rceil - 1 & k^* \text{ odd} \\ -\lfloor\frac{\ell+1}{2}\rfloor - 1 & k^* \text{ even} \end{cases}
\end{aligned}$$

Consulting Proposition 3.1.2, the exponent difference is indeed $-h(k^*;c) - 1$, so the action of $\widetilde{\mu}_{\mathrm{RS}}$ is to rearrange the vertices of Q_{Lin} according to σ_G. From here, the action of $\widetilde{\mu}_{\mathrm{RS}}$ generalizes to larger quivers as in the proof for $\widetilde{\mu}_{\mathrm{CS}}$. □

7.3 Rotation mutations

We now turn to verifying the longer mutations, starting with μ_{rot}. First, we recall work of Zickert to describe the effect of rot in terms of canonical forms, taking (h_1, h_2, h_3, u) to $(h_3, h_1, h_2, \widetilde{u})$. We show, using an identity of Fomin–Zelevinsky, that when all h_i are trivial, μ^*_{rotTw} takes coordinates of u to those of $\widetilde{u}$. Then we show that the monomial map m of $\mathcal{M}$ is exactly what is necessary to extend to the general case. The final pieces $\widetilde{\mu}_{\mathrm{O1}}$ and $\widetilde{\mu}_{\mathrm{O2}}$, rearrange the vertices without changing their coordinates according to Lemmas 7.2.1 and 7.2.2.

Recall that by Proposition 4.1.5, we have

$$\widetilde{u} = h_2^{-1}(w_0(h_1))^{-1}(\Phi\Psi\Phi\Psi)(u)(w_0(h_1))h_2.$$

So we must consider $(\Phi\Psi)^2(u)$ in terms of minor coordinates, then show that the result agrees in general with the cluster action of μ_{rot}. In other words, we must show that μ_{rot}^* fits into the diagram of Figure 7.2. We interpret the actions of the maps Φ and Ψ on generalized minors, then appeal to an identity of Fomin–Zelevinsky that applies at every step of the mutation sequence μ_{rotTw}.

Lemma 7.3.1. *Item 4 of Definition 4.2.9 holds for $h_1 = h_2 = h_3$ are all trivial.*

Proof. By Remark 2.7.4, we need only consider the commuting diagrams for $\mathcal{A}$-coordinates; the diagrams for $\mathcal{X}$-coordinates will then follow immediately by applying p.

Let $\widetilde{u} = (w_0(h_1))^{-1}(\Phi\Psi)^2(u)w_0(h_1)h_2$ for notation. The core idea is that, modulo coordinates of h_i, $\Delta_{w_k\omega_{i_k}, e\omega_{i_k}}(u)$ is approximately $\Delta_{w_0\omega_{i_k}, w_k\omega_{i_k}}(\widetilde{u})$. The mutation sequence μ_{rot} transforms $\Delta_{w_0\omega_{i_k}, w_k\omega_{i_k}}$ to $\Delta_{w_k\omega_{i_k}, e\omega_{i_k}}$ by [FZ99, Theorem 1.17], therefore changing coordinates of u into coordinates of $\widetilde{u}$. Later, we will show that the monomial map exactly compensates for the coordinates of h_i.

To be more precise, the coordinate assigned to interior vertex $v_{i,j}$ is (letting $k = k(i,j)$ as in Definition 5.6.1)

$$v_{i,j} = \Delta_{w_k\omega_{i_k}, e\omega_{i_k}}(u) \cdot \Delta^{\omega_i}(\sigma_G(h_1^{-1})h_2) \cdot \Delta_{\oplus}^{w_{k(i,j)}\omega_i}(\sigma_G(h_1))$$

Focusing on the first term, consider $(\Phi\Psi)^2(u) = [\Psi([\Psi(u)\overline{w_0}]_-)\overline{w_0}]_-$. Letting $x_\bullet \in N_+$, $h_\bullet \in H$, and $y_\bullet \in N_-$,

$$y_0 = u \qquad x_1 = \Psi(y_0) \qquad y_2 H_2 x_2 = x_1\overline{w_0}$$

$$y_2 = (\Phi\Psi)(u) \qquad x_3 = \Psi(y_2) \qquad y_4 H_4 x_4 = x_3\overline{w_0}$$

$$y_4 = (\Phi\Psi)^2(u).$$

Expanding those terms, we have

$$\begin{aligned}
y_4 H_4 x_4 &= x_3\overline{w_0} \\
&= \Psi(y_2)\overline{w_0} \\
&= \Psi(x_1\overline{w_0}x_2^{-1}H_2^{-1})\overline{w_0} \\
&= \Psi(\Psi(y_0)\overline{w_0}x_2^{-1}H_2^{-1})\overline{w_0} \\
&= \Psi(H_2^{-1})\Psi(x_2^{-1})\Psi(\overline{w_0})y_0\overline{w_0} \\
&= H_2^{-1}\Psi(x_2^{-1})\overline{w_0}y_0\overline{w_0} \\
&= \overbrace{H_2^{-1}\Psi(x_2^{-1})H_2}^{\in N_-} \cdot \overbrace{H_2^{-1}s_G}^{\in H} \cdot \overbrace{\overline{w_0}^{-1}y_0\overline{w_0}}^{\in N_+}
\end{aligned}$$

Extracting the N_- term, $(\Phi\Psi)^2(u) = y_4 = H_2^{-1}\Psi(x_2^{-1})H_2$, which we can rewrite via

$$H_2 = [\Psi(y_0)\overline{w_0}]_0, \qquad x_2 = \Psi([\Psi(y_0)\overline{w_0}]_+^{-1}).$$

Now, consider the minor $\Delta_{w_0\omega_{i_k}, w_k\omega_{i_k}}(y_4)$. Denote by w_k^* the word defined by

Definition 2.4.8 using fixed presentation $\mathbf{i}^* = \sigma_G(\mathbf{i})$. By relying on a number of identities from [FZ99],

$$\Delta_{w_0\omega_{i_k},w_k\omega_{i_k}}(H_2^{-1}\Psi(x_2^{-1})H_2) = \frac{\Delta_{w_0\omega_{i_k},w_k\omega_{i_k}}(\Psi(x_2^{-1}))\cdot\Delta_{w_k\omega_{i_k},w_k\omega_{i_k}}(H_2)}{\Delta_{w_0\omega_{i_k},w_0\omega_{i_k}}(H_2)}$$

([FZ99, Equation 2.14])

$$\begin{aligned}
\Delta_{w_0\omega_{i_k},w_k\omega_{i_k}}(\Psi(x_2^{-1})) &= \Delta_{w_k\omega_{i_k},w_0\omega_{i_k}}(x_2^{-1}) \qquad \text{([FZ99, Equation 2.25])}\\
&= \Delta_{w_k\omega_{i_k},w_0\omega_{i_k}}(\Psi([\Psi(y_0)\overline{w_0}]_+^{-1})^{-1})\\
&= \Delta_{w_k^*\omega_{i_k^*},w_0\omega_{i_k^*}}(\overline{w_0}[\Psi(y_0)\overline{w_0}]_+^{-1}\overline{w_0}^{-1})\\
&= \Delta_{w_0w_k^*\omega_{i_k^*},e\omega_{i_k^*}}([\Psi(y_0)\overline{w_0}]_+^{-1})\\
&= \Delta_{w_0w_k^*\omega_{i_k^*},e\omega_{i_k^*}}([\overline{w_0}y_0]_-)\\
&= \Big(\Delta_{w_0w_k^*\omega_{i_k^*},e\omega_{i_k^*}}(\overline{w_0}y_0)\Big)\Big/\Big(\Delta_{e\omega_{i_k^*},e\omega_{i_k^*}}(\overline{w_0}y_0)\Big)\\
&\qquad \text{([FZ99, Equation 2.23])}\\
&= \Big(\Delta_{w_k^*\omega_{i_k^*},e\omega_{i_k^*}}(y_0)\Big)\Big/\Big(\Delta^{\omega_{i_k^*}}([\overline{w_0}^{-1}y_0]_0)\Big)
\end{aligned}$$

Also, by Lemma 4.1.3, $H_2 = [\Psi(y_0)\overline{w_0}]_0 = [\overline{w_0}^{-1}y_0]_0 = (w_0(h_3h_1)h_2)^{-1}$, where the h_i in the last term refer to the elements of H in the canonical form for $\mathrm{Conf}_3^*(G/N_+)$. Writing solely in terms of (h_1,h_2,h_3,u)), we have

$$\begin{aligned}
\Delta_{w_0\omega_{i_k},w_k\omega_{i_k}}((\Phi\Psi)^2(u)) &= \Delta_{w_k^*\omega_{i_k^*},e\omega_{i_k^*}}(u)\cdot\frac{\Delta_{w_0\omega_{i_k},w_0\omega_{i_k}}(w_0(h_3h_1)h_2)\Delta_{e\omega_{i_k^*},e\omega_{i_k^*}}(w_0(h_3h_1)h_2)}{\Delta_{w_k\omega_{i_k},w_k\omega_{i_k}}(w_0(h_3h_1)h_2)}\\
&= \Delta_{w_k^*\omega_{i_k^*},e\omega_{i_k^*}}(u)\cdot\frac{1}{\Delta_{w_k\omega_{i_k},w_k\omega_{i_k}}(w_0(h_3h_1)h_2)}
\end{aligned}$$

Now, introducing the factor of $w_0(h_1)h_2$, we have

$$\begin{aligned}\Delta_{w_0\omega_{i_k},w_k\omega_{i_k}}(\widetilde{u}) &= \Delta_{w_0\omega_{i_k},w_k\omega_{i_k}}((w_0(h_1))^{-1}(\Phi\Psi)^2(u)w_0(h_1)h_2)\\ &= \Delta_{w_0\omega_{i_k},w_k\omega_{i_k}}((\Phi\Psi)^2(u))\cdot\Delta_{w_0\omega_{i_k},w_0\omega_{i_k}}(w_0(h_1)^{-1}h_2^{-1})\cdot\Delta_{w_k\omega_{i_k},w_k\omega_{i_k}}(w_0(h_1)h_2)\\ &= \Delta_{w_k^*\omega_{i_k^*},e\omega_{i_k^*}}(u)\cdot\frac{1}{\Delta_{w_k\omega_{i_k},w_k\omega_{i_k}}(w_0(h_3))\Delta_{w_0\omega_{i_k},w_0\omega_{i_k}}(w_0(h_1)h_2)}\end{aligned}$$

Thus, including the assignment of the monomial map m, the coordinate at $v_{i,j}$ is given by

$$\begin{aligned}v_{i,j} &= \Delta_{w_0\omega_{i_k},w_k\omega_{i_k}}(\widetilde{u})\cdot\overbrace{\Delta_{e\omega_{i_i},e\omega_{i_i}}(w_0(h_1)h_2)\cdot\Delta_{w_0\omega_{i_k},w_0\omega_{i_k}}(w_0(h_1)h_2)}^{1}\cdot\Delta_{w_k\omega_{i_k},w_k\omega_{i_k}}(w_0(h_3))\cdot\Delta_{\oplus}^{w_k\omega_{i_k}}(w_0(h_1^{-1}))\\ &= \Delta_{w_0\omega_{i_k},w_k\omega_{i_k}}(\widetilde{u}\cdot w_0(h_3))\cdot\Delta_{\oplus}^{w_k\omega_{i_k}}(w_0(h_1^{-1}))\end{aligned}$$

In the case that $h_1 = h_2 = h_3 = e$, we may ignore the frozen vertices of the quiver. The proof is completed by appealing to the identity of [FZ99, Theorem 1.17], which states that when $\text{length}(us_i) = \text{length}(u)+1$ and $\text{length}(vs_i) = \text{length}(v)+1$,

$$\Delta_{u\omega_i,v\omega_i}\Delta_{us_i\omega_i,vs_i\omega_i} = \Delta_{us_i\omega_i,v\omega_i}\Delta_{v\omega_i,us_i\omega_i} + \prod_{j\neq i}\Delta_{u\omega_i,v\omega_i}^{-a_{ji}}.$$

This identity exactly matches the quiver mutation relation at each step of μ_{rotTw}, so each $\widetilde{\mu}_{\text{Col}}(i)$ in μ_{rot} transforms the coordinates of $v_{i,\bullet}$ from

$$\Delta_{c^n\omega_j,c^m\omega_j}(\widetilde{u})$$

to

$$\Delta_{c^{n-1}\omega_j, c^{m-1}\omega_j}(\widetilde{u}).$$

Thus, after applying μ_{rotTw} the vertex $v_{i,j}$ has coordinate

$$\Delta_{w^*_{m+r-k}\omega_{i^*_k}, e\omega_{i^*_k}}(\widetilde{u}).$$

By Lemmas 7.2.1 and 7.2.2, $\widetilde{\mu}_{\mathrm{O}1}$ converts the w_{m+r-k} to w_k, and $\widetilde{\mu}_{\mathrm{O}2}$ removes the $\cdot^*$. Thus μ_{rot} changes coordinates for u into those for $h_2^{-1}w_0(h_1)^{-1}(\Phi\Psi)^2(u)w_0(h_1)h_2$ as desired. □

Proof of Lemma 5.3.4. Since we ignored frozen vertices above, the result applies to Q_0. Since the quiver mutation changes coordinates of α to those of $\mathrm{rot}(\alpha)$, it is of order 3 and induces a graph isomorphism. □

Lemma 7.3.2. *Item 4 of Definition 4.2.9 holds for arbitrary h_1, h_2, h_3.*

Proof. In the case that h_1, h_2, h_3 are not trivial, we must verify that the equation of [FZ99, Theorem 1.17] still holds with the frozen vertices considered, which introduce factors not included in the equation. Again, we only consider $\mathcal{A}$-coordinates by Remark 2.7.4. We note the following:

- The vertices B_k take the place of

$$\Delta_{w_0\omega_{i_k}, e\omega_{i_k}}(\widetilde{u}) = \chi_{i^*_k}([\overline{w_0}^{-1}\widetilde{u}]_0) = \chi_{i^*_k}(w_0(h_2h_3)h_1)^{-1}$$

in the equation. Therefore, they should be considered as

$$B_k = \Delta_{w_0\omega_{i_k}, e\omega_{i_k}}(\widetilde{u}) \cdot \frac{A_{k^*}}{C_k} = \Delta_{w_0\omega_{i_k}, e\omega_{i_k}}(\widetilde{u} \cdot w_0(h_3)) \cdot A_{k^*}.$$

- If, at each step of the mutation, the frozen vertices and the monomial map m induce equal extra factors on the $\Delta_{us_i\omega_i, v\omega_i}\Delta_{v\omega_i, us_i\omega_i}$ and $\prod_{j\neq i} \Delta^{-a_{ji}}_{u\omega_i, v\omega_i}$ terms, then the difference of the final terms from coordinates of $\widetilde{u}$ will also be a monomial map.

- If this final difference monomial map matches m, the result will be proven.

- Since the extra factors at each coordinate are given by a monomial map, each frozen vertex may be checked individually.

For any particular group, these results may be verified by a few numerical calculations: to verify the monomial identity $x_1^{a_1} \cdots x_m^{a_m} \stackrel{?}{=} x_1^{b_1} \cdots x_m^{b_m}$, taking log of both sides reduces to $\sum_i \log(x_i)(a_i - b_i) \stackrel{?}{=} 0$. Therefore, evaluating at $m+1$ linearly independent choices of vector $\langle \log(x_i) \rangle$ verifies the result. This has been carried out for the exceptional groups. The `test-murot` sub-program of [Gil20] may be used for this purpose.

We present an argument that the coordinates of h_1 are treated correctly by the monomial map and μ_{rot} for type D_n; other arguments are similar.

We will ignore all terms other than $h_{1,j} = A_j$, and annotate vertices with these. Half the Coxeter number minus one will be denoted ℓ, which in the case of D_n is equal to $n-2$. Then we will trace the effects of μ_{rotTw} in general. The objective is to show that, starting with the assignments given by the monomial factors attached to $\Delta_{w_0\omega_{i_k}, w_k\omega_{i_k}}(\widetilde{u})$ above, we end with those factors of given by the monomial map m following rot (i.e. the powers associated to h_2 by m). Before applying any mutations, Q_{D_n} is as in Figure 7.3. The formula for A_j factors is, following $\Delta_{\oplus}^{w_k\omega_{i_k}}(w_0(h_1^{-1}))$,

$$\text{at } v_{i,j} : \begin{cases} A_j & i \geq n-1 \\ A_j & i < n-1, i+j < n \\ A_j A_{i+j+1-n} & i < n-1, i+j \geq n \end{cases} \qquad \begin{array}{l} \text{at } A_k\text{: } A_k, \\ \text{at } B_k\text{: } A_{k^*}, \\ \text{at } C_k\text{: } 1 \end{array}$$

After one iteration of $\widetilde{\mu}_{\mathrm{Col}}(1)$, by straight-forward induction the factor at $v_{i,j}$ matches that at $v_{i,j+1}$, as in Figure 7.4.

After $\prod_{y=1}^{\ell} \widetilde{\mu}_{\mathrm{Col}}(y)$, this pattern continues as shown in Figure 7.5: for $j < \ell$, we have the factor at $v_{i,j}$ matching the original factor at $v_{i,j+1}$. At $v_{i,\ell}$, however, the factor matches the original factor associated to B_i. Therefore, the new formula for A_j factors at $v_{i,j}$ is

$$\text{at } v_{i,j} : \begin{cases} A_{i^*} & j = \ell \\ A_{j+1} & j < \ell, i \geq n-1 \\ A_{j+1} & j < \ell, i < n-1, i+j < n \\ A_{j+1} A_{i+j+2-n} & j < \ell, i < n-1, i+j \geq n \end{cases}$$

Each successive $\prod_{y=1}^{\cdots} \widetilde{\mu}_{\mathrm{Col}}(y)$ performs the same adjustment: factors of A_j are copied from left to right. As can be shown by induction, after $\prod_{x=1}^{k} \prod_{y=1}^{\ell+1-x} \widetilde{\mu}_{\mathrm{Col}}(y)$, the formula for A_j factors at $v_{i,j}$ is

$$\text{at } v_{i,j} : \begin{cases} A_{i^*} & j > \ell - k \\ A_{j+k} & j \leq \ell - k, i \geq n-1 \\ A_{j+k} & j \leq \ell - k, i < n-1, i+j < n \\ A_{j+k}A_{i+j+1+k-n} & j \leq \ell - k, i < n-1, i+j \geq n \end{cases}$$

Thus, after applying μ_{rotTw}, each $v_{i,j}$ has a factor of A_{i^*} attached. Since the monomial map m associates a factor of $\Delta^{\omega_i}(h_2) = B_i$, rotation renames $A_\bullet$ to $B_\bullet$, and $\widetilde{\mu}_{\text{O1}}$ renames $v_{i,j}$ to $v_{i^*,j}$, the result is proven.

Proofs that the monomial map works correctly for factors of h_2 and h_3 for type D, and for all factors of type A, B, and C, have similar forms, and the result has been numerically checked for exceptional types. This proves the result in all cases. □

7.4 Flip mutations

We now verify the longest flip mutation: the flip. The idea of the proof is similar to that of μ_{rot}. First, we put the coordinates of $Q^{0\text{–}2}$ into a rectangular pattern, then apply mutations that respect [FZ99, Theorem 1.17], showing that the resulting coordinates are of $Q^{1\text{–}3}$. The most significant difficulty is to construct the element of G holding the appropriate generalized minors, which we obtain by a result of Zickert.

Lemma 7.4.1. *Item 5 of Definition 4.2.9 holds.*

Proof. This proof almost entirely focuses on $\widetilde{\mu}_{\text{Flipcore}}$. The prefix, $\widetilde{\mu}_{\text{P}}$, is a sequence of rotations as depicted in Figure 5.5. The suffix, $\widetilde{\mu}_{\text{O3}}\widetilde{\mu}_{\text{O4}}\widetilde{\mu}_{\text{O5}}$, is a reordering that switches the left and right sides.

Let $\alpha = (g_0N_+, g_1N_+, g_2N_+, g_3N_+) \in \text{Conf}_4^*(G/N_+)$, and recall Definition 4.1.6. For notation, let

$$\alpha_{012} = (g_0 s_G N_+, g_1N_+, g_2N_+) \qquad \alpha_{023} = (g_0N_+, g_2N_+, g_3N_+)$$

$$\alpha_{123} = (g_1N_+, g_2N_+, g_3N_+) \qquad \alpha_{013} = (g_0N_+, g_1 s_G N_+, g_3N_+),$$

with other α_{ijk} defined by rotation, and let u_{ijk} be the element of N_- in the canonical form of α_{ijk}. As usual, let $h_{ij} = [\overline{w_0}^{-1} g_i^{-1} g_j]_0$.

We must show that μ_{flip} takes $Q^{0\text{–}2}$ to $Q^{1\text{–}3}$. By an argument entirely analogous to that of Lemma 7.3.2, we assume that all edge coordinates of α_{123} and α_{013} are trivial. That is, every h_{ij} is trivial except h_{20} and h_{02}. We also appeal to Remark 2.7.4, and only consider $\mathcal{A}$-coordinates.

By Definition 4.2.7, coordinates for $Q^{0\text{–}2}$ come from α_{012} and α_{023}. After applying $\widetilde{\mu}_{\text{P}}$, these are rotated to α_{120} and α_{230}. Treating α_{230} as the rotation of α_{302}, the coordinates are given by

$$\Delta_{w_k\omega_{\mathbf{i}_k},\omega_{\mathbf{i}_k}}(u_{120}) \quad \text{and} \quad \Delta_{w_k\omega_{\mathbf{i}_k},\omega_{\mathbf{i}_k}}(u_{230}).$$

We must show that $\widetilde{\mu}_{\text{Flipcore}}$ takes these to

$$\Delta_{w_k\omega_{\mathbf{i}_k},\omega_{\mathbf{i}_k}}(u_{123}) \quad \text{and} \quad \Delta_{w_k\omega_{\mathbf{i}_k},\omega_{\mathbf{i}_k}}(u_{013}).$$

Now consider $g = \Phi^{-1}(u_{123})u_{130}h_{31}^{-1}h_{30}$. We also have

$$g = u_{120}h_{20}w_0(h_{12})\Phi^{-1}(u_{023})$$

by [Zic19, Proposition 5.14], and this will be identity which relates the two sides of the flip. All the coordinates we need are, up to edge coordinates, minors of g.

$$\begin{aligned}
\Delta_{w_k\omega_{\mathbf{i}_k},\omega_{\mathbf{i}_k}}(u_{120}) &= \chi_{\omega_{\mathbf{i}_k}}\left(\left[\overline{\overline{w_k^{-1}}}[g]_-\right]_0\right) \\
&= \chi_{\omega_{\mathbf{i}_k}}\left(\left[\overline{\overline{w_k^{-1}}}g[g]_0^{-1}\left([g]_0^{-1}[g]_+^{-1}[g]_0\right)\right]_0\right) \\
&= \chi_{\omega_{\mathbf{i}_k}}\left(\left[\overline{\overline{w_k^{-1}}}g\right]_0\right)/\chi_{\omega_{\mathbf{i}_k}}\left([g]_0\right) \\
&= \Delta_{w_k\omega_{\mathbf{i}_k},\omega_{\mathbf{i}_k}}(g)\frac{1}{\chi_{\omega_{\mathbf{i}_k}}(h_{20}w_0(h_{12}))} \\
&= \Delta_{w_k\omega_{\mathbf{i}_k},\omega_{\mathbf{i}_k}}(g)\frac{1}{\chi_{\omega_{\mathbf{i}_k}}(h_{20})\chi_{\omega_{\mathbf{i}_k}}(w_0(h_{12}))}
\end{aligned}$$

$$
\begin{aligned}
\Delta_{w_k\omega_{\mathbf{i}_k},\omega_{\mathbf{i}_k}}(u_{302}) &= \Delta_{w_k\omega_{\mathbf{i}_k},\omega_{\mathbf{i}_k}}((\Phi\Psi\Phi\Psi)(u_{023}))\frac{\Delta_{\omega_{\mathbf{i}_k},\omega_{\mathbf{i}_k}}(w_0(h_{02})h_{23})}{\Delta_{w_k\omega_{\mathbf{i}_k},w_k\omega_{\mathbf{i}_k}}(w_0(h_{02})h_{23})} \\
&= \Delta_{w_k\omega_{\mathbf{i}_k},\omega_{\mathbf{i}_k}}([[\overline{w_0}u_{023}]_+\overline{w_0}]_-)\frac{\Delta_{w_k\omega_{\mathbf{i}_k},w_k\omega_{\mathbf{i}_k}}(w_0(h_{02})h_{23})}{\Delta_{w_0\omega_{\mathbf{i}_k},w_0\omega_{\mathbf{i}_k}}(w_0(h_{02})h_{23})} \\
&= \Delta_{w_k\omega_{\mathbf{i}_k},\omega_{\mathbf{i}_k}}([\overline{w_0}u_{023}]_-^{-1})\frac{\Delta_{\omega_{\mathbf{i}_k},\omega_{\mathbf{i}_k}}([\overline{w_0}u_{023}]_0)\Delta_{w_k\omega_{\mathbf{i}_k},w_k\omega_{\mathbf{i}_k}}(w_0(h_{02})h_{23})}{\Delta_{w_k\omega_{\mathbf{i}_k},w_k\omega_{\mathbf{i}_k}}([\overline{w_0}u_{023}]_0)\Delta_{w_0\omega_{\mathbf{i}_k},w_0\omega_{\mathbf{i}_k}}(w_0(h_{02})h_{23})} \\
&= \Delta_{\omega_{\mathbf{i}_k^*},w_k^*\omega_{\mathbf{i}_k^*}}([g]_+)\frac{\Delta_{\omega_{\mathbf{i}_k},\omega_{\mathbf{i}_k}}([\overline{w_0}u_{023}]_0)\Delta_{w_k\omega_{\mathbf{i}_k},w_k\omega_{\mathbf{i}_k}}(w_0(h_{02})h_{23})}{\Delta_{w_k\omega_{\mathbf{i}_k},w_k\omega_{\mathbf{i}_k}}([\overline{w_0}u_{023}]_0)\Delta_{w_0\omega_{\mathbf{i}_k},w_0\omega_{\mathbf{i}_k}}(w_0(h_{02})h_{23})} \\
&= \Delta_{\omega_{\mathbf{i}_k^*},w_k^*\omega_{\mathbf{i}_k^*}}(g)\frac{\Delta_{\omega_{\mathbf{i}_k},\omega_{\mathbf{i}_k}}([\overline{w_0}u_{023}]_0)\Delta_{w_k\omega_{\mathbf{i}_k},w_k\omega_{\mathbf{i}_k}}(w_0(h_{02})h_{23})}{\Delta_{\omega_{\mathbf{i}_k^*},\omega_{\mathbf{i}_k^*}}([g]_0)\Delta_{w_k\omega_{\mathbf{i}_k},w_k\omega_{\mathbf{i}_k}}([\overline{w_0}u_{023}]_0)\Delta_{w_0\omega_{\mathbf{i}_k},w_0\omega_{\mathbf{i}_k}}(w_0(h_{02})h_{23})} \\
&= \Delta_{\omega_{\mathbf{i}_k^*},w_k^*\omega_{\mathbf{i}_k^*}}(g)\frac{\chi_{\omega_{\mathbf{i}_k}}\left(w_0(h_{20}h_{30}^{-1})h_{12}\right)}{\Delta_{w_k\omega_{\mathbf{i}_k},w_k\omega_{\mathbf{i}_k}}(h_{03})} \\
&= \Delta_{\omega_{\mathbf{i}_k^*},w_k^*\omega_{\mathbf{i}_k^*}}(g)\frac{\chi_{\omega_{\mathbf{i}_k}}\left(w_0(h_{30}^{-1})h_{12}\right)}{\chi_{\omega_{\mathbf{i}_k}}(h_{02})\Delta_{w_k\omega_{\mathbf{i}_k},w_k\omega_{\mathbf{i}_k}}(h_{03})}
\end{aligned}
$$

$$
\begin{aligned}
\Delta_{w_k\omega_{\mathbf{i}_k},\omega_{\mathbf{i}_k}}(u_{123}) &= \Delta_{w_k\omega_{\mathbf{i}_k},\omega_{\mathbf{i}_k}}([g\overline{w_0}]_-) \\
&= \chi_{\omega_{\mathbf{i}_k}}\left(\left[\overline{\overline{w_k^{-1}}}g\overline{w_0}[g\overline{w_0}]_+^{-1}[g\overline{w_0}]_0^{-1}\right]_0\right) \\
&= \Delta_{w_k\omega_{\mathbf{i}_k},w_0\omega_{\mathbf{i}_k}}(g)\frac{1}{\Delta_{\omega_{\mathbf{i}_k},w_0\omega_{\mathbf{i}_k}}(g)} \\
&= \Delta_{w_k\omega_{\mathbf{i}_k},w_0\omega_{\mathbf{i}_k}}(g)\frac{1}{\chi_{\omega_{\mathbf{i}_k}}([\overline{w_0}[\overline{w_0}^{-1}u_{123}]_-]_0 w_0(h_{31}^{-1}h_{30}))} \\
&= \Delta_{w_k\omega_{\mathbf{i}_k},w_0\omega_{\mathbf{i}_k}}(g)\frac{1}{\chi_{\omega_{\mathbf{i}_k}}([\overline{w_0}^{-1}u_{123}]_0^{-1} w_0(h_{31}^{-1}h_{30}))} \\
&= \Delta_{w_k\omega_{\mathbf{i}_k},w_0\omega_{\mathbf{i}_k}}(g)\frac{1}{\chi_{\omega_{\mathbf{i}_k}}(w_0(h_{12}h_{30})h_{23})}
\end{aligned}
$$

$$
\begin{aligned}
\Delta_{w_0\omega_{\mathbf{i}_k},w_k\omega_{\mathbf{i}_k}}(u_{130}) &= \Delta_{w_0\omega_{\mathbf{i}_k},w_k\omega_{\mathbf{i}_k}}(\Phi^{-1}(u_{123})u_{130}h_{31}^{-1}h_{30})\frac{1}{\Delta_{w_k\omega_{\mathbf{i}_k},w_k\omega_{\mathbf{i}_k}}(h_{31}^{-1}h_{30})} \\
&= \Delta_{w_0\omega_{\mathbf{i}_k},w_k\omega_{\mathbf{i}_k}}(g)\frac{1}{\Delta_{w_k\omega_{\mathbf{i}_k},w_k\omega_{\mathbf{i}_k}}(h_{31}^{-1}h_{30})}
\end{aligned}
$$

Amalgamating α_{120} and α_{302} and labeling coordinates by minors of g (up to edge coordinates) gives Figure 7.6. This includes the central edge because $[g]_0 = h_{20}w_0(h_{12})$. This is not $Q^{0\text{–}2}$, but the difference is only a matter of rotations and twistings.

Note that the factors of $\chi_{\omega_{\mathbf{i}_k}}(h_{20})$ and $\chi_{\omega_{\mathbf{i}_k}}(h_{02})$ in the assignments for

$$\Delta_{w_k\omega_{\mathbf{i}_k},\omega_{\mathbf{i}_k}}(u_{120}) \qquad \text{and} \qquad \Delta_{w_k\omega_{\mathbf{i}_k},\omega_{\mathbf{i}_k}}(u_{302})$$

are exactly cancelled out by the $\Delta^{\omega_i}(h_2)$ factor of m from Definition 5.6.1. Therefore we are justified in ignoring edge coordinates for the remainder of the proof since we assume the others to be trivial.

After applying $\left(\mu^R_{\mathrm{rotTw}}\right)$ the result is $\widetilde{\mu}_{\mathrm{P}}(Q^{0\text{–}2})$, with the coordinates of α_{120} and α_{230}, as in Figure 7.7. In this figure, since each $\Delta_{w_j\omega_{\mathbf{i}_k},w_k\omega_{\mathbf{i}_k}}(g)$ is equivalent to some $\Delta_{c^n\omega_{\mathbf{i}_k},c^m\omega_{\mathbf{i}_k}}(g)$, we label vertices by the pair (n, m). Further, taking into account that $c^h = w_0^2$ which lifts to an element of H, we may consider n and m modulo h.

As with μ_{rot}, each mutation in $\widetilde{\mu}_{\mathrm{Flipcore}}$ takes $\Delta_{c^a\omega_{\mathbf{i}_k},c^b\omega_{\mathbf{i}_k}}(g)$ to $\Delta_{c^{a-1}\omega_{\mathbf{i}_k},c^{b-1}\omega_{\mathbf{i}_k}}(g)$ by [FZ99, Theorem 1.17] and Lemma 3.2.4. That results in Figure 7.8.

Converting (n, m) back to $\Delta_{c^n\omega_{\mathbf{i}_k},c^m\omega_{\mathbf{i}_k}}(g)$, we obtain Figure 7.9. By the calculations above, these are coordinates of u_{123} on the right and u_{013} (as rotated u_{130}) on the left.

The final combination of $\widetilde{\mu}_{\mathrm{O3}}\widetilde{\mu}_{\mathrm{O4}}\widetilde{\mu}_{\mathrm{O5}}$ serve to swap the left and right halves by Lemma 7.2.2. This completes the proof. $\square$

Chapter 8: Proof of Theorem 4.2.12 for semisimple G

We now expand from "simple" to "semisimple". Since the algorithm of Chapter 5 really only needed the information of G's Dynkin diagram, and Dynkin diagrams behave very nicely with respect to products, the construction easily generalizes.

Lemma 8.0.1. *Let G, a semisimple Lie group over $\mathbb{C}$, be $G = G_1 \oplus G_2 \oplus \cdots \oplus G_n$, with each G_i a simple Lie group over $\mathbb{C}$ admitting quivers Q_i with Fock–Goncharov coordinate structures $(\mu_{rot})_i$, $(\mu_{flip})_i$, and $\mathcal{M}_i$.*

Then for G, there is a quiver Q which also carries a Fock–Goncharov coordinate structure.

Proof. We must construct Q, μ_{rot}, μ_{flip}, $\mathcal{M}$, then show that all requirements of Definition 4.2.9 are satisfied. This follows entirely from the fact that the Dynkin diagram of the direct product of groups is the disjoint union of Dynkin diagrams of the factors.

For Q, take the disjoint union of each Q_i, with each $\mathrm{d}_\Delta(Q)$ being the disjoint union of all $\mathrm{d}_\Delta(Q_i)$. This is triangular since each component is.

The mutations μ_{rot} and μ_{flip} are concatenations of all $(\mu_{\mathrm{rot}})_i$ and $(\mu_{\mathrm{flip}})_i$ respectively. The ordering of these components are irrelevant, because quiver mutations at disconnected vertices commute and Q is exactly a disjoint union. Any concatenation will operate componentwise on the pieces of the disjoint union. This satisfies Item 1.

By the aforementioned fact of Dynkin diagrams, Item 2 is also satisfied.

The map $\mathcal{M}$ is defined as a product of each $\mathcal{M}_i$ in the following manner. Let α_i and α_j be any two elements of the roots system for G associated to different factors. Then the nodes corresponding to those roots are disconnected in the Dynkin diagram for G, so the $A_{ij} = 0$ in the Cartan matrix.

By e.g. [Kna96, Proposition 2.95], the bracket of any $\{ e_i, f_i, h_i \}$ and $\{ e_j, f_j, h_j \}$ vanishes, so any $\{ x_i, y_i, \chi_i^* \}$ and $\{ x_j, y_j, \chi_j^* \}$ commute. Letting

$$\alpha = (g_0 N_+, g_1 N_+, g_2 N_+, g_3 N_+),$$

by sufficient genericity we may factorize any g_i as a product of x_k, y_k, χ_k^* elements, and may therefore rearrange these factors so that

$$g_i = (g_i)_1 (g_i)_2 \cdots (g_i)_n, \qquad (g_i)_j \in G_j.$$

Then the maps rot and Ψ_{ij} trivially factor through the decomposition of α into

$$\prod_{i=1}^{n} \left[\alpha_i = ((g_0)_i, (g_1)_i N_+, (g_2)_i N_+, (g_3)_i N_+) \right].$$

That is, it is obvious that $\prod_i \operatorname{rot}(\alpha_i) = \operatorname{rot}(\prod_i \alpha_i)$. This, together with the construction of Q, shows that Items 3 to 5 are satisfied since they are satisfied for each

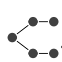
G_i. □

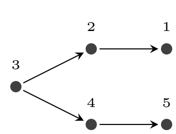
Chapter 9: Examples

Here we provide examples, in consistent notation, of Q, μ_{rot}, μ_{flip}, and $\mathcal{M}$ for various types (for F_4 see the examples of Chapter 5). Our notation is intended to agree with that of [Zic19], with the caveat that the mutations we present are longer but do not require vertex renaming (see Section 5.7). All Dynkin diagrams are taken from [Kna96, Figure 2.4], equivalently [Bou02, Plates II–IX] (ignoring the extending nodes). In each example, $\alpha = (h_1, h_2, h_3, u)$, where the coordinates of h_i are $\{ h_{ij} \}$, as in Example 5.6.3.

These results were obtained by the `latex-Q`, `print-murot`, `print-muflip`, and `print-M` commands of [Gil20].

9.1 A_5

We use the Dynkin diagram . The Coxeter element is $c = \{ 3, 2, 4, 1, 5 \}$, and the partitions are $T_0 = \{ 3 \}$, $T_1 = \{ 2, 4 \}$, $T_2 = \{ 1, 5 \}$. The Dynkin-type quiver follows.



Q_{A_5} is given in Figure 9.1.

$$
\begin{aligned}
\mu_{\mathrm{rot}} = \{ \; & v_{31}, v_{21}, v_{41}, v_{11}, v_{51}, v_{32}, v_{22}, v_{42}, v_{12}, v_{52}, v_{31}, v_{21}, v_{41}, v_{11}, v_{51}, v_{31}, v_{21}, v_{41}, \\
& v_{11}, v_{51}, v_{31}, v_{21}, v_{41}, v_{11}, v_{51}, v_{31}, v_{21}, v_{41}, v_{11}, v_{51}, v_{31}, v_{21}, v_{41}, v_{11}, v_{51}, v_{32}, \\
& v_{22}, v_{42}, v_{12}, v_{52}, v_{32}, v_{22}, v_{42}, v_{12}, v_{52}, v_{32}, v_{22}, v_{42}, v_{12}, v_{52}, v_{32}, v_{22}, v_{42}, v_{12}, \\
& v_{52}, v_{31}, v_{32}, v_{31}, v_{32}, v_{31}, v_{21}, v_{41}, v_{22}, v_{42}, v_{21}, v_{41}, v_{22}, v_{42}, v_{21}, v_{41}, v_{11}, v_{51}, \\
& v_{12}, v_{52}, v_{11}, v_{51}, v_{12}, v_{52}, v_{11}, v_{51} \; \}
\end{aligned}
$$

$$
\begin{aligned}
\mu_{\mathrm{flip}} = \{ \; & w_{15}, w_{55}, w_{25}, w_{45}, w_{35}, w_{14}, w_{54}, w_{24}, w_{44}, w_{34}, w_{15}, w_{55}, w_{25}, w_{45}, w_{35}, w_{12}, \\
& w_{52}, w_{22}, w_{42}, w_{32}, w_{11}, w_{51}, w_{21}, w_{41}, w_{31}, w_{12}, w_{52}, w_{22}, w_{42}, w_{32}, w_{35}, w_{25}, \\
& w_{45}, w_{15}, w_{55}, w_{34}, w_{24}, w_{44}, w_{14}, w_{54}, w_{33}, w_{23}, w_{43}, w_{13}, w_{53}, w_{32}, w_{22}, w_{42}, \\
& w_{12}, w_{52}, w_{31}, w_{21}, w_{41}, w_{11}, w_{51}, w_{35}, w_{25}, w_{45}, w_{15}, w_{55}, w_{34}, w_{24}, w_{44}, w_{14}, \\
& w_{54}, w_{33}, w_{23}, w_{43}, w_{13}, w_{53}, w_{32}, w_{22}, w_{42}, w_{12}, w_{52}, w_{35}, w_{25}, w_{45}, w_{15}, w_{55}, \\
& w_{34}, w_{24}, w_{44}, w_{14}, w_{54}, w_{33}, w_{23}, w_{43}, w_{13}, w_{53}, w_{35}, w_{25}, w_{45}, w_{15}, w_{55}, w_{34}, \\
& w_{24}, w_{44}, w_{14}, w_{54}, w_{35}, w_{25}, w_{45}, w_{15}, w_{55}, w_{11}, w_{51}, w_{12}, w_{52}, w_{13}, w_{53}, w_{14}, \\
& w_{54}, w_{15}, w_{55}, w_{11}, w_{51}, w_{12}, w_{52}, w_{13}, w_{53}, w_{14}, w_{54}, w_{11}, w_{51}, w_{12}, w_{52}, w_{13}, \\
& w_{53}, w_{11}, w_{51}, w_{12}, w_{52}, w_{11}, w_{51}, w_{15}, w_{55}, w_{14}, w_{54}, w_{13}, w_{53}, w_{12}, w_{52}, w_{11}, \\
& w_{51}, w_{21}, w_{41}, w_{22}, w_{42}, w_{23}, w_{43}, w_{24}, w_{44}, w_{25}, w_{45}, w_{21}, w_{41}, w_{22}, w_{42}, w_{23}, \\
& w_{43}, w_{24}, w_{44}, w_{21}, w_{41}, w_{22}, w_{42}, w_{23}, w_{43}, w_{21}, w_{41}, w_{22}, w_{42}, w_{21}, w_{41}, w_{25}, \\
& w_{45}, w_{24}, w_{44}, w_{23}, w_{43}, w_{22}, w_{42}, w_{21}, w_{41}, w_{31}, w_{32}, w_{33}, w_{34}, w_{35}, w_{31}, w_{32}, \\
& w_{33}, w_{34}, w_{31}, w_{32}, w_{33}, w_{31}, w_{32}, w_{31}, w_{35}, w_{34}, w_{33}, w_{32}, w_{31}, w_{14}, w_{54}, w_{15}, \\
& w_{55}, w_{14}, w_{54}, w_{15}, w_{55}, w_{14}, w_{54}, w_{24}, w_{44}, w_{25}, w_{45}, w_{24}, w_{44}, w_{25}, w_{45}, w_{24}, \\
& w_{44}, w_{34}, w_{35}, w_{34}, w_{35}, w_{34}, w_{11}, w_{51}, w_{12}, w_{52}, w_{11}, w_{51}, w_{12}, w_{52}, w_{11}, w_{51}, \\
& w_{21}, w_{41}, w_{22}, w_{42}, w_{21}, w_{41}, w_{22}, w_{42}, w_{21}, w_{41}, w_{31}, w_{32}, w_{31}, w_{32}, w_{31} \; \}
\end{aligned}
$$

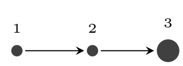
$\mathcal{M}(\alpha)$ gives

$$
\begin{array}{lllll}
A_1 = h_{11} & A_2 = h_{12} & A_3 = h_{13} & A_4 = h_{14} & A_5 = h_{15} \\
B_1 = h_{21} & B_2 = h_{22} & B_3 = h_{23} & B_4 = h_{24} & B_5 = h_{25} \\
C_1 = h_{35} & C_2 = h_{34} & C_3 = h_{33} & C_4 = h_{32} & C_5 = h_{31}
\end{array}
$$

$$
\begin{array}{ll}
v_{31} = \Delta^{w_6\omega_3}(u) \cdot h_{23} & v_{32} = \Delta^{w_{11}\omega_3}(u) \cdot h_{23} \\
v_{21} = \Delta^{w_7\omega_2}(u) \cdot \dfrac{h_{13}h_{22}}{h_{14}} & v_{22} = \Delta^{w_{12}\omega_2}(u) \cdot \dfrac{h_{13}h_{22}}{h_{14}} \\
v_{41} = \Delta^{w_8\omega_4}(u) \cdot \dfrac{h_{13}h_{24}}{h_{12}} & v_{42} = \Delta^{w_{13}\omega_4}(u) \cdot \dfrac{h_{13}h_{24}}{h_{12}} \\
v_{11} = \Delta^{w_9\omega_1}(u) \cdot \dfrac{h_{13}h_{21}}{h_{15}} & v_{12} = \Delta^{w_{14}\omega_1}(u) \cdot \dfrac{h_{14}h_{21}}{h_{15}} \\
v_{51} = \Delta^{w_{10}\omega_5}(u) \cdot \dfrac{h_{13}h_{25}}{h_{11}} & v_{52} = \Delta^{w_{15}\omega_5}(u) \cdot \dfrac{h_{12}h_{25}}{h_{11}}
\end{array}
$$

9.2 B_3

We use the Dynkin diagram ●–●⇒●. This is linear, so the Coxeter element and partitions are trivially $c = \{1, 2, 3\}$ and $T_0 = \{1\}, T_1 = \{2\}, T_2 = \{3\}$. The Dynkin-type quiver follows (recall Remark 5.1.3).



Q_{B_3} is given in Figure 9.2.

$\mu_{\text{rot}} = \{\ v_{11}, v_{21}, v_{31}, v_{12}, v_{22}, v_{32}, v_{11}, v_{21}, v_{31}, v_{11}, v_{12}, v_{11}, v_{12}, v_{11}, v_{21}, v_{22}, v_{21}, v_{22},$
$v_{21}, v_{31}, v_{32}, v_{31}, v_{32}, v_{31}\ \}$

$\mu_{\text{flip}} = \{\ w_{35}, w_{25}, w_{15}, w_{34}, w_{24}, w_{14}, w_{35}, w_{25}, w_{15}, w_{32}, w_{22}, w_{12}, w_{31}, w_{21}, w_{11}, w_{32},$
$w_{22}, w_{12}, w_{15}, w_{25}, w_{35}, w_{14}, w_{24}, w_{34}, w_{13}, w_{23}, w_{33}, w_{12}, w_{22}, w_{32}, w_{11}, w_{21},$
$w_{31}, w_{15}, w_{25}, w_{35}, w_{14}, w_{24}, w_{34}, w_{13}, w_{23}, w_{33}, w_{12}, w_{22}, w_{32}, w_{15}, w_{25}, w_{35},$

$w_{14},\, w_{24},\, w_{34},\, w_{13},\, w_{23},\, w_{33},\, w_{15},\, w_{25},\, w_{35},\, w_{14},\, w_{24},\, w_{34},\, w_{15},\, w_{25},\, w_{35},\, w_{31},$

$w_{32},\, w_{33},\, w_{34},\, w_{35},\, w_{31},\, w_{32},\, w_{33},\, w_{34},\, w_{31},\, w_{32},\, w_{33},\, w_{31},\, w_{32},\, w_{31},\, w_{35},\, w_{34},$

$w_{33},\, w_{32},\, w_{31},\, w_{21},\, w_{22},\, w_{23},\, w_{24},\, w_{25},\, w_{21},\, w_{22},\, w_{23},\, w_{24},\, w_{21},\, w_{22},\, w_{23},\, w_{21},$

$w_{22},\, w_{21},\, w_{25},\, w_{24},\, w_{23},\, w_{22},\, w_{21},\, w_{11},\, w_{12},\, w_{13},\, w_{14},\, w_{15},\, w_{11},\, w_{12},\, w_{13},\, w_{14},$

$w_{11},\, w_{12},\, w_{13},\, w_{11},\, w_{12},\, w_{11},\, w_{15},\, w_{14},\, w_{13},\, w_{12},\, w_{11},\, w_{34},\, w_{35},\, w_{34},\, w_{35},\, w_{34},$

$w_{24},\, w_{25},\, w_{24},\, w_{25},\, w_{24},\, w_{14},\, w_{15},\, w_{14},\, w_{15},\, w_{14},\, w_{31},\, w_{32},\, w_{31},\, w_{32},\, w_{31},\, w_{21},$

$w_{22},\, w_{21},\, w_{22},\, w_{21},\, w_{11},\, w_{12},\, w_{11},\, w_{12},\, w_{11} \;\}$

$\mathcal{M}(\alpha)$ gives

$$A_1 = h_{11} \qquad A_2 = h_{12} \qquad A_3 = h_{13}$$

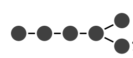
$$B_1 = h_{11} \qquad B_2 = h_{12} \qquad B_3 = h_{13}$$

$$C_1 = h_{11} \qquad C_2 = h_{12} \qquad C_3 = h_{13}$$

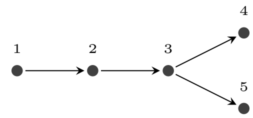
$$v_{11} = \Delta^{w_4\omega_1}(u) \cdot h_{21} \qquad v_{21} = \Delta^{w_5\omega_2}(u) \cdot \frac{h_{11}h_{22}}{h_{12}} \qquad v_{31} = \Delta^{w_6\omega_3}(u) \cdot \frac{h_{11}h_{23}}{h_{13}}$$

$$v_{12} = \Delta^{w_7\omega_1}(u) \cdot \frac{h_{12}h_{21}}{h_{11}} \qquad v_{22} = \Delta^{w_8\omega_2}(u) \cdot h_{11}h_{22} \qquad v_{32} = \Delta^{w_9\omega_3}(u) \cdot \frac{h_{12}h_{23}}{h_{13}}$$

9.3 D_5

Based on the Dynkin diagram , the following quiver is of well-rooted Dynkin type for D_5 (recall that σ_G is trivial for type D_{2n} and non-trivial for type D_{2n+1}). It admits an induced Coxeter element $c = \{1, 2, 3, 4, 5\}$, with partitions

$$T_0 = \{1\}, \quad T_1 = \{2\}, \quad T_2 = \{3\}, \quad T_3 = \{4, 5\}.$$



Q_{D_5} is given in Figure 9.3.

$$\mu_{\text{rot}} = \{\ v_{11},\ v_{21},\ v_{31},\ v_{41},\ v_{51},\ v_{12},\ v_{22},\ v_{32},\ v_{42},\ v_{52},\ v_{11},\ v_{21},\ v_{31},\ v_{41},\ v_{51},\ v_{13},\ v_{23},\ v_{33},$$

$$v_{43},\ v_{53},\ v_{12},\ v_{22},\ v_{32},\ v_{42},\ v_{52},\ v_{11},\ v_{21},\ v_{31},\ v_{41},\ v_{51},\ v_{11},\ v_{21},\ v_{31},\ v_{41},\ v_{51},\ v_{11},$$

$$v_{21},\ v_{31},\ v_{41},\ v_{51},\ v_{11},\ v_{21},\ v_{31},\ v_{41},\ v_{51},\ v_{11},\ v_{21},\ v_{31},\ v_{41},\ v_{51},\ v_{11},\ v_{21},\ v_{31},\ v_{41},$$

$$v_{51},\ v_{12},\ v_{22},\ v_{32},\ v_{42},\ v_{52},\ v_{12},\ v_{22},\ v_{32},\ v_{42},\ v_{52},\ v_{12},\ v_{22},\ v_{32},\ v_{42},\ v_{52},\ v_{12},\ v_{22},$$

$$v_{32},\ v_{42},\ v_{52},\ v_{12},\ v_{22},\ v_{32},\ v_{42},\ v_{52},\ v_{13},\ v_{23},\ v_{33},\ v_{43},\ v_{53},\ v_{13},\ v_{23},\ v_{33},\ v_{43},\ v_{53},$$

$$v_{13},\ v_{23},\ v_{33},\ v_{43},\ v_{53},\ v_{13},\ v_{23},\ v_{33},\ v_{43},\ v_{53},\ v_{13},\ v_{23},\ v_{33},\ v_{43},\ v_{53},\ v_{11},\ v_{12},\ v_{13},$$

$$v_{11},\ v_{12},\ v_{11},\ v_{13},\ v_{12},\ v_{11},\ v_{21},\ v_{22},\ v_{23},\ v_{21},\ v_{22},\ v_{21},\ v_{23},\ v_{22},\ v_{21},\ v_{31},\ v_{32},\ v_{33},$$

$$v_{31},\ v_{32},\ v_{31},\ v_{33},\ v_{32},\ v_{31},\ v_{41},\ v_{51},\ v_{42},\ v_{52},\ v_{43},\ v_{53},\ v_{41},\ v_{51},\ v_{42},\ v_{52},\ v_{41},\ v_{51},$$

$$v_{43},\ v_{53},\ v_{42},\ v_{52},\ v_{41},\ v_{51}\ \}$$

$$\mu_{\text{flip}} = \{\ w_{47},\ w_{57},\ w_{37},\ w_{27},\ w_{17},\ w_{46},\ w_{56},\ w_{36},\ w_{26},\ w_{16},\ w_{47},\ w_{57},\ w_{37},\ w_{27},\ w_{17},\ w_{45},$$

$$w_{55},\ w_{35},\ w_{25},\ w_{15},\ w_{46},\ w_{56},\ w_{36},\ w_{26},\ w_{16},\ w_{47},\ w_{57},\ w_{37},\ w_{27},\ w_{17},\ w_{43},\ w_{53},$$

$$w_{33},\ w_{23},\ w_{13},\ w_{42},\ w_{52},\ w_{32},\ w_{22},\ w_{12},\ w_{43},\ w_{53},\ w_{33},\ w_{23},\ w_{13},\ w_{41},\ w_{51},\ w_{31},$$

$$w_{21},\ w_{11},\ w_{42},\ w_{52},\ w_{32},\ w_{22},\ w_{12},\ w_{43},\ w_{53},\ w_{33},\ w_{23},\ w_{13},\ w_{17},\ w_{27},\ w_{37},\ w_{47},$$

$$w_{57},\ w_{16},\ w_{26},\ w_{36},\ w_{46},\ w_{56},\ w_{15},\ w_{25},\ w_{35},\ w_{45},\ w_{55},\ w_{14},\ w_{24},\ w_{34},\ w_{44},\ w_{54},$$

$$w_{13},\ w_{23},\ w_{33},\ w_{43},\ w_{53},\ w_{12},\ w_{22},\ w_{32},\ w_{42},\ w_{52},\ w_{11},\ w_{21},\ w_{31},\ w_{41},\ w_{51},\ w_{17},$$

$$w_{27},\ w_{37},\ w_{47},\ w_{57},\ w_{16},\ w_{26},\ w_{36},\ w_{46},\ w_{56},\ w_{15},\ w_{25},\ w_{35},\ w_{45},\ w_{55},\ w_{14},\ w_{24},$$

$$w_{34},\ w_{44},\ w_{54},\ w_{13},\ w_{23},\ w_{33},\ w_{43},\ w_{53},\ w_{12},\ w_{22},\ w_{32},\ w_{42},\ w_{52},\ w_{17},\ w_{27},\ w_{37},$$

$$w_{47},\ w_{57},\ w_{16},\ w_{26},\ w_{36},\ w_{46},\ w_{56},\ w_{15},\ w_{25},\ w_{35},\ w_{45},\ w_{55},\ w_{14},\ w_{24},\ w_{34},\ w_{44},$$

$$w_{54},\ w_{13},\ w_{23},\ w_{33},\ w_{43},\ w_{53},\ w_{17},\ w_{27},\ w_{37},\ w_{47},\ w_{57},\ w_{16},\ w_{26},\ w_{36},\ w_{46},\ w_{56},$$

$$w_{15},\ w_{25},\ w_{35},\ w_{45},\ w_{55},\ w_{14},\ w_{24},\ w_{34},\ w_{44},\ w_{54},\ w_{17},\ w_{27},\ w_{37},\ w_{47},\ w_{57},\ w_{16},$$

$$w_{26},\ w_{36},\ w_{46},\ w_{56},\ w_{15},\ w_{25},\ w_{35},\ w_{45},\ w_{55},\ w_{17},\ w_{27},\ w_{37},\ w_{47},\ w_{57},\ w_{16},\ w_{26},$$

w_{36}, w_{46}, w_{56}, w_{17}, w_{27}, w_{37}, w_{47}, w_{57}, w_{41}, w_{51}, w_{42}, w_{52}, w_{43}, w_{53}, w_{44}, w_{54},

w_{45}, w_{55}, w_{46}, w_{56}, w_{47}, w_{57}, w_{41}, w_{51}, w_{42}, w_{52}, w_{43}, w_{53}, w_{44}, w_{54}, w_{45}, w_{55},

w_{46}, w_{56}, w_{41}, w_{51}, w_{42}, w_{52}, w_{43}, w_{53}, w_{44}, w_{54}, w_{45}, w_{55}, w_{41}, w_{51}, w_{42}, w_{52},

w_{43}, w_{53}, w_{44}, w_{54}, w_{41}, w_{51}, w_{42}, w_{52}, w_{43}, w_{53}, w_{41}, w_{51}, w_{42}, w_{52}, w_{41}, w_{51},

w_{47}, w_{57}, w_{46}, w_{56}, w_{45}, w_{55}, w_{44}, w_{54}, w_{43}, w_{53}, w_{42}, w_{52}, w_{41}, w_{51}, w_{31}, w_{32},

w_{33}, w_{34}, w_{35}, w_{36}, w_{37}, w_{31}, w_{32}, w_{33}, w_{34}, w_{35}, w_{36}, w_{31}, w_{32}, w_{33}, w_{34}, w_{35},

w_{31}, w_{32}, w_{33}, w_{34}, w_{31}, w_{32}, w_{33}, w_{31}, w_{32}, w_{31}, w_{37}, w_{36}, w_{35}, w_{34}, w_{33}, w_{32},

w_{31}, w_{21}, w_{22}, w_{23}, w_{24}, w_{25}, w_{26}, w_{27}, w_{21}, w_{22}, w_{23}, w_{24}, w_{25}, w_{26}, w_{21}, w_{22},

w_{23}, w_{24}, w_{25}, w_{21}, w_{22}, w_{23}, w_{24}, w_{21}, w_{22}, w_{23}, w_{21}, w_{22}, w_{21}, w_{27}, w_{26}, w_{25},

w_{24}, w_{23}, w_{22}, w_{21}, w_{11}, w_{12}, w_{13}, w_{14}, w_{15}, w_{16}, w_{17}, w_{11}, w_{12}, w_{13}, w_{14}, w_{15},

w_{16}, w_{11}, w_{12}, w_{13}, w_{14}, w_{15}, w_{11}, w_{12}, w_{13}, w_{14}, w_{11}, w_{12}, w_{13}, w_{11}, w_{12}, w_{11},

w_{17}, w_{16}, w_{15}, w_{14}, w_{13}, w_{12}, w_{11}, w_{45}, w_{55}, w_{46}, w_{56}, w_{47}, w_{57}, w_{45}, w_{55}, w_{46},

w_{56}, w_{45}, w_{55}, w_{47}, w_{57}, w_{46}, w_{56}, w_{45}, w_{55}, w_{35}, w_{36}, w_{37}, w_{35}, w_{36}, w_{35}, w_{37},

w_{36}, w_{35}, w_{25}, w_{26}, w_{27}, w_{25}, w_{26}, w_{25}, w_{27}, w_{26}, w_{25}, w_{15}, w_{16}, w_{17}, w_{15}, w_{16},

w_{15}, w_{17}, w_{16}, w_{15}, w_{41}, w_{51}, w_{42}, w_{52}, w_{43}, w_{53}, w_{41}, w_{51}, w_{42}, w_{52}, w_{41}, w_{51},

w_{43}, w_{53}, w_{42}, w_{52}, w_{41}, w_{51}, w_{31}, w_{32}, w_{33}, w_{31}, w_{32}, w_{31}, w_{33}, w_{32}, w_{31}, w_{21},

w_{22}, w_{23}, w_{21}, w_{22}, w_{21}, w_{23}, w_{22}, w_{21}, w_{11}, w_{12}, w_{13}, w_{11}, w_{12}, w_{11}, w_{13}, w_{12},

w_{11} $\}$

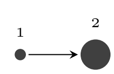
$\mathcal{M}(\alpha)$ gives

$$
\begin{array}{lllll}
A_1 = h_{11} & A_2 = h_{12} & A_3 = h_{13} & A_4 = h_{14} & A_5 = h_{15} \\
B_1 = h_{21} & B_2 = h_{22} & B_3 = h_{23} & B_4 = h_{24} & B_5 = h_{25} \\
C_1 = h_{31} & C_2 = h_{32} & C_3 = h_{33} & C_4 = h_{35} & C_5 = h_{34}
\end{array}
$$

$$
\begin{array}{lll}
v_{11} = \Delta^{w_6\omega_1}(u) \cdot h_{21} & v_{12} = \Delta^{w_{11}\omega_1}(u) \cdot \dfrac{h_{12}h_{21}}{h_{11}} & v_{13} = \Delta^{w_{16}\omega_1}(u) \cdot \dfrac{h_{13}h_{21}}{h_{11}} \\
v_{21} = \Delta^{w_7\omega_2}(u) \cdot \dfrac{h_{11}h_{22}}{h_{12}} & v_{22} = \Delta^{w_{12}\omega_2}(u) \cdot h_{22} & v_{23} = \Delta^{w_{17}\omega_2}(u) \cdot \dfrac{h_{11}h_{13}h_{22}}{h_{12}} \\
v_{31} = \Delta^{w_8\omega_3}(u) \cdot \dfrac{h_{11}h_{23}}{h_{13}} & v_{32} = \Delta^{w_{13}\omega_3}(u) \cdot \dfrac{h_{11}h_{12}h_{23}}{h_{13}} & v_{33} = \Delta^{w_{18}\omega_3}(u) \cdot h_{12}h_{23} \\
v_{41} = \Delta^{w_9\omega_4}(u) \cdot \dfrac{h_{11}h_{24}}{h_{15}} & v_{42} = \Delta^{w_{14}\omega_4}(u) \cdot \dfrac{h_{12}h_{24}}{h_{15}} & v_{43} = \Delta^{w_{19}\omega_4}(u) \cdot \dfrac{h_{13}h_{24}}{h_{15}} \\
v_{51} = \Delta^{w_{10}\omega_5}(u) \cdot \dfrac{h_{11}h_{25}}{h_{14}} & v_{52} = \Delta^{w_{15}\omega_5}(u) \cdot \dfrac{h_{12}h_{25}}{h_{14}} & v_{53} = \Delta^{w_{20}\omega_5}(u) \cdot \dfrac{h_{13}h_{25}}{h_{14}}
\end{array}
$$

9.4 G_2

Based on the Dynkin diagram ●⇚●, (with $c = \{1, 2\}$ and trivial linear partitioning) we use the following quiver for Dynkin type.



Q_{G_2} is given in Figure 9.4.

$$\mu_{\text{rot}} = \{\ v_{11}, v_{21}, v_{12}, v_{22}, v_{11}, v_{21}, v_{11}, v_{12}, v_{11}, v_{12}, v_{11}, v_{21}, v_{22}, v_{21}, v_{22}, v_{21}\ \}$$

$$
\begin{aligned}
\mu_{\text{flip}} = \{\ & w_{25}, w_{15}, w_{24}, w_{14}, w_{25}, w_{15}, w_{22}, w_{12}, w_{21}, w_{11}, w_{22}, w_{12}, w_{15}, w_{25}, w_{14}, w_{24}, \\
& w_{13}, w_{23}, w_{12}, w_{22}, w_{11}, w_{21}, w_{15}, w_{25}, w_{14}, w_{24}, w_{13}, w_{23}, w_{12}, w_{22}, w_{15}, w_{25}, \\
& w_{14}, w_{24}, w_{13}, w_{23}, w_{15}, w_{25}, w_{14}, w_{24}, w_{15}, w_{25}, w_{21}, w_{22}, w_{23}, w_{24}, w_{25}, w_{21}, \\
& w_{22}, w_{23}, w_{24}, w_{21}, w_{22}, w_{23}, w_{21}, w_{22}, w_{21}, w_{25}, w_{24}, w_{23}, w_{22}, w_{21}, w_{11}, w_{12}, \\
& w_{13}, w_{14}, w_{15}, w_{11}, w_{12}, w_{13}, w_{14}, w_{11}, w_{12}, w_{13}, w_{11}, w_{12}, w_{11}, w_{15}, w_{14}, w_{13},
\end{aligned}
$$

$w_{12},\ w_{11},\ w_{24},\ w_{25},\ w_{24},\ w_{25},\ w_{24},\ w_{14},\ w_{15},\ w_{14},\ w_{15},\ w_{14},\ w_{21},\ w_{22},\ w_{21},\ w_{22},$

$w_{21},\ w_{11},\ w_{12},\ w_{11},\ w_{12},\ w_{11}\ \}$

$\mathcal{M}(\alpha)$ gives

$$A_1 = h_{11} \qquad A_2 = h_{12}$$

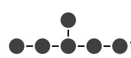
$$B_1 = h_{21} \qquad B_2 = h_{22}$$

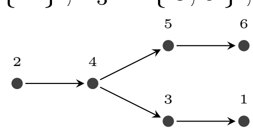
$$C_1 = h_{31} \qquad C_2 = h_{32}$$

$$v_{11} = \Delta^{w_3\omega_1}(u) \cdot h_{21} \qquad v_{21} = \Delta^{w_4\omega_2}(u) \cdot \frac{h_{11}^3 h_{22}}{h_{12}}$$

$$v_{12} = \Delta^{w_5\omega_1}(u) \cdot h_{11}h_{21} \quad v_{22} = \Delta^{w_6\omega_2}(u) \cdot \frac{h_{11}^3 h_{22}}{h_{12}}$$

9.5 E_6

Based on the Dynkin diagram , the following quiver is of well-rooted Dynkin type for E_6. It admits an induced Coxeter element $c = \{2, 4, 3, 5, 1, 6\}$, with partitions $T_0 = \{2\}, T_1 = \{4\}, T_3 = \{3, 5\}, T_4 = \{1, 6\}$.



Q_{E_6} is given in Figure 9.5.

$\mu_{\text{rot}} = \{\ v_{21},\ v_{41},\ v_{31},\ v_{51},\ v_{11},\ v_{61},\ v_{22},\ v_{42},\ v_{32},\ v_{52},\ v_{12},\ v_{62},\ v_{21},\ v_{41},\ v_{31},\ v_{51},\ v_{11},\ v_{61},$

$v_{23},\ v_{43},\ v_{33},\ v_{53},\ v_{13},\ v_{63},\ v_{22},\ v_{42},\ v_{32},\ v_{52},\ v_{12},\ v_{62},\ v_{21},\ v_{41},\ v_{31},\ v_{51},\ v_{11},\ v_{61},$

$v_{24},\ v_{44},\ v_{34},\ v_{54},\ v_{14},\ v_{64},\ v_{23},\ v_{43},\ v_{33},\ v_{53},\ v_{13},\ v_{63},\ v_{22},\ v_{42},\ v_{32},\ v_{52},\ v_{12},\ v_{62},$

$v_{21},\ v_{41},\ v_{31},\ v_{51},\ v_{11},\ v_{61},\ v_{25},\ v_{45},\ v_{35},\ v_{55},\ v_{15},\ v_{65},\ v_{24},\ v_{44},\ v_{34},\ v_{54},\ v_{14},\ v_{64},$

$v_{23},\ v_{43},\ v_{33},\ v_{53},\ v_{13},\ v_{63},\ v_{22},\ v_{42},\ v_{32},\ v_{52},\ v_{12},\ v_{62},\ v_{21},\ v_{41},\ v_{31},\ v_{51},\ v_{11},\ v_{61},$

$v_{21},\ v_{41},\ v_{31},\ v_{51},\ v_{11},\ v_{61},\ v_{21},\ v_{41},\ v_{31},\ v_{51},\ v_{11},\ v_{61},\ v_{21},\ v_{41},\ v_{31},\ v_{51},\ v_{11},\ v_{61},$

$$v_{21},\, v_{41},\, v_{31},\, v_{51},\, v_{11},\, v_{61},\, v_{21},\, v_{41},\, v_{31},\, v_{51},\, v_{11},\, v_{61},\, v_{21},\, v_{41},\, v_{31},\, v_{51},\, v_{11},\, v_{61},$$

$$v_{21},\, v_{41},\, v_{31},\, v_{51},\, v_{11},\, v_{61},\, v_{22},\, v_{42},\, v_{32},\, v_{52},\, v_{12},\, v_{62},\, v_{22},\, v_{42},\, v_{32},\, v_{52},\, v_{12},\, v_{62},$$

$$v_{22},\, v_{42},\, v_{32},\, v_{52},\, v_{12},\, v_{62},\, v_{22},\, v_{42},\, v_{32},\, v_{52},\, v_{12},\, v_{62},\, v_{22},\, v_{42},\, v_{32},\, v_{52},\, v_{12},\, v_{62},$$

$$v_{22},\, v_{42},\, v_{32},\, v_{52},\, v_{12},\, v_{62},\, v_{22},\, v_{42},\, v_{32},\, v_{52},\, v_{12},\, v_{62},\, v_{23},\, v_{43},\, v_{33},\, v_{53},\, v_{13},\, v_{63},$$

$$v_{23},\, v_{43},\, v_{33},\, v_{53},\, v_{13},\, v_{63},\, v_{23},\, v_{43},\, v_{33},\, v_{53},\, v_{13},\, v_{63},\, v_{23},\, v_{43},\, v_{33},\, v_{53},\, v_{13},\, v_{63},$$

$$v_{23},\, v_{43},\, v_{33},\, v_{53},\, v_{13},\, v_{63},\, v_{23},\, v_{43},\, v_{33},\, v_{53},\, v_{13},\, v_{63},\, v_{23},\, v_{43},\, v_{33},\, v_{53},\, v_{13},\, v_{63},$$

$$v_{24},\, v_{44},\, v_{34},\, v_{54},\, v_{14},\, v_{64},\, v_{24},\, v_{44},\, v_{34},\, v_{54},\, v_{14},\, v_{64},\, v_{24},\, v_{44},\, v_{34},\, v_{54},\, v_{14},\, v_{64},$$

$$v_{24},\, v_{44},\, v_{34},\, v_{54},\, v_{14},\, v_{64},\, v_{24},\, v_{44},\, v_{34},\, v_{54},\, v_{14},\, v_{64},\, v_{24},\, v_{44},\, v_{34},\, v_{54},\, v_{14},\, v_{64},$$

$$v_{24},\, v_{44},\, v_{34},\, v_{54},\, v_{14},\, v_{64},\, v_{25},\, v_{45},\, v_{35},\, v_{55},\, v_{15},\, v_{65},\, v_{25},\, v_{45},\, v_{35},\, v_{55},\, v_{15},\, v_{65},$$

$$v_{25},\, v_{45},\, v_{35},\, v_{55},\, v_{15},\, v_{65},\, v_{25},\, v_{45},\, v_{35},\, v_{55},\, v_{15},\, v_{65},\, v_{25},\, v_{45},\, v_{35},\, v_{55},\, v_{15},\, v_{65},$$

$$v_{25},\, v_{45},\, v_{35},\, v_{55},\, v_{15},\, v_{65},\, v_{25},\, v_{45},\, v_{35},\, v_{55},\, v_{15},\, v_{65},\, v_{21},\, v_{22},\, v_{23},\, v_{24},\, v_{25},\, v_{21},$$

$$v_{22},\, v_{23},\, v_{24},\, v_{21},\, v_{22},\, v_{23},\, v_{21},\, v_{22},\, v_{21},\, v_{25},\, v_{24},\, v_{23},\, v_{22},\, v_{21},\, v_{41},\, v_{42},\, v_{43},\, v_{44},$$

$$v_{45},\, v_{41},\, v_{42},\, v_{43},\, v_{44},\, v_{41},\, v_{42},\, v_{43},\, v_{41},\, v_{42},\, v_{41},\, v_{45},\, v_{44},\, v_{43},\, v_{42},\, v_{41},\, v_{31},\, v_{51},$$

$$v_{32},\, v_{52},\, v_{33},\, v_{53},\, v_{34},\, v_{54},\, v_{35},\, v_{55},\, v_{31},\, v_{51},\, v_{32},\, v_{52},\, v_{33},\, v_{53},\, v_{34},\, v_{54},\, v_{31},\, v_{51},$$

$$v_{32},\, v_{52},\, v_{33},\, v_{53},\, v_{31},\, v_{51},\, v_{32},\, v_{52},\, v_{31},\, v_{51},\, v_{35},\, v_{55},\, v_{34},\, v_{54},\, v_{33},\, v_{53},\, v_{32},\, v_{52},$$

$$v_{31},\, v_{51},\, v_{11},\, v_{61},\, v_{12},\, v_{62},\, v_{13},\, v_{63},\, v_{14},\, v_{64},\, v_{15},\, v_{65},\, v_{11},\, v_{61},\, v_{12},\, v_{62},\, v_{13},\, v_{63},$$

$$v_{14},\, v_{64},\, v_{11},\, v_{61},\, v_{12},\, v_{62},\, v_{13},\, v_{63},\, v_{11},\, v_{61},\, v_{12},\, v_{62},\, v_{11},\, v_{61},\, v_{15},\, v_{65},\, v_{14},\, v_{64},$$

$$v_{13},\, v_{63},\, v_{12},\, v_{62},\, v_{11},\, v_{61} \;\Big\}$$

$$\mu_{\text{flip}} = \Big\{\; w_{1b},\, w_{6b},\, w_{3b},\, w_{5b},\, w_{4b},\, w_{2b},\, w_{1a},\, w_{6a},\, w_{3a},\, w_{5a},\, w_{4a},\, w_{2a},\, w_{1b},\, w_{6b},\, w_{3b},\, w_{5b},$$

$$w_{4b},\, w_{2b},\, w_{19},\, w_{69},\, w_{39},\, w_{59},\, w_{49},\, w_{29},\, w_{1a},\, w_{6a},\, w_{3a},\, w_{5a},\, w_{4a},\, w_{2a},\, w_{1b},\, w_{6b},$$

$$w_{3b},\, w_{5b},\, w_{4b},\, w_{2b},\, w_{18},\, w_{68},\, w_{38},\, w_{58},\, w_{48},\, w_{28},\, w_{19},\, w_{69},\, w_{39},\, w_{59},\, w_{49},\, w_{29},$$

$$w_{1a},\, w_{6a},\, w_{3a},\, w_{5a},\, w_{4a},\, w_{2a},\, w_{1b},\, w_{6b},\, w_{3b},\, w_{5b},\, w_{4b},\, w_{2b},\, w_{17},\, w_{67},\, w_{37},\, w_{57},$$

$$w_{47},\, w_{27},\, w_{18},\, w_{68},\, w_{38},\, w_{58},\, w_{48},\, w_{28},\, w_{19},\, w_{69},\, w_{39},\, w_{59},\, w_{49},\, w_{29},\, w_{1a},\, w_{6a},$$

$$w_{3a},\, w_{5a},\, w_{4a},\, w_{2a},\, w_{1b},\, w_{6b},\, w_{3b},\, w_{5b},\, w_{4b},\, w_{2b},\, w_{15},\, w_{65},\, w_{35},\, w_{55},\, w_{45},\, w_{25},$$

w_{14}, w_{64}, w_{34}, w_{54}, w_{44}, w_{24}, w_{15}, w_{65}, w_{35}, w_{55}, w_{45}, w_{25}, w_{13}, w_{63}, w_{33}, w_{53}, w_{43}, w_{23}, w_{14}, w_{64}, w_{34}, w_{54}, w_{44}, w_{24}, w_{15}, w_{65}, w_{35}, w_{55}, w_{45}, w_{25}, w_{12}, w_{62}, w_{32}, w_{52}, w_{42}, w_{22}, w_{13}, w_{63}, w_{33}, w_{53}, w_{43}, w_{23}, w_{14}, w_{64}, w_{34}, w_{54}, w_{44}, w_{24}, w_{15}, w_{65}, w_{35}, w_{55}, w_{45}, w_{25}, w_{11}, w_{61}, w_{31}, w_{51}, w_{41}, w_{21}, w_{12}, w_{62}, w_{32}, w_{52}, w_{42}, w_{22}, w_{13}, w_{63}, w_{33}, w_{53}, w_{43}, w_{23}, w_{14}, w_{64}, w_{34}, w_{54}, w_{44}, w_{24}, w_{15}, w_{65}, w_{35}, w_{55}, w_{45}, w_{25}, w_{2b}, w_{4b}, w_{3b}, w_{5b}, w_{1b}, w_{6b}, w_{2a}, w_{4a}, w_{3a}, w_{5a}, w_{1a}, w_{6a}, w_{29}, w_{49}, w_{39}, w_{59}, w_{19}, w_{69}, w_{28}, w_{48}, w_{38}, w_{58}, w_{18}, w_{68}, w_{27}, w_{47}, w_{37}, w_{57}, w_{17}, w_{67}, w_{26}, w_{46}, w_{36}, w_{56}, w_{16}, w_{66}, w_{25}, w_{45}, w_{35}, w_{55}, w_{15}, w_{65}, w_{24}, w_{44}, w_{34}, w_{54}, w_{14}, w_{64}, w_{23}, w_{43}, w_{33}, w_{53}, w_{13}, w_{63}, w_{22}, w_{42}, w_{32}, w_{52}, w_{12}, w_{62}, w_{21}, w_{41}, w_{31}, w_{51}, w_{11}, w_{61}, w_{2b}, w_{4b}, w_{3b}, w_{5b}, w_{1b}, w_{6b}, w_{2a}, w_{4a}, w_{3a}, w_{5a}, w_{1a}, w_{6a}, w_{29}, w_{49}, w_{39}, w_{59}, w_{19}, w_{69}, w_{28}, w_{48}, w_{38}, w_{58}, w_{18}, w_{68}, w_{27}, w_{47}, w_{37}, w_{57}, w_{17}, w_{67}, w_{26}, w_{46}, w_{36}, w_{56}, w_{16}, w_{66}, w_{25}, w_{45}, w_{35}, w_{55}, w_{15}, w_{65}, w_{24}, w_{44}, w_{34}, w_{54}, w_{14}, w_{64}, w_{23}, w_{43}, w_{33}, w_{53}, w_{13}, w_{63}, w_{22}, w_{42}, w_{32}, w_{52}, w_{12}, w_{62}, w_{2b}, w_{4b}, w_{3b}, w_{5b}, w_{1b}, w_{6b}, w_{2a}, w_{4a}, w_{3a}, w_{5a}, w_{1a}, w_{6a}, w_{29}, w_{49}, w_{39}, w_{59}, w_{19}, w_{69}, w_{28}, w_{48}, w_{38}, w_{58}, w_{18}, w_{68}, w_{27}, w_{47}, w_{37}, w_{57}, w_{17}, w_{67}, w_{26}, w_{46}, w_{36}, w_{56}, w_{16}, w_{66}, w_{25}, w_{45}, w_{35}, w_{55}, w_{15}, w_{65}, w_{24}, w_{44}, w_{34}, w_{54}, w_{14}, w_{64}, w_{23}, w_{43}, w_{33}, w_{53}, w_{13}, w_{63}, w_{2b}, w_{4b}, w_{3b}, w_{5b}, w_{1b}, w_{6b}, w_{2a}, w_{4a}, w_{3a}, w_{5a}, w_{1a}, w_{6a}, w_{29}, w_{49}, w_{39}, w_{59}, w_{19}, w_{69}, w_{28}, w_{48}, w_{38}, w_{58}, w_{18}, w_{68}, w_{27}, w_{47}, w_{37}, w_{57}, w_{17}, w_{67}, w_{26}, w_{46}, w_{36}, w_{56}, w_{16}, w_{66}, w_{25}, w_{45}, w_{35}, w_{55}, w_{15}, w_{65}, w_{24}, w_{44}, w_{34}, w_{54}, w_{14}, w_{64}, w_{2b}, w_{4b}, w_{3b}, w_{5b}, w_{1b}, w_{6b}, w_{2a}, w_{4a}, w_{3a}, w_{5a}, w_{1a}, w_{6a}, w_{29}, w_{49}, w_{39}, w_{59}, w_{19}, w_{69}, w_{28}, w_{48}, w_{38}, w_{58}, w_{18}, w_{68}, w_{27}, w_{47}, w_{37}, w_{57}, w_{17}, w_{67}, w_{26}, w_{46}, w_{36}, w_{56}, w_{16}, w_{66}, w_{25}, w_{45}, w_{35}, w_{55}, w_{15}, w_{65}, w_{2b}, w_{4b}, w_{3b}, w_{5b}, w_{1b}, w_{6b}, w_{2a}, w_{4a}, w_{3a}, w_{5a}, w_{1a}, w_{6a}, w_{29}, w_{49}, w_{39}, w_{59}, w_{19}, w_{69}, w_{28}, w_{48}, w_{38}, w_{58}, w_{18}, w_{68}, w_{27}, w_{47}, w_{37}, w_{57}, w_{17}, w_{67},

w_{26}, w_{46}, w_{36}, w_{56}, w_{16}, w_{66}, w_{2b}, w_{4b}, w_{3b}, w_{5b}, w_{1b}, w_{6b}, w_{2a}, w_{4a}, w_{3a}, w_{5a},
w_{1a}, w_{6a}, w_{29}, w_{49}, w_{39}, w_{59}, w_{19}, w_{69}, w_{28}, w_{48}, w_{38}, w_{58}, w_{18}, w_{68}, w_{27}, w_{47},
w_{37}, w_{57}, w_{17}, w_{67}, w_{2b}, w_{4b}, w_{3b}, w_{5b}, w_{1b}, w_{6b}, w_{2a}, w_{4a}, w_{3a}, w_{5a}, w_{1a}, w_{6a},
w_{29}, w_{49}, w_{39}, w_{59}, w_{19}, w_{69}, w_{28}, w_{48}, w_{38}, w_{58}, w_{18}, w_{68}, w_{2b}, w_{4b}, w_{3b}, w_{5b},
w_{1b}, w_{6b}, w_{2a}, w_{4a}, w_{3a}, w_{5a}, w_{1a}, w_{6a}, w_{29}, w_{49}, w_{39}, w_{59}, w_{19}, w_{69}, w_{2b}, w_{4b},
w_{3b}, w_{5b}, w_{1b}, w_{6b}, w_{2a}, w_{4a}, w_{3a}, w_{5a}, w_{1a}, w_{6a}, w_{2b}, w_{4b}, w_{3b}, w_{5b}, w_{1b}, w_{6b},
w_{11}, w_{61}, w_{12}, w_{62}, w_{13}, w_{63}, w_{14}, w_{64}, w_{15}, w_{65}, w_{16}, w_{66}, w_{17}, w_{67}, w_{18}, w_{68},
w_{19}, w_{69}, w_{1a}, w_{6a}, w_{1b}, w_{6b}, w_{11}, w_{61}, w_{12}, w_{62}, w_{13}, w_{63}, w_{14}, w_{64}, w_{15}, w_{65},
w_{16}, w_{66}, w_{17}, w_{67}, w_{18}, w_{68}, w_{19}, w_{69}, w_{1a}, w_{6a}, w_{11}, w_{61}, w_{12}, w_{62}, w_{13}, w_{63},
w_{14}, w_{64}, w_{15}, w_{65}, w_{16}, w_{66}, w_{17}, w_{67}, w_{18}, w_{68}, w_{19}, w_{69}, w_{11}, w_{61}, w_{12}, w_{62},
w_{13}, w_{63}, w_{14}, w_{64}, w_{15}, w_{65}, w_{16}, w_{66}, w_{17}, w_{67}, w_{18}, w_{68}, w_{11}, w_{61}, w_{12}, w_{62},
w_{13}, w_{63}, w_{14}, w_{64}, w_{15}, w_{65}, w_{16}, w_{66}, w_{17}, w_{67}, w_{11}, w_{61}, w_{12}, w_{62}, w_{13}, w_{63},
w_{14}, w_{64}, w_{15}, w_{65}, w_{16}, w_{66}, w_{11}, w_{61}, w_{12}, w_{62}, w_{13}, w_{63}, w_{14}, w_{64}, w_{15}, w_{65},
w_{11}, w_{61}, w_{12}, w_{62}, w_{13}, w_{63}, w_{14}, w_{64}, w_{11}, w_{61}, w_{12}, w_{62}, w_{13}, w_{63}, w_{11}, w_{61},
w_{12}, w_{62}, w_{11}, w_{61}, w_{1b}, w_{6b}, w_{1a}, w_{6a}, w_{19}, w_{69}, w_{18}, w_{68}, w_{17}, w_{67}, w_{16}, w_{66},
w_{15}, w_{65}, w_{14}, w_{64}, w_{13}, w_{63}, w_{12}, w_{62}, w_{11}, w_{61}, w_{31}, w_{51}, w_{32}, w_{52}, w_{33}, w_{53},
w_{34}, w_{54}, w_{35}, w_{55}, w_{36}, w_{56}, w_{37}, w_{57}, w_{38}, w_{58}, w_{39}, w_{59}, w_{3a}, w_{5a}, w_{3b}, w_{5b},
w_{31}, w_{51}, w_{32}, w_{52}, w_{33}, w_{53}, w_{34}, w_{54}, w_{35}, w_{55}, w_{36}, w_{56}, w_{37}, w_{57}, w_{38}, w_{58},
w_{39}, w_{59}, w_{3a}, w_{5a}, w_{31}, w_{51}, w_{32}, w_{52}, w_{33}, w_{53}, w_{34}, w_{54}, w_{35}, w_{55}, w_{36}, w_{56},
w_{37}, w_{57}, w_{38}, w_{58}, w_{39}, w_{59}, w_{31}, w_{51}, w_{32}, w_{52}, w_{33}, w_{53}, w_{34}, w_{54}, w_{35}, w_{55},
w_{36}, w_{56}, w_{37}, w_{57}, w_{38}, w_{58}, w_{31}, w_{51}, w_{32}, w_{52}, w_{33}, w_{53}, w_{34}, w_{54}, w_{35}, w_{55},
w_{36}, w_{56}, w_{37}, w_{57}, w_{31}, w_{51}, w_{32}, w_{52}, w_{33}, w_{53}, w_{34}, w_{54}, w_{35}, w_{55}, w_{36}, w_{56},
w_{31}, w_{51}, w_{32}, w_{52}, w_{33}, w_{53}, w_{34}, w_{54}, w_{35}, w_{55}, w_{31}, w_{51}, w_{32}, w_{52}, w_{33}, w_{53},
w_{34}, w_{54}, w_{31}, w_{51}, w_{32}, w_{52}, w_{33}, w_{53}, w_{31}, w_{51}, w_{32}, w_{52}, w_{31}, w_{51}, w_{3b}, w_{5b},

w_{3a}, w_{5a}, w_{39}, w_{59}, w_{38}, w_{58}, w_{37}, w_{57}, w_{36}, w_{56}, w_{35}, w_{55}, w_{34}, w_{54}, w_{33}, w_{53}, w_{32}, w_{52}, w_{31}, w_{51}, w_{41}, w_{42}, w_{43}, w_{44}, w_{45}, w_{46}, w_{47}, w_{48}, w_{49}, w_{4a}, w_{4b}, w_{41}, w_{42}, w_{43}, w_{44}, w_{45}, w_{46}, w_{47}, w_{48}, w_{49}, w_{4a}, w_{41}, w_{42}, w_{43}, w_{44}, w_{45}, w_{46}, w_{47}, w_{48}, w_{49}, w_{41}, w_{42}, w_{43}, w_{44}, w_{45}, w_{46}, w_{47}, w_{48}, w_{41}, w_{42}, w_{43}, w_{44}, w_{45}, w_{46}, w_{47}, w_{41}, w_{42}, w_{43}, w_{44}, w_{45}, w_{46}, w_{41}, w_{42}, w_{43}, w_{44}, w_{45}, w_{41}, w_{42}, w_{43}, w_{44}, w_{41}, w_{42}, w_{43}, w_{41}, w_{42}, w_{41}, w_{4b}, w_{4a}, w_{49}, w_{48}, w_{47}, w_{46}, w_{45}, w_{44}, w_{43}, w_{42}, w_{41}, w_{21}, w_{22}, w_{23}, w_{24}, w_{25}, w_{26}, w_{27}, w_{28}, w_{29}, w_{2a}, w_{2b}, w_{21}, w_{22}, w_{23}, w_{24}, w_{25}, w_{26}, w_{27}, w_{28}, w_{29}, w_{2a}, w_{21}, w_{22}, w_{23}, w_{24}, w_{25}, w_{26}, w_{27}, w_{28}, w_{29}, w_{21}, w_{22}, w_{23}, w_{24}, w_{25}, w_{26}, w_{27}, w_{28}, w_{21}, w_{22}, w_{23}, w_{24}, w_{25}, w_{26}, w_{27}, w_{21}, w_{22}, w_{23}, w_{24}, w_{25}, w_{26}, w_{21}, w_{22}, w_{23}, w_{24}, w_{25}, w_{21}, w_{22}, w_{23}, w_{24}, w_{21}, w_{22}, w_{23}, w_{21}, w_{22}, w_{21}, w_{2b}, w_{2a}, w_{29}, w_{28}, w_{27}, w_{26}, w_{25}, w_{24}, w_{23}, w_{22}, w_{21}, w_{17}, w_{67}, w_{18}, w_{68}, w_{19}, w_{69}, w_{1a}, w_{6a}, w_{1b}, w_{6b}, w_{17}, w_{67}, w_{18}, w_{68}, w_{19}, w_{69}, w_{1a}, w_{6a}, w_{17}, w_{67}, w_{18}, w_{68}, w_{19}, w_{69}, w_{17}, w_{67}, w_{18}, w_{68}, w_{17}, w_{67}, w_{1b}, w_{6b}, w_{1a}, w_{6a}, w_{19}, w_{69}, w_{18}, w_{68}, w_{17}, w_{67}, w_{37}, w_{57}, w_{38}, w_{58}, w_{39}, w_{59}, w_{3a}, w_{5a}, w_{3b}, w_{5b}, w_{37}, w_{57}, w_{38}, w_{58}, w_{39}, w_{59}, w_{3a}, w_{5a}, w_{37}, w_{57}, w_{38}, w_{58}, w_{39}, w_{59}, w_{37}, w_{57}, w_{38}, w_{58}, w_{37}, w_{57}, w_{3b}, w_{5b}, w_{3a}, w_{5a}, w_{39}, w_{59}, w_{38}, w_{58}, w_{37}, w_{57}, w_{47}, w_{48}, w_{49}, w_{4a}, w_{4b}, w_{47}, w_{48}, w_{49}, w_{4a}, w_{47}, w_{48}, w_{49}, w_{47}, w_{48}, w_{47}, w_{4b}, w_{4a}, w_{49}, w_{48}, w_{47}, w_{27}, w_{28}, w_{29}, w_{2a}, w_{2b}, w_{27}, w_{28}, w_{29}, w_{2a}, w_{27}, w_{28}, w_{29}, w_{27}, w_{28}, w_{27}, w_{2b}, w_{2a}, w_{29}, w_{28}, w_{27}, w_{11}, w_{61}, w_{12}, w_{62}, w_{13}, w_{63}, w_{14}, w_{64}, w_{15}, w_{65}, w_{11}, w_{61}, w_{12}, w_{62}, w_{13}, w_{63}, w_{14}, w_{64}, w_{11}, w_{61}, w_{12}, w_{62}, w_{13}, w_{63}, w_{11}, w_{61}, w_{12}, w_{62}, w_{11}, w_{61}, w_{15}, w_{65}, w_{14}, w_{64}, w_{13}, w_{63}, w_{12}, w_{62}, w_{11}, w_{61}, w_{31}, w_{51}, w_{32}, w_{52}, w_{33}, w_{53}, w_{34}, w_{54}, w_{35}, w_{55}, w_{31}, w_{51}, w_{32}, w_{52}, w_{33}, w_{53}, w_{34}, w_{54}, w_{31}, w_{51}, w_{32}, w_{52}, w_{33}, w_{53}, w_{31}, w_{51}, w_{32}, w_{52}, w_{31}, w_{51}, w_{35}, w_{55}, w_{34}, w_{54}, w_{33}, w_{53}, w_{32}, w_{52}, w_{31}, w_{51}, w_{41}, w_{42}, w_{43}, w_{44}, w_{45}, w_{41}, w_{42}, w_{43}, w_{44}, w_{41},

$w_{42},\, w_{43},\, w_{41},\, w_{42},\, w_{41},\, w_{45},\, w_{44},\, w_{43},\, w_{42},\, w_{41},\, w_{21},\, w_{22},\, w_{23},\, w_{24},\, w_{25},\, w_{21},$

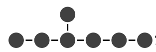
$w_{22},\, w_{23},\, w_{24},\, w_{21},\, w_{22},\, w_{23},\, w_{21},\, w_{22},\, w_{21},\, w_{25},\, w_{24},\, w_{23},\, w_{22},\, w_{21} \quad \}$

$\mathcal{M}(\alpha)$ gives

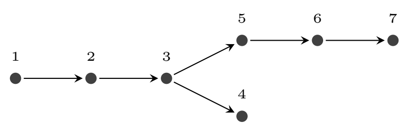
$$
\begin{array}{llllll}
A_1 = h_{11} & A_2 = h_{12} & A_3 = h_{13} & A_4 = h_{14} & A_5 = h_{15} & A_6 = h_{16} \\
B_1 = h_{21} & B_2 = h_{22} & B_3 = h_{23} & B_4 = h_{24} & B_5 = h_{25} & B_6 = h_{26} \\
C_1 = h_{36} & C_2 = h_{32} & C_3 = h_{35} & C_4 = h_{34} & C_5 = h_{33} & C_6 = h_{31}
\end{array}
$$

$$
\begin{array}{lll}
v_{21} = \Delta^{w_7\omega_2}(u) \cdot h_{22} & v_{41} = \Delta^{w_8\omega_4}(u) \cdot \dfrac{h_{12}h_{24}}{h_{14}} & v_{31} = \Delta^{w_9\omega_3}(u) \cdot \dfrac{h_{12}h_{23}}{h_{15}} \\
v_{51} = \Delta^{w_{10}\omega_5}(u) \cdot \dfrac{h_{12}h_{25}}{h_{13}} & v_{11} = \Delta^{w_{11}\omega_1}(u) \cdot \dfrac{h_{12}h_{21}}{h_{16}} & v_{61} = \Delta^{w_{12}\omega_6}(u) \cdot \dfrac{h_{12}h_{26}}{h_{11}} \\
v_{22} = \Delta^{w_{13}\omega_2}(u) \cdot \dfrac{h_{14}h_{22}}{h_{12}} & v_{42} = \Delta^{w_{14}\omega_4}(u) \cdot h_{12}h_{24} & v_{32} = \Delta^{w_{15}\omega_3}(u) \cdot \dfrac{h_{12}h_{14}h_{23}}{h_{15}} \\
v_{52} = \Delta^{w_{16}\omega_5}(u) \cdot \dfrac{h_{12}h_{14}h_{25}}{h_{13}} & v_{12} = \Delta^{w_{17}\omega_1}(u) \cdot \dfrac{h_{14}h_{21}}{h_{16}} & v_{62} = \Delta^{w_{18}\omega_6}(u) \cdot \dfrac{h_{14}h_{26}}{h_{11}} \\
v_{23} = \Delta^{w_{19}\omega_2}(u) \cdot h_{22} & v_{43} = \Delta^{w_{20}\omega_4}(u) \cdot h_{12}^2 h_{24} & v_{33} = \Delta^{w_{21}\omega_3}(u) \cdot \dfrac{h_{12}h_{14}h_{23}}{h_{15}} \\
v_{53} = \Delta^{w_{22}\omega_5}(u) \cdot \dfrac{h_{12}h_{14}h_{25}}{h_{13}} & v_{13} = \Delta^{w_{23}\omega_1}(u) \cdot \dfrac{h_{12}h_{21}}{h_{16}} & v_{63} = \Delta^{w_{24}\omega_6}(u) \cdot \dfrac{h_{12}h_{26}}{h_{11}} \\
v_{24} = \Delta^{w_{25}\omega_2}(u) \cdot h_{14}h_{22} & v_{44} = \Delta^{w_{26}\omega_4}(u) \cdot h_{12}h_{14}h_{24} & v_{34} = \Delta^{w_{27}\omega_3}(u) \cdot \dfrac{h_{12}h_{14}h_{23}}{h_{15}} \\
v_{54} = \Delta^{w_{28}\omega_5}(u) \cdot \dfrac{h_{12}h_{14}h_{25}}{h_{13}} & v_{14} = \Delta^{w_{29}\omega_1}(u) \cdot \dfrac{h_{14}h_{21}}{h_{16}} & v_{64} = \Delta^{w_{30}\omega_6}(u) \cdot \dfrac{h_{14}h_{26}}{h_{11}} \\
v_{25} = \Delta^{w_{31}\omega_2}(u) \cdot \dfrac{h_{14}h_{22}}{h_{12}} & v_{45} = \Delta^{w_{32}\omega_4}(u) \cdot h_{12}h_{24} & v_{35} = \Delta^{w_{33}\omega_3}(u) \cdot \dfrac{h_{14}h_{23}}{h_{15}} \\
v_{55} = \Delta^{w_{34}\omega_5}(u) \cdot \dfrac{h_{14}h_{25}}{h_{13}} & v_{15} = \Delta^{w_{35}\omega_1}(u) \cdot \dfrac{h_{15}h_{21}}{h_{16}} & v_{65} = \Delta^{w_{36}\omega_6}(u) \cdot \dfrac{h_{13}h_{26}}{h_{11}}
\end{array}
$$

9.6 E_7

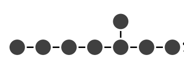
Based on the Dynkin diagram , the following quiver is of well-rooted Dynkin type for E_7. It admits an induced Coxeter element $c = \{1,2,3,4,5,6,7\}$, with partitions

$$T_0 = \{1\}, \quad T_1 = \{2\}, \quad T_2 = \{3\}, \quad T_3 = \{4,5\}, \quad T_4 = \{6\}, \quad T_5 = \{7\}.$$

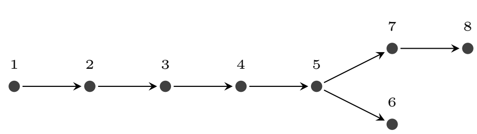


Q_{E_7} is given in Figure 9.6. The mutations are too long to be reasonably presented.

9.7 E_8

Based on the Dynkin diagram , the following quiver is of well-rooted Dynkin type for E_8. It admits an induced Coxeter element $c = \{1,2,3,4,5,6,7,8\}$, with partitions

$$T_0 = \{1\}, T_1 = \{2\}, T_2 = \{3\}, T_3 = \{4\}, T_4 = \{5\}, T_5 = \{6,7\}, T_6 = \{8\}.$$



Q_{E_8} is given in Figure 9.7. The mutations are too long to be reasonably presented.

9.8 $D_2 = A_1 \times A_1$

Using the Dynkin diagram • •, we use the following quiver.

The quiver Q_G is shown in Figure 9.8.

$\mu_{\text{rot}} = \{ \quad \}$

$\mu_{\text{flip}} = \{ \ w_{11}, w_{21} \ \}$

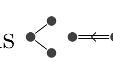
And $\mathcal{M}$ is given the edge coordinate assignment only. Since σ_{D_2} is trivial, this is direct.

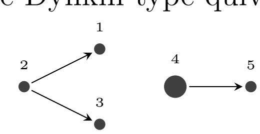
$$A_1 = h_{11} \qquad A_2 = h_{12}$$

$$B_1 = h_{21} \qquad B_2 = h_{22}$$

$$C_1 = h_{31} \qquad C_2 = h_{32}$$

9.9 $A_3 \times C_2$

To demonstrate Lemma 8.0.1 for a less trivial type, consider $G = G_1 \oplus G_2$, with G_1 of type A_3 and G_2 of type C_2. The Dynkin diagram for G is . The quiver Q_G is the disjoint union of Q_{A_3} and Q_{C_2} as shown in Figure 9.9. We label the simple roots as shown in the Dynkin type quiver.



$\mu_{\text{rot}} = \{ \ v_{21}, v_{11}, v_{31}, v_{21}, v_{11}, v_{31}, v_{21}, v_{11}, v_{31}, v_{21}, v_{11}, v_{31}, \qquad v_{41}, v_{51} \ \}$

$\mu_{\text{flip}} = \{ \ w_{13}, w_{33}, w_{23}, w_{11}, w_{31}, w_{21}, w_{23}, w_{13}, w_{33}, w_{22}, w_{12}, w_{32}, w_{21}, w_{11}, w_{31}, w_{23},$

$w_{13}, w_{33}, w_{22}, w_{12}, w_{32}, w_{23}, w_{13}, w_{33}, w_{11}, w_{31}, w_{12}, w_{32}, w_{13}, w_{33}, w_{11}, w_{31},$

$w_{12}, w_{32}, w_{11}, w_{31}, w_{13}, w_{33}, w_{12}, w_{32}, w_{11}, w_{31}, w_{21}, w_{22}, w_{23}, w_{21}, w_{22}, w_{21},$

$w_{23}, w_{22}, w_{21}, \quad w_{53}, w_{43}, w_{51}, w_{41}, w_{43}, w_{53}, w_{42}, w_{52}, w_{41}, w_{51}, w_{43}, w_{53},$

$w_{42}, w_{52}, w_{43}, w_{53}, w_{51}, w_{52}, w_{53}, w_{51}, w_{52}, w_{51}, w_{53}, w_{52}, w_{51}, w_{41}, w_{42}, w_{43},$

$w_{41}, w_{42}, w_{41}, w_{43}, w_{42}, w_{41} \ \}$

To construct $\mathcal{M}$, as described in Lemma 8.0.1, the coordinates on u are exactly divided between two elements of N_- corresponding to A_3 and C_2. Let $(w_k)_1$ be the word w_k for A_3, and $(w_k)_2$ be the word w_k for C_2 (with the indices of each component s_{α_i} increased by 3). Then appropriate i and j in $\Delta^{(w_k)_i\omega_j}(u)$ will allow extracting coordinates of u in ways corresponding to the factors G_i.

$$A_1 = h_{11} \quad A_2 = h_{12} \quad A_3 = h_{13} \quad A_4 = h_{14} \quad A_5 = h_{15}$$

$$B_1 = h_{21} \quad B_2 = h_{22} \quad B_3 = h_{23} \quad B_4 = h_{24} \quad B_5 = h_{25}$$

$$C_1 = h_{33} \quad C_2 = h_{32} \quad C_3 = h_{31} \quad C_4 = h_{34} \quad C_5 = h_{35}$$

$$v_{21} = \Delta^{(w_4)_1\omega_2}(u) \cdot h_{22} \qquad v_{11} = \Delta^{(w_5)_1\omega_1}(u) \cdot \frac{h_{12}h_{21}}{h_{13}} \qquad v_{31} = \Delta^{(w_6)_1\omega_3}(u) \cdot \frac{h_{12}h_{23}}{h_{11}}$$

$$v_{41} = \Delta^{(w_3)_2\omega_4}(u) \cdot h_{24} \qquad v_{51} = \Delta^{(w_4)_2\omega_5}(u) \cdot \frac{h_{14}^2 h_{25}}{h_{15}}$$

$$
\widetilde{\mu}_{\mathrm{P}} = \mu \left\{ \begin{array}{r}
w_{4b},w_{3b},w_{2b},w_{1b}, \\
w_{4a},w_{3a},w_{2a},w_{1a},\; w_{4b},w_{3b},w_{2b},w_{1b}, \\
w_{49},w_{39},w_{29},w_{19},\; w_{4a},w_{3a},w_{2a},w_{1a},\; w_{4b},w_{3b},w_{2b},w_{1b}, \\
w_{48},w_{38},w_{28},w_{18},\; w_{49},w_{39},w_{29},w_{19},\; w_{4a},w_{3a},w_{2a},w_{1a},\; w_{4b},w_{3b},w_{2b},w_{1b}, \\
w_{47},w_{37},w_{27},w_{17},\; w_{48},w_{38},w_{28},w_{18},\; w_{49},w_{39},w_{29},w_{19},\; w_{4a},w_{3a},w_{2a},w_{1a},\; w_{4b},w_{3b},w_{2b},w_{1b}, \\
w_{45},w_{35},w_{25},w_{15}, \\
w_{44},w_{34},w_{24},w_{14},\; w_{45},w_{35},w_{25},w_{15}, \\
w_{43},w_{33},w_{23},w_{13},\; w_{44},w_{34},w_{24},w_{14},\; w_{45},w_{35},w_{25},w_{15}, \\
w_{42},w_{32},w_{22},w_{12},\; w_{43},w_{33},w_{23},w_{13},\; w_{44},w_{34},w_{24},w_{14},\; w_{45},w_{35},w_{25},w_{15}, \\
w_{41},w_{31},w_{21},w_{11},\; w_{42},w_{32},w_{22},w_{12},\; w_{43},w_{33},w_{23},w_{13},\; w_{44},w_{34},w_{24},w_{14},\; w_{45},w_{35},w_{25},w_{15}
\end{array} \right\}
$$

$$
\widetilde{\mu}_{\mathrm{Flipcore}} = \mu \left\{ \begin{array}{l}
w_{1b},w_{2b},w_{3b},w_{4b},w_{1a},w_{2a},w_{3a},w_{4a},w_{19},w_{29},w_{39},w_{49},w_{18},w_{28},w_{38},w_{48},w_{17},w_{27},w_{37},w_{47},w_{16}, \\
\quad w_{26},w_{36},w_{46},w_{15},w_{25},w_{35},w_{45},w_{14},w_{24},w_{34},w_{44},w_{13},w_{23},w_{33},w_{43},w_{12},w_{22},w_{32}, \\
\quad w_{42},w_{11},w_{21},w_{31},w_{41}, \\
w_{1b},w_{2b},w_{3b},w_{4b},w_{1a},w_{2a},w_{3a},w_{4a},w_{19},w_{29},w_{39},w_{49},w_{18},w_{28},w_{38},w_{48},w_{17},w_{27},w_{37},w_{47},w_{16}, \\
\quad w_{26},w_{36},w_{46},w_{15},w_{25},w_{35},w_{45},w_{14},w_{24},w_{34},w_{44},w_{13},w_{23},w_{33},w_{43},w_{12},w_{22},w_{32}, \\
\quad w_{42}, \\
w_{1b},w_{2b},w_{3b},w_{4b},w_{1a},w_{2a},w_{3a},w_{4a},w_{19},w_{29},w_{39},w_{49},w_{18},w_{28},w_{38},w_{48},w_{17},w_{27},w_{37},w_{47},w_{16}, \\
\quad w_{26},w_{36},w_{46},w_{15},w_{25},w_{35},w_{45},w_{14},w_{24},w_{34},w_{44},w_{13},w_{23},w_{33},w_{43}, \\
w_{1b},w_{2b},w_{3b},w_{4b},w_{1a},w_{2a},w_{3a},w_{4a},w_{19},w_{29},w_{39},w_{49},w_{18},w_{28},w_{38},w_{48},w_{17},w_{27},w_{37},w_{47},w_{16}, \\
\quad w_{26},w_{36},w_{46},w_{15},w_{25},w_{35},w_{45},w_{14},w_{24},w_{34},w_{44}, \\
w_{1b},w_{2b},w_{3b},w_{4b},w_{1a},w_{2a},w_{3a},w_{4a},w_{19},w_{29},w_{39},w_{49},w_{18},w_{28},w_{38},w_{48},w_{17},w_{27},w_{37},w_{47},w_{16}, \\
\quad w_{26},w_{36},w_{46},w_{15},w_{25},w_{35},w_{45}, \\
w_{1b},w_{2b},w_{3b},w_{4b},w_{1a},w_{2a},w_{3a},w_{4a},w_{19},w_{29},w_{39},w_{49},w_{18},w_{28},w_{38},w_{48},w_{17},w_{27},w_{37},w_{47},w_{16}, \\
\quad w_{26},w_{36},w_{46}, \\
w_{1b},w_{2b},w_{3b},w_{4b},w_{1a},w_{2a},w_{3a},w_{4a},w_{19},w_{29},w_{39},w_{49},w_{18},w_{28},w_{38},w_{48},w_{17},w_{27},w_{37},w_{47}, \\
w_{1b},w_{2b},w_{3b},w_{4b},w_{1a},w_{2a},w_{3a},w_{4a},w_{19},w_{29},w_{39},w_{49},w_{18},w_{28},w_{38},w_{48}, \\
w_{1b},w_{2b},w_{3b},w_{4b},w_{1a},w_{2a},w_{3a},w_{4a},w_{19},w_{29},w_{39},w_{49}, \\
w_{1b},w_{2b},w_{3b},w_{4b},w_{1a},w_{2a},w_{3a},w_{4a}, \\
w_{1b},w_{2b},w_{3b},w_{4b},
\end{array} \right\}
$$

$$
\widetilde{\mu}_{\mathrm{O3}} = \mu \left\{ \begin{array}{l}
w_{41},w_{42},w_{43},w_{44},w_{45},w_{46},w_{47},w_{48},w_{49},w_{4a},w_{4b}, \\
w_{41},w_{42},w_{43},w_{44},w_{45},w_{46},w_{47},w_{48},w_{49},w_{4a}, \\
w_{41},w_{42},w_{43},w_{44},w_{45},w_{46},w_{47},w_{48},w_{49}, \\
w_{41},w_{42},w_{43},w_{44},w_{45},w_{46},w_{47},w_{48}, \\
w_{41},w_{42},w_{43},w_{44},w_{45},w_{46},w_{47}, \\
w_{41},w_{42},w_{43},w_{44},w_{45},w_{46}, \\
w_{41},w_{42},w_{43},w_{44},w_{45}, \\
w_{41},w_{42},w_{43},w_{44}, \\
w_{41},w_{42},w_{43}, \\
w_{41},w_{42}, \\
w_{41}, \\
w_{4b},w_{4a},w_{49},w_{48},w_{47},w_{46},w_{45},w_{44},w_{43},w_{42},w_{41}, \\
\\
w_{31},w_{32},w_{33},w_{34},w_{35},w_{36},w_{37},w_{38},w_{39},w_{3a},w_{3b}, \\
w_{31},w_{32},w_{33},w_{34},w_{35},w_{36},w_{37},w_{38},w_{39},w_{3a}, \\
w_{31},w_{32},w_{33},w_{34},w_{35},w_{36},w_{37},w_{38},w_{39}, \\
w_{31},w_{32},w_{33},w_{34},w_{35},w_{36},w_{37},w_{38}, \\
w_{31},w_{32},w_{33},w_{34},w_{35},w_{36},w_{37}, \\
w_{31},w_{32},w_{33},w_{34},w_{35},w_{36}, \\
w_{31},w_{32},w_{33},w_{34},w_{35}, \\
w_{31},w_{32},w_{33},w_{34}, \\
w_{31},w_{32},w_{33}, \\
w_{31},w_{32}, \\
w_{31}, \\
w_{3b},w_{3a},w_{39},w_{38},w_{37},w_{36},w_{35},w_{34},w_{33},w_{32},w_{31}, \\
\\
w_{21},w_{22},w_{23},w_{24},w_{25},w_{26},w_{27},w_{28},w_{29},w_{2a},w_{2b}, \\
w_{21},w_{22},w_{23},w_{24},w_{25},w_{26},w_{27},w_{28},w_{29},w_{2a}, \\
w_{21},w_{22},w_{23},w_{24},w_{25},w_{26},w_{27},w_{28},w_{29}, \\
w_{21},w_{22},w_{23},w_{24},w_{25},w_{26},w_{27},w_{28}, \\
w_{21},w_{22},w_{23},w_{24},w_{25},w_{26},w_{27}, \\
w_{21},w_{22},w_{23},w_{24},w_{25},w_{26}, \\
w_{21},w_{22},w_{23},w_{24},w_{25}, \\
w_{21},w_{22},w_{23},w_{24}, \\
w_{21},w_{22},w_{23}, \\
w_{21},w_{22}, \\
w_{21}, \\
w_{2b},w_{2a},w_{29},w_{28},w_{27},w_{26},w_{25},w_{24},w_{23},w_{22},w_{21}, \\
\\
w_{11},w_{12},w_{13},w_{14},w_{15},w_{16},w_{17},w_{18},w_{19},w_{1a},w_{1b}, \\
w_{11},w_{12},w_{13},w_{14},w_{15},w_{16},w_{17},w_{18},w_{19},w_{1a}, \\
w_{11},w_{12},w_{13},w_{14},w_{15},w_{16},w_{17},w_{18},w_{19}, \\
w_{11},w_{12},w_{13},w_{14},w_{15},w_{16},w_{17},w_{18}, \\
w_{11},w_{12},w_{13},w_{14},w_{15},w_{16},w_{17}, \\
w_{11},w_{12},w_{13},w_{14},w_{15},w_{16}, \\
w_{11},w_{12},w_{13},w_{14},w_{15}, \\
w_{11},w_{12},w_{13},w_{14}, \\
w_{11},w_{12},w_{13}, \\
w_{11},w_{12}, \\
w_{11}, \\
w_{1b},w_{1a},w_{19},w_{18},w_{17},w_{16},w_{15},w_{14},w_{13},w_{12},w_{11},
\end{array} \right\},
$$

$$
\widetilde{\mu}_{\mathrm{O4}} = \mu \left\{ \begin{array}{l}
w_{47},w_{48},w_{49},w_{4a},w_{4b}, \\
w_{47},w_{48},w_{49},w_{4a}, \\
w_{47},w_{48},w_{49}, \\
w_{47},w_{48}, \\
w_{47}, \\
w_{4b},w_{4a},w_{49},w_{48},w_{47}, \\
\\
w_{37},w_{38},w_{39},w_{3a},w_{3b}, \\
w_{37},w_{38},w_{39},w_{3a}, \\
w_{37},w_{38},w_{39}, \\
w_{37},w_{38}, \\
w_{37}, \\
w_{3b},w_{3a},w_{39},w_{38},w_{37}, \\
\\
w_{27},w_{28},w_{29},w_{2a},w_{2b}, \\
w_{27},w_{28},w_{29},w_{2a}, \\
w_{27},w_{28},w_{29}, \\
w_{27},w_{28}, \\
w_{27}, \\
w_{2b},w_{2a},w_{29},w_{28},w_{27}, \\
\\
w_{17},w_{18},w_{19},w_{1a},w_{1b}, \\
w_{17},w_{18},w_{19},w_{1a}, \\
w_{17},w_{18},w_{19}, \\
w_{17},w_{18}, \\
w_{17}, \\
w_{1b},w_{1a},w_{19},w_{18},w_{17},
\end{array} \right\}
\quad
\widetilde{\mu}_{\mathrm{O5}} = \mu \left\{ \begin{array}{l}
w_{41},w_{42},w_{43},w_{44},w_{45}, \\
w_{41},w_{42},w_{43},w_{44}, \\
w_{41},w_{42},w_{43}, \\
w_{41},w_{42}, \\
w_{41}, \\
w_{45},w_{44},w_{43},w_{42},w_{41}, \\
\\
w_{31},w_{32},w_{33},w_{34},w_{35}, \\
w_{31},w_{32},w_{33},w_{34}, \\
w_{31},w_{32},w_{33}, \\
w_{31},w_{32}, \\
w_{31}, \\
w_{35},w_{34},w_{33},w_{32},w_{31}, \\
\\
w_{21},w_{22},w_{23},w_{24},w_{25}, \\
w_{21},w_{22},w_{23},w_{24}, \\
w_{21},w_{22},w_{23}, \\
w_{21},w_{22}, \\
w_{21}, \\
w_{25},w_{24},w_{23},w_{22},w_{21}, \\
\\
w_{11},w_{12},w_{13},w_{14},w_{15}, \\
w_{11},w_{12},w_{13},w_{14}, \\
w_{11},w_{12},w_{13}, \\
w_{11},w_{12}, \\
w_{11}, \\
w_{15},w_{14},w_{13},w_{12},w_{11},
\end{array} \right\}
$$

Figure 5.7: The flip for F_4.

i	1	2	3	4		1	2	3	4		1	2	3	4
$k=1$	$\frac{1}{t_1}$	$\frac{1}{t_2}$	$\frac{1}{t_3}$	$\frac{1}{t_4}$	9	$\frac{t_2}{t_3}$	$\frac{t_1t_2}{t_3t_4}$	$\frac{t_1^2t_2^2}{t_3^2t_4}$	$\frac{t_2^2}{t_3t_4}$	17	$\frac{t_1t_2}{t_3}$	$\frac{t_1t_2^2}{t_3t_4}$	$\frac{t_1^2t_2^2}{t_3t_4}$	$\frac{t_2^2}{t_3}$
2	$\frac{t_1}{t_2}$	$\frac{1}{t_2}$	$\frac{1}{t_3}$	$\frac{1}{t_4}$	10	$\frac{t_1}{t_4}$	$\frac{t_1t_2}{t_3t_4}$	$\frac{t_1^2t_2^2}{t_3^2t_4}$	$\frac{t_2^2}{t_3t_4}$	18	$\frac{t_2}{t_4}$	$\frac{t_1t_2^2}{t_3t_4}$	$\frac{t_1^2t_2^2}{t_3t_4}$	$\frac{t_2^2}{t_3}$
3	$\frac{t_1}{t_2}$	$\frac{t_1}{t_3}$	$\frac{1}{t_3}$	$\frac{1}{t_4}$	11	$\frac{t_1}{t_4}$	$\frac{t_1^2t_2}{t_3t_4}$	$\frac{t_1^2t_2^2}{t_3^2t_4}$	$\frac{t_2^2}{t_3t_4}$	19	$\frac{t_2}{t_4}$	$\frac{t_1t_2}{t_4}$	$\frac{t_1^2t_2^2}{t_3t_4}$	$\frac{t_2^2}{t_3}$
4	$\frac{t_1}{t_2}$	$\frac{t_1}{t_3}$	$\frac{t_1^2}{t_3t_4}$	$\frac{1}{t_4}$	12	$\frac{t_1}{t_4}$	$\frac{t_1^2t_2}{t_3t_4}$	$\frac{t_1^2t_2^2}{t_3t_4^2}$	$\frac{t_2^2}{t_3t_4}$	20	$\frac{t_2}{t_4}$	$\frac{t_1t_2}{t_4}$	$\frac{t_2^2}{t_4}$	$\frac{t_2^2}{t_3}$
5	$\frac{t_1}{t_2}$	$\frac{t_1}{t_3}$	$\frac{t_1^2}{t_3t_4}$	$\frac{t_1^2}{t_3}$	13	$\frac{t_1}{t_4}$	$\frac{t_1^2t_2}{t_3t_4}$	$\frac{t_1^2t_2^2}{t_3t_4^2}$	$\frac{t_1^2}{t_4}$	21	$\frac{t_2}{t_4}$	$\frac{t_1t_2}{t_4}$	$\frac{t_2^2}{t_4}$	$\frac{t_3}{t_4}$
6	$\frac{t_2}{t_3}$	$\frac{t_1}{t_3}$	$\frac{t_1^2}{t_3t_4}$	$\frac{t_1^2}{t_3}$	14	$\frac{t_1t_2}{t_3}$	$\frac{t_1^2t_2}{t_3t_4}$	$\frac{t_1^2t_2^2}{t_3t_4^2}$	$\frac{t_1^2}{t_4}$	22	t_1	$\frac{t_1t_2}{t_4}$	$\frac{t_2^2}{t_4}$	$\frac{t_3}{t_4}$
7	$\frac{t_2}{t_3}$	$\frac{t_1t_2}{t_3t_4}$	$\frac{t_1^2}{t_3t_4}$	$\frac{t_1^2}{t_3}$	15	$\frac{t_1t_2}{t_3}$	$\frac{t_1t_2^2}{t_3t_4}$	$\frac{t_1^2t_2^2}{t_3t_4^2}$	$\frac{t_1^2}{t_4}$	23	t_1	t_2	$\frac{t_2^2}{t_4}$	$\frac{t_3}{t_4}$
8	$\frac{t_2}{t_3}$	$\frac{t_1t_2}{t_3t_4}$	$\frac{t_1^2t_2^2}{t_3^2t_4}$	$\frac{t_1^2}{t_3}$	16	$\frac{t_1t_2}{t_3}$	$\frac{t_1t_2^2}{t_3t_4}$	$\frac{t_1^2t_2^2}{t_3t_4}$	$\frac{t_1^2}{t_4}$	24	t_1	t_2	t_3	$\frac{t_3}{t_4}$

Figure 5.8: For type F_4, $\Delta_{w_k\omega_i,w_k\omega_i}(A)$ for abstract $A = \prod_{i=1}^r \chi_i^*(t_i)$.

$$A_1 = h_{11} \qquad A_2 = h_{12} \qquad A_3 = h_{13} \qquad A_4 = h_{14}$$
$$B_1 = h_{11} \qquad B_2 = h_{12} \qquad B_3 = h_{13} \qquad B_4 = h_{14}$$
$$C_1 = h_{11} \qquad C_2 = h_{12} \qquad C_3 = h_{13} \qquad C_4 = h_{14}$$

$$v_{11} = \Delta^{w_5\omega_1}(u) \cdot h_{21} \qquad v_{21} = \Delta^{w_6\omega_2}(u) \cdot \frac{h_{11}h_{22}}{h_{12}} \qquad v_{31} = \Delta^{w_7\omega_3}(u) \cdot \frac{h_{11}^2h_{23}}{h_{13}}$$
$$v_{41} = \Delta^{w_8\omega_4}(u) \cdot \frac{h_{11}^2h_{24}}{h_{14}}$$
$$v_{12} = \Delta^{w_9\omega_1}(u) \cdot \frac{h_{12}h_{21}}{h_{11}} \qquad v_{22} = \Delta^{w_{10}\omega_2}(u) \cdot h_{11}h_{22} \qquad v_{32} = \Delta^{w_{11}\omega_3}(u) \cdot \frac{h_{11}^2h_{12}^2h_{23}}{h_{13}}$$
$$v_{42} = \Delta^{w_{12}\omega_4}(u) \cdot \frac{h_{12}^2h_{24}}{h_{14}}$$
$$v_{13} = \Delta^{w_{13}\omega_1}(u) \cdot h_{21} \qquad v_{23} = \Delta^{w_{14}\omega_2}(u) \cdot h_{11}^2h_{22} \qquad v_{33} = \Delta^{w_{15}\omega_3}(u) \cdot \frac{h_{11}^2h_{12}^2h_{23}}{h_{13}}$$
$$v_{43} = \Delta^{w_{16}\omega_4}(u) \cdot \frac{h_{11}^2h_{24}}{h_{14}}$$
$$v_{14} = \Delta^{w_{17}\omega_1}(u) \cdot h_{12}h_{21} \qquad v_{24} = \Delta^{w_{18}\omega_2}(u) \cdot h_{11}h_{12}h_{22} \qquad v_{34} = \Delta^{w_{19}\omega_3}(u) \cdot \frac{h_{11}^2h_{12}^2h_{23}}{h_{13}}$$
$$v_{44} = \Delta^{w_{20}\omega_4}(u) \cdot \frac{h_{12}^2h_{24}}{h_{14}}$$
$$v_{15} = \Delta^{w_{21}\omega_1}(u) \cdot \frac{h_{12}h_{21}}{h_{11}} \qquad v_{25} = \Delta^{w_{22}\omega_2}(u) \cdot h_{11}h_{22} \qquad v_{35} = \Delta^{w_{23}\omega_3}(u) \cdot \frac{h_{12}^2h_{23}}{h_{13}}$$
$$v_{45} = \Delta^{w_{24}\omega_4}(u) \cdot \frac{h_{13}h_{24}}{h_{14}}$$

Figure 5.9: $\mathcal{M}(\alpha)$ for type F_4.

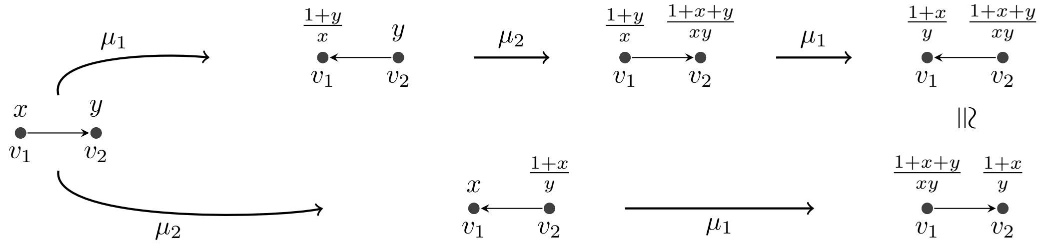


Figure 5.10: The pentagon recurrence.

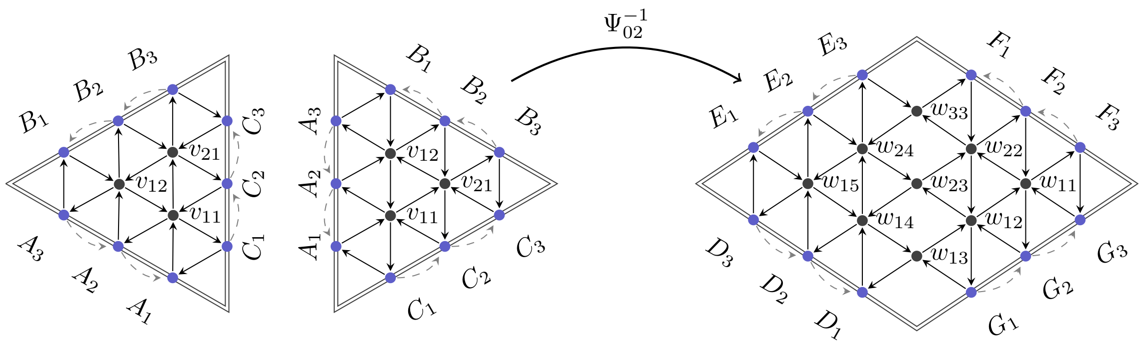


Figure 6.1: Quivers for type A_3, also showing naming convention for Q^{0-2}.

Figure 7.1: Q_{Alt} on the left, Q_{Lin} on the right, both quivers of Dynkin type A_ℓ.

$$\begin{array}{ccccc}
\mathrm{Conf}_3^*(G/N) & \hookleftarrow & N_- \cap \overline{w_0}G_0 & \xrightarrow{Y_V} & (\mathbb{C}^*)^m \\
\downarrow{\scriptstyle \mathrm{cyc}_3} & & \downarrow{\scriptstyle (w_0(h1)h2)\left((\Phi\Psi)^2(\bullet)\right)} & & \downarrow{\scriptstyle \mu^*_{\mathrm{rot}}} \\
\mathrm{Conf}_3^*(G/N) & \hookleftarrow & N_- \cap \overline{w_0}G_0 & \xrightarrow{Y_V} & (\mathbb{C}^*)^m
\end{array}$$

Figure 7.2: Maps Φ and Ψ by Proposition 4.1.5, Y_V by Definition 5.6.1.

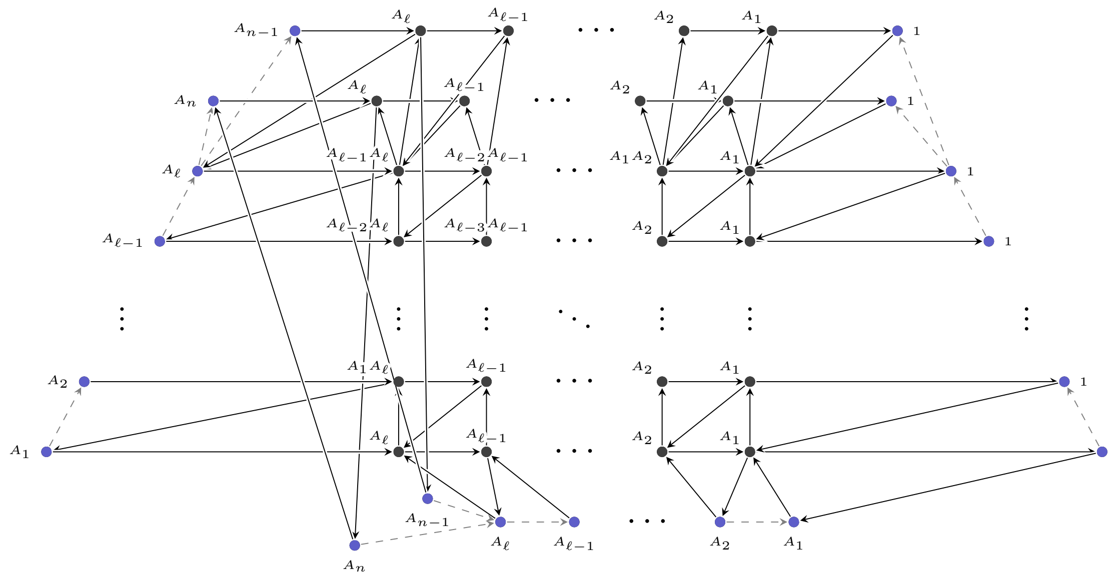


Figure 7.3: Q_{D_n} before any mutation.

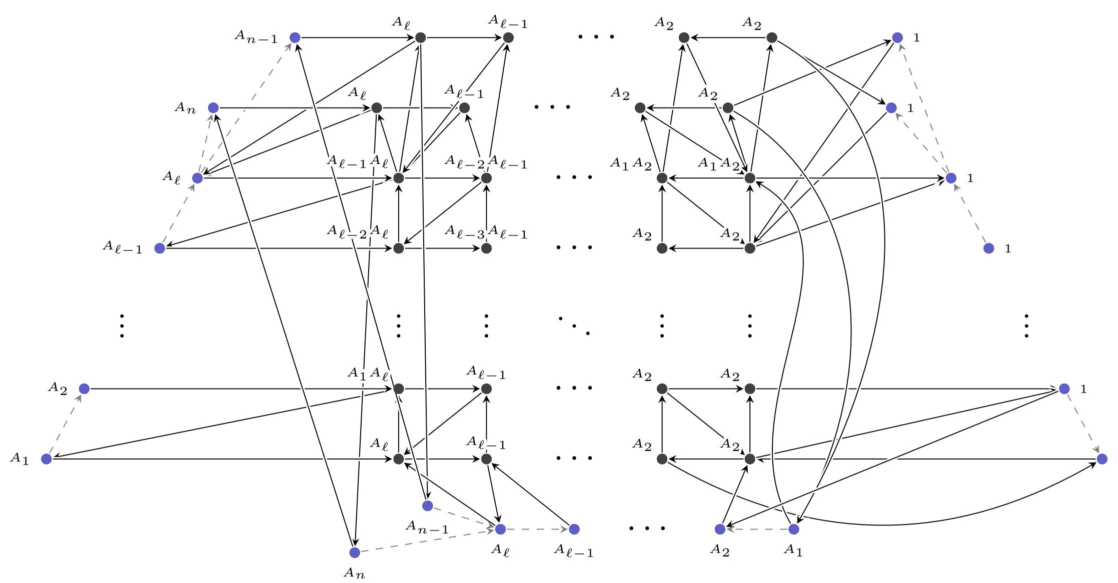


Figure 7.4: Q_{D_n} after $\widetilde{\mu}_{\mathrm{Col}}(1)$.

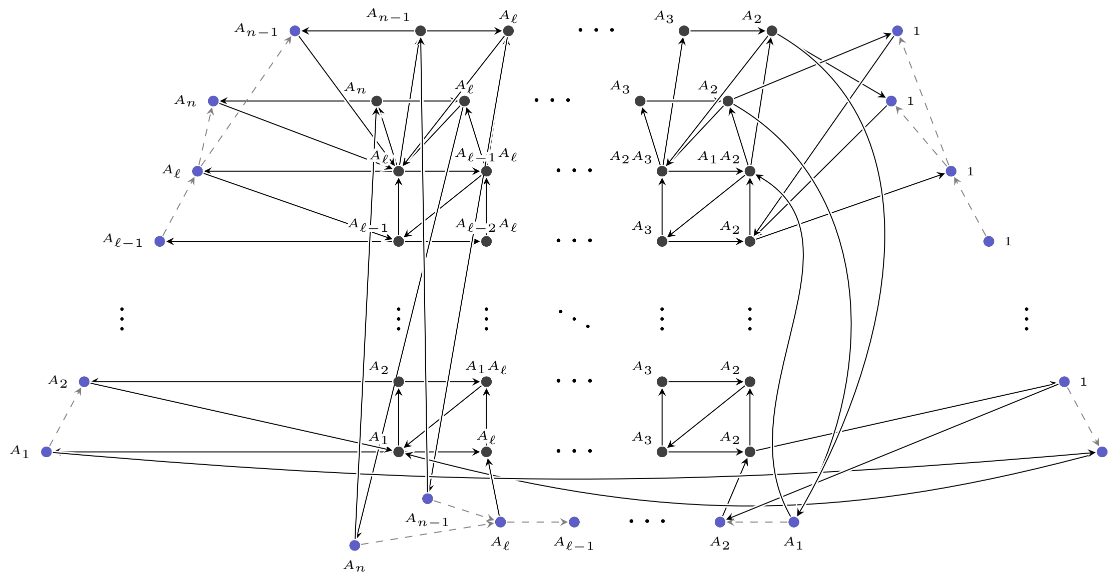


Figure 7.5: Q_{D_n} after $\prod_{y=1}^{\ell} \widetilde{\mu}_{\mathrm{Col}}(y)$.

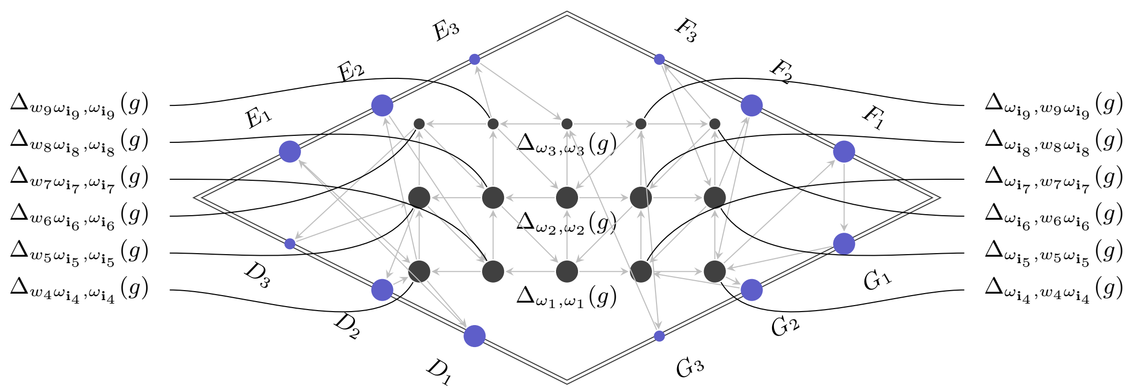


Figure 7.6: Coordinates from α_{120} and α_{302}.

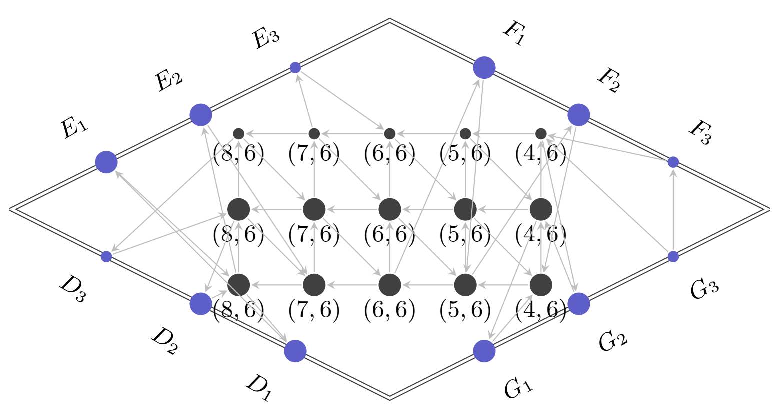


Figure 7.7: Coordinates from α_{120} and α_{230}, labeled by (n, m).

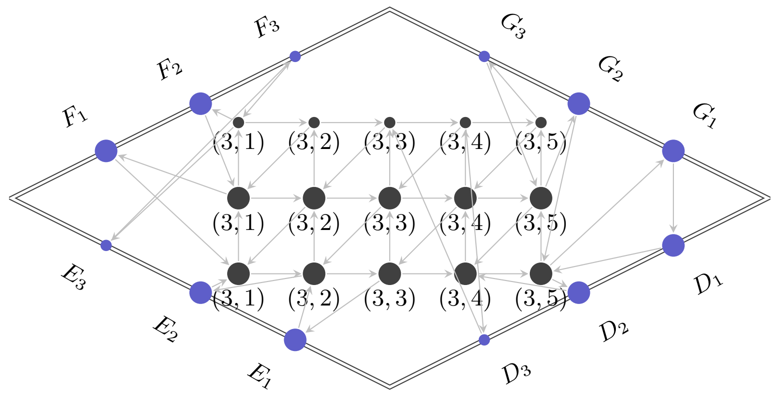


Figure 7.8: Coordinates after applying $\widetilde{\mu}_{\text{Flipcore}}$, labeled by (n, m).

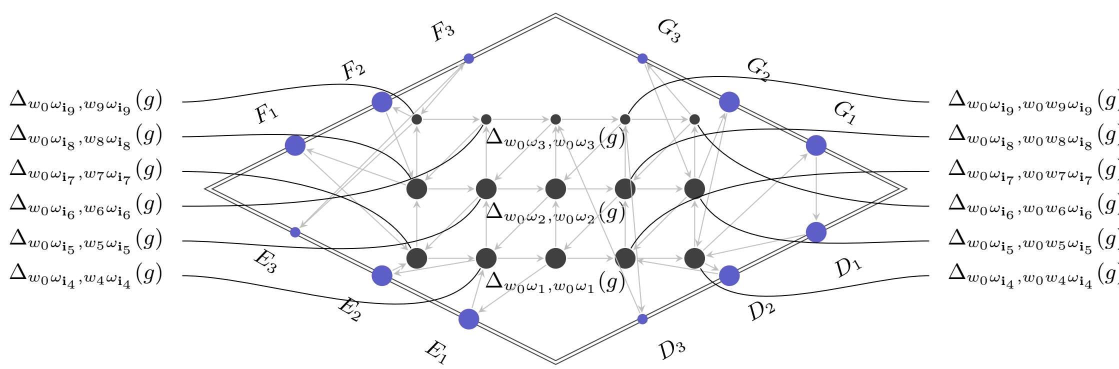


Figure 7.9: Coordinates after applying $\widetilde{\mu}_{\text{Flipcore}}$.

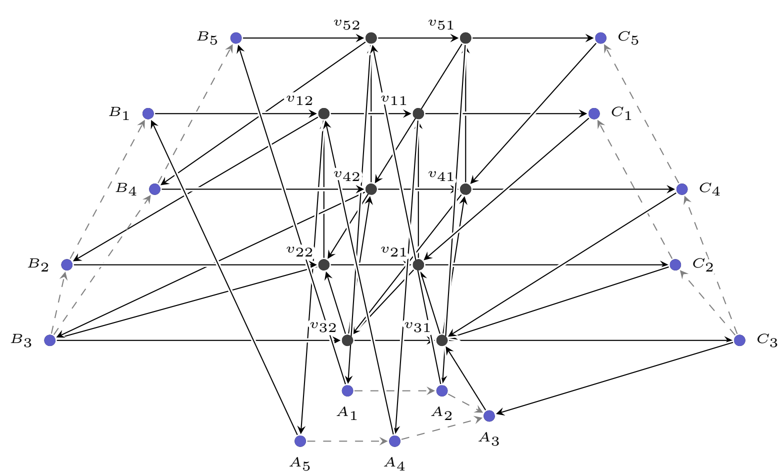


Figure 9.1: Q for A_5.

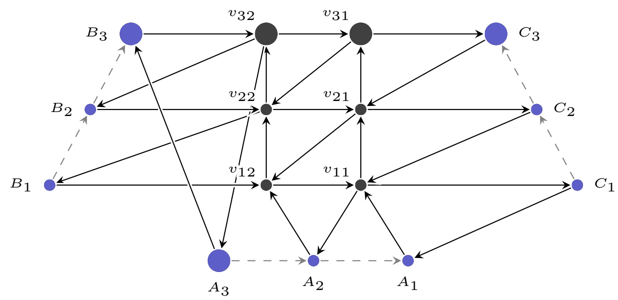


Figure 9.2: Q for B_3.

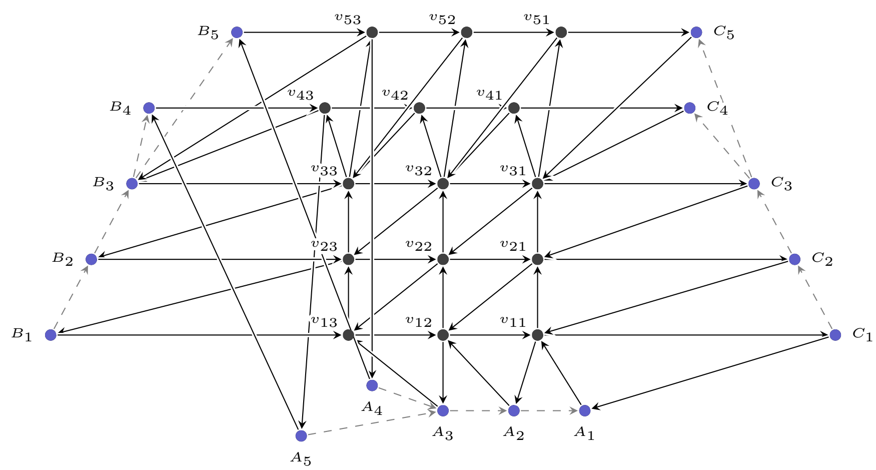


Figure 9.3: Q for D_5.

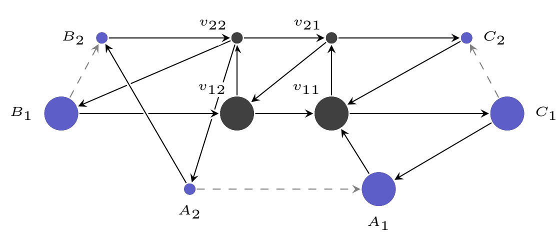


Figure 9.4: Q for G_2.

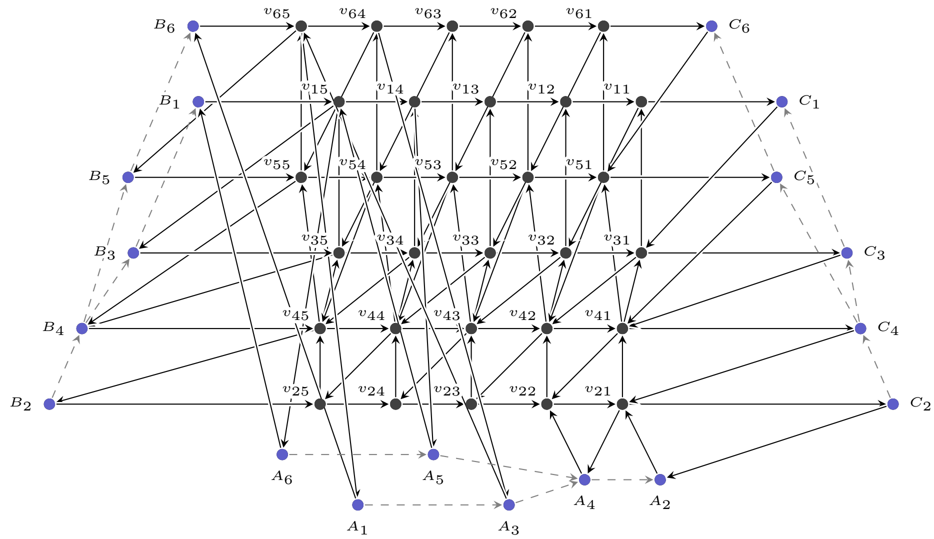


Figure 9.5: Q for E_6.

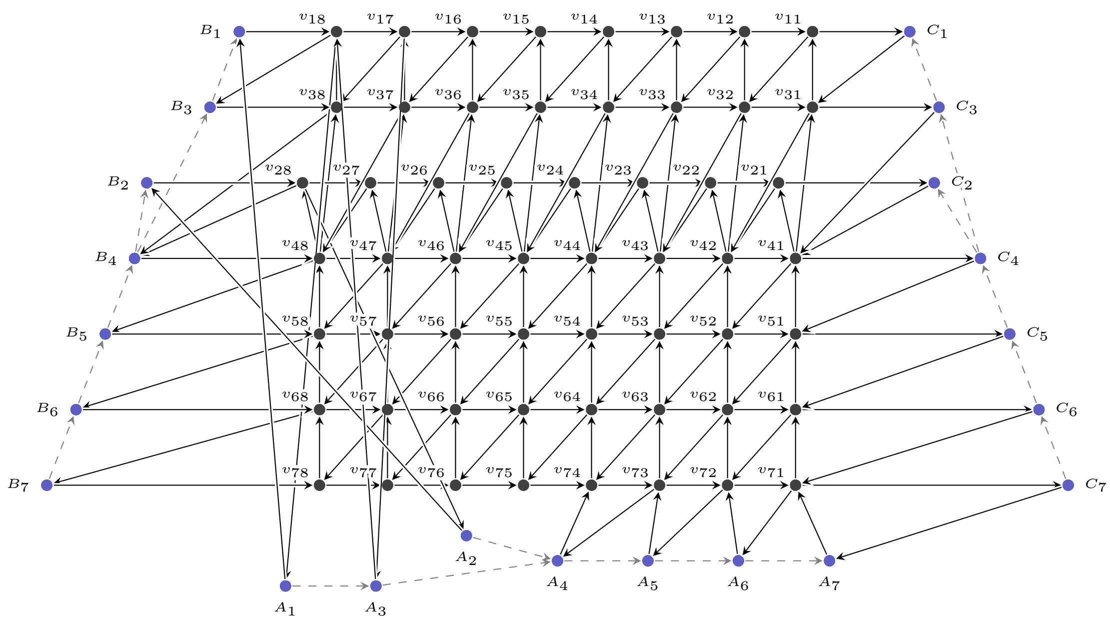


Figure 9.6: Q for E_7.

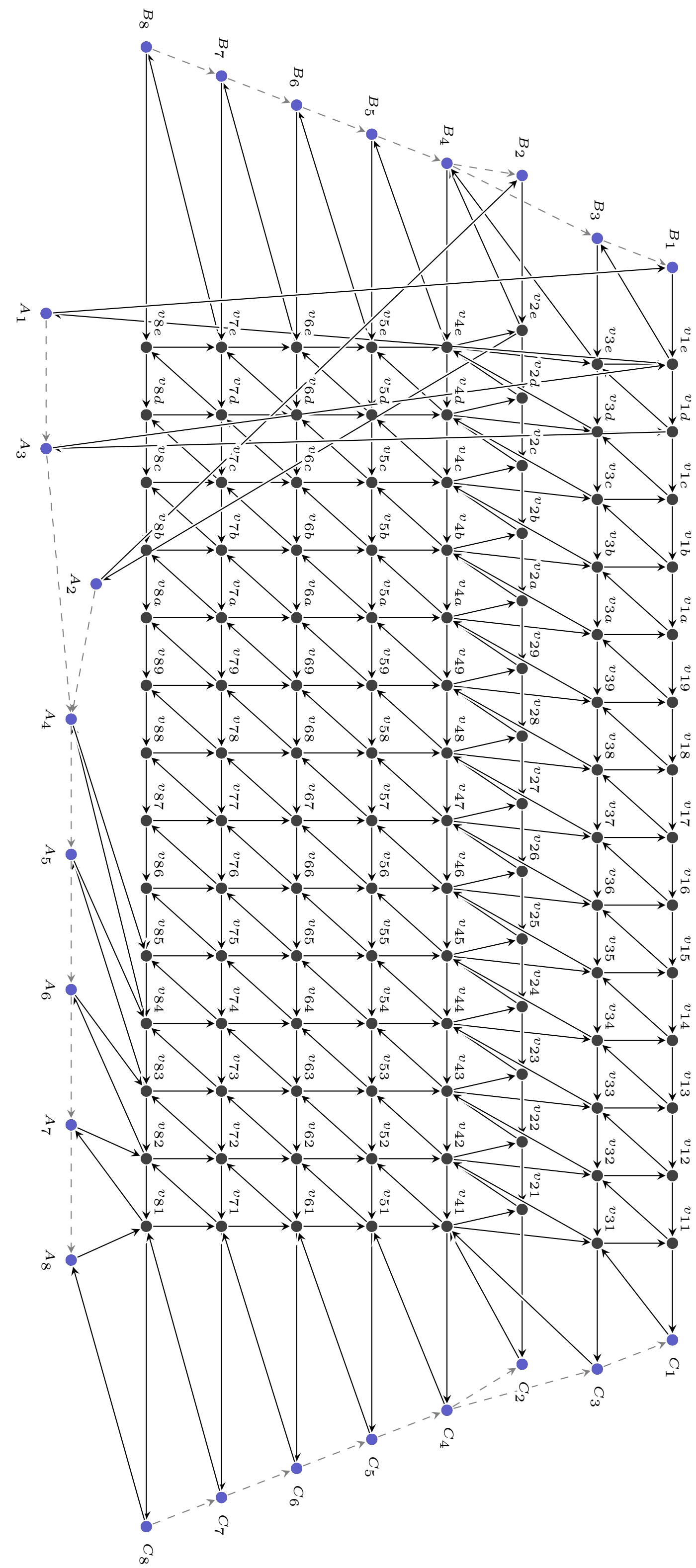


Figure 9.7: Q for E_8.

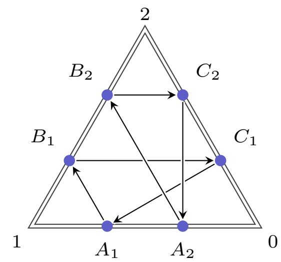


Figure 9.8: Q for $D_2 = A_1 \times A_1$.

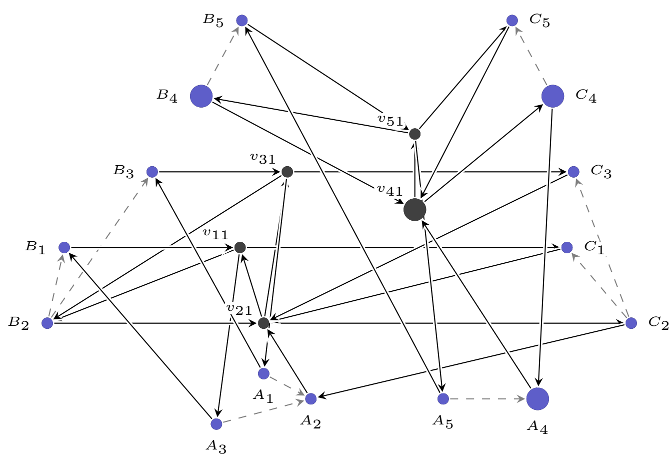


Figure 9.9: Q for $A_3 \times C_2$.

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