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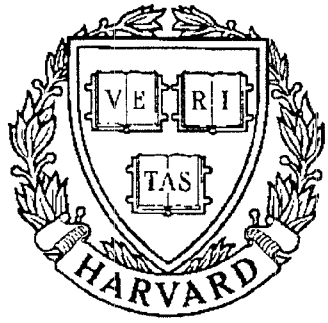

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Wronskians and Linear Dependency of Entire Functions in $\mathbb{C}^n$

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WRONSKIANS AND LINEAR DEPENDENCY OF ENTIRE FUNCTIONS IN $\mathbb{C}^n$

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While the Wronskian criterion for linear dependency for smooth (or holomorphic) functions of one real (or complex) variable is taught in our undergraduate courses, the corresponding result for functions of several variables does not seem to be so well-known. In this note we describe necessary and sufficient conditions for the linear dependence of N entire functions in $\mathbb{C}^n$, $f_1, \dots, f_N$, in terms of vanishing of several, conveniently generalized, Wronskians. This criterion is also correct for meromorphic functions in open domains of $\mathbb{C}^n$ and sufficiently smooth functions in domains in $\mathbb{R}^n$.

Let $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$ and $\alpha = (a_1, \dots, a_n) \in (\mathbb{Z}_+)^n$ be a multiindex with $|\alpha| = a_1 + \cdots + a_n$. We shall use the standard notations

$$\partial^\alpha f = \frac{\partial^{|\alpha|} f}{\partial^{a_1} z_1 \cdots \partial^{a_n} z_n} \qquad \text{and} \qquad f_{z_j^m} = \frac{\partial^m f}{\partial z_j^m},$$

where $1 \le j \le n$ and $m \in \mathbb{Z}_+$. A determinant with the form

$$\mathcal{W} = \mathcal{W}_{\alpha_1 \cdots \alpha_{N-1}} := \begin{vmatrix} f_1 & f_2 & \cdots & f_N \\ \partial^{\alpha_1} f_1 & \partial^{\alpha_1} f_2 & \cdots & \partial^{\alpha_1} f_N \\ \cdots & & & \\ \partial^{\alpha_{N-1}} f_1 & \partial^{\alpha_{N-1}} f_2 & \cdots & \partial^{\alpha_{N-1}} f_N \end{vmatrix}$$

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is called a Wronskian of functions $f_1, \ldots, f_N$, where $\alpha_k \in (\mathbb{Z}_+)^n$, $1 \le k \le N-1$. In particular, when $n = 1$, the determinant

$$\begin{vmatrix} f_1 & f_2 & \cdots & f_N \\ f_1' & f_2' & \cdots & f_N' \\ \cdots & & & \\ f_1^{(N-1)} & f_2^{(N-1)} & \cdots & f_N^{(N-1)} \end{vmatrix} \tag{1}$$

is the standard Wronskian of $f_1, \ldots, f_N$, which plays a special role in the theory of differential equations (see e.g. [BG], [CL]).

In our recent work ([BCL]) on Nevanlinna type uniqueness theorem for meromorphic functions in $\mathbb{C}^n$, we need to know when N entire functions $f_1, \ldots, f_N$ are linearly dependent. For $n = 1$, it is well known that $f_1, \ldots, f_N$ are linearly dependent if and only if the Wronskian (1) is identically zero in $\mathbb{C}$ (see e.g. [K]). This result has been frequently used in the theory of meromorphic functions of one complex variable (see e.g. [Ch], [DY], [L], and [N]). For several complex variables, it turns out that the conditions are $\mathcal{W}_{\alpha_1 \cdots \alpha_{N-1}} \equiv 0$ for $\alpha_k \in (\mathbb{Z}_+)^n$, $|\alpha_k| \le k$, $1 \le k \le N-1$. Moreover, it is no loss of generality to assume that for some fixed multiindices $\alpha_k \in (\mathbb{Z}_+)^n$, $|\alpha_k| \le k$, $1 \le k \le N-2$,

$$\Delta := \begin{vmatrix} f_1 & f_2 & \cdots & f_{N-1} \\ \partial^{\alpha_1} f_1 & \partial^{\alpha_1} f_2 & \cdots & \partial^{\alpha_1} f_{N-1} \\ \cdots & & & \\ \partial^{\alpha_{N-2}} f_1 & \partial^{\alpha_{N-2}} f_2 & \cdots & \partial^{\alpha_{N-2}} f_{N-1} \end{vmatrix} \not\equiv 0. \tag{2}$$

(Otherwise we just need to consider $N-1$ functions $f_1, \ldots, f_{N-1}$). Then the conditions can be weakened to only $n \times (N-1)$ Wronskians vanishing identically as shown in the main theorem below.

We refer the reader to [Kr] for the basic facts about holomorphic functions of several complex variables.

Theorem. *Let $f_1 \ldots, f_N$ be N entire functions in $\mathbb{C}^n$ satisfying (2) for some fixed $\alpha_k \in (\mathbb{Z}_+)^n$, $|\alpha_k| \le k$, $1 \le k \le N-2$. Then*

$$\begin{vmatrix} f_1 & f_2 & \cdots & f_N \\ \partial^{\alpha_1} f_1 & \partial^{\alpha_1} f_2 & \cdots & \partial^{\alpha_1} f_N \\ \cdots & & & \\ \partial^{\alpha_{N-2}} f_1 & \partial^{\alpha_{N-2}} f_2 & \cdots & \partial^{\alpha_{N-2}} f_N \\ (\partial^{\alpha_\iota} f_1)_{z_j} & (\partial^{\alpha_\iota} f_2)_{z_j} & \cdots & (\partial^{\alpha_\iota} f_N)_{z_j} \end{vmatrix} \equiv 0 \tag{3}$$

for $0 \leq \ell \leq N-2$ $(\partial^{\alpha_0} f := f)$ *and* $1 \leq j \leq n$, *implies that* $f_1 \ldots, f_N$ *are linearly dependent.*

Proof. Since (3) holds and $\Delta \not\equiv 0$, we can find $N-1$ functions $C_j(z) : \mathbb{C}^n \to \mathbb{C}$, $1 \leq j \leq N-1$, such that

$$(4) \qquad \begin{cases} f_N = C_1 f_1 + \cdots + C_{N-1} f_{N-1} \\ \partial^{\alpha_1} f_N = C_1 \partial^{\alpha_1} f_1 + \cdots + C_{N-1} \partial^{\alpha_1} f_{N-1} \\ \cdots \\ \partial^{\alpha_{N-2}} f_N = C_1 \partial^{\alpha_{N-2}} f_1 + \cdots + C_{N-1} \partial^{\alpha_{N-2}} f_{N-1}. \end{cases}$$

Moreover, we can solve this system to get

$$C_j = \frac{\Delta_j}{\Delta},$$

where Δ_j is the determinant of the matrix by replacing the j-the column of Δ by the column vector

$$\begin{bmatrix} f_N \\ \partial^{\alpha_1} f_N \\ \cdots \\ \partial^{\alpha_{N-2}} f_N \end{bmatrix}.$$

The function C_j is holomorphic outside the set $\mathcal{Z} := \{z \in \mathbb{C}^n : \Delta(z) = 0\}$. It is well-known that $\mathcal{Z}$ is a $2n$-dimensional measure zero closed set and does not disconnect $\mathbb{C}^n$. For that reason, it is enough to prove that $(C_\ell)_{z_j}(z) = 0$ for each $j \in \{1, 2, \ldots, n\}$ and $\ell \in \{1, \ldots, N-1\}$, whenever $\Delta(z) \neq 0$. This implies that C_ℓ will be a constant in $\mathbb{C}^n \setminus \mathcal{Z}$. In fact, we just need to check $\ell = 1$. Using the same method, we can deal with other C_ℓ, $2 \leq \ell \leq N-1$. Now we have for each $j \in \{1, 2, \ldots, n\}$,

$$(C_1)_{z_j} = \frac{(\Delta_1)_{z_j} \Delta - \Delta_1 (\Delta)_{z_j}}{\Delta^2}.$$

Hence we just need to prove

$$\bigtriangledown := (\Delta_1)_{z_j} \Delta - \Delta_1 (\Delta)_{z_j} \equiv 0.$$

Indeed,

$$
\begin{aligned}
\nabla = & \\
& \begin{vmatrix} f_N & f_2 & \cdots & f_{N-1} \\ \partial^{\alpha_1} f_N & \partial^{\alpha_1} f_2 & \cdots & \partial^{\alpha_1} f_{N-1} \\ \cdots & & & \\ \partial^{\alpha_{N-2}} f_N & \partial^{\alpha_{N-2}} f_2 & \cdots & \partial^{\alpha_{N-2}} f_{N-1} \end{vmatrix}_{z_j} \begin{vmatrix} f_1 & f_2 & \cdots & f_{N-1} \\ \partial^{\alpha_1} f_1 & \partial^{\alpha_1} f_2 & \cdots & \partial^{\alpha_1} f_{N-1} \\ \cdots & & & \\ \partial^{\alpha_{N-2}} f_1 & \partial^{\alpha_{N-2}} f_2 & \cdots & \partial^{\alpha_{N-2}} f_{N-1} \end{vmatrix} \\
& - \begin{vmatrix} f_N & f_2 & \cdots & f_{N-1} \\ \partial^{\alpha_1} f_N & \partial^{\alpha_1} f_2 & \cdots & \partial^{\alpha_1} f_{N-1} \\ \cdots & & & \\ \partial^{\alpha_{N-2}} f_N & \partial^{\alpha_{N-2}} f_2 & \cdots & \partial^{\alpha_{N-2}} f_{N-1} \end{vmatrix} \begin{vmatrix} f_1 & f_2 & \cdots & f_{N-1} \\ \partial^{\alpha_1} f_1 & \partial^{\alpha_1} f_2 & \cdots & \partial^{\alpha_1} f_{N-1} \\ \cdots & & & \\ \partial^{\alpha_{N-2}} f_1 & \partial^{\alpha_{N-2}} f_2 & \cdots & \partial^{\alpha_{N-2}} f_{N-1} \end{vmatrix}_{z_j} \\
= & \\
& \Bigg(\begin{vmatrix} (f_N)_{z_j} & (f_2)_{z_j} & \cdots & (f_{N-1})_{z_j} \\ \partial^{\alpha_1} f_N & \partial^{\alpha_1} f_2 & \cdots & \partial^{\alpha_1} f_{N-1} \\ \cdots & & & \\ (\partial^{\alpha_{N-2}} f_N & \partial^{\alpha_{N-2}} f_2 & \cdots & \partial^{\alpha_{N-2}} f_{N-1} \end{vmatrix} + \\
& + \begin{vmatrix} f_N & f_2 & \cdots & f_{N-1} \\ (\partial^{\alpha_1} f_N)_{z_j} & (\partial^{\alpha_1} f_2)_{z_j} & \cdots & (\partial^{\alpha_1} f_{N-1})_{z_j} \\ \cdots & & & \\ \partial^{\alpha_{N-2}} f_N & \partial^{\alpha_{N-2}} f_2 & \cdots & \partial^{\alpha_{N-2}} f_{N-1} \end{vmatrix} + \cdots + \\
& + \begin{vmatrix} f_N & f_2 & \cdots & f_{N-1} \\ \partial^{\alpha_1} f_N & \partial^{\alpha_1} f_2 & \cdots & \partial^{\alpha_1} f_{N-1} \\ \cdots & & & \\ (\partial^{\alpha_{N-2}} f_N)_{z_j} & (\partial^{\alpha_{N-2}} f_2)_{z_j} & \cdots & (\partial^{\alpha_{N-2}} f_{N-1})_{z_j} \end{vmatrix} \Bigg) \begin{vmatrix} f_1 & \cdots & f_{N-1} \\ \partial^{\alpha_1} f_1 & \cdots & \partial^{\alpha_1} f_{N-1} \\ \cdots & & \\ \partial^{\alpha_{N-2}} f_1 & \cdots & \partial^{\alpha_{N-2}} f_{N-1} \end{vmatrix} \\
& - \begin{vmatrix} f_N & f_2 & \cdots & f_{N-1} \\ \partial^{\alpha_1} f_N & \partial^{\alpha_1} f_2 & \cdots & \partial^{\alpha_1} f_{N-1} \\ \cdots & & & \\ \partial^{\alpha_{N-2}} f_N & \partial^{\alpha_{N-2}} f_2 & \cdots & \partial^{\alpha_{N-2}} f_{N-1} \end{vmatrix} \Bigg(\begin{vmatrix} (f_1)_{z_j} & (f_2)_{z_j} & \cdots & (f_{N-1})_{z_j} \\ \partial^{\alpha_1} f_1 & \partial^{\alpha_1} f_2 & \cdots & \partial^{\alpha_1} f_{N-1} \\ \cdots & & & \\ \partial^{\alpha_{N-2}} f_1 & \partial^{\alpha_{N-2}} f_2 & \cdots & \partial^{\alpha_{N-2}} f_{N-1} \end{vmatrix} \\
& + \begin{vmatrix} f_1 & \cdots & f_{N-1} \\ (\partial^{\alpha_1} f_1)_{z_j} & \cdots & (\partial^{\alpha_1} f_{N-1})_{z_j} \\ \cdots & & \\ \partial^{\alpha_{N-2}} f_1 & \cdots & \partial^{\alpha_{N-2}} f_{N-1} \end{vmatrix} + \cdots + \begin{vmatrix} f_1 & \cdots & f_{N-1} \\ \partial^{\alpha_1} f_1 & \cdots & \partial^{\alpha_1} f_{N-1} \\ \cdots & & \\ (\partial^{\alpha_{N-2}} f_1)_{z_j} & \cdots & (\partial^{\alpha_{N-2}} f_{N-1})_{z_j} \end{vmatrix} \Bigg) \\
:= & (\Lambda_0 + \Lambda_1 + \cdots + \Lambda_{N-2})\,\Delta - \Delta_1\,(\Gamma_0 + \Gamma_1 + \cdots + \Gamma_{N-2}),
\end{aligned}
$$

the definitions of Λ_ℓ and Γ_ℓ, $0 \le \ell \le N-2$, is clear from above.

It suffices to prove that each $\Lambda_\ell \Delta - \Delta_1 \Gamma_\ell \equiv 0$ for $0 \le \ell \le N-2$. Let $\Lambda_{\ell,k}$, $1 \le k \le N-1$, be the algebraic complement minor of Λ_ℓ with respect to $(\partial^{\alpha_\ell} f_N)_{z_j}$, $(\partial^{\alpha_\ell} f_2)_{z_j}$,..., $(\partial^{\alpha_\ell} f_{N-1})_{z_j}$ and let $\gamma_{\ell,k}$, $1 \le k \le N-1$, be the algebraic complement minor of Γ_ℓ with respect to $(\partial^{\alpha_\ell} f_1)_{z_j}$, $(\partial^{\alpha_\ell} f_2)_{z_j}, \ldots, (\partial^{\alpha_\ell} f_{N-1})_{z_j}$. Here $\partial^{\alpha_0} f_i := f_i$ for $1 \le i \le N$. Clearly

$\lambda_{\ell,1} = \gamma_{\ell,1}$. Let us denote it by M_ℓ. Hence we deduce that

$$\begin{aligned}\Lambda_\ell\Delta - \Delta_1\Gamma_\ell = &\left\{(\partial^{\alpha_\ell} f_N)_{z_j} M_\ell + (\partial^{\alpha_\ell} f_2)_{z_j}\lambda_{\ell,2} + \cdots + (\partial^{\alpha_\ell} f_{N-1})_{z_j}\lambda_{\ell,N-1}\right\}\Delta \\ &- \Delta_1\left\{(\partial^{\alpha_\ell} f_1)_{z_j} M_\ell + (\partial^{\alpha_\ell} f_2)_{z_j}\gamma_{\ell,2} + \cdots + (\partial^{\alpha_\ell} f_{N-1})_{z_j}\gamma_{\ell,N-1}\right\}.\end{aligned} \tag{5}$$

We will show that

$$\begin{aligned}\Lambda_\ell\Delta - \Delta_1\Gamma_\ell &= M_\ell \begin{vmatrix} f_1 & f_2 & \cdots & f_{N-1} & f_N \\ \partial^{\alpha_1} f_1 & \partial^{\alpha_1} f_2 & \cdots & \partial^{\alpha_1} f_{N-1} & \partial^{\alpha_1} f_N \\ \cdots & & & & \\ \partial^{\alpha_{N-2}} f_1 & \partial^{\alpha_{N-2}} f_2 & \cdots & \partial^{\alpha_{N-2}} f_{N-1} & \partial^{\alpha_{N-2}} f_N \\ (\partial^{\alpha_\ell} f_1)_{z_j} & (\partial^{\alpha_\ell} f_2)_{z_j} & \cdots & (\partial^{\alpha_\ell} f_{N-1})_{z_j} & (\partial^{\alpha_\ell} f_N)_{z_j} \end{vmatrix} \\ &:= M_\ell \nabla_\ell = M_\ell \mathcal{W}_{\alpha_1\cdots\alpha_{N-2}\tilde{\alpha}_\ell}\end{aligned} \tag{6}$$

where $\tilde{\alpha}_\ell = \alpha_\ell + \delta_j$ with $\delta_j = (0,\ldots,0,\overset{j}{\check{1}},0\ldots,0,\ldots,0)$, which is zero by the hypothesis (3). For this purpose, let $\nabla_{\ell,k}$, $1 \le k \le N$, be the algebraic complement minor of ∇_ℓ with respect to

$$(\partial^{\alpha_\ell} f_1)_{z_j}, (\partial^{\alpha_\ell} f_2)_{z_j}, \ldots, (\partial^{\alpha_\ell} f_N)_{z_j},$$

respectively. Clearly, $\nabla_{\ell,1} = -\Delta_1$ and $\nabla_{\ell,N} = \Delta$. Therefore

$$\nabla_\ell = (\partial^{\alpha_\ell} f_1)_{z_j}(-\Delta_1) + (\partial^{\alpha_\ell} f_2)_{z_j}\ \nabla_{\ell,2} + \cdots + (\partial^{\alpha_\ell} f_{N-1})_{z_j}\ \nabla_{\ell,N-1} + (\partial^{\alpha_\ell} f_N)_{z_j}\Delta.$$

Now by (5), we have

$$\begin{aligned}&\Lambda_\ell\Delta - \Delta_1\Gamma_\ell \\ =&M_\ell\left\{(\partial^{\alpha_\ell} f_N)_{z_j}\Delta - \Delta_1(\partial^{\alpha_\ell} f_1)_{z_j}\right\} + (\partial^{\alpha_\ell} f_2)_{z_j}\left\{\lambda_{\ell,2}\Delta - \gamma_{\ell,2}\Delta_1\right\} + \cdots \\ &+ (\partial^{\alpha_\ell} f_{N-1})_{z_j}\left\{\lambda_{\ell,N-1}\Delta - \gamma_{\ell,N-1}\Delta_1\right\} \\ =&M_\ell\left\{\nabla_\ell - (\partial^{\alpha_\ell} f_2)_{z_j}\ \nabla_{\ell,2} - (\partial^{\alpha_\ell} f_3)_{z_j}\ \nabla_{\ell,3} - \cdots - (\partial^{\alpha_\ell} f_{N-1})_{z_j}\nabla_{\ell,N-1}\right\} \\ &+ (\partial^{\alpha_\ell} f_2)_{z_j}\left\{\lambda_{\ell,2}\Delta - \gamma_{\ell,2}\Delta_1\right\} + \cdots + (\partial^{\alpha_\ell} f_{N-1})_{z_j}\left\{\lambda_{\ell,N-1}\Delta - \gamma_{\ell,N-1}\Delta_1\right\} \\ =&M_\ell\ \nabla_\ell + (\partial^{\alpha_\ell} f_2)_{z_j}\left\{\lambda_{\ell,2}\Delta - \gamma_{\ell,2}\Delta_1 - \nabla_{\ell,2}M_\ell\right\} + \cdots + \\ &+ (\partial^{\alpha_\ell} f_{N-1})_{z_j}\left\{\lambda_{\ell,N-1}\Delta - \gamma_{\ell,N-1}\Delta_1 - \nabla_{\ell,N-1}M_\ell\right\}.\end{aligned} \tag{7}$$

Let us define

$$\begin{aligned}\mathbf{X}_\ell &= \begin{vmatrix} Y_0 & f_3 & \cdots & f_{N-1} \\ Y_1 & \partial^{\alpha_1} f_3 & \cdots & \partial^{\alpha_1} f_{N-1} \\ \cdots & & & \\ \widehat{Y_\ell} & \widehat{\partial^{\alpha_\ell} f_3} & \cdots & \widehat{\partial^{\alpha_\ell} f_{N-1}} \\ \cdots & & & \\ Y_{N-2} & \partial^{\alpha_{N-2}} f_3 & \cdots & \partial^{\alpha_{N-2}} f_{N-1} \end{vmatrix} \\ &= Y_0\Psi_0 + Y_1\Psi_1 + \cdots + Y_{\ell-1}\Psi_{\ell-1} + Y_{\ell+1}\Psi_{\ell+1} + \cdots + Y_{N-2}\Psi_{N-2},\end{aligned}$$

where $\widehat{Y_\ell}$, $\widehat{\partial^{\alpha_\ell} f_3}, \ldots, \widehat{\partial^{\alpha_\ell} f_{N-1}}$ mean that X_ℓ does not contain such entries and Ψ_j, $0 \le j \le N-2$, $j \ne \ell$, is the algebraic complement minor of $\mathbf{X}_\ell$ with respect to $Y_0, \ldots, \widehat{Y_\ell}, \ldots, Y_{N-2}$. Let

$$A_1 = \begin{vmatrix} f_1 & f_3 & \cdots & f_{N-1} \\ \partial^{\alpha_1} f_1 & \partial^{\alpha_1} f_3 & \cdots & \partial^{\alpha_1} f_{N-1} \\ \cdots & & & \\ \widehat{\partial^{\alpha_\ell} f_1} & \widehat{\partial^{\alpha_\ell} f_3} & \cdots & \widehat{\partial^{\alpha_\ell} f_{N-1}} \\ \cdots & & & \\ \partial^{\alpha_{N-2}} f_1 & \partial^{\alpha_{N-2}} f_3 & \cdots & \partial^{\alpha_{N-2}} f_{N-1} \end{vmatrix}$$
$$= f_1\Psi_0 + \partial^{\alpha_1} f_1 \Psi_1 + \cdots + \partial^{\alpha_{\ell-1}} f_1 \Psi_{\ell-1} + \partial^{\alpha_{\ell+1}} f_1 \Psi_{\ell+1} + \cdots + \partial^{\alpha_{N-2}} f_1 \Psi_{N-2},$$

$$A_2 = \begin{vmatrix} f_2 & f_3 & \cdots & f_{N-1} \\ \partial^{\alpha_1} f_2 & \partial^{\alpha_1} f_3 & \cdots & \partial^{\alpha_1} f_{N-2} \\ \cdots & & & \\ \widehat{\partial^{\alpha_\ell} f_2} & \widehat{\partial^{\alpha_\ell} f_3} & \cdots & \widehat{\partial^{\alpha_\ell} f_{N-1}} \\ \cdots & & & \\ \partial^{\alpha_{N-2}} f_2 & \partial^{\alpha_{N-2}} f_3 & \cdots & \partial^{\alpha_{N-2}} f_{N-1} \end{vmatrix}$$
$$= f_2\Psi_0 + \partial^{\alpha_1} f_2 \Psi_1 + \cdots + \partial^{\alpha_{\ell-1}} f_2 \Psi_{\ell-1} + \partial^{\alpha_{\ell+1}} f_2 \Psi_{\ell+1} + \cdots + \partial^{\alpha_{N-2}} f_2 \Psi_{N-2},$$

and

$$A_3 = \begin{vmatrix} f_N & f_3 & \cdots & f_{N-1} \\ \partial^{\alpha_1} f_N & \partial^{\alpha_1} f_3 & \cdots & \partial^{\alpha_1} f_{N-1} \\ \cdots & & & \\ \widehat{\partial^{\alpha_\ell} f_N} & \widehat{\partial^{\alpha_\ell} f_3} & \cdots & \widehat{\partial^{\alpha_\ell} f_{N-1}} \\ \cdots & & & \\ \partial^{\alpha_{N-2}} f_N & \partial^{\alpha_{N-2}} f_3 & \cdots & \partial^{\alpha_{N-2}} f_{N-1} \end{vmatrix}$$
$$= f_N\Psi_0 + \partial^{\alpha_1} f_N \Psi_1 + \cdots + \partial^{\alpha_{\ell-1}} f_N \Psi_{\ell-1} + \partial^{\alpha_{\ell+1}} f_N \Psi_{\ell+1} + \cdots + \partial^{\alpha_{N-2}} f_N \Psi_{N-2}.$$

It is not difficult to verify that

$$A_1 = (-1)^{\ell+1}\gamma_{\ell,2}, \qquad A_2 = (-1)^{\ell} M_\ell, \qquad A_3 = (-1)^{\ell+1}\lambda_{\ell,2}.$$

Thus,

$$\lambda_{\ell,2}\Delta - \gamma_{\ell,2}\Delta_1 - \bigtriangledown_{\ell,2} M_\ell$$
$$= (-1)^{\ell+1} \begin{vmatrix} f_1 & f_2 & f_3 & \cdots & f_{N-1} & f_N \\ \partial^{\alpha_1} f_1 & \partial^{\alpha_1} f_2 & \partial^{\alpha_1} f_3 & \cdots & \partial^{\alpha_1} f_{N-1} & \partial^{\alpha_1} f_N \\ \cdots & & & & & \\ \partial^{\alpha_{N-2}} f_1 & \partial^{\alpha_{N-2}} f_2 & \partial^{\alpha_{N-2}} f_3 & \cdots & \partial^{\alpha_{N-2}} f_{N-1} & \partial^{\alpha_{N-2}} f_N \\ A_1 & A_2 & 0 & \cdots & 0 & A_3 \end{vmatrix}. \tag{8}$$

On the other hand, for $3 \le k \le N-1$,

$$0 = \begin{vmatrix} f_k & f_3 & \cdots & f_{N-1} \\ \partial^{\alpha_1} f_k & \partial^{\alpha_1} f_3 & \cdots & \partial^{\alpha_1} f_{N-1} \\ \cdots & & & \\ \widehat{\partial^{\alpha_\ell} f_k} & \widehat{\partial^{\alpha_\ell} f_3} & \cdots & \widehat{\partial^{\alpha_\ell} f_{N-1}} \\ \cdots & & & \\ \partial^{\alpha_N - 2} f_k & \partial^{\alpha_N - 2} f_3 & \cdots & \partial^{\alpha_N - 2} f_{N-1} \end{vmatrix}$$
$$= f_k \Psi_0 + \partial^{\alpha_1} f_k \Psi_1 + \cdots + \partial^{\alpha_{\ell-1}} f_k \Psi_{\ell-1} + \partial^{\alpha_{\ell+1}} f_k \Psi_{\ell+1} + \cdots + \partial^{\alpha_N - 2} f_k \Psi_{N-2}.$$

Therefore the last line of the determinant (8) is a linear combination of the previous lines, namely the i-th line, $1 \le i \le N-1$, $i \ne \ell+1$, multiplied by Ψ_{i-1}. By the property of determinants, we have that

$$\lambda_{\ell,2}\Delta - \gamma_{\ell,2}\Delta_1 - \nabla_{\ell,2} M_\ell \equiv 0,$$

as we wanted. In other words, the second term of (7) is zero. Exactly same argument will show that in (7) each i-th term is zero for $3 \le i \le N-1$. Hence (6) holds, *i.e.*, $\Lambda_\ell \Delta - \Delta_1 \Gamma_\ell \equiv 0$. We have thus proved that $(C_1)_{z_j} \equiv 0$ in $\mathbb{C}^n \setminus \mathcal{Z}$. This implies that $C_1(z)$ is a constant in $\mathbb{C}^n \setminus \mathcal{Z}$. Using the same method, we can show that all C_ℓ, $1 \le \ell \le N-1$ are constants in $\mathbb{C}^n \setminus \mathcal{Z}$. By continuity, the relation (4) holds everywhere in $\mathbb{C}^n$ with this choice of constants.

The proof of the theorem is therefore complete. □

Corollary. *Let $f_1, \ldots, f_N$ be N entire functions in $\mathbb{C}^n$. Then the necessary and sufficient condition of linear dependence of these N functions is that all the Wronskians satisfy the following conditions:*

$$\mathcal{W}_{\alpha_1 \cdots \alpha_{N-1}} \equiv 0$$

for $\alpha_k \in (\mathbb{Z}_+)^n$, $|\alpha_k| \le k$, $1 \le k \le N-1$.

Proof. (*Necessary conditon*): The proof of the necessary conditon is immediate. In fact, if $f_1, \ldots, f_N$ are linearly dependent then there are N constants $C_1, \ldots, C_N$ not all zero such that

$$C_1 f_1 + \cdots + C_N f_N = 0.$$

Now for any $\alpha_k \in (\mathbb{Z}_+)^n$, $1 \le k \le N-1$, we may differentiate the above identity to obtain the following system:

$$\begin{cases} C_1 f_1 + \cdots + C_N f_N = 0 \\ C_1 \partial^{\alpha_1} f_1 + \cdots + C_N \partial^{\alpha_1} f_N = 0 \\ \cdots \\ C_1 \partial^{\alpha_{N-1}} f_1 + \cdots + C_N \partial^{\alpha_{N-1}} f_N = 0. \end{cases}$$

Since this system has nontrivial solution $C_1, \ldots, C_N$, we must have $\mathcal{W}_{\alpha_1 \cdots \alpha_{N-1}} \equiv 0$.

(*Sufficient conditon*): Without loss of generality, we may assume that one of the Wronskian of $(N-1)$ functions is not zero. Otherwise, we just need to consider $N-1$ functions $f_1, \ldots, f_{N-1}$. Thus we may assume that for some $\alpha_k \in (\mathbb{Z}_+)^n$, $|\alpha_k| \le k$, $1 \le k \le N-1$, $\Delta \not\equiv 0$ (see (2)). Then the sufficiency follows from the above theorem. □

Remarks.

(1). For $n = 1$, our condition is exactly $|\alpha_k| = k$, since if for some k, $|\alpha_k| < k$, then the determinant equals to zero because a row is repeated. So, our condition is exactly the usual one:

$$\begin{vmatrix} f_1 & f_2 & \cdots & f_N \\ f_1' & f_2' & \cdots & f_N' \\ \cdots & & & \\ f_1^{(N-1)} & f_2^{(N-1)} & \cdots & f_N^{(N-1)} \end{vmatrix} \equiv 0$$

for linear dependence of $f_1, \ldots, f_N$.

(2). We observe that when $n \ge 2$, we cannot replace the condition $|\alpha_k| \le k$ by the weaker condition $|\alpha_k| = k$. For instance, let $f_1 = 1$, $f_2 = z_1$, and $f_3 = z_2$ in $\mathbb{C}^2$, then

$$\mathcal{W}_{\alpha_1 \alpha_2} \equiv 0$$

for any $|\alpha_1| = 1$ and $|\alpha_2| = 2$, but

$$\mathcal{W}_{\delta_1 \delta_2} = \begin{vmatrix} f_1 & f_2 & f_3 \\ (f_1)_{z_1} & (f_2)_{z_1} & (f_3)_{z_1} \\ (f_1)_{z_2} & (f_2)_{z_2} & (f_3)_{z_2} \end{vmatrix} = \begin{vmatrix} 1 & z_1 & z_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \not\equiv 0.$$

(3). In fact, the proof of our main theorem does not need to assume that $f_1, \ldots, f_N$ are entire functions in $\mathbb{C}^n$. Exactly same argument shows that $f_1, \ldots, f_N$ can be N meromorphic functions in a domain $\Omega \subseteq \mathbb{C}^n$.

(4). We may just assume that $f_1, \ldots, f_N$ are functions in $C^{N-1}(\Omega)$, Ω is a domain in $\mathbb{R}^n$ and then the theorem and corollary hold locally. Note that the main difference is that the set $\mathcal{Z} := \{x \in \mathbb{R}^n : \Delta(x) = 0\}$ could be disconnected in Ω.

(5). After this note was written, we learned of Roth's work (see [C, pp. 112-113]) on diophantine approximation where the same criteria as our corollary (not our theorem) was found for rational functions of several real variables by a completely different method.

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