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# Network Connectivity with Heterogeneous Mobility

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***Abstract*—We study the issue of mobile wireless network (MWN) connectivity. In particular, we investigate the smallest communication or transmission range of the nodes necessary for connectivity of MWNs, which we call the *critical transmission range* (CTR). Unlike many of existing studies, however, the mobilities of the nodes are *not* assumed homogeneous, and the locations of the nodes are not identically distributed. We examine the distribution of CTR when the number of nodes in the network is large. We show that, under some conditions, the CTR is inversely proportional to the infimum of the *average* spatial density of the nodes in the network and its distribution goes through a *phase transition* over a small range.**

## I. Introduction

Mobile ad-hoc networks (MANETs) or multi-hop wireless networks (MHWNs) have attracted much interest from the networking community, due to their potential for numerous applications. In a traditional wired network, traffic generated by so-called end nodes is routed through the network by dedicated routers. However, in a MANET wireless nodes form and maintain the network and share the responsibility of routing packets from sources to destinations. Moreover, when (some of) nodes are mobile, the one-hop connectivity, hence topology, of the network varies with time. This requires the network protocols to cope with potentially frequent changes in network topology.

When information to be transferred by a MHWN cannot tolerate large delays, timely delivery of information demands that the network be able to find an end-to-end route between a source and a destination. In order for such an end-to-end route to exist when one is needed, the network should be connected (with a high probability). For this reason the issue of network connectivity enjoyed much attention in recent years.

A natural question that arises for MHWNs, in particular, when (some of) nodes are battery powered is: "What is the smallest communication range needed for network connectivity?" In order to study the connectivity properties of MHWNs, researchers often represent the one-hop connectivity of the network as a random graph and investigate the connectivity of the graph. Study of connectivity property of random graphs dates back to late 1950's, starting with the pioneering work by Erdös and Rényi [4], [5].

More recently, another line of research more related to the connectivity of MHWNs examined various properties of *geometric* random graphs (GRGs), including their connectivity (e.g., [1], [10], [9], [14]). We refer interested readers to a monograph by Penrose [16]. In a GRG, one-hop connectivity between a pair of nodes is determined by the distance between them. In other words, there exists an edge between two nodes $i$ and $j$ if and only if their distance is smaller than some threshold $\gamma$. This threshold $\gamma$ can be interpreted as a proxy to a common communication or transmission range of the nodes, which depends on the employed transmit power, in the context of MHWNs.

Most of existing studies on connectivity of GRG models, however, focus on the scenarios where the locations of the nodes are independent of each other with *identical* spatial distribution (e.g., [1], [9], [10], [14]). The dynamic case studied in [3] also assumes independent and *homogeneous* node mobility. Unfortunately, when either of these assumptions is relaxed, little is known about the connectivity property of random graphs. La and Seo [12] investigated the network connectivity under a class of *group mobility* models similar to the reference point group mobility [11] for one-dimensional cases where the nodes lie on a unit circle. They showed that the distribution of the smallest communication range necessary for network connectivity, which we call *critical transmission range* (CTR), exhibits a form of parametric sensitivity with respect the space occupied by each group over a certain regime.

We take another step towards better understanding connectivity when nodes' mobility is *heterogeneous* and, hence, the locations of the nodes have different spatial distributions. In a nutshell, our findings reveal that, in large networks with $n$ $(n \gg 1)$ nodes, the distribution of CTR is concentrated around $\sqrt{\log(n)/(\pi \ \phi_\star \ n)}$, where $\phi_\star$ is the infimum of the *average* spatial density of the nodes.[1] While the *qualitative* nature of our findings is similar to those by Penrose for *homogeneous* spatial distributions [14], [15], our settings are more general and allow the support of spatial density of the nodes to be different. Therefore, some nodes may *never* be close to each other when the support of their spatial density does not overlap, which may be the case in many scenarios. We elaborate on this point in Section III.

Throughout the rest of the paper we assume that all random variables (rvs) and random/stochastic processes of interest are defined on some common probability space $(\mathcal{S}, \mathcal{F}, \mathbb{P})$. The rest of the paper is organized as follows: Section II explains the setup, mobility model and parametric scenario we introduce for carrying out an asymptotic analysis. We summarize some of well known results for homogeneous mobility cases and present our main results in Section III. Numerical results are provided in Section IV. We conclude and suggest some future directions in Section V.

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[1]Throughout the paper $\log(\cdot)$ denotes the natural logarithm.

## II. Setup

In this section we first explain the assumed mobility model of nodes and the GRGs for capturing the one-hop connectivity between nodes. Then, we describe the parametric scenario we assume for our asymptotic analysis as the number of nodes in the network increases.

### A. Node mobility and network connectivity

Suppose that there are $N$, $N \geq 1$, nodes in the network which move on a domain $\Omega \subset \mathbb{R}^2$. The *mobility process* or *trajectory* of node $k$ ($k = 1, 2, \ldots, N$) is denoted by $\mathbb{X}_k := \{X_k(t);\ t \in \mathbb{R}_+\}$, where $\mathbb{R}_+ := [0, \infty)$. For $t \in \mathbb{R}_+$, the random variable (rv) $X_k(t) \in \Omega$ indicates the location of node $k$ at time $t$.

Given a common communication or transmission range[2] $\gamma > 0$, two nodes $j$ and $k$ are said to be *immediate neighbors*, or simply *neighbors*, at time $t \in \mathbb{R}_+$ if and only if $D(X_j(t), X_k(t)) \leq \gamma$, where $D(X_j(t), X_k(t))$ denotes the Euclidean distance between the two nodes, i.e., $D(X_j(t), X_k(t)) = ||X_j(t) - X_k(t)||_2$. There is a bi-directional (communication) link between two neighbors $j$ and $k$, which we denote by $j \leftrightarrow k$.

*Definition 1:* A network is said to be *connected* at time $t \in \mathbb{R}_+$ if and only if, for every pair of nodes $j$ and $k$, we can find $M \in \mathbb{N} := \{1, 2, \ldots\}$ and a sequence of nodes $k_1, k_2, \ldots, k_M$ such that

C1. $k_1 = j$ and $k_M = k$, and
C2. $k_\ell \leftrightarrow k_{\ell+1}$ for all $\ell = 1, 2, \ldots, M-1$.

The above definition of (network) connectivity simply means that, given any two nodes in the network, we can find a sequence of intermediate nodes that can provide the end-to-end connectivity between the two nodes.

### B. Parametric scenario

Given a network with $N \in \mathbb{N}$ nodes, the CTR at time $t \in \mathbb{R}_+$ is denoted by $\gamma^c(t; N)$. Obviously, this CTR depends on the number of nodes in the network, $N$, and their locations (which are given by rvs), and computing the exact distribution of CTR is challenging.

For this reason, researchers often turn to an asymptotic theory for $\gamma^c(t; N)$ as the number of nodes $N$ becomes large: Oftentimes, as the number of nodes grows (i.e., for large networks), the distribution of CTR concentrates over a (short) interval we can identify or approximate more easily. Following this spirit we are interested in examining how $\gamma^c(t; N)$ behaves as $N$ increases. To this end, we introduce the following parametric scenario:

For each $n \in \mathbb{N}$, there are $n \geq 1$ nodes in the network. These $n$ nodes move on (a subset of) $\Omega \subset \mathbb{R}^2$ according to mobility processes $\mathbb{X}_k^{(n)} = \{X_k^{(n)}(t);\ t \in \mathbb{R}_+\}$, $k \in \mathcal{N}_n := \{1, 2, \ldots, n\}$. The mobility domain $\Omega$ is compact and connected in $\mathbb{R}^2$. In addition, its boundary $\partial\Omega$ is a one-dimensional $C^2$ submanifold of $\mathbb{R}^2$.

[2]The communication range of a node is defined to be the maximum distance another node can be at, while maintaining a communication link with the node.

In order to make progress we introduce the following assumptions on the mobility processes:

A1. The processes are $\mathbb{X}_k^{(n)},\ k \in \mathcal{N}_n$, are mutually independent;
A2. they are stationary and ergodic; and
A3. $\mathbb{X}_k^{(n)}, k \in \mathcal{N}_n$, yields a spatial distribution $F_k^{(n)}$ with a continuous density $f_k^{(n)}$.

Note that our assumptions allow for the possibility that $f_k^{(n)}$ have different and even non-overlapping support. In other words, nodes can lie in different strict *subsets* of $\Omega$. Hence, unlike in homogeneous mobility cases, some nodes may never be neighbors of each other when the support of their spatial densities does not overlap. We denote the support of $f_k^{(n)}$ by $S_k^{(n)} \subset \Omega$ and assume that $S_k^{(n)}, n \in \mathbb{N}$ and $k \in \mathcal{N}_n$, satisfy the same assumptions as $\Omega$.

We assume that the spatial distribution of the nodes is sufficiently smooth, which we capture by the following assumption.

A4. There exists $\kappa < \infty$ such that, for all $n \in \mathbb{N}$ and $k \in \mathcal{N}_n$,

$$\left| f_k^{(n)}(\mathbf{x}_1) - f_k^{(n)}(\mathbf{x}_2) \right| \leq \kappa \, ||\mathbf{x}_1 - \mathbf{x}_2||_2 \quad \text{for all } \mathbf{x}_1, \mathbf{x}_2 \in \Omega. \tag{1}$$

Note that Assumption A4 implies that the spatial density functions $f_k^{(n)}$ are uniformly bounded by some finite constant $f^\star$.

For each $n \in \mathbb{N}$, we define the *average* spatial density function of the nodes $\psi^{(n)} : \Omega \to \mathbb{R}_+$, where

$$\psi^{(n)}(\mathbf{x}) = \frac{\sum_{k=1}^{n} f_k^{(n)}(\mathbf{x})}{n} \quad \text{for all } \mathbf{x} \in \Omega. \tag{2}$$

We assume $\lim_{n\to\infty} \left( \inf_{\mathbf{x}\in\Omega} \psi^{(n)}(\mathbf{x}) \right)$ exists and denote it by $\psi_{\inf}$. From the above definition, for any $\delta > 0$, for all sufficiently large $n$, $\inf_{\mathbf{x}\in\Omega} \psi^{(n)}(\mathbf{x}) \geq \psi_{\inf} - \delta$. Throughout the paper we assume $\psi_{\inf} > 0$.

## III. Main results

As mentioned in Section I, in order for a network to be able to provide an end-to-end route between arbitrary sources and destinations (when a connection is requested), the network should be connected most of the time. From the assumed ergodicity and stationarity of the mobility processes, this implies that the network sampled at some time should be connected with high probability. Therefore, we examine the connectivity of the sampled *static* graph instead.

Suppose that we sample the network at time $t_s \in \mathbb{R}_+$. From the stated stationarity assumption, without loss of generality, we can assume $t_s = 0$. Furthermore, for notational simplicity we omit the dependence on time, e.g., we write $X_k^{(n)}$ in place of $X_k^{(n)}(0)$.

Let $\mathbb{G}(n; \gamma)$ be the GRG representing the one-hop connectivity of the network with $n$ nodes sampled at $t = 0$, where each node employs a common communication range of $\gamma$, according to the setup described in the previous section. We define

$$\mathbf{P}^{(n)}(\gamma) := \mathbb{P}\left[\mathbb{G}(n; \gamma) \text{ is connected}\right].$$

It is obvious that $\mathbf{P}^{(n)}(\gamma)$ is increasing in $\gamma$.

Let us define $\rho_n(a) := \sqrt{a\ \log(n)/(\pi\ n)}$ for all $n \in \mathbb{N}$. We first describe the existing result for the special case where $f_k^{(n)} = g$, where $g$ is some fixed density function, for all $n \in \mathbb{N}$ and $k \in \mathcal{N}_n$.

$\diamond$ **Independent and identically distributed case –** Consider the special case where the locations of the $n$ nodes are homogeneous, i.e., $X_k^{(n)}, k \in \mathcal{N}_n$, are given by independent and identically distributed (i.i.d.) rvs with a common distribution $G$ and density $g$. Under the assumption that $g_\star := \inf_{\mathbf{x}\in\Omega} g(\mathbf{x}) > 0$, Penrose proved the following result [14], [15]: Suppose that the nodes select their common transmission ranges according to $\gamma(n) = \rho_n(t)$ for all $n \in \mathbb{N}$, where $t \in (0, \infty)$.

i. If $t > 1/g_\star$, $\lim_{n\to\infty} \mathbf{P}^{(n)}(\gamma(n)) = 1$; and
ii. If $t < 1/g_\star$, $\lim_{n\to\infty} \mathbf{P}^{(n)}(\gamma(n)) = 0$.

This finding tells us that, for all sufficiently large $n$, the necessary CTR will be close to $\sqrt{\log(n)/(\pi\ g_\star\ n)} =: \gamma^\dagger(n)$ with high probability. Hence, the distribution of CTR goes through what is commonly known as a *phase transition* around $\gamma^\dagger(n)$.

While this finding is remarkable in that the distribution of CTR becomes concentrated around $\gamma^\dagger(n)$ under very mild conditions, it assumes that the locations of the nodes are *identically distributed*. However, in many cases the nodes may have different spatial distributions due to heterogeneous mobility. Our findings below generalize this result by Penrose to the cases where the nodes have different spatial distributions under the following assumption.

*Assumption 1:* There exist $\zeta > 0$ and $n_0(\zeta) < \infty$ such that, for all $n \geq n_0(\zeta)$, we can find $\mathbf{x}^{(n)} \in \{\mathbf{x} \in \Omega \mid \psi^{(n)}(\mathbf{x}) = \inf_{\mathbf{y}\in\Omega} \psi^{(n)}(\mathbf{y})\}$ that satisfies the following:

1) $B(\mathbf{x}^{(n)}, \zeta) \subset \Omega$, where $B(\mathbf{x}^{(n)}, \zeta) = \{\mathbf{y} \in \mathbb{R}^2 \mid D(\mathbf{x}^{(n)}, \mathbf{y}) \leq \zeta\}$, and
2) for every $k \in \mathcal{N}_n$, either (i) $B(\mathbf{x}^{(n)}, \zeta) \subset S_k^{(n)}$ and $f_k^{(n)}(\mathbf{x}) \geq \zeta$ for all $\mathbf{x} \in B(\mathbf{x}^{(n)}, 0.5\zeta)$ or (ii) $B(\mathbf{x}^{(n)}, \zeta) \cap S_k^{(n)} = \emptyset$.

This is a technical assumption we introduce to simplify the proof of our results, but can be relaxed at the expense of a more cumbersome proof. It implies that, for all sufficiently large $n$, we can find a small disk (or a neighborhood) where the *average* spatial density of the nodes is close to the infimum and, for every node, either the support of its spatial density includes the disk or does not overlap with the disk.

The first theorem below tells us that, for all sufficiently large $n$, if the nodes choose their communication range to be smaller than $\rho_n(1/\psi_{\inf})$, then with high probability, the network will *not* be connected.

*Theorem 1:* Suppose that the nodes select their communication ranges according to $\gamma(n) = \rho_n(t)$ for all $n \in \mathbb{N}$. If $t < \frac{1}{\psi_{\inf}}$, then $\mathbf{P}^{(n)}(\gamma(n)) \to 0$ as $n \to \infty$.

*Proof:* A proof is provided in Appendix A. ■

The second theorem, which complements the first theorem, states that if the communication range is chosen to be larger than $\rho_n(1/\psi_{\inf})$, for all sufficiently large $n$, the network will be connected with high probability.

*Theorem 2:* Suppose that the nodes choose their communication ranges according to $\gamma(n) = \rho_n(t)$ for all $n \in \mathbb{N}$. If $t > \frac{1}{\psi_{\inf}}$, then $\mathbf{P}^{(n)}(\gamma(n)) \to 1$ as $n \to \infty$.

*Proof:* A proof is given in Appendix B. ■

Theorems 1 and 2 tell us that, in large networks consisting of many nodes (i.e,. $n \gg 1$), assuming that the *average* spatial density of the nodes is non-vanishing, the CTR will be in the neighborhood of $\gamma^\star := \sqrt{\log(n)/(\pi\ \phi_\star\ n)}$ with high probability, where $\phi_\star$ is the infimum of the average spatial density. As a result, the probability of (network) connectivity as a function of the communication range of the nodes goes through a *phase transition* around $\gamma^\star$, i.e., the probability of connectivity rises rapidly from (close to) zero to (close to) one around $\gamma^\star$.

## IV. Numerical results

In this section we provide a numerical example. In our example, there are $N = 500$ nodes in the network. These nodes belong to five different classes. Nodes 1 through 100 are uniformly distributed on the disk centered at the origin with radius 2. Of the remaining 400 nodes, 100 nodes are uniformly distributed on each of four disks centered at $(\pm 1/\sqrt{2}, \pm 1/\sqrt{2})$ with radius $1/\sqrt{2}$. We plot the locations of the 500 nodes in one realization in Fig. 1.

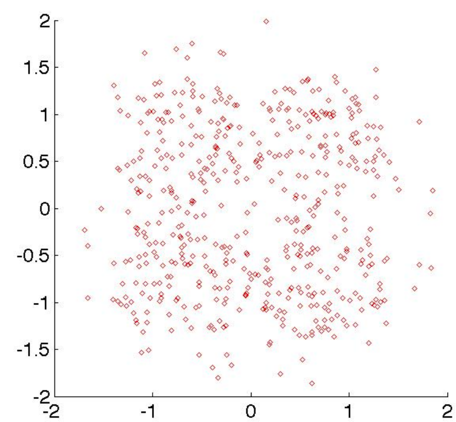


Fig. 1. Plot of nodes' locations in one realization ($N = 500$).

From the given spatial distributions of the nodes, the infimum of the *average* spatial density of the nodes, $\phi_\star$, is $1/20\pi$, and our findings (Theorems 1 and 2) suggest that the phase transition in the probability of connectivity should take place around $\sqrt{\log(500)/(500\ \pi\ \phi_\star)} = 0.4986$.

We generated 150 realizations and computed the fraction of time the corresponding GRG is connected as the communication range of the nodes is varied. Fig. 2 plots this probability of connectivity ($y$-axis) as a function of the communication range of the nodes ($x$-axis). We also plot a red dotted vertical line at $x = 0.4986$ to indicate where we expect the phase transition to occur. As the figure illustrates, indeed the probability of connectivity increases sharply around 0.4986 as predicted by Theorems 1 and 2.

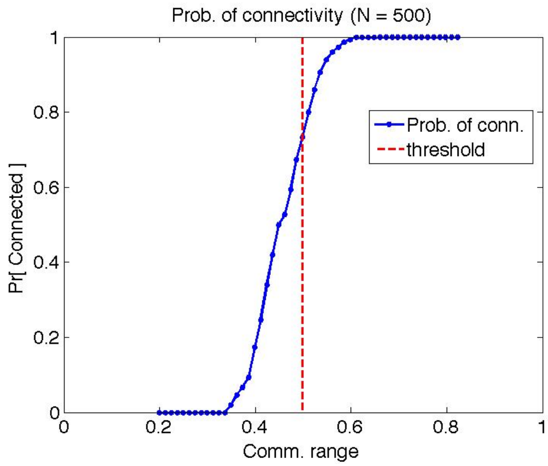


Fig. 2. Probability of network connectivity ($N = 500$).

## V. Conclusion

We studied the issue of connectivity of multi-hop wireless networks. In particular, we focused on (the distribution of) the smallest communication range necessary for network connectivity, called the critical transmission range, when the network comprises many nodes with varying spatial distributions. We showed that, similar to the homogeneous mobility cases studied by Penrose, even under heterogeneous mobility of the nodes, as the number of nodes in the network increases, the distribution of the critical transmission range becomes concentrated around some threshold that depends on the number of nodes in the network and the infimum of the average spatial density of the nodes. Hence, the probability of connectivity as a function of the communication range goes through a phase transition around the aforementioned threshold.

While our findings add to our understanding of the critical transmission ranges and the issue of network connectivity, our model still assumes that the mobilities of the nodes are mutually independent. In many real-life scenarios, the mobilities of some nodes may be correlated, violating the assumption of mutual independence. We are currently investigating how the critical transmission ranges change as the mobility of the nodes becomes correlated via group mobility in 2-dimensional and 3-dimensional cases.

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## Appendix A
## Proof of Theorem 1

Our proof of Theorem 1 is an adaptation of the arguments used in both the proof of Theorem 1 in [2] and that of Proposition 3.1 in [15]: For fixed $r > 0$, let $\sigma(r)$ be the maximum number of *disjoint* disks with radius $r$ whose union is contained in a disk with radius $\tau(\leq 0.5\zeta)$, where $\zeta$ was defined in Assumption 1 Fix $\alpha > 1$ and choose $\epsilon < \gamma < \beta$ that satisfy

a1. $\epsilon < 1/(2 + \alpha\ \psi_{\inf})$ and
a2. $\sqrt{\epsilon} + \sqrt{t} < \sqrt{\gamma}$.

For each $n \in \mathbb{N}$, let $\sigma_n := \sigma(\rho_n(\beta))$ and $\{\mathbf{z}_i^{(n)}; i = 1, 2, \ldots, \sigma_n\} \subset \Omega$ such that $B(\mathbf{z}_i^{(n)}, \rho_n(\beta))$ are disjoint and $\cup_{i=1}^{\sigma_n} B(\mathbf{z}_i^{(n)}, \rho_n(\beta)) \subset B(\mathbf{x}^{(n)}, \tau)$. For any $A \subset \Omega$, we denote the number of nodes from $\mathcal{N}_n$ in $A$ by $\#N^{(n)}(A)$.

For each $i = 1, 2, \ldots, \sigma_n$, define an event

$$\begin{aligned} E_i^{(n)} = \Big\{ &\#N^{(n)}(B(\mathbf{z}_i^{(n)}, \rho_n(\epsilon))) \geq 1, \\ &\#N^{(n)}(B(\mathbf{z}_i^{(n)}, \rho_n(\beta)) \setminus B(\mathbf{z}_i^{(n)}, \rho_n(\gamma))) \geq 1, \text{ and} \\ &\#N^{(n)}(B(\mathbf{z}_i^{(n)}, \rho_n(\gamma)) \setminus B(\mathbf{z}_i^{(n)}, \rho_n(\epsilon))) = 0 \Big\}. \end{aligned} \quad (3)$$

From the condition a2 above, if $E_i^{(n)}$ holds, the nodes in $B(\mathbf{z}_i^{(n)}, \rho_n(\epsilon))$ are not connected to the nodes in the annulus $B(\mathbf{z}_i^{(n)}, \rho_n(\beta)) \setminus B(\mathbf{z}_i^{(n)}, \rho_n(\gamma))$. Hence, the GRG $\mathbb{G}(n; \rho_n(t))$ is not connected and $\mathbb{P}\left[\mathbb{G}(n; \rho_n(t)) \text{ is connected}\right] \leq \mathbb{P}\left[\left(\cup_{i=1}^{\sigma_n} E_i^{(n)}\right)^c\right]$.

We can prove the theorem by first showing that if $t < 1/\psi_{\inf}$, $\sum_{n \in \mathbb{N}} \mathbb{P}\left[\cap_{i=1}^{\sigma_n} \left(E_i^{(n)}\right)^c\right] < \infty$ and then invoking Borel-Cantelli lemma [7], which tells us that $\cap_{i=1}^{\sigma_n} \left(E_i^{(n)}\right)^c =: \mathcal{E}^{(n)}$ happens only for finitely many $n \in \mathbb{N}$ with probability 1. This is done by rewriting the event $\mathcal{E}^{(n)}$ as a union of three events: For notational simplicity, we denote $\#N^{(n)}(B(\mathbf{z}_i^{(n)}, \rho_n(\beta)))$ by $N_i^{(n)}$. For each $n \in \mathbb{N}$ and $i \in \{1, 2, \ldots, \sigma_n\}$, define the following events.

- $A^{(n)} = \cup_{i=1}^{\sigma_n} \{N_i^{(n)} \leq 1\}$,
- $B^{(n)} = \cup_{i=1}^{\sigma_n} \{N_i^{(n)} \geq \alpha\beta \tilde{f}_i^{(n)} \log(n)\}$, where $\tilde{f}_i^{(n)} = \sum_{k=1}^{n} \int_{B(\mathbf{z}_i^{(n)}, \rho_n(\beta))} f_i^{(n)}(\mathbf{z})\ d\mathbf{z} / (\beta\ \log(n))$, and
- $D^{(n)} = \cap_{i=1}^{\sigma_n} \{2 \leq N_i^{(n)} < \alpha\beta \tilde{f}_i^{(n)} \log(n)\}$.

Then, we can rewrite $\mathcal{E}^{(n)}$ as $\left(\mathcal{E}^{(n)} \cap A^{(n)}\right) \cup \left(\mathcal{E}^{(n)} \cap B^{(n)}\right) \cup \left(\mathcal{E}^{(n)} \cap D^{(n)}\right)$, and prove that $\mathbb{P}\left[\mathcal{E}^{(n)} \cap A^{(n)}\right]$, $\mathbb{P}\left[\mathcal{E}^{(n)} \cap B^{(n)}\right]$, and $\mathbb{P}\left[\mathcal{E}^{(n)} \cap D^{(n)}\right]$ are all summable. We examine these three probabilities below.

- $\mathbb{P}\left[\mathcal{E}^{(n)} \cap A^{(n)}\right]$ : We prove that the upper bound $\mathbb{P}\left[\mathcal{E}^{(n)} \cap A^{(n)}\right] \leq \mathbb{P}\left[A^{(n)}\right]$ is summable. Using the union bound,

$$\mathbb{P}\left[A^{(n)}\right] \leq \sum_{i=1}^{\sigma_n} \mathbb{P}\left[N_i^{(n)} \leq 1\right]. \tag{4}$$

For $n \in \mathbb{N}$, $i \in \{1, 2, \ldots, \sigma_n\}$ and $k \in \{1, 2, \ldots, n\}$, define $p_k^{n,i} = \int_{B(\mathbf{z}_i^{(n)}, \rho_n(\beta))} f_k^{(n)}(\mathbf{z})\ d\mathbf{z}$ to be the probability that node $k$ will be in the disk $B(\mathbf{z}_i^{(n)}, \rho_n(\beta))$. Note that $p_k^{n,i} \downarrow 0$ as $n \to \infty$ for all $k \in \mathbb{N}$. Thus, for any $\delta > 0$, there exists $n_1 := n_1(\delta)$ such that, for all $n \geq n_1$, $(1 - p_k^{n,i})^{-1} \leq (1 + \delta)$ for all $k \in \mathbb{N}$.

Defining $\tilde{f}_i^{(n)} := \sum_{k=1}^{n} p_k^{n,i} / (\beta\ \log(n))$ and using the inequality $(1 - x) \leq \exp(-x)$ for all $x \in [0, 1)$,

$$\begin{aligned}
&\mathbb{P}\left[N_i^{(n)} \leq 1\right] = \mathbb{P}\left[N_i^{(n)} = 0\right] + \mathbb{P}\left[N_i^{(n)} = 1\right] \\
&\leq \exp\left(-\sum_{k=1}^{n} p_k^{n,i}\right) + (1+\delta) \sum_{k=1}^{n} p_k^{n,i} \exp\left(-\sum_{k=1}^{n} p_k^{n,i}\right) \\
&= \exp\left(-\beta\ \tilde{f}_i^{(n)}\ \log(n)\right) \\
&\qquad + (1+\delta)\beta\ \tilde{f}_i^{(n)}\ \log(n) \exp\left(-\beta\ \tilde{f}_i^{(n)}\ \log(n)\right) \\
&= \exp\left(-\beta\ \tilde{f}_i^{(n)}\ \log(n)\right) (1 + (1+\delta)\beta\ \tilde{f}_i^{(n)}\ \log(n)).
\end{aligned} \tag{5}$$

Substituting (5) in (4), we obtain

$$\begin{aligned}
&\mathbb{P}\left[A^{(n)}\right] \\
&\leq \sum_{i=1}^{\sigma_n} \exp\left(-\beta\ \tilde{f}_i^{(n)}\ \log(n)\right) (1 + (1+\delta)\beta\ \tilde{f}_i^{(n)}\ \log(n)) \\
&\leq \sigma_n \exp\left(-\beta\psi_{\inf}\ \log(n)\right) (1 + (1+\delta)\beta\ f^\star\ \log(n)) \\
&= \sigma_n n^{-\beta\psi_{\inf}} (1 + (1+\delta)\beta\ f^\star\ \log(n)).
\end{aligned} \tag{6}$$

Therefore $\mathbb{P}\left[A^{(n)}\right]$ is summable if $\beta\ \psi_{\inf} > 2$.

- $\mathbb{P}\left[\mathcal{E}^{(n)} \cap B^{(n)}\right]$ : Again, we prove that the upper bound $\mathbb{P}\left[B^{(n)}\right]$ is summable instead. Using the Boole's inequality,

$$\mathbb{P}\left[B^{(n)}\right] \leq \sum_{i=1}^{\sigma_n} \mathbb{P}\left[N_i^{(n)} \geq \alpha\beta \tilde{f}_i^{(n)}\ \log(n)\right]. \tag{7}$$

We upper bound each summand in (7) as follows. First, we can rewrite $N_i^{(n)}$ as a sum of independent Bernoulli rvs, i.e,.

$$N_i^{(n)} = \sum_{k=1}^{n} \mathbf{1}\left\{\text{node } k \text{ is in } B(\mathbf{z}_i^{(n)}, \rho_n(\beta))\right\},$$

where $\mathbf{1}\left\{\text{node } k \text{ is in } B(\mathbf{z}_i^{(n)}, \rho_n(\beta))\right\}$ is a Bernoulli rv with mean $p_k^{n,i}$. Thus, by Theorem 5.1 in [17],

$$\begin{aligned}
&\mathbb{P}\left[N_i^{(n)} \geq \alpha\beta \tilde{f}_i^{(n)}\ \log(n)\right] \\
&\leq \mathbb{P}\left[Poisson\left(\sum_{k=1}^{n} \lambda_k^{n,i}\right) \geq \alpha\beta \tilde{f}_i^{(n)}\ \log(n)\right],
\end{aligned} \tag{8}$$

where $\lambda_k^{n,i} = -\log(1 - p_k^{n,i}) \geq p_k^{n,i}$ and $Poisson(\sum_{k=1}^{n} \lambda_k^{n,i})$ denotes a Poisson rv with mean $\sum_{k=1}^{n} \lambda_k^{n,i}$.

Define $\Lambda_i^{(n)} := \sum_{k=1}^{n} \lambda_k^{n,i} = -\sum_{k=1}^{n} \log(1 - p_k^{n,i})$. We can upper bound the right-hand side of (8) using Proposition 1 in [6].

$$\begin{aligned}
(8) &\leq \left(1 - \left(\frac{\Lambda_i^{(n)}}{\alpha\beta \tilde{f}_i^{(n)}\ \log(n) + 1}\right)\right)^{-1} \\
&\quad \times \frac{\left(\Lambda_i^{(n)}\right)^{\alpha\beta \tilde{f}_i^{(n)}\ \log(n)}}{\left(\alpha\beta \tilde{f}_i^{(n)}\ \log(n)\right)!} \exp\left(-\Lambda_i^{(n)}\right) \\
&= \left(1 - \left(\frac{\Lambda_i^{(n)}}{\alpha\beta \tilde{f}_i^{(n)}\ \log(n) + 1}\right)\right)^{-1} \\
&\quad \times \frac{n^{\alpha\beta \tilde{f}_i^{(n)}\ \log(\Lambda_i^{(n)})}}{\left(\alpha\beta \tilde{f}_i^{(n)}\ \log(n)\right)!} \exp\left(-\Lambda_i^{(n)}\right).
\end{aligned} \tag{9}$$

For $n \geq n_1$, where $n_1$ is defined earlier, $-\log(1 - p_k^{n,i}) \leq p_k^{n,i} + (1+\delta)\left(p_k^{n,i}\right)^2$. Thus,

$$\begin{aligned}
\sum_{k=1}^{n} \lambda_k^{n,i} &\leq \sum_{k=1}^{n} \left(p_k^{n,i} + (1+\delta)\left(p_k^{n,i}\right)^2\right) \\
&= \beta \tilde{f}_i^{(n)}\ \log(n) + O(\log^2(n)/n).
\end{aligned} \tag{10}$$

Substituting (10) in (9), we obtain

$$\begin{aligned}
(9) &\sim \left(1 - \left(\frac{1}{\alpha}\right)\right)^{-1} \frac{n^{\alpha\beta \tilde{f}_i^{(n)}\ \log(\beta \tilde{f}_i^{(n)}\ \log(n))}}{\left(\alpha\beta \tilde{f}_i^{(n)}\ \log(n)\right)!} \\
&\quad \times \exp\left(-\beta \tilde{f}_i^{(n)}\ \log(n)\right).
\end{aligned} \tag{11}$$

By Stirling's formula [7],

$$\begin{aligned}
\left(\alpha\beta \tilde{f}_i^{(n)}\ \log(n)\right)! &\sim n^{\alpha\beta \tilde{f}_i^{(n)}\ \log(\alpha\beta \tilde{f}_i^{(n)}\ \log(n))} \cdot n^{-\alpha\beta \tilde{f}_i^{(n)}} \\
&\quad \times \sqrt{2\pi\alpha\beta \tilde{f}_i^{(n)}\ \log(n)}.
\end{aligned} \tag{12}$$

From (11) and (12),

$$\begin{aligned}
(9) &\sim \left(1 - \left(\frac{1}{\alpha}\right)\right)^{-1} n^{-\alpha\beta \tilde{f}_i^{(n)}\ \log(\alpha) + \beta \tilde{f}_i^{(n)}(\alpha - 1)} \\
&\quad \times \left(2\pi\alpha\beta \tilde{f}_i^{(n)}\ \log(n)\right)^{-0.5}
\end{aligned} \tag{13}$$

Therefore, $\mathbb{P}\left[B^{(n)}\right]$ is summable if the right-hand side of (13) times $\sigma_n$ is summable.

Lemma 2.1 of [15] states that there exists $c_1 > 0$ such that, for all sufficiently large $n$, $\sigma_n \geq c_1 \cdot (\rho_n(\beta))^{-2} =$

$c_1 \ \pi \ n/(\beta \ \log(n))$. Therefore, a sufficient condition for the summability of $\mathbb{P}\left[B^{(n)}\right]$ is

$$-\alpha\beta\tilde{f}_i^{(n)} \ \log(\alpha) + \beta\tilde{f}_i^{(n)}(\alpha - 1) < -2$$

or, equivalently,

$$\beta\tilde{f}_i^{(n)} \left(\alpha(\log(\alpha) - 1) + 1\right) > 2. \quad (14)$$

The first-order derive of $\alpha(\log(\alpha) - 1)$ is $\log(\alpha)$. Hence, the minimum of the left-hand side of (14) is achieved at $\alpha = 0$, which is equal to zero, and for all $\alpha > 1$, $\alpha(\log(\alpha) - 1) > 0$. Thus, for any fixed $\alpha > 1$, we can select large enough $\beta$ so that the inequality in (14) is satisfied.

• $\mathbb{P}\left[\mathcal{E}^{(n)} \cap D^{(n)}\right]$ : Recall that $D^{(n)} = \cap_{i=1}^{\sigma_n}\{2 \le N_i^{(n)} < \alpha\beta\tilde{f}_i^{(n)} \ \log(n)\}$. Instead of dealing with the probabilities $\mathbb{P}\left[\mathcal{E}^{(n)} \cap D^{(n)}\right]$, we show that their upper bounds are summable: For each $n \in \mathbb{N}$ and $i \in \{1, 2, \ldots, \sigma_n\}$, define

$$\begin{aligned}\tilde{E}_i^{(n)} = \Big\{ &\#N^{(n)}(B(\mathbf{z}_i^{(n)}, \rho_n(\epsilon))) = 1, \\ &\#N^{(n)}(B(\mathbf{z}_i^{(n)}, \rho_n(\beta)) \setminus B(\mathbf{z}_i^{(n)}, \rho_n(\gamma))) \ge 1, \text{ and} \\ &\#N^{(n)}(B(\mathbf{z}_i^{(n)}, \rho_n(\gamma)) \setminus B(\mathbf{z}_i^{(n)}, \rho_n(\epsilon))) = 0 \Big\} \ . \end{aligned} \quad (15)$$

It is clear from the definition that $\tilde{E}_i^{(n)} \subset E_i^{(n)}$. Thus, $\mathbb{P}\left[\cap_{i=1}^{\sigma_n}\left(E_i^{(n)}\right)^c \cap D^{(n)}\right] \le \mathbb{P}\left[\cap_{i=1}^{\sigma_n}\left(\tilde{E}_i^{(n)}\right)^c \cap D^{(n)}\right]$.

We first rewrite the probability $\mathbb{P}\left[\cap_{i=1}^{\sigma_n}\left(\tilde{E}_i^{(n)}\right)^c \cap D^{(n)}\right]$ by conditioning on the possible values of the vector $(N_1^{(n)}, \ldots, N_{\sigma_n}^{(n)}) =: \mathbf{N}^{(n)}$. Let $\mathbb{N}_*^{(n)} := \prod_{i=1}^{\sigma_n}\{2, 3, \ldots, \alpha\beta\tilde{f}_i^{(n)} \ \log(n)\}$.

$$\begin{aligned}&\mathbb{P}\left[\cap_{i=1}^{\sigma_n}\left(\tilde{E}_i^{(n)}\right)^c \cap D^{(n)}\right] \\ &= \sum_{\mathbf{n}\in\mathbb{N}_*^{(n)}} \Big(\mathbb{P}\left[\cap_{i=1}^{\sigma_n}\left(\tilde{E}_i^{(n)}\right)^c \cap D^{(n)} \mid \mathbf{N}^{(n)} = \mathbf{n}\right] \\ &\qquad \times \mathbb{P}\left[\mathbf{N}^{(n)} = \mathbf{n}\right]\Big). \end{aligned} \quad (16)$$

From Assumptions 1 and A4 (in Section II), for any $\delta_2 > 0$, there exists finite $n_3 := n_3(\delta_2)$ such that, for all $n \ge n_3$ and for all $k \in \{1, 2, \ldots, n\}$,

$$\begin{aligned}&\mathbb{P}\left[\mathbf{x}_k^{(n)} \in B(\mathbf{z}_i^{(n)}, \rho_n(\epsilon)) \mid \mathbf{x}_k^{(n)} \in B(\mathbf{z}_i^{(n)}, \rho_n(\beta))\right] \\ &\ge (1 - \delta_2)\frac{\epsilon}{\beta}\end{aligned}$$

and, letting $A_i^{(n)} := B(\mathbf{z}_i^{(n)}, \rho_n(\beta)) \setminus B(\mathbf{z}_i^{(n)}, \rho_n(\gamma))$,

$$\mathbb{P}\left[\mathbf{x}_k^{(n)} \in A_i^{(n)} \mid \mathbf{x}_k^{(n)} \in B(\mathbf{z}_i^{(n)}, \rho_n(\beta))\right] = (1 - \delta_2)\frac{\beta - \gamma}{\beta}.$$

Since the mobility of the nodes is assumed mutually independent, these observations tell us that, for all $n \ge n_3$,

$$\begin{aligned}&\mathbb{P}\left[\cap_{i=1}^{\sigma_n}\left(\tilde{E}_i^{(n)}\right)^c \cap D^{(n)} \mid \mathbf{N}^{(n)} = \mathbf{n}\right] \\ &\le \prod_{i=1}^{\sigma_n}\left(1 - n_i(1 - \delta_2)^{n_i}\frac{\epsilon}{\beta}\left(1 - \frac{\gamma}{\beta}\right)^{n_i - 1}\right). \end{aligned} \quad (17)$$

Using the inequality $(1 - x) \le \exp(-x)$ for all $x \in (0, 1]$,

$$\begin{aligned}&\mathbb{P}\left[\cap_{i=1}^{\sigma_n}\left(\tilde{E}_i^{(n)}\right)^c \cap D^{(n)} \mid \mathbf{N}^{(n)} = \mathbf{n}\right] \\ &\le \exp\left(-\frac{\epsilon}{\beta}\sum_{i=1}^{\sigma_n} n_i(1 - \delta_2)^{n_i}\left(1 - \frac{\gamma}{\beta}\right)^{n_i - 1}\right). \end{aligned} \quad (18)$$

Because $n_i \in \{1, 2, \ldots, \alpha\beta\tilde{f}_i^{(n)} \ \log(n)\}$ for all $i \in \{1, 2, \ldots, \sigma_n\}$ when $D^{(n)}$ is true,

$$\begin{aligned}&\frac{\epsilon}{\beta}\sum_{i=1}^{\sigma_n} n_i(1 - \delta_2)^{n_i}\left(1 - \frac{\gamma}{\beta}\right)^{n_i - 1} \\ &= \frac{\epsilon}{\beta - \gamma}\sum_{i=1}^{\sigma_n} n_i(1 - \delta_2)^{n_i}\left(1 - \frac{\gamma}{\beta}\right)^{n_i} \\ &\ge \frac{2\epsilon}{\beta - \gamma}\sum_{i=1}^{\sigma_n}(1 - \delta_2)^{n_i}\left(1 - \frac{\gamma}{\beta}\right)^{n_i} \\ &\ge \frac{2\epsilon}{\beta - \gamma}\sum_{i=1}^{\sigma_n}\left((1 - \delta_2)\left(1 - \frac{\gamma}{\beta}\right)\right)^{\alpha\beta\tilde{f}_i^{(n)} \log(n)}. \end{aligned} \quad (19)$$

Therefore, by lower bounding the exponent in (18) using (19), we obtain the following upper bound.

$$\begin{aligned}&\mathbb{P}\left[\cap_{i=1}^{\sigma_n}\left(\tilde{E}_i^{(n)}\right)^c \cap D^{(n)} \mid \mathbf{N}^{(n)} = \mathbf{n}\right] \\ &\le \exp\left(-\frac{2\epsilon}{\beta - \gamma}\sum_{i=1}^{\sigma_n}\left((1 - \delta_2)\left(1 - \frac{\gamma}{\beta}\right)\right)^{\alpha\beta\tilde{f}_i^{(n)} \log(n)}\right) \\ &= \exp\left(-\frac{2\epsilon}{\beta - \gamma}\sum_{i=1}^{\sigma_n} n^{\alpha\beta\tilde{f}_i^{(n)} \log((1-\delta_2)(1-\gamma/\beta))}\right) \end{aligned} \quad (20)$$

Since $\sigma_n \ge c_2 \ n/\log(n)$ for some $c_2 > 0$ from Lemma 2.1. of [15], for any $\delta_3 > 0$ for all sufficiently large $n$,

$$\begin{aligned}&(20) \\ &\le \exp\left(-\frac{2\epsilon c_2}{(\beta - \gamma)\log(n)} n^{1+\alpha\beta(\psi_{\inf}-\delta_3)\log((1-\delta_2)(1-\gamma/\beta))}\right)\end{aligned}$$

From the above inequality, a sufficient condition for (20) to be summable is that $1+\alpha\beta(\psi_{\inf}-\delta_3)\log((1-\delta_3)(1-\gamma/\beta)) > 0$ or, equivalently,

$$\gamma < \beta\left(1 - \frac{1}{1 - \delta_2}\exp\left(-\frac{1}{\alpha\beta(\psi_{\inf} - \delta_3)}\right)\right). \quad (21)$$

Note that, as $\beta \uparrow \infty$, the right-hand side of (21) converges to $\frac{1}{(1-\delta_2)\alpha(\psi_{\inf}-\delta_3)} - \frac{\beta\delta_2}{1-\delta_2}$. Thus, for sufficiently small $\delta_2$ and sufficiently large $\beta$, we have

$$\frac{1 - \epsilon}{\alpha\psi_{\inf}} < \beta\left(1 - \frac{1}{1 - \delta_2}\exp\left(-\frac{1}{\alpha\beta\psi_{\inf}}\right)\right). \quad (22)$$

Choose $\gamma = \frac{1-2\epsilon}{\alpha\psi_{\inf}}$. Then,

$$\gamma < \frac{1 - \epsilon}{\alpha\psi_{\inf}} < \beta\left(1 - \frac{1}{1 - \delta_2}\exp\left(-\frac{1}{\alpha\beta\psi_{\inf}}\right)\right).$$

and $\mathbb{P}\left[\mathcal{E}^{(n)} \cap D^{(n)}\right]$ is summable.

## Appendix B
## Proof of Theorem 2

The *outline* of the proof of Theorem 2 is similar to that of the proof of Theorem 1.1 in [14, p.247] with differences in key steps in the proof of Lemmas 3.3 and 3.4 (used to prove Proposition 3.1) and Proposition 3.2 in [14] that form the basis of the proof of Theorem 1.1 in [14].

For $\mathbf{x} = (x_1, x_2)^T \in \mathbb{R}^2$ and $a > 0$, let $C(\mathbf{x}; a) := \prod_{i=1}^{2} \left[x_i - \frac{a}{2}, x_i + \frac{a}{2}\right)$, i.e., the hypercube centered at $\mathbf{x}$ with side $a$. The *diameter* of $S \subset \mathbb{R}^2$ is defined to be

$$\begin{aligned}&\mathrm{diam}(S)\\ &:= \inf\{a \geq 0 \mid \text{there exists } \mathbf{x} \in \mathbb{R}^2 \text{ with } S \subset C(\mathbf{x}; a)\}.\end{aligned}$$

A *separating set* for $\mathbb{G}(n; \rho_n(t))$ is defined to be a nonempty subset $U \subsetneq \mathcal{N}_n$ that is a connected component of $\mathbb{G}(n; \rho_n(t))$. If $\mathbb{G}(n; \rho_n(t))$ is not connected, there exist at least two disjoint separating sets of the graph and, for any $K > 0$, at least one of the following two events is true:

i. $E^{(n)}(K)$ – there is at least one separating set of diameter at most $K\rho_n(t)$;
ii. $H^{(n)}(K)$ – there are at least two separating sets of diameter greater than $K\rho_n(t)$.

Thus, in order to prove Theorem 1, it suffices to show that there exists $K > 0$ such that, with probability 1, $\sum_{n\in\mathbb{N}} E^{(n)}(K) < \infty$ and $\sum_{n\in\mathbb{N}} H^{(n)}(K) < \infty$. This implies that, with probability 1, the events $E^{(n)}(K)$ and $H^{(n)}(K)$, $n \in \mathbb{N}$, occur finitely many times and $\mathbb{G}(n; \gamma_n(t))$ is connected for all sufficiently large $n$. This will be proved with the help of three lemmas.

First, let us introduce some notation and a proposition from [14]. Define $\mathcal{U} := \{\mathbf{y} \in \mathbb{R}^2 \mid ||\mathbf{y}|| = 1\}$ to be the unit circle centered at the origin. Given $\mathbf{x} \in \mathbb{R}^2$, $r > 0$, $\mathbf{e} \in \mathcal{U}$ and $\eta > 0$,

$$\begin{aligned}&B^+(\mathbf{x}; r, \eta, \mathbf{e}) := \{\mathbf{y} \in B(\mathbf{x}; r) \mid (\mathbf{y} - \mathbf{x})^T\mathbf{e} > \eta r\} \quad \text{and}\\ &B^-(\mathbf{x}; r, \eta, \mathbf{e}) := \{\mathbf{y} \in B(\mathbf{x}; r) \mid (\mathbf{y} - \mathbf{x})^T\mathbf{e} < -\eta r\}.\end{aligned}$$

For $\mathbf{x} \in \mathbb{R}^2$, $\mathbf{e} \in \mathcal{U}$ and $r > 0$, let $L(\mathbf{x}; \mathbf{e}) := \{\mathbf{x}+\lambda\mathbf{e} \mid \lambda \in R\}$ and $D(\mathbf{x}; r, \mathbf{e}) := \{\mathbf{x} + \mathbf{y} + \lambda\mathbf{e} \mid \mathbf{y}^T\mathbf{e} = 0,\ ||\mathbf{y}|| < r, -r < \lambda < r\}$. In addition, for $\eta > 0$,

$$\begin{aligned}&D^+(\mathbf{x}; r, \eta, \mathbf{e}) := \{\mathbf{y} \in D(\mathbf{x}; r, \mathbf{e}) \mid (\mathbf{y} - \mathbf{x})^T\mathbf{e} > \eta r\} \text{ and}\\ &D^-(\mathbf{x}; r, \eta, \mathbf{e}) := \{\mathbf{y} \in D(\mathbf{x}; r, \mathbf{e}) \mid (\mathbf{y} - \mathbf{x})^T\mathbf{e} < -\eta r\}.\end{aligned}$$

*Proposition 2.1 [14, p.250] There exist a constant $\delta_1 > 0$ and a finite set $\{(\xi_i, \mathbf{e}_i),\ i = 1, 2, \ldots, \mu\}$ with $\xi_i \in \partial\Omega$ and $\mathbf{e}_i \in \mathcal{U}$ such that*

i. $\partial\Omega \subset \cup_{i=1}^{\mu} D(\xi_i; \delta_1, \mathbf{e}_i)$,
ii. *if* $\mathbf{y} \in D(\xi_i; 10\delta_1, \mathbf{e}_i) \cap \Omega$ *for some* $i \in \{1, 2, \ldots, \mu\}$, *then for all* $r \in (0,\ 10\delta_1)$, $D^+(\mathbf{y}; r, 0.1, \mathbf{e}_i) \subset \Omega$, *and*
iii. *if* $\mathbf{y} \in D(\xi_i; 10\delta_1, \mathbf{e}_i) \cap \partial\Omega$ *for some* $i \in \{1, 2, \ldots, \mu\}$, *then for all* $r \in (0,\ 10\delta_1)$, $D^-(\mathbf{y}; r, 0.1, \mathbf{e}_i) \subset \Omega^c$.

Following the same steps in [14], choose

C1. $\psi_1 \in (0, \bar{\psi}_{\inf})$ and $q < r < s < t^* =: t/\sqrt{\pi}$ such that $\pi\ \psi_1\ q^2 > 1$, and
C2. $\varepsilon > 0$ that satisfies
  C2-a. $\varepsilon < \min\left(\frac{t^*-s}{\sqrt{2}},\ \frac{s}{2\sqrt{2}},\ \frac{\delta_1}{2}\right)$, where $\delta_1$ is the constant in Proposition 2.1 from [14] stated above,
  C2-b. $\frac{t^*-4\varepsilon}{t^*+4\varepsilon} \geq 0.1$, and
  C2-c. for all $\mathbf{x} \in \mathbb{R}^2$ and $\mathbf{e} \in \mathcal{U}$, $\mathrm{Leb}\left(B^+(\mathbf{x}; s, 2\varepsilon/s, \mathbf{e})\right) \geq \pi r^2/2$, where $\mathrm{Leb}(A)$ denotes the Lebesgue measure of subset $A \subset \mathbb{R}^2$.

Since $\pi\ \psi_1\ q^2 > 1$ from condition C1, we can find a positive integer $\alpha$ such that $\alpha(\pi\ \psi_1\ q^2 - 1) > 2$. For each $n \in \mathbb{N}$, let $\nu(n) = n^\alpha \in \mathbb{Z}$. We first show that, for any $K > 0$, with probability 1, $\sum_{n\in\mathbb{N}} E^{\nu(n)}(K) < \infty$.

We introduce following notation needed to state the necessary lemmas:

- For each $n \in \mathbb{N}$, let $\mathcal{L}_n := \{\varepsilon\chi_n\mathbf{z} \mid \mathbf{z} \in \mathbb{Z}^2\}$, where $\chi_n = \sqrt{\log(n)/n}$. For simplicity of notation, for $\mathbf{z} \in \mathcal{L}_n$, we denote the hypercube $C(\mathbf{z}; \varepsilon\chi_n)$ by $C_n(\mathbf{z})$.
- For any $K > 0$,
$$\begin{aligned}\mathcal{T}_n(K) := \{\tau \subset \mathcal{L}_{\nu(n)} \mid \mathrm{diam}(\tau) \leq (K + \varepsilon)\chi_{\nu(n)} \text{ and},\\ \text{for all } \mathbf{z} \in \tau,\ C_{\nu(n)}(\mathbf{z}) \cap \Omega \neq \emptyset\}. \qquad (23)\end{aligned}$$
- Given $\tau \in \mathcal{T}_n(K)$,
$$A_n(\tau) := \left(\cup_{\mathbf{z}\in\tau} B(\mathbf{z}; s\ \chi_{\nu(n+1)})\right) \setminus \left(\cup_{\mathbf{z}\in\tau} C_{\nu(n)}(\mathbf{z})\right). \quad (24)$$
- $F_n(\tau) := \{\#N^{\nu(n)}(A_n(\tau)) = 0\}$.
- $\mathcal{T}_n^I(K) := \{\tau \in \mathcal{T}_n(K) \mid A_n(\tau) \subset \Omega\}$, and $\mathcal{T}_n^B(K) := \mathcal{T}_n(K) \setminus \mathcal{T}_n^I(K)$.

The first lemma stated below is proved in [14]. It allows us to bound the probability of the events $E^{(n)}(K)$, using the events $F_n(\tau)$ defined above.

*Lemma 3.1 [14] There exists $n_0$ such that if $n \geq n_0$ and $\nu(n) \leq m < \nu(n + 1)$, then $E^{(m)}(K) \subset \cup_{\tau\in\mathcal{T}_n(K)} F_n(\tau)$.*

Although the claims of the next lemma are the same as those of Lemmas 3.3 and 3.4 in [14], their proofs are quite different, which are provided in Appendix C. Fix $K > 0$.

*Lemma 1:* Define $G_n^I = \cup_{\tau\in\mathcal{T}_n^I(K)} F_n(\tau)$ and $G_n^B = \cup_{\tau\in\mathcal{T}_n^B(K)} F_n(\tau)$. With probability 1, $\sum_{n\in\mathbb{N}} \mathbf{1}\left\{G_n^I\right\} < \infty$ and $\sum_{n\in\mathbb{N}} \mathbf{1}\left\{G_n^B\right\} < \infty$, where $\mathbf{1}\{A\}$ denotes the indicator function of the event $A$.

Lemma 1 along with Lemma 3.1 in [14] stated above prove that the events $E^{\nu(n)}(K), n \in \mathbb{N}$, occur only finitely many times with probability 1. The gaps between $\nu(n)$ can be filled using the argument in [1].

The next lemma proves that there exists $K^\star > 0$ such that, with probability 1, the events $H^{(n)}(K^\star)$, $n \in \mathbb{N}$, are true for finitely many $n$. Its proof is provided in Appendix **??**.

*Lemma 2:* There exists some $K^\star > 0$ such that, with probability 1, $\sum_{n\in\mathbb{N}} \mathbf{1}\left\{H^{(n)}(K^\star)\right\} < \infty$.

## Appendix C
## Proof of Lemma 1

Recall from the definition of $\mathcal{T}_n^I(K)$ that $A_n(\tau) \subset \Omega$ for all $\tau \in \mathcal{T}_n^I(K)$. Therefore, for all $\tau \in \mathcal{T}_n^I(K)$,

$$\begin{aligned}\mathbb{P}\left[F_n(\tau)\right] &= \prod_{k=1}^{\nu(n)} \left(1 - \int_{A_n(\tau)} f_k^{\nu(n)}(\mathbf{x})\ d\mathbf{x}\right)\\ &\leq \exp\left(-\sum_{k=1}^{\nu(n)} \int_{A_n(\tau)} f_k^{\nu(n)}(\mathbf{x})\ d\mathbf{x}\right). \quad (25)\end{aligned}$$

Note that, for any $\epsilon > 0$, for all sufficiently large $n$,

$$\sum_{k=1}^{\nu(n)} \int_{A_n(\tau)} f_k^{\nu(n)}(\mathbf{x})\ d\mathbf{x} \geq \nu(n) \int_{A_n(\tau)} (\psi_{\inf} - \epsilon)\ d\mathbf{x} \\ = \nu(n)\ \mathrm{Leb}(A_n(\tau))(\psi_{\inf} - \epsilon), \tag{26}$$

where $\mathrm{Leb}(A_n(\tau))$ is the Lebesgue measure of $A_n(\tau)$. Using the inequality in (26) in (25), we obtain

$$\mathbb{P}\left[F_n(\tau)\right] \leq \exp\left(-\nu(n)\mathrm{Leb}(A_n(\tau))(\psi_{\inf} - \epsilon)\right). \tag{27}$$

From condition C1, since $\chi_{\nu(n+1)}/\chi_{\nu(n)} \to 1$ as $n \to \infty$, for all sufficiently large $n$,

$$\mathrm{Leb}(A_n(\tau)) \geq \pi \left(r\ \chi_{\nu(n+1)}\right)^2 \geq \pi q^2\ \alpha\ \log(n)/\nu(n).$$

Using this lower bound in (27),

$$(27) \leq \exp\left(-\pi q^2\ \alpha\ \log(n)(\psi_{\inf} - \epsilon)\right). \tag{28}$$

We need the following lemma from [14] to complete the proof of the first part of Lemma 1, which provides us with upper bounds for the cardinality of $\mathcal{T}_n^I(K)$ and $\mathcal{T}_n^B(K)$.

**Lemma 3.2 [14]** *Let $K > 0$. There exists $c_3 := c_3(K, \varepsilon) > 0$ such that, for all sufficiently large $n$,*

$$\left|\mathcal{T}_n^I(K)\right| \leq c_3\ \chi_{\nu(n)}^{-2} = c_3\ \frac{n^\alpha}{\alpha\ \log(n)} \tag{29}$$

*and*

$$\left|\mathcal{T}_n^B(K)\right| \leq c_3\ \chi_{\nu(n)}^{-1} = c_3\sqrt{\frac{n^\alpha}{\alpha\ \log(n)}}. \tag{30}$$

Now, using a union bound,

$$\begin{aligned}\mathbb{P}\left[G_n^I\right] &= \mathbb{P}\left[\cup_{\tau \in \mathcal{T}_n^I(K)} F_n(\tau)\right] \\ &\leq \left|\mathcal{T}_n^I(K)\right|\ \exp\left(-\pi q^2\ \alpha\ \log(n)(\psi_{\inf} - \epsilon)\right) \\ &\leq c_3\ \frac{n^\alpha}{\alpha\ \log(n)} \times n^{-(\psi_{\inf} - \epsilon)\pi q^2 \alpha} \\ &= \frac{c_3}{\alpha}\ \frac{n^{\alpha\ (1 - (\psi_{\inf} - \epsilon)\pi q^2)}}{\log(n)}.\end{aligned} \tag{31}$$

Recall that $\alpha$ is chosen to satisfy $\alpha(\pi\ \psi_1\ q^2 - 1) > 2$ with $\psi_1 < \psi_{\inf}$. Hence, for sufficiently small $\epsilon > 0$, we have $\alpha\ (1 - (\psi_{\inf} - \epsilon)\ \pi\ q^2) < -1$, and $\sum_{n \in \mathbb{N}} \mathbb{P}\left[G_n^I\right] < \infty$. Borel-Cantelli lemma [7] now tells us that the events $G_n^I$, $n \in \mathbb{N}$, occur only for finitely many $n$ with probability 1.

We follow similar steps for the proof of the first part of Lemma 1.

$$\begin{aligned}\mathbb{P}\left[F_n(\tau)\right] &= \prod_{k=1}^{\nu(n)} \left(1 - \int_{A_n(\tau) \cap \Omega} f_k^{\nu(n)}(\mathbf{x})\ d\mathbf{x}\right) \\ &\leq \exp\left(-\sum_{k=1}^{\nu(n)} \int_{A_n(\tau) \cap \Omega} f_k^{\nu(n)}(\mathbf{x})\ d\mathbf{x}\right).\end{aligned} \tag{32}$$

It is shown in [14] that $\mathrm{Leb}(A_n(\tau) \cap \Omega) \geq \pi q^2 \alpha \log(n)/(2\nu(n))$. Therefore, for any $\epsilon > 0$, for all sufficiently large $n$,

$$\begin{aligned}(32) &\leq \exp\left(-\pi q^2 \alpha \log(n)(\psi_{\inf} - \epsilon)/2\right) \\ &= n^{-(\psi_{\inf} - \epsilon)\pi q^2 \alpha/2}.\end{aligned} \tag{33}$$

From (32) and (33), a union bound gives us

$$\begin{aligned}\mathbb{P}\left[G_n^B\right] &= \mathbb{P}\left[\cup_{\tau \in \mathcal{T}_n^B(K)} F_n(\tau)\right] \\ &\leq \left|\mathcal{T}_n^B(K)\right|\ e^\epsilon \cdot n^{-(\psi_{\inf} - \epsilon)\pi q^2 \alpha/2}.\end{aligned} \tag{34}$$

Now, applying the upper bound for $\left|\mathcal{T}_n^B(K)\right|$ in (30), we obtain

$$\begin{aligned}\mathbb{P}\left[G_n^B\right] &\leq c_3\sqrt{\frac{n^\alpha}{\alpha\ \log(n)}}\ n^{-(\psi_{\inf} - \epsilon)\pi q^2 \alpha/2} \\ &= c_3\ \frac{n^{\alpha(1 - (\psi_{\inf} - \epsilon)\pi q^2)/2}}{\sqrt{\alpha\ \log(n)}}.\end{aligned} \tag{35}$$

Recall that $\alpha$ is chosen to satisfy $\alpha(\pi\ \psi_1\ q^2 - 1) > 2$ with $\psi_1 < \psi_{\inf}$. Hence, for sufficiently small $\epsilon > 0$, we have $\alpha\ (1 - (\psi_{\inf} - \epsilon)\pi q^2) < -2$, and $\sum_{n \in \mathbb{N}} \mathbb{P}\left[G_n^B\right] < \infty$. Hence, by Borel-Cantelli lemma [7], the events $G_n^B$, $n \in \mathbb{N}$, take place only finitely many times with probability 1.

## Appendix D
## Proof of Lemma 2

We begin with some preliminaries we need to prove the lemma. Let $\delta_1 > 0$ and $\{(\xi_i, \mathbf{e}_i),\ i = 1, 2, \ldots, \mu\}$ be the constant and the set of pairs to construct a covering of the boundary $\partial\Omega$ in Proposition 2.1 from [14]. We define $\Omega^I = \Omega \setminus (\cup_{i=1}^{\mu} D(\xi_i; \delta_1, \mathbf{e}_i))$, which is shown to be nonempty in [14, pp.250-251].

For any set $A \subset \mathbb{R}^2$, we denote by cl$(A)$ (resp. int$(A)$) the closure (resp. interior) of $A$. For each $n \in \mathbb{N}$, let

$$\mathcal{A}_n := \{\Gamma = \cup_{i=1}^{J} C(2^{-n}\ \mathbf{z}_i; 2^{-n}) \mid \{\mathbf{z}_1, \ldots, \mathbf{z}_J\} \subset \mathbb{Z}^2, \\ J \in \mathbb{N},\ \text{and int}(\Gamma) \text{ is connected}\}.$$

Fix $\mathbf{x}_0 \in \Omega^I$. For each $n \in \mathbb{N}$, denote by $\Gamma_n$ the maximal element $\Gamma$ of $\mathcal{A}_n$ such that (i) $\mathbf{x}_0 \in \Gamma$ and (ii) cl$(\Gamma) \subset$ int$(\Omega)$. Then, from the definition of $\mathcal{A}_n$, for all $n \in \mathbb{N}$, $\Gamma_n \subseteq \Gamma_{n+1}$, and it is shown [14, p.251] that $\cup_{n \in \mathbb{N}}$ int$(\Gamma_n) =$ int$(\Omega)$. Choose $m_1 < m_2 < m_3 < \infty$ such that $\Omega^I \subset \mathrm{int}(\Gamma_{m_1}) \subset \mathrm{cl}(\Gamma_{m_1}) \subset \mathrm{int}(\Gamma_{m_2}) \subset \mathrm{cl}(\Gamma_{m_2}) \subset \mathrm{int}(\Gamma_{m_3}) \subset \mathrm{cl}(\Gamma_{m_3}) \subset \mathrm{int}(\Omega)$, and define $\eta_1 := 2^{-m_3}$.

For fixed $n \in \mathbb{N}$, a set $\sigma \subset \mathcal{L}_n$ is said to be $*$-connected if $\mathrm{cl}\big(\sum_{\mathbf{z} \in \sigma} C_n(\mathbf{z})\big)$ is connected. We define

$$\mathcal{C}_{n,i}(\eta) := \{\sigma \in \mathcal{L}_n \mid \sigma \text{ is } *\text{-connected},\ |\sigma| = i, \\ \text{and} \sum_{\mathbf{z} \in \sigma} \mathbf{1}\{C_n(\mathbf{z}) \subset \Omega\} \geq \eta\ i\}.$$

We first state a lemma from [14] which will be used to complete the proof.

**Lemma 3.5 [14]** *There exist some constant $\eta_2 > 0$ and finite $n_4$ such that, for all $n \geq n_4$, if $U$ and $V$ are two separating sets for $\mathbb{G}(n; \rho_n(t))$, then there exists $\sigma \in \mathcal{C}_{n,i}(\eta_2)$ for some $i \in \mathbb{N}$ such that*

*i. $\#N^{(n)}\left(\cup_{\mathbf{z} \in \sigma} C_n(\mathbf{z})\right) = 0$, and*

*ii. $i\ \varepsilon\ \chi_n \geq \min\left(diam(U),\ diam(V),\ \eta_1/2\right)$.*

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Choose some $K^\star$ such that $K^\star \eta_2 \psi_{\inf} \varepsilon \sqrt{t/\pi} > 2$. Let $n_5 := \inf\{n \in \mathbb{N} \mid K^\star\ \rho_n(t) \leq \eta_1/2\}$. For all $n \geq \max(n_4, n_5)$, if there exist two separating sets $U$ and $V$ with

$\min\{\text{diam}(U),\ \text{diam}(V)\} \geq K^\star\ \rho_n(t)$, i.e., $H^{(n)}(K^\star)$ is true, then there exists $\sigma \in \mathcal{C}_{n,i}(\eta_2)$ for some $i$ that satisfies the conditions in Lemma 3.5 in [14]. In particular, since $K^\star\ \rho_n(t) \leq \eta_1/2$ for all $n \geq \max(n_4, n_5)$, we have

$$\begin{aligned} i\ \varepsilon\ \chi_n &\geq \min\left(\text{diam}(U),\ \text{diam}(V),\ \eta_1/2\right) \\ &\geq \min\left(\text{diam}(U),\ \text{diam}(V),\ K^\star\ \chi_n\right) \\ &= K^\star\ \rho_n(t), \end{aligned}$$

where the equality follows from that $\min\{\text{diam}(U),\ \text{diam}(V)\} \geq K^\star\ \rho_n(t)$ when $H^{(n)}(K^\star)$ is true. Thus, it implies $i\ \varepsilon \geq K^\star\sqrt{t/\pi} =: K^\dagger$.

Making use of this observation and a simple union bound, for all $n \geq \max(n_4, n_5)$,

$$\begin{aligned} &\mathbb{P}\left[H_n(K^\star)\right] \\ &\leq \sum_{i \geq K^\dagger/\varepsilon} \left( \sum_{\sigma \in \mathcal{C}_{n,i}(\eta_2)} \mathbb{P}\left[\# N^{(n)}\left(\cup_{\mathbf{z}\in\sigma} C_n(\mathbf{z})\right) = 0\right] \right). \end{aligned} \tag{36}$$

By Peierls argument [8], there exist $\gamma > 0$ and $c > 0$ such that $|\mathcal{C}_{n,i}(\eta_2)| \leq c\ n\ \exp(\gamma\ i)/\log(n)$. Using the upper bound in (36) and following the same argument in (25) through (27) in the proof of Lemma 1, for any $\epsilon > 0$ and $\delta > 0$, there exists $n_6(\delta, \epsilon)$ such that, for all $n \geq n_6(\delta, \epsilon)$,

$$\begin{aligned} &\mathbb{P}\left[H_n(K^\star)\right] \\ &\leq \frac{c\ n}{\log(n)} \sum_{i \geq K^\dagger/\varepsilon} \Big( \exp\left(\gamma\ i\right) \\ &\qquad\qquad \times \exp\Big(-\eta_2\ i\ n\left(\psi_{\inf} - \delta\right)\left(\varepsilon\ \chi_n\right)^2\Big)\Big) \\ &= \frac{c\ n}{\log(n)} \sum_{i \geq K^\dagger/\varepsilon} \exp\Big(i\big(\gamma \\ &\qquad\qquad -\eta_2\ n\ \left(\psi_{\inf} - \delta\right)\varepsilon^2\ \log(n)/n\big)\Big) \end{aligned} \tag{37}$$

where the equality follows from the definition $\chi_n = \sqrt{\log(n)/n}$. After a little algebra, we get

$$\begin{aligned} (37) &\leq \frac{2\ c\ n}{\log(n)}\ \exp\left(\frac{K^\dagger}{\varepsilon}\left(\gamma - \eta_2\left(\psi_{\inf} - \delta\right)\varepsilon^2\log(n)\right)\right) \\ &= \frac{2\ c}{\log(n)}\ \exp\left(\frac{K^\dagger\ \gamma}{\varepsilon}\right) n^{1-\eta_2(\psi_{\inf}-\delta)\varepsilon K^\dagger}. \end{aligned}$$

Recall that $K^\star \eta_2 \psi_{\inf} \varepsilon \sqrt{t/\pi} = K^\dagger \eta_2 \psi_{\inf} \varepsilon > 2$. Hence, for sufficiently small $\delta$, $1 - \eta_2\left(\psi_{\inf} - \delta\right)\varepsilon K^\dagger < -1$, and $\sum_{n\in\mathbb{N}} \mathbb{P}\left[H_n(K^\star)\right] < \infty$. The lemma now follows from Borel-Cantelli lemma.