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Integral Geometry in Hyperbolic Spaces and Electrical Impedance Tomography

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INTEGRAL GEOMETRY IN HYPERBOLIC SPACES AND ELECTRICAL IMPEDANCE TOMOGRAPHY

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ABSTRACT. We study the relation between convolution operators and the totally geodesic Radon transform on hyperbolic spaces. As an application we show that the linearized inverse conductivity problem in the disk can be interpreted exactly in terms of the X-ray transform with respect to the Poincaré metric and of a simple convolution operator.

0. INTRODUCTION

Electrical impedance tomography (EIT) has been proposed by D. H. Barber and B. H. Brown [BB1], [BB2] to image lungs, blood flow, and other features of the human body where conventional tomography might not be so successful. Its mathematical model is a two-dimensional inverse conductivity problem that has attracted considerable interest in recent years, especially because it is equivalent to other important inverse problems, such as inverse scattering and seismology (cf. [SU], [N]): namely, the problem on the unit disk D given by

$$\begin{cases} \operatorname{div}(\beta \operatorname{grad} u) = 0 & \text{in } D, \\ \beta \dfrac{\partial u}{\partial n} = \psi & \text{on } \partial D, \\ u = \phi & \text{on } \partial D, \end{cases} \tag{0.1}$$

where the input ψ, the boundary current to be applied (typically a dipole at a boundary point ω), is such that $\int_{\partial D} \psi = 0$, and the measured output is the boundary potential ϕ (unique up to an additive constant). The goal is the recovery of the conductivity β, which is a strictly positive function on D. Due to the nonlinear nature of this inverse problem it is standard to try to solve it approximately by linearization, performed here around the constant solution. In essence, this is what Barber and Brown did, as explained by F. Santosa and M. Vogelius [SV], though in fact all of them work with a further approximation of the linearized problem. As announced in [BC4] we have succeeded in representing the exact linearized problem

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as an equation in the hyperbolic plane involving the X-ray transform and a simple-looking convolution operator. This will be a consequence of our work below on the relation between the totally geodesic Radon transform R in the real hyperbolic space $\mathbf{H}^n$ and convolution operators.

1. Preliminaries

Let us recall the definitions and notation for the totally geodesic Radon transform in $\mathbf{H}^n$ as given in [H2]. We shall use the 'conformal disk' model for $\mathbf{H}^n$, viz., the open unit ball of $\mathbf{R}^n$ with the metric

$$ds^2 = \frac{4dx^2}{(1-\|x\|^2)^2} = \frac{4\sum_{j=1}^n dx_j^2}{(1-\sum_{j=1}^n x_j^2)^2},$$

where $\|\cdot\|$ denotes the Euclidean norm in $\mathbf{R}^n$. Such metric is in fact conformal to the Euclidean one dx^2 and has constant curvature -1 (in some chapters of [H2] the curvature is -4). The induced distance between $x \in \mathbf{H}^n$ and the origin o is related to the Euclidean norm of x by

$$\|x\| = \tanh \frac{d(x,o)}{2}. \tag{1.1}$$

The geodesics and the totally geodesic hypersurfaces of $\mathbf{H}^n$ are arcs of circle, respectively spherical caps, which intersect the Euclidean unit sphere $\mathbf{S}^{n-1}$ perpendicularly.

In geodesic polar coordinates write $x \in \mathbf{H}^n$ as $x = (\omega, r)$, where $r = d(x,o)$ and $\omega \in \mathbf{S}^{n-1}$. The hyperbolic metric is then expressed by

$$ds^2 = dr^2 + \sinh^2 r\, d\omega^2,$$

where $d\omega^2$ is the usual metric in $\mathbf{S}^{n-1}$. Correspondingly, the $(n-1)$-dimensional area of a geodesic sphere of radius r is

$$A_n(r) = \Omega_n \sinh^{n-1} r, \qquad \text{where } \Omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)} \text{ is the Euclidean area of } \mathbf{S}^{n-1}.$$

The Laplace-Beltrami operator on $\mathbf{H}^n$ is

$$\Delta_H = \frac{(1-\|x\|^2)^n}{4} \sum_{j=1}^n \frac{\partial}{\partial x_j}\left[(1-\|x\|^2)^{2-n} \frac{\partial}{\partial x_j}\right],$$

which specializes to $\Delta_H = (1-|z|^2)^2 \partial^2/\partial z \partial \bar{z}$ (with $z = x_1 + ix_2$) in the case of $n = 2$. In polar coordinates

$$\Delta_H = \frac{\partial^2}{\partial r^2} + (n-1)\coth r \frac{\partial}{\partial r} + \sinh^{-2} r\, \Delta_S,$$

where Δ_S is the Laplace-Beltrami operator on $\mathbf{S}^{n-1}$.

The space $\mathcal{D}(\mathbf{H}^n)$ denotes as usual the space of all $\mathcal{C}^\infty$ functions with compact support in $\mathbf{H}^n$, i.e., it coincides with the space of $\mathcal{C}^\infty$ functions f on $\mathbf{R}^n$ whose support is in the open unit ball. In this paper all operators will act on the space $\mathcal{D}$, even though they can clearly be extended to larger spaces of functions on $\mathbf{H}^n$ by continuity.

The space $\Gamma = \Gamma_{n,n-1}$ of totally geodesic hypersurfaces ($(n-1)$-*geodesics* for short) of $\mathbf{H}^n$ is a homogeneous space under an action of the group $SO(1,n)$ of isometries of $\mathbf{H}^n$. As a 'distance' on Γ we will use the continuous function κ on $\Gamma \times \Gamma$ given by

$$\kappa(\gamma,\xi) = \begin{cases} \cosh d(\gamma,\xi) & \text{if } \gamma\cap\xi \text{ is empty,} \\ \cos\widehat{\gamma\xi} & \text{otherwise,} \end{cases}$$

where $\widehat{\gamma\xi}$ is the nonobtuse dihedral angle between γ, ξ. Each $\gamma \in \Gamma$ carries the $(n-1)$-dimensional area element $d\sigma_\gamma$ induced by the volume element dv in $\mathbf{H}^n$. Hence the totally geodesic $(n-1)$-dimensional Radon transform R is defined on the space $\mathcal{S}(\mathbf{H}^n)$ by

$$Rf(\gamma) = \int_\gamma f(x)\, d\sigma_\gamma(x) \quad \text{for all } \gamma \in \Gamma.$$

The family Γ_x of elements of Γ passing through a fixed point x is a homogeneous space for the isotropy group $SO(1,n)_x$ of x, which is isomorphic to $SO(n)$. Hence Γ_x carries a normalized measure dm_x which is invariant under $SO(1,n)_x$, and is 'independent' of x in an obvious sense. For a continuous function ϕ on Γ we can define the backprojection operator R^* by

$$R^*\phi(x) = \int_{\Gamma_x} \phi(\gamma)\, dm_x(\gamma) = \int_{SO(n)} \phi(gh\cdot\gamma)\, dh \quad \text{for all } x \in \mathbf{H}^n, \tag{1.2}$$

where g is a fixed element of $SO(1,n)$ such that $g \cdot o = x$, while h runs in $SO(n)$ and dh is the normalized invariant measure in $SO(n)$. One of the uses of the backprojection operator is to find an inversion formula for the Radon transform. This is based on the fact that, denoting by $d\theta$ the area element on the geodesic sphere $S(x,r)$ of center x and radius r, we have

$$\begin{aligned} R^*Rf(x) &= \int_{\mathbf{H}^n} f(y)\mathcal{R}(d(x,y))\, dv(y) \\ &= \int_0^\infty \mathcal{R}(r)\left[\int_{S(x,r)} f(y)\, d\theta(y)\right] dr, \end{aligned}$$

for a function $\mathcal{R}$ on $[0,+\infty)$ (cf. [H2, Theorem I.4.5]): interpreting $\mathcal{R}$ as a radial function on $\mathbf{H}^n$ through $\mathcal{R}(x) = \mathcal{R}(d(x,o))$ (the same abuse of notation will be extended to all radial functions), we write this integral as $(\mathcal{R}\circ d) * f(x)$—note that the inner integral is not normalized—: in fact both $\mathcal{R}$ and f can be pulled back as functions on the group $SO(1,n)$, convolved there, and the result, pushed to $\mathbf{H}^n$ again, coincides with the middle term of (1.2).

The function $\mathcal{R}$ turns out to be [H1]

$$\mathcal{R}(r) = \frac{\Gamma(n/2)}{\sqrt{\pi}\,\Gamma((n-1)/2)\sinh r}.$$

In [BC1] it is shown that if S is the operator of convolution associated with the radial function $\mathcal{S}(r) = \coth r$, then for an explicit polynomial p of degree k one has

$$p(\Delta_H)SR^*R = I$$

(where I is the identity operator), a filtered backprojection inversion formula. For instance if $n = 2$ this formula reduces to $-(4\pi)^{-1}SR^*R = I$: the kernel $\mathcal{S}$ can be replaced by $\coth r - 1$ to obtain an integrable kernel. Other inversion formulas, which do not factor through R^*R, can be found in [H3]. An extension of the inversion formulas to more general Riesz transforms, and a characterization of the range of R, were obtained in [BC2], [BC3].

An $(n-1)$-geodesic $\gamma \in \Gamma$ is determined by the pair of polar coordinates $(\omega', r') \in [0, +\infty) \times \mathbf{S}^{n-1}$, where $r' = d(\gamma, o)$ is the distance from o to γ, achieved at a unique point $x_o \in \gamma$, and ω' is the Euclidean unit vector $x_o/\|x_o\|$. The volume measures on $\mathbf{H}^n$, Γ are then expressed by

$$dv(\omega, r) = \sinh^{n-1} r\, dr\, d\omega, \tag{1.3}$$

$$dm(\omega', r') = \cosh^{n-1} r'\, dr'\, d\omega', \tag{1.4}$$

respectively (cf. [BC3]).

We shall also need different parametrizations ('cylindrical coordinates') of $\mathbf{H}^n$, Γ and the expressions of the invariant measures in such coordinates. Fix an $(n-1)$-geodesic γ and parametrize $\mathbf{H}^n$ identifying a point y with (x_y, u), where u is the signed distance from y to γ (consistent with the choice of an exterior normal to γ) and x_y is the closest point of γ to y. The analogous coordinatization of Γ is somewhat more complicated. We first partition Γ into three subsets $\Gamma^+, \Gamma^0, \Gamma^-$, placing $\gamma \in \Gamma$ according to the sign of $\kappa(\gamma, \xi) - 1$. Observe that Γ^+, Γ^- are open; on the other hand, we shall not need to parametrize Γ^0, since its measure is zero. (In the Euclidean case, the Γ^+ part is not open either, and has measure zero.) The coordinates of $\xi \in \Gamma^+$ are x_ξ, r, where $r = d(\xi, \gamma)$ (signed distance) and x_ξ is the closest point of γ to ξ. Let Λ_γ be the set of $(n-2)$-geodesics of γ: in Γ^- identify ξ with (λ, ρ), where $\lambda = \gamma \cap \xi \in \Lambda_\gamma$ and ρ is the angle $\widehat{\gamma\xi}$ or its complementary—the choice of ρ can be made locally in Λ_γ in a continuous fashion.

Lemma 1.1. *In these coordinates the invariant measures are*

$$dv(x_y, u) = \cosh^{n-1} u\, du\, d\sigma_\gamma(x_y) \qquad \text{in } \mathbf{H}^n, \tag{1.5}$$

$$dm(x_\xi, r) = \sinh^{n-1} |r|\, dr\, d\sigma_\gamma(x_\xi) \qquad \text{in } \Gamma^+, \tag{1.6}$$

$$dm(\lambda, \rho) = \sin^{n-1} \rho\, dm_\gamma(\lambda)\, d\rho \qquad \text{in } \Gamma^-, \tag{1.7}$$

where dm_γ is the measure on Λ_γ.

Proof. The map $y \mapsto (x_y, u)$ is a diffeomorphism of $\mathbf{H}^n$ with $\gamma \times \mathbf{R}$. Furthermore, (x_y, u) is a system of orthogonal coordinates, and u is the arc length parameter of the geodesic $\{x_y = \text{constant}\}$. To compute the Jacobian of $(x_y, u) \mapsto y$ we first assume that $n = 2$: up to automorphisms, we can suppose that γ is the real interval $(-1, 1)$ and $x_y = o$ in the conformal disk model—consequently $y = i \tanh u/2$. The one-parameter group of automorphisms

$$g_t(z) = \frac{z + \tanh t/2}{1 + z \tanh t/2}$$

fixes γ, therefore the absolute value of the Jacobian at (x_y, u) equals the ratio of hyperbolic lengths of the vectors $(dg_t/dt)(i \tanh u/2)$ and $(dg_t/dt)(0)$, which an elementary computations shows to be $\cosh u$. Formula (1.5) for general n follows by orthogonality.

Assume again that $o \in \gamma$, $n = 2$, and $\gamma = (-1, 1)$. Given $\xi \in \Gamma^+$, let (ω', r'), (x_ξ, r) be its polar, respectively cylindrical coordinates with respect to γ; identify ω' with the corresponding angle with the positive imaginary half-axis. Letting y_o, y_γ be respectively the closest points of ξ to o, γ, the angle at o of the geodesic quadrilateral (lying in a 2-geodesic) of vertices o, y_o, y_γ, x_ξ is ω' (or its complementary), and the other three are right angles. We can apply hyperbolic trigonometry (cf. [M, Theorem 32.21]) to obtain

$$\begin{aligned} \sinh r' &= \sinh r \, \cosh t, \\ \cot \omega' &= \tanh r \, \sinh t, \end{aligned}$$

where $t = d(x_\xi, o)$ (with sign). The Jacobian of r', ω' with respect to r, t equals $\sinh r \,/\, \cosh r'$. We conclude with (1.6) as before.

If $n = 2$ and $o \in \gamma$, for $\xi \in \Gamma^-$ consider the geodesic triangle of vertices o, y_o, λ: the angles at o, λ are respectively ω', ρ (or complementaries). Again by hyperbolic trigonometry ([M, Corollary 32.13]) we have

$$\begin{aligned} \sinh r' &= \sin \rho \, \sinh t, \\ \tan \omega' &= \tan \rho \, \cosh t. \end{aligned}$$

The Jacobian of r', ω' with respect to λ, ρ is therefore $\sin \rho \,/\, \cosh r'$. Formula (1.7) follows. □

2. Convolution operators

Let $\mathcal{A}, \mathcal{C}$ be functions on $[1, +\infty)$, and $\mathcal{B}$ a function on $[0, +\infty)$ (sufficient smoothness and decay will be assumed throughout). Let A be the convolution operator on $\mathbf{H}^n$ given by $Af = (\mathcal{A} \circ \cosh d) * f$ (see Section 1), for f a function on $\mathbf{H}^n$. Similarly, using the 'distance' function κ define a convolution operator B on Γ by $B\phi = (\mathcal{B} \circ \kappa) * \phi$, with ϕ a function on Γ. Finally consider a mixed situation: let C be the operator $Cf = (\mathcal{C} \circ \cosh d) * f$, where f is a function on $\mathbf{H}^n$, but the value Cf is a function on Γ (in this case d is the distance between points and $(n-1)$-geodesics); notice that for every $g \in \mathrm{Aut}(\mathbf{H}^n)$ we have $C(gf)(g\gamma) = Cf(\gamma)$, where $gf(x) = f(g^{-1}x)$. The purpose of this section is to establish the relations among the kernels $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ for which $RA = C = BR$.

Theorem 2.1. *The identities*

$$RA = C, \tag{2.1}$$

$$BR = C \tag{2.2}$$

are respectively equivalent to the Abel integral equations

$$\mathcal{C}(U) = \Omega_{n-1} U^{2-n} \int_U^\infty (T^2 - U^2)^{(n-3)/2} \mathcal{A}(T)\, dT \qquad \text{for } U \geq 1, \tag{2.3}$$

$$\mathcal{C}(U) = \Omega_{n-1} U^{2-n} \int_0^U (U^2 - T^2)^{(n-3)/2} \mathcal{B}(T)\, dT \qquad \text{for } U \geq 1. \tag{2.4}$$

These are inverted: for odd $n = 2m + 3$ by

$$\mathcal{A}(T) = \frac{(-\pi)^{-m-1}}{2} \frac{d}{dT} \frac{d^m}{d(T^2)^m} [T^{2m+1}\mathcal{C}(T)] \qquad \text{for } T \geq 1,$$

$$\mathcal{B}(T) = \frac{\pi^{-m-1}}{2} \frac{d}{dT} \frac{d^m}{d(T^2)^m} [T^{2m+1}\mathcal{C}(T)] \qquad \text{for } T \geq 0;$$

for even $n = 2m + 2$ by

$$\mathcal{A}(T) = (-\pi)^{-m-1} T^{-1} \int_T^\infty \frac{1}{\sqrt{U^2 - T^2}} \left(U + U^2 \frac{d}{dU}\right) \frac{d^m}{d(U^2)^m} [U^{2m}\mathcal{C}(U)]\, dU \qquad \text{for } T \geq 1,$$

$$\mathcal{B}(T) = \pi^{-m-1} T \int_0^T \frac{1}{\sqrt{T^2 - U^2}} \frac{d}{dU} \frac{d^m}{d(U^2)^m} [U^{2m}\mathcal{C}(U)]\, dU \qquad \text{for } T \geq 0.$$

In both expressions of $\mathcal{B}$, the function $\mathcal{C}$ is extended to $[0, 1)$ arbitrarily.

Note that the restriction of $\mathcal{B}$ to $[1, \infty)$ does or does not depend on such extension of $\mathcal{C}$ according if n is even or odd.

Proof. Applying both sides of (2.1) to a function f on $\mathbf{H}^n$, then evaluating at $\gamma \in \Gamma$ gives

$$\int_\gamma \int_{\mathbf{H}^n} \mathcal{A}(\cosh d(x, y)) f(y)\, dv(y)\, d\sigma_\gamma(x) = \int_{\mathbf{H}^n} \mathcal{C}(\cosh d(y, \gamma)) f(y)\, dv(y).$$

Exchanging the order of integration in the left-hand side and letting f vary in $\mathcal{D}(\mathbf{H}^n)$ we obtain the identity

$$\mathcal{C}(\cosh d(y, \gamma)) = \int_\gamma \mathcal{A}(\cosh d(x, y))\, d\sigma_\gamma(x) \qquad \text{for all } y \in \mathbf{H}^n,\ \gamma \in \Gamma.$$

Taking cylindrical coordinates $y = (x_y, u)$ in $\mathbf{H}^n$ with respect to γ, and setting $t = d(x, x_y)$, by hyperbolic trigonometry [M, Corollary 32.13] we have

$$\cosh d(x, y) = \cosh t\, \cosh u;$$

parametrizing γ in polar coordinates $x = (\omega, t)$ around $o = x_y$ we then obtain (cf. (1.3))

$$\mathcal{C}(\cosh u) = \Omega_{n-1} \int_0^\infty \mathcal{A}(\cosh t \, \cosh u) \sinh^{n-2} t \, dt \qquad \text{for every } u \in \mathbf{R}.$$

The change of variables $U = \cosh u$ and $T = U \cosh t$ yields (2.3).

As to (2.2), its left-hand side applied to f and evaluated at γ is

$$\left[\int_{\Gamma+} + \int_{\Gamma-}\right] \int_\xi \mathcal{B}(\kappa(\gamma, \xi)) f(y) \, d\sigma_\xi(y) \, dm(\xi) \tag{2.5}$$

(recall that Γ^0 has measure zero). In Γ^+ take cylindrical coordinates $\xi = (x_\xi, r)$, and parametrize ξ in polar coordinates $y = (\theta, s)$ around $o = y_\gamma$ (the closest point of ξ to γ). Using (1.6), (1.3) and identifying with $\mathbf{S}^{n-2}$ the intersection of $\mathbf{S}^{n-1}$ with the tangent space of ξ at y_γ, the first summand of (2.5) is

$$\int_\gamma \int_{-\infty}^{+\infty} \int_{\mathbf{S}^{n-2}} \int_0^\infty \mathcal{B}(\cosh r) f(y) \sinh^{n-1} |r| \, \sinh^{n-2} s \, ds \, d\theta \, dr \, d\sigma_\gamma(x_\xi).$$

Parametrize $\mathbf{H}^n$ in cylindrical coordinates $y = (x_y, u)$ with respect to γ: the geodesic quadrilateral (lying in a 2-geodesic) of vertices y, x_y, x_ξ, y_γ has right angles except at y, so by [M, Theorem 32.21]

$$\sinh u = \sinh r \, \cosh s,$$

which allows us to replace the parameter s with u, and (observing that the Jacobian of this change of one coordinate reduces to $|\partial u / \partial s|$) to obtain

$$\int_\gamma \int_{-\infty}^{+\infty} \int_{\mathbf{S}^{n-2}} \left[\pm \int_r^{\pm\infty}\right] \mathcal{B}(\cosh r) f(y) \sinh |r| \\ (\cosh^2 u - \cosh^2 r)^{(n-3)/2} \cosh u \, du \, d\theta \, dr \, d\sigma_\gamma(x_\xi)$$

(the sign in the rightmost integral is that of r). Now replace x_ξ with x_y, which can be written as $g(x_\xi)$ with an automorphism g of $\mathbf{H}^n$ that only depends on the other coordinates and on γ—thus contributing a factor 1 in the Jacobian of this change of coordinates, which is therefore 1 as a whole. Exchanging the order of integration, applying (1.5) again, and using the evenness in r in order to dispose of the sign, we then have

$$\Omega_{n-1} \int_{\mathbf{H}^n} \left[\int_0^{|u|} \mathcal{B}(\cosh r)(\cosh^2 u - \cosh^2 r)^{(n-3)/2} \sinh r \, dr\right] f(y) \cosh^{2-n} u \, dm(y).$$

In Γ^- also take cylindrical coordinates $\xi = (\lambda, \rho)$, and parametrize ξ itself in cylindrical coordinates $y = (z_y, s)$ around λ (so z_y is the closest point of λ to y). By (1.7), (1.5) the second summand of (2.5) is

$$\int_{\Lambda_\gamma} \int_0^\pi \int_\lambda \int_{-\infty}^\infty \mathcal{B}(\cos \rho) f(y) \sin^{n-1} \rho \, \cosh^{n-2} s \, ds \, d\sigma_\lambda(z_y) \, d\rho \, dm_\gamma(\lambda).$$

The geodesic triangle of vertices y, x_y, z_y is right at x_y, so, again by [M, Corollary 32.13],

$$\sinh u = \sin\rho \, \sinh s :$$

proceeding as before, first replace s with u and obtain

$$\int_{\Lambda_\gamma}\int_0^\pi \int_\lambda \int_{-\infty}^{\infty} \mathcal{B}(\cos\rho) f(y) \sin\rho \\ (\cosh^2 u - \cos^2\rho)^{(n-3)/2} \cosh u \, du \, d\sigma_\lambda(z_y) \, d\rho \, dm_\gamma(\lambda);$$

subsequently replace z_y with x_y, then λ with $\theta \in \mathbf{S}^{n-2}$ (the direction from z_y to x_y)—both Jacobians are 1—and rearrange the order of integration to get

$$\Omega_{n-1} \int_{\mathbf{H}^n} \left[\int_0^{\pi/2} \mathcal{B}(\cos\rho)(\cosh^2 u - \cos^2\rho)^{(n-3)/2} \sin\rho \, d\rho \right] f(y) \cosh^{2-n} u \, dm(y).$$

Putting together the two summands of (2.5) thus rewritten, we can apply (2.2) to every $f \in \mathcal{D}(\mathbf{H}^n)$ to dispose of the integral over $\mathbf{H}^n$: with the changes of variables $U = \cosh u$, and $T = \cosh r$ or $T = \cos\rho$, the summands equal the integral of (2.4) on the intervals $[1, U]$, $[0, 1]$ respectively.

The inversion formulas for $n = 2m + 3$ are obtained by differentiating m times the integral in (2.3), (2.4) respectively, with respect to U^2, then once with respect to U. For $n = 2m + 2$, the same m-fold differentiations yield the equations

$$\int_U^\infty \frac{\mathcal{A}(T)}{\sqrt{T^2 - U^2}} \, dT = \frac{(-\pi)^{-m}}{2} \frac{d^m}{d(U^2)^m} [U^{2m} \mathcal{C}(U)],$$
$$\int_0^U \frac{\mathcal{B}(T)}{\sqrt{U^2 - T^2}} \, dT = \frac{\pi^{-m}}{2} \frac{d^m}{d(U^2)^m} [U^{2m} \mathcal{C}(U)],$$

both equivalent to the standard Abel integral equation, and solved by the inversion formulas in the statement. □

Corollary 2.2. *If $n = 2$ and $\mathcal{C}(U) = U^{-\alpha}$ for $\Re e\, \alpha > 1$, then*

$$\mathcal{A}(T) = \frac{\Gamma((a+1)/2)}{\Gamma(\alpha/2)\sqrt{\pi}} T^{-\alpha}.$$

Proof. With the variable $V = U/T$, use [GR, 3.251.3, 8.384.1]. □

3. Electrical impedance tomography

Setting $\beta = 1 + \delta\beta$, with corresponding decomposition $u = U + \delta U$, and assuming that $\delta\beta$ vanishes on ∂D, the linearization of problem (0.1) is

$$\begin{cases} \Delta_E(\delta U) = -\langle \operatorname{grad} \delta\beta, \operatorname{grad} U \rangle & \text{in } D, \\ \dfrac{\partial(\delta U)}{\partial n} = 0 & \text{on } \partial D: \end{cases}$$

here $\langle \cdot, \cdot \rangle$, Δ_E are the Euclidean scalar product, respectively Laplacian, and U is the solution to the unperturbed problem

$$\begin{cases} \Delta_E U = 0 & \text{in } D, \\ \dfrac{\partial U}{\partial n} = -\pi \dfrac{\partial \delta_\omega}{\partial \tau} & \text{on } \partial D, \end{cases}$$

where $\partial/\partial\tau$ is the counterclockwise tangential derivative, and δ_ω is the Dirac delta function at the dipole location $\omega \in \partial D$.

Up to a rotation we can assume $\omega = i$. Take a new complex variable $\zeta = \xi + i\eta = (1-iz)/(z-i)$: the Cayley transformation $\psi : z \mapsto \zeta$ maps D holomorphically (hence conformally) onto the upper half plane $\Pi = \{\eta > 0\}$ of $\mathbf{C}$, taking $\partial D \setminus \omega$ to the real line. As shown in [SV], [BC4] (although, for convenience, the above change of coordinates is slightly different), we get the equation

$$\frac{dg}{du}(u) = \frac{1}{4\pi} \int_\Pi \frac{[(\xi-u)/\eta]^2 - 1}{([(\xi-u)/\eta]^2+1)^2} b(\zeta) \frac{d\xi\, d\eta}{\eta^2}, \tag{3.1}$$

where g, b are the perturbations, in such coordinates, of the boundary potential and of the internal conductivity, respectively. The pullback via ψ^{-1} of the hyperbolic metric on D is $ds^2 = |\zeta|^2/\eta^2$, which therefore makes Π into another conformal model of $\mathbf{H}^2$; the isometry ψ takes the geodesics through ω to the vertical half lines $\gamma(\omega, u) = \{\xi = u \text{ and } \eta > 0\}$, for $u \in \mathbf{R}$, geodesics in Π. The distance of ζ from $\gamma(\omega, u)$, i.e., from its closest point $u + i|\zeta|$ (the generic geodesics of Π are arcs of circles centered on the real line) is easily computed, for example via ψ and (1.1), and satisfies

$$\sinh d_\Pi(\zeta, \gamma(\omega, u)) = |\xi - u|/\eta :$$

thus the right-hand side of (3.1) equals, with the notation of Section 2,

$$Cb(\gamma(\omega, u)), \qquad \text{where } C(U) = \frac{U^{-2} - 2U^{-4}}{4\pi},$$

whereas the left-hand side can be written as $\chi(\gamma(\omega, u))$, for a function χ on Γ which is assumed known (from the data), and is a posteriori independent of the orientation of the geodesic. Using now Corollary 2.2 we have thus proved:

Proposition 3.1. *The function b satisfies the equation*

$$R[(\mathcal{A} \circ \cosh d) * b] = \chi, \qquad \textit{where } \mathcal{A}(T) = \frac{T^{-2} - 3T^{-4}}{8\pi}. \quad \square$$

(The discrepancy of a factor -2 from the formula in [BC4, Proposition 3.1] is due to the abovementioned difference in the coordinate change.)

In trying to explain Barber and Brown's approximate solution, Santosa and Vogelius used in [SV] the method of the Beylkin generalized Radon transform [B]. Their approach consisted in approximating the right-hand side of (3.1) with a one-variable convolution operator composed with a transform that integrates along the geodesics of D after multiplication by an exponential factor: the integral is performed against the Euclidean arc length along vertical geodesics in the half plane model Π (while the convolution is in the variable u). This transform distinguishes the orientation of the geodesics, whereas physical considerations indicate that it is not the case. Barber and Brown's approximate inversion is obtained by simply applying the dual X-ray transform R^* (see Section 1) to the data χ, thus getting

$$R^*\chi = R^*RAb. \tag{3.2}$$

The product R^*R is a radial convolution operator, so it commutes with A. If $\mathcal{Z} \circ \cosh d = (1/4\pi)\Delta_H(1 - \coth T)$ is the convolution inverse of R^*R, we have

$$\mathcal{Z}(T) \approx \mathcal{A}(T) \qquad \text{for } T \to \infty;$$

yet, $\mathcal{Z}$ is singular at $T = 1$ while $\mathcal{A}$ is not. The Barber-Brown procedure $R^*\chi \approx b$ can be understood by replacing $\mathcal{A}$ with $\mathcal{Z}$ in (3.2). Setting $\Gamma(a \pm b) = \Gamma(a+b)\,\Gamma(a-b)$, recalling that $\hat{\Delta}_H(\lambda) = -\lambda^2 - 1/4$ (with respect to Helgason's spherical Fourier transform [H2]), and using [GR, 7.132.7], the symbol of A turns out to be

$$\begin{aligned} \hat{A}(\lambda) &= 2\pi \int_1^\infty \mathcal{A}(T)\, P_{i\lambda - 1/2}(T)\, dT \\ &= \frac{1}{4\sqrt{\pi}}\left[\Gamma\left(\frac{3}{4} \pm \frac{i\lambda}{2}\right) - 2\Gamma\left(\frac{7}{4} \pm \frac{i\lambda}{2}\right)\right] \\ &= \frac{1}{8\sqrt{\pi}}\hat{\Delta}_H(\lambda)\,\Gamma\left(\frac{3}{4} \pm \frac{i\lambda}{2}\right), \end{aligned}$$

where P_ν is the associated Legendre function. We have not succeeded in finding the convolution inverse of A.

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