

ABSTRACT

Title of Dissertation: FLOQUET THEORY ANALYSIS
OF ROTOR BLADE AEROELASTIC STABILITY
AT HIGH ADVANCE RATIOS

Spencer Manin Fishman
Doctor of Philosophy, 2025

Dissertation Directed by: Inderjit Chopra
Distinguished University Professor
Clark Distinguished Chair
Director Alfred Gessow Rotorcraft Center
Aerospace Engineering Department

A compound rotorcraft operating at high forward speed may slow its rotor, resulting in a high advance ratio (μ). While this mitigates compressibility effects, it creates stability challenges, which are explored in the present work. The aeroelastic stability characteristics of rotor blades are analyzed in the rotating frame using Floquet theory for high- μ flight. Linearized coupled equations of motion are derived for the flap, lag, and torsion degrees of freedom for an isolated rotor system, with the assistance of symbolic manipulation software. Periodic coefficients, reverse flow, unsteady aerodynamics, pitch-flap/ pitch-lag/ structural coupling, chordwise center of gravity, aerodynamic center, and other structural and aerodynamic properties are considered. The flap trim state is found through periodic shooting, and a stall model provides bounding for stability predictions. Through numerical integration of the state transition matrix, the resultant eigenvalues and eigenvector motion

allow for the determination of the type of response to perturbations from trim. A novel eigenvalue trajectory tracking method using a modal analysis of the periodic eigenvectors allows for a consistent identification of roots as parameters are changed. Damping and frequency response behaviors are evaluated, and rotor stability boundaries are presented for numerous configurations, to include various articulated and hingeless blade types. Advance ratios in excess of $\mu = 3$ are explored, with evaluations of both physical dynamic stability and numerical stability. Trim and stability predictions are validated using historical analyses and experimental results, a fundamental understanding of associated phenomena such as parametric resonance is achieved, and recommendations regarding stable configurations are provided.

FLOQUET THEORY ANALYSIS
OF ROTOR BLADE AEROELASTIC STABILITY
AT HIGH ADVANCE RATIOS

by

Spencer Manin Fishman

Dissertation submitted to the Faculty of the Graduate School of the
University of Maryland, College Park in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
2025

Advisory Committee:

Professor Inderjit Chopra, Chair/Advisor
Professor James Baeder
Professor Anubhav Datta
Assistant Professor Umberto Saetti
Professor Maria Cameron, Dean's Representative

© Copyright by
Spencer M. Fishman
2025

Dedication

To my wife Amy.

Acknowledgments

This completed dissertation exists due to the guidance and patience of my Advisor, Professor Inder Chopra. I am honored to have been the latest apprentice to the master of modern rotorcraft research. I am thankful to my Committee members for supporting me through the Ph.D. milestones.

There exists a long line of tough and forgiving educators and instructors from primary school through college, the military, and graduate school (both times) who gave me the foundation I needed to get here. I have been humbled by the helpfulness and intelligence of my fellow graduate students here at Maryland.

I am grateful for the hard work of my Sailors, friendship of my peers, and the charity of my superiors with whom I've served with during my years as a Naval Officer. As a Permanent Military Professor selectee, I am privileged to have been supported by the U.S. Naval Academy and the American taxpayer. I am also indebted for the extra time I was provided for this program. Now I can give back to the next generation of Naval Officers and aerospace engineering students.

I have been blessed with a wonderful family: my brother Ted, mother Sheila, and father Bob. They all have big hearts and big brains. My Dad provided the inspiration and inception for my career: we share USNA and UMD diplomas, time in uniform, a love of aviation and space, and have returned to our alma mater as colleagues.

I could not have made it here (or anywhere) without the love and support of my dearest wife, Amy. She has always provided me with encouragement, prayers, and a reminder that “everything is figure-outable”. She too suffered through my poorly planned all-nighters. With this done, I can finally treat her to a well-deserved vacation. Thank you my love.

It is with God’s grace that I have completed this work.

Table of Contents

Dedication	ii
Acknowledgements	iii
Table of Contents	v
List of Figures	vii
Abbreviations and Acronyms	xx
Notation and Symbols	xxii
Chapter 1: Introduction	1
1.1 The Challenge of High Advance Ratio Flight	1
1.2 Floquet Theory and Stability Analysis	6
1.3 Rotorcraft High- μ and Aeroelastic Stability Research	11
1.4 General Approach and Novel Contributions of this Work	20
Chapter 2: Flap Stability	23
2.1 Equations of Motion	23
2.2 Floquet Theory and Numerical Integration	31
2.3 Eigenvector Motion and Modal Participation	37
2.4 Numerical Stability of the Modal Waveform	42
2.5 The Effect of Advance Ratio and Reverse Flow	45
2.6 Parametric Evaluation of Flap Stability	77
2.7 Flap Stability Conclusions	89
Chapter 3: High- μ Flap Trim	93
3.1 Forcing and Stall	93
3.2 Periodic Shooting	101
3.3 Trim and Blade Stall for Various Configurations	104
3.4 Flap Trim Conclusions	113
Chapter 4: Flap-Lag Stability	114
4.1 Equations of Motion	114
4.2 Assembly of the Coefficient Matrix	118
4.3 Stability Analysis	119

4.4	Verification and Validation of the Coupled Lag Stability Model	124
4.5	Parametric Evaluation of Coupled Flap-Lag Damping	130
4.6	Eigenvalue Trajectory Analysis	130
4.7	Flap-Lag Stability Conclusions	137
Chapter 5:	Flap-Lag-Torsion Stability	230
5.1	Equations of Motion	230
5.2	Incorporation of Unsteady Aerodynamics	232
5.3	Dynamic Inflow and the Lift Deficiency Function	234
5.4	Assembly of the Coefficient Matrix	236
5.5	Verification of the Coupled Torsion Stability Model	244
5.6	Parametric Evaluation of Coupled Flap-Lag-Torsion Damping	253
5.7	Eigenvalue Trajectory Analysis	253
5.8	Flap-Lag-Torsion Stability Conclusions	262
Chapter 6:	Conclusions and Future Work	337
6.1	Summary of Stability Studies	337
6.2	Novel Developments	339
6.3	Future Research	339
Appendix A:	High- μ Flap Trim Validation	341
A.1	Articulated Rotor Trim Validation	341
A.2	Hingeless Rotor Trim Validation	355
Appendix B:	Flap-Lag-Torsion Aerodynamic Forcing Moment Equations	358
B.1	Region I, Normal Flow	359
B.2	Region II, Partial Reverse Flow	365
B.3	Region III, Complete Reverse Flow	377
Bibliography		383

List of Figures

1.1	Examples of high advance ratio compound rotorcraft.	4
1.2	Types of root instability behaviors in the z-plane and s-plane.	8
1.3	Stability map of Mathieu function.	10
1.4	Linear aerodynamic coefficient approximations overlaid on wind tunnel data of a NACA 0012 airfoil in normal and reverse flow, $Re = 1.8 \times 10^6$	13
1.5	Flap root locus with constant and periodic coefficients; $\nu_\beta = 1/\text{rev}$, $\gamma = 12$, $0 \leq \mu \leq 0.5$, $N_b = 3$	16
1.6	Rotor blade parameters.	21
2.1	Blade elements in normal and reverse flow.	25
2.2	Normal flow, partial reverse flow, and complete reverse flow regions.	27
2.3	Comparison of ODE algorithms when calculating flap damping, 200 time steps, $\gamma = 12$, $\nu_\beta = 1/\text{rev}$	34
2.4	Floquet theory approach to determine root loci.	42
2.5	Numerical stability for Euler, Backward Euler, and RK4 integration methods.	43
2.6	Moving block approach to measure physical damping or numerical dissipation.	44
2.7	Damping (C) and stiffness (K) coefficients, without reverse flow.	46
2.8	Damping (C) and stiffness (K) coefficients, with reverse flow.	47
2.9	Flap characteristic multipliers, $\nu_\beta = 1$, $\gamma = 12$, $\mu=0$ to 0.22.	48
2.10	Flap characteristic multipliers (F1), $\nu_\beta = 1$, $\gamma = 12$, $\mu=0$ to 1.73.	49
2.11	Flap characteristic multipliers (F2), $\nu_\beta = 1$, $\gamma = 12$, $\mu=0$ to 1.73.	50
2.12	Flap characteristic multipliers, $\nu_\beta = 1$, $\gamma = 12$, $\mu=0$ to 3.	51
2.13	Flap damping and principal frequency, $\nu_\beta = 1$, $\gamma = 12$, without reverse flow.	52
2.14	Flap damping and principal frequency, $\nu_\beta = 1$, $\gamma = 12$, with reverse flow.	54
2.15	Eigenvector motion and modal participation, F1, $\nu_\beta = 1$, $\gamma = 12$, $\mu = 0$	55
2.16	Eigenvector motion and modal participation, F2, $\nu_\beta = 1$, $\gamma = 12$, $\mu = 0$	56
2.17	Flap root locus, $\nu_\beta = 1$, $\gamma = 12$, $\mu = 0$	57
2.18	Eigenvector motion and modal participation, F1, $\nu_\beta = 1$, $\gamma = 12$, $\mu = 0.22$	58
2.19	Eigenvector motion and modal participation, F2, $\nu_\beta = 1$, $\gamma = 12$, $\mu = 0.22$	59
2.20	Flap root locus, $\nu_\beta = 1$, $\gamma = 12$, $\mu = 0.22$	61
2.21	Eigenvector motion and modal participation, F1, $\nu_\beta = 1$, $\gamma = 12$, $\mu = 1.73$	62
2.22	Eigenvector motion and modal participation, F2, $\nu_\beta = 1$, $\gamma = 12$, $\mu = 1.73$	63
2.23	Flap root locus, $\nu_\beta = 1$, $\gamma = 12$, $\mu = 1.73$	64

2.24	Eigenvector motion and modal participation, F1, $\nu_\beta = 1$, $\gamma = 12$, $\mu = 3$. . .	65
2.25	Eigenvector motion and modal participation, F2, $\nu_\beta = 1$, $\gamma = 12$, $\mu = 3$. . .	66
2.26	Flap root locus, $\nu_\beta = 1$, $\gamma = 12$, $\mu = 1.73$	67
2.27	Modal participation method verification.	68
2.28	Modal participation vs. μ , F1, $\nu_\beta = 1$, $\gamma = 12$	70
2.29	Modal participation vs. μ , F2, $\nu_\beta = 1$, $\gamma = 12$	71
2.30	Effect of blade configuration on numerical stability, $\nu_\beta = 1$, $\gamma = 12$	72
2.31	Dominant flap frequency responses vs. μ , $\nu_\beta = 1$, $\gamma = 12$	73
2.32	Flap root locus (dominant modes), $\nu_\beta = 1$, $\gamma = 12$, $\mu=0$ to 0.22.	74
2.33	Flap root locus (dominant modes), $\nu_\beta = 1$, $\gamma = 12$, $\mu=0$ to 1.73.	75
2.34	Flap root locus (dominant modes), $\nu_\beta = 1$, $\gamma = 12$, $\mu=0$ to 3.	76
2.35	Flap root locus (dominant modes), $\nu_\beta = 1$, $\gamma = 6$, $\mu=0$ to 3.	78
2.36	Flap root locus (dominant modes), $\nu_\beta = 1$, $\gamma = 6$, $\delta_3 = 15^\circ$, $\mu=0$ to 3.	79
2.37	Flap root locus (dominant modes), $\nu_\beta = 1.15$, $\gamma = 6$, $\mu = 0$ to 3.	80
2.38	Flap root locus (dominant modes), $\nu_\beta = 1.52$, $\gamma = 4.74$, $\mu = 0$ to 3.	81
2.39	Flap root locus (dominant modes), teetering rotor, $\nu_\beta = 1$, $\gamma = 6$, $\mu = 0$ to 3.	82
2.40	Flap stability map verification (γ vs. μ), articulated rotor, $\nu_\beta = 1$	83
2.41	Flap stability map (γ vs. μ), articulated rotor, $\nu_\beta = 1$	84
2.42	Flap stability map (γ vs. μ), articulated rotor, $\nu_\beta = 1$, $\delta_3 = 15^\circ$	85
2.43	Flap stability map (γ vs. μ), hingeless rotor, $\nu_\beta = 1.15$	86
2.44	Flap stability map (γ vs. μ), UMD hingeless model rotor, $\nu_\beta = 1.52$	87
2.45	Flap stability map (γ vs. μ), teetering rotor, $\nu_\beta = 1$	88
2.46	Expanded flap stability map (γ vs. μ), $\nu_\beta = 1$	90
2.47	Numerical stability of expanded flap stability map.	91
3.1	Angle of attack distribution and blade stall, $\nu_\beta = 1/\text{rev}$, $\gamma = 6$, $\mu = 0.75$, $C_T/\sigma = 0.02$	96
3.2	Angle of attack distribution and blade stall, $\nu_\beta = 1/\text{rev}$, $\gamma = 6$, $\mu = 0.75$, $C_T/\sigma = 0.08$	97
3.3	Flight control inputs and flap trim.	104
3.4	Trimmed flight control positions and blade stall, $\nu_\beta = 1$, $\gamma = 12$	106
3.5	Trimmed flight control positions and blade stall, $\nu_\beta = 1$, $\gamma = 6$	107
3.6	Trimmed flight control positions and blade stall, $\nu_\beta = 1$, $\gamma = 6$, $\alpha_s = 4^\circ$	108
3.7	Trimmed flight control positions and blade stall, $\nu_\beta = 1$, $\gamma = 6$, $\alpha_s = -4^\circ$	109
3.8	Trimmed flight control positions and blade stall, $\nu_\beta = 1$, $\gamma = 6$, $C_T/\sigma = 0.03$ to 0.07, $\alpha_s = -2^\circ$ to -5° (avoids thrust reversal discontinuity).	111
3.9	Autorotational trim with aft shaft tilt, $\nu_\beta = 1$, $\gamma = 6$, $C_T/\sigma = 0.04$	112
4.1	Flap-Lag damping matrix ($\underline{\underline{C}}$), $\nu_\beta = 1$, $\nu_\zeta = 0.3/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0$	120
4.2	Flap-Lag stiffness matrix ($\underline{\underline{K}}$), $\nu_\beta = 1$, $\nu_\zeta = 0.3/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0$	121
4.3	Flap-Lag damping matrix ($\underline{\underline{C}}$), $\nu_\beta = 1$, $\nu_\zeta = 0.3/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$	122
4.4	Flap-Lag stiffness matrix ($\underline{\underline{K}}$), $\nu_\beta = 1$, $\nu_\zeta = 0.3/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$	123
4.5	Flap-lag stability map verification (ν_ζ vs. ν_β), $C_T/\sigma = 0.2$, $\gamma = 5$, $\mu = 0$ to 0.4.	125

4.6	Validation of lag stability in hover with varying collective, $\nu_\beta = 1.17$, $\nu_\zeta = 1.21/\text{rev}$, $\gamma = 2.84$, soft pitch flexure: $R_s = 0.08$, $\zeta_L = 0.001$, $\theta_{tw} = -10.8^\circ$.	126
4.7	Validation of lag stability in hover with varying collective, $\nu_\beta = 1.05$, $\nu_\zeta = 1.44/\text{rev}$, $\gamma = 5.1$, stiff pitch flexure: $R_s = 1$.	127
4.8	Lag stability validation with wind tunnel model testing, $\nu_\beta = 1.105$, $\nu_\zeta = 0.61/\text{rev}$, $\gamma = 7.54$, $R_s = 0$, $\zeta_L = 0.001$, NACA 23012 airfoil, $\theta_0 = 0^\circ$, $\alpha_s = 0^\circ$, untrimmed.	128
4.9	Lag stability validation with wind tunnel model testing, $\theta_0 = 3^\circ$, $\alpha_s = 0^\circ$ to 8° , untrimmed.	129
4.10	Flap-lag stability map (ν_ζ vs. ν_β), $\gamma = 6$, $C_T/\sigma = 0.04$, $\mu = 0.6$ to 1 .	131
4.11	Flap-lag stability map (ν_ζ vs. ν_β), $\gamma = 6$, $C_T/\sigma = 0.04$, $R_s = 1$, $\mu = 0.6$ to 1 .	132
4.12	Flap-lag stability map (ν_ζ vs. ν_β), $\gamma = 6$, $C_T/\sigma = 0.04$, $\mu = 1.25$ to 2.5 .	133
4.13	Flap-lag stability map (ν_ζ vs. ν_β), $\gamma = 6$, $C_T/\sigma = 0.04$, $R_s = 1$, $\mu = 1.25$ to 2.5 .	134
4.14	Flap-lag damping, trim, blade stall, and numerical stability, very soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.3/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$.	138
4.15	Flap-lag characteristic multipliers, very soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.3/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $\mu = 0$ to 1.5 .	139
4.16	Flap-lag characteristic multipliers, very soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.3/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $\mu = 1.5$ to 3 .	140
4.17	Flap-lag characteristic multipliers, F1 and F2 roots, very soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.3/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $\mu = 0$ to 1.5 .	141
4.18	Flap-lag characteristic multipliers, F1 and F2 roots, very soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.3/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $\mu = 1.5$ to 3 .	142
4.19	Flap-lag characteristic multipliers, L1 root, very soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.3/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $\mu = 0$ to 1.5 .	143
4.20	Flap-lag characteristic multipliers, L1 root, very soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.3/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $\mu = 1.5$ to 3 .	144
4.21	Flap-lag characteristic multipliers, L2 root, very soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.3/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $\mu = 0$ to 1.5 .	145
4.22	Flap-lag characteristic multipliers, L2 root, very soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.3/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $\mu = 1.5$ to 3 .	146
4.23	Flap-lag root locus, very soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.3/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $\mu = 0$ to 1.5 .	147
4.24	Flap-lag root locus, very soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.3/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $\mu = 1.5$ to 3 .	148
4.25	Flap-lag root locus, F1 root, very soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.3/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $\mu = 0$ to 1.5 .	149
4.26	Flap-lag root locus, F1 root, very soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.3/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $\mu = 1.5$ to 3 .	150
4.27	Flap-lag root locus, F2 root, very soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.3/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $\mu = 0$ to 1.5 .	151

4.28	Flap-lag root locus, F2 root, very soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.3/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $\mu = 1.5$ to 3	152
4.29	Flap-lag root locus, L1 root, very soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.3/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $\mu = 0$ to 1.5	153
4.30	Flap-lag root locus, L1 root, very soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.3/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $\mu = 1.5$ to 3	154
4.31	Flap-lag root locus, L2 root, very soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.3/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $\mu = 0$ to 1.5	155
4.32	Flap-lag root locus, L2 root, very soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.3/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $\mu = 1.5$ to 3	156
4.33	Flap-lag damping, trim, blade stall, and numerical stability, soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.7/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$. .	157
4.34	Flap-lag characteristic multipliers, soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.7/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $\mu = 0$ to 1.5	158
4.35	Flap-lag characteristic multipliers, soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.7/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $\mu = 1.5$ to 3	159
4.36	Flap-lag characteristic multipliers, F1 and F2 roots, soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.7/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $\mu = 0$ to 1.5	160
4.37	Flap-lag characteristic multipliers, F1 and F2 roots, soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.7/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $\mu = 1.5$ to 3	161
4.38	Flap-lag characteristic multipliers, L1 root, soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.7/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $\mu = 0$ to 1.5	162
4.39	Flap-lag characteristic multipliers, L1 root, soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.7/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $\mu = 1.5$ to 3	163
4.40	Flap-lag characteristic multipliers, L2 root, soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.7/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $\mu = 0$ to 1.5	164
4.41	Flap-lag characteristic multipliers, L2 root, soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.7/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $\mu = 1.5$ to 3	165
4.42	Flap-lag root locus, soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.7/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $\mu = 0$ to 1.5	166
4.43	Flap-lag root locus, soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.7/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $\mu = 1.5$ to 3	167
4.44	Flap-lag root locus, F1 and F2 roots, soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.7/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $\mu = 0$ to 1.5	168
4.45	Flap-lag root locus, F1 root, soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.7/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $\mu = 1.5$ to 3	169
4.46	Flap-lag root locus, F2 root, soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.7/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $\mu = 1.5$ to 3	170
4.47	Flap-lag root locus, L1 root, soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.7/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $\mu = 0$ to 1.5	171
4.48	Flap-lag root locus, L1 root, soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.7/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $\mu = 1.5$ to 3	172
4.49	Flap-lag root locus, L2 root, soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.7/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $\mu = 0$ to 1.5	173

4.50	Flap-lag root locus, L2 root, soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.7/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $\mu = 1.5$ to 3	174
4.51	Flap-lag damping, trim, blade stall, and numerical stability, soft-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 0.7/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$	175
4.52	Flap-lag characteristic multipliers, soft-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 0.7/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $\mu = 0$ to 1.5	176
4.53	Flap-lag characteristic multipliers, soft-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 0.7/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $\mu = 1.5$ to 3	177
4.54	Flap-lag characteristic multipliers, F1 and F2 roots, soft-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 0.7/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $\mu = 0$ to 1.5	178
4.55	Flap-lag characteristic multipliers, L1 root, soft-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 0.7/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $\mu = 0$ to 1.5	179
4.56	Flap-lag characteristic multipliers, L1 root, soft-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 0.7/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $\mu = 1.5$ to 3	180
4.57	Flap-lag characteristic multipliers, L2 root, soft-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 0.7/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $\mu = 0$ to 1.5	181
4.58	Flap-lag characteristic multipliers, L2 root, soft-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 0.7/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $\mu = 1.5$ to 3	182
4.59	Flap-lag root locus, soft-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 0.7/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $\mu = 0$ to 1.5	183
4.60	Flap-lag root locus, soft-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 0.7/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $\mu = 1.5$ to 3	184
4.61	Flap-lag root locus, F1 root, soft-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 0.7/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $\mu = 0$ to 1.5	185
4.62	Flap-lag root locus, F1 root, soft-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 0.7/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $\mu = 1.5$ to 3	186
4.63	Flap-lag root locus, F2 root, soft-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 0.7/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $\mu = 0$ to 1.5	187
4.64	Flap-lag root locus, F2 root, soft-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 0.7/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $\mu = 1.5$ to 3	188
4.65	Flap-lag root locus, L1 root, soft-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 0.7/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $\mu = 0$ to 1.5	189
4.66	Flap-lag root locus, L1 root, soft-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 0.7/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $\mu = 1.5$ to 3	190
4.67	Flap-lag root locus, L2 root, soft-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 0.7/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $\mu = 0$ to 1.5	191
4.68	Flap-lag root locus, L2 root, soft-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 0.7/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $\mu = 1.5$ to 3	192
4.69	Flap-lag damping, trim, blade stall, and numerical stability, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$	193

4.70	Flap-lag characteristic multipliers, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $\mu = 0$ to 1.5	194
4.71	Flap-lag characteristic multipliers, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $\mu = 1.5$ to 3	195
4.72	Flap-lag characteristic multipliers, F1 and F2 roots, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $\mu = 0$ to 1.5	196
4.73	Flap-lag characteristic multipliers, L1 root, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $\mu = 0$ to 1.5	197
4.74	Flap-lag characteristic multipliers, L1 root, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $\mu = 1.5$ to 3	198
4.75	Flap-lag characteristic multipliers, L2 root, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $\mu = 0$ to 1.5	199
4.76	Flap-lag characteristic multipliers, L2 root, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $\mu = 1.5$ to 3	200
4.77	Flap-lag root locus, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $\mu = 0$ to 1.5	201
4.78	Flap-lag root locus, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $\mu = 1.5$ to 3	202
4.79	Flap-lag root locus, F1 root, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $\mu = 0$ to 1.5	203
4.80	Flap-lag root locus, F1 root, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $\mu = 1.5$ to 3	204
4.81	Flap-lag root locus, F2 root, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $\mu = 0$ to 1.5	205
4.82	Flap-lag root locus, F2 root, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $\mu = 1.5$ to 3	206
4.83	Flap-lag root locus, L1 root, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $\mu = 0$ to 1.5	207
4.84	Flap-lag root locus, L1 root, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $\mu = 1.5$ to 3	208
4.85	Flap-lag root locus, L2 root, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $\mu = 0$ to 1.5	209
4.86	Flap-lag root locus, L2 root, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $\mu = 1.5$ to 3	210
4.87	Flap-lag damping, trim, blade stall, and numerical stability, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $R_s = 1$	211
4.88	Flap-lag characteristic multipliers, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $R_s = 1$, $\mu = 0$ to 1.5	212
4.89	Flap-lag characteristic multipliers, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $R_s = 1$, $\mu = 1.5$ to 3	213

4.90	Flap-lag characteristic multipliers, F1 and F2 roots, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $R_s = 1$, $\mu = 0$ to 1.5 .	214
4.91	Flap-lag characteristic multipliers, L1 root, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $R_s = 1$, $\mu = 0$ to 1.5 .	215
4.92	Flap-lag characteristic multipliers, L1 root, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $R_s = 1$, $\mu = 1.5$ to 3 .	216
4.93	Flap-lag characteristic multipliers, L2 root, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $R_s = 1$, $\mu = 0$ to 1.5 .	217
4.94	Flap-lag characteristic multipliers, L2 root, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $R_s = 1$, $\mu = 1.5$ to 3 .	218
4.95	Flap-lag root locus, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $R_s = 1$, $\mu = 0$ to 1.5 .	219
4.96	Flap-lag root locus, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $R_s = 1$, $\mu = 1.5$ to 3 .	220
4.97	Flap-lag root locus, F1 root, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $R_s = 1$, $\mu = 0$ to 1.5 .	221
4.98	Flap-lag root locus, F1 root, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $R_s = 1$, $\mu = 1.5$ to 3 .	222
4.99	Flap-lag root locus, F2 root, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $R_s = 1$, $\mu = 0$ to 1.5 .	223
4.100	Flap-lag root locus, F2 root, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $R_s = 1$, $\mu = 1.5$ to 3 .	224
4.101	Flap-lag root locus, L1 root, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $R_s = 1$, $\mu = 0$ to 1.5 .	225
4.102	Flap-lag root locus, L1 root, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $R_s = 1$, $\mu = 1.5$ to 3 .	226
4.103	Flap-lag root locus, L2 root, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $R_s = 1$, $\mu = 0$ to 1.5 .	227
4.104	Flap-lag root locus, L2 root, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $R_s = 1$, $\mu = 1.5$ to 3 .	228
5.1	Effect of lift deficiency function and dynamic inflow on flap-lag-torsion damping, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$.	236
5.2	Flap-torsion damping matrix ($\underline{\underline{C}}$), $\nu_\beta = 1$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0$, $\alpha_s = 0^\circ$, $C(k) = 1$.	240
5.3	Flap-torsion stiffness matrix ($\underline{\underline{K}}$), $\nu_\beta = 1$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0$, $\alpha_s = 0^\circ$, $C(k) = 1$.	241
5.4	Flap-torsion damping matrix ($\underline{\underline{C}}$), $\nu_\beta = 1$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$.	242

5.5	Flap-torsion stiffness matrix ($\underline{K}$), $\nu_\beta = 1$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$.	243
5.6	Flap-lag-torsion damping matrix ($\underline{C}$), $\nu_\beta = 1$, $\nu_\zeta = 0.3$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0$, $\alpha_s = 0^\circ$, $C(k) = 1$.	244
5.7	Flap-lag-torsion stiffness matrix ($\underline{K}$), $\nu_\beta = 1$, $\nu_\zeta = 0.3$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0$, $\alpha_s = 0^\circ$, $C(k) = 1$.	245
5.8	Flap-lag-torsion damping matrix ($\underline{C}$), $\nu_\beta = 1$, $\nu_\zeta = 0.3$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$.	246
5.9	Flap-lag-torsion stiffness matrix ($\underline{K}$), $\nu_\beta = 1$, $\nu_\zeta = 0.3$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$.	247
5.10	Comparison of coupled flap stability model with multimodal nonlinear forward flight analysis of stiff-in-plane hingeless rotor, quasisteady aero., propulsive trim.	249
5.11	Comparison of coupled lag stability model with multimodal nonlinear forward flight analysis of stiff-in-plane hingeless rotor, quasisteady aero., propulsive trim.	250
5.12	Comparison of coupled torsion stability model with multimodal nonlinear forward flight analysis of stiff-in-plane hingeless rotor, quasisteady aero., propulsive trim.	251
5.13	Comparison of coupled lag stability model with elastic FEM forward flight analysis of soft and stiff-in-plane hingeless rotors, dynamic inflow, propulsive trim.	252
5.14	Flap-lag-torsion stability map (ν_ζ vs. ν_β), $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $C(k) = 1$, $\mu = 0.6$ to 1 .	254
5.15	Flap-lag-torsion stability map (ν_ζ vs. ν_β), $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $C(k) = 1$, $\mu = 1.1$.	255
5.16	Flap-lag-torsion stability map (ν_ζ vs. ν_β), $\nu_\theta = 10/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $C(k) = 1$, $\mu = 0.6$ to 1 .	256
5.17	Flap-lag-torsion stability map (ν_ζ vs. ν_β), $\nu_\theta = 10/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $C(k) = 1$, $\mu = 1.1$ to 1.4 .	257
5.18	Flap-lag-torsion stability map (ν_ζ vs. ν_β), $\nu_\theta = 10/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $C(k) = 1$, $\mu = 0.6$ to 1 .	258
5.19	Flap-lag-torsion stability map (ν_ζ vs. ν_β), $\nu_\theta = 15/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $C(k) = 1$, $\mu = 1.2$ to 1.8 .	259
5.20	Comparison of default eig order with root sorting algorithm, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $C(k) = 1$.	261
5.21	Flap-lag damping, trim, blade stall, and numerical stability, very soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.3$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$.	263
5.22	Flap-lag-torsion characteristic multipliers, very soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.3$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8 .	264

5.23	Flap-lag-torsion characteristic multipliers, very soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.3$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$, $\mu = 0.8$ to 1.6	265
5.24	Flap-lag-torsion characteristic multipliers, F1, F2, T1, and T2 roots, very soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.3$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8	266
5.25	Flap-lag-torsion characteristic multipliers, F1, F2, T1, and T2 roots, very soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.3$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$, $\mu = 0.8$ to 1.6	267
5.26	Flap-lag-torsion characteristic multipliers, L1 root, very soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.3$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8	268
5.27	Flap-lag-torsion characteristic multipliers, L1 root, very soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.3$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$, $\mu = 0.8$ to 1.6	269
5.28	Flap-lag-torsion characteristic multipliers, L2 root, very soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.3$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8	270
5.29	Flap-lag-torsion characteristic multipliers, L2 root, very soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.3$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$, $\mu = 0.8$ to 1.6	271
5.30	Flap-lag-torsion root locus, very soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.3$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8	272
5.31	Flap-lag-torsion root locus, very soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.3$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$, $\mu = 0.8$ to 1.6	273
5.32	Flap-lag-torsion root locus, F1 root, very soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.3$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8	274
5.33	Flap-lag-torsion root locus, F2 root, very soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.3$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8	275
5.34	Flap-lag-torsion root locus, T1 root, very soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.3$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8	276
5.35	Flap-lag-torsion root locus, T2 root, very soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.3$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8	277
5.36	Flap-lag-torsion root locus, L1 root, very soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.3$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8	278
5.37	Flap-lag-torsion root locus, L2 root, very soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.3$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8	279

5.38	Flap-lag-torsion root locus, F1, L1, and T1 roots, very soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.3$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$, $\mu = 0.8$ to 1.6 .	280
5.39	Flap-lag-torsion root locus, F2, L2, and T2 roots, very soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.3$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$, $\mu = 0.8$ to 1.6 .	281
5.40	Flap-lag-torsion damping, trim, blade stall, and numerical stability, soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.7$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$.	282
5.41	Flap-lag-torsion characteristic multipliers, soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.7$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8 .	283
5.42	Flap-lag-torsion characteristic multipliers, soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.7$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$, $\mu = 0.8$ to 1.58 .	284
5.43	Flap-lag-torsion characteristic multipliers, F1, F2, T1, and T2 roots, soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.7$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8 .	285
5.44	Flap-lag-torsion characteristic multipliers, F1, F2, T1, and T2 roots, soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.7$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$, $\mu = 0.8$ to 1.58 .	286
5.45	Flap-lag-torsion characteristic multipliers, L1 root, soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.7$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8 .	287
5.46	Flap-lag-torsion characteristic multipliers, L1 root, soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.7$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$, $\mu = 0.8$ to 1.58 .	288
5.47	Flap-lag-torsion characteristic multipliers, L2 root, soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.7$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8 .	289
5.48	Flap-lag-torsion characteristic multipliers, L2 root, soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.7$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$, $\mu = 0.8$ to 1.58 .	290
5.49	Flap-lag-torsion root locus, soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.7$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8 .	291
5.50	Flap-lag-torsion root locus, soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.7$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$, $\mu = 0.8$ to 1.58 .	292
5.51	Flap-lag-torsion root locus, F1 root, soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.7$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8 .	293
5.52	Flap-lag-torsion root locus, F2 root, soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.7$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8 .	294

5.53	Flap-lag-torsion root locus, T1 root, soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.7$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8 .	295
5.54	Flap-lag-torsion root locus, T2 root, soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.7$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8 .	296
5.55	Flap-lag-torsion root locus, L1 root, soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.7$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8 .	297
5.56	Flap-lag-torsion root locus, L2 root, soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.7$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8 .	298
5.57	Flap-lag-torsion root locus, F1, L1, and T1 roots, soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.7$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$, $\mu = 0.8$ to 1.58 .	299
5.58	Flap-lag-torsion root locus, F2, L2, and T2 roots, soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.7$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$, $\mu = 0.8$ to 1.58 .	300
5.59	Flap-lag-torsion damping, trim, blade stall, and numerical stability, soft-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 0.7$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $C(k) = 1$.	301
5.60	Flap-lag-torsion characteristic multipliers, soft-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 0.7$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8 .	302
5.61	Flap-lag-torsion characteristic multipliers, soft-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 0.7$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $C(k) = 1$, $\mu = 0.8$ to 1.6 .	303
5.62	Flap-lag-torsion characteristic multipliers, F1, F2, T1, and T2 roots, soft-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 0.7$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8 .	304
5.63	Flap-lag-torsion characteristic multipliers, F1, F2, T1, and T2 roots, soft-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 0.7$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $C(k) = 1$, $\mu = 0.8$ to 1.6 .	305
5.64	Flap-lag-torsion characteristic multipliers, L1 root, soft-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 0.7$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8 .	306
5.65	Flap-lag-torsion characteristic multipliers, L1 root, soft-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 0.7$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $C(k) = 1$, $\mu = 0.8$ to 1.6 .	307
5.66	Flap-lag-torsion characteristic multipliers, L2 root, soft-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 0.7$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8 .	308
5.67	Flap-lag-torsion characteristic multipliers, L2 root, soft-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 0.7$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $C(k) = 1$, $\mu = 0.8$ to 1.6 .	309

5.68	Flap-lag-torsion root locus, soft-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 0.7$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8	310
5.69	Flap-lag-torsion root locus, soft-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 0.7$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $C(k) = 1$, $\mu = 0.8$ to 1.6 . . .	311
5.70	Flap-lag-torsion root locus, F1 and T1 roots, soft-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 0.7$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8	312
5.71	Flap-lag-torsion root locus, F2 and T2 roots, soft-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 0.7$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8	313
5.72	Flap-lag-torsion root locus, L1 and L2 roots, soft-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 0.7$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8	314
5.73	Flap-lag-torsion root locus, F1, L1, and T1 roots, soft-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 0.7$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $C(k) = 1$, $\mu = 0.8$ to 1.6	315
5.74	Flap-lag-torsion root locus, F2, L2, and T2 roots, soft-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 0.7$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $C(k) = 1$, $\mu = 0.8$ to 1.6	316
5.75	Flap-lag-torsion damping, trim, blade stall, and numerical stability, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $C(k) = 1$	317
5.76	Flap-lag-torsion characteristic multipliers, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8	318
5.77	Flap-lag-torsion characteristic multipliers, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $C(k) = 1$, $\mu = 0.8$ to 1.58	319
5.78	Flap-lag-torsion characteristic multipliers, F1, F2, T1, and T2 roots, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8	320
5.79	Flap-lag-torsion characteristic multipliers, F1, F2, T1, and T2 roots, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $C(k) = 1$, $\mu = 0.8$ to 1.58	321
5.80	Flap-lag-torsion characteristic multipliers, L1 root, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8	322
5.81	Flap-lag-torsion characteristic multipliers, L1 root, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $C(k) = 1$, $\mu = 0.8$ to 1.58	323
5.82	Flap-lag-torsion characteristic multipliers, L2 root, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8	324

5.83	Flap-lag-torsion characteristic multipliers, L2 root, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $C(k) = 1$, $\mu = 0.8$ to 1.58	325
5.84	Flap-lag-torsion root locus, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8	326
5.85	Flap-lag-torsion root locus, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $C(k) = 1$, $\mu = 0.8$ to 1.58 . . .	327
5.86	Flap-lag-torsion root locus, F1 root, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8	328
5.87	Flap-lag-torsion root locus, F2 root, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8	329
5.88	Flap-lag-torsion root locus, L1 root, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8	330
5.89	Flap-lag-torsion root locus, L2 root, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8	331
5.90	Flap-lag-torsion root locus, T1 root, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8	332
5.91	Flap-lag-torsion root locus, T2 root, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8	333
5.92	Flap-lag-torsion root locus, F1, L1, and T1 roots, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $C(k) = 1$, $C(k) = 1$, $\mu = 0.8$ to 1.58	334
5.93	Flap-lag-torsion root locus, F2, L2, and T2 roots, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $C(k) = 1$, $\mu = 0.8$ to 1.58	335
A.1	Trim comparison with NASA high- μ articulated rotor tests, blade loading vs. collective, $\mu = 1.45$, $\alpha_s = 0$	342
A.2	Trim comparison with NASA high- μ articulated rotor tests, lateral cyclic vs. collective, $\mu = 1.45$, $\alpha_s = 0$	343
A.3	Trim comparison with NASA high- μ articulated rotor tests, longitudinal cyclic vs. collective, $\mu = 1.45$, $\alpha_s = 0$	344
A.4	Trim omparison with UMD high- μ articulated rotor tests, blade loading vs. collective, $\mu = 0.25$	345
A.5	Trim omparison with UMD high- μ articulated rotor tests, blade loading vs. collective. $\mu = 0.41$	346
A.6	Trim omparison with UMD high- μ articulated rotor tests, blade loading vs. collective, $\mu = 0.62$	347

A.7	Trim omparison with UMD high- μ articulated rotor tests, blade loading vs. collective, $\mu = 0.83$.	348
A.8	Trim omparison with UMD high- μ articulated rotor tests, blade loading vs. collective, $\mu = 1.03$.	349
A.9	Trim comparison with UMD high- μ articulated rotor tests, blade loading vs. collective, $\mu = 1.20$.	350
A.10	Trim comparison with UMD high- μ articulated rotor tests, lateral cyclic vs. collective, low μ .	351
A.11	Trim comparison with UMD high- μ articulated rotor tests, lateral cyclic vs. collective, high μ .	352
A.12	Trim comparison with UMD high- μ articulated rotor tests, longitudinal cyclic vs. collective, low μ .	353
A.13	Trim comparison with UMD high- μ articulated rotor tests, longitudinal cyclic vs. collective, high μ .	354
A.14	Trim comparison with UMD high- μ hingeless rotor tests, blade loading vs. collective.	355
A.15	Trim omparison with UMD high- μ hingeless rotor tests, lateral cyclic vs. collective.	356
A.16	Trim comparison with UMD high- μ hingeless rotor tests, longitudinal cyclic vs. collective.	357

Abbreviations and Acronyms

AC	Aerodynamic center
AOA	Angle of attack
BEMT	Blade element momentum theory
CAMRAD	Comprehensive Analytical Model of Rotorcraft Aerodynamics and Dynamics
C.G.	Center of gravity
EA	Elastic axis
F1, F2	Flap roots
FEM	Finite element method
FET	Finite element in time
F-L	Flap-lag analysis
F-T	Flap-torsion analysis
F-L-T	Flap-lag-torsion analysis
FFT	Fast Fourier transform
IC	Initial condition
L1, L2	Lag roots
LHS	Left hand side of equation
LTI	Linear Time Invariant
LTP	Linear Time Periodic
MATLAB	Matrix Laboratory (programming language/ computing environment)
NACA	National Advisory Committee for Aeronautics
NASA	National Aeronautics and Space Administration
ODE	Ordinary differential equation
PA	Pitch axis
QR	QR algorithm (eigenvalue calculation method)
RCAS	Rotorcraft Comprehensive Analysis System
rev	Rotor revolution (e.g. n per rev response)
RHS	Right hand side of equation
RK4	Runge-Kutta method, fourth order
T1, T2	Torsion roots
UMARC	University of Maryland Advanced Rotorcraft Code
UMD	University of Maryland, College Park

Notation and Symbols

A	Rotor disk area (πR^2), m ²
$\underline{\underline{A}}$	Coefficient matrix
a	Lift curve slope, $c_{l\alpha}$, per rad; Mathieu equation parameter
$\underline{b}$	Forcing vector
C	Damping matrix
C'	Loewy's lift deficiency function; Pitt-Peters dynamic inflow lift deficiency
$C(k)$	Theodorsen's lift deficiency function
C_T	Thrust coefficient
c	Rotor blade chord, m
c_d	Sectional drag coefficient
c_l	Sectional lift coefficient
c_m	Sectional pitching moment coefficient
$c_{m\alpha}$	Pitching moment slope, per rad
F	Non-dimensional aerodynamic force
f	Equivalent flat plate drag area for propulsive trim, m^2
function	Numeric computing function, e.g. <code>atan2</code> , <code>eig</code> , <code>fft</code> , <code>ode</code>
$\underline{\underline{I}}$	Identity matrix
$\underline{\underline{I}}_n$	$n \times n$ identity matrix
I_b	Blade moment of inertia about hub
I_f	Blade moment of inertia about feathering axis
I_x	Blade moment of inertia due to chordwise C.G. offset from feathering axis
i	Unit imaginary number
Im	Imaginary part of complex number
K	Stiffness matrix
k	Reduced frequency; linearization type
$k_{p\beta}$	Pitch-flap coupling, $\tan(\delta_3)$
$k_{p\zeta}$	Pitch-lag coupling, $\tan(\alpha_1)$
M	Aerodynamic flap moment; Mass Matrix
L_1	Lift at 1/4-chord (circulatory), unsteady thin airfoil theory
L_2	Lift at 1/2-chord (non-circulatory), unsteady thin airfoil theory
L_3	Lift at 3/4-chord (non-circulatory), unsteady thin airfoil theory
m	Integer (modal) addition to principal frequency, per rev
N	Integration steps; samples (per rev)
N_b	Number of blades
n	Harmonic or root number
Q	Aerodynamic lag moment
$\underline{\underline{Q}}$	Modal waveform matrix

$\underline{q}$	State vector; Mathieu equation parameter
R	Blade radius, m
Re	Real part of complex number
R_s	Structural coupling parameter
r	Blade spanwise position, m
s_n	Characteristic exponent for root n (eigenvalue in s -plane)
s_{np}	Principal value of characteristic exponent
T	Aerodynamic torsion moment; period, rad
t	Time, s
U	Local velocity of blade section, m/s
u	Non-dimensional velocity, $U/(\Omega R)$
V	True airspeed of aircraft, m/s; eigenvector at $\psi = 0$ or T
V_{eff}	Effective velocity (mass-flow parameter for dynamic inflow)
x	Non-dimensional blade spanwise location, r/R ; Mathieu equation independent variable
x_a	Chordwise pitch axis location, m
x_c	Root cutout, r_c/R
x_I	Chordwise center of mass location, m
y	Mathieu equation dependent variable
Z	Flap-lag structural stiffness parameter
z_n	Characteristic multiplier for root n (eigenvalue in z -plane)
$\mathbb{Z}$	Set of integers
α	Blade section angle of attack, rad
α_i	Rotor disk incidence angle, $\alpha_s + \beta_{1c}$, rad
α_{max}	2D blade stall angle, rad
α_s	Longitudinal (forward) rotor shaft tilt, rad
α_1	Pitch-lag coupling hinge angle, rad
β	Flap angle, rad
β_T	Trimmed flap angle, rad
γ	Lock number, $\rho a c R^4 / I_b$
$\delta()$	Perturbation term
δ_3	Pitch-flap coupling hinge angle, rad
ζ	Lag angle, rad
ζ_β	Flap damping ratio
ζ_ζ	Lag damping ratio
ζ_θ	Torsion damping ratio
ζ_L	Structural lag damping ratio
ζ_T	Structural torsion damping ratio
η_m	Modal participation factor for mode m
θ	Blade pitch or torsion angle, rad
θ_C	Blade pitch control Input, rad
θ_s	Blade pitch at pitch bearing, rad
θ_{tw}	Blade twist angle, rad/span
θ_T	Trimmed blade pitch, rad

Λ	Yawed flow angle (rad)
λ	Inflow ratio
λ_i	Induced inflow ratio
μ	Advance ratio, $V \cos \alpha_i / (\Omega R)$
μ_z	Axial velocity ratio, $V \sin \alpha_i / (\Omega R)$
$\nu_{()}$	Non-dimensional rotating natural frequency, $\omega_{()} / \Omega$, per rev
$\nu_{()0}$	Non-dimensional non-rotating natural frequency, $\omega_{()0} / \Omega$, per rev
ρ	Air Density, kg/m ³
σ	Rotor solidity
σ_Q	Numerical dissipation of modal waveform, $\underline{\underline{Q}}(T \rightarrow 2T)$
ϕ_Q	Numerical phase error of modal waveform, $\underline{\underline{Q}}(T \rightarrow 2T)$
$\underline{\underline{\Phi}}$	State transition matrix
ψ	Blade azimuth angle, Ωt , rad
ω	Frequency (angular velocity), rad/s
Ω	Rotor speed (angular velocity), rad/s
$\in$	Element of
$()_f$	Forcing term
$()_{me}$	Complex Fourier coefficient for $e^{im\psi}$
$()_{nc}$	Fourier coefficient for $\cos(n\psi)$
$()_{ns}$	Fourier coefficient for $\sin(n\psi)$
$()_P$	Perpendicular component
$()_R$	Radial component
$()_T$	Tangential component
$()_\beta$	Flap term
$()_\zeta$	Lag term
$()_\theta$	Torsion term
$()_{\theta_{tw}}$	Blade twist term
$()_\lambda$	Inflow term
$()^*$	For β, ζ, θ : $()^* = \frac{\partial}{\partial \psi}() = \frac{1}{\Omega} \frac{\partial}{\partial t}() = \frac{1}{\Omega} \dot{()}$; for moments of inertia: $I_0^* = I_0 / I_b$
$\bar{()}$	Mean or normalized value
$\hat{()}$	Value in reverse flow region
$\vec{()}$	Vector notation
$\underline{()}$	Matrix notation
$()^\top$	Transpose
$[\diagdown()_n \diagdown]$	Diagonal matrix

Chapter 1: Introduction

1.1 The Challenge of High Advance Ratio Flight

The concept of an aircraft that can hover efficiently and fly fast has been a dream since the dawn of aviation. The most successful approach has been the helicopter. At zero forward speed, the rotor's airflow is axial, much like a steerable propeller oriented skyward. Its large area and low induced velocity requires less power than higher disk loading approaches. But in forward flight the flow becomes partially edgewise, moving transversely across the disk, causing the rotor blades to encounter varying flow in the rotating reference frame.

Factors limiting the maximum speed of the conventional helicopter rotor include compressibility effects on the advancing blades as they approach the speed of sound and the stall of the retreating blades. While these are significant issues, aeroelastic instability caused by the periodically interacting aerodynamics and structural properties of the rotor is another limiting factor. This instability is in part dependent on the ratio between the airspeed and blade tip speed relative to the rotor hub. This non-dimensional parameter is

known as the advance ratio, μ :

$$\text{Advance Ratio} = \mu = \frac{\text{True Airspeed}}{\text{Rotor Tip Speed}} = \frac{V}{\Omega R} \quad (1.1)$$

where Ω is the rotational speed of the rotor and R is its radius. This definition is only strictly true for the edgewise component of V ; forward or aft incidence of the rotor disk introduces a cosine term in the numerator. Because they are designed for axial flight, propellers use a different (perpendicular) definition for advance ratio.

Flapping is the up and down motion of a rotor blade, denoted by the angle β , which affects the pitch and roll of the aircraft. In normal flow, the influence of μ on blade flapping is linear. As advance ratio increases, a circle of reverse flow forms on the retreating side of the rotor disk, and in this region the effect of advance ratio on flapping is of order μ^4 . This indicates that large deviations in blade motion can be caused by small perturbations of this parameter.

Tilt rotors avoid the challenges of high- μ flight by transitioning to axial flow at high speeds, but come with their own set of complex engineering challenges and compromises. Thus, the compound helicopter design is appealing due to its higher hover efficiency and potentially greater mechanical simplicity (no need to rotate nacelles or rotors). Lift and propulsive forces are partially offloaded from the main rotor through the employment of wings and propellers or jets. Some of the most successful early rotorcraft were autogyros, which are compound designs with rotors powered through autorotation. As for conventional helicopters, blades can be structurally and aerodynamically improved to address Mach effects, but the maximum speed is effectively limited to about 200 knots, with increasing

performance and stability penalties. This can be exceeded by compounding lift and thrust and slowing the rotor, which increases advance ratio. The goal of this work is to examine the effects of increasing μ on blade stability.

Some historical high- μ compound single main rotor designs are illustrated in Fig. 1.1 and are discussed below. The aircraft images are sourced from Refs. 1, 2, 3, 4, and 5.

In the mid 1950s, the McDonnell XV-1 had limited success at high speed, but demonstrated high- μ capabilities. In part due to immature turboshaft engine technology, the designers elected to use rotor tip jets, which were very noisy and had a high fuel consumption. Thrust was provided by jet engines and the lift was augmented with wings. The rotor was an innovative gimbaled design with three blades, which were rigidly connected to the hub and tilted (flapped) together. The rotor was tested to an advance ratio of 0.95 in flight and exceeded 1 in the wind tunnel. One stability issue to come out of these experiments was what was termed the “1/2-p” oscillation. This was a rotor flapping response with a frequency equal to half of the rotor’s rotational speed, or “1/2 per rev” (Refs. 6, 7).

In the 1960s, tests of the Sikorsky NH-3A, a compound modification of a Sea King, also encountered 1/2-p self excitation at around $\mu = 0.5$. Like many classic and modern helicopters, the H-3 has articulated rotor blades, which are hinged near the hub and flap independently with a natural frequency at or just above 1 per revolution. Other issues were torsional excitations due to shifting pitch moments as Mach numbers exceeded 0.9 (Ref. 2).

Lockheed elected to design two of its high-speed compounds (the XH-51 and AH-56A Cheyenne) with novel hingeless rotors. The term hingeless is a broad term that refers to various combinations of physical hinges or bearings and blade flexure mechanisms. The AH-



McDonnell XV-1
 $\mu = 0.95$ (1955)



Sikorsky NH-3A
 $\mu = 0.53$ (1968)



Lockheed XH-51
 $\mu = 0.55$ (1967)



Lockheed AH-56A
 $\mu = 0.57$ (1968)



CarterCopter
 $\mu = 1.02$ (2005)



Carter PAV
 $\mu = 1.16$ (2013)



Eurocopter X³
 $\mu = 0.68$ (2013)

Figure 1.1: Examples of high advance ratio compound rotorcraft.

56A incorporated a “door-hinge” mechanism for flapping (natural frequency of $> 1.1/\text{rev}$ with a unique gyro control) but was very stiff in lag ($1.4/\text{rev}$). Advantages of hingeless designs include increased control power, reduced drag, and anticipated lower maintenance. The stability differences of hingeless blades compared to articulated blades of the same dimensions can be approximated by considering their differing natural frequencies and Lock numbers.

Unlike the previous experimental rotorcraft, the AH-56A was designed to be a combat aircraft and its flight tests were geared for meeting operational requirements and specifications. There were many setbacks in the program, but two notable stability problems were caused by resonance. The first was the “1P X 2P” excitation, encountered during a reduced rotor RPM (70%) autorotational landing. To avoid resonance, aircraft designers normally ensure that the natural frequencies of the rotor (as well as tail rotor, propeller, and fuselage modes) are not harmonic multiples of one another. However, while briefly operating at this off-design point rotor speed, the in-plane (or lag) natural frequency was exactly twice the rotor speed, and this cyclic-lag resonance caused the aircraft to rollover. The second problem was known as “half-P hop”. This was the same phenomenon as the previously mentioned “1/2-p” high- μ parametric excitations. Unfortunately this caused inadvertent collective actuation, resulting in a fatal mishap that contributed to the Cheyenne’s eventual cancellation (Ref. 3).

More recently, the first manned aircraft to exceed $\mu = 1$ in flight was the lightweight CarterCopter in 2005. The aircraft was destroyed shortly afterwards due to an unrelated engine failure. The recent Carter PAV is an improved design which has since raised the record to $\mu = 1.16$. Both of these aircraft employed a two-bladed teetering rotor (Ref. 4).

The Eurocopter (now Airbus Helicopters) X³ was a dual-dual propeller high-speed compound test aircraft that has paved the way for the RACER program now in development. 0.5/rev resonance was also found during its flights. Their engineers have published flap-lag stability data from the X³ and are leveraging this for their new design (Ref. 5).

Although these historical examples have all been single main rotor aircraft in edgewise flight, it should be noted that other rotor configurations can be suitable for high speed. For example, Sikorsky has long worked on coaxial designs: the XH-59, X2, S-97, Raider X, and the now-cancelled collaboration with Boeing, the SB>1 Defiant.

1.2 Floquet Theory and Stability Analysis

In 1868, Mathieu proposed a second order ordinary differential equation with periodic coefficients to model the vibrations of a drum (Ref. 8). This function is frequently expressed in the form of

$$\frac{d^2y}{dx^2} + \left(a - 2q \cos(2x)\right)y = 0 \quad (1.2)$$

Because the simplified form of the rotor flapping equation in forward flight is a Mathieu function, the methods used to solve this problem are of interest to the present work.

Floquet published a form for the solutions to this equation based on its period (π) and parameters (a, q) in Ref. 9. Several years later, Hill developed a broader form of differential equation, of which the Mathieu equation is a special type, to model the harmonic behavior of the moon (Ref. 10). Approaching the end of the 19th century, in Ref. 11 Lyapunov developed a theory of stability for this T -periodic system: through a state transition, there exists a matrix $\Phi(T)$ whose eigenvalues determine the response of the system. Today,

Floquet theory and the contributions of Lyapunov are still foundational to the study of the behavior of periodic systems. The development of the computer and numerical integration, which is needed to find the state transition matrix in practice, made Floquet theory more relevant in the modern age of dynamical systems analysis.

Modern references with good backgrounds on Floquet theory include Refs. [12](#), [13](#), [14](#), [15](#), and [16](#). Implementations specific to the rotor blade stability problem include Refs. [17](#) and [18](#); many others are discussed in the following overview of aeroelastic stability analyses.

For a constant coefficient ODE, a Fourier transform simplifies the problem into an algebraic one. The eigenvalues (poles) lie in the complex s -plane, where the real part corresponds to damping (a negative sign indicates stability) and the imaginary part is the damped frequency. As for the solutions found from Floquet theory, there are several important differences. The aforementioned eigenvalues of $\Phi(T)$, known as the characteristic multipliers, exist in the complex z -plane, where stability lies in the unit circle and the frequency is represented by the angle (argument). When transforming these eigenvalues into the s -plane, the solutions (characteristic exponents) are not distinct due to the complex logarithm involved. While the damping is known, only the principal value of the frequency is immediately apparent, and it is limited to the set $[-\pi, \pi]$ ($[-1/2, 1/2]/\text{rev}$ if $T = 2\pi$). To determine the overall frequency in the s -plane, m/rev integer(s) must be found and added to the principal frequency. The Floquet eigenvectors are periodic, and an analysis of their motion yields the participation of each frequency multiple m . For the time-invariant or constant coefficient case, there is a unique integer m for each root, and thus a unique overall frequency response. The introduction of periodic terms leads to multiple integers with different modal participation factors.

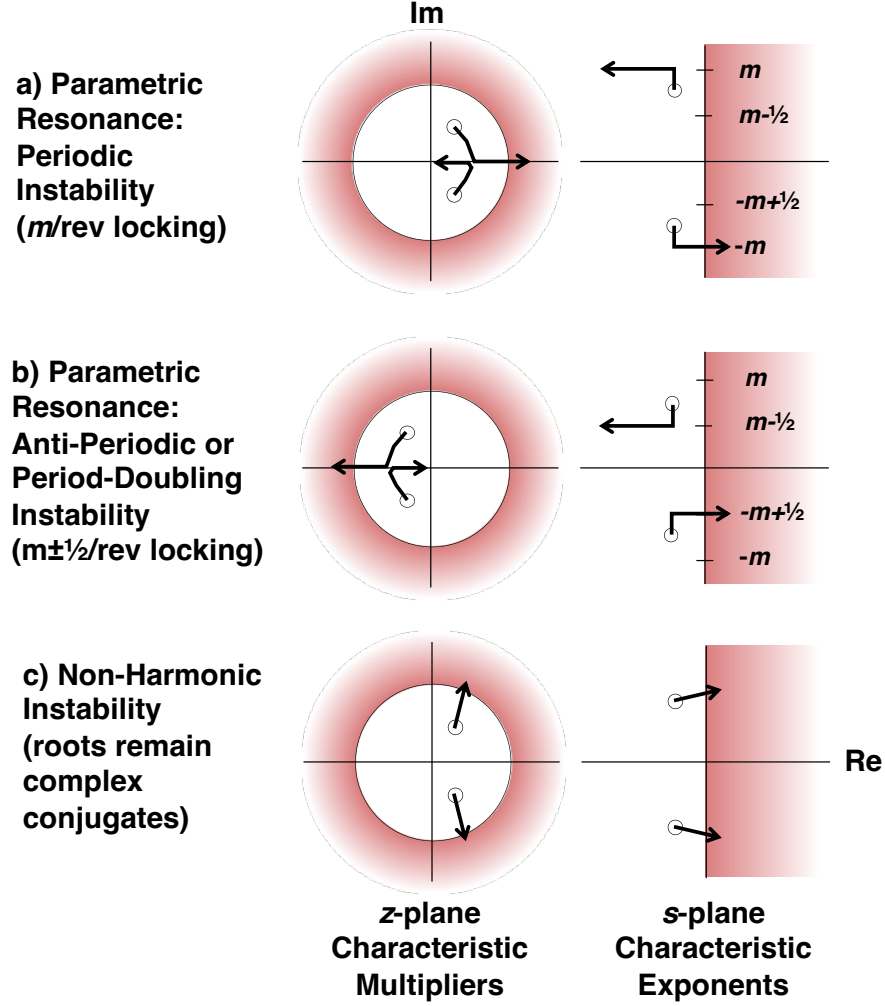


Figure 1.2: Types of root instability behaviors in the z-plane and s-plane.

The often chaotic behavior of eigenvalues when parameters (such as μ) are varied can be interpreted using bifurcation theory. There are several ways that a periodic system can become unstable as shown in Fig. 1.2, which combines depictions from Refs. 19 and 20. Cases (a) and (b) in Fig. 1.2 show parametric resonances ultimately leading to instability. As a notional parameter is increased, case (a) begins with complex conjugate roots and transitions to an integer ($\mathbb{Z}$) per revolution response with a period of T . Case (b) becomes a half integer ($\mathbb{Z} + 1/2/\text{rev}$) response with period $2T$. These are also referred to as Hill-type

instabilities and demonstrate that the roots of a periodic system do not necessarily exist in complex conjugate pairs. One root loses stability and the other becomes more damped in these cases. The response type of Case (b) has been variously described as anti-periodic, period doubling, flip/ subharmonic bifurcation, and as the aforementioned 1/2-P instability observed in high- μ helicopter tests. Case (c) appears like the more “conventional” instability that might be encountered in a constant coefficient system, where the roots remain complex conjugates and are not locked onto harmonics. In Ref. 13, Yakubovich offers this explanation: compared to ordinary resonance, parametric resonance is more “dangerous.” Discrete natural frequencies drive normal resonance, which has a power or linear growth rate, whereas parametric resonance has critical frequencies varying based on the perturbations and increases exponentially. The term frequency locking will also be used throughout this work in reference to the parametric resonance phenomenon.

Applying these principles to the Mathieu equation, a stability map can be created by varying the parameters a and q and plotting the damping and frequency. This map is shown in Fig. 1.3. The stability is symmetric for corresponding positive and negative values of a . The stable regions have very low levels of damping and do not have any parametric resonance. The unstable parameter combinations result in alternating lobes of integer and half-integer frequency locked regions. Similar types of stability maps will be generated for the rotor equations in this work.

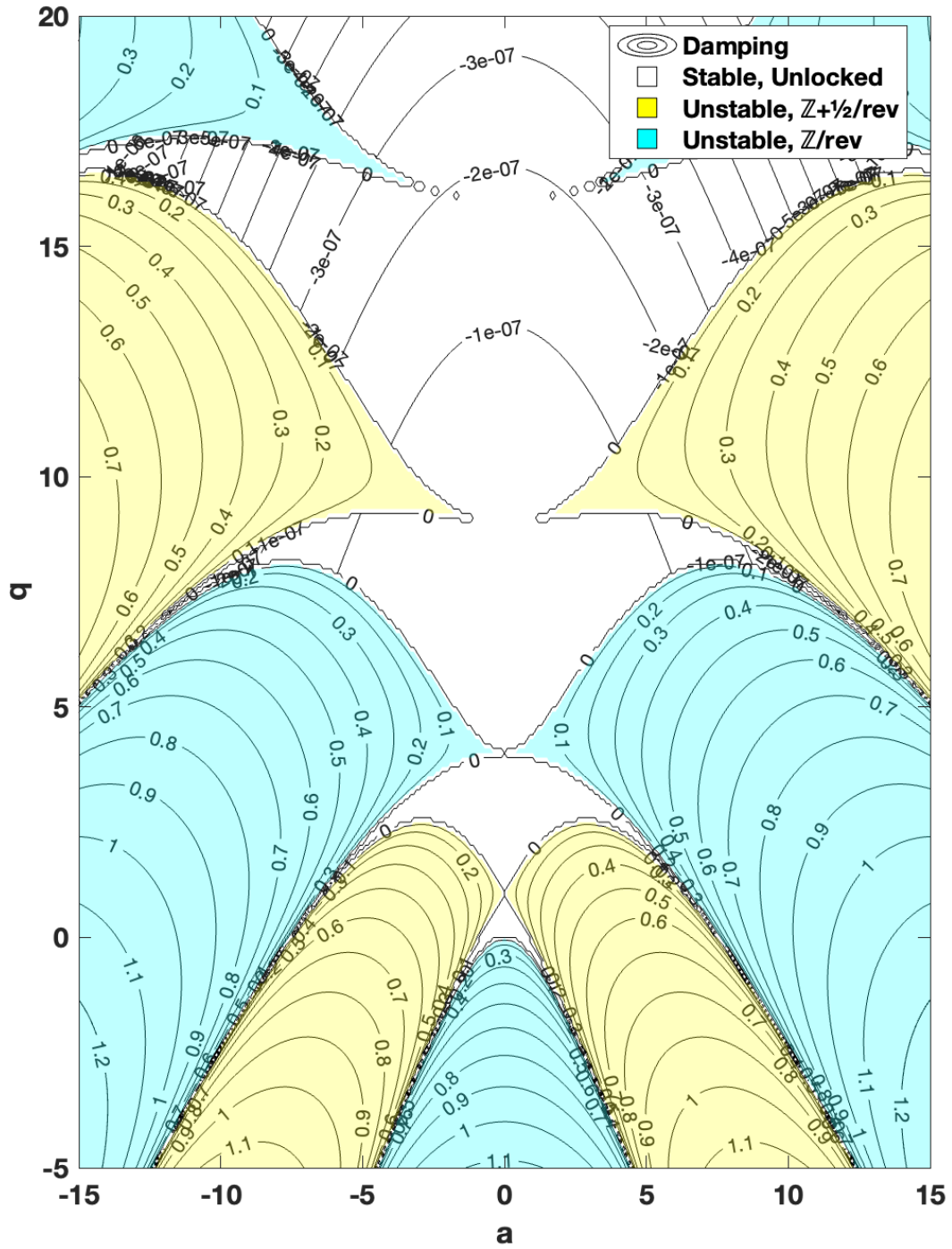


Figure 1.3: Stability map of Mathieu function.

1.3 Rotorcraft High- μ and Aeroelastic Stability Research

Blade element momentum theory has its origins in the 19th century, motivated by research into turbines and ship propellers (Ref. 21). Rankine began work on momentum theory, which considers the power and thrust for axial flow through a disk (Ref. 22). In Ref. 23), Froude introduced the concepts of Blade Element Theory, which provides for the determination the forces at spanwise stations. Modern BEMT was developed by C.N.H. Lock in the 1920s, notably in Ref 24. The Lock number γ , a parameter which is foundational to rotorcraft aerodynamics, was named after him. It is a non-dimensional ratio between aerodynamic and inertial forces on a rotor blade. Ref. 25 provides some common Lock numbers for main rotors and tail rotors. At sea level on a standard day, values can range from about 1.0 for a Hughes 300C tail rotor to 11.5 for main rotor blades of the the CH-53E. Articulated rotors generally have Lock numbers around 8, and hingeless or rigid (including teetering) rotors are closer to 5. At the 3,500 ft on a standard day, Lock Numbers decrease by 10% due to the drop in air density. I n a vacuum, the Lock number is zero.

Among his many contributions to aerodynamics, Glauert improved rotor BEMT by showing how forward flight affects inflow compared to hover (Ref. 26). He also performed an early flap stability investigation in 1933 (Ref. 27). After World War II, Horvay introduced the concept of a γ vs. μ stability map (Ref. 28). Horvay was also aware of the subharmonic 1/2-p phenomenon.

Just as in the case of propellers, in the early days of rotorcraft it was understood that the inflow and lift distribution of rotor disks were not uniform, even for ideally twisted

rotors. The effects of tip vortices were recognized, but it was too difficult to model at the time. An early way to approach this was Glauert’s concept of tip loss (Ref. 29), which was developed from earlier work by Prandtl (Ref. 30). Avoiding discontinuities and accounting for the time-varying nature of skewed inflow due to advance ratio, Mangler proposed a smooth Fourier series function in Ref. 31.

Theodorsen explored aerodynamic instability, flutter mechanisms, and unsteady aerodynamics. In Ref. 32, published in 1935, he introduced the concept of a lift deficiency function. In 1957, Loewy expanded lift deficiency modeling for unsteady rotary wing aerodynamics (Ref. 33). This was two-dimensional and did not account for compressibility or viscous effects, but modeled returning wake (vortices) from previous revolutions and blades.

Several unique experiments were performed in the 1950s and 1960s that are still relevant for high-speed rotors. In Ref. 34, the aerodynamic coefficients of a NACA 0012 airfoil were evaluated at angles of attack from 0 to $\pm 180^\circ$. Historically, wind tunnel data for most airfoils are only available to just past stall ($\alpha < 25^\circ$). However, a substantial portion of a high- μ rotor is in reverse flow. The experiments showed that linear lift behavior is still somewhat present in reverse, albeit with a reduction in stall angle and higher drag. Because the center of lift has shifted from the 1/4-chord to the 3/4-chord position in reverse flow, the pitching moment linearly varies with angle of attack. At low Reynold’s numbers, the lift coefficient at 180° is steep and has hysteresis due to vortices forming at the sharp “leading” (normally trailing) edge. Linear approximations of the aerodynamic coefficients are overlaid on the test data in Fig. 1.4.

Jenkins led wind tunnel tests of an articulated rotor up to $\mu = 1.45$ (Ref. 35). Unique

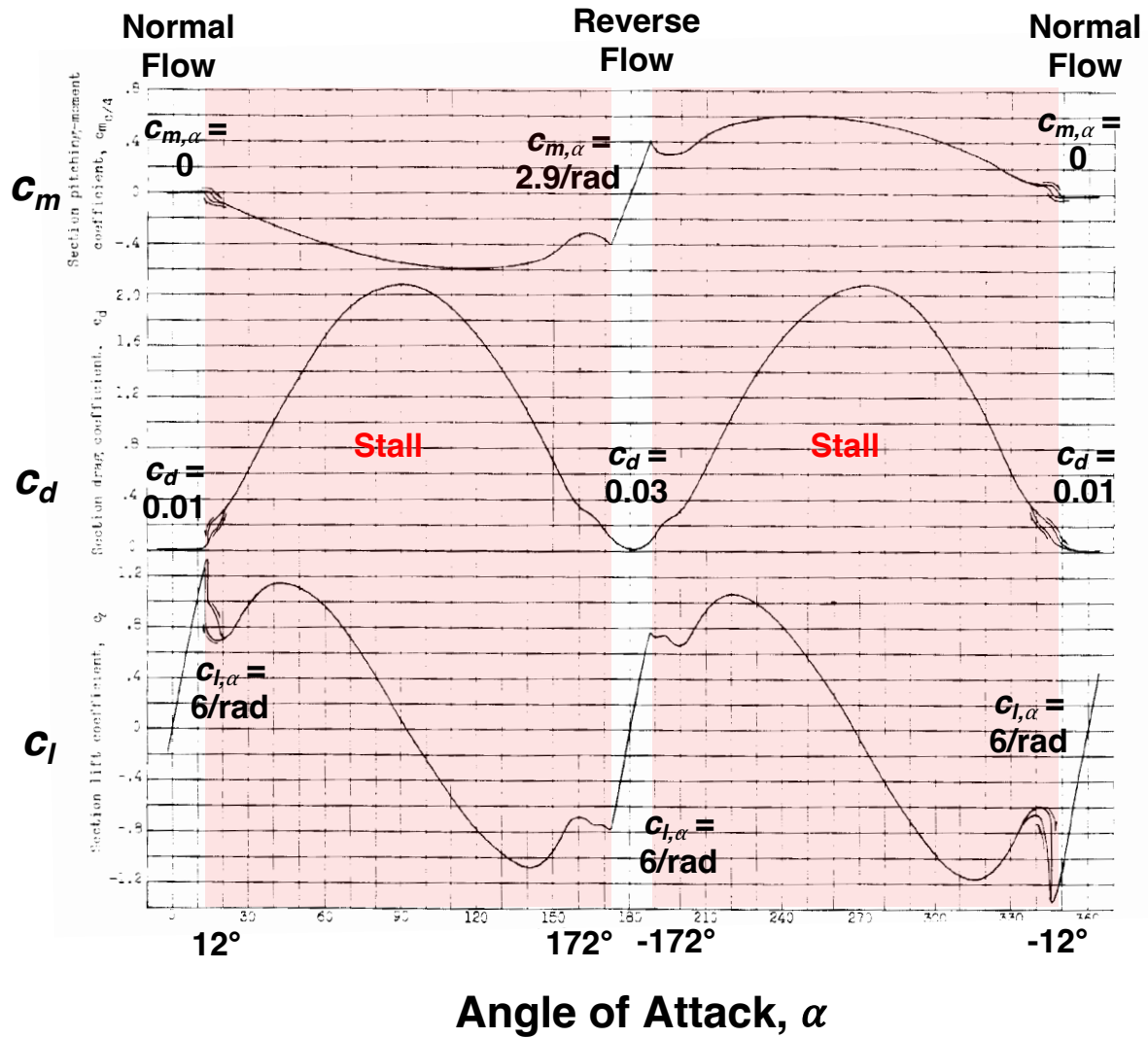


Figure 1.4: Linear aerodynamic coefficient approximations overlaid on wind tunnel data of a NACA 0012 airfoil in normal and reverse flow, $Re = 1.8 \times 10^6$.

stability tests of a stopped, full-scale two-bladed rotor were performed at the NASA Ames 40×80 ft wind tunnel in 1968 (Ref. 36).

Early rotor stability analyses were usually carried out using constant coefficient approximations and for $\mu < 0.5$. Spurred by an interest in high speed military helicopters like AH-56A, with the advancement of computers in the late 1960s, transient analysis (numerical perturbation of equations and evaluation of the response) allowed for the solutions of forward flight equations to be found, including nonlinear analysis. Operating through the U.S. Army and NASA, numerous publications explored rotor stability, including flap-lag-torsion (Refs. 37, 38, 39, 40, 41, 42, and 43). In Ref. 44, Sissingh modeled the reverse flow regions, accounted for the periodic flap moments, and correctly predicted the promise of teetering rotors for high- μ flight.

In the early 1970s, it was found that computers not only assisted with transient numerical analyses, but also were a practical way to apply Floquet theory for the evaluation of differential equations with periodic coefficients. High-speed flap stability, including the essential effect of reverse flow, was investigated in Ref. 45 up to $\mu = 2.5$ for a centrally hinged, rigid blade. Around the same time, Johnson found a closed form solution to the flap perturbation equation using asymptotic analysis in Ref. 46, but since μ must remain the small perturbation parameter (ϵ), this method is also only valid at low speeds. Similar Floquet analysis was performed in Ref. 47, although it was limited to $\mu \leq 0.5$.

A classical problem for rotorcraft is the difference between the rotating frame (blade reference) and fixed frame (aircraft reference). Flight controls in the fixed frame must manipulate the blade pitch change links using stationary and rotating swashplates, and blade forces and vibrations transfer back to the fuselage. Hohenemser completed the development

of the Multiblade Coordinate (MBC) system in Ref. 48, which uses Fourier transforms to move the forces from the rotor to the non-rotating frame. Without using the MBC during stability analysis of the flap equation evaluating constant coefficient equations, the stability roots will not change when μ is increased. Evaluating the constant coefficient equations in the fixed frame with MBC does provide a moderately accurate trend of root motion for low μ , but a separate equation must be solved for each blade. Fig. 1.5, adapted from Ref. 20, compares the MBC/ constant coefficient approach to Floquet for an articulated rotor.

The “corners” in Fig. 1.5 where the roots lock into a frequency of 0.5/rev are a real physical response: the aforementioned “1/2-p” parametric resonance encountered in historical high- μ tests. This was not captured using the constant coefficient approximation. A feature of Floquet theory is that the stability of independent, identical blades (articulated or hingeless rotors, $N_b > 3$) in the rotating frame can be transformed to the fixed frame by adding roots displaced vertically by $\pm i$. The collective (coning) roots are the same as the rotating frame root; the cyclic modes are shifted by $\pm 1/\text{rev}$. Note the the solutions for gimbaled rotors are different because the blade motions are not independent; MBC must be applied in this case.

An actuator-disk model accounting for wake effects was developed in Ref. 49. A perennial question is what needs to be included in an analytical model to provide reasonably accurate predictions for a real aircraft’s performance and stability. Ref. 50 compared actual periodic blade loading from an H-34 flown up to 194 kts. As long as stall conditions were avoided, the classical method of rigid blades, quasi-steady aerodynamics, and uniform inflow provided “good correlation” with the flight tests. Accurately modeling the chordwise center of gravity appeared important for torsion (blade pitch) deflections. Interestingly, the

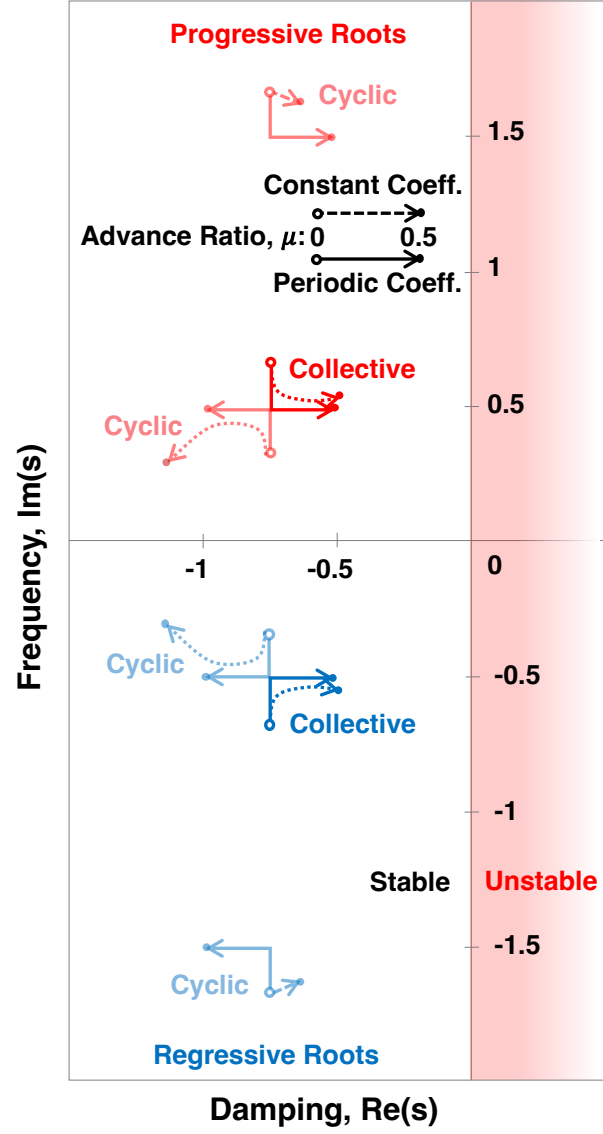


Figure 1.5: Flap root locus with constant and periodic coefficients; $\nu_\beta = 1/\text{rev}$, $\gamma = 12$, $0 \leq \mu \leq 0.5$, $N_b = 3$.

unsteady aerodynamics model under-predicted the actual performance, where the expected lift deficiency was possibly overcome by the opposite effect of “stall delay”.

The lag degree of freedom was included in Refs. 51, 52, 53, and 54, and refined multi-mode elastic blade models were used, but these analyses were limited to hover or moderate forward speed flights. The results of Ref. 53 included reverse flow, which is especially important for high- μ analysis. Hingeless and bearingless rotors were becoming more popular in the 1970s, and new methods of analysis (particularly accounting for bending modes) were explained in Refs. 55 and 56.

Floquet theory also provides an elegant solution for finding the trimmed state (sustained periodic motion) of a rotor. Peters explained this method, called periodic shooting, in Ref. 57. Ref. 58 explored an active rotor control system also using a Floquet approach.

Johnson developed a more modern form of unsteady rotary wing aerodynamics in Ref. 59. The dynamic inflow models developed by Pitt and Peters in the early 1980s (Refs. 60, 61, and 62) were seminal contributions to this field. In Ref. 63, Gaonkar and Peters evaluated the effectiveness of these inflow models for hover and forward flight.

A limited number of flap-lag stability experiments in the wind tunnel were conducted during the 1980s (Refs. 64, 65, and 66). Hover flap-lag-torsion stability testing of hingeless rotors were performed in Ref. 67 (small scale) and Ref. 68 (full scale BO-105). A follow-on analysis predicted flap-lag-torsion loads and lag damping up to $\mu = 0.35$ in anticipation of future wind tunnel tests (Ref. 69).

Early flap-lag-torsion analyses include Refs. 70, 71, and 72. Chopra, along with Sivaneri and Panda, began to employ a finite element approach in Refs. 73, 74, and 75 that laid the groundwork for what would become the UMARC comprehensive code.

Surveys of aeroelastic stability research include those completed by Friedmann (Refs. 76, 77), Ormiston (Ref. 78), and Chopra (Ref. 79). Similar to this present work, Ref. 80 was a rare example of using symbolic manipulation software to develop equations of motion.

Modern rotorcraft analysis is dominated by comprehensive codes, such as UMARC, CAMRAD, and RCAS, which are constantly being improved (Ref. 81). The University of Maryland's Advanced Rotorcraft Code (UMARC) uses finite elements in space and time (FEM/FET) and iterates until trim convergence. It can model elastic blades with multiple degrees of freedom and non-linear aerodynamics, along with dynamic stall and free-wake models such as those developed by Leishman and Beddoes (Ref. 82, Bagai and Leishman (Ref. 83), and Bhagwat and Leishman (Ref. 84).

As for stability analysis, as explained in Ref. 85, UMARC's methods are generally similar to other approaches: 1) Linear Time Invariant: Constant coefficient approximation, which requires using MBC to convert the equations into the fixed frame and average the coefficients over one revolution (suitable for $\mu < 0.25$). 2) Linear Time Periodic: Floquet solution with periodic coefficients, with roots found from the eigenvalues. 3) Non-Linear: Transient analysis; numerical perturbation followed by Moving-Block analysis of the response to determine damping (see Ref. 86).

As noted in Ref. 87, Milgram and Chopra employed a method of analyzing the periodic eigenvector response and choosing an integer multiple that best represented the harmonic behavior in Ref. 88. They studied the air resonance of full aircraft using FET/UMARC and Floquet analysis up to $\mu = 0.4$.

In Ref. 89, Floros and Johnson revisited the high- μ flap mode stability problem using Floquet theory, also addressing teetering and gimbaled rotors. Stability maps (γ vs.

μ plots) were specifically developed for the CarterCopter, and CAMRAD II was used to perform a comprehensive analysis of its rotor and wing. The initial conference paper was presented the year before the aircraft exceeded $\mu = 1$.

Although not stability tests per se, there have been notable recent high- μ wind tunnel experiments investigating trim, performance, and loads. In Ref. 90, Datta et al. evaluated a full-scale H-60 rotor slowed to achieve $\mu = 1$ at the NASA Ames 40×80 ft wind tunnel. The University of Maryland performed experiments in the Glenn L. Martin Wind Tunnel up to $\mu = 1.2$ for an articulated rotor (Ref. 91) and $\mu = 0.7$ for a hingeless rotor (Ref. 92).

Due to longstanding misconceptions about the frequencies of the characteristic exponents and how integer multiples are selected, Peters et al. revisited the characteristics of periodic Floquet eigenvectors and the resultant modal waveform matrix in Ref. 87. Using Mathieu's equation and the flap equation (without reverse flow) as examples, frequency branching behaviors and modal participations were shown. This general approach to harmonic participation is utilized in the present work, albeit with more complex coupled equations and parametric combinations.

There are alternatives to Floquet and averaging approaches for solving periodic systems. Ref. 93 is a recent review of Linear Time Invariant (LTI) and Linear Time Periodic (LTP) systems applied to rotorcraft. Ref. 94 show how LTPs can be linearized and transformed into LTIs using harmonic decomposition. Ref. 95 presents a numerical simulation method which models nonlinear systems using higher order LTI approximations, the results of which indicate computational advantages. Nevertheless, Floquet theory remains a very suitable method for this work.

Mohan and Gaonkar performed an assessment that showed how fast Floquet theory

and the Fourier analysis of the periodic eigenvector was suitable for hundreds of states (Ref. 96). In fact, the authors bemoaned that the state of art comprehensive codes like RCAS still did not include the robust mode identification capabilities features of Floquet, and proposed that it should be applied more routinely to rotor stability analyses.

Some content of this Dissertation builds upon two conference papers, Refs. 97 and 98, which were presented by the author at Vertical Flight Society Forums 79 and 80.

1.4 General Approach and Novel Contributions of this Work

Flap-lag-torsion stability equations for centrally-hinged rotors as applicable to high advance ratios are derived, stability roots are found after calculating Floquet transition matrices, and various rotor configurations are evaluated. The linearized equations with periodic coefficients are derived for a centrally restrained blade modeled with hub springs, with reverse flow and dynamic inflow effects included. Unsteady pitching moments as well as pitch axis and center of mass offsets from aerodynamic center are considered. Flap trim is found using periodic shooting. Perturbation versions of the equations are derived and manipulated to determine the periodic stiffness and damping matrices. Fig. 1.6 shows the general parameters involved in the rotating frame.

Floquet theory is used to evaluate stability roots: the homogeneous versions of the equations are integrated around the rotor azimuth, and the eigenvalues are used to determine the frequencies and damping. Eigenvector motion analysis is performed to determine the modal participation. Root loci are calculated, with the real part determining the damping and the imaginary parts showing the frequency content. Numerous parametric studies

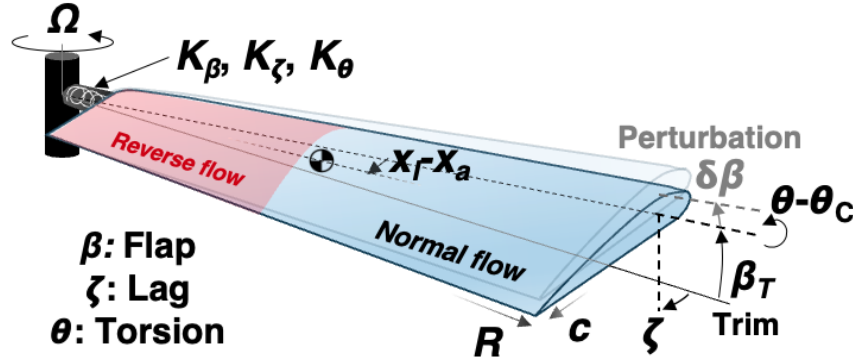


Figure 1.6: Rotor blade parameters.

are completed to show the effects of various configurations and conditions. The necessary calculations are performed using MATLAB.

This work has resulted in new and unique contributions to the fields of rotary wing aeroelastic stability and the study of periodic systems in general. These include:

1. An expansion of coupled flap-lag-torsion stability analysis to higher advance ratios ($\mu = 3$). Previous studies of lag and torsion were generally limited to $\mu < 0.5$ and neglected reverse flow.
2. Expanded derivations of the linearized flap-lag-torsion perturbation equations including terms applicable to high speed flight.
3. Creation of an accessible rotor stability analysis computer code that allows for rapid assessments of the influence of parameters and configuration comparisons.
4. Utilization of a first-principles, hybrid symbolic manipulation/ numerical integration method that bridges the gap between simple models and high-end comprehensive analysis.

5. A new eigenvalue trajectory analysis scheme has been developed. This algorithm automatically assists in identifying which roots and modes correspond to each degree of freedom. Previous analyses often included damping but not frequency information. This method is extensible to Floquet analyses of any system, not just rotorcraft aeroelastic stability.
6. New conclusions regarding parametric resonance, frequency branching and locking behavior for multiple coupled degrees of freedom.
7. Concurrent calculations of flight control positions, blade stall, and numerical stability allow for a confidence assessment that the linearized stability results are actually representing the physical system.

Overall, this research has provided a fundamental understanding and baseline results for expanding comprehensive stability analysis codes, followed by high-advance ratio stability testing in the wind tunnel, and the eventual design of high- μ aircraft.

Chapter 2: Flap Stability

2.1 Equations of Motion

Consider a centrally-hinged rigid rotor blade analyzed in the rotating reference frame with quasi-steady aerodynamics, uniform inflow, constant chord, and uniform blade mass distribution. The equation of motion for the flap angle β is

$$\ddot{\beta} + \nu_\beta^2 \beta = \gamma \bar{M}_\beta \quad (2.1)$$

where the derivatives are with respect to the blade azimuth angle $\psi = \Omega t$.

The natural frequency in this equation is the first rotating flap frequency normalized by the rotor speed:

$$\nu_\beta = \omega_\beta / \Omega \quad (2.2)$$

which is non-dimensional and expressed in “per rev” or “/rev”. It is related to the non-rotating natural frequency by the relation

$$\nu_\beta^2 = 1 + \frac{\omega_{\beta 0}^2}{\Omega^2} \quad (2.3)$$

While the rotating natural frequency may be presented as fixed for a given rotor in subse-

quent discussions, because it is normalized to the rotor speed, it will increase if the rotor is slowed from its reference value. This effect is more important for hingeless rotors, which have higher non-rotating frequencies ($\omega_{\beta 0}$), whereas articulated rotors have flap frequencies near 1.

On the right hand side (RHS) of the equation, the Lock number is defined as

$$\gamma = \frac{\rho a c R^4}{I_b} \quad (2.4)$$

where ρ is the air density, a is the lift curve slope, c is the blade chord, and R is the rotor radius. For a uniform blade, the moment of inertia about the hub is

$$I_b = \int_0^R m r^2 dr \quad (2.5)$$

where m is the mass per unit span and r is the spanwise position. The Lock number represents a ratio of the aerodynamic and inertial forces.

For the centrally-hinged rotor approximation, the flap and blade moments of inertia are the same ($I_\beta = I_b$), hence the coefficient for β^{**} is unity.

The non-dimensional aerodynamic forcing moment is

$$\bar{M}_\beta = \frac{M_\beta}{\rho a c \Omega^2 R^4} = \int_{x_c}^1 x \bar{F}_\beta dx \quad (2.6)$$

where $x = r/R$ and x_c is the non-dimensional root cutout.

The transverse and perpendicular non-dimensional velocities needed for the aerody-

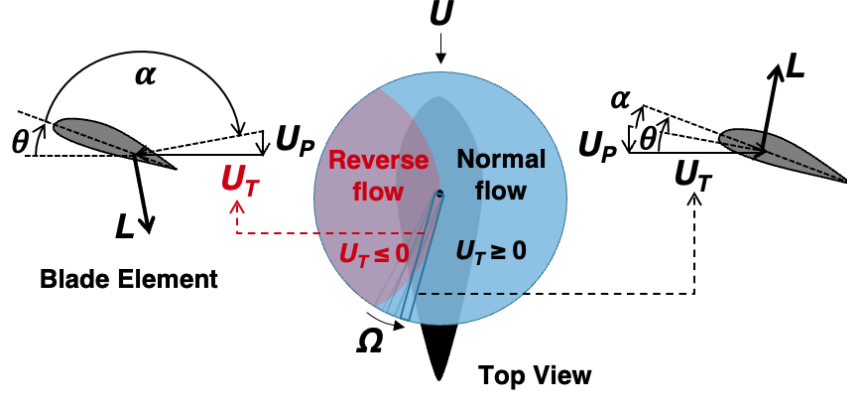


Figure 2.1: Blade elements in normal and reverse flow.

dynamic forces are

$$\begin{aligned} u_T &= x + \mu \sin(\psi) \\ u_P &= \lambda + x\beta^* + \beta\mu \cos(\psi) \end{aligned} \tag{2.7}$$

The radial velocity u_R will be addressed later when discussing flap trim and stall; for now its effect is neglected.

A reverse flow region with a diameter equal to the advance ratio, μ , is located on the retreating (left) side of the rotor disk. In hover, the entire rotor disk is in normal flow, but the reverse flow region takes up a large portion of the retreating side in high advance ratio flight. As a portion of the blade transitions through the narrow stalled boundary of the reverse flow region, u_T becomes negative. The process repeats as the blade element leaves the reverse flow region and u_T becomes positive again. Fig. 2.1 depicts the blade elements in the different flow regimes.

Following the derivation in Ref. 51, assuming small angles of attack and blade pitch,

the flapping force is

$$\bar{F}_\beta = \frac{1}{2} \left(u_T^2 \theta - \left(1 + \frac{c_d}{a} \right) u_P u_T \right) \quad (2.8)$$

which applies for the advancing side of the rotor disk ($0^\circ < \psi < 180^\circ$). In the reverse flow region,

$$\bar{\hat{F}}_\beta = -\frac{1}{2} \frac{\hat{a}}{a} \left(u_T^2 \theta + \left(1 + \frac{\hat{c}_d}{\hat{a}} \right) u_P u_T \right) \quad (2.9)$$

The negative sign results from the convention of re-baselining the angle of attack (AOA) from 0° in normal flow to 180° in reverse flow, and allows for the small angle assumption to remain valid.

If present, pitch-flap coupling will cause a difference between the control inputs and blade pitch:

$$\theta = \theta_C - k_{p\beta} \beta \quad (2.10)$$

This coupling can be defined by a “delta-3” hinge angle:

$$k_{p\beta} = \tan \delta_3 \quad (2.11)$$

The delta-3 hinge angle is zero or positive in most rotor designs for stability considerations. A notable exception is the case of gimbaled tilt rotors, which are designed for axial flow in forward flight.

As introduced in Ref. 44, applied to the Floquet approach in Ref. 45, and further explained in Ref. 20, the rotor disk can be divided into three azimuthal regions to address the periodic contributions of reverse flow. Fig. 2.2 displays how advance ratio, blade span position, and azimuth define the geometry of these flow regions. It is drawn to scale for

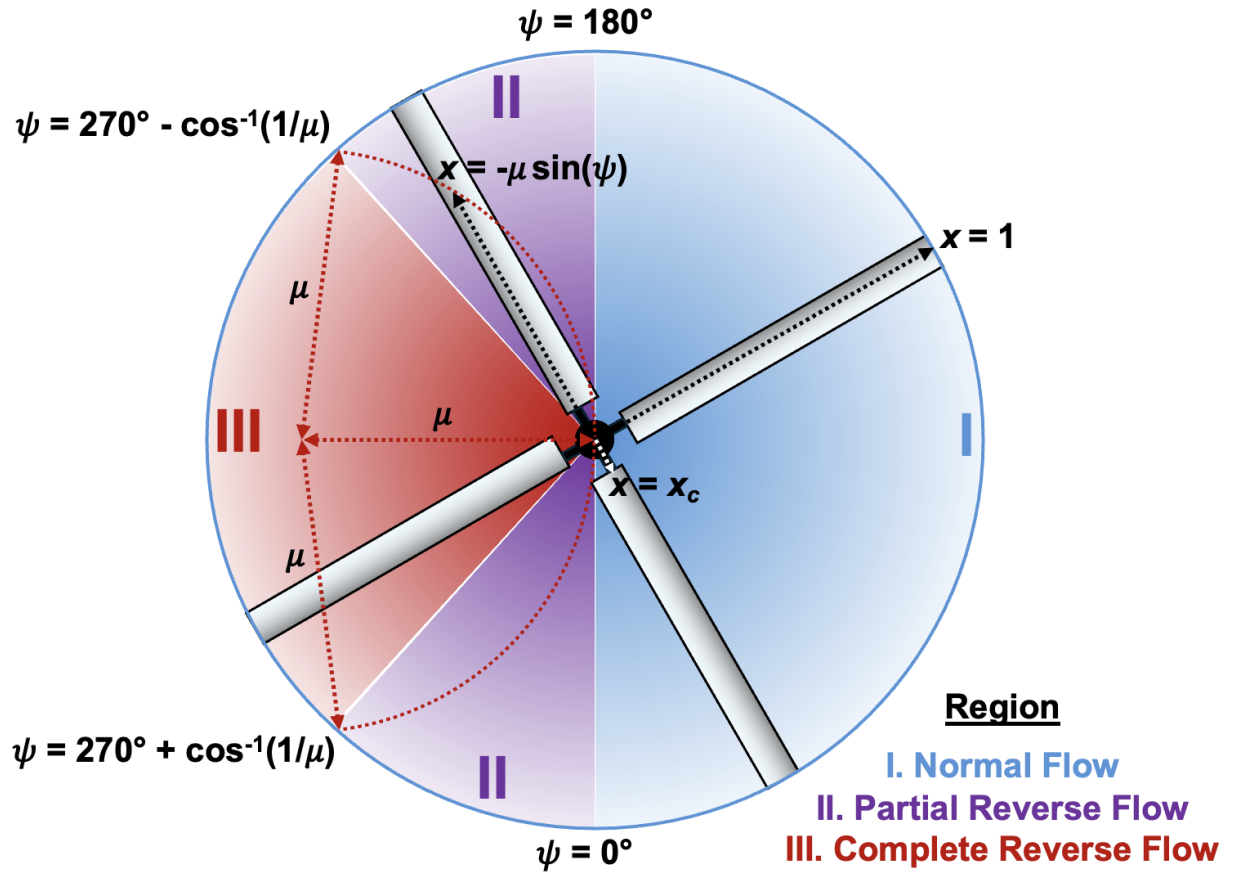


Figure 2.2: Normal flow, partial reverse flow, and complete reverse flow regions.

$\mu = 1.5$ and a 10% blade root cutout.

In region I (the advancing side), the flow is normal and the non-dimensional flapping moment is

$$\bar{M}_\beta = \int_{x_c}^1 x \bar{F}_\beta dx \quad (2.12)$$

In region II (retreating side), the blade is in partial reverse flow. Reverse flow exists from the root ($x = 0$) to $x = -\mu \sin(\psi)$; and from this point to the tip ($x = 1$) the blade is in normal flow. When $\mu \leq 1$, the flap moment on the retreating side ($180^\circ \leq \psi \leq 360^\circ$)

is therefore

$$\bar{M}_\beta = \int_{x_c}^{-\mu \sin(\psi)} x \bar{\bar{F}}_\beta dx + \int_{-\mu \sin(\psi)}^1 x \bar{F}_\beta dx \quad (2.13)$$

When the reverse flow region is within the root cutout ($\mu u < x_c$), a conditional statement is needed to ensure the first integral in the above equation is not evaluated.

Region III exists when $\mu > 1$, which is portion of the retreating side where the entire blade is in reverse flow: $270^\circ - \cos^{-1}(1/\mu) < \psi < 270^\circ + \cos^{-1}(1/\mu)$. Note that the arccosine must be consistent with the azimuth units (degrees or radians). The flapping moment is

$$\bar{M}_\beta = \int_{x_c}^1 x \bar{\bar{F}}_\beta dx \approx - \int_{x_c}^1 x \bar{F}_\beta dx \quad (2.14)$$

which is equal to the negative of the region I moment, assuming equal aerodynamic coefficients.

For each of these azimuth ranges, the flap moment functions are integrated and the terms are grouped:

$$\bar{M}_\beta = M_{\beta}^* \beta + M_\beta \beta + M_\theta \theta + M_{\theta_{tw}} \theta_{tw} + M_\lambda \lambda \quad (2.15)$$

The moments due to blade pitch, twist, and inflow ($M_\theta, M_{\theta_{tw}}, M_\lambda$) remain on the RHS side of the flap equation, while the flap and flap rate moments (M_β, M_β^*) are subtracted from the left hand side (LHS). Only the homogeneous form of the equation is needed when calculating stability; these RHS side terms will be revisited later when discussing flap trim.

When pitch-flap coupling is present, it is important to differentiate between θ and θ_C . M_θ will be grouped with the θ_C terms, and $k_{p\beta}$ will be collected in the M_β terms.

Neglecting root cutout, on the advancing side (region I, $0^\circ \leq \psi \leq 180^\circ$):

$$\begin{aligned} M_\beta^* &= -\frac{c_d}{8a} - \frac{\mu \sin(\psi)}{6} \left(\frac{c_d}{a} + 1 \right) - \frac{1}{8} \\ M_\beta &= -\frac{k_{p\beta} \mu^2 \sin(\psi)^2}{4} - \frac{\cos(\psi)}{4} \left(\frac{c_d}{a} + 1 \right) \mu^2 \sin(\psi) \\ &\quad - \frac{k_{p\beta} \mu \sin(\psi)}{3} - \frac{\cos(\psi)}{6} \left(\frac{c_d}{a} + 1 \right) \mu - \frac{k_{p\beta}}{8} \end{aligned} \quad (2.16)$$

For the partial reverse flow region (region II, $180^\circ < \psi < 360^\circ$):

$$\begin{aligned} M_\beta^* &= -\left(\frac{a + \hat{a} + c_d + \hat{c}_d}{24a} \right) \mu^4 \sin(\psi)^4 - \frac{c_d}{8a} - \frac{\mu \sin(\psi)}{6} - \frac{c_d \mu \sin(\psi)}{6a} - \frac{1}{8} \\ M_\beta &= \left(\frac{a + \hat{a} + c_d + \hat{c}_d}{12a} \right) \mu^4 \cos(\psi) \sin(\psi)^3 - \frac{\mu \cos(\psi)}{6} - \frac{k_{p\beta} \mu \sin(\psi)}{3} - \frac{k_{p\beta}}{8} \\ &\quad - \frac{k_{p\beta} \mu^2 \sin(\psi)^2}{4} + \frac{k_{p\beta} \mu^4 \sin(\psi)^4}{24} - \frac{\mu^2 \cos(\psi) \sin(\psi)}{4} - \frac{c_d \mu \cos(\psi)}{6a} \\ &\quad - \frac{c_d \mu^2 \cos(\psi) \sin(\psi)}{4a} + \frac{\hat{a} k_{p\beta} \mu^4 \sin(\psi)^4}{24a} \end{aligned} \quad (2.17)$$

When $\mu > 1$, the moments in the total reverse flow region (region III, $270^\circ - \cos^{-1}(1/\mu) < \psi < 270^\circ + \cos^{-1}(1/\mu)$) are:

$$\begin{aligned} M_\beta^* &= \frac{\hat{a} \left(\frac{c_d}{\hat{a}} + 1 \right)}{8a} + \frac{\hat{a} \mu \sin(\psi) \left(\frac{c_d}{\hat{a}} + 1 \right)}{6a} \\ M_\beta &= \frac{\hat{a} \left(2 k_{p\beta} \mu \sin(\psi) + \mu \cos(\psi) \left(\frac{c_d}{\hat{a}} + 1 \right) \right)}{6a} \\ &\quad + \frac{\hat{a} \left(k_{p\beta} \mu^2 \sin(\psi)^2 + \cos(\psi) \left(\frac{c_d}{\hat{a}} + 1 \right) \mu^2 \sin(\psi) \right)}{4a} + \frac{\hat{a} k_{p\beta}}{8a} \end{aligned} \quad (2.18)$$

which are simply the opposite sign of the moments on the advancing side if $\hat{a} = a$ and $\hat{c}_d = c_d$. Ref. 20 uses this convention, along with assuming that $c_d/a = 0$, which results in

simpler equations.

For $\mu = 0$, the terms from region I alone define the equation of motion. For $\mu > 0$, the differential equation is piecewise. If $0 < \mu \leq 1$, the equation uses the region I terms for the first half of the azimuth range and region II for the second half. When $\mu > 1$, the region sequence is I-II-III-II, swapping between equations at the aforementioned values of ψ .

Reframing the homogeneous flap equation in mass-spring-damper form,

$$M\beta^{**}(\psi) + C(\psi)\beta^*(\psi) + K(\psi)\beta(\psi) = 0 \quad (2.19)$$

The corresponding mass, damping, and stiffness coefficients are

$$M = 1, \quad C(\psi) = -\gamma M_\beta^*, \quad K(\psi) = \nu_\beta^2 - \gamma M_\beta \quad (2.20)$$

In first order form, the homogeneous equation becomes

$$\begin{pmatrix} \beta^* \\ \beta^{**} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -K(\psi) & -C(\psi) \end{pmatrix} \begin{pmatrix} \beta \\ \beta^* \end{pmatrix} \quad (2.21)$$

Defining the coefficient matrix $A(\psi)$ and the flap state vector $\underline{q} = [\beta \ \beta^*]^\top$, the equation can be expressed a more general form:

$$\underline{\dot{q}}^* = \underline{A}q \quad (2.22)$$

2.2 Floquet Theory and Numerical Integration

For this T-periodic second order system, the stability can be evaluated using Floquet theory. The differential equation needed to find the state transition matrix $\underline{\underline{\Phi}}$ is

$$\underline{\underline{\Phi}}^*(\psi) = \underline{\underline{A}}(\psi)\underline{\underline{\Phi}}(\psi) \quad (2.23)$$

where for a T-periodic system,

$$\underline{\underline{\Phi}}(\psi + T) = \underline{\underline{\Phi}}(\psi) \quad (2.24)$$

The differential equation is solved by numerically integrating from 0 to T (2π for one rotor revolution) with the initial condition $\underline{\underline{\Phi}}(0) = \underline{\underline{I}}$. The state transition matrix at one period, $\underline{\underline{\Phi}}(T) = \underline{\underline{\Phi}}(0)$, is also called the monodromy matrix (Ref. 16). Its eigenvalues (or characteristic multipliers), z_n , describe the stability of the system's response to perturbations. A root is stable if $|z_n| \leq 1$.

The choice of numerical integration algorithm is a balance of computational cost, accuracy, numerical stability, and ease of implementation. In Refs. 17 and 99, Friedmann provides a general Runge-Kutta method designed for obtaining $\underline{\underline{\Phi}}(T)$. Despite having to program this manually, an advantage of this approach is that integrations can be completed in a single “pass.”

MATLAB's Ordinary Differential Equation suite, including solvers such as `ode45`, features high order precision and automatically assigned variable time steps that decrease computation time (Ref. 100). Higher order equations must be converted to systems of first order equations, which has already been accomplished here by using state variables

and the transition matrix. Only one set of initial conditions can be processed per function call, so two integration passes are required for a 2×2 matrix. Recent software releases have included the `ode` object, which includes features like automatic solver selection and stiffness detection (Ref. [101](#)).

For this multiple pass approach, the columns of the state transition matrix are solved sequentially: $\underline{\Phi} = [\underline{q}_1 \ \underline{q}_2]$. For the first pass, the initial condition input is $\underline{q}_1^T(0) = [1 \ 0]$, and for each azimuth step the MATLAB ODE output is $[\Phi_{11}(\psi) \ \Phi_{21}(\psi)]$. Note that the ODE IC and outputs are in row form, even though they correspond to a column of $\underline{\Phi}$. The initial condition for the second pass is $\underline{q}_2^T(0) = [0 \ 1]$, and the output is $[\Phi_{12}(\psi) \ \Phi_{22}(\psi)]$. These two outputs must be combined with those from the first pass and transposed to assemble the state transition matrix.

In general, the adaptive variable time increments assigned by MATLAB's ODE solvers provide a good balance of precision and efficiency. However, even with event detection, piecewise-defined equations present a challenge. By default, the ODEs only require a `tspan` vector providing the initial and final time values, which would be $[0 \ 2\pi]$ if each pass were solved using a single function call. But as was noted earlier, the ODEs are defined by 1, 2, or 4 segments (region I for $\mu = 0$, I-II for $0 < \mu \leq 1$, or I-II-III-II for $\mu > 1$). So for an optimum implementation, there would be 2, 4, or 8 function calls depending on μ to complete both passes, with the final states of each segment forming the initial conditions of the next segment.

Additionally, constant time steps are desired for frequency analysis. If additional times are included within `tspan`, MATLAB will adjust its maximum step size accordingly, but still calculate solutions at its own “internal” points. The solutions are then continuously

extended (interpolated) to the manually assigned steps with accuracy of the same order as the original steps (Refs. 100 and 102). When implementing the multiple segment approach to the piecewise problem, the overlapping `tspan` indices would need to be removed when the $\underline{q}$ arrays are concatenated.

Although not officially supported by MATLAB, simple ODE functions that are natively fixed step solvers have been presented by MathWorks. These include versions of `ode4`, which uses a 4th-order Runge-Kutta method (Refs. 103 and 104). Aside from the lack of options like error tolerance, the general input and output formats are the same as the built-in ODE solvers, except that the `tspan` array must include all of the integration time increments. As shown in Fig. 2.3, `ode4` provides suitable results for the analyses in this work. In Ref. 74, which employed a similar Floquet approach, it was found that at least 120 time steps for a Runge-Kutta integrator provided convergence within four significant digits.

After the monodromy matrix is calculated, the eigenvalues (z) and eigenvectors (V) must be found. Software packages, such as MATLAB's `eig`, will typically employ an iterative algorithm (e.g. QR) and an ordering scheme (e.g. ascending values) to solve the eigenproblem. Because characteristic equation is only 2nd order in the flap equation, an iterative numerical method is not required in this case. In general, eigenvalue equation is

$$(\underline{\underline{\Phi}}(T) - \underline{\underline{zI}})\underline{\underline{V}} = \underline{\underline{0}} \quad (2.25)$$

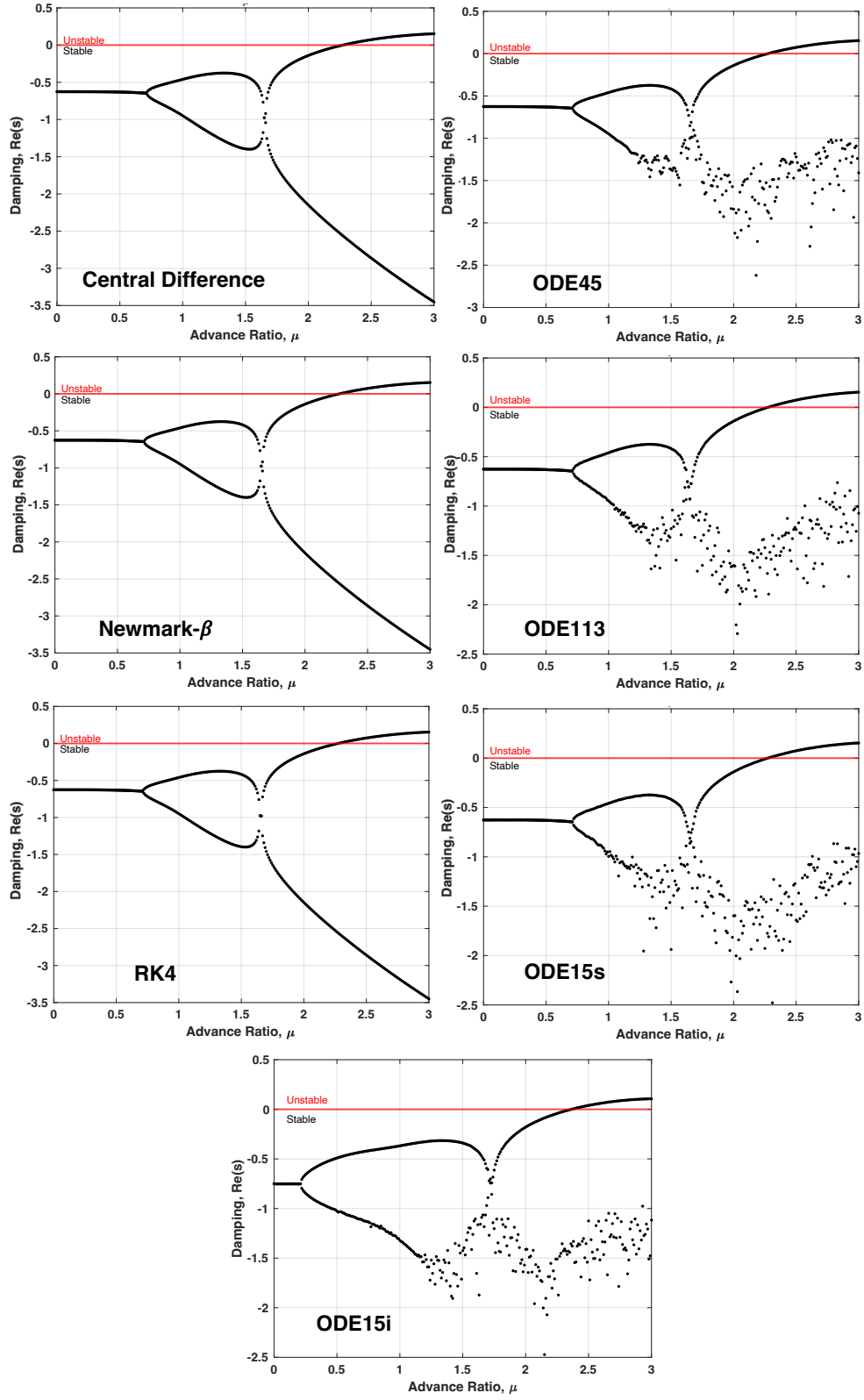


Figure 2.3: Comparison of ODE algorithms when calculating flap damping, 200 time steps, $\gamma = 12$, $\nu_\beta = 1/\text{rev}$.

and the eigenvalues can be found by solving

$$|\underline{\underline{\Phi}}(T) - \underline{z}I| = 0 \quad (2.26)$$

for $\underline{z}$.

The monodromy matrix is 2×2 for a single 2nd order ODE. Decomposing the matrix into its elements,

$$\underline{\underline{\Phi}}(T) = \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix} \quad (2.27)$$

The determinant is evaluated and the quadratic characteristic equation solved for z . The eigenvalues are

$$\begin{aligned} z_1 &= \frac{\Phi_{11}}{2} + \frac{\Phi_{22}}{2} + \frac{\sqrt{\Phi_{11}^2 - 2\Phi_{11}\Phi_{22} + \Phi_{22}^2 + 4\Phi_{12}\Phi_{21}}}{2} \\ z_2 &= \frac{\Phi_{11}}{2} + \frac{\Phi_{22}}{2} - \frac{\sqrt{\Phi_{11}^2 - 2\Phi_{11}\Phi_{22} + \Phi_{22}^2 + 4\Phi_{12}\Phi_{21}}}{2} \end{aligned} \quad (2.28)$$

and the unnormalized eigenvectors are

$$\begin{aligned} \underline{V}_1 &= \begin{pmatrix} (z_1 - \Phi_{22})/\Phi_{21} \\ 1 \end{pmatrix} \\ \underline{V}_2 &= \begin{pmatrix} (z_2 - \Phi_{22})/\Phi_{21} \\ 1 \end{pmatrix} \end{aligned} \quad (2.29)$$

An eigenvalue is stable when its magnitude is less than 1, within the unit circle of the

complex z -plane. The argument (angle between the complex and real part) determines its principal frequency. It should be noted that the abbreviated notation for the eigenvectors actually corresponds to their value at beginning and end of the period, $\underline{\underline{V}}(0)$.

The choice of assigning z_1 as the “+” conjugate and z_2 as the “−” conjugate of the quadratic equation is arbitrary. The unnormalized vectors have the same values as the Jordan canonical form (textttjordan in MATLAB). The eigenvectors can be normalized to a magnitude of unity, which is how they are normally displayed by packages such as eig.

For the flap only case, the root ordering initially seems unimportant because both roots correspond to the same degree of freedom. Aside from using the upper or lower conjugate, the assignment could also be driven by which root is the least stable. But even without the addition of another variable (e.g. lag motion), consistently tracing roots provides insights, such as what parameters drive instability. Despite the choice of eigenvalue solution method, apparent root swaps will occur as parameters such as μ and γ are varied. Some assistance can be provided by a shuffling program such as Ref. 105, but “unnatural” jumping will still be apparent. Before fully investigating this issue, the root locus must be determined.

The characteristic exponents (or roots), s_n , are found by transforming the characteristic multipliers from the z -plane to the s -plane:

$$s_n = \frac{1}{T} \ln|z_n| + i \left(\frac{1}{T} \tan^{-1} \left(\frac{\text{Im}(z_n)}{\text{Re}(z_n)} \right) + m \right) \quad (2.30)$$

The real component of the characteristic multiplier corresponds to the damping, and the imaginary component is the frequency response to the perturbation. The multiple m is

added to the principal value of the root, s_{np} . The frequency and principal frequency are related by

$$\text{Im}(s_n) = \text{Im}(s_{np}) + m \quad (2.31)$$

assuming a unique value for m , which is only strictly true when the equation is time invariant ($\mu = 0$). For $\mu > 0$, this serves as an approximation for the frequency response using the most dominant value for m .

The non-dimensional damped frequency ω_d/Ω is equal to $\text{Im}(s_n)$. The natural frequency is

$$\omega_n/\Omega = |s_n| \quad (2.32)$$

The damping, σ/Ω , is equal to $\text{Re}(s_n)$. In this convention, damping is positive when the real part of a root is in the left hand plane, so $\sigma < 0$ for stability. The damping ratio is $\zeta = \frac{-\sigma}{\omega_n}$, which is a fraction of critical damping. The response of the perturbation (per revolution) is equal to the damped frequency, $\text{Im}(s_n)$. The damping ratio determines how quickly the system returns to trim in a stable system ($\zeta > 0$) or how quickly it diverges in an unstable system ($\zeta < 0$).

2.3 Eigenvector Motion and Modal Participation

The domain of the modal addition m depends on how $\tan^{-1}$ is evaluated. For the single argument arctangent (`atan`), the range of $\text{Im}(s_{np})$ is $[-1/4, 1/4]$, making m a multiple of 1/2 or 1. If $\text{Re}(z_n) > 0$, then m is an element in the set of integers: $m \in \mathbb{Z}$ (e.g., -2, -1, 0, 1, 2). If $\text{Re}(z_n) < 0$, then $m \in \mathbb{Z} + 1/2$ (e.g., -3/2, -1/2, 1/2, 3/2). With the two-

argument arctangent (`atan2`), the quadrant ambiguity is removed, giving $\text{Im}(s_{np})$ a range of $[-1/2, 1/2]$, and $m \in \mathbb{Z}$. As in Ref. 87, the latter convention is used, so m will be described as an integer (or modal) addition to the principal frequency.

Another subtle ambiguity arises with `atan2` when the argument is equal to $\pm\pi$. In this case, the principal frequency output is 0.5, but it could also be -0.5. Trend analysis could determine if the frequency is tending toward -0.5, and use this context to determine the sign, but rapid eigenvalue changes could make this difficult. Regardless, the only effect of this is that the values of m might change sign to preserve the overall frequency. There is also the similar IEEE “minus 0” convention difference, but in any case as long as one method is applied consistently, the overall eigenvector motion is unaffected.

The constant coefficient solution can be used to find m exactly, but only for $\mu = 0$. The full solution contains of a weighted array of m values, which requires an analysis of the eigenvector motion. The periodic eigenvectors are

$$\underline{\underline{V}}(\psi) = \underline{\underline{\Phi}}(\psi)\underline{\underline{V}}(0)[\setminus \exp(\text{Re}(-s_n\psi))\setminus] \quad (2.33)$$

Through a similar relation, the modal waveform matrix is

$$\underline{\underline{Q}}(\psi) = \underline{\underline{\Phi}}(\psi)\underline{\underline{V}}(0)[\setminus \exp(-s_n\psi)\setminus] \quad (2.34)$$

using a similar convention to Peters et al. in Ref. 87.

The initial conditions for $\underline{\underline{\Phi}}(\psi)$ are the eigenvectors found from the first Floquet integration of the state transition matrix, $\underline{\underline{\Phi}}(0) = \underline{\underline{V}}(0)$. The diagonal matrix $[\setminus \exp(-s_n p\psi)\setminus]$

contains the characteristic exponents in the principal frequency form. The negative sign on the exponents cancels out the damping, allowing pure frequency content to be revealed. In the $\underline{\underline{Q}}(\psi)$ equation the principal frequency is also cancelled, meaning that the remaining waveforms consist of superpositions of the modal additions for that root. The waveform of $\underline{\underline{V}}(\psi)$ shows the full frequency response, but because it may consist of non-integer frequencies, it is harder to analyze via fast Fourier transform (FFT) than $\underline{\underline{Q}}(\psi)$.

In the event of identical eigenvalues, the eigenvectors would be indistinct. As a result, the diagonal matrix would need to be modified to include the term $-\psi \exp(-s_n \psi)$ in the upper-right element; see Refs. 12 and 15 on the Jordan canonical form in general. Ref. 87 also includes this caveat in regards to flap stability eigenvector motion. In theory this only occurs at the precise onset of parametric resonance (frequency locking), and is not practically observed when solving this problem numerically.

Performing FFTs on the diagonal elements of $\underline{\underline{Q}}(\psi)$ provides the frequency content and the strength of each mode, and thus the values of m for each root. For a root number n , its corresponding eigenvector motion vs. azimuth is found in the element Q_{nn} (i.e. $Q_{11}(\psi)$ is the waveform for Root 1, $Q_{22}(\psi)$ corresponds to Root 2). Considering this waveform as a complex Fourier series (see Refs. 106 and 107),

$$Q(\psi) = \sum_{m=-m_{\max}}^{m_{\max}} Q_{me} e^{im\psi} \quad (2.35)$$

The complex Fourier coefficients are

$$Q_{me} = \frac{1}{2\pi} \int_0^{2\pi} Q(\psi) e^{-im\psi} d\psi \quad (2.36)$$

which can be evaluated numerically using tools such as MATLAB's `fft` function.

To completely reconstruct a signal with unlimited bandwidth, the maximum evaluated frequency would need to be $m_{\max} = \infty$ per rev. However, the frequency range of interest is fairly limited in the rotor stability problem. For a numerical integration with N equally spaced points in time, the maximum number of samples for the FFT without interpolation is also N . The highest resolvable frequency, and thus $m_{\max}$, is $N/2$ due to the Nyquist-Shannon sampling theorem (Ref. 108). The minimum resolvable frequency is the inverse of the evaluated number of periods, $1/T$ (i.e. 1 per rev for 1 period, $1/2$ per rev for 2 periods). The output of MATLAB's `fft` is divided by the number of samples, which provides an array of Fourier coefficients for each frequency. The default order for `fft` is to lead with the resultant zero-frequency element at the beginning: $[Q_{0e}, Q_{1e}, \dots, Q_{m_{\max}e}, Q_{-m_{\max}e}, \dots, Q_{-1e}]$. Using the function `fftshift` reindexes the array to place it in the middle, as is typical with two-sided FFTs: $[Q_{-m_{\max}e}, \dots, Q_{-1e}, Q_{0e}, Q_{1e}, \dots, Q_{m_{\max}e}]$.

For a given frequency m per rev, the modal participation factor is the norm of the Fourier coefficient:

$$\eta_m = |Q_{me}| = \sqrt{\text{Re}(Q_{me})^2 + \text{Im}(Q_{me})^2} \quad (2.37)$$

Each factor is divided by the sum of all factors to normalize the total to unity (100%). The frequency with the highest value of η is the dominant value of m . When $\mu = 0$, there is a distinct integer value of m for each root, where $\eta = 1$ (within the precision of the integration method and the FFT solver). As μ increases, the frequency spectrum expands,

and soon no participation factor exceeds 50%. The resulting roots can be expressed as

$$s_n(m) = s_{np} + im \quad (2.38)$$

When parametric resonance occurs, the principal frequencies for the root pairs are identical, and there will be pairs of frequencies locked with equal magnitudes, η . When $\text{Im}(s_{np}) = 0$, the m pairs are of the form $(-1, +1), (-2, +2)$, etc. The total response is a superposition of integer ($\mathbb{Z}/\text{rev}$) frequencies, which makes it periodic. Note that $m = 0$ could still be the most dominant in this case; a 0 per rev response indicates exponential decay (if stable) or growth (if unstable). If $\text{Im}(s_{np}) = \pm 1/2$, the locked integer pairs are of the form $(0, +1), (-1, +2)$ or $(-1, 0), (-2, +1)$, and the response is a superposition of $\mathbb{Z} + 1/2$ per rev responses, or antiperiodic.

Fig. 2.4 depicts the overall Floquet theory approach to determine stability and root loci that has been described above.

The convention employed in this work is to assign the first root, (Flap 1 or F1) to the most damped root ($\text{Re}(s_1) < \text{Re}(s_2)$). That being equal (when the roots are complex conjugates), the first root is the most “upper” (progressive) value and the second root is the “lower” or most regressive value based on the weighted array of frequencies. The difference is subtle at high- μ due the high harmonic content and when frequencies are locked. The lowest residual between the eigenvalues and the eigenvector waveforms for a change in parameter (μ) determines the root order, providing the smoothest trajectory in the s -plane.

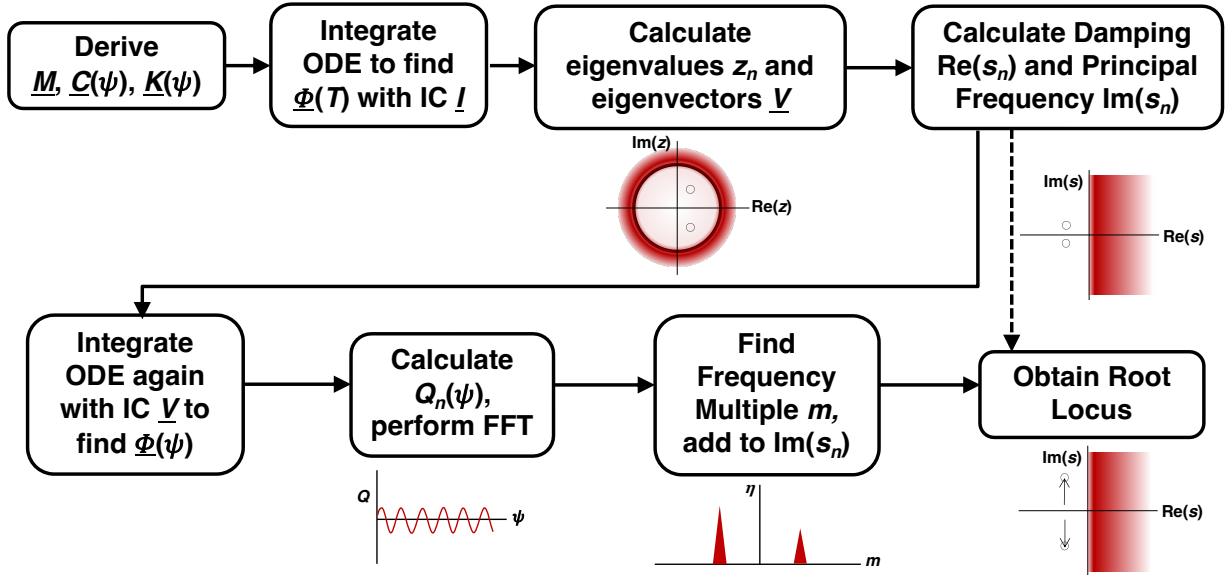


Figure 2.4: Floquet theory approach to determine root loci.

2.4 Numerical Stability of the Modal Waveform

Numerical stability can obfuscate the physical stability of the system. Fig. 2.5 shows the s -plane stability regions for the Euler, Backward Euler, and RK4 integration methods. The implicit Backward Euler looks more appealing than the explicit methods from a numerical stability standpoint, but stability does not necessarily mean accuracy or suitability for a certain problem. Earlier it was seen that the RK4 method had better accuracy for the flap damping solution compared to all of the adaptive time step solvers, including ODE15i, which is implicit like the Backward Euler. Nevertheless, it is apparent from Fig. 2.5 that in the case of a using an RK4 scheme, very well damped ($\text{Re}(s) < -3$), high frequency ($|\text{Im}(s)| > 3$), or physically unstable roots ($\text{Re}(s) > 0$) may not be numerically stable. The explicit time-marching RK4 method is fourth-order accurate but “zero stable.” Finer time (azimuth) steps can mitigate numerical stability, up to a point.

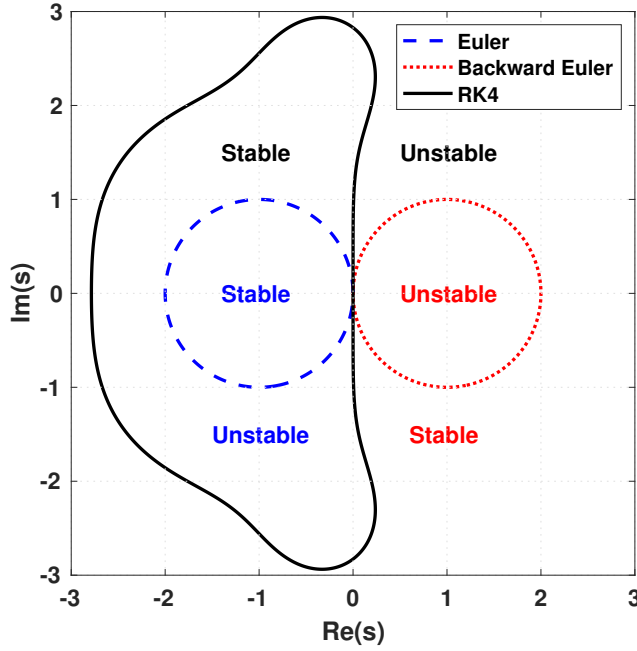


Figure 2.5: Numerical stability for Euler, Backward Euler, and RK4 integration methods.

Having a periodic eigenvector solution means that any change in frequency content between periods is due to numerical instability. Numerical dissipation is similar to physical damping; negative values for certain modes will suppress those frequencies and positive values will enhance others. Fig. 2.6 depicts a moving block approach to determine damping. The slope of the change in the logarithm of the magnitudes is the dissipation. This is the same approach used for numerical transient analysis. Thus the dissipation of the modal waveform for a given mode is

$$\sigma_Q(m) = \ln(\eta_m(2T)) - \ln(\eta_m(T)) \quad (2.39)$$

Phase error is found by calculated changes in the relative contributions of the real and

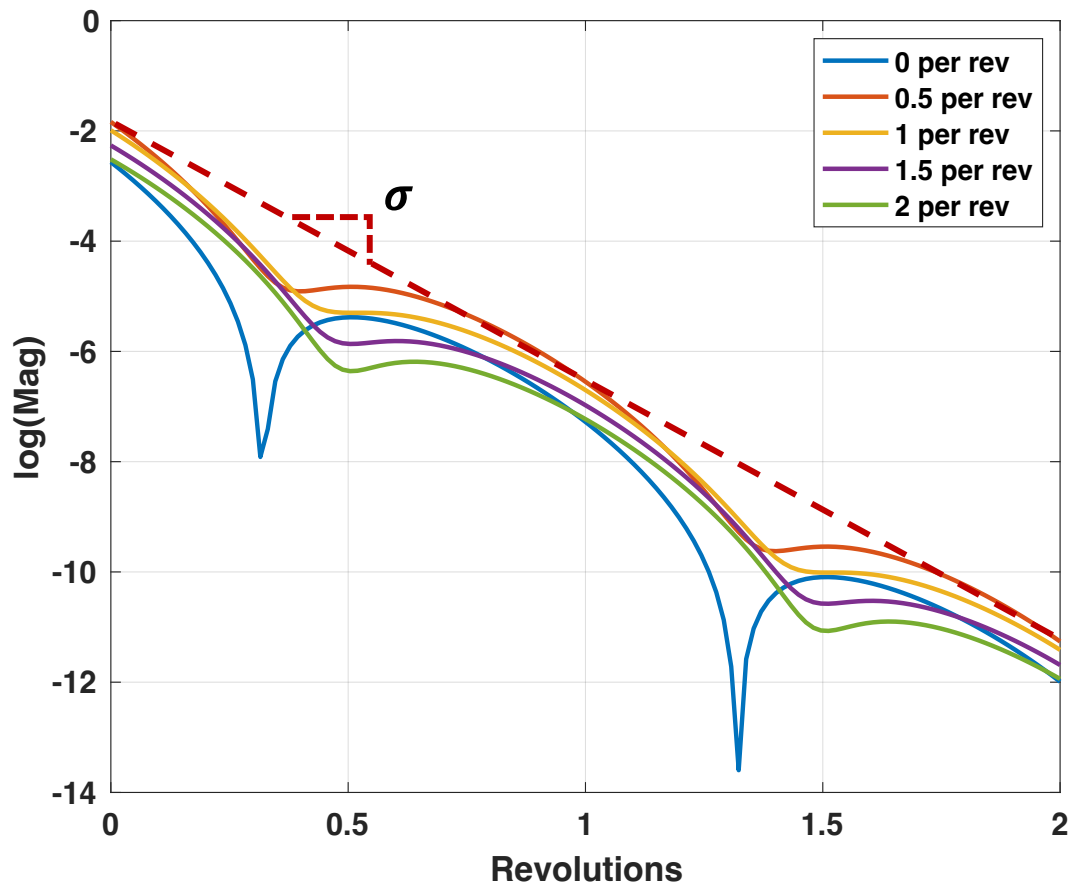


Figure 2.6: Moving block approach to measure physical damping or numerical dissipation.

imaginary complex Fourier coefficients between periods:

$$\begin{aligned}
\phi_Q(m) &= \arg(Q_{me}(2T)) - \arg(Q_{me}(T)) \\
&= \tan^{-1}\left(\text{Im}(Q_{me}(2T))/\text{Re}(Q_{me}(2T))\right) - \tan^{-1}\left(\text{Im}(Q_{me}(T))/\text{Re}(Q_{me}(T))\right)
\end{aligned}
\tag{2.40}$$

2.5 The Effect of Advance Ratio and Reverse Flow

Consider an articulated-like rotor with a Lock number of $\gamma = 12$ and a rotating flap frequency of 1 per rev. A lift curve slope of $a = 2\pi$ and drag coefficient of $c_d = 0.01$ are used. Next, the damping and stiffness matrices are assembled. Fig. 2.7 shows the periodic damping and stiffness coefficients up to $\mu = 3$ without including the reverse flow terms.

Fig. 2.8 shows the coefficients with reverse flow. The reverse flow aerodynamic coefficients are assumed to be the same as the normal flow values. Note the asymmetrical values on the retreating side of the rotor disk ($\psi > 180^\circ$) at high- μ .

The trajectories of the eigenvalues (characteristic multipliers) can be plotted in the z -plane. These are shown in Figs. 2.9, 2.10, 2.11, and 2.12.

Fig. 2.13 shows the damping and principal frequency of the flap roots without reverse flow. They begin at $\mu = 0$ as complex conjugates, having the same damping and opposite frequencies. At $\mu = 0.21$, they lock into a half per rev multiple and the damping values split. They re-converge at $\mu = 0.96$ only to lock into a 1 per rev frequency multiple at $\mu = 1.03$. converge and split again around $\mu = 1$. The 2nd flap root becomes unstable at $\mu = 1.4$, while the 1st root continues to become more stable.

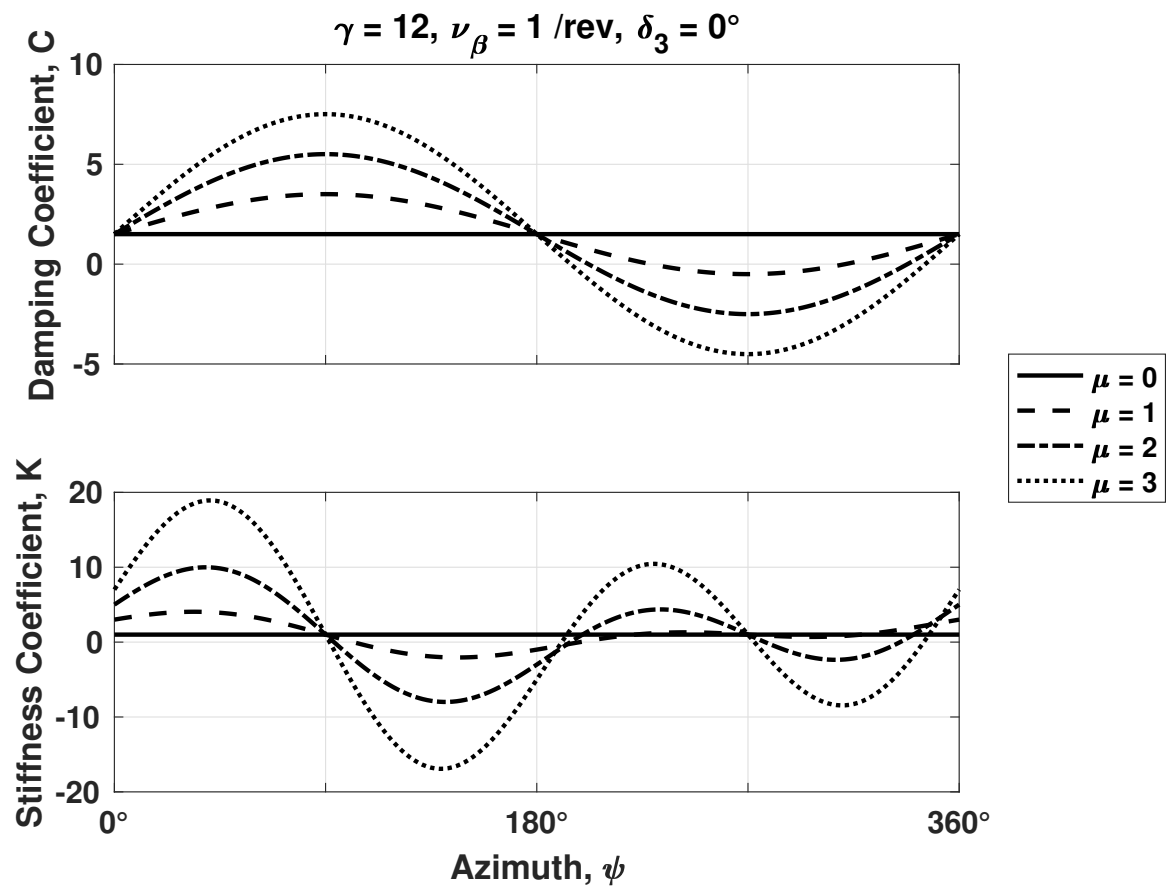


Figure 2.7: Damping (C) and stiffness (K) coefficients, without reverse flow.

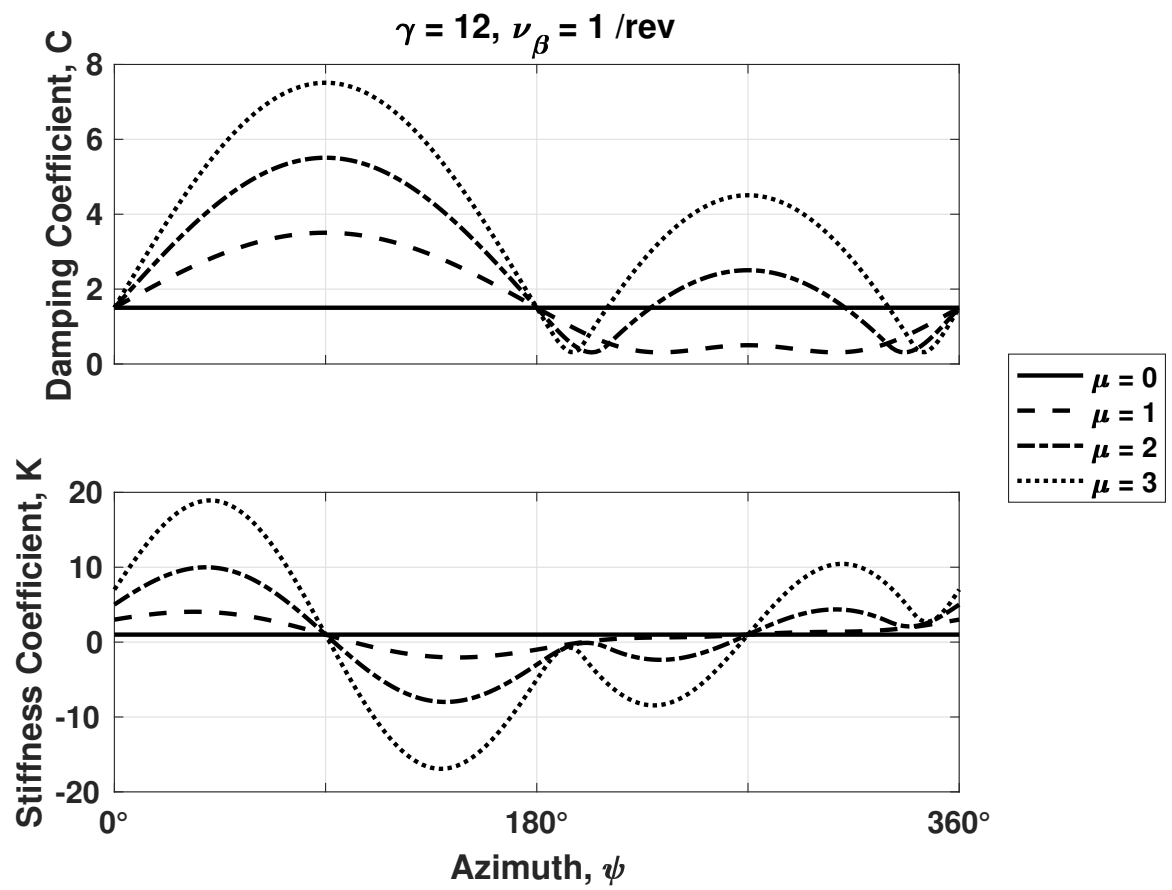


Figure 2.8: Damping (C) and stiffness (K) coefficients, with reverse flow.

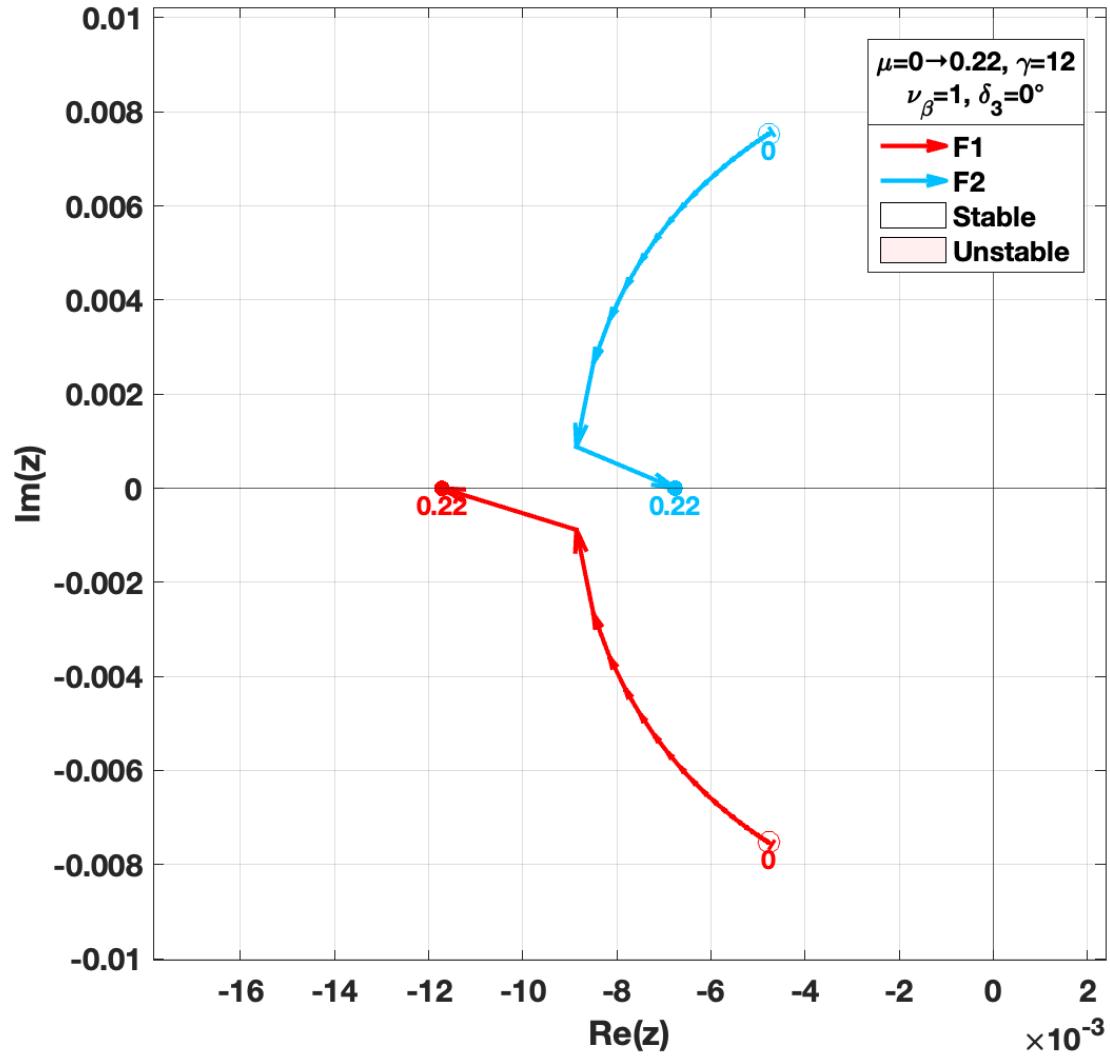


Figure 2.9: Flap characteristic multipliers, $\nu_\beta = 1$, $\gamma = 12$, $\mu=0$ to 0.22.

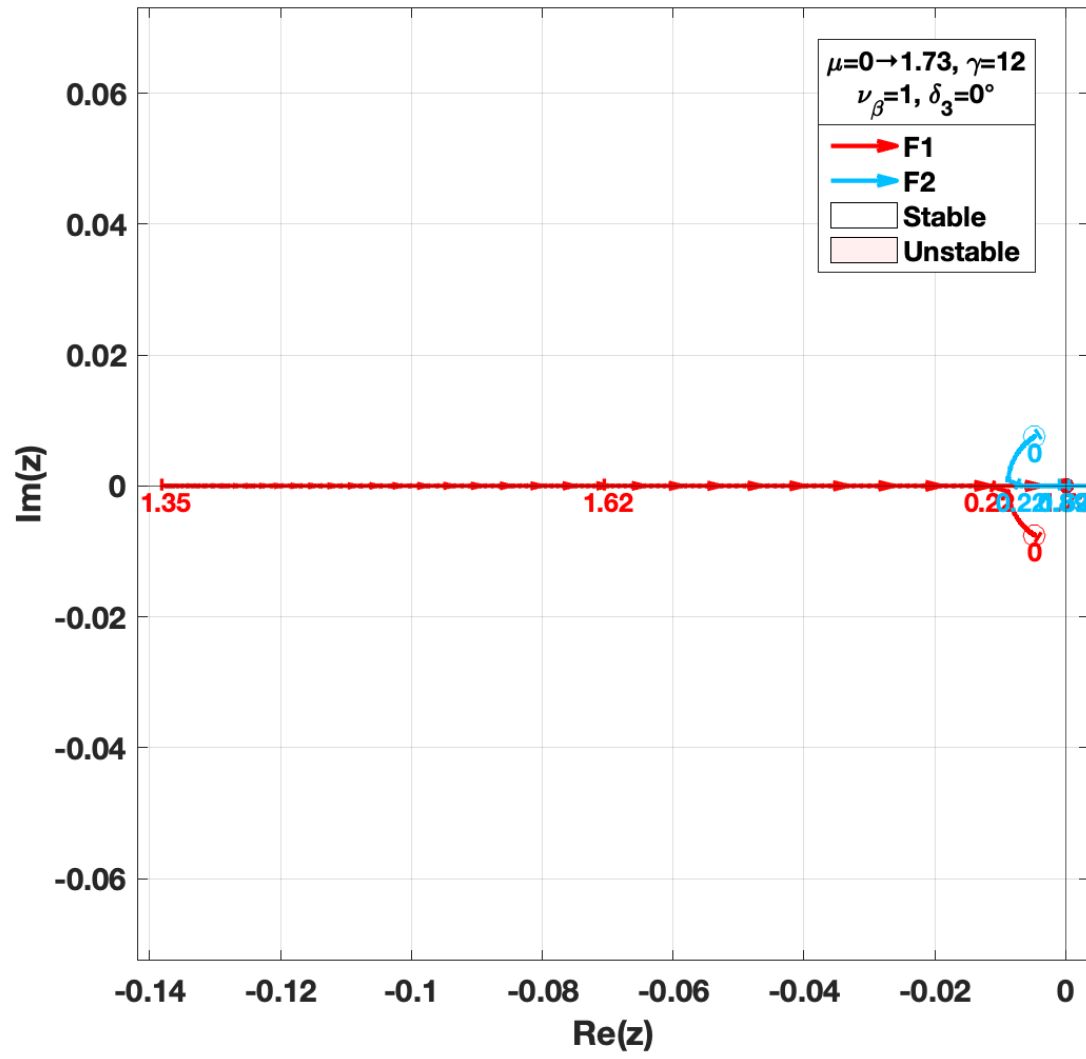


Figure 2.10: Flap characteristic multipliers ($F1$), $\nu_\beta = 1$, $\gamma = 12$, $\mu=0$ to 1.73.

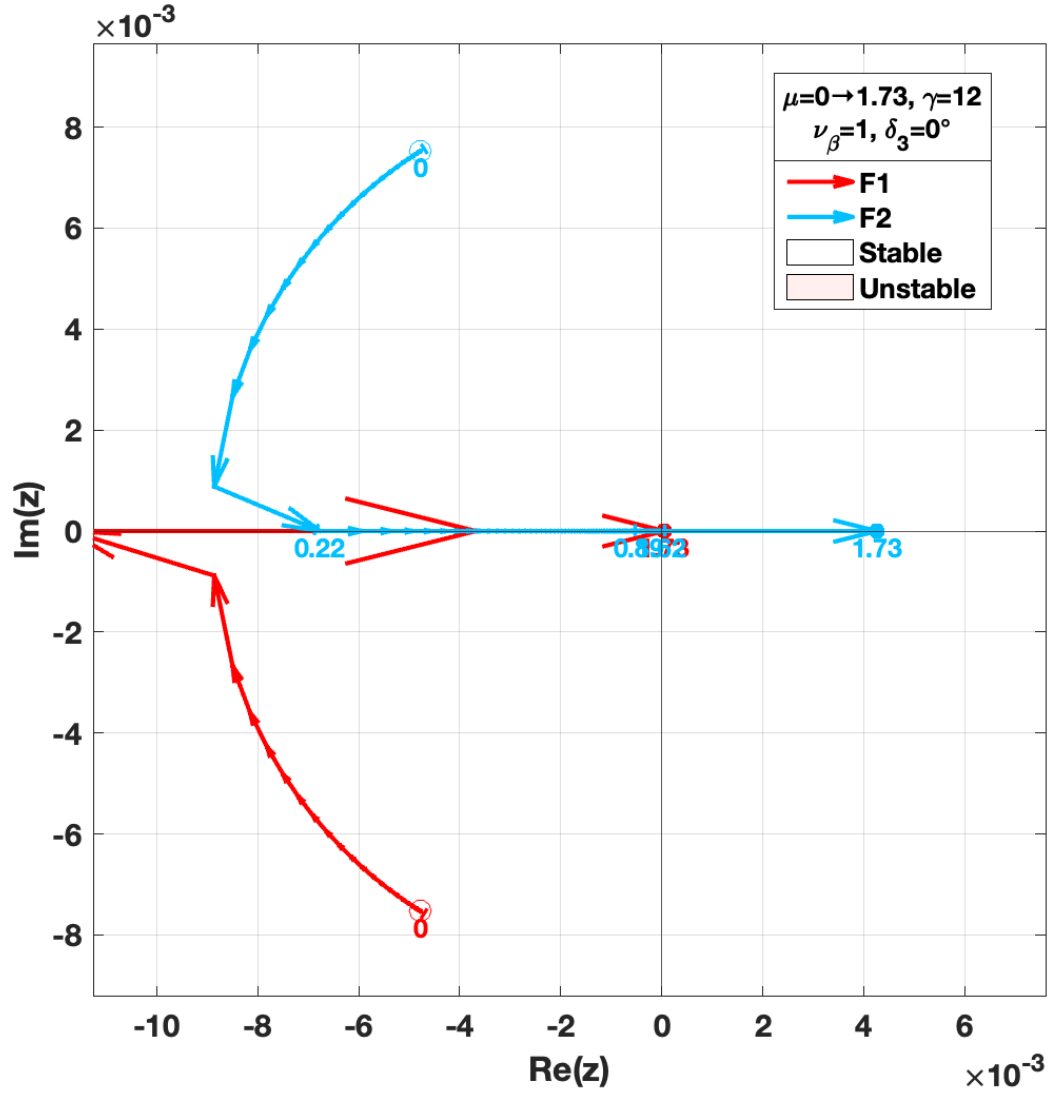


Figure 2.11: Flap characteristic multipliers ($F2$), $\nu_\beta = 1$, $\gamma = 12$, $\mu=0$ to 1.73.

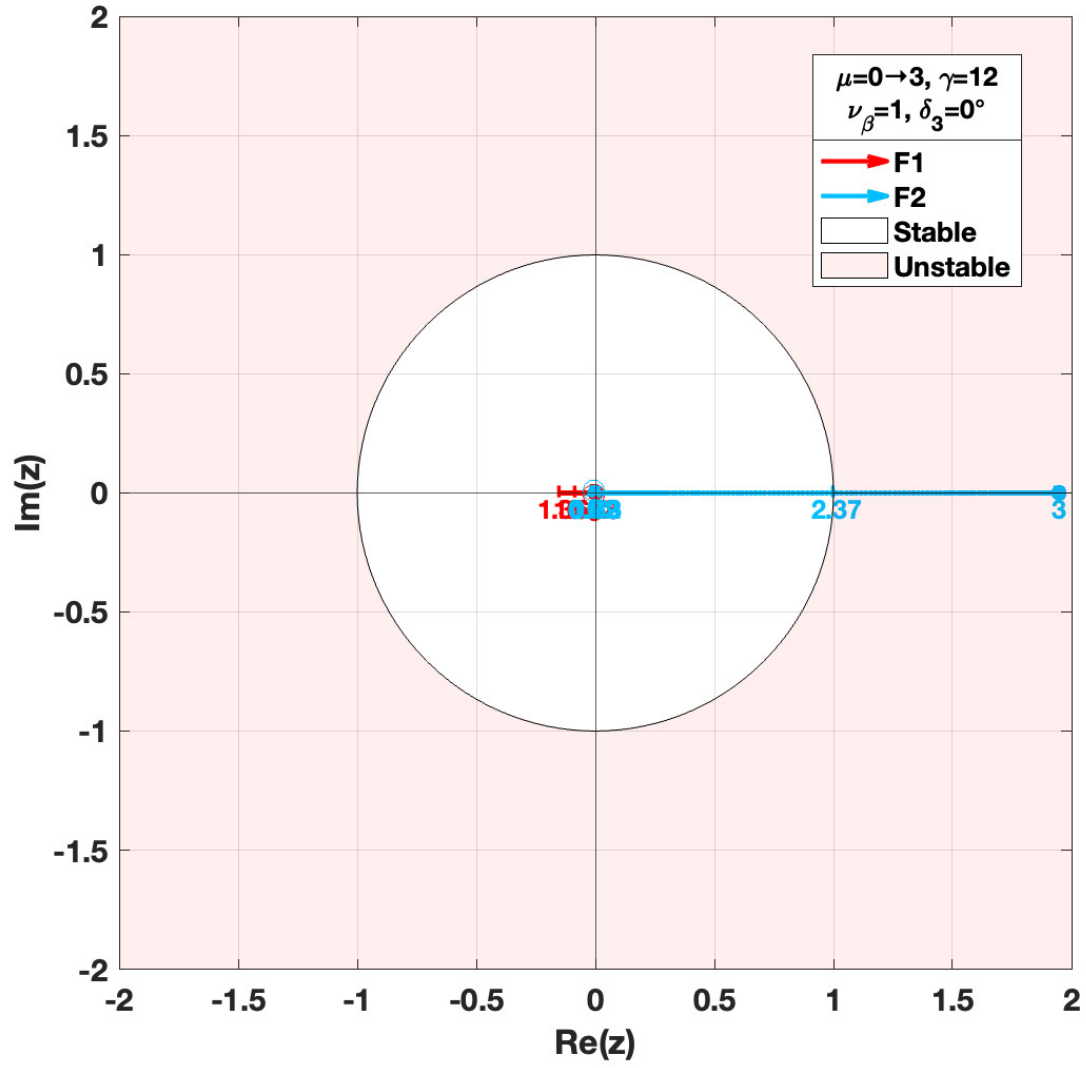


Figure 2.12: Flap characteristic multipliers, $\nu_\beta = 1$, $\gamma = 12$, $\mu=0$ to 3.

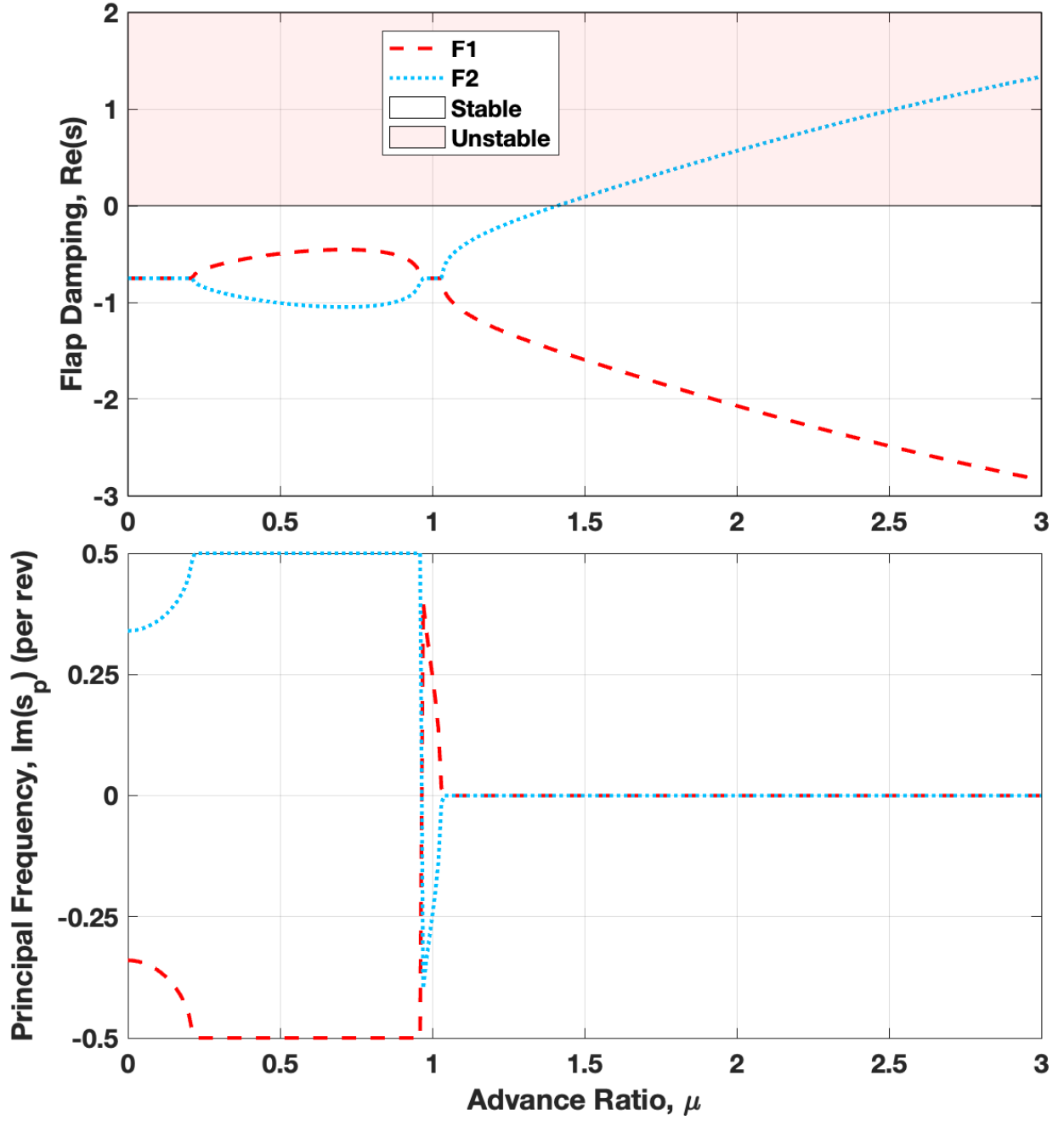


Figure 2.13: Flap damping and principal frequency, $\nu_\beta = 1$, $\gamma = 12$, without reverse flow.

Fig. 2.14 shows the damping with reverse flow included, which has a stabilizing effect in addition to more accurately modeling the aerodynamics of the system. The low- μ behavior is the nearly the same as expected, and the high- μ stability appears stretched to the right. The damping values meet and split at a higher speed of $\mu = 1.73$ where the frequency transitions from $1/2$ to $1/\text{rev}$ multiples. The 2nd flap root becomes less damped becomes unstable at $\mu = 2.4$, while the 1st root again continues to become more stable.

Continuing with the reverse flow model and calculating the eigenvector motion $Q_{nn}(\psi)$, the frequency content for each root can be found. Plotting the response of the 1st flap root for $\mu = 0$ (hover) in Fig. 2.15 reveals a clear $1/\text{rev}$ response, and the FFT of the first period reveals the $1/\text{rev}$ multiple. There is no difference between the motion in the first and second periods, resulting in zero dissipation and phase error, and indicating numerical stability. Because the principal frequency is $-0.34/\text{rev}$, adding the $1/\text{rev}$ multiple means that the actual frequency is $0.66/\text{rev}$. This is of course the same as the constant coefficient solution.

The motion of the 2nd flap root in hover is shown in Fig. 2.16. The dominant multiple is -1 , which means F2 is a regressive root with a frequency of $-0.66/\text{rev}$.

The damping and frequencies are combined to plot the first root locus in Fig. 2.17.

At $\mu = 0.22$, the flap roots enter $1/2$ rev locking. Fig. 2.18 shows that the 0 and $1/\text{rev}$ multiples are dominant and have equal magnitudes. Adding these to the principal frequency of $-0.5/\text{rev}$, the response is $80\% \pm 0.5/\text{rev}$. The higher harmonics are beginning to have an influence as well, as their model participation indicates a combined 14% for $\pm 0.15/\text{rev}$.

Like the F1 root, F2 is also locked into $\pm 0.5/\text{rev}$ at $\mu = 0.22$ as shown in Fig. 2.19.

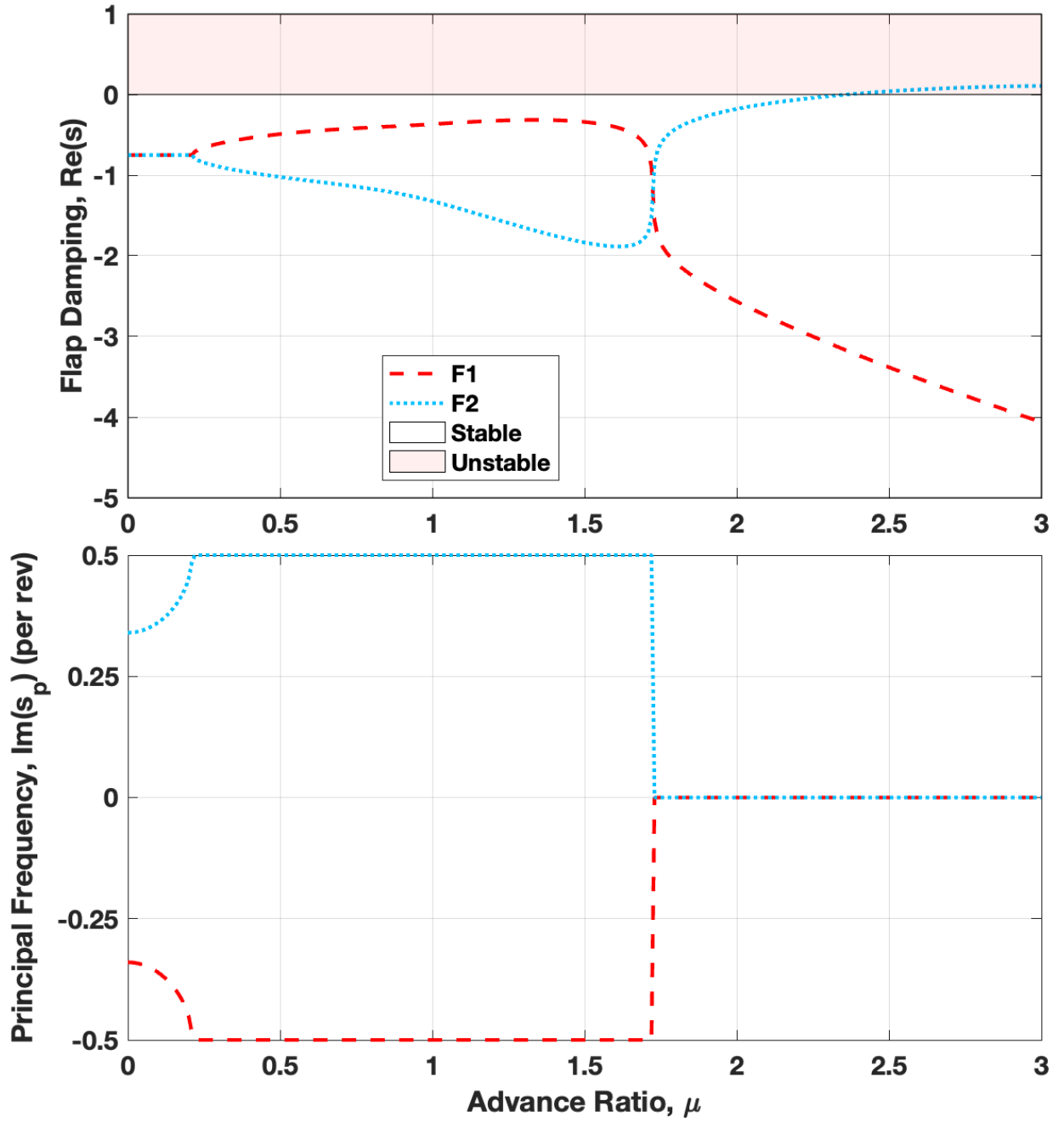


Figure 2.14: Flap damping and principal frequency, $\nu_\beta = 1$, $\gamma = 12$, with reverse flow.

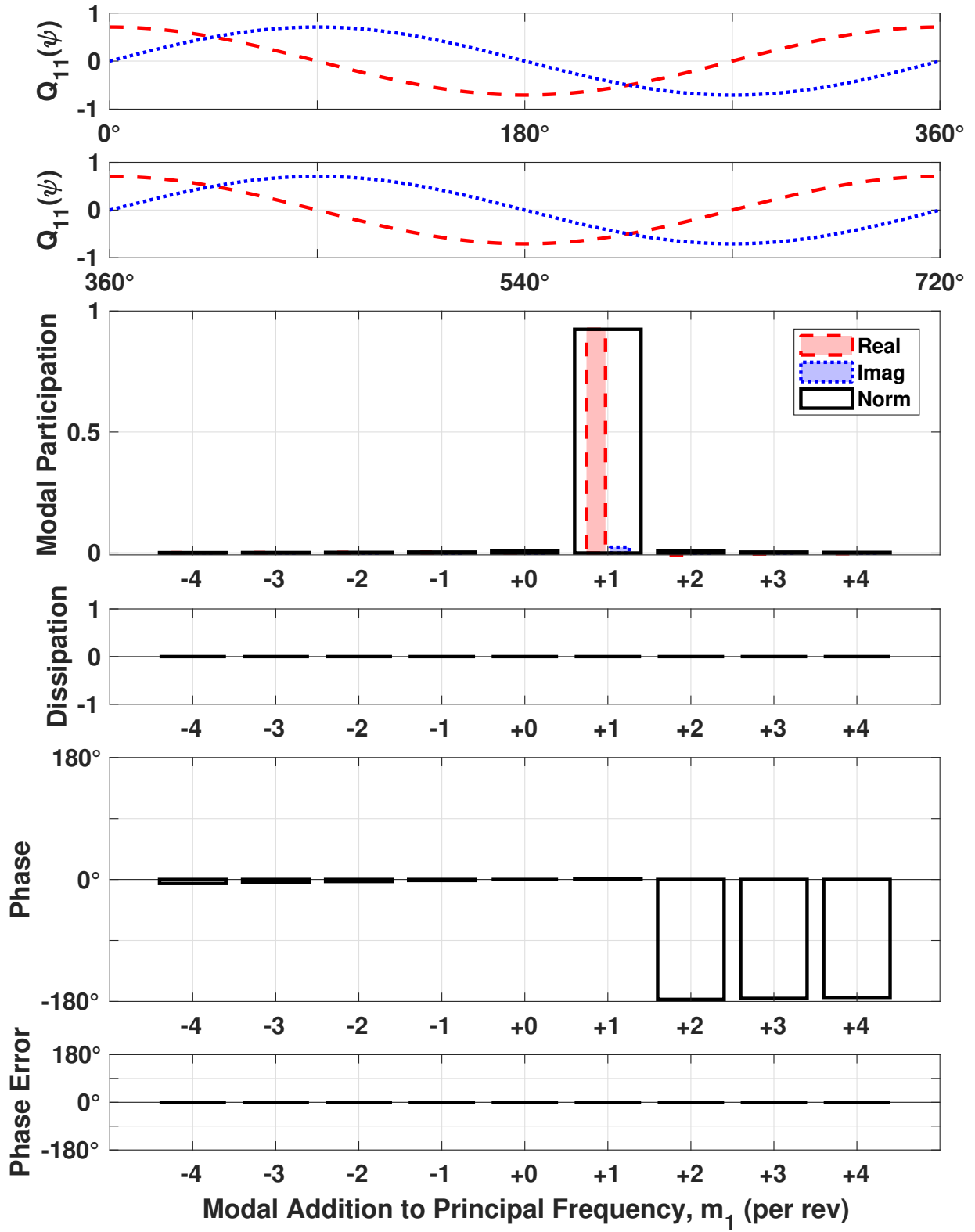


Figure 2.15: Eigenvector motion and modal participation, F1, $\nu_\beta = 1$, $\gamma = 12$, $\mu = 0$.

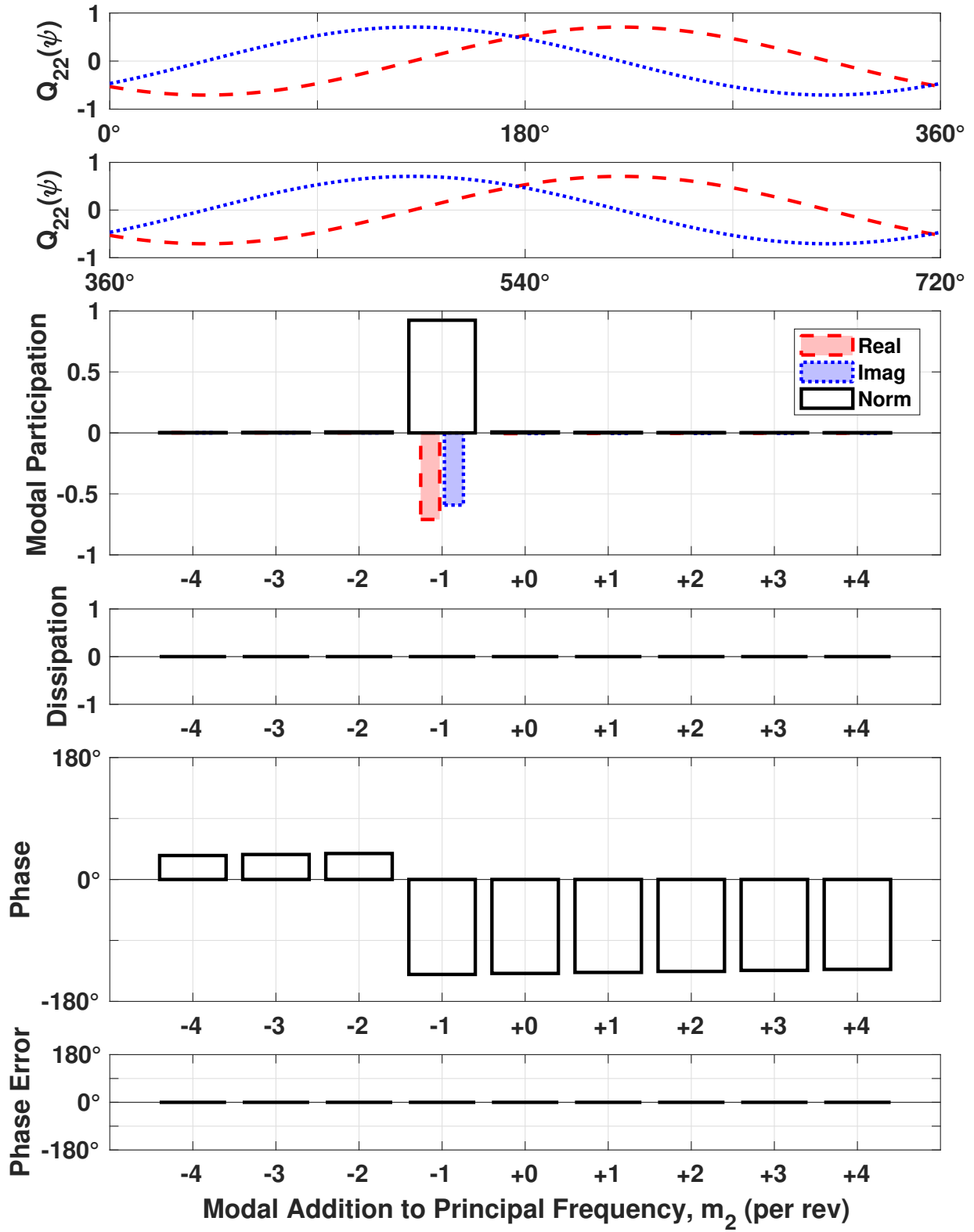


Figure 2.16: Eigenvector motion and modal participation, F2, $\nu_\beta = 1$, $\gamma = 12$, $\mu = 0$.

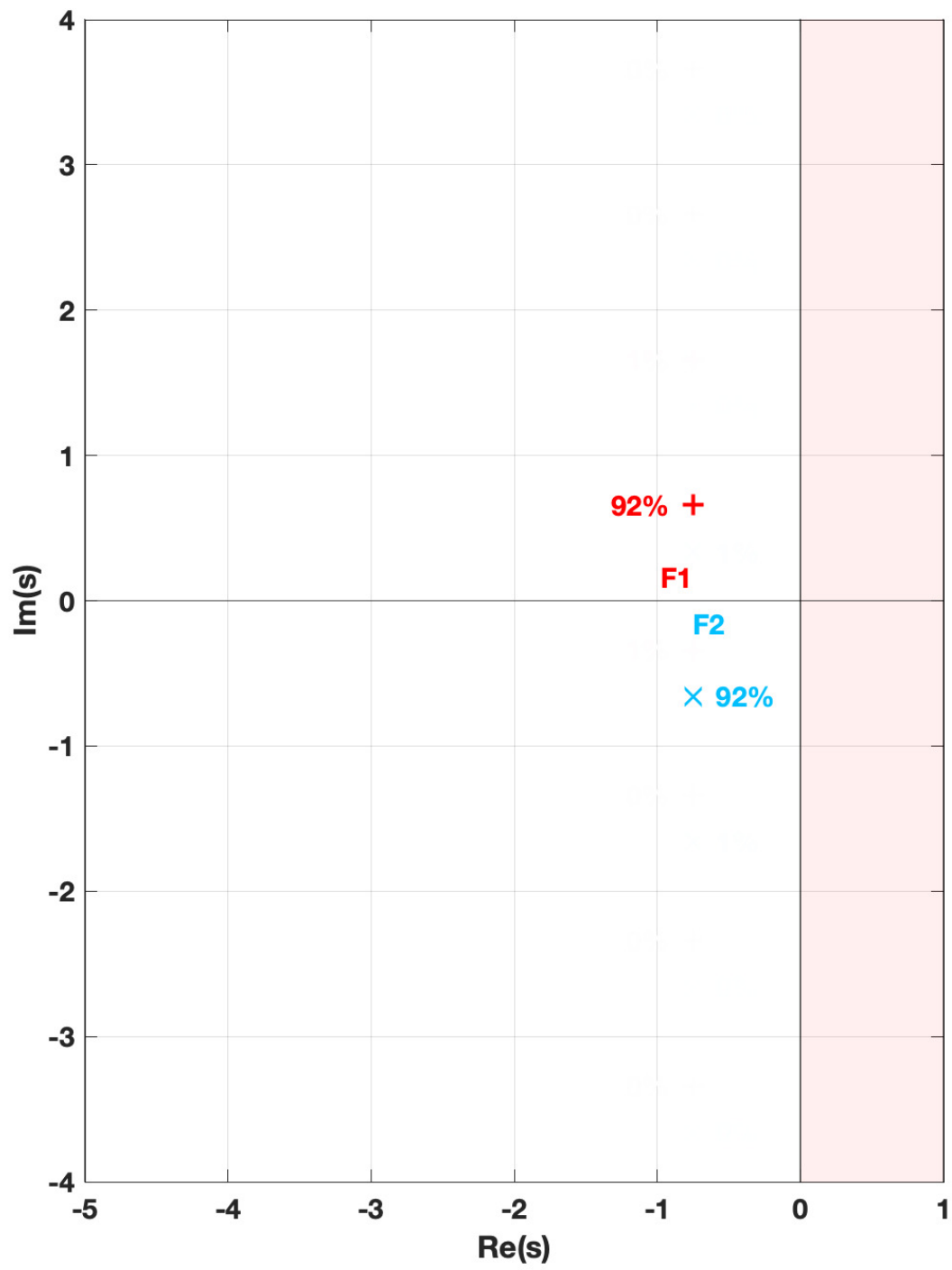


Figure 2.17: Flap root locus, $\nu_\beta = 1$, $\gamma = 12$, $\mu = 0$.

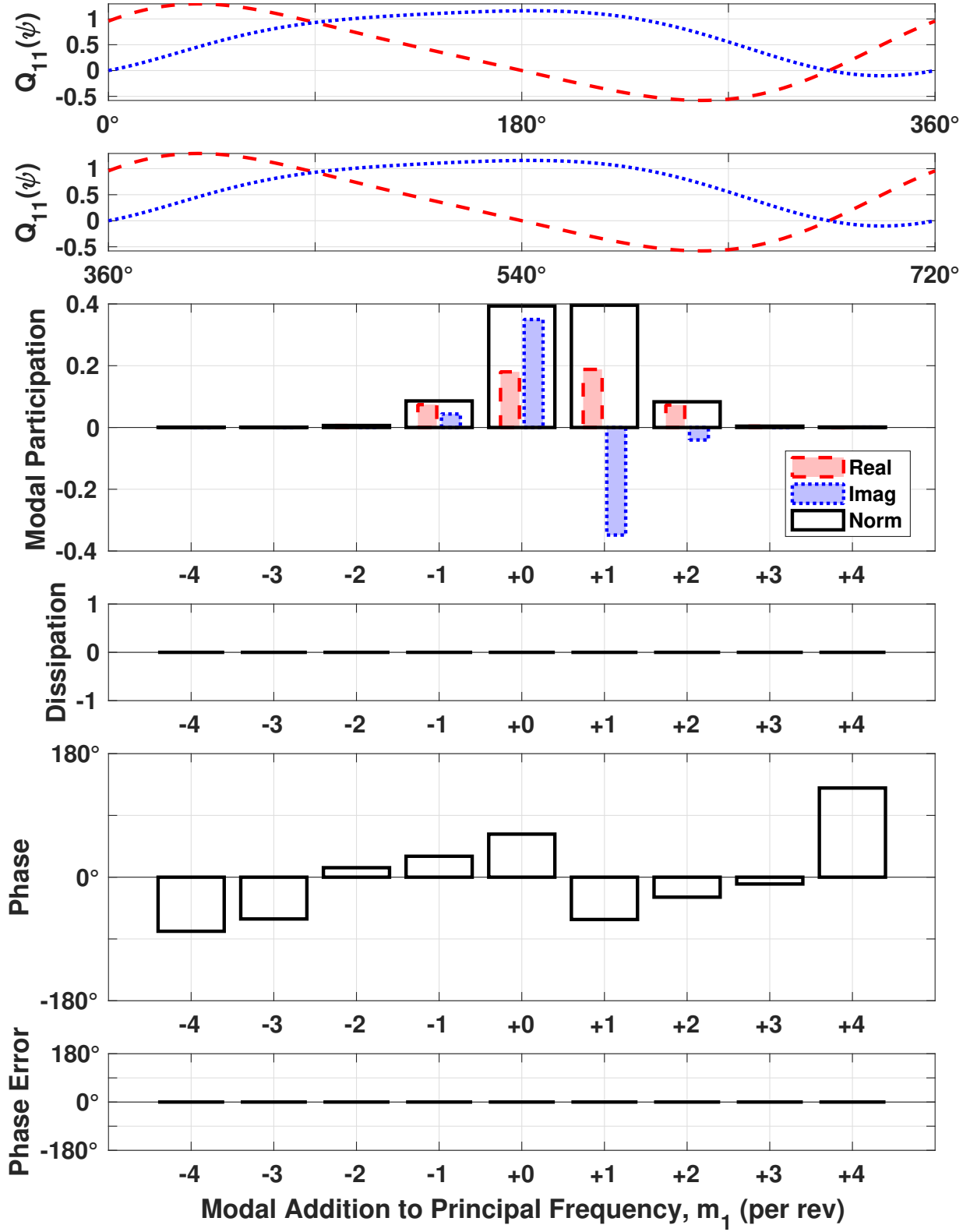


Figure 2.18: Eigenvector motion and modal participation, F1, $\nu_\beta = 1$, $\gamma = 12$, $\mu = 0.22$.

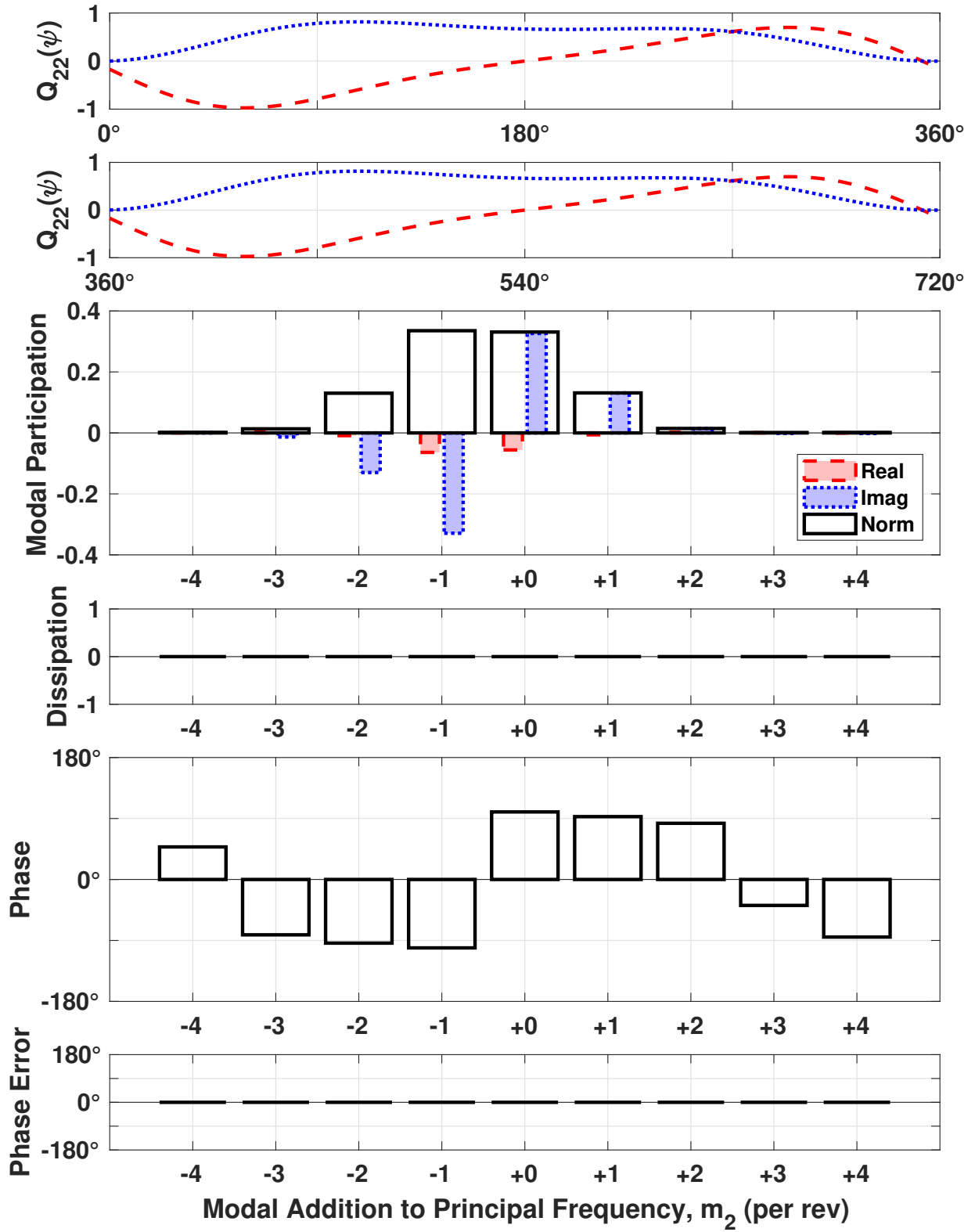


Figure 2.19: Eigenvector motion and modal participation, F2, $\nu_\beta = 1$, $\gamma = 12$, $\mu = 0.22$.

The root locus for $\mu = 0.22$ can be plotted. The flap roots significantly overlap at the onset 1/2 per rev locking. This is shown in Fig 2.20.

At $\mu = 1.73$, the flap roots lock into 1/rev multiples as shown in Figs. 2.21 and 2.22. F1 has the greatest contribution from ± 1 and F2 has a plurality of $\pm 2/\text{rev}$; in both cases the other modes have increasing influence. An important conclusion is that the frequency responses for the two roots are not identical, as was usually assumed to be the case (Ref. 87). The nature of the reverse flow regions and the resultant discrete ODE problem contributes to this phenomenon.

For the root locus at $\mu = 1.73$, the flap roots are locked into 1 per rev multiples are shown in Fig 2.23.

At $\mu = 3$, F2 has become unstable and remains in parametric resonance with a dominant $\pm 2/\text{rev}$ response (Fig. 2.25). Although F1 has become even more damped, and thus is physically very stable, it appears to be outside of the boundaries for a stable RK4 integration. In the second period of Q_{11} , its amplitude has diverged by 100 million times. This numerical instability is evident by the massive dissipation values and varying phase errors for each mode. Thus the modal participation values at this point are not necessarily accurate.

The multi-modal root locus for $\mu = 3$ is shown in Fig 2.26. F1 is overdamped and F2 is stable; the modal distribution is also different between the two roots.

Repeating the FFT process for each root and various advance ratios shows interesting stability behavior resulting from the periodic coefficients. To verify this method with previous results, Fig. 2.27 compares the 1st flap root's modal participation vs. μ with those from Ref. 87 for the same rotor configuration and no reverse flow.

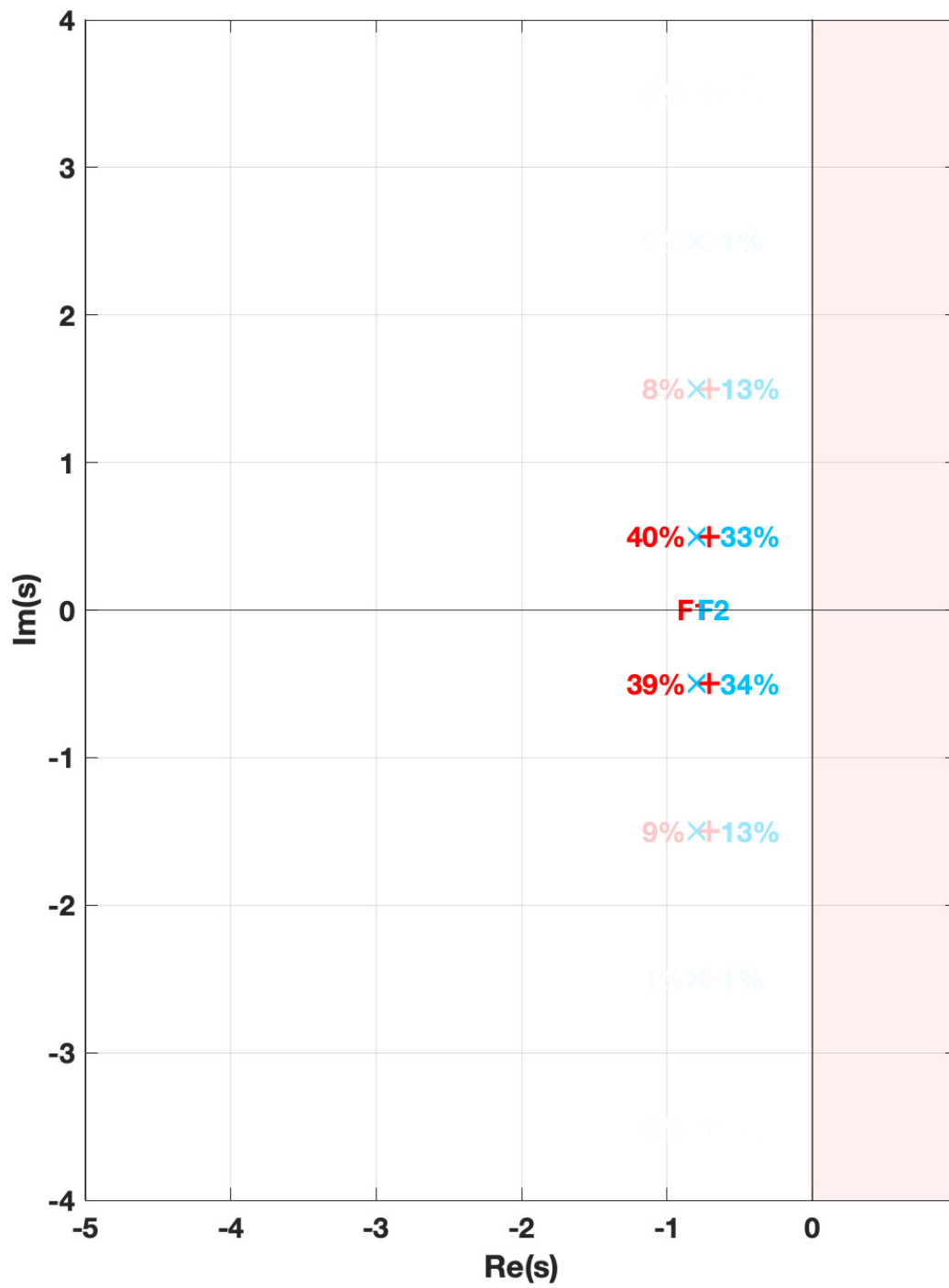


Figure 2.20: Flap root locus, $\nu_\beta = 1$, $\gamma = 12$, $\mu = 0.22$.

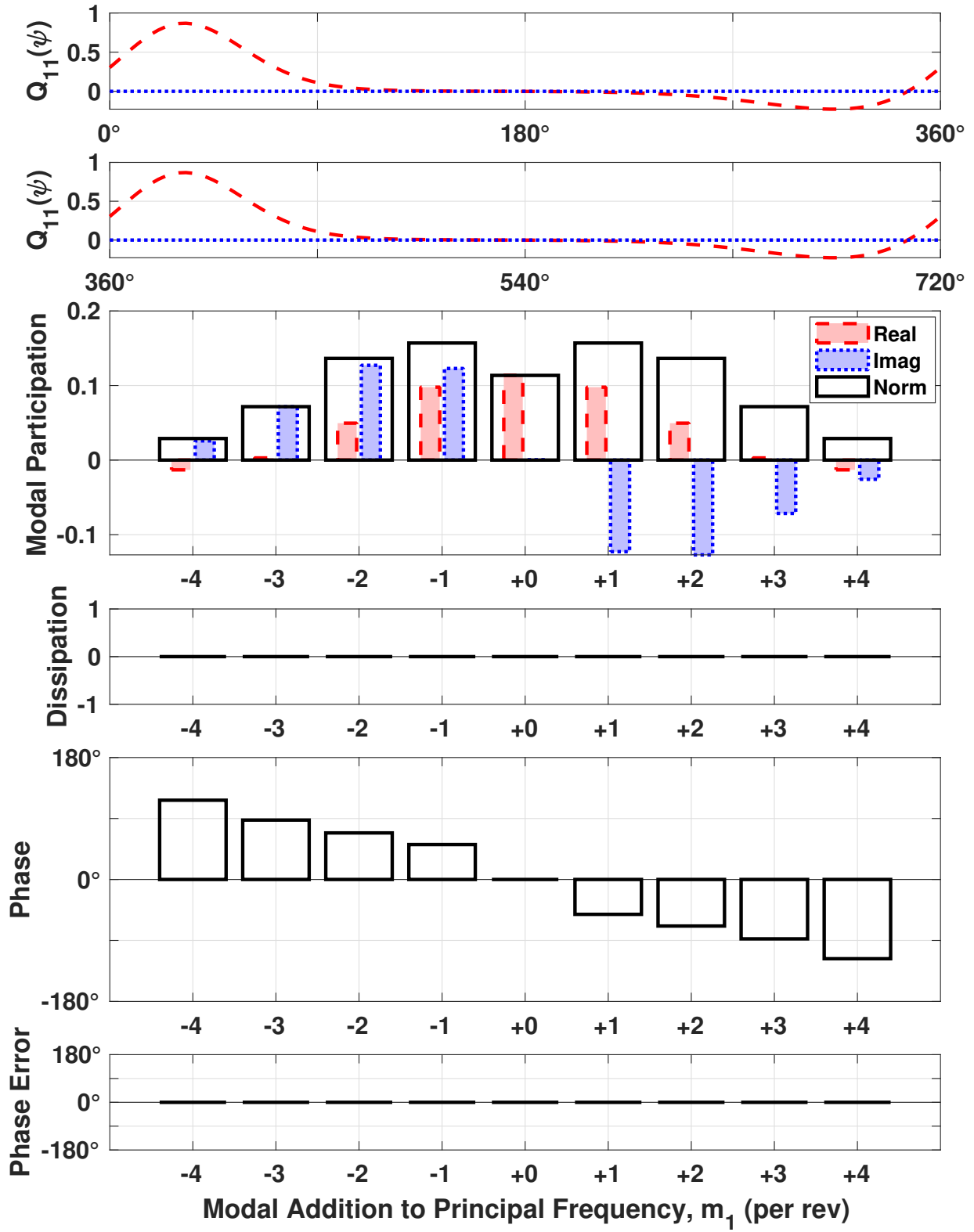


Figure 2.21: Eigenvector motion and modal participation, F1, $\nu_\beta = 1$, $\gamma = 12$, $\mu = 1.73$.

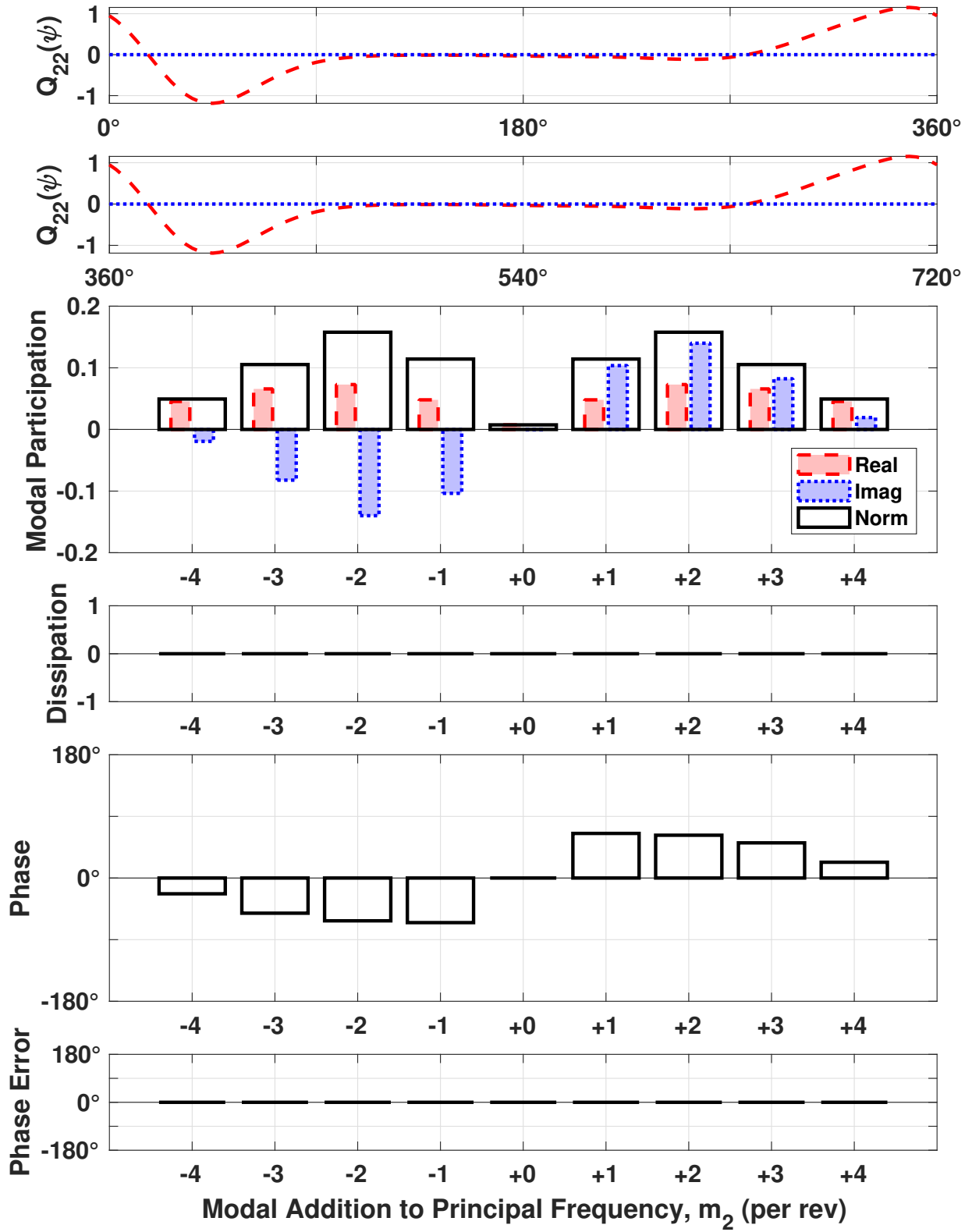


Figure 2.22: Eigenvector motion and modal participation, F2, $\nu_\beta = 1$, $\gamma = 12$, $\mu = 1.73$.

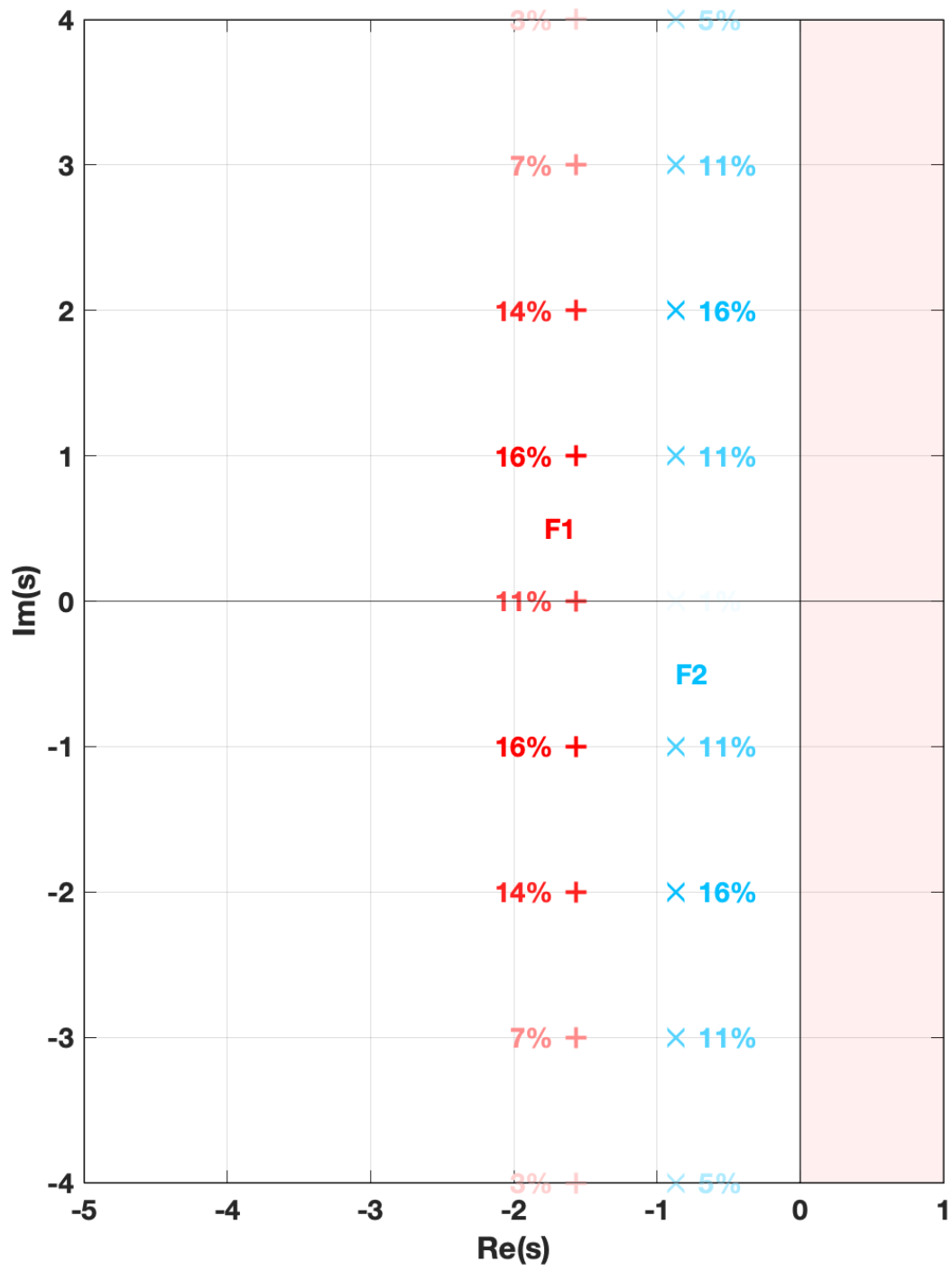


Figure 2.23: Flap root locus, $\nu_\beta = 1$, $\gamma = 12$, $\mu = 1.73$.

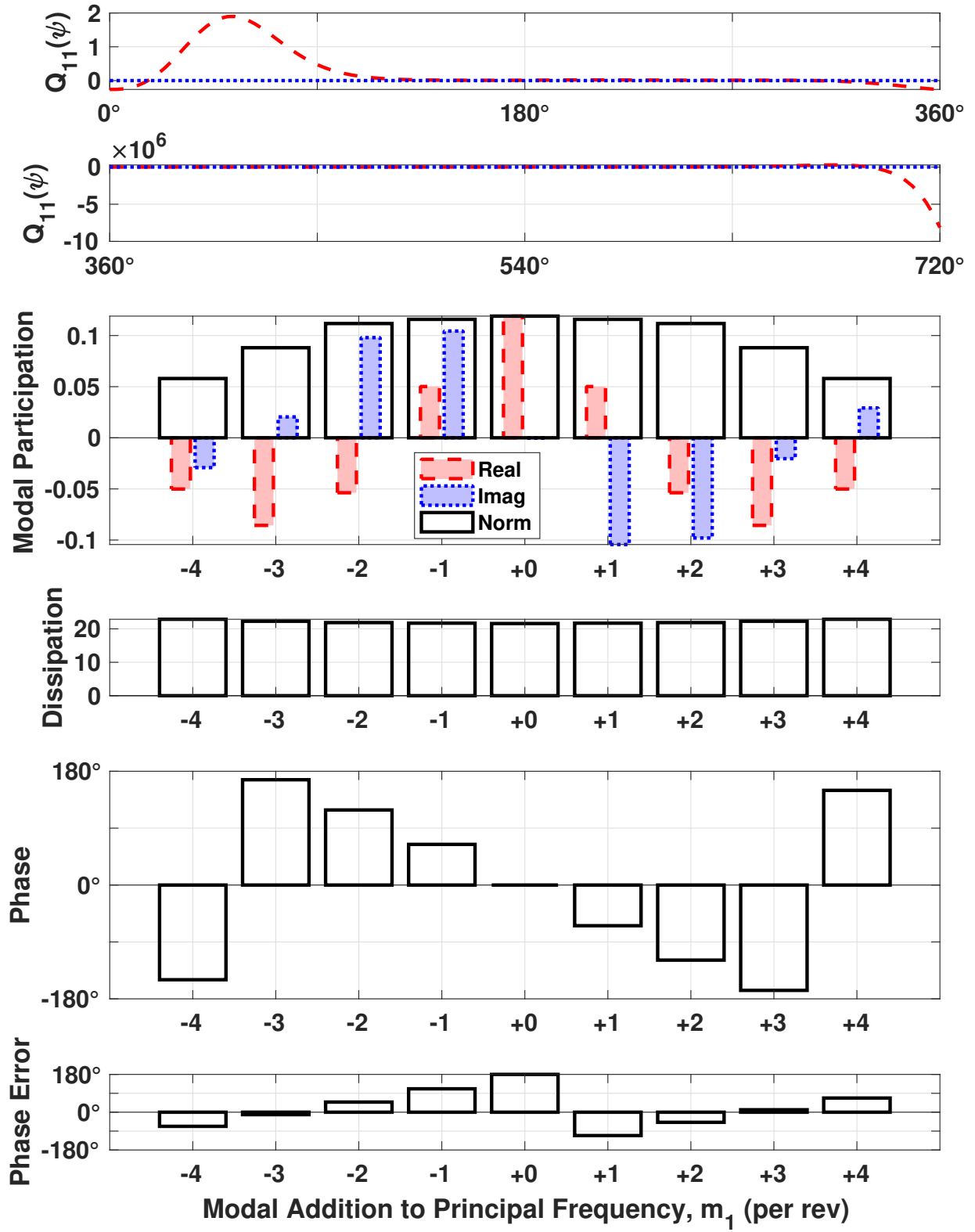


Figure 2.24: Eigenvector motion and modal participation, $F1$, $\nu_\beta = 1$, $\gamma = 12$, $\mu = 3$.

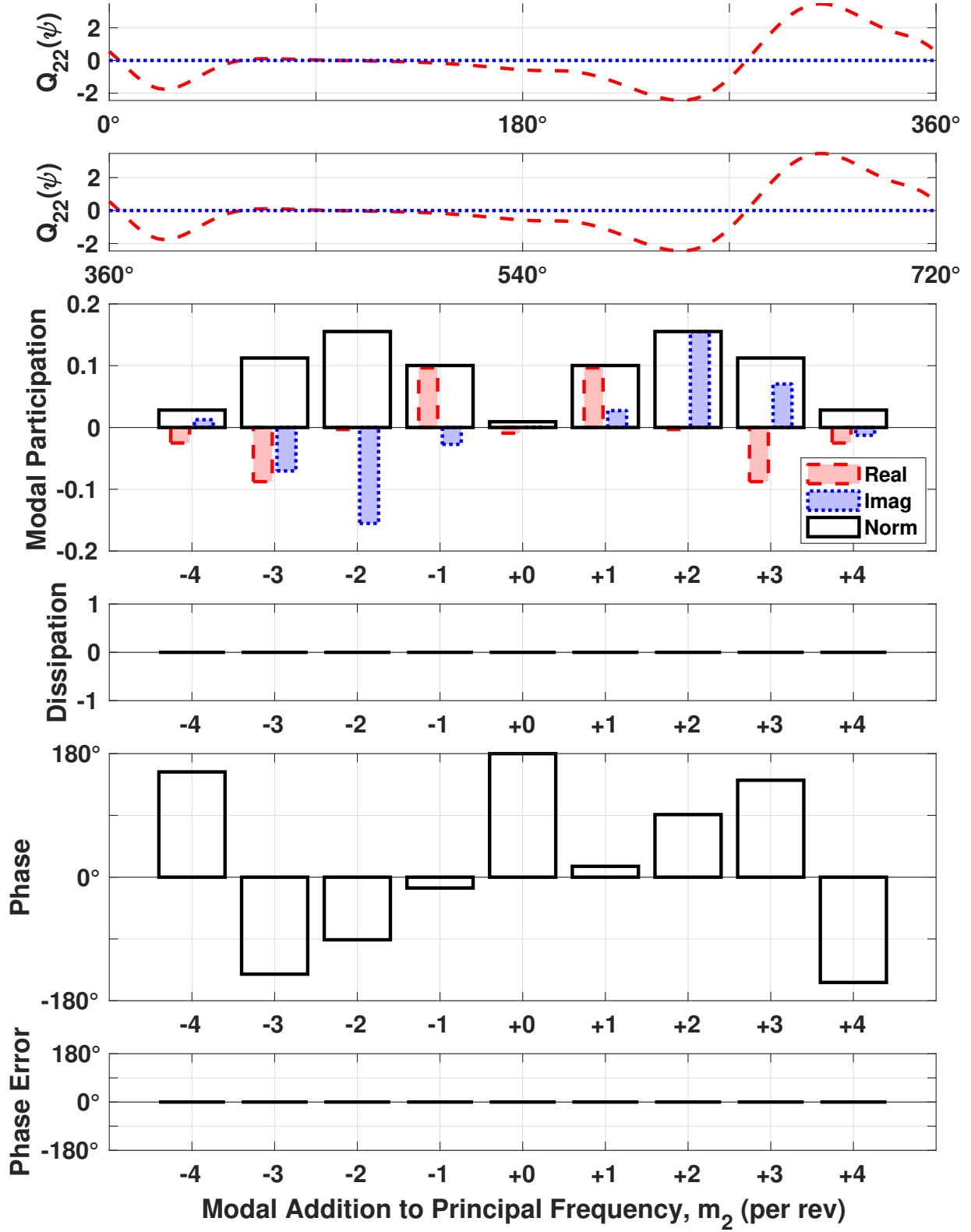


Figure 2.25: Eigenvector motion and modal participation, F2, $\nu_\beta = 1$, $\gamma = 12$, $\mu = 3$.

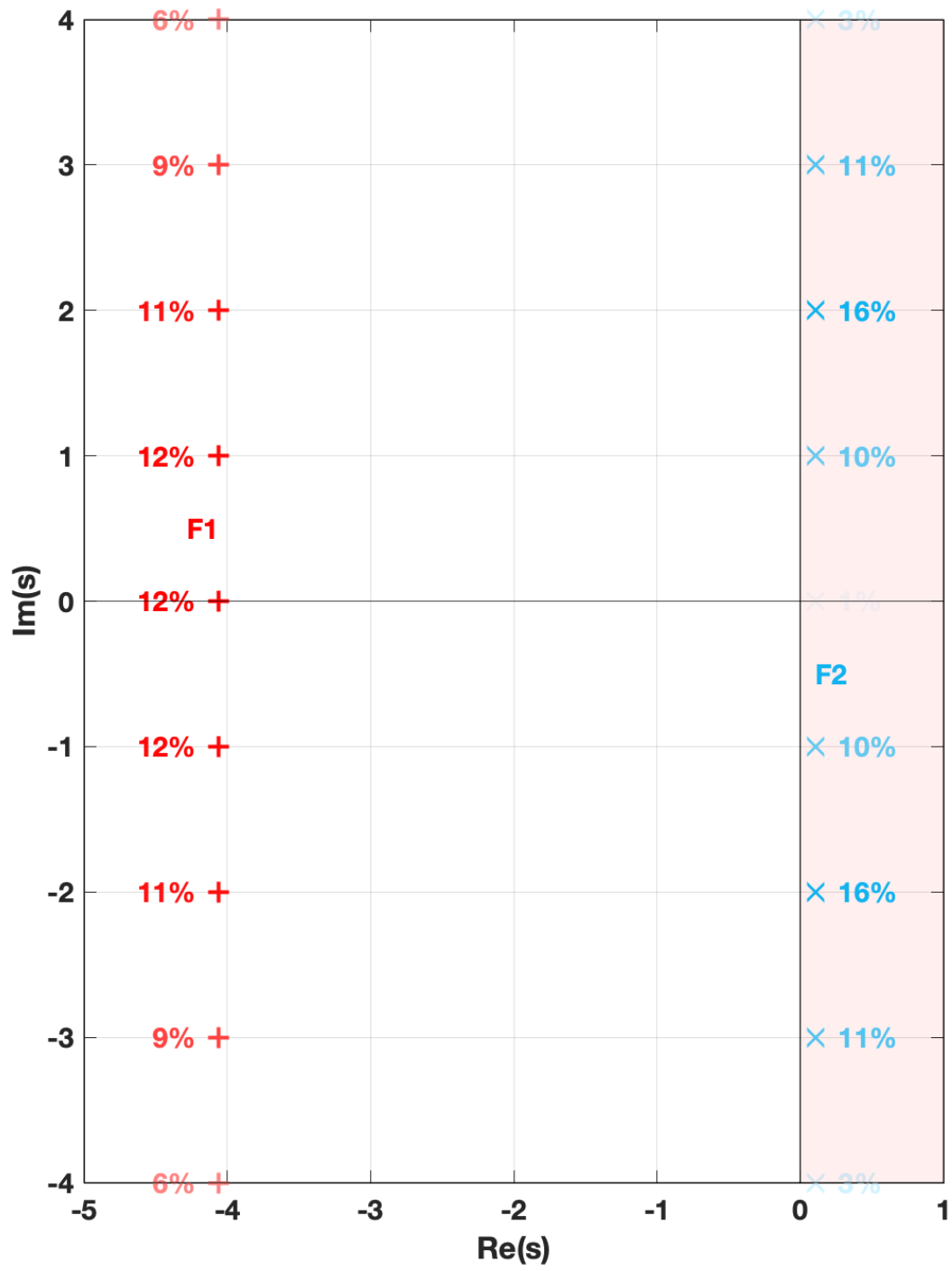


Figure 2.26: Flap root locus, $\nu_\beta = 1$, $\gamma = 12$, $\mu = 3$.

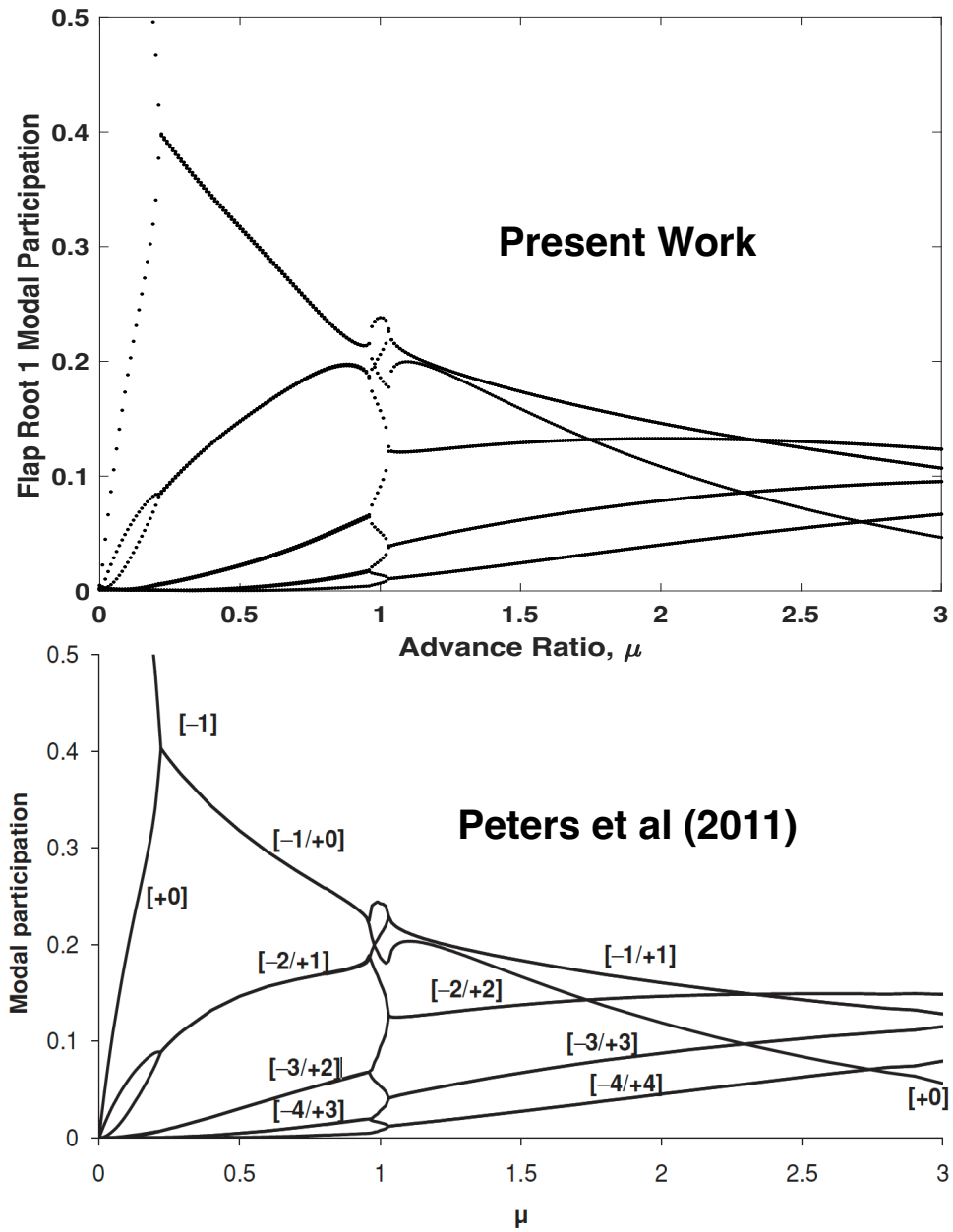


Figure 2.27: Modal participation method verification.

Fig. 2.28 shows the modal participation vs. μ for F1 with reverse flow. The positive and negative modal factors converging at $\mu = 0.22$ and 1.73 illustrate frequency coalescence and locking. The increasing numerical instability (dissipation and phase error) when $\mu > 2.4$ also corresponds to when F2 becomes physically unstable.

Fig. 2.28 shows the modal participation vs. μ for F2. As hinted at by the sample FFT cases earlier, the dominant frequencies trend slightly differently than F1. Although F2 is physically unstable, it is numerically stable.

One set of parameters that could affect stability are the aerodynamic coefficients and root cutout. The simple thin airfoil assumed at the beginning of this section differs slightly from the NACA 0012 values depicted in Fig. 1.4. Fig. 2.30 shows the differences in numerical dissipation for three blade configurations. The only noticeable difference, albeit small, was the root cutout's slight increase in numerical stability (dissipation) between $\mu = 1$ and 2 . The flap damping, principal frequency, and modal participation were found to be effectively identical regardless of configuration and are not shown.

Although there are multiple frequencies for each root, it is desirable to choose one to provide a cleaner visualization of trajectories in the s -plane. In the cases where parametric resonance means there are two opposite frequencies, the closest one is chosen in order to avoid jumping between progressive and regressive modes. This is shown in Fig. 2.31 for the present $\nu_\beta = 1$, $\gamma = 12$ case.

Finally a “smooth” root loci can be constructed, as shown in Figs. 2.32, 2.33, and 2.34.

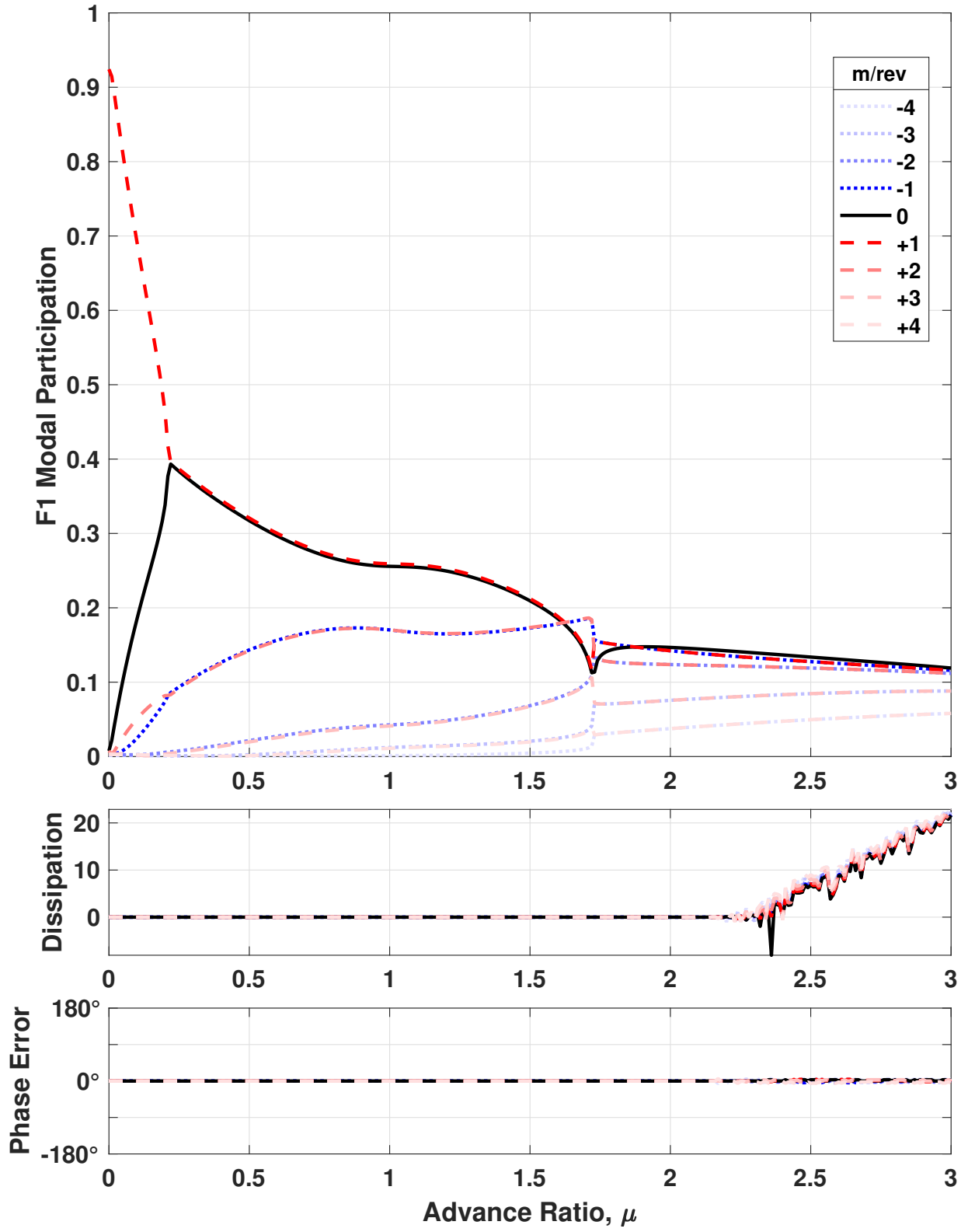


Figure 2.28: Modal participation vs. μ , $F1$, $\nu_\beta = 1$, $\gamma = 12$.

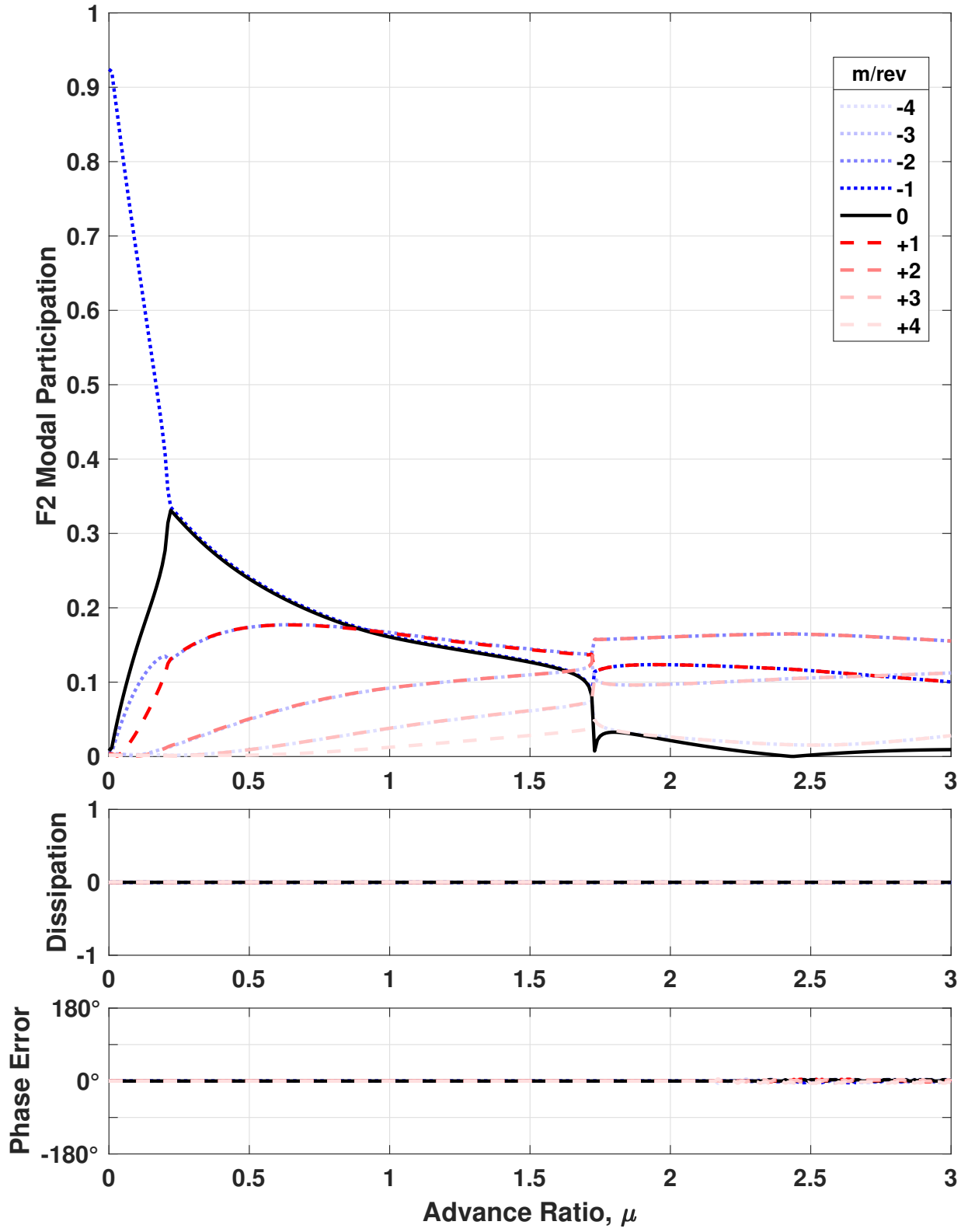


Figure 2.29: Modal participation vs. μ , F2, $\nu_\beta = 1$, $\gamma = 12$.

Blade A: $a = 2\pi$, $c_d = 0.01$, $x_c = 0$

Blade B: $a = 6$, $\hat{a} = 6$, $c_d = 0.01$, $\hat{c}_d = 0.03$, $x_c = 0$

Blade C: $a = 6$, $\hat{a} = 6$, $c_d = 0.01$, $\hat{c}_d = 0.03$, $x_c = 0.1$

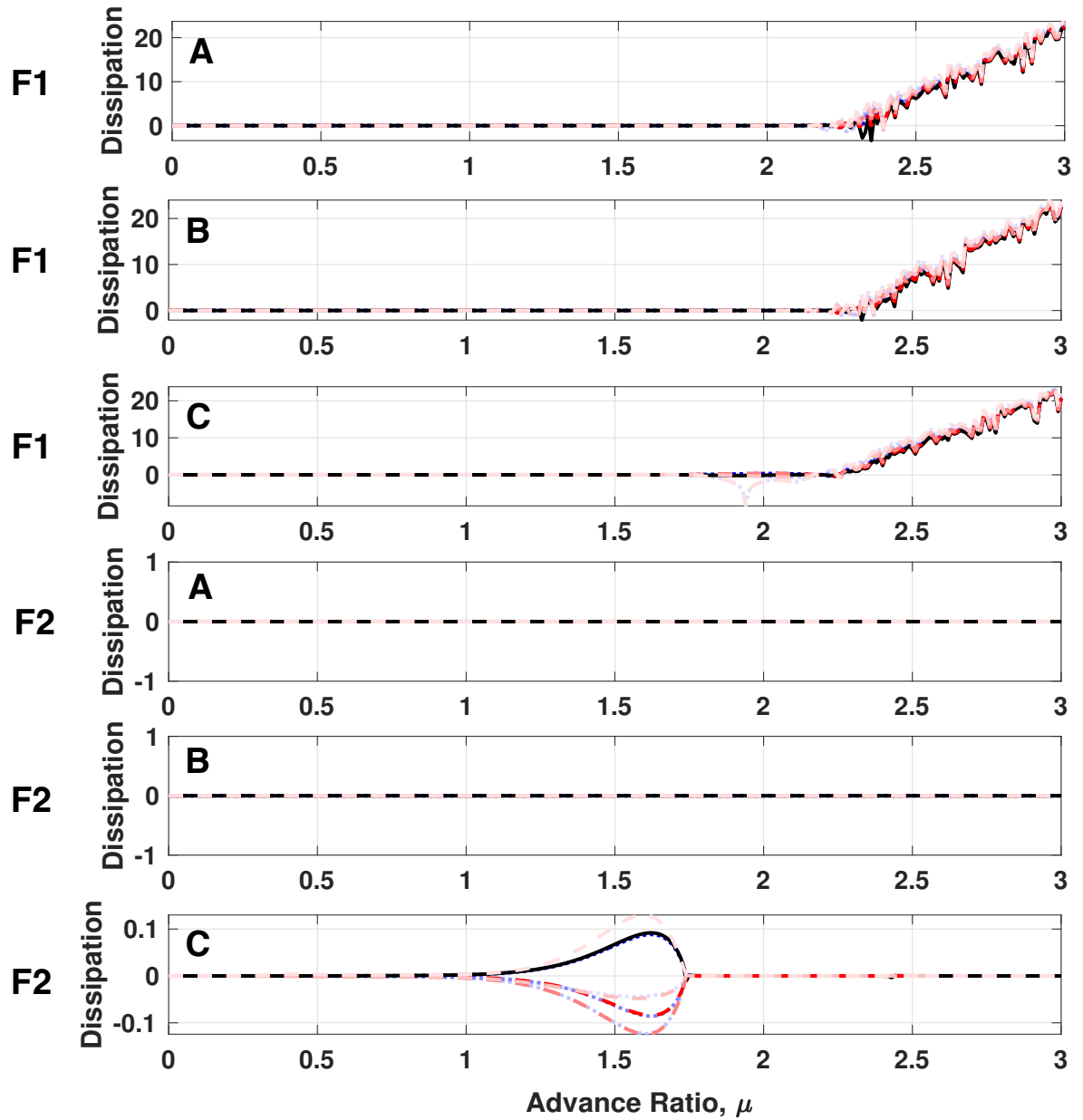


Figure 2.30: Effect of blade configuration on numerical stability, $\nu_\beta = 1$, $\gamma = 12$.

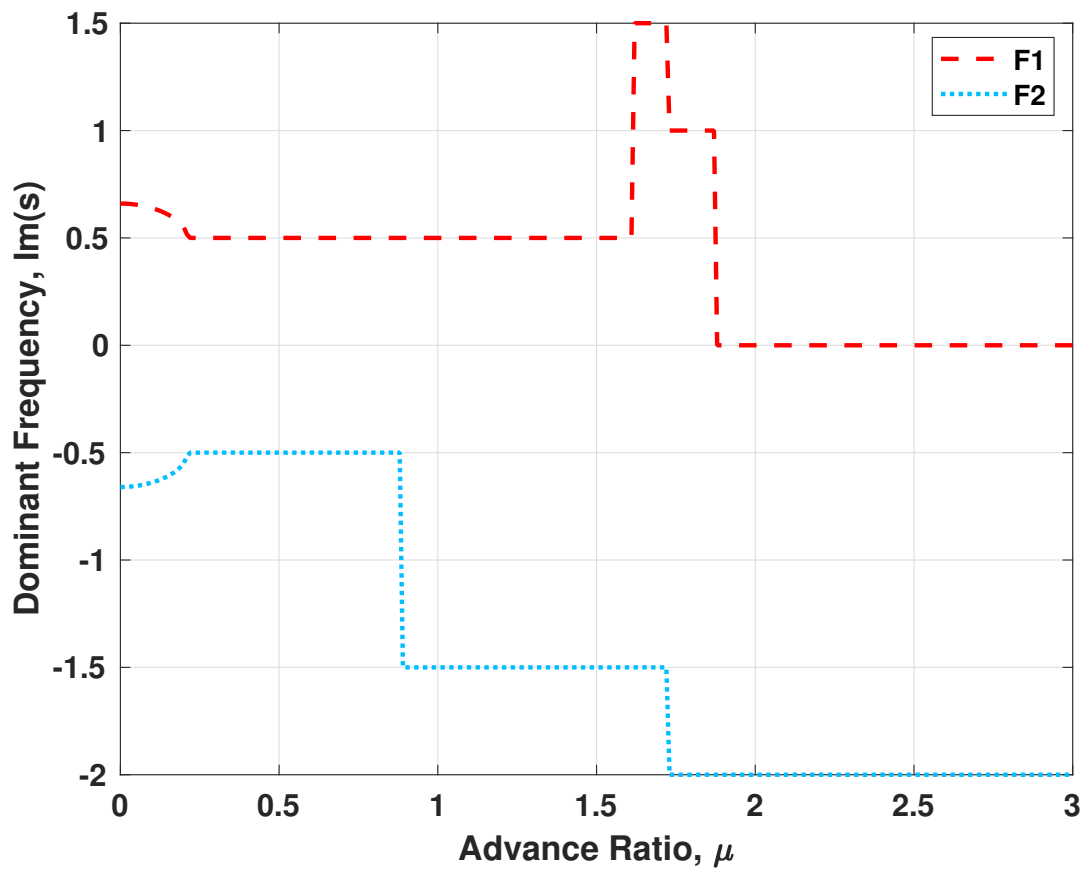


Figure 2.31: Dominant flap frequency responses vs. μ , $\nu_\beta = 1$, $\gamma = 12$.

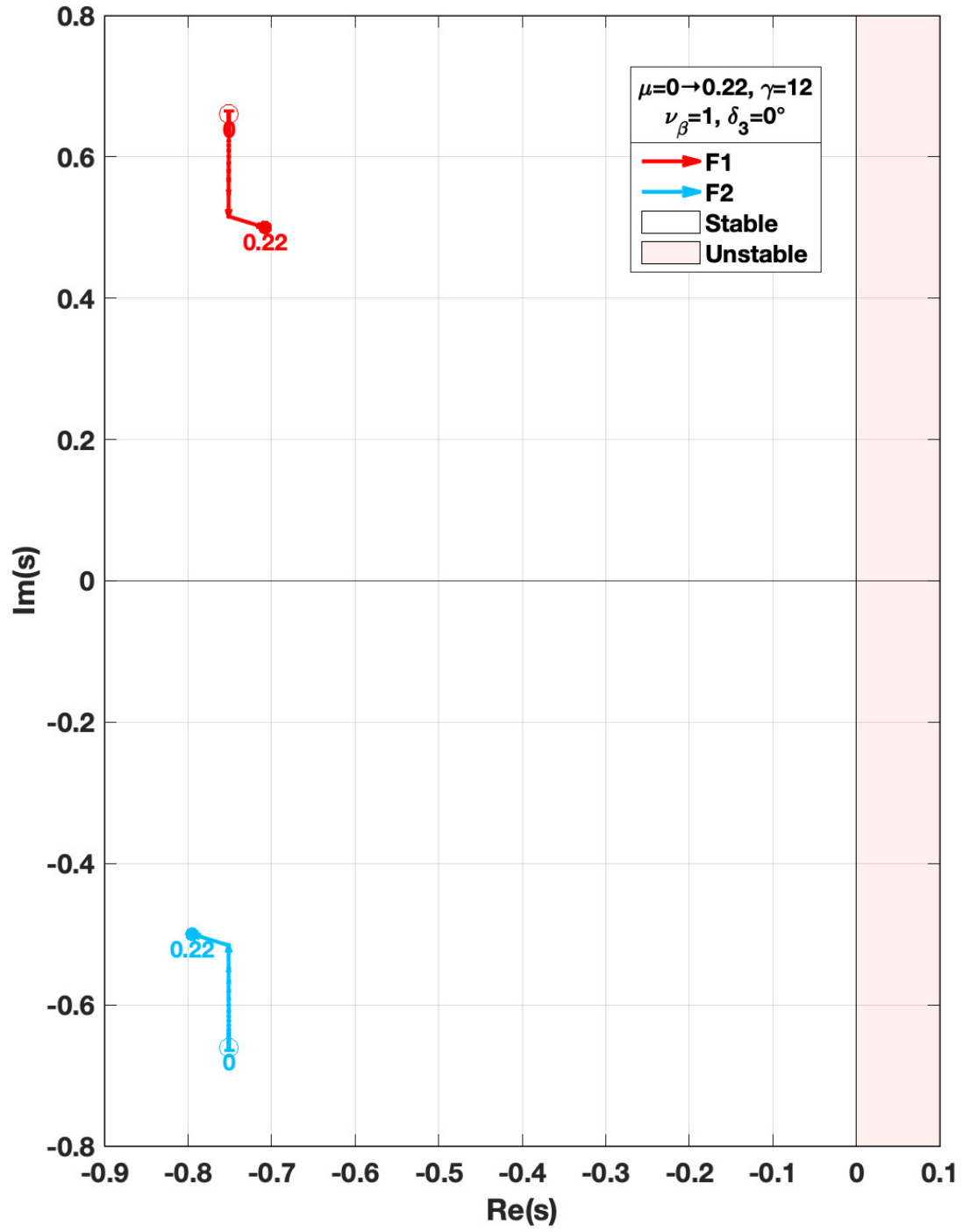


Figure 2.32: Flap root locus (dominant modes), $\nu_\beta = 1$, $\gamma = 12$, $\mu=0$ to 0.22.

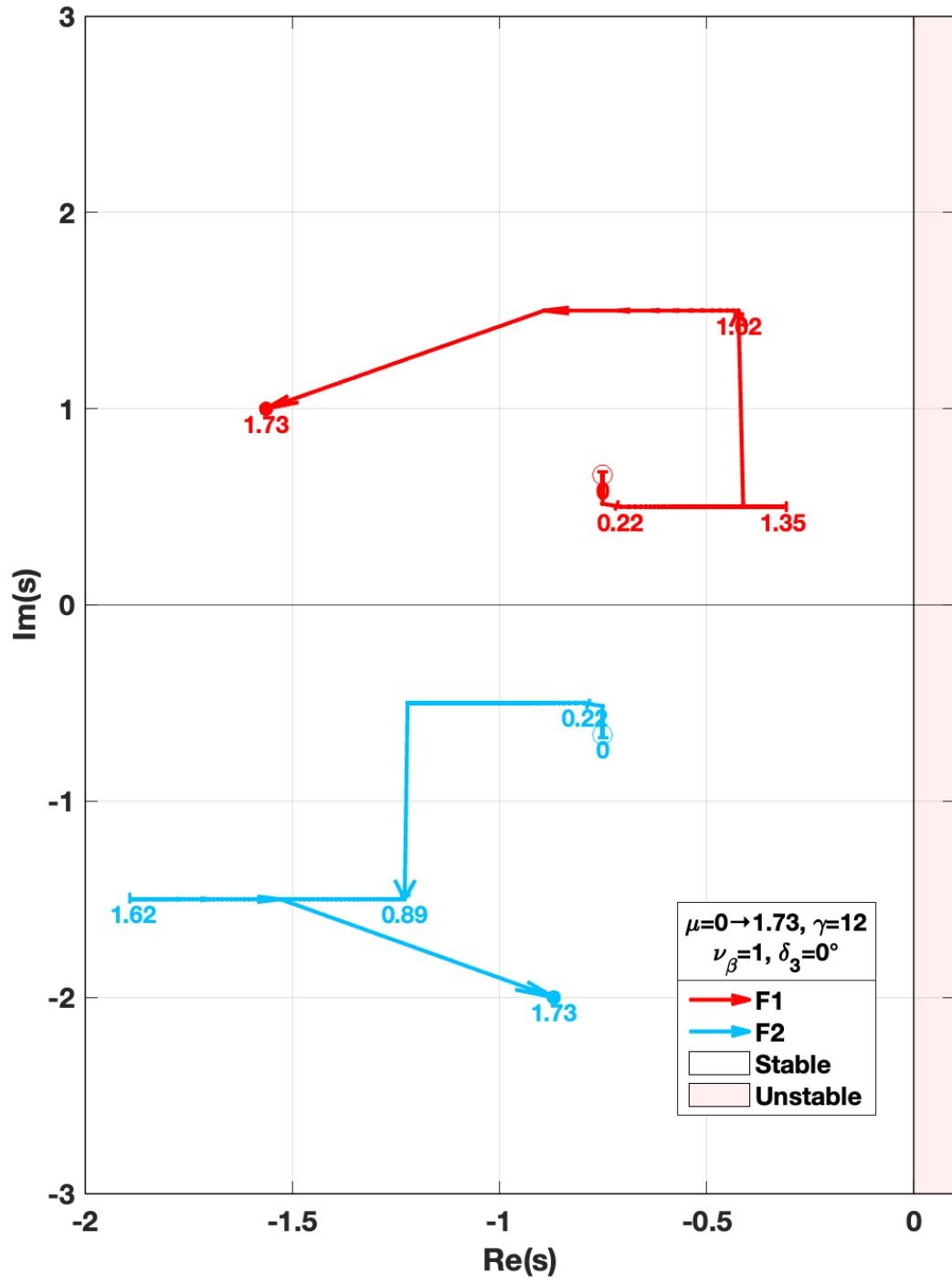


Figure 2.33: Flap root locus (dominant modes), $\nu_\beta = 1$, $\gamma = 12$, $\mu=0$ to 1.73.

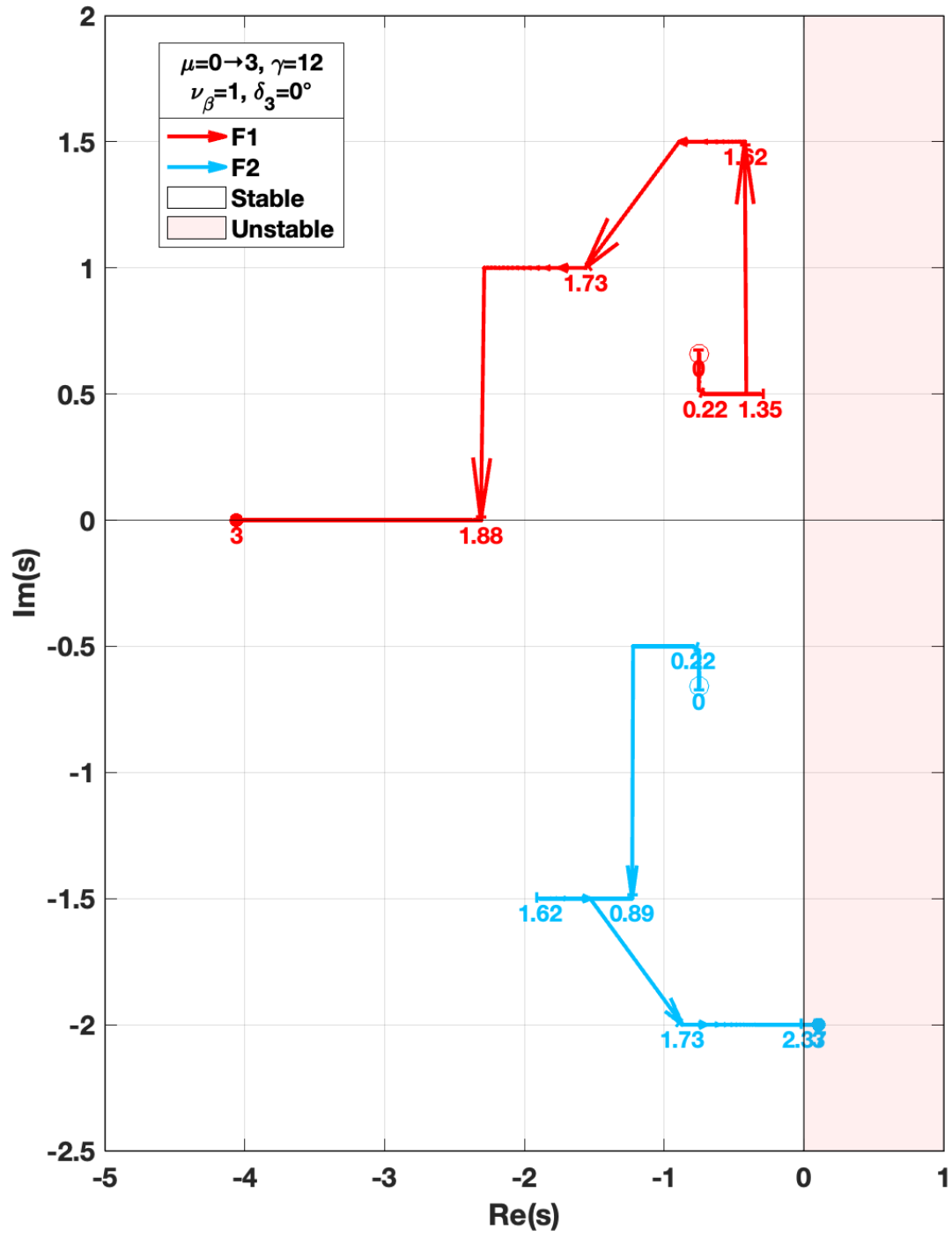


Figure 2.34: Flap root locus (dominant modes), $\nu_\beta = 1$, $\gamma = 12$, $\mu=0$ to 3.

2.6 Parametric Evaluation of Flap Stability

The prior examples assumed a single, fairly high Lock number of 12 to verify this method with prior analyses. Now that the general method has been demonstrated, a value of $\gamma = 6$ will now be investigated. The root locus in Fig. 2.35 shows that the lower (regressive) root F2 eventually becomes unstable in a -2 per rev response, just as in the higher Lock number case.

Now exploring the effect of pitch-flap coupling, a δ_3 angle of 15° is introduced. This root locus is seen in Fig. 2.36. This is a more stable configuration; the damping never becomes positive.

The flap frequency is increased to $\nu_\beta = 1.15$, approximating a hingeless rotor. The root locus is presented in Fig. 2.37. The root motion is similar to the previous δ_3 example, but the F1 root becomes unstable in this case.

The configuration for the stiff hingeless model rotor recently tested in the University of Maryland wind tunnel is analyzed. This design has a very high flap frequency of $\nu_\beta = 1.52/\text{rev}$ and a Lock Number of $\gamma = 4.74$. The root locus is seen in Fig. 2.38. It does not become unstable.

A two-bladed teetering rotor requires simultaneously accounting for the moments on both blades during the ODE integration process. The root locus is seen in Fig. 2.39. As indicated in previous analysis and experiments, is an extremely stable configuration for high- μ .

Flap stability can be parametrically analyzed with Lock number vs. advance ratio (γ vs. μ) stability maps. These maps are contour plots assembled by extracting the least

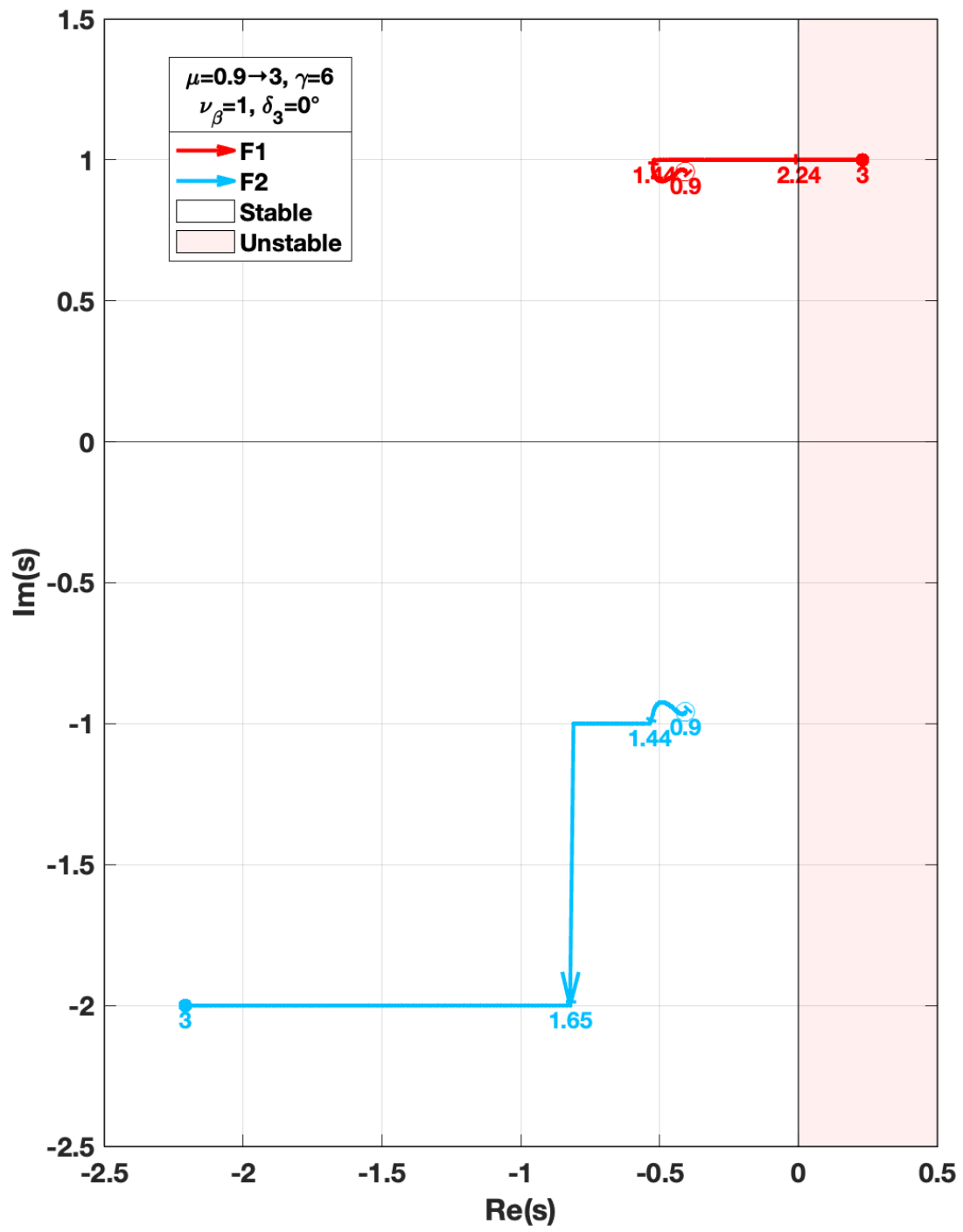


Figure 2.35: Flap root locus (dominant modes), $\nu_\beta = 1$, $\gamma = 6$, $\mu=0$ to 3.

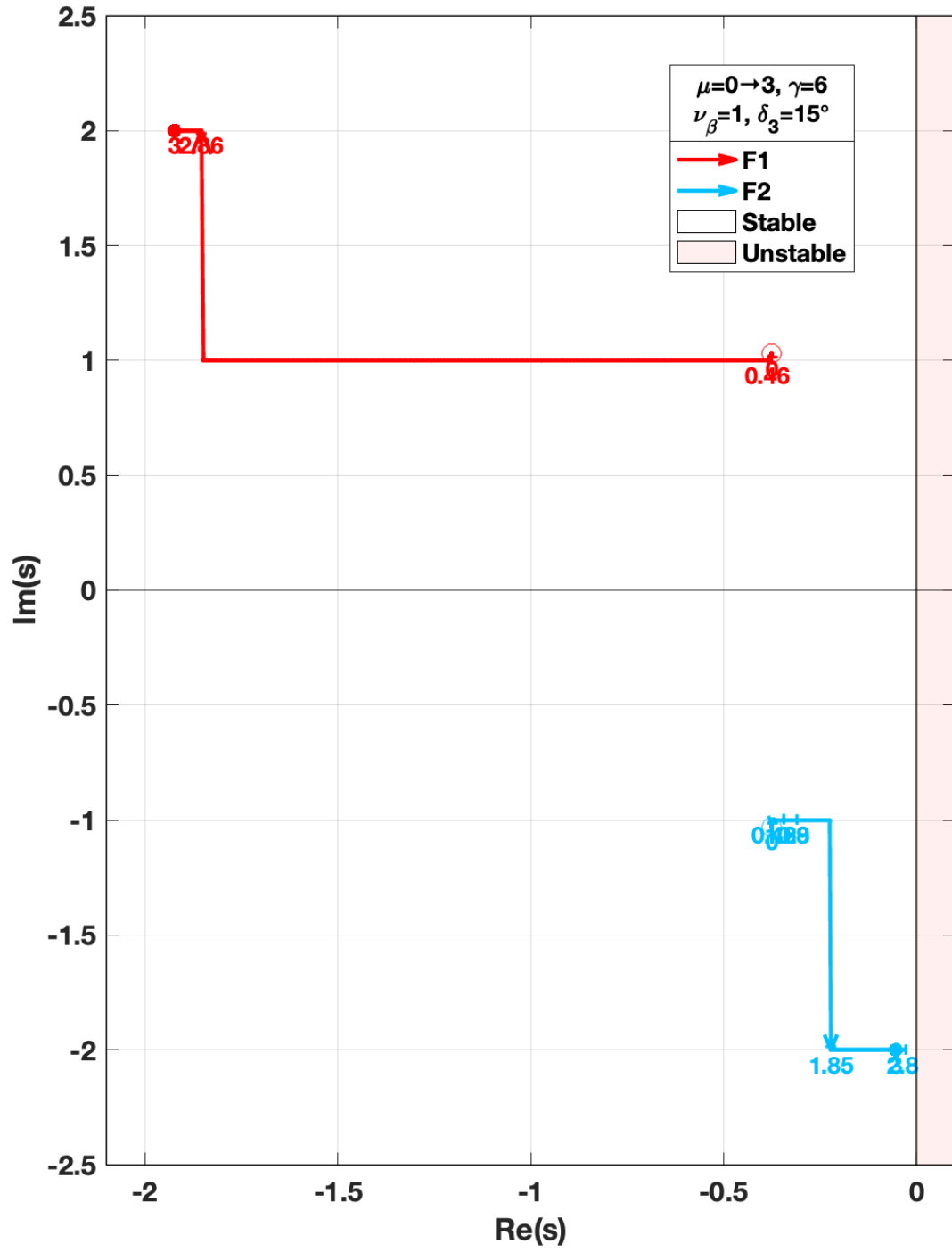


Figure 2.36: Flap root locus (dominant modes), $\nu_\beta = 1$, $\gamma = 6$, $\delta_3 = 15^\circ$, $\mu=0$ to 3.

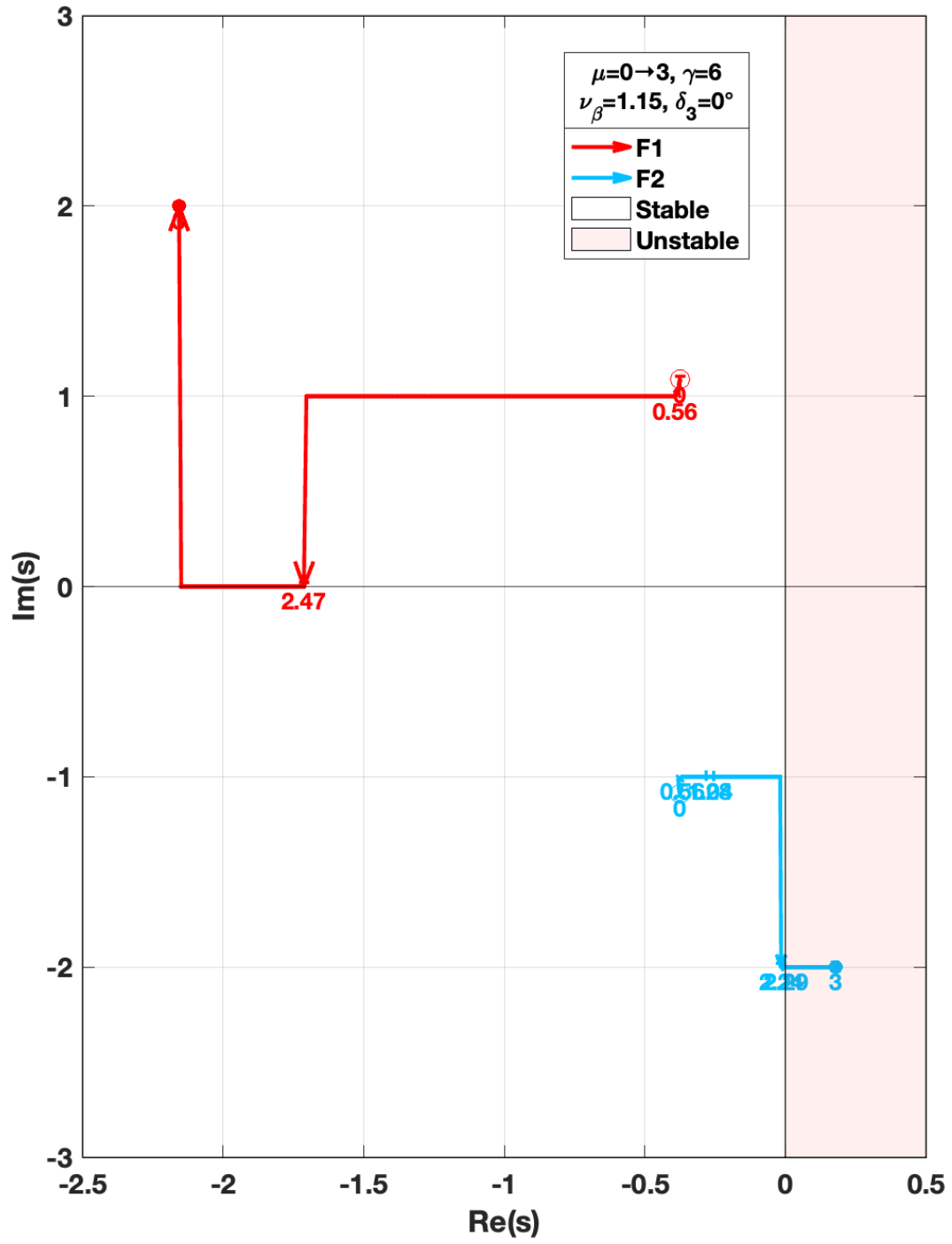


Figure 2.37: Flap root locus (dominant modes), $\nu_\beta = 1.15$, $\gamma = 6$, $\mu = 0$ to 3.

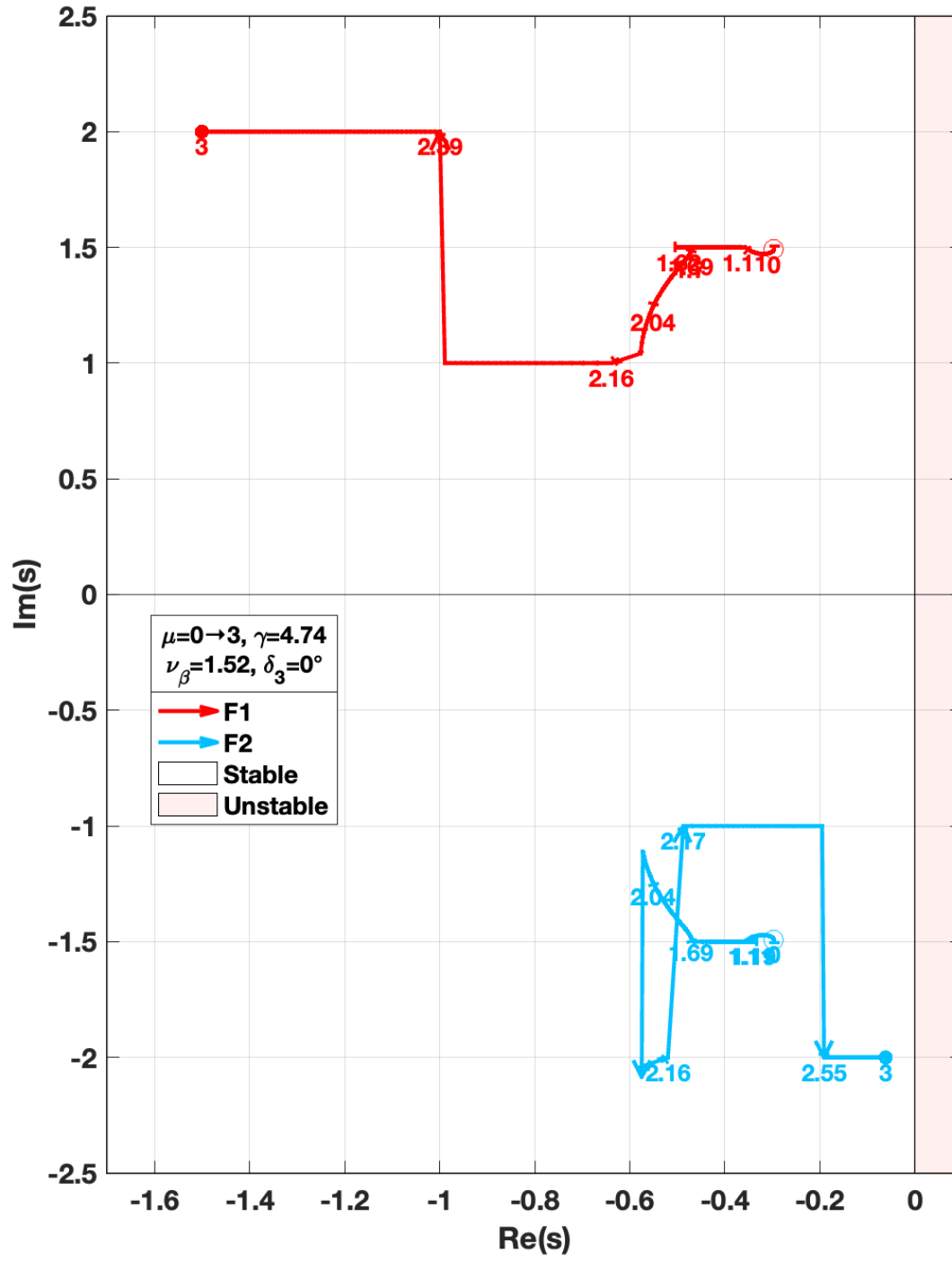


Figure 2.38: Flap root locus (dominant modes), $\nu_\beta = 1.52$, $\gamma = 4.74$, $\mu = 0$ to 3.

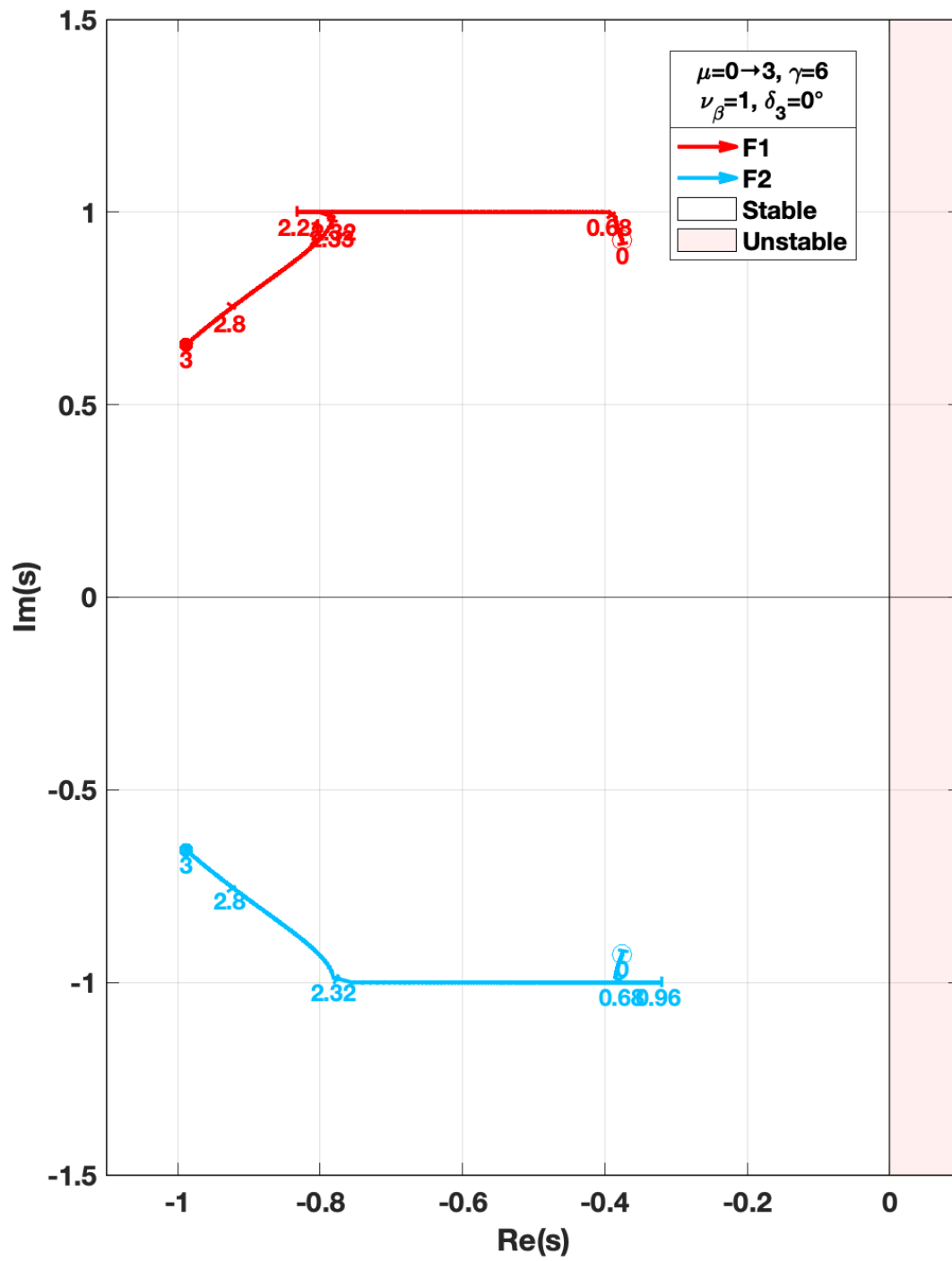


Figure 2.39: Flap root locus (dominant modes), teetering rotor, $\nu_\beta = 1$, $\gamma = 6$, $\mu = 0$ to 3.

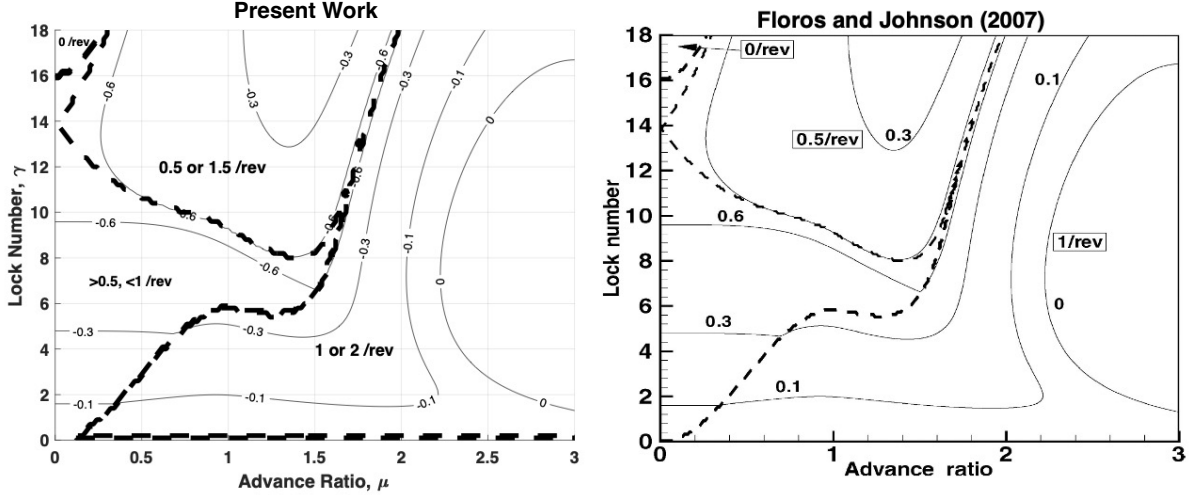


Figure 2.40: Flap stability map verification (γ vs. μ), articulated rotor, $\nu_\beta = 1$.

stable roots from thousands of root loci. Verification of the method is found through a comparison with Ref. 89, as shown in Fig. 2.40. Despite the opposite sign convention for damping, the results are consistent.

The following flap stability maps have been generated to show the frequency and instability regions for a variety of rotor types. The configurations are an articulated rotor (Fig. 2.41), an articulated rotor with δ_3 (Fig. 2.42), a notional hingeless rotor (Fig. 2.43), the UMD hingeless model rotor (Fig. 2.44,) and a teetering rotor (Ref. 2.45). The “upward” frequency region shifting effects of increasing flap frequency are noted. As in the root locus example, a positive δ_3 angle provides an effective flap frequency increase and is stabilizing. The teetering rotor appears to be the most desirable configuration. This, along with the pitch-flap coupling γ - μ maps, also match the results from Ref. 89.

The choice of limiting the analysis to $\mu = 3$ follows earlier conventions in the literature. It is also a consideration of how much a rotor can be slowed and still be subject to the

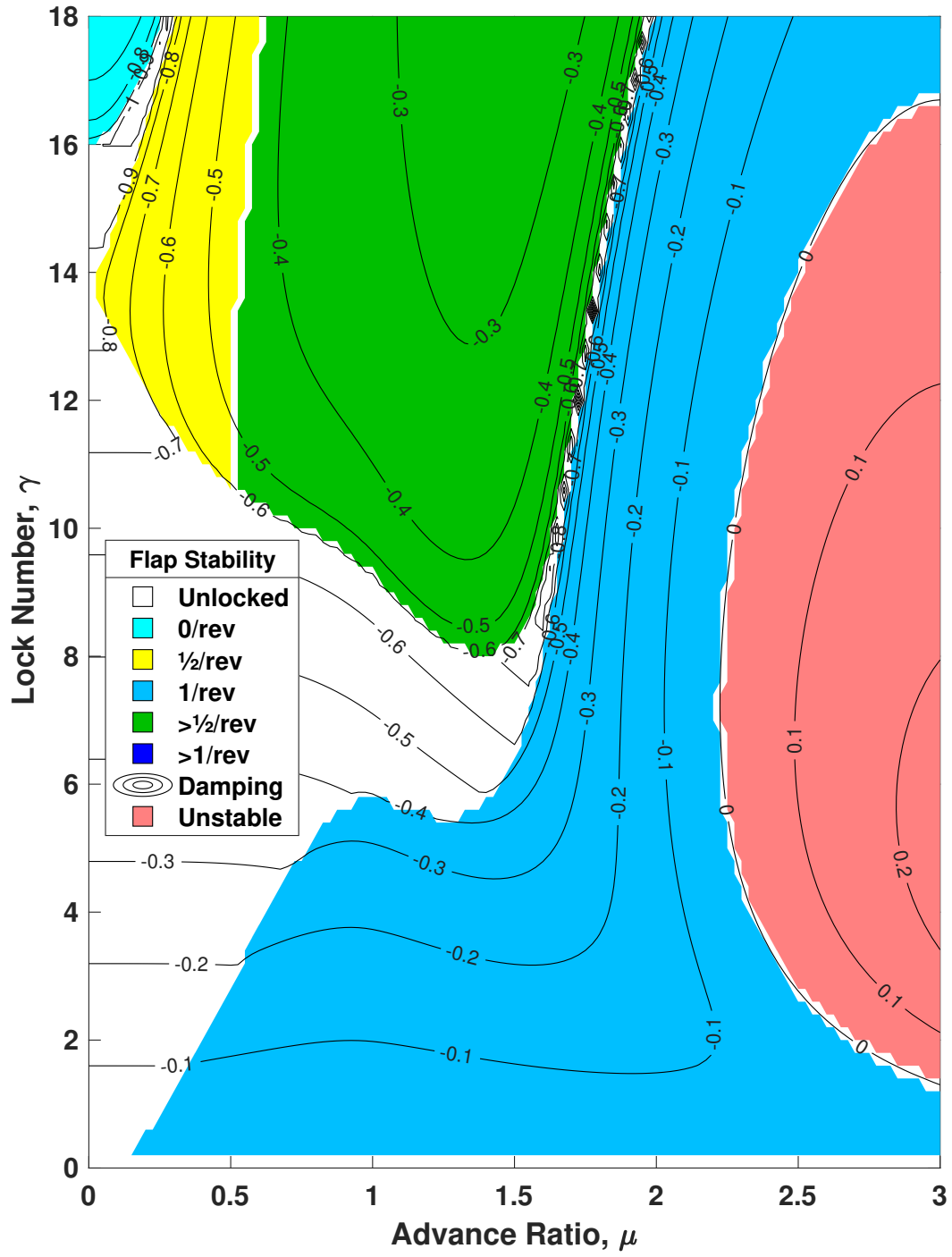


Figure 2.41: Flap stability map (γ vs. μ), articulated rotor, $\nu_\beta = 1$.

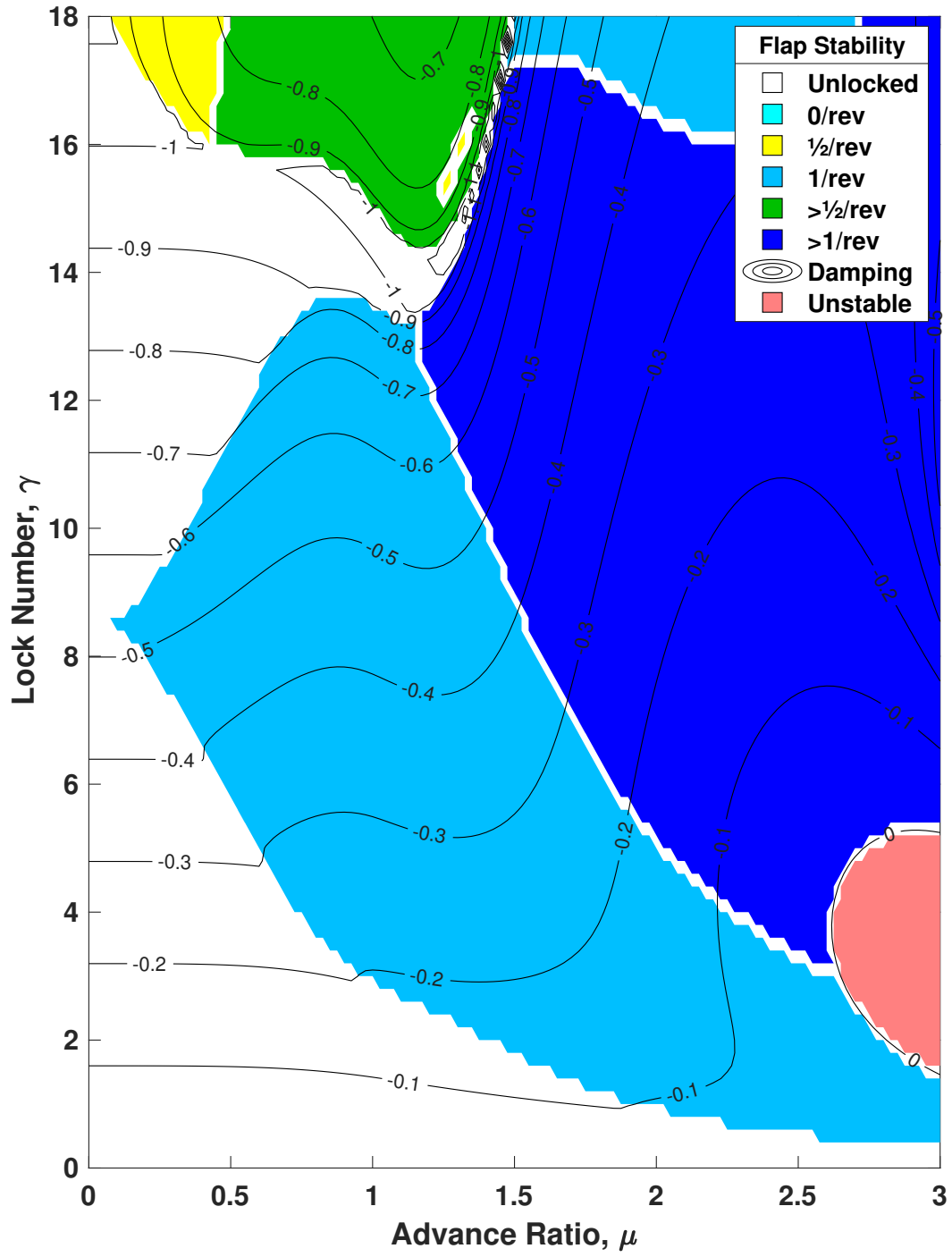


Figure 2.42: Flap stability map (γ vs. μ), articulated rotor, $\nu_\beta = 1$, $\delta_3 = 15^\circ$.

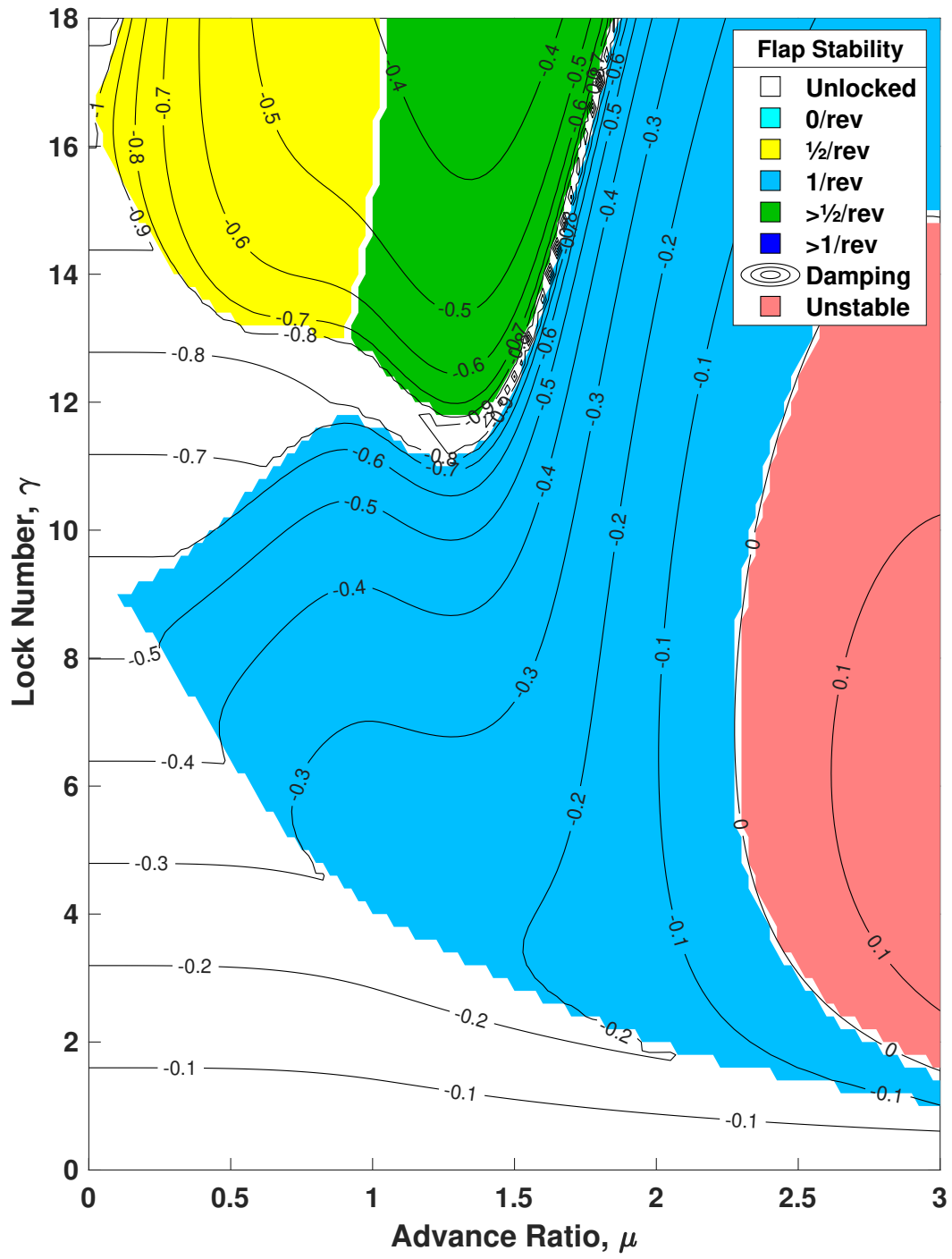


Figure 2.43: Flap stability map (γ vs. μ), hingeless rotor, $\nu_\beta = 1.15$.

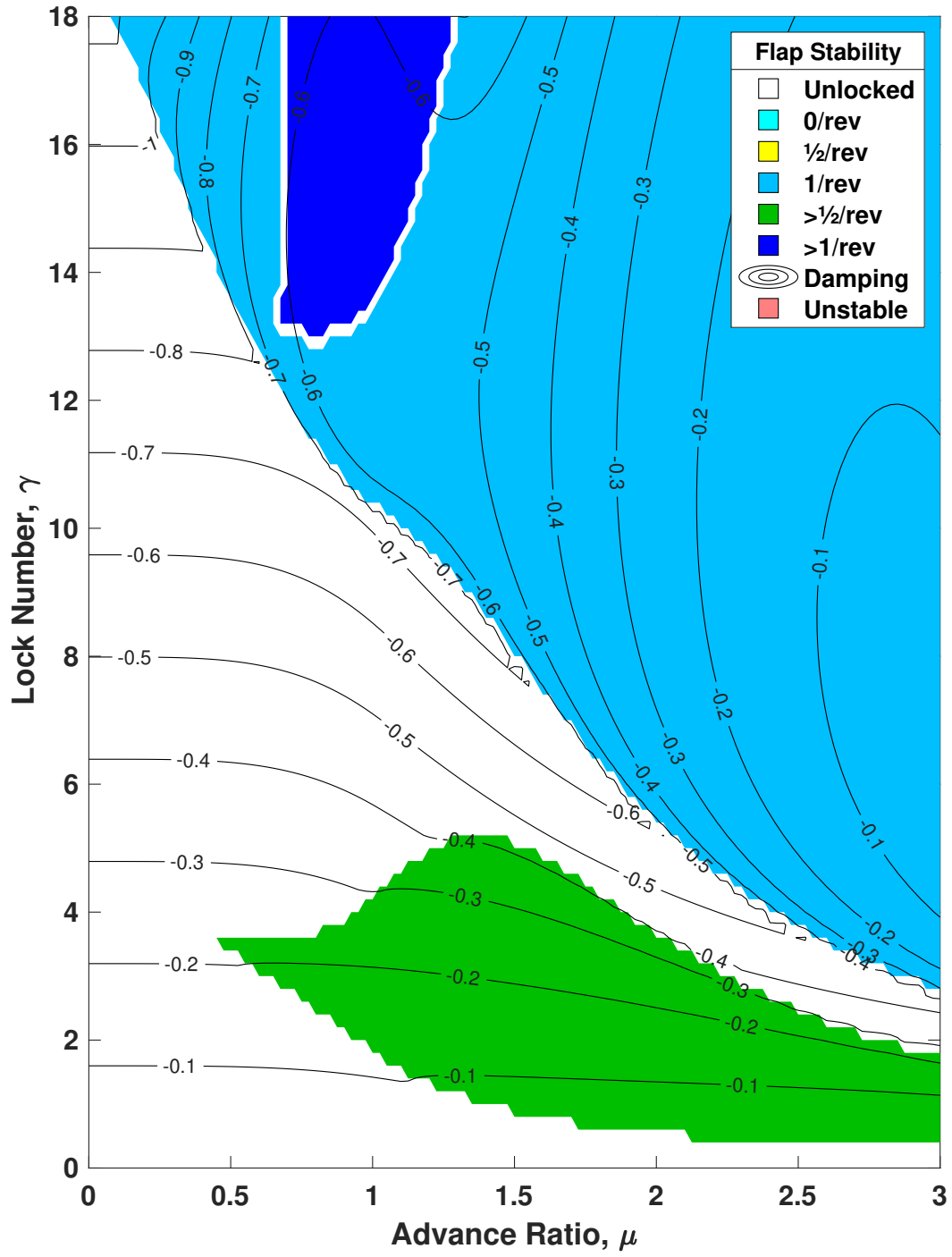


Figure 2.44: Flap stability map (γ vs. μ), UMD hingeless model rotor, $\nu_\beta = 1.52$.

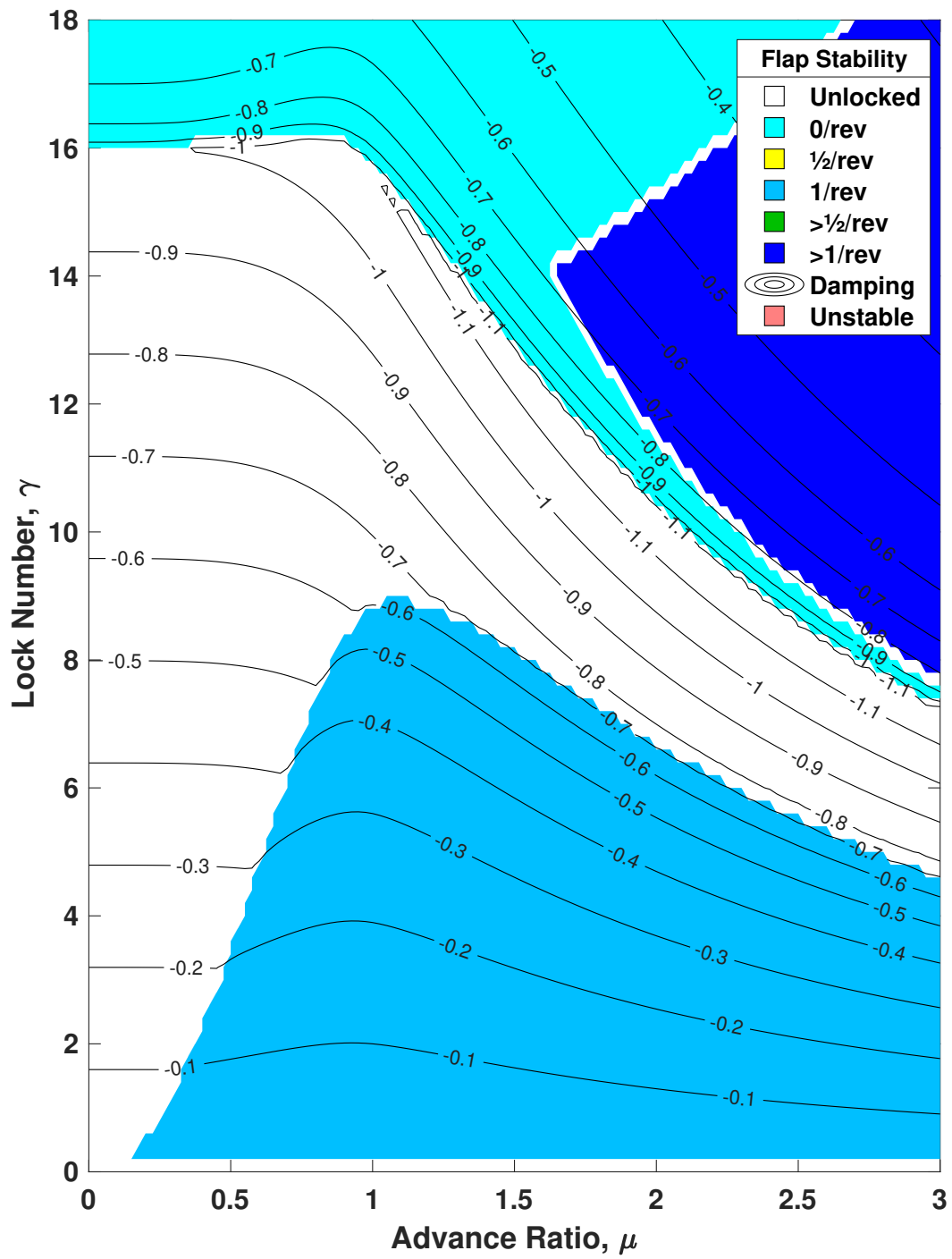


Figure 2.45: Flap stability map (γ vs. μ), teetering rotor, $\nu_\beta = 1$.

same aerodynamic assumptions. The transverse flow seen by the advancing blade tip is

$$U_T(\psi = 90^\circ, r = R) = \Omega R + V = \Omega R + \mu \Omega R = (1 + \mu) \Omega R \quad (2.41)$$

For example, if this tip speed is chosen so that it equates to Mach number 0.8 at $\mu = 3$, then at $\mu = 0$ this would result in a tip Mach of only 0.2, which is low for rotors.

Despite this practical concern, it is of interest to see what sort of numerical limits might be encountered at even higher advance ratios. In Fig. 2.46, the $\gamma - \mu$ envelope is expanded to $\gamma = 36$ (an extremely low-inertia rotor) and $\mu = 6$, quadrupling the map's area. The numerical stability for this map (dissipation magnitude of the dominant modal waveform) is shown in Fig. 2.47. The least stable flap root is numerically stable for all $\gamma - \mu$ combinations (note the absence of red contours), but as hinted at by the earlier fixed- γ modal analyses, the least stable root has diverging dissipation. This is significant in the upper right of the $\gamma \leq 18, \mu \leq 3$ quadrant (the limits of the previous maps) and for most of the three new quadrants. Despite this, the damping and frequency of the least stable root can be found for the entire map. Just as in the solution of the Mathieu equation, there are repeating patterns of frequency lobes and islands of stability.

2.7 Flap Stability Conclusions

A rigid hub-hinged rotor model with quasi-steady reverse flow aerodynamics has been created to analyze stability in the rotating frame using Floquet theory. In a new application to the high- μ problem, a modal waveform analysis of the periodic eigenvectors was performed. This provides insights into the multi-spectral frequency response and para-

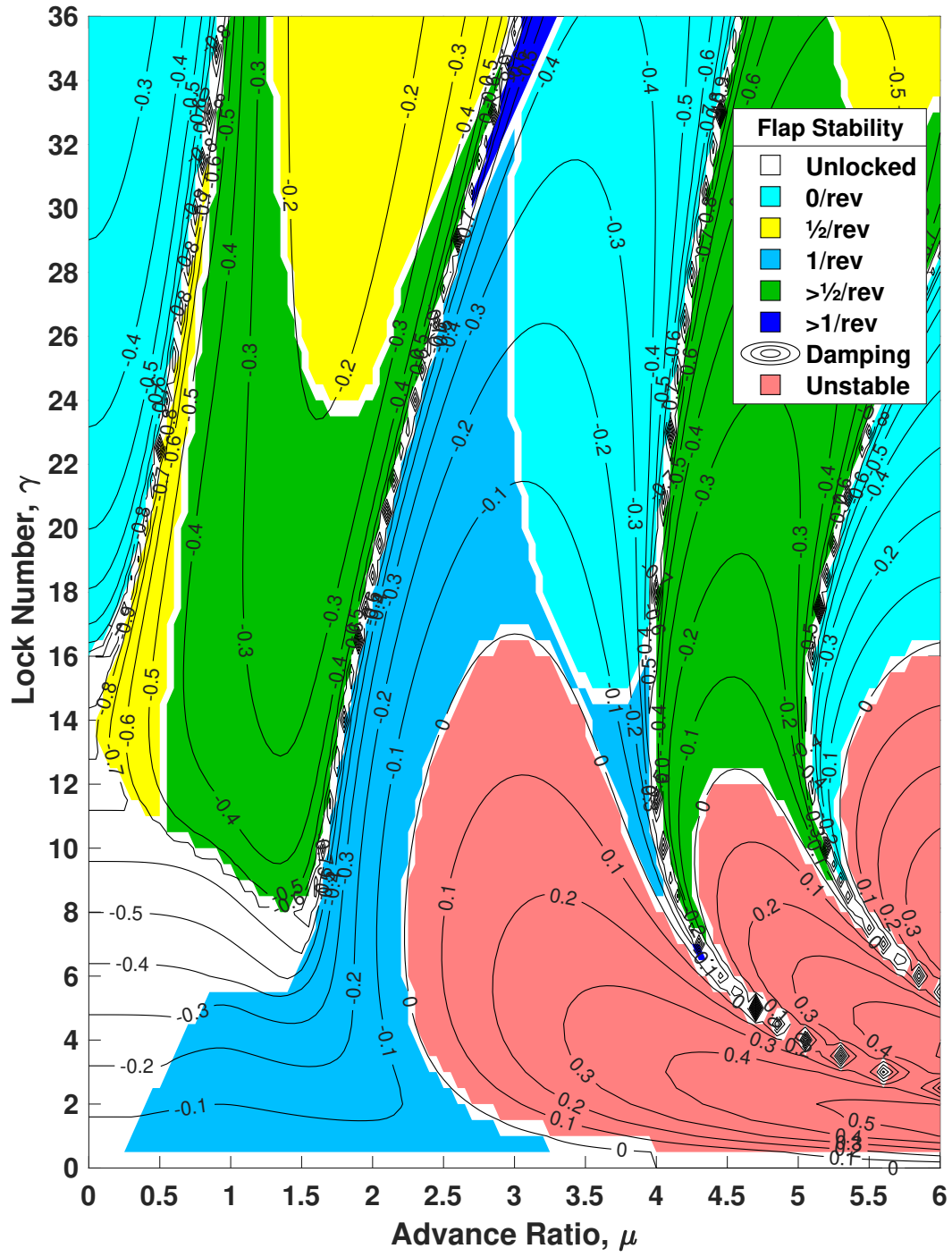


Figure 2.46: Expanded flap stability map (γ vs. μ), $\nu_\beta = 1$.

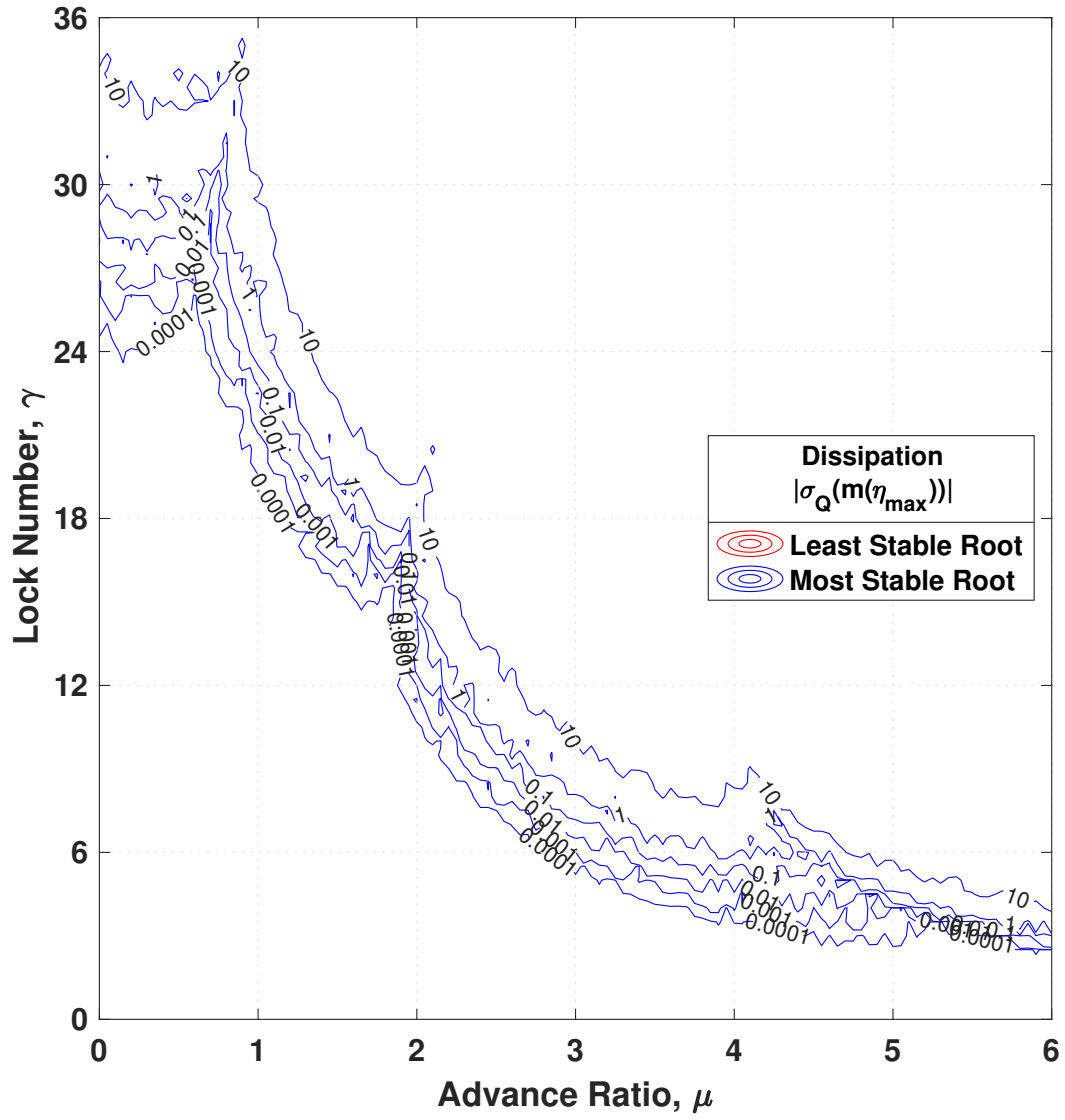


Figure 2.47: Numerical stability of expanded flap stability map.

metric resonance, which has been previously under-explored. Numerical instability is also quantified, in addition to finding the physical stability of the physical system.

Many configurations with varying parameters were analyzed. While the teetering rotor has promising stability results, as evidenced by previous analyses and real world flight tests, the subsequent analyses in this work will focus on articulated and hingeless rotors. These are of more immediate interest for wind tunnel testing at the University of Maryland.

Chapter 3: High- μ Flap Trim

3.1 Forcing and Stall

As shown in the previous chapter, the non-dimensional flap equation is

$$\beta^{**} + \nu_\beta^2 \beta = \gamma \bar{M}_\beta \quad (3.1)$$

with the aerodynamic forcing moment on the RHS expanding to

$$\bar{M}_\beta = M_{\beta}^* \beta + M_\beta \beta + M_\theta \theta + M_{\theta_{tw}} \theta_{tw} + M_\lambda \lambda \quad (3.2)$$

The M_{β}^* and M_β terms were derived and subtracted from the LHS of the equation, leaving the M_θ , $M_{\theta_{tw}}$, and M_λ moments on the RHS to be revisited now.

The blade pitch θ is a function of ψ . It is affected by control inputs, coupling, and blade twist. The collective θ_0 , lateral cyclic θ_{1c} , and longitudinal cyclic θ_{1s} are combined to form the periodic control input:

$$\theta_C = \theta_0 + \theta_{1c} \cos \psi + \theta_{1s} \sin \psi \quad (3.3)$$

Combining this with pitch-flap coupling,

$$\theta_s = \theta_C - k_{p\beta}\beta \quad (3.4)$$

Considering blade twist, the blade pitch at a given spanwise location is

$$\theta = \theta_s + \theta_{tw} x \quad (3.5)$$

The total inflow ratio is the sum of the inflow due to the axial component of the advance ratio and the induced inflow due to the rotor thrust:

$$\lambda = \mu_z + \lambda_i = \mu \tan \alpha_i + \frac{C_T}{2 \sqrt{\lambda^2 + \mu^2}} \quad (3.6)$$

The forward shaft tilt and longitudinal flapping are combined into the rotor disk incidence angle:

$$\alpha_i = \alpha_s + \beta_{1c} \quad (3.7)$$

Because λ is a recursive function in forward flight, it is found using a Newton-Raphson iteration. Periodic and spanwise-varying inflow models such as those found in Refs. [31](#) and [60](#) could be implemented with a mean value of $\bar{\lambda}$ determined as above.

The non-dimensional velocities are

$$\begin{aligned}
u_T &= x + \mu \sin \psi \\
u_P &= \lambda + x\beta^* + \beta\mu \cos \psi \\
u_R &= \mu \cos \psi
\end{aligned} \tag{3.8}$$

The effect of the spanwise (yawed) flow is not directly included in the lift and drag equations, but is used to determine blade stall regions. While u_R can be considerable at high advance ratios, the primary effect is to create an equivalent sweep:

$$\cos \Lambda = \sqrt{\frac{u_T^2 + u_P^2}{u_T^2 + u_P^2 + u_R^2}} \tag{3.9}$$

Within the linear lift curve slope region, there is no change to a or c_d due to Λ . The yawed flow effect of u_R provides a 3D flow correction to stall, where the stall angle of attack (AOA) increases by the relation

$$\alpha_{max_{3D}} = \alpha_{max} / \cos^2 \Lambda \tag{3.10}$$

The reverse flow region takes up a large portion of the retreating side of the rotor disk in high advance ratio flight. As a portion of the blade transitions through the narrow stalled boundary of the reverse flow region, u_T “instantly” becomes negative. The process repeats as the blade leaves the region and u_T becomes positive again. An illustration of AOA and retreating blade stall in this regime is shown in Figs. 3.1 and 3.2.

Following the derivation in Ref. 51, assuming small angles of attack and blade pitch,

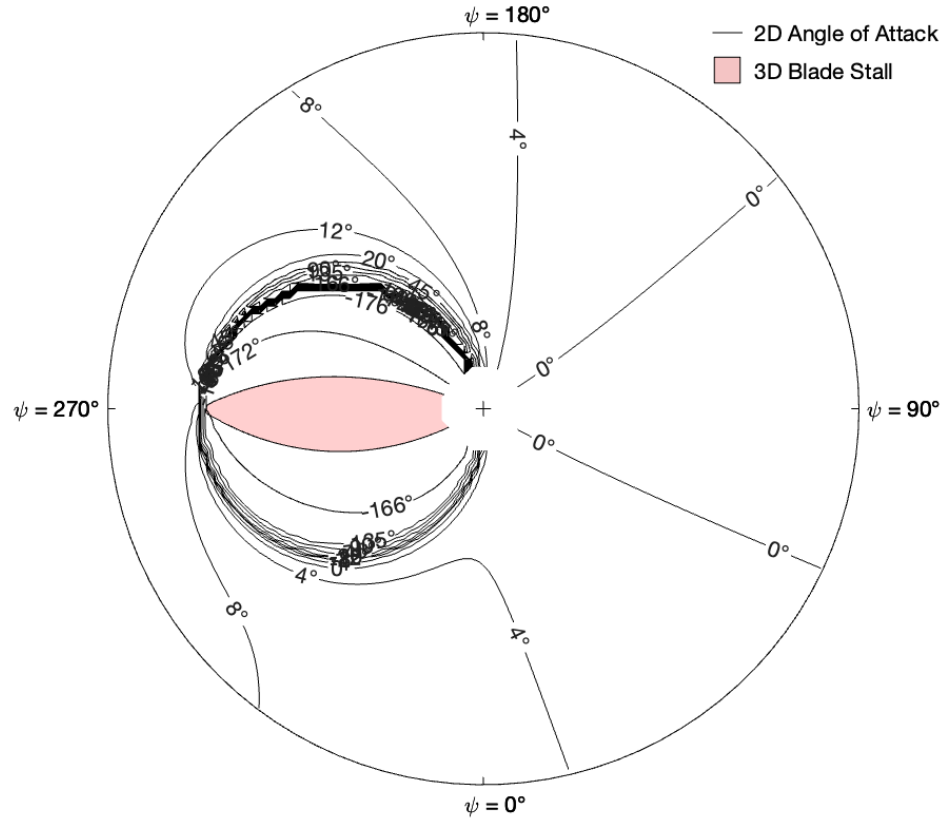


Figure 3.1: Angle of attack distribution and blade stall, $\nu_\beta = 1/\text{rev}$, $\gamma = 6$, $\mu = 0.75$, $C_T/\sigma = 0.02$.

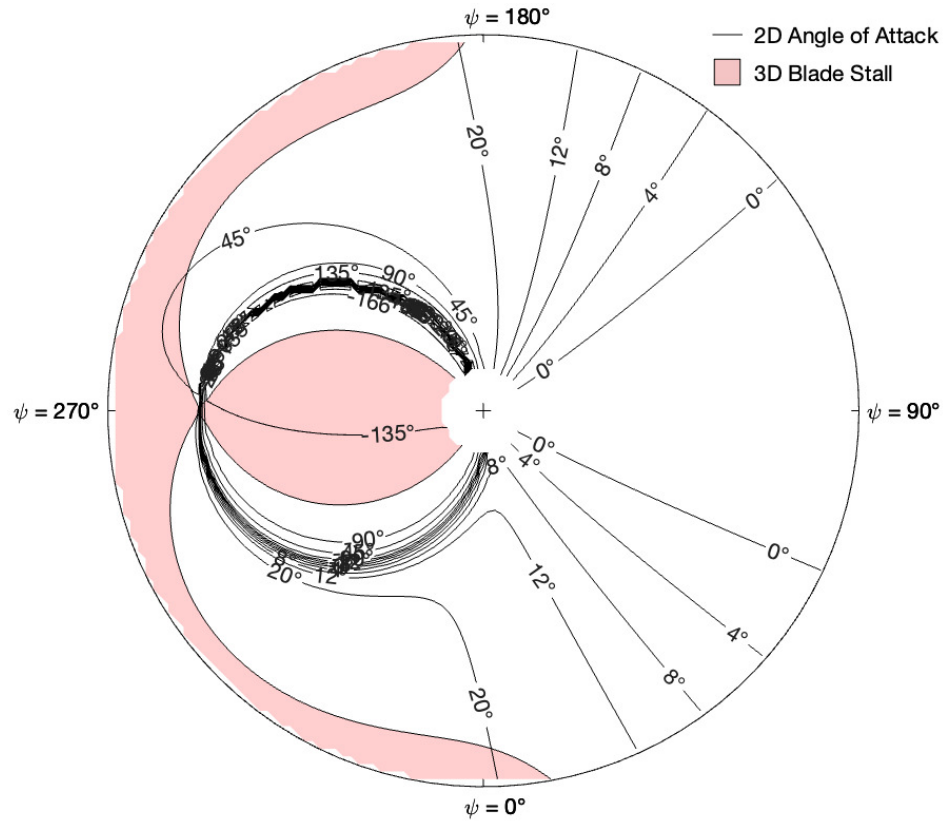


Figure 3.2: Angle of attack distribution and blade stall, $\nu_\beta = 1/\text{rev}$, $\gamma = 6$, $\mu = 0.75$, $C_T/\sigma = 0.08$.

the non-dimensional flapping force is

$$\bar{F}_\beta = \frac{1}{2} \left(u_T^2 \theta - \left(1 + \frac{c_d}{a} \right) u_P u_T \right) \quad (3.11)$$

which is always valid for $0^\circ < \psi < 180^\circ$. In the reverse flow region,

$$\bar{\bar{F}}_\beta = -\frac{1}{2} \frac{\hat{a}}{a} \left(u_T^2 \theta + \left(1 + \frac{\hat{c}_d}{\hat{a}} \right) u_P u_T \right) \quad (3.12)$$

The portion of the blade in reverse flow is from $x = 0$ to $-\mu \sin \psi$. When $\mu \leq 1$, the flap moment on the retreating side ($180^\circ \leq \psi \leq 360^\circ$) is therefore

$$\bar{M}_\beta = \int_{-\mu \sin \psi}^1 x \bar{F}_\beta dx + \int_0^{-\mu \sin \psi} x \bar{\bar{F}}_\beta dx \quad (3.13)$$

When $\mu > 1$, there is a section of the retreating side where the entire blade is in reverse flow: $270^\circ - \cos^{-1}(1/\mu) < \psi < 270^\circ + \cos^{-1}(1/\mu)$. For each of these azimuth segments, the flap moment functions are integrated and the terms are grouped with like terms $(\beta^*, \beta, \theta, \theta_{tw}, \lambda)$.

Even if numerically evaluated, the equations can become extremely cumbersome. Symbolic manipulation software (MATLAB's `int`, `collect`, and `subs` functions within `syms`) are used to prevent algebraic errors when isolating these terms.

Neglecting root cutout, on the advancing side ($0^\circ \leq \psi \leq 180^\circ$), the RHS forcing moments

are:

$$\begin{aligned}
M_\theta &= \frac{\mu^2 \sin(\psi)^2}{4} + \frac{\mu \sin(\psi)}{3} + \frac{1}{8} \\
M_{\theta_{tw}} &= \frac{\mu^2 \sin(\psi)^2}{6} + \frac{\mu \sin(\psi)}{4} + \frac{1}{10} \\
M_\lambda &= -\frac{c_d}{6a} - \frac{\mu \sin(\psi)}{4} - \frac{c_d \mu \sin(\psi)}{4a} - \frac{1}{6}
\end{aligned} \tag{3.14}$$

For the partial reverse flow region ($180^\circ < \psi < 360^\circ$):

$$\begin{aligned}
M_\theta &= \frac{\mu^2 \sin(\psi)^2}{4} - \frac{\mu^4 \sin(\psi)^4}{24} + \frac{\mu \sin(\psi)}{3} - \frac{\hat{a} \mu^4 \sin(\psi)^4}{24a} + \frac{1}{8} \\
M_{\theta_{tw}} &= \frac{\mu^2 \sin(\psi)^2}{6} + \frac{\mu^5 \sin(\psi)^5}{60} + \frac{\mu \sin(\psi)}{4} + \frac{\hat{a} \mu^5 \sin(\psi)^5}{60a} + \frac{1}{10} \\
M_\lambda &= \frac{\mu^3 \sin(\psi)^3}{12} - \frac{c_d}{6a} - \frac{\mu \sin(\psi)}{4} + \frac{\hat{a} \mu^3 \sin(\psi)^3}{12a} + \frac{c_d \mu^3 \sin(\psi)^3}{12a} \\
&\quad + \frac{c_{\hat{d}} \mu^3 \sin(\psi)^3}{12a} - \frac{c_d \mu \sin(\psi)}{4a} - \frac{1}{6}
\end{aligned} \tag{3.15}$$

When $\mu > 1$, the moments in the total reverse flow region ($270^\circ - \cos^{-1}(1/\mu) < \psi < 270^\circ + \cos^{-1}(1/\mu)$) are:

$$\begin{aligned}
M_\theta &= -\frac{\hat{a}}{8a} - \frac{\hat{a} \mu^2 \sin(\psi)^2}{4a} - \frac{\hat{a} \mu \sin(\psi)}{3a} \\
M_{\theta_{tw}} &= -\frac{\hat{a}}{10a} - \frac{\hat{a} \mu^2 \sin(\psi)^2}{6a} - \frac{\hat{a} \mu \sin(\psi)}{4a} \\
M_\lambda &= \frac{\hat{a}}{6a} + \frac{c_{\hat{d}}}{6a} + \frac{\hat{a} \mu \sin(\psi)}{4a} + \frac{c_{\hat{d}} \mu \sin(\psi)}{4a}
\end{aligned} \tag{3.16}$$

As before, these reduce to the simpler forms found in Ref. 20 when assuming $\hat{a} = a$, $\hat{c}_d = c_d$, and $c_d/a = 0$.

The thrust coefficient is

$$C_T = \sigma a \int_0^1 \frac{1}{2} |u_T| (u_T \theta - u_P) dx \quad (3.17)$$

which is the average lift force (approximately equal to the flapping force) averaged over the rotor azimuth and in the normal and reverse flow regions.

As in the homogeneous equation, the mass, damping, and stiffness terms on the LHS are

$$M = 1, \quad C(\psi) = -\gamma M_\beta^*, \quad K(\psi) = \nu_\beta^2 - \gamma M_\beta \quad (3.18)$$

where the moments M_β^* and M_β were derived earlier for the flap stability analysis.

The sum of the RHS forcing moments is

$$M_f = \gamma (M_\theta \theta + M_{\theta_{tw}} \theta_{tw} + M_\lambda \lambda) \quad (3.19)$$

Thus, the inhomogeneous flap equation can be expressed as

$$\beta^{**} + C(\psi)\beta^* + K(\psi)\beta = M_f(\psi) \quad (3.20)$$

Recall the flap state vector, $\underline{q} = [\beta^* \ \beta]^T$. With forcing, the equation of motion in state variable form is

$$\underline{\dot{q}}^* = \underline{A} \underline{q} + \underline{b} \quad (3.21)$$

The forcing vector is

$$\underline{b} = \begin{pmatrix} 0 \\ M_f(\psi) \end{pmatrix} \quad (3.22)$$

3.2 Periodic Shooting

A challenge of the trim problem is that even if the control inputs $(\theta_0, \theta_{1c}, \theta_{1s})$ required to obtain given C_T/σ and flapping targets are known, the initial flap state $\underline{q}(0)$ is unknown. The solution is found using periodic shooting, which also leverages Floquet theory. Originally applied to the rotorcraft trim problem in Ref. 57, the method here mostly follows the form of the equations found in Ref. 20. One of the advantages of periodic shooting is that it provides a trim solution even if the flap motion is unstable. If the initial conditions for the differential equations were instead sought through iterations, a numerical solution would only be convergent when both roots are stable.

The state transition matrix $\underline{\Phi}(T)$ is found in the same manner as during the flap stability analysis for the homogeneous case. Again, for the coefficient matrix

$$\underline{\underline{A}}(\psi) = \begin{pmatrix} 0 & 1 \\ -K(\psi) & -C(\psi) \end{pmatrix} \quad (3.23)$$

the equation $\underline{\dot{q}}^* = \underline{\underline{A}}\underline{q}$ is numerically integrated with the initial condition matrix I_2 .

The flap state at one period due to forcing alone, $\underline{q}_f(T)$, is found by integrating the inhomogeneous equation of motion with a zero initial condition vector, $\underline{q}_f(0) = \underline{0} = [0 \ 0]^T$.

Because there is only a single initial condition vector (as opposed to the matrix of dual vectors needed for the transition matrix), this is a “single pass” integration. Consider the motion of the unforced system:

$$\underline{q}(\psi) = \underline{\Phi}(\psi)\underline{q}(0) \quad (3.24)$$

Adding the forced motion,

$$\underline{q}(\psi) = \underline{q}_f(\psi) + \underline{\Phi}(\psi)\underline{q}(0) \quad (3.25)$$

For a periodic system, $\underline{q}(0) = \underline{q}(T)$, so the initial condition vector is then

$$\underline{q}(0) = (\underline{I} - \underline{\Phi}(T))^{-1}\underline{q}_f(T) \quad (3.26)$$

The motion of the flap state vector is

$$\underline{q}(\psi) = \underline{q}_f(\psi) + \underline{\Phi}(\psi)(\underline{I} - \underline{\Phi}(T))^{-1}\underline{q}_f(T) \quad (3.27)$$

The above equation requires that the arrays $\underline{\Phi}(\psi)$ and $\underline{q}_f(\psi)$ were stored during each step of their integrations (not just the final values at T). Alternatively, the flap state can be found by integrating $\underline{\dot{q}} = \underline{A}\underline{q} + \underline{b}$ over one period with the initial condition $\underline{q}(0)$.

Recall β is the first element of $\underline{q}$; $\beta(\psi)$ with consistent initial conditions is now known. For a MATLAB ODE, this is the first column of the solution array $\mathbf{y}$ ($\mathbf{t}=\psi$, $\mathbf{y}=\underline{q}(\psi)^T$). The flap motion expressed as a Fourier series is

$$\beta(\psi) = \sum_{n=-n_{\max}}^{n_{\max}} \beta_{ne} e^{in\psi} = \beta_0 + \sum_{n=1}^{n_{\max}} \beta_{nc} \cos(n\psi) + \beta_{ns} \sin(n\psi) \quad (3.28)$$

A Fast Fourier Transform of the calculated $\beta(\psi)$ array is performed, which approximates this series. For each integer frequency of n per rev, there will be a complex Fourier coefficient found from the FFT. For a real flap waveform, its sine-cosine Fourier coefficients can be converted from the exponential forms through the relations:

$$\left. \begin{aligned} \beta_0 &= \beta_{0e} \end{aligned} \right\} \text{ for } n = 0$$

$$\left. \begin{aligned} \beta_{nc} &= \beta_{-ne} + \beta_{ne} = 2 \operatorname{Re}(\beta_{ne}) \\ \beta_{ns} &= i(-\beta_{-ne} + \beta_{ne}) = -2 \operatorname{Im}(\beta_{ne}) \end{aligned} \right\} \text{ for } n > 0 \quad (3.29)$$

Therefore, only a single-sided frequency analysis ($n \geq 0$) is needed due to the Hermitian symmetry of the real signal. Versions of these relations can be seen in Ref. 107 and in standard texts covering Fourier series and transforms. Just as in the Modal Waveform FFT discussion from Chapter 2, for a sampling rate of N equal steps per period, $n_{\max}$ is $N/2$. Regardless, most of the high flap frequencies can be ignored, especially if N is on the order of 100. Again, the real practical benefit of a higher number integration steps is the accuracy of the eigensolution. Due to the excitation expected at higher advance ratios, flap trim harmonics up to 4/rev are considered in this work ($n_{\max} = 4$).

Wind tunnel trim in the rotating frame is used for most of the results. A prescribed blade loading (C_T/σ) is selected and zero flap moments are targeted ($\beta_{1c}, \beta_{1s} = 0$). The trim Jacobian matrices are iterated until steady state convergence. The resultant coning (β_0) and higher harmonics (β_{2c}, β_{2s} , etc.) are accepted. This is depicted in Fig. 3.3.

C_T must be the same as the corresponding value used for the mean inflow at the

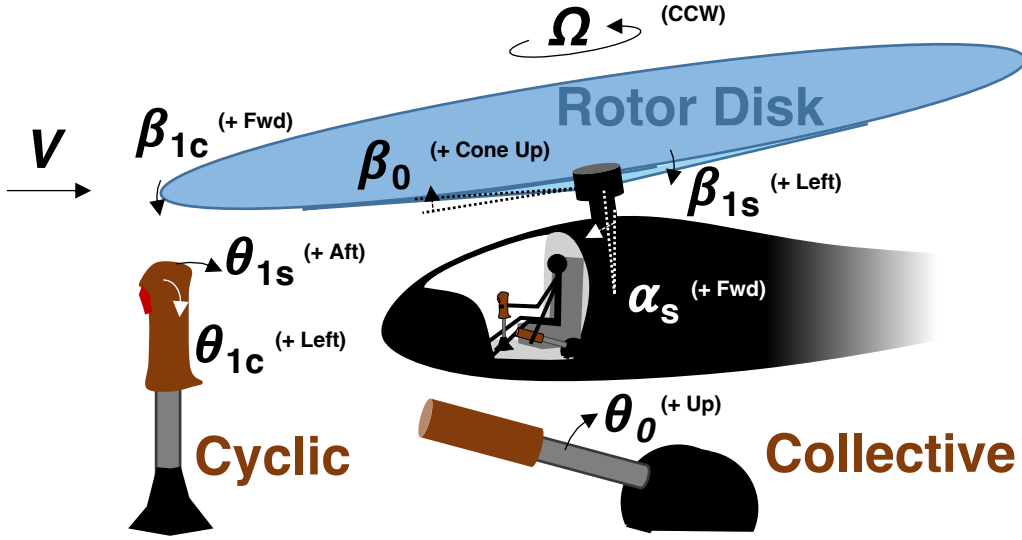


Figure 3.3: Flight control inputs and flap trim.

beginning of the routine. The values are iterated until the blade loading and cyclic (θ_{1c}, θ_{1s}) values converge.

In the untrimmed case, the cyclic angles are 0 or no cyclic controls are installed, such as in a simple wind tunnel rotor apparatus. The collective (θ_0) and shaft tilt (α_s) may be targeted or fixed, but

3.3 Trim and Blade Stall for Various Configurations

A non-zero C_T/σ target while transitioning between $\mu = 0.75$ and 1 is difficult to achieve due to the phenomenon of thrust reversal, where increasing collective decreases thrust. Figs. 3.4 through 3.8 show the trimmed flight control positions for varying advance ratios, lock numbers, flap frequencies, and blade loadings. Also depicted are stalled blade area and flap stability boundaries. Blade stall is presented both as a percentage of the total blade area and as a percentage of the outer 50% of the blade span, which is 75% of the disk area. The latter is of greater concern.

For the configurations presented, a simple symmetrical airfoil model is used, with aerodynamic coefficients $a = 2\pi$ and $c_d = 0.01$. A solidity of $\sigma = 0.05$ is used and root cutout is neglected. Control limits of $\pm 20^\circ$ are depicted. The values are chosen to provide a reasonable range, real rotors may not be able to achieve these values due to constraints such as rigging limits or blade loads.

Figs. 3.4 and 3.5 show the control positions and blade stall for a flap frequency of $\nu_\beta = 1/\text{rev}$ and Lock numbers of $\gamma = 12$ and 6, respectively. The general trends are very similar; small differences include lateral cyclic displacements and that thrust reversal discontinuity occurs at $\mu = 0.8$ for $\gamma = 12$ and at $\mu = 0.85$ for $\gamma = 6$. The subsequent trim figures will continue to use the $\gamma = 6$ rotor.

The trim for other configurations can be found, such as for pitch-flap coupling or a hingeless rotor. The solutions for $\delta_3 = 15^\circ$ and $\nu_\beta = 1.15/\text{rev}$ were very similar to Fig. 3.5. The hingeless rotor has a subtly different lateral cyclic and the pitch-flap coupling case has slightly greater collective magnitude; both have marginally earlier thrust reversal.

Because of the effect from inflow, shaft tilt (α_s) has a great influence on trim. Figs. 3.6 and 3.7 show the results for $\alpha_s = 4^\circ$ (forward tilt) and -4° (aft tilt), respectively. Forward shaft tilt aggravates the issue of control limits being reached. Aft shaft tilt has the interesting feature of reversing collective and cyclic directions depending on the blade loading. It appears that between $C_T/\sigma = 0.04$ and 0.08, the asymptotic trend reverses, which means that near $C_T/\sigma = 0.06$, there might not be a discontinuity.

Thus, a carefully selected shaft tilt, blade loading, and advance ratio transition profile (reducing rotor speed or increasing airspeed) could mitigate the thrust reversal discontinuity. Such combinations are shown in Fig. 3.8. For $\nu_\beta = 1/\text{rev}$, $\gamma = 6$, and $C_T\sigma = 0.04$, the

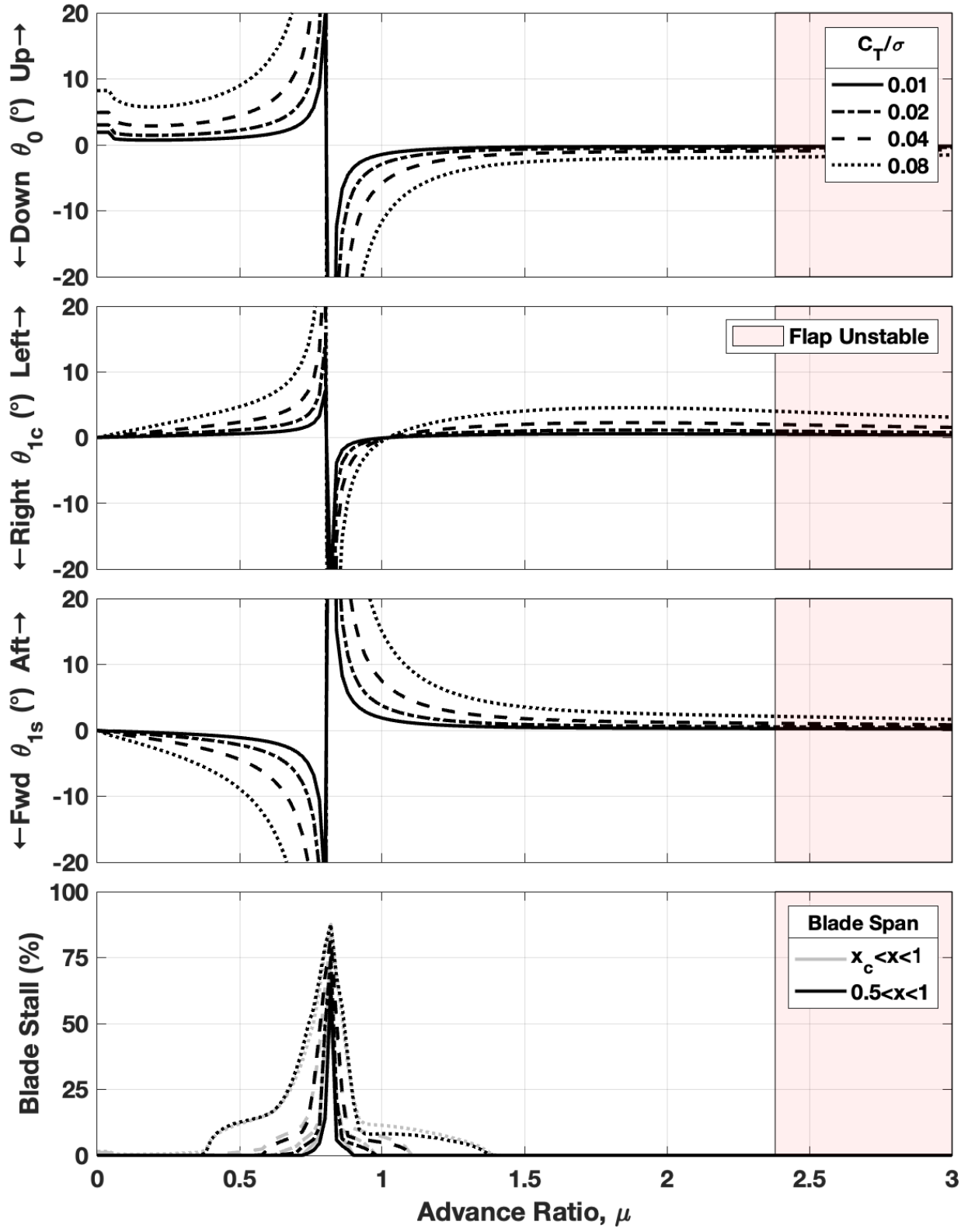


Figure 3.4: Trimmed flight control positions and blade stall, $\nu_\beta = 1$, $\gamma = 12$.

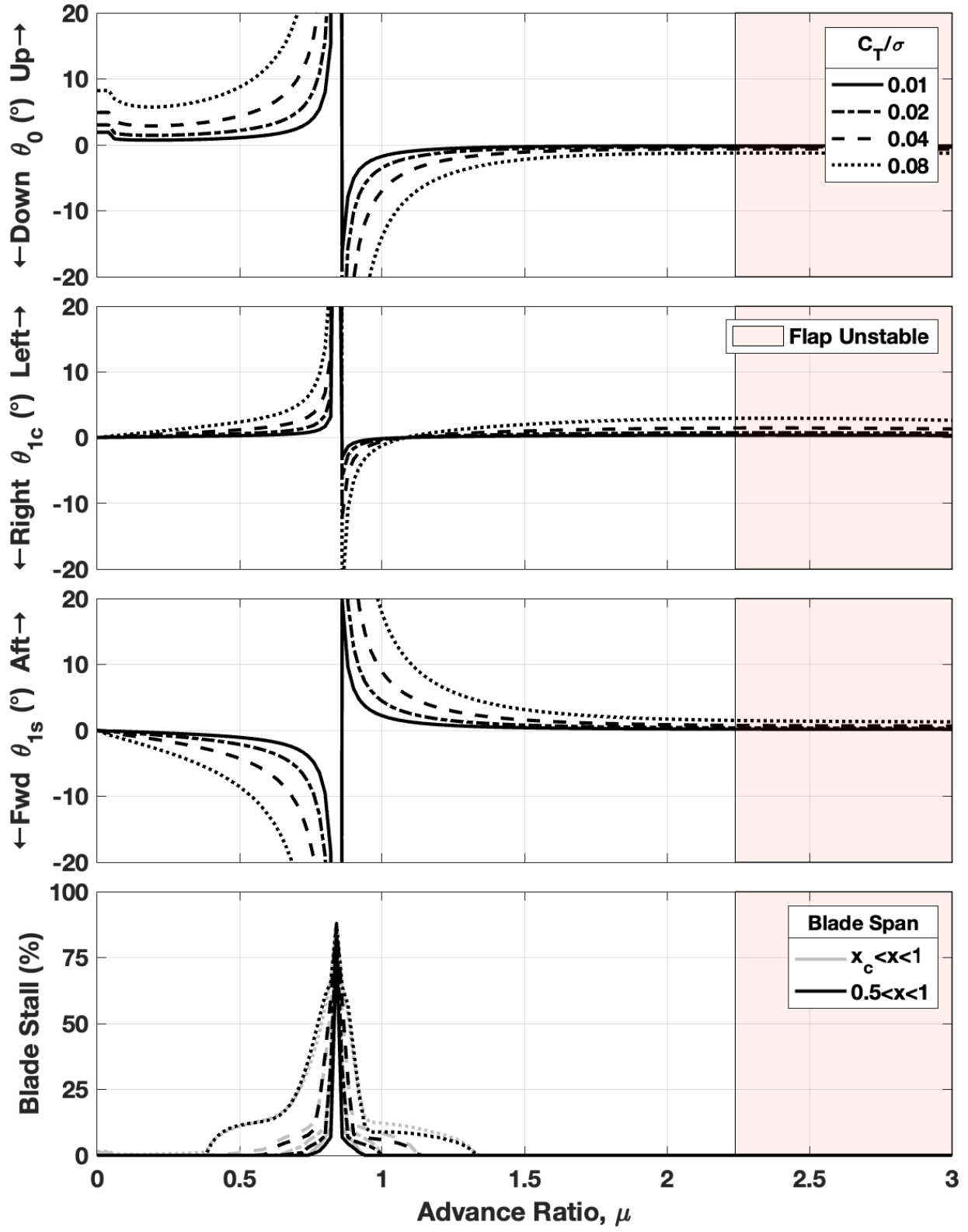


Figure 3.5: Trimmed flight control positions and blade stall, $\nu_\beta = 1$, $\gamma = 6$.

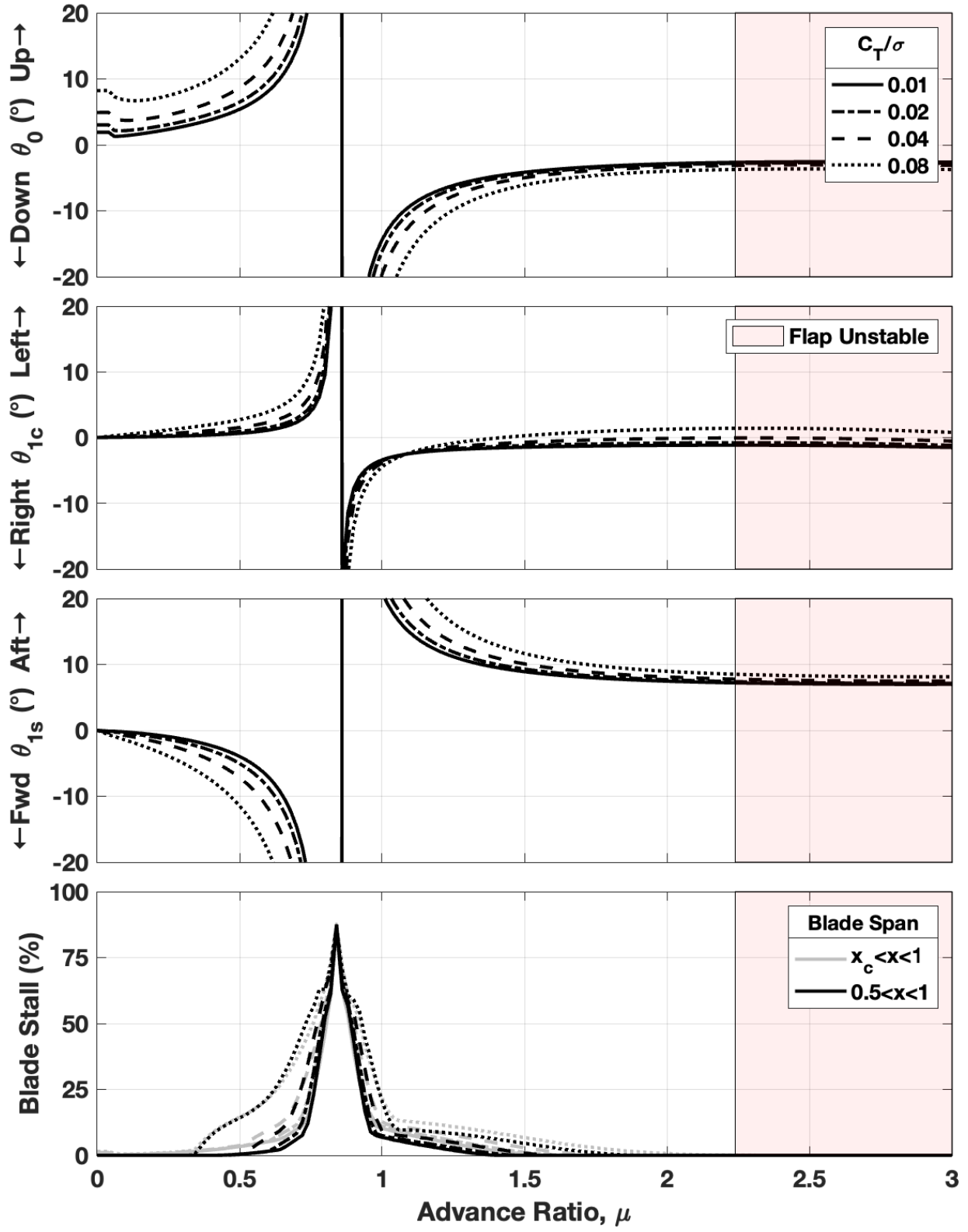


Figure 3.6: Trimmed flight control positions and blade stall, $\nu_\beta = 1$, $\gamma = 6$, $\alpha_s = 4^\circ$.

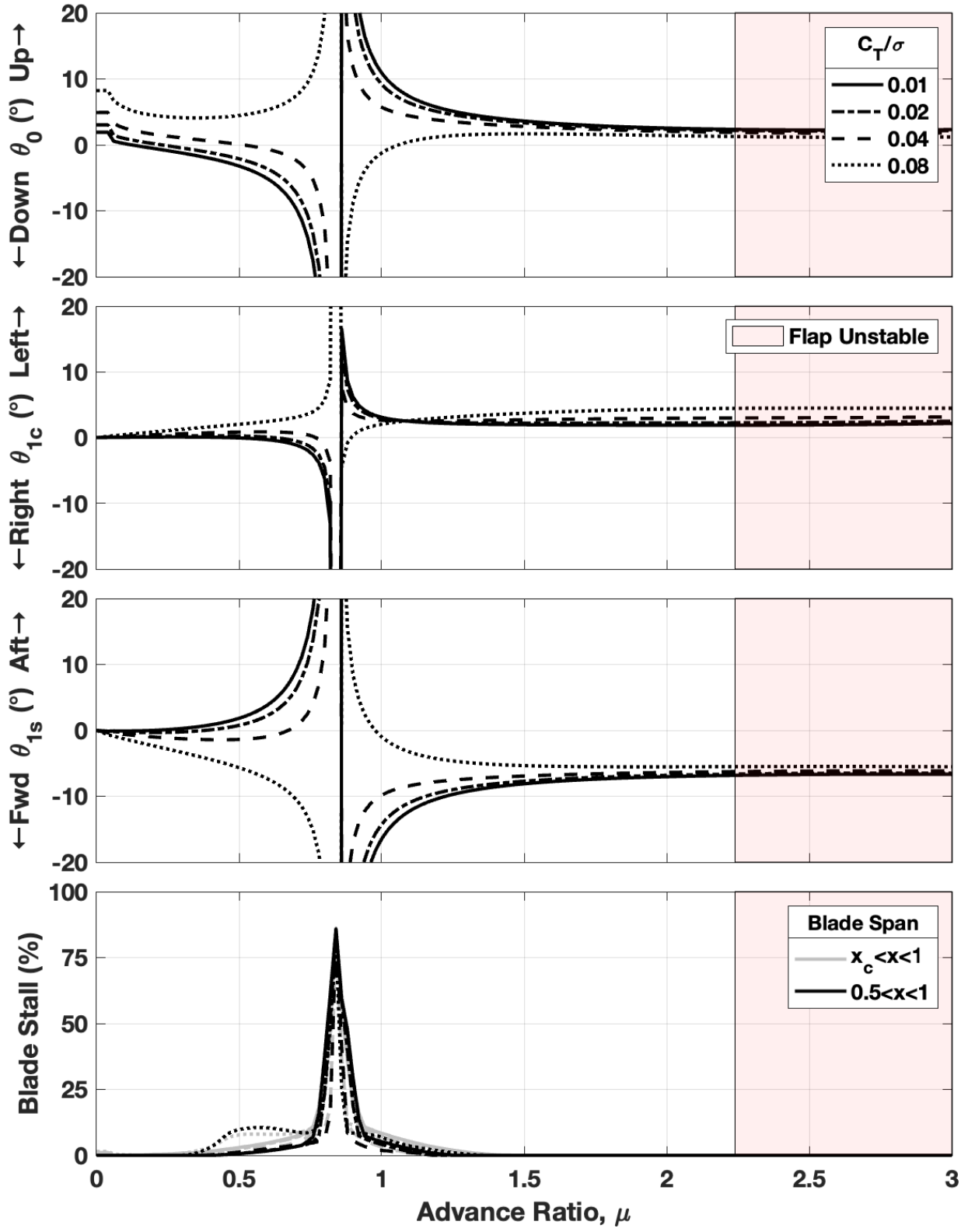


Figure 3.7: Trimmed flight control positions and blade stall, $\nu_\beta = 1$, $\gamma = 6$, $\alpha_s = -4^\circ$.

shaft tilt needed to avoid the thrust reversal discontinuity is $\alpha_s = -2.767^\circ$.

Another trim approach could be to vary the shaft tilt in order to minimize coning at all advance ratios. The resulting small flap motion would minimize the flap-lag Coriolis coupling terms (to be discussed in Chapter 4) and thus potentially increase stability. An example of this autogyro-like solution is shown in Fig. 3.9. At low advance ratios, targeting $\beta_0 = 0$ may not be achievable at realistic shaft tilt values (here limited to $\pm 20^\circ$, still quite a large range) because there is insufficient negative inflow, so a transition between trim schemes would be required. Regardless, the sensitivity of the trim solution can be seen by the spike in $\partial\beta_0/\partial\alpha_s$ near $\mu = 0.8$.

Propulsive trim is frequently used to model the performance and stability of helicopters at low advance ratios. In order to provide thrust to overcome the parasite drag of an equivalent flat plate area f , the shaft tilt must be

$$\alpha_s = \left(\frac{f}{A}\right) \frac{\mu^2}{2C_T} \quad (3.30)$$

Because of its limited applicability for a compound rotorcraft at high- μ , propulsive trim via forward shaft or fuselage tilt is only used in this work during comparisons with historical aeroelastic analyses where it was prescribed. For full-aircraft trim, the height of the rotor above the center of gravity, anti-torque thrust, and other factors must also be considered when determining the thrust and moment targets. A compound design may have some or all of these forces and moments provided by auxiliary control and propulsion systems in lieu of the rotor in forward flight.

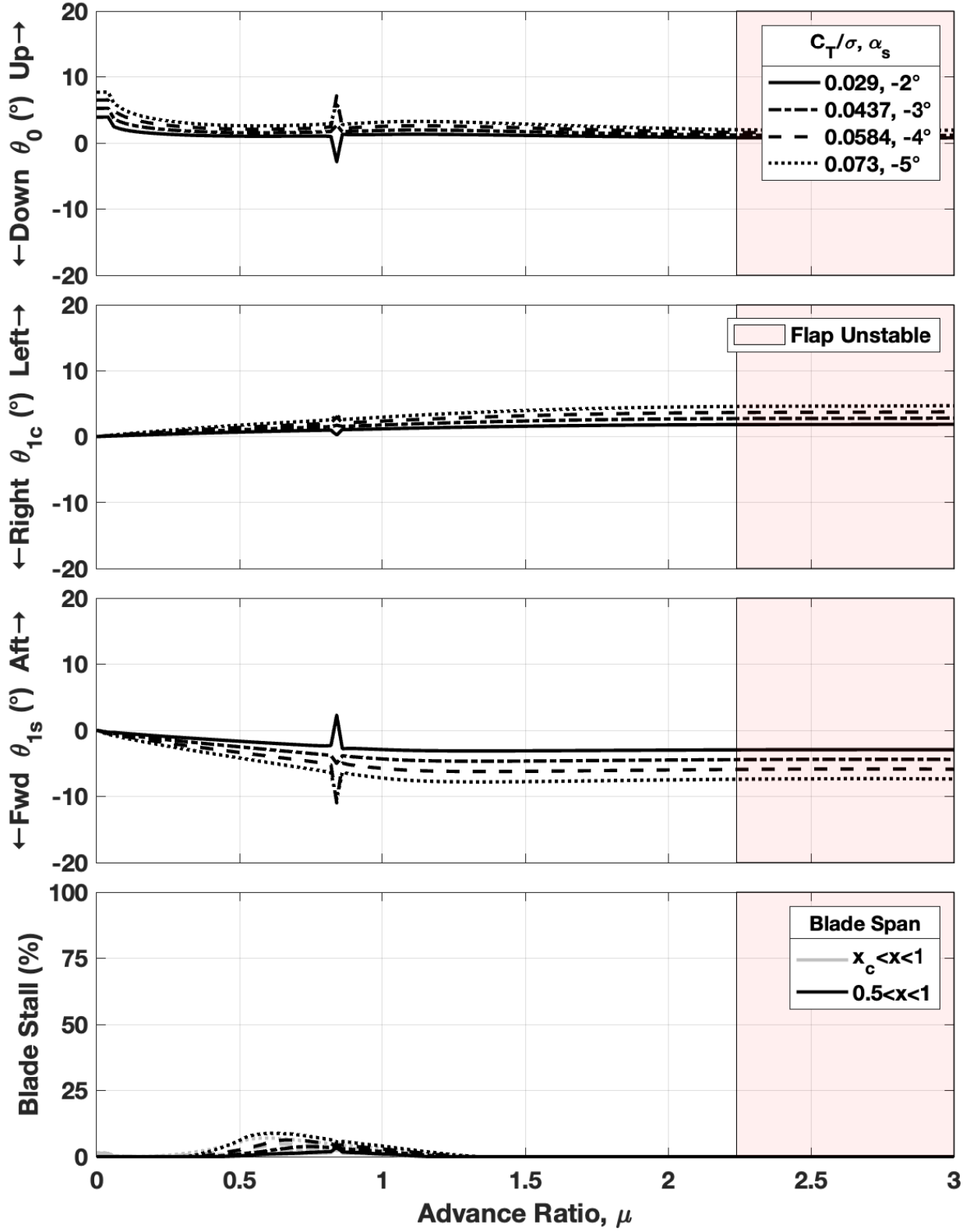


Figure 3.8: Trimmed flight control positions and blade stall, $\nu_\beta = 1$, $\gamma = 6$, $C_T/\sigma = 0.03$ to 0.07, $\alpha_s = -2^\circ$ to -5° (avoids thrust reversal discontinuity).

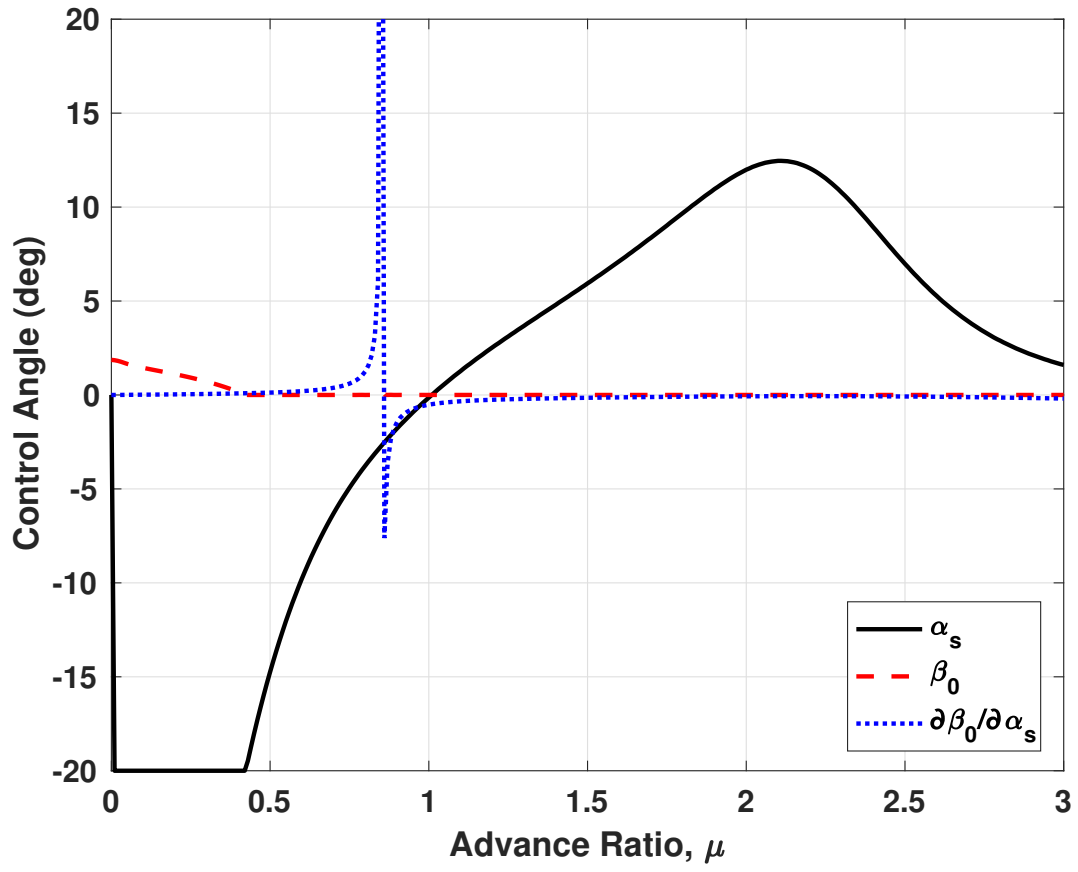


Figure 3.9: Autorotational trim with aft shaft tilt, $\nu_\beta = 1$, $\gamma = 6$, $C_T/\sigma = 0.04$.

3.4 Flap Trim Conclusions

The flap trim implementation using periodic shooting suitably complements the Floquet approach. It allows for the determination of the sustained periodic motion from which the lag and torsion perturbations will deviate. It is also computationally efficient and allows for the trim state to be determined even when the flap motion is unstable. The stall calculations provide a ceiling to assess the applicability of the linear quasi-steady aerodynamics assumptions when needed.

In the region of thrust reversal, between $\mu = 0.75$ to 1, it is difficult to trim the rotor without reaching control limits or blade stall for non-zero blade loadings. This can be mitigated by employing aft shaft tilt and pairing the blade loading accordingly.

In Appendix A, the linear model in this work is validated against historical high advance ratio wind tunnel experiments. Two articulated rotor cases are presented: Jenkins' NASA tests up to $\mu = 1.45$ (Ref. 35) and Bowen-Davies and Chopra's experiments at UMD up to $\mu = 1.2$ (Ref. 91). The hingeless rotor comparison is with the UMD wind tunnel tests performed by Wang et al. as high as $\mu = 0.7$ (Ref. 92).

Note that the trim results for the previously discussed teetering rotor are not depicted, and further discussion of this promising configuration is beyond the scope of this work. One issue with the teetering rotor is that low or zero blade loadings ($C_T/\sigma = 0$) are discouraged because of the risk of mast bumping. A transition through the thrust reversal barrier and requires operating in the vicinity of this very trim condition. Thus the aft shaft tilt configuration to maintain positive blade loading would be especially important in this case.

Chapter 4: Flap-Lag Stability

4.1 Equations of Motion

Analysis of flap-lag stability requires two coupled equations of motion. For a centrally hinged uniform rotor blade, the non-dimensional coupled flap equation is

$$\beta^{**} + \nu_{\beta}^2 \beta - 2\beta^* \zeta + Z\zeta = \gamma \bar{M}_{\beta} \quad (4.1)$$

with the flap aerodynamic forcing moment defined as in the flap-only case. The coupled lag equation is

$$\zeta^{**} + 2\nu_{\zeta 0} \zeta_L^* \zeta + \nu_{\zeta}^2 \zeta + 2\beta^* \beta + Z\beta = \gamma \bar{Q}_{\zeta} \quad (4.2)$$

Flap and lag motion are dependent on one another due to Coriolis $(-2\beta^* \zeta, 2\beta \beta^*)$ and structural coupling $(Z\zeta, Z\beta)$ terms. It is assumed that the moments of inertia are effectively the same $(I_{\zeta} = I_{\beta} = I_b)$, hence the second derivative and Coriolis term coefficients are all unity.

The lag aerodynamic forcing moment is

$$\bar{Q}_{\zeta} = \int_{x_c}^1 x \bar{F}_{\zeta} dx \quad (4.3)$$

where the lag force is

$$\bar{F}_\zeta = \frac{1}{2} \left(\frac{c_d}{a} u_T^2 + \theta u_P u_T - u_P^2 \right) \quad (4.4)$$

. which are both non-dimensionalized in the same manner as flap in Chapter 2.

In the reverse flow region,

$$\bar{\hat{F}}_\zeta = -\frac{1}{2} \frac{\hat{a}}{a} \left(\frac{\hat{c}_d}{\hat{a}} u_T^2 + \theta u_P u_T - u_P^2 \right) \quad (4.5)$$

For a rigid hub-hinged blade, the non-dimensional first rotating and non-rotating natural frequencies are related by

$$\nu_\beta^2 = 1 + \nu_{\beta 0}^2 \quad (4.6)$$

$$\nu_\zeta = \nu_{\zeta 0}$$

It is important to note that when specifying a blade's rotating natural frequency, it is not only normalized to a certain rotor speed Ω , but also tied to a specific pitch angle θ . Unless otherwise noted, that is the zero pitch condition.

Lag structural damping is included via the term ζ_L , which is on the order of ≈ 0.01 . Unless otherwise noted, this lag structural damping ratio is set to 0 for the following analyses.

Now including the effect of pitch-lag coupling, blade pitch at the feathering bearing is

$$\theta_s = \theta_C - k_{p\beta} \beta - k_{p\zeta} \zeta \quad (4.7)$$

The pitch-lag coupling can be defined by an “alpha-1” hinge angle:

$$k_{p\zeta} = \tan \alpha_1 \quad (4.8)$$

Combining the conventions used in Refs. 51, 74, and 20, for small blade pitch angles, the structural elastic coupling is defined by

$$Z = -R_s(\nu_{\zeta 0}^2 - \nu_{\beta 0}^2)\theta_s \quad (4.9)$$

The parameter R_s may vary from 0 (no coupling, flexible hub, rigid blade) to 1 (rigid hub, flexible blade). Z is added to the off-diagonals of the flap-lag stiffness matrix to account for the coupling. At zero blade pitch ($\theta_s = 0$) or in the case of a matched stiffness rotor ($\nu_{\zeta 0} = \nu_{\beta 0}$), structural coupling vanishes. For large pitch angles, ν_β and ν_ζ will change slightly and the diagonal terms will be affected as well.

The equations will now be converted to their perturbation forms. Stability is analyzed about the flap trim state, thus the flap, lag, and pitch perturbations are

$$\begin{aligned} \beta &= \beta_T + \delta\beta \\ \zeta &= \zeta_T + \delta\zeta \approx \delta\zeta \\ \theta &= \theta_T + \delta\theta \end{aligned} \quad (4.10)$$

The perturbation to blade pitch due to pitch-flap and pitch-lag coupling is

$$\delta\theta = -k_{p\beta}\delta\beta - k_{p\zeta}\delta\zeta \quad (4.11)$$

The Coriolis terms become

$$\begin{aligned}\delta(\beta\zeta)^* &= \beta_T\delta\zeta^* \approx \beta_0\delta\zeta^* \\ \delta(\beta\beta)^* &= \beta_T\delta\beta^* + \beta_T^*\delta\beta_T\end{aligned}\tag{4.12}$$

and products of other terms are discarded (e.g. $\beta^2 = 0$). The aerodynamic force functions are also converted into their perturbation forms,

$$\begin{aligned}\delta\bar{F}_\beta &= \frac{1}{2}\left(\delta u_T\left(2u_T\theta_T - \left(1 + \frac{c_d}{a}\right)u_P\right) \right. \\ &\quad \left. - \left(1 + \frac{c_d}{a}\right)\delta u_P u_T + \delta\theta u_T^2\right) \\ \delta\bar{F}_\zeta &= \frac{1}{2}\left(\delta u_T\left(2u_T\frac{c_d}{a} + u_P\theta_T\right) \right. \\ &\quad \left. + \delta u_P\left(u_T\theta_T - 2u_P\right) + \delta\theta\left(u_P u_T\right)\right)\end{aligned}\tag{4.13}$$

In reverse flow,

$$\begin{aligned}\delta\bar{\hat{F}}_\beta &= -\frac{1}{2}\frac{\hat{a}}{a}\left(\delta u_T\left(2u_T\theta_T - \left(1 + \frac{\hat{c}_d}{\hat{a}}\right)u_P\right) \right. \\ &\quad \left. - \left(1 + \frac{c_d}{a}\right)\delta u_P u_T + \delta\theta u_T^2\right) \\ \delta\bar{\hat{F}}_\zeta &= -\frac{1}{2}\frac{\hat{a}}{a}\left(\delta u_T\left(2u_T\frac{\hat{c}_d}{\hat{a}} + u_P\theta_T\right) \right. \\ &\quad \left. + \delta u_P\left(u_T\theta_T - 2u_P\right) + \delta\theta\left(u_P u_T\right)\right)\end{aligned}\tag{4.14}$$

The perturbation velocities are

$$\begin{aligned}\delta u_T &= -x\zeta^* - \delta\zeta\mu\cos\psi \\ \delta u_P &= x\delta\beta^* + \delta\beta\mu\cos\psi\end{aligned}\tag{4.15}$$

4.2 Assembly of the Coefficient Matrix

The reverse flow regions and integration limits are the same as the flap only case. The aerodynamic forces are integrated to find the moments for each azimuth range, and grouped into the following terms:

$$\begin{aligned}\bar{M}_\beta &= M_{\beta}^* \beta + M_{\zeta}^* \zeta + M_{\beta\beta} + M_{\zeta\zeta} \\ \bar{Q}_\zeta &= Q_{\beta}^* \beta + Q_{\zeta}^* \zeta + Q_{\beta\beta} + Q_{\zeta\zeta}\end{aligned}\tag{4.16}$$

Re-arranging the system of equations into its homogeneous perturbation form,

$$\underline{\underline{M}}\delta\underline{q}^{**} + \underline{\underline{C}}\delta\underline{q}^* + \underline{\underline{K}}\delta\underline{q} = \underline{0}\tag{4.17}$$

Where

$$\delta\underline{q}^{**} = \begin{pmatrix} \delta\beta^{**} \\ \delta\zeta^{**} \end{pmatrix}, \delta\underline{q}^* = \begin{pmatrix} \delta\beta^* \\ \delta\zeta^* \end{pmatrix}, \delta\underline{q} = \begin{pmatrix} \delta\beta \\ \delta\zeta \end{pmatrix}\tag{4.18}$$

The linearized coefficient matrix is

$$\underline{\underline{A}} = \begin{pmatrix} \underline{\underline{0}} & \underline{\underline{I}} \\ -\underline{\underline{M}}^{-1}\underline{\underline{K}} & -\underline{\underline{M}}^{-1}\underline{\underline{C}} \end{pmatrix}\tag{4.19}$$

The matrix $\underline{\underline{0}}$ is 2 by 2, as are the other matrices. The mass, damping, and stiffness

are

$$\begin{aligned}
\underline{\underline{M}} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
\underline{\underline{C}} &= \begin{pmatrix} -\gamma M_{\beta}^* & -2\beta_T - \gamma M_{\zeta}^* \\ 2\beta_T - \gamma Q_{\beta}^* & 2\nu_{\zeta_0} \zeta_L - \gamma Q_{\zeta}^* \end{pmatrix} \\
\underline{\underline{K}} &= \begin{pmatrix} \nu_{\beta}^2 - \gamma M_{\beta} & -\gamma M_{\zeta} + Z \\ 2\beta_T^* - \gamma Q_{\beta} + Z & \nu_{\zeta}^2 - \gamma Q_{\zeta} \end{pmatrix}
\end{aligned} \tag{4.20}$$

C_{11} and K_{11} are equal to C and K from the uncoupled flap-only case.

4.3 Stability Analysis

The periodic damping and stiffness matrices are now 2×2 . Examples of $\underline{\underline{C}}$ and $\underline{\underline{K}}$ for flap-lag perturbation are shown in Figs. 4.1 through 4.4. Figs. 4.1 and 4.2 show that when $C_T/\sigma = 0$, the diagonal flap-lag coupling terms vanish. The effects of adding blade loading (and thus flap trim) are shown in Figs. 4.3 and 4.4.

The stability roots (now four) are found using Floquet theory just as in the flap-only case, with the initial condition $\underline{\underline{I}}$ now being a 4-by-4 matrix.

To examine flap-lag stability for a given configuration, a trim state is also specified: target blade loading (C_T/σ) or collective (θ_0), and longitudinal and lateral cyclic (θ_{1s}, θ_{1c}) are adjusted to achieve the required longitudinal and lateral flapping (β_{1c}, β_{1s}), which are zero for wind tunnel trim.

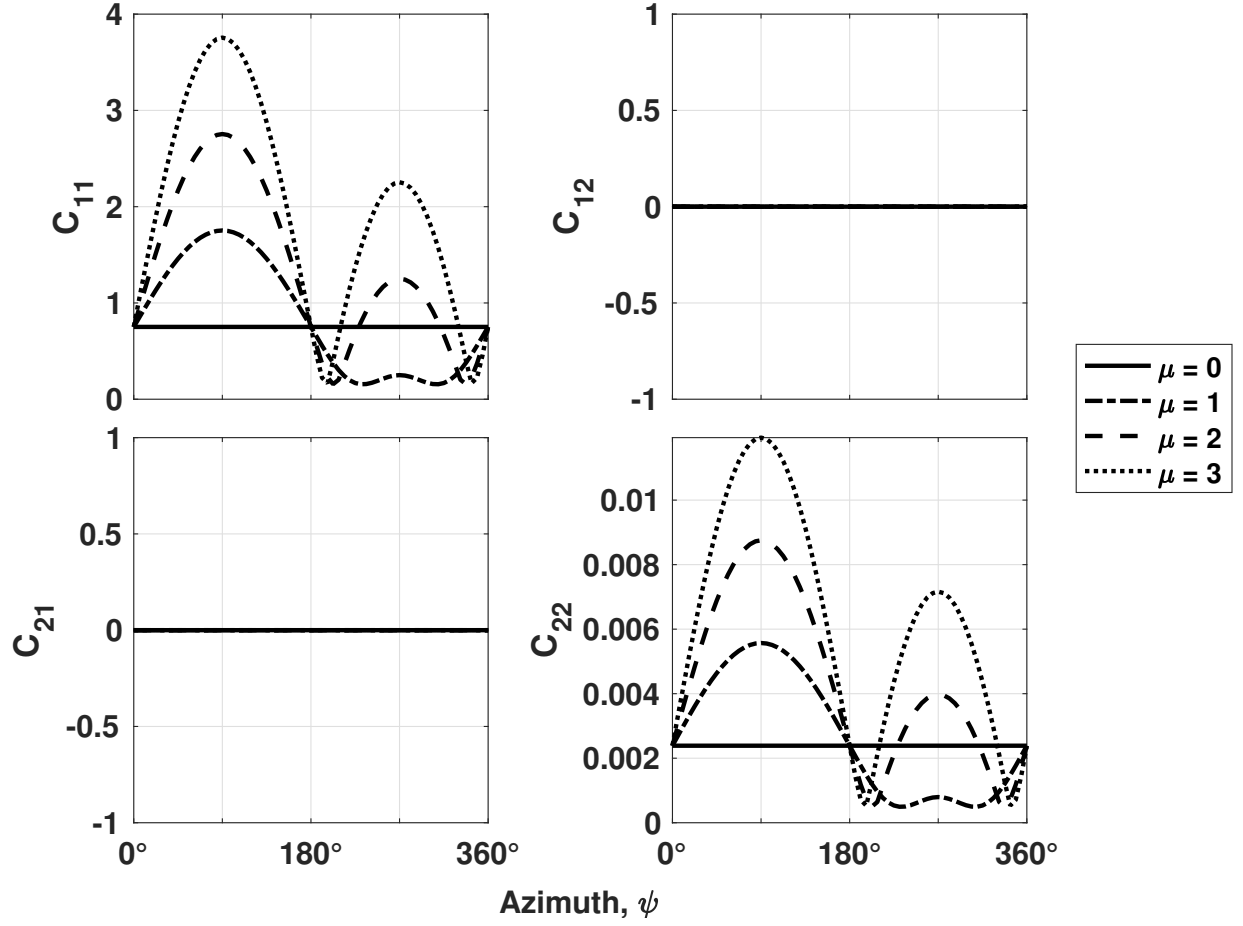


Figure 4.1: Flap-Lag damping matrix ($\underline{\underline{C}}$), $\nu_\beta = 1$, $\nu_\zeta = 0.3/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0$.

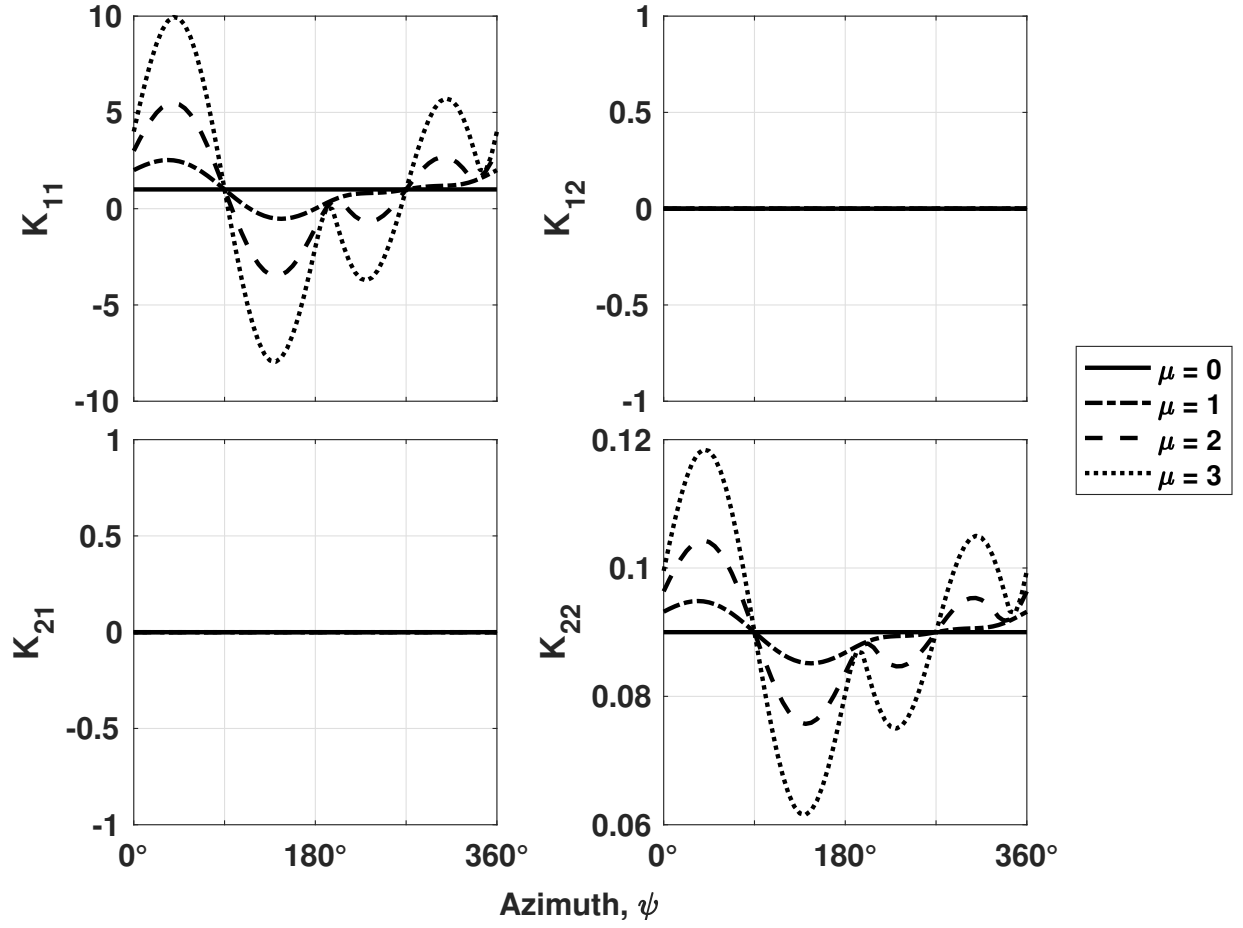


Figure 4.2: Flap-Lag stiffness matrix ($\underline{\underline{K}}$), $\nu_\beta = 1$, $\nu_\zeta = 0.3/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0$.

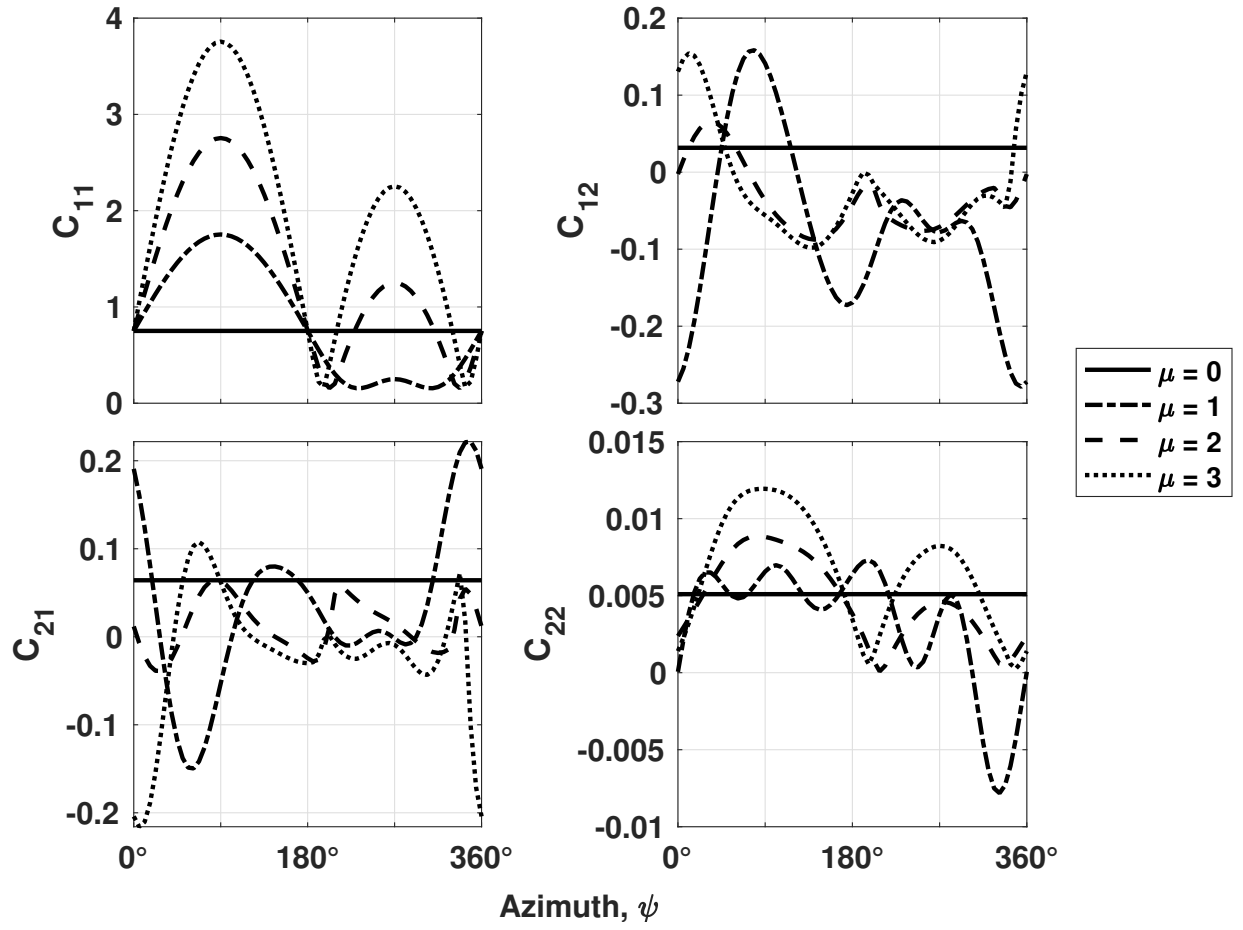


Figure 4.3: Flap-Lag damping matrix ($\underline{\underline{C}}$), $\nu_\beta = 1$, $\nu_\zeta = 0.3/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$.

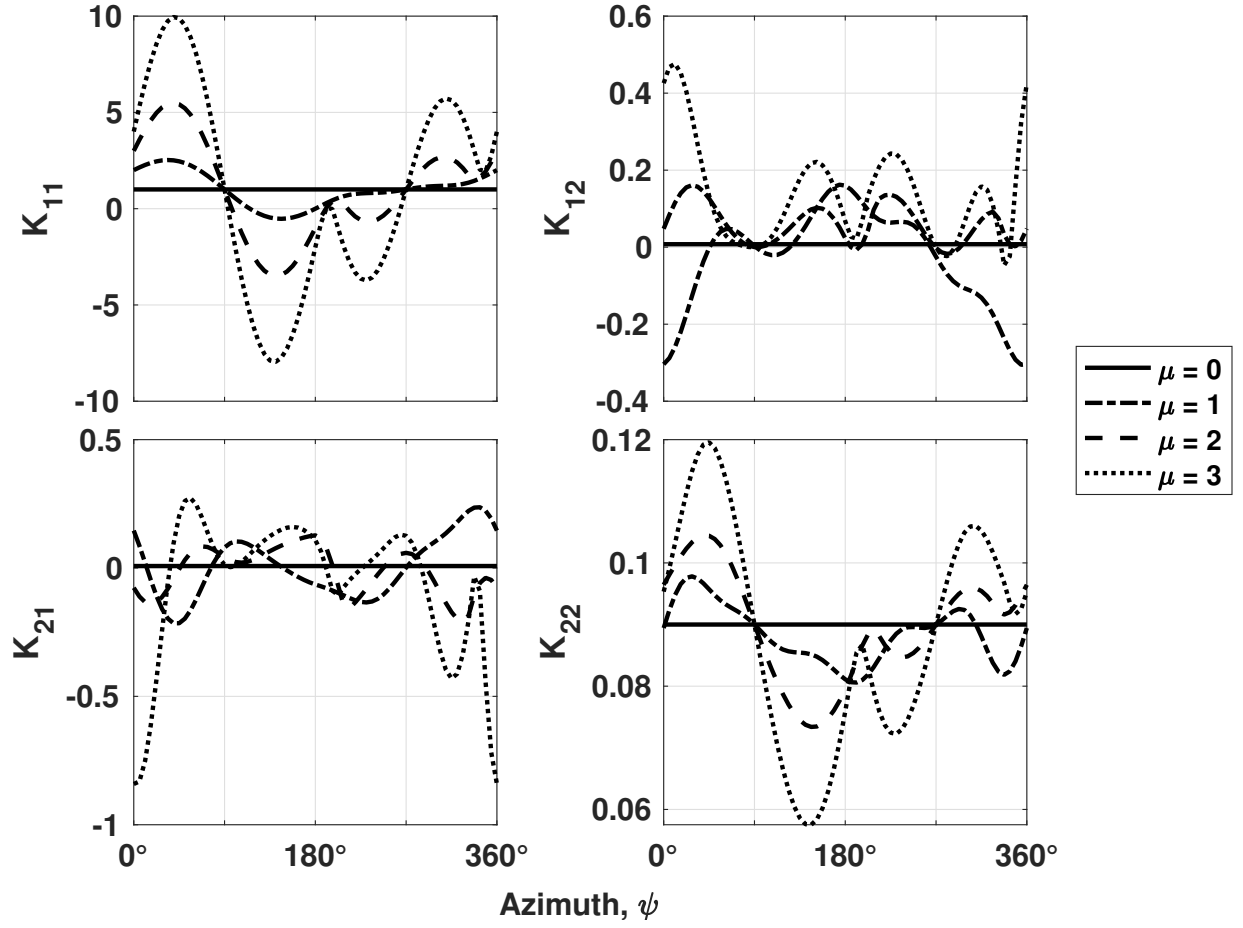


Figure 4.4: Flap-Lag stiffness matrix ($\underline{\underline{K}}$), $\nu_\beta = 1$, $\nu_\zeta = 0.3/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$.

4.4 Verification and Validation of the Coupled Lag Stability Model

The approach to flap-lag stability analysis is verified with past efforts found in the literature (Refs. 53 and 54) up to $\mu = 0.4$. The lag vs. flap frequency “egg” plots in Fig. 4.5 show the instability regions for a trimmed rotor with a high blade loading of $C_T/\sigma = 0.2$ and $\gamma = 5$. Note that Ref. 54) explored both flap-lag and lag-flap hinge sequences; in the present analysis, the hinge locations are coincident at the rotor hub.

In addition to comparing the current work with previous analyses, it is also desirable to confirm that the flap-lag model comports with actual flights and experiments. Fig. 4.6 presents an overlay of the lag model with hover data from Ormiston and Bousman’s tests in Ref. 52. The model matches well with Ormiston’s hover experiments, except at high collective pitch likely due to blade stall.

Sharpe also conducted hover testing measuring the lag stability of a two-bladed stiff-in-plane hingeless rotor with configurable control rigidity in Ref. 67. In addition to lag stability, this allows for a verification of how well the structural coupling parameter R_s models a real system. The current model is overlaid on the data in Fig. 4.7.

For forward flight validation, the flap-lag stability model was compared to wind tunnel experiments conducted by Gaonkar et al and presented in Refs. 64 and 66. The experiment investigated advance ratios up to $\mu = 0.55$. Note that in these tests, longitudinal and lateral flapping were not trimmed with cyclic. Only the collective pitch and shaft tilt were varied. These data with the present results overlaid are shown in Figs. 4.8 and 4.9.

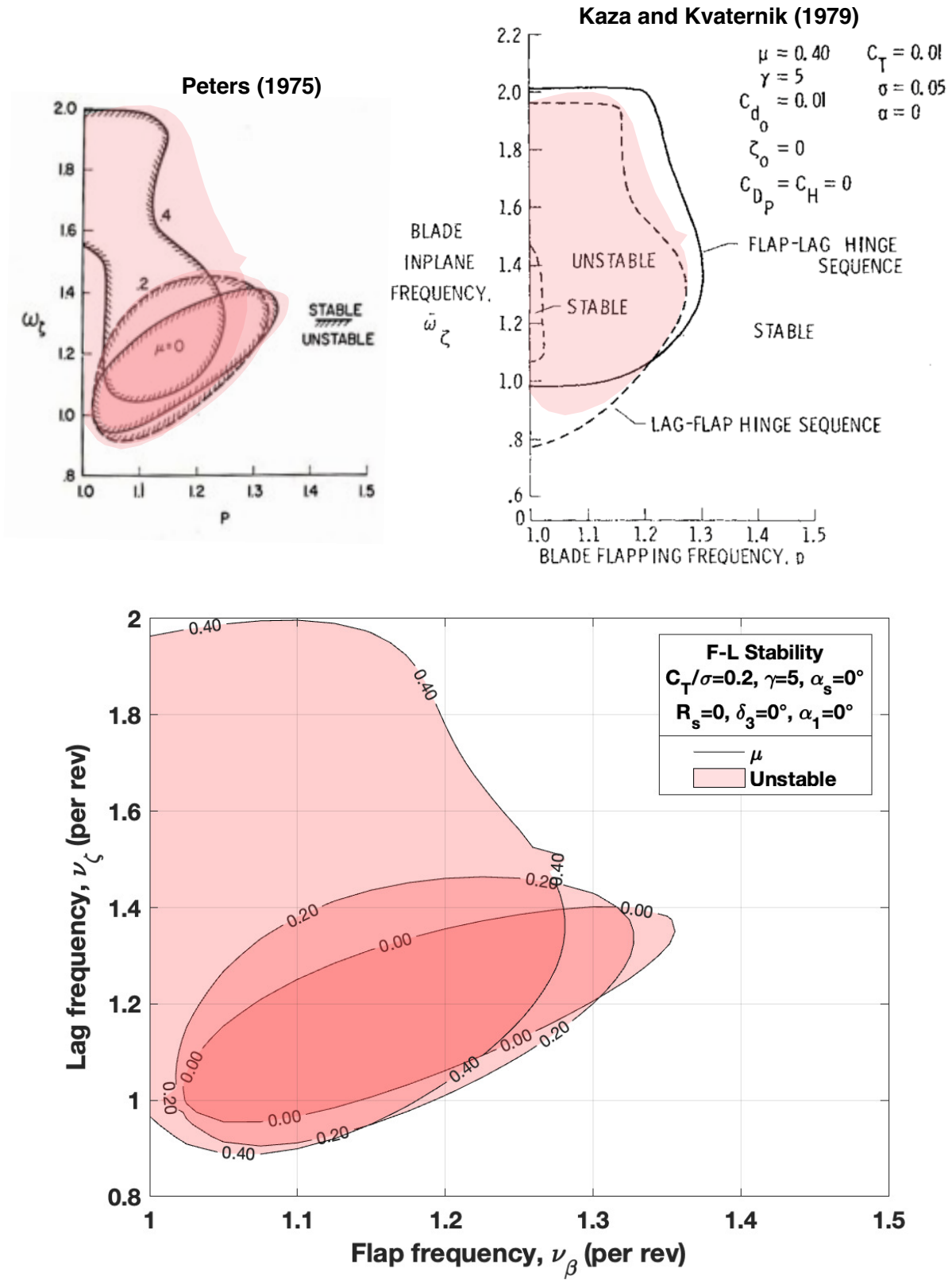


Figure 4.5: Flap-lag stability map verification (ν_{ζ} vs. ν_{β}), $C_T/\sigma = 0.2$, $\gamma = 5$, $\mu = 0$ to 0.4 .

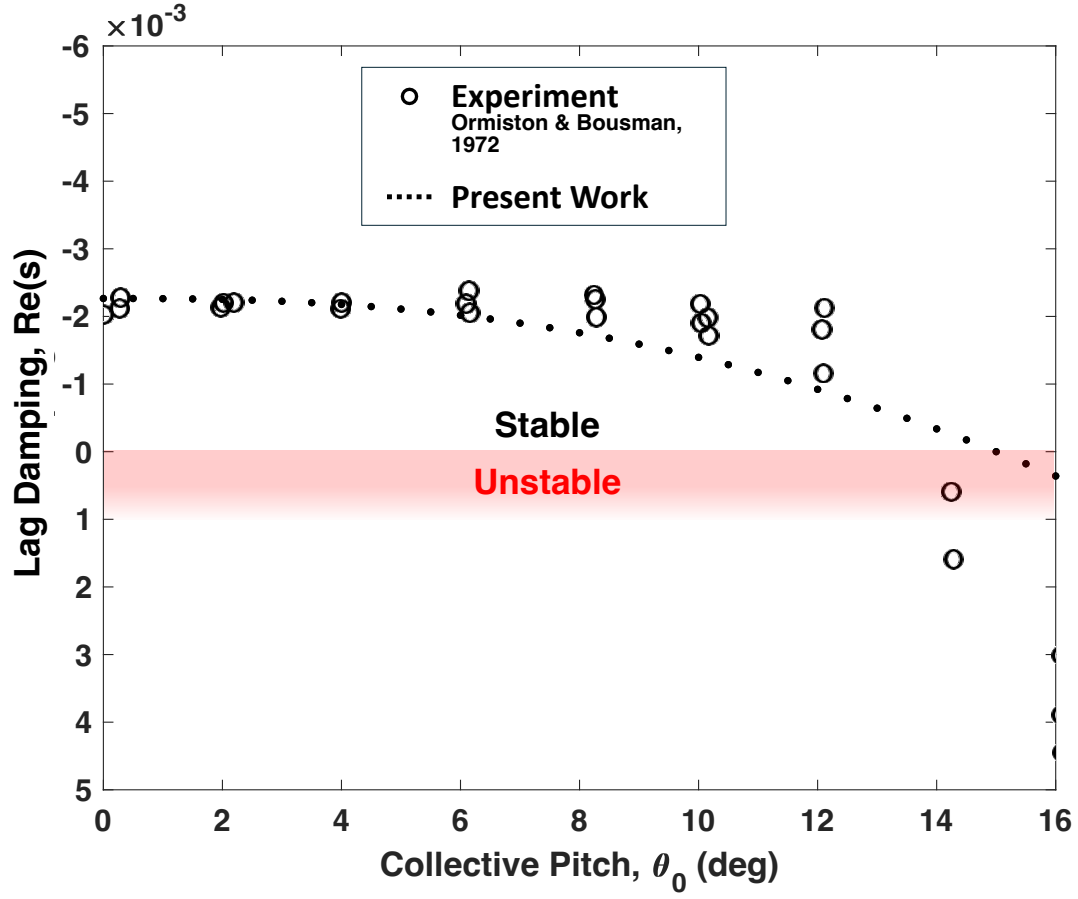


Figure 4.6: Validation of lag stability in hover with varying collective, $\nu_\beta = 1.17$, $\nu_\zeta = 1.21/\text{rev}$, $\gamma = 2.84$, soft pitch flexure: $R_s = 0.08$, $\zeta_L = 0.001$, $\theta_{tw} = -10.8^\circ$.

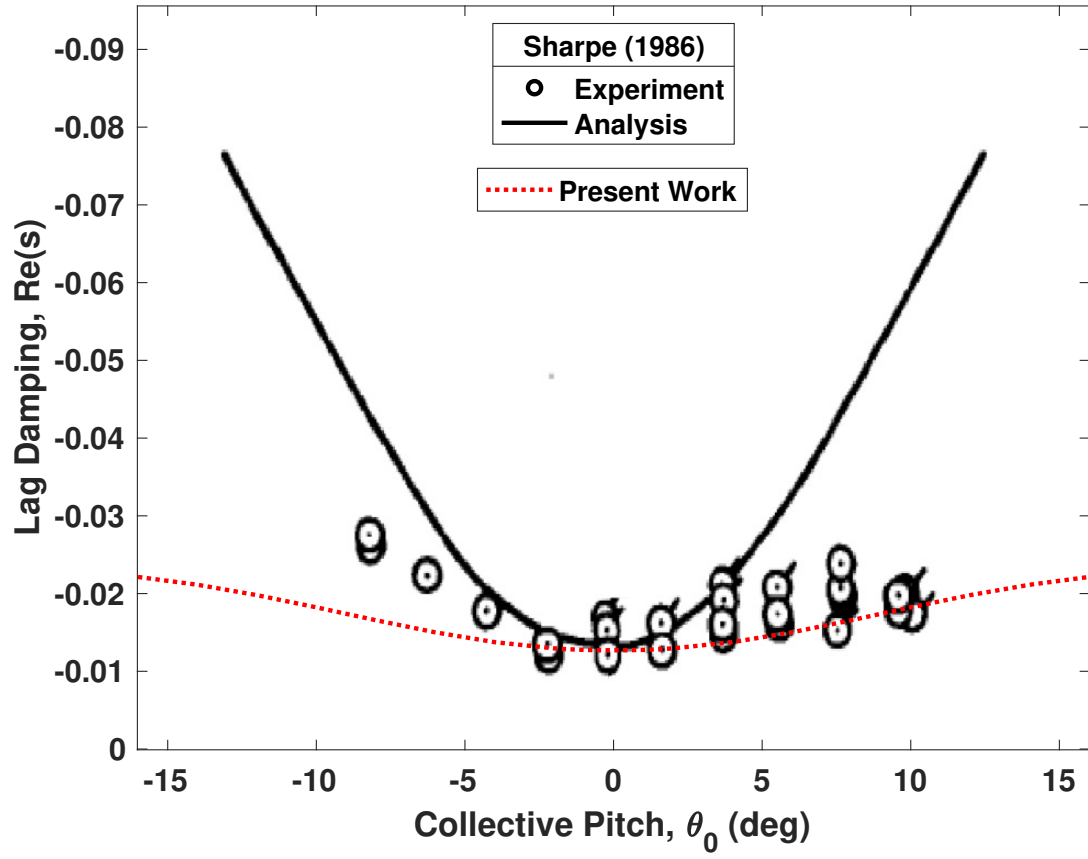


Figure 4.7: Validation of lag stability in hover with varying collective, $\nu_\beta = 1.05$, $\nu_\zeta = 1.44/\text{rev}$, $\gamma = 5.1$, stiff pitch flexure: $R_s = 1$.

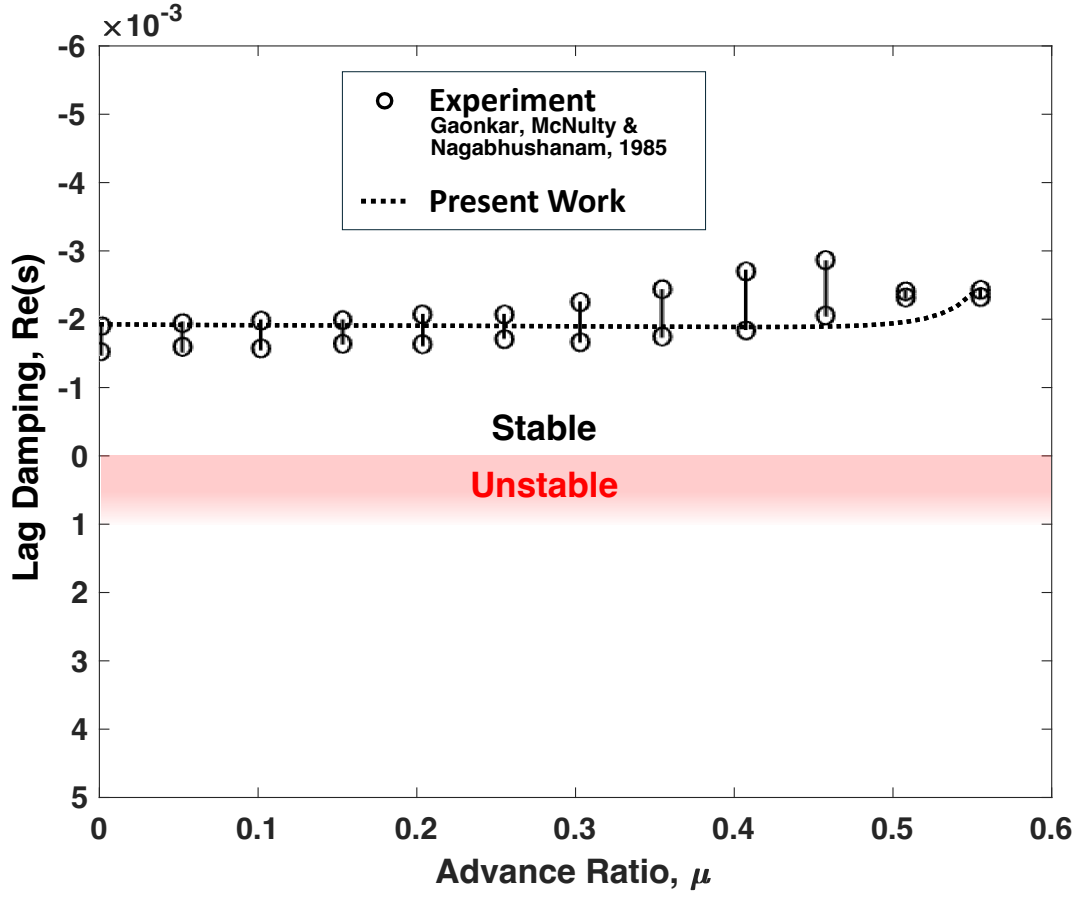


Figure 4.8: Lag stability validation with wind tunnel model testing, $\nu_\beta = 1.105$, $\nu_\zeta = 0.61/\text{rev}$, $\gamma = 7.54$, $R_s = 0$, $\zeta_L = 0.001$, NACA 23012 airfoil, $\theta_0 = 0^\circ$, $\alpha_s = 0^\circ$, untrimmed.

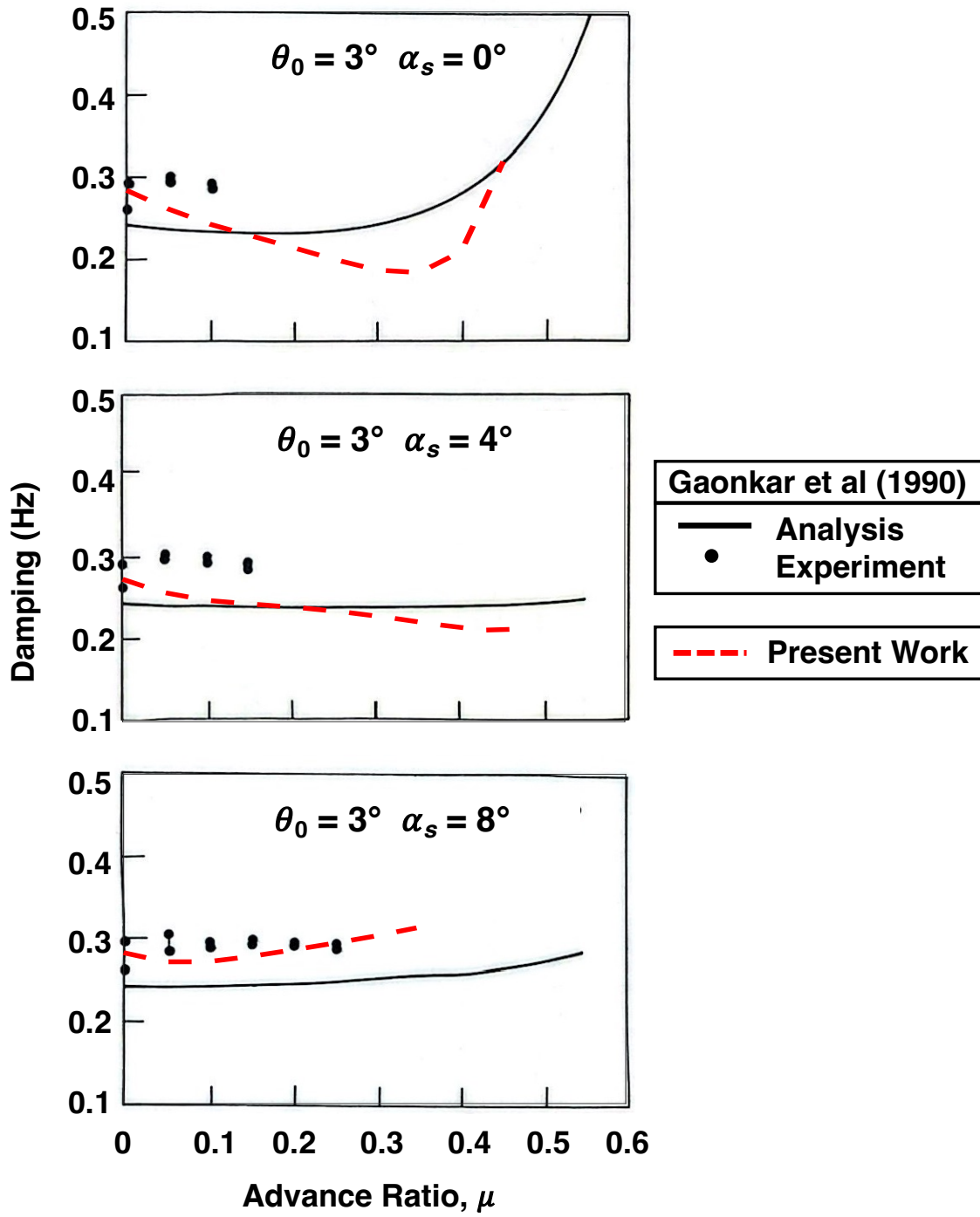


Figure 4.9: Lag stability validation with wind tunnel model testing, $\theta_0 = 3^\circ$, $\alpha_s = 0^\circ$ to 8° , untrimmed.

4.5 Parametric Evaluation of Coupled Flap-Lag Damping

The following parametric analysis will show the flap-lag stability for advance ratios between 0 and 2.5 and a Lock number of $\gamma = 6$. Flap and lag frequencies are varied between $\nu_\beta = 1$ to 1.5 and $\nu_\zeta = 0.1$ to 2. Blade loading is set at $C_T/\sigma = 0.04$ and the elastic coupling is set to $R_s = 0$ or 1. Figs. 4.10 through 4.13 depict the stability of the least stable flap or lag roots. Shaded unstable regions (where damping > 0) correspond to the numbered μ values.

Shaft tilt is targeted to minimize flapping at the thrust reversal advance ratio. For a blade loading of $C_T/\sigma = 0.04$ and a Lock number of $\gamma = 6$, this results in shaft tilts of approximately $\alpha_s = -2.7$ to -2.8° (aft). The flap frequency will slightly affect this value.

The major conclusions from this parametric analysis is that aft shaft tilt is needed to maintain stability through thrust reversal, as was predicted during the flap trim discussion in Chapter 3. Full elastic coupling is destabilizing in the cases explored, except for the matched stiffness cases where the non-rotating flap and lag frequencies cancel this term. Low lag frequencies (very soft-in-plane), specifically those around 0.5/rev, are consistently unstable. Half and one per rev frequency multiples should also be avoided when selecting the blade properties and rotor speed that determines ν_ζ .

4.6 Eigenvalue Trajectory Analysis

An automated eigenvector trajectory analysis is used to identify the flap and lag roots. Like in the flap-only case, by finding the minimum residual of the eigenvalues $\underline{z}$

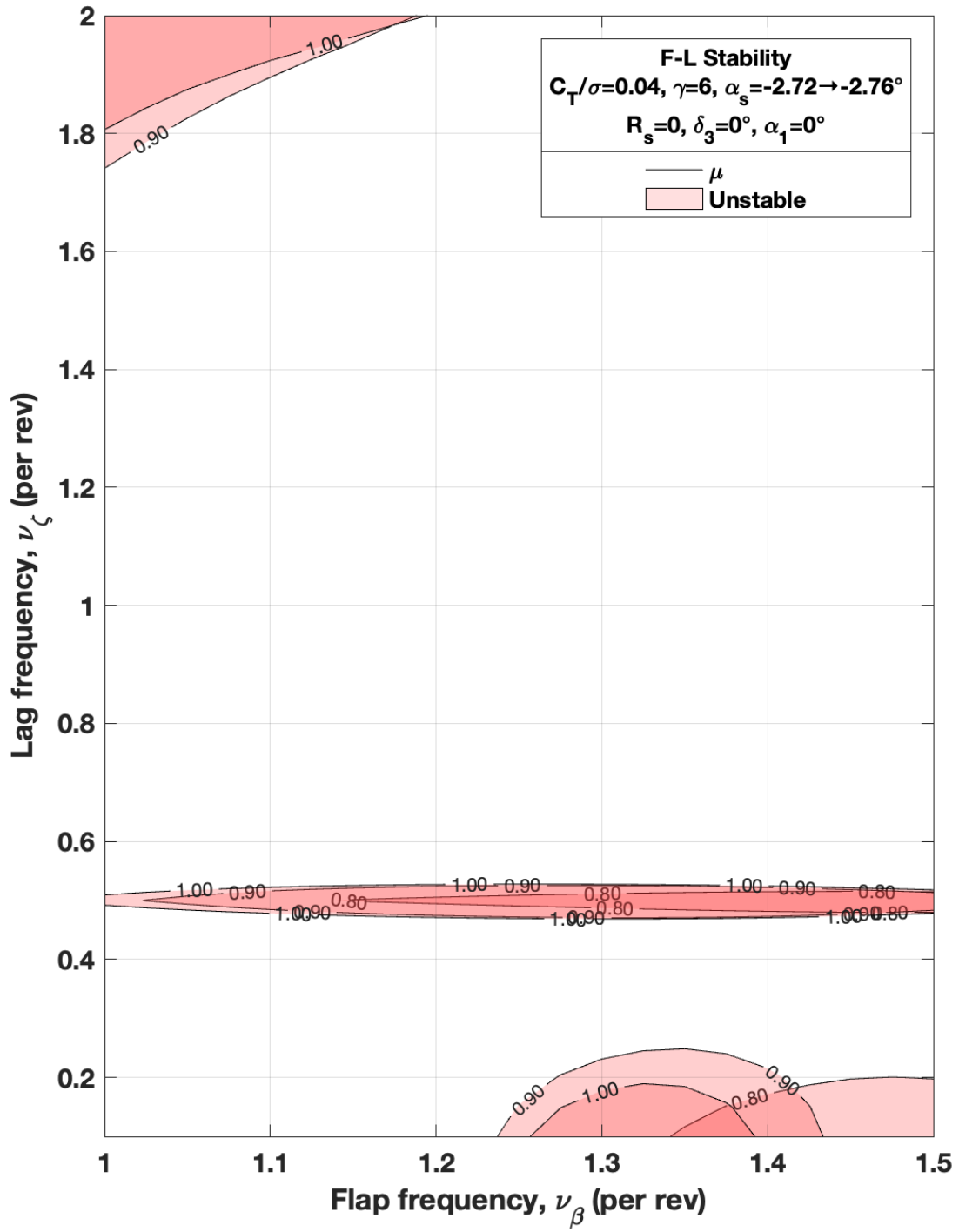


Figure 4.10: Flap-lag stability map (ν_ζ vs. ν_β), $\gamma = 6$, $C_T/\sigma = 0.04$, $\mu = 0.6$ to 1.0 .

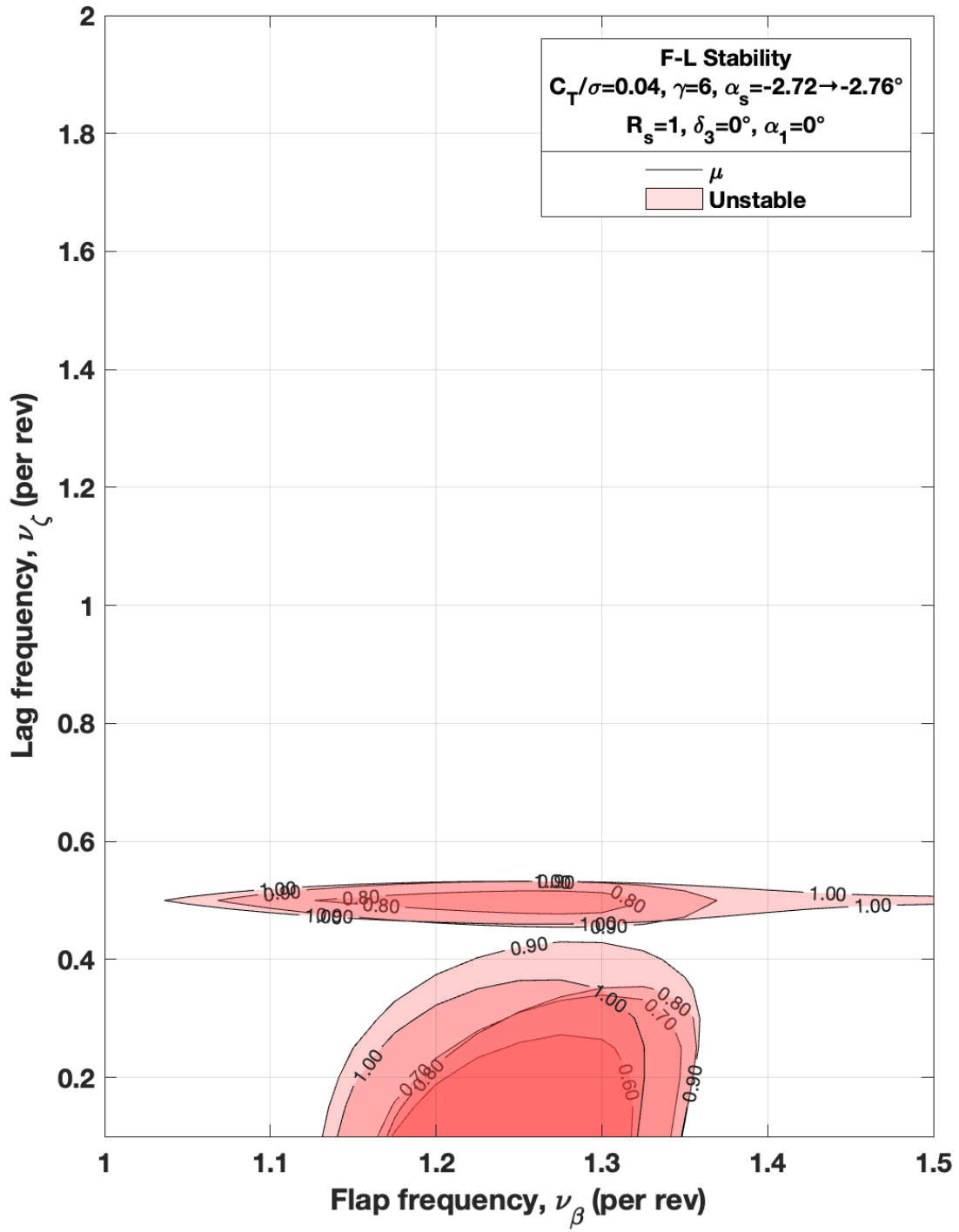


Figure 4.11: Flap-lag stability map (ν_ζ vs. ν_β), $\gamma = 6$, $C_T/\sigma = 0.04$, $R_s = 1$, $\mu = 0.6$ to 1 .

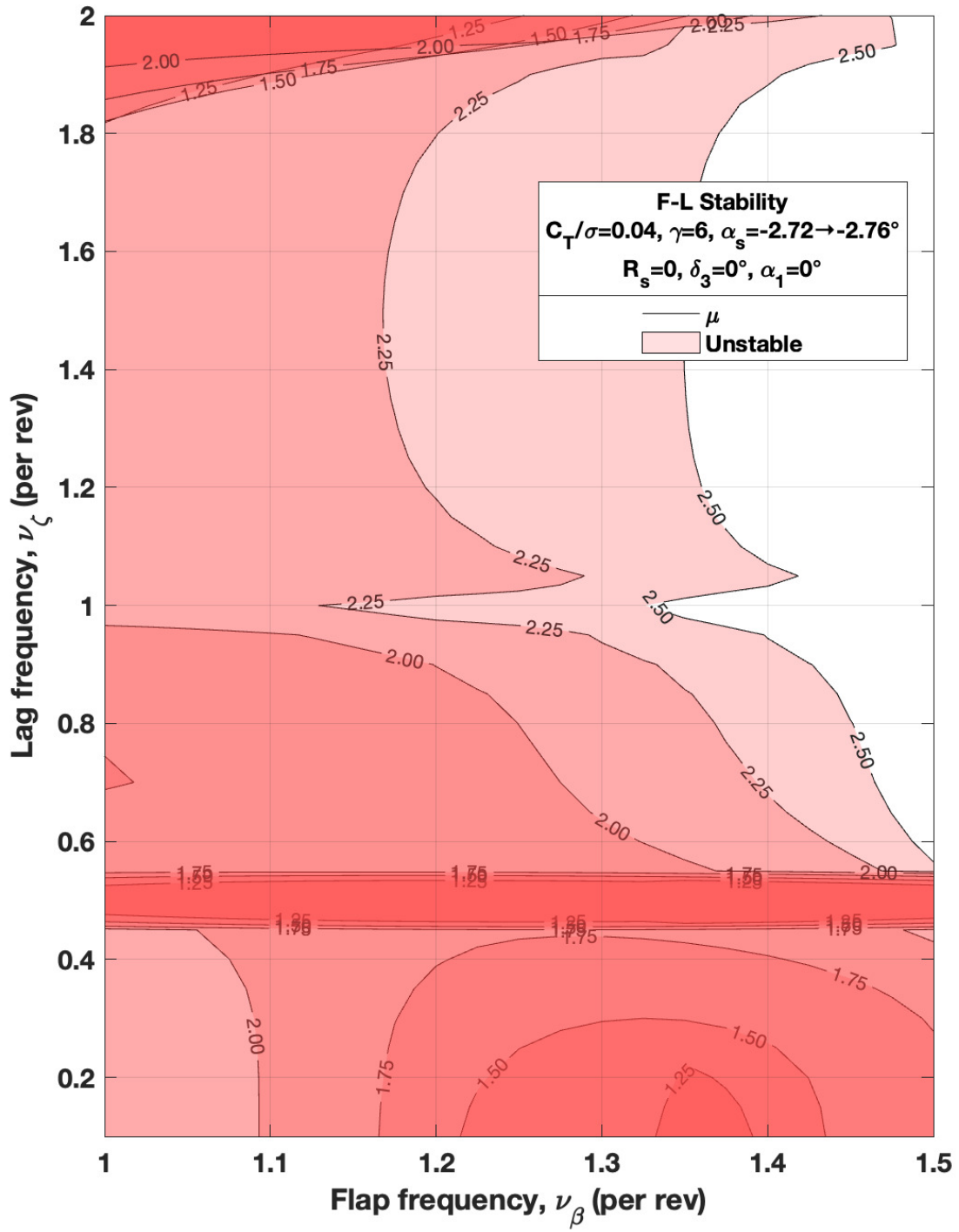


Figure 4.12: Flap-lag stability map (ν_ζ vs. ν_β), $\gamma = 6$, $C_T/\sigma = 0.04$, $\mu = 1.25$ to 2.5 .

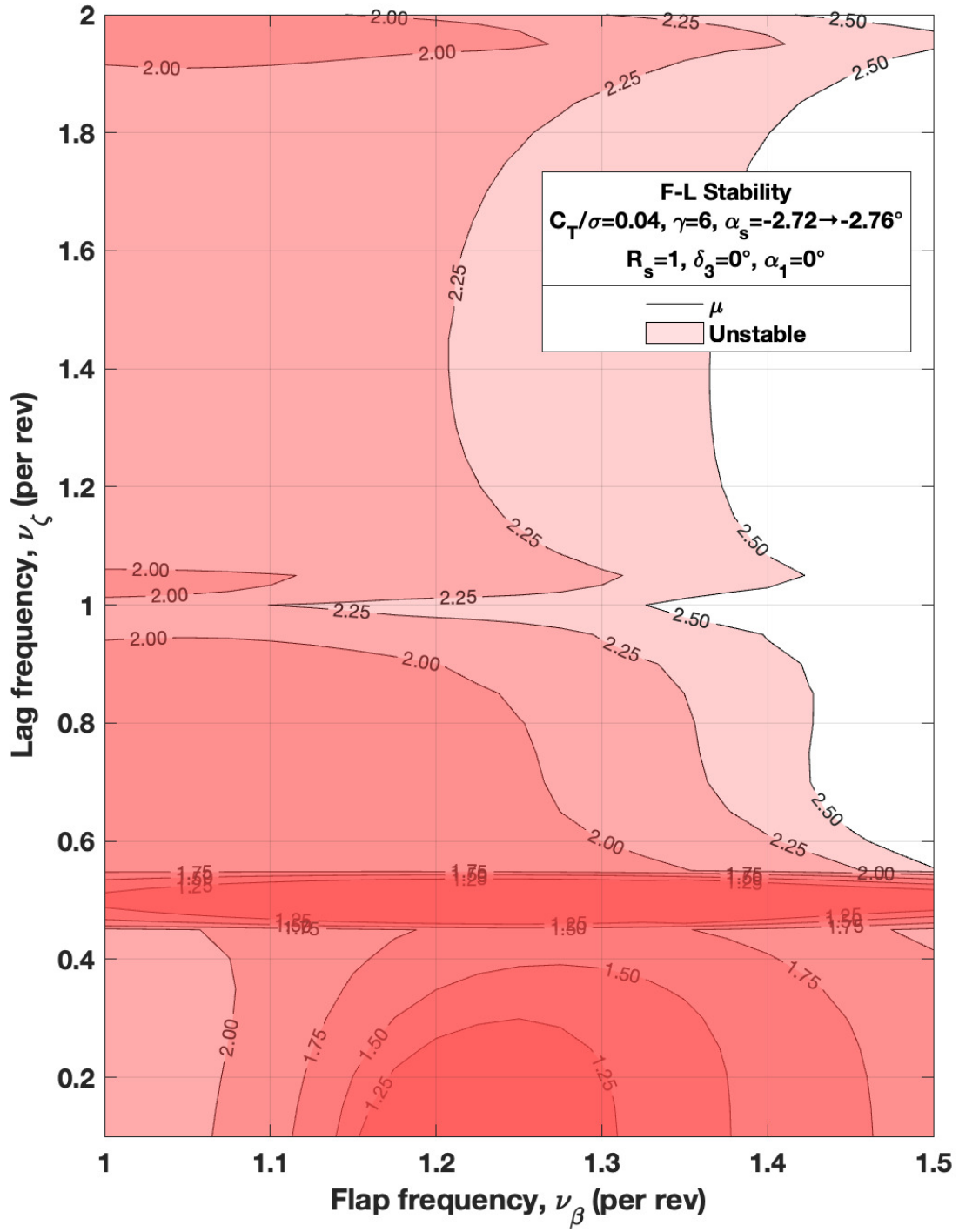


Figure 4.13: Flap-lag stability map (ν_ζ vs. ν_β), $\gamma = 6$, $C_T/\sigma = 0.04$, $R_s = 1$, $\mu = 1.25$ to 2.5.

and the participation factors η from the FFT of the eigenvector motion $\underline{\underline{V}}(\psi)$ over a small advance ratio increment $\Delta\mu$, a likely eigenvalue trajectory can be determined. However, in the flap-lag case, there are now $4! = 24$ permutations, making incorrect root swaps more significant. Fortunately, the eligible combinations can be reduced by applying the following rules regarding the eigenvalues:

1. For correct root pairs (i.e. F1/F2 and L1/L2), the values of $\text{Re}(z_n)$ have the same sign. Positive real values (right half of the z -plane) correspond to $\mathbb{Z}/\text{rev } T$ -periodic frequency responses, and negative real values (left half of the z -plane) correspond to $\mathbb{Z} + 1/2\text{rev } 2T$ -periodic perturbations.
2. Root pairs have the same magnitude of $\text{Im}(z_n)$ with opposite signs. The eigenvalues are complex conjugates when $|\text{Im}(z_n)| > 0$; in this case their real values are identical. When $\text{Im}(z_n) = 0$, the roots are frequency locked and their real values will diverge from one another. Discerning between frequency locked roots can be challenging because they have nearly the same progressive and regressive frequency components, but the more important differentiation is between degrees of freedom.

Only those trial root pairs meeting these compatibility criteria need to go through the FFT process, saving some computation time. If all of the roots are frequency locked in the same real half of the z -plane, then no combinations can be eliminated categorically without performing the eigenvector motion analysis. There is also the possibility that all of the trial combinations have incompatible eigenvalues, which has been observed to occur at values of μ after the onset of numerical instability. This typically manifests itself when the extremely damped root of a frequency-locked pair develops high dissipation, its eigenvalue

shifts to the opposite half of the real z -plane, and its frequency becomes undefined. While the least stable root is more important for the stability of the physical system, and the motion of its highly damped (though numerically unstable) pair might not be practically measurable, this nevertheless makes eigenvalue tracking more difficult.

The root selection frequently varies as advance ratio, Lock number, blade loading, and other parameters are changed. At $\mu = 0$ and $C_T/\sigma = 0$, the flap and lag equations are time-invariant and uncoupled, which allows for the initial stability of each degree of freedom to be solved independently. The eigenvalue trajectories are then incrementally projected from this zero-thrust hover point.

Using this automatic root identification algorithm, the motion of the characteristic multipliers and characteristic exponents (root loci) are plotted. The following results are for 120 time steps and $\Delta\mu = 0.01$. The Lock number is set to $\gamma = 6$ and the blade loading is $C_T/\sigma = 0.04$. The shaft tilt is again trimmed aft to achieve zero coning at the thrust reversal point. Damping, trim, blade stall, and numerical stability are also compared as a function of μ . Structural coupling is initially set to 0.

Four rotor configurations will be explored. These cases are:

1. Articulated rotor, very soft-in-plane: $\nu_\beta = 1$, $\nu_\zeta = 0.3/\text{rev}$ (Figs. 4.14 through 4.32).
2. Articulated rotor, soft-in-plane: $\nu_\beta = 1$, $\nu_\zeta = 0.7/\text{rev}$ (Figs. 4.33 through 4.50).
3. Hingeless rotor, soft-in-plane: $\nu_\beta = 1.15$, $\nu_\zeta = 0.7/\text{rev}$ (Figs. 4.51 through 4.86).
4. Hingeless rotor, stiff-in-plane: $\nu_\beta = 1.15$, $\nu_\zeta = 1.4/\text{rev}$ (Figs. 4.69 through 4.78).

The effect of structural coupling is evaluated against the stiff-in-plane hingeless rotor (Case

4); R_s is increased from 0 (elastic hub, rigid blade) to 1 (rigid hub, elastic blade); see Figs. 4.87 through 4.104.

4.7 Flap-Lag Stability Conclusions

It is seen in most cases that the primary driver of instability is lag. At high- μ , this instability is closely tied to the asymptotic flap trim divergence found earlier, which occurs while transitioning through thrust reversal ($\mu = 0.75$ to 0.95). Lag stability is especially sensitive to the trim solution due to the Coriolis terms in the equation of motion. Aft shaft tilt and closely paired blade loading can mitigate this problem. Only in the cases where there is low or zero blade loading (and thus minimal activation of the Coriolis terms) is flap instability reached in the same manner as the uncoupled case.

Very soft-in-plane ($\nu_\zeta < 0.5/\text{rev}$) rotor configurations are unstable when in combination with hingeless rotors having higher flap frequencies (admittedly a rare configuration). Unsurprisingly, avoiding $0.5/\text{rev}$ and $1/\text{rev}$ multiples for the lag natural frequency was beneficial for stability. The effect of structural coupling on the lag-flap stability map appeared small and was varyingly stabilizing or destabilizing depending on the configuration.

Blades that are stiff in both flap and lag generally exhibit better stability as μ increases. For soft-in-plane articulated and hingeless rotors in the configuration shown, lag instability occurs before flap instability, at approximately $\mu = 1.8$. The very soft-in-plane articulated rotor can maintain lag stability until flap becomes unstable, and the stiff-in-plane hingeless blade remains stable through $\mu = 3$. It can be seen that lag was conventionally unstable (no frequency locking/ parametric resonance) in these examples.

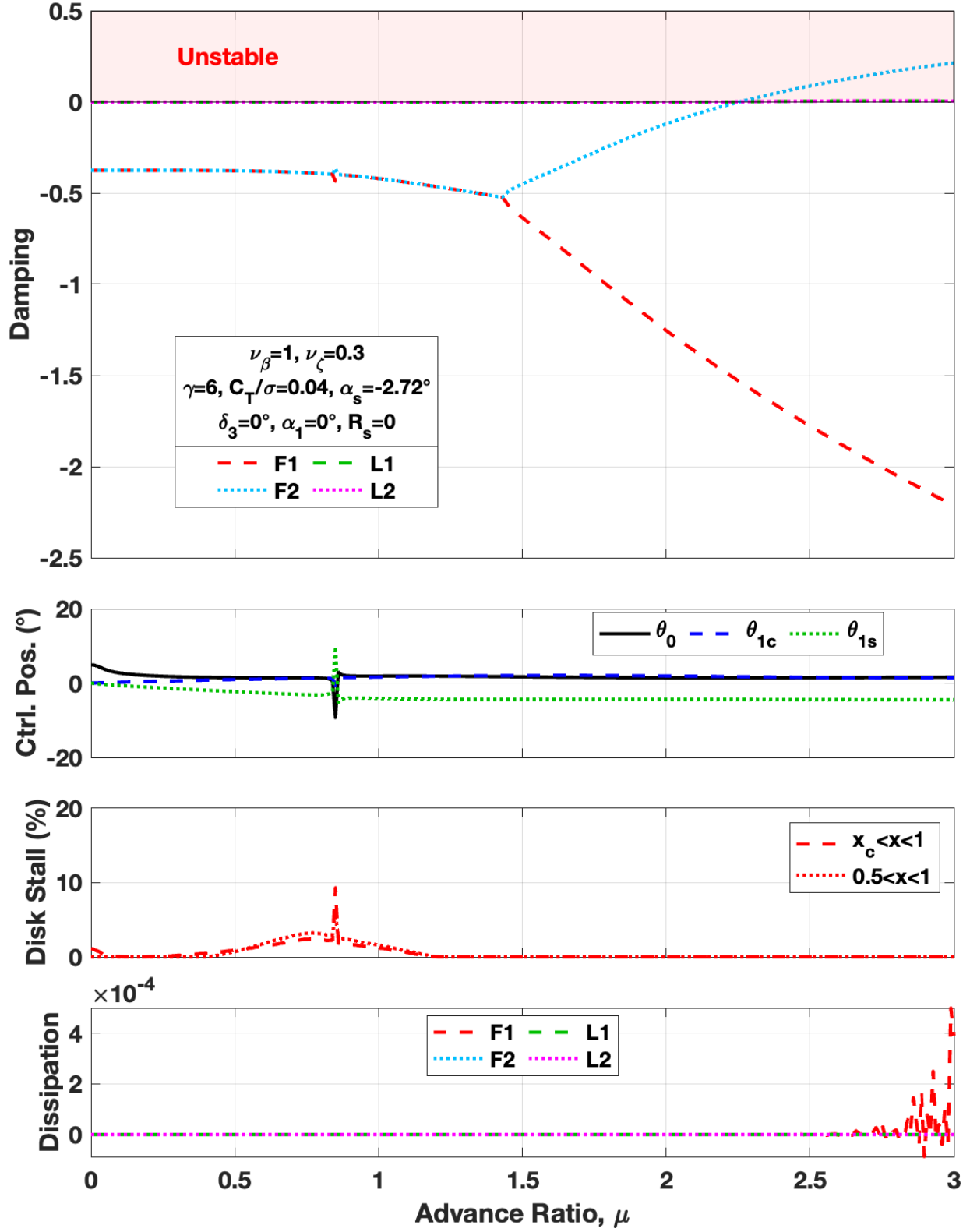


Figure 4.14: Flap-lag damping, trim, blade stall, and numerical stability, very soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.3/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$.

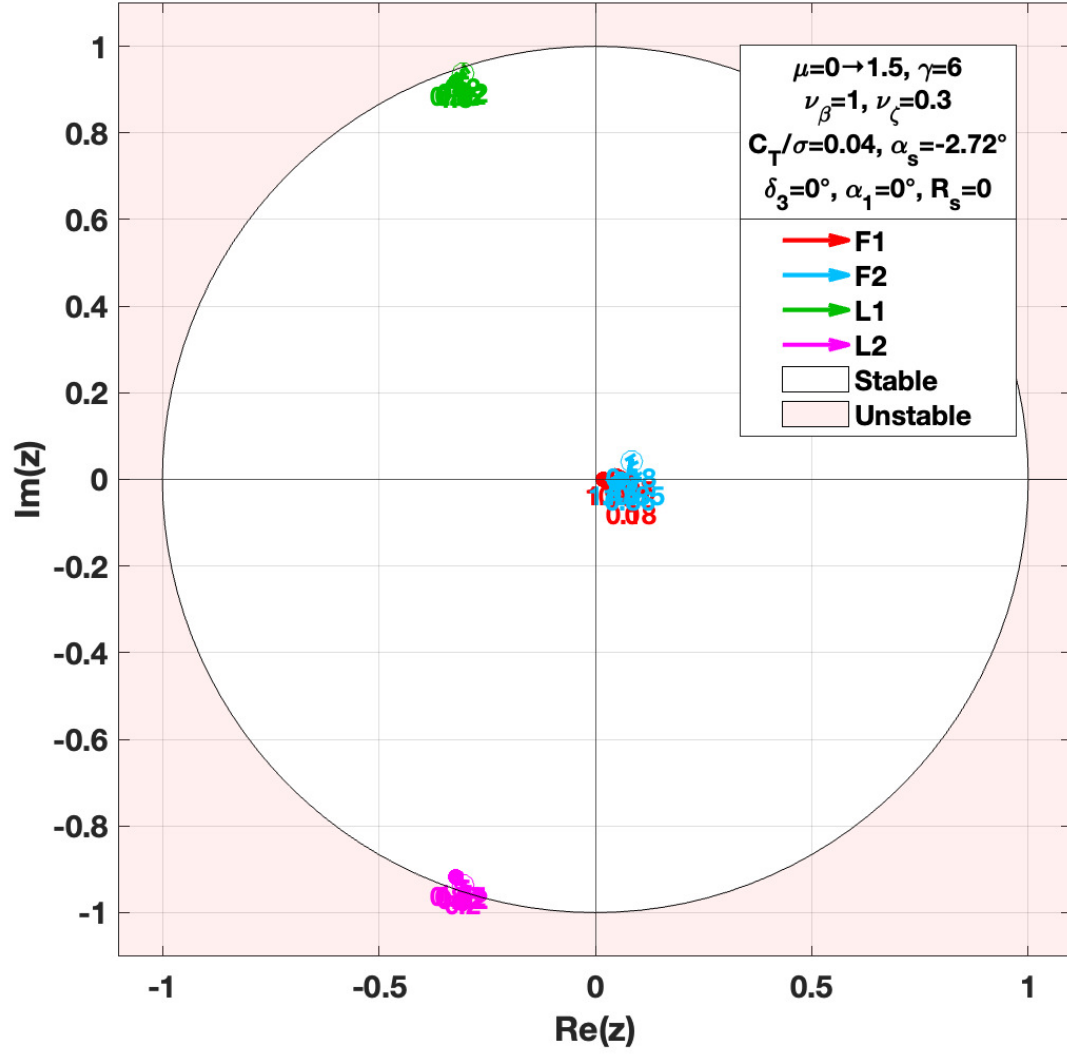


Figure 4.15: Flap-lag characteristic multipliers, very soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.3/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $\mu = 0$ to 1.5.

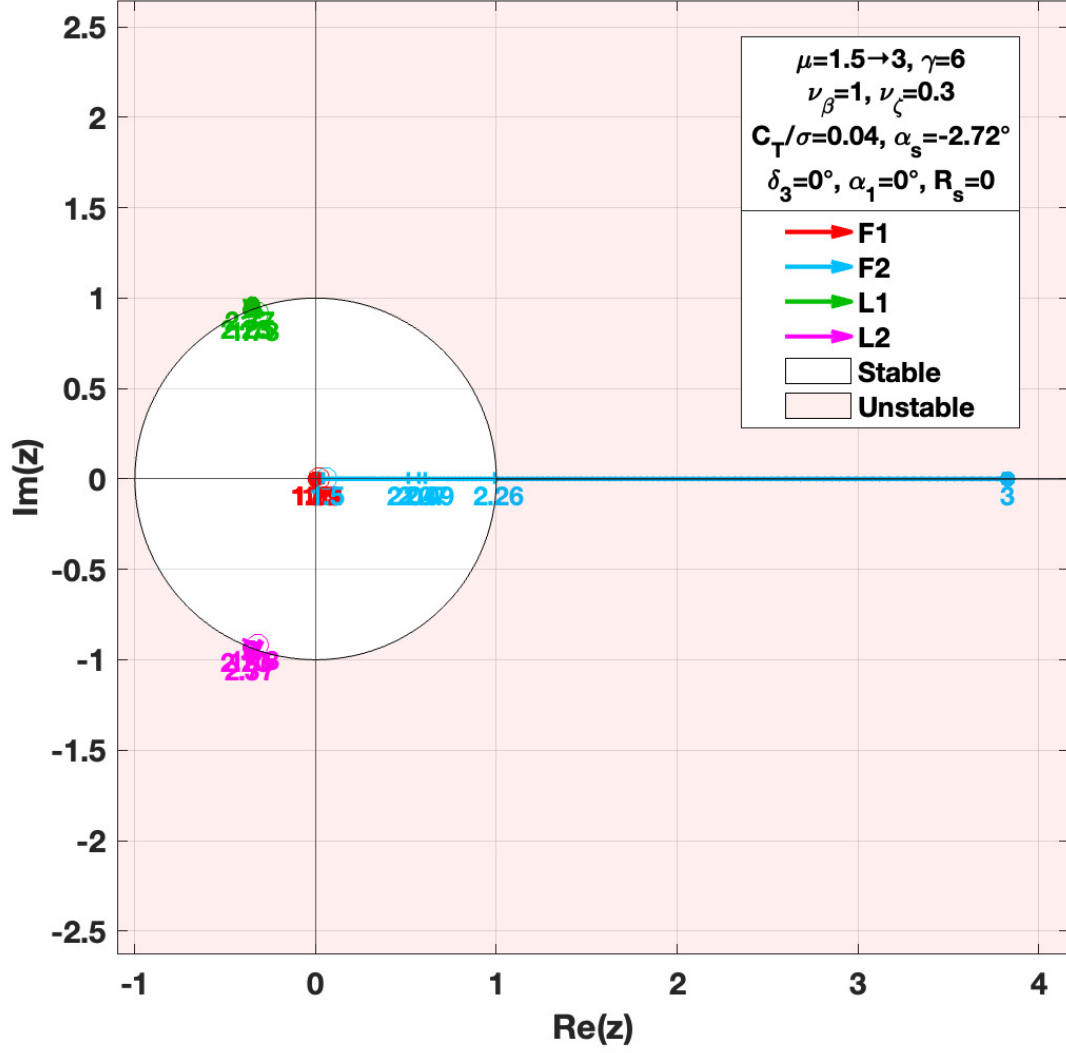


Figure 4.16: Flap-lag characteristic multipliers, very soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.3/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $\mu = 1.5$ to 3 .

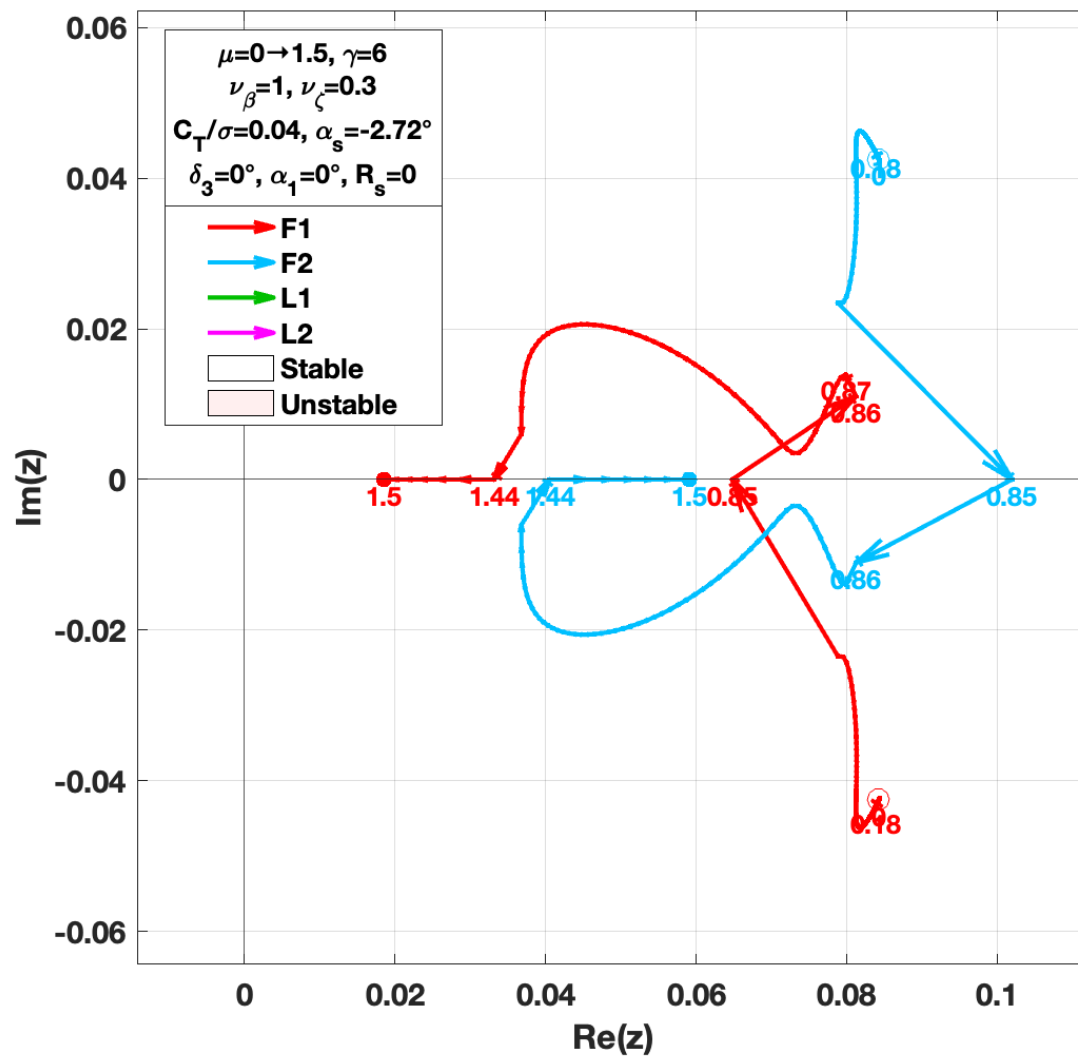


Figure 4.17: Flap-lag characteristic multipliers, F1 and F2 roots, very soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.3/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $\mu = 0$ to 1.5.

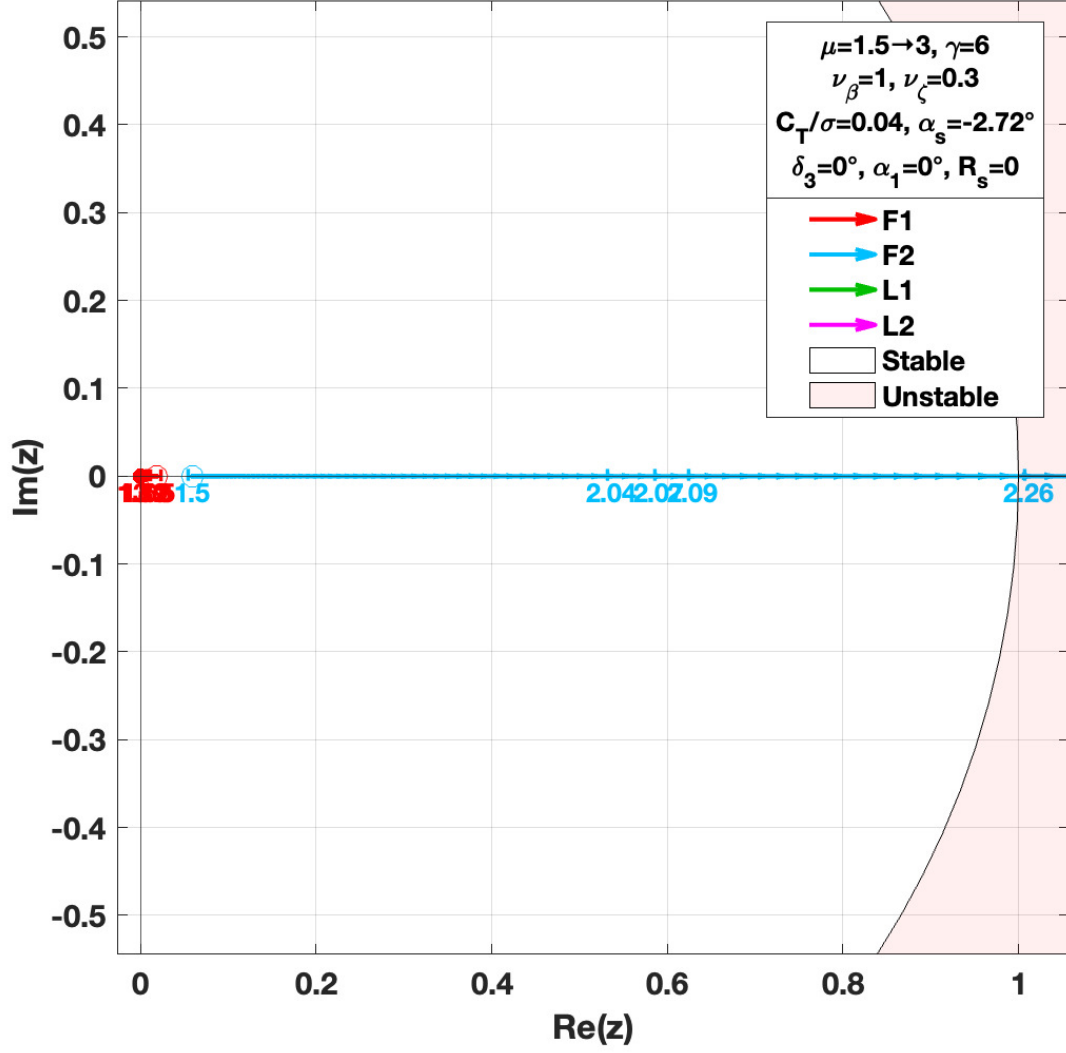


Figure 4.18: Flap-lag characteristic multipliers, F1 and F2 roots, very soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.3/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $\mu = 1.5$ to 3.

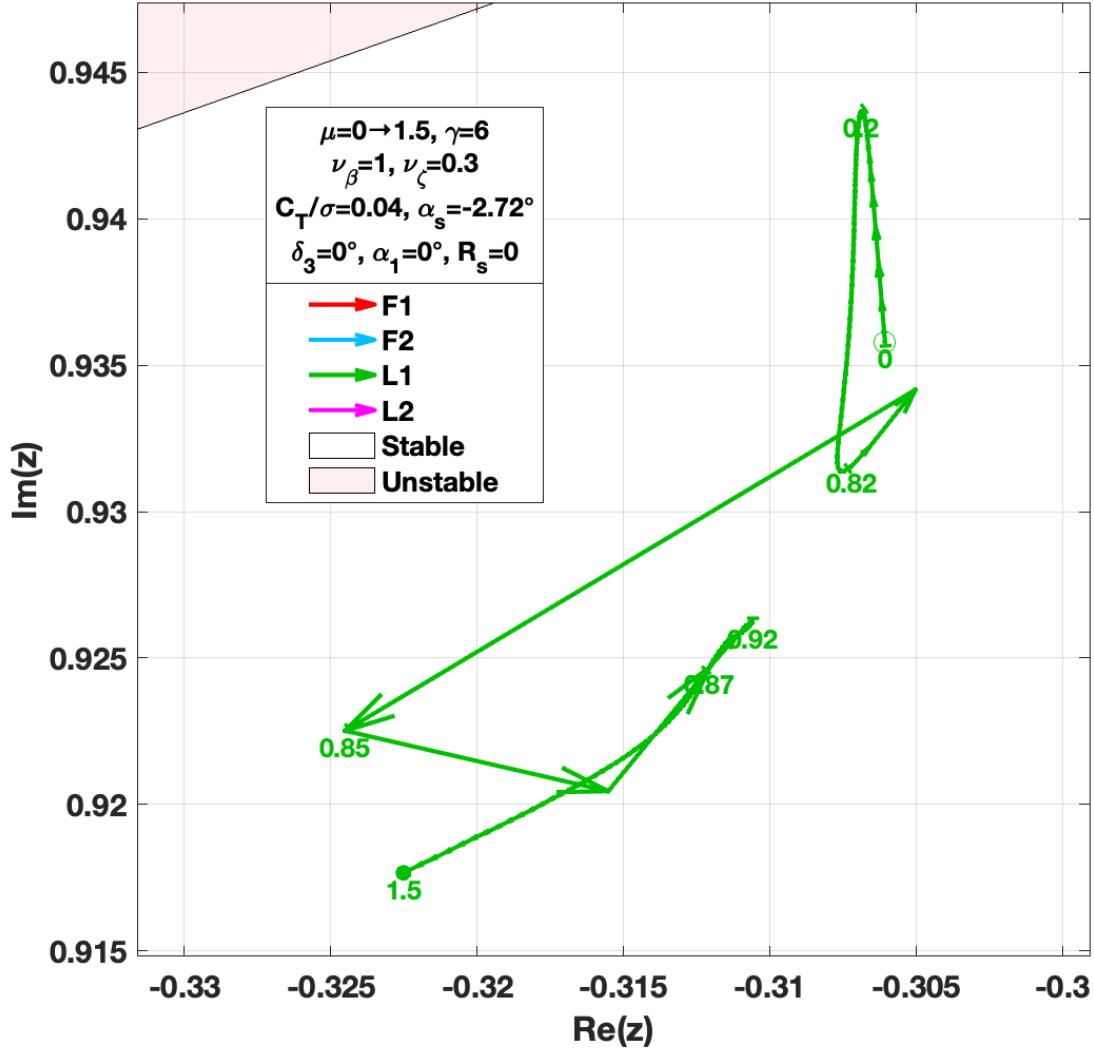


Figure 4.19: Flap-lag characteristic multipliers, L1 root, very soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.3/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $\mu = 0$ to 1.5 .

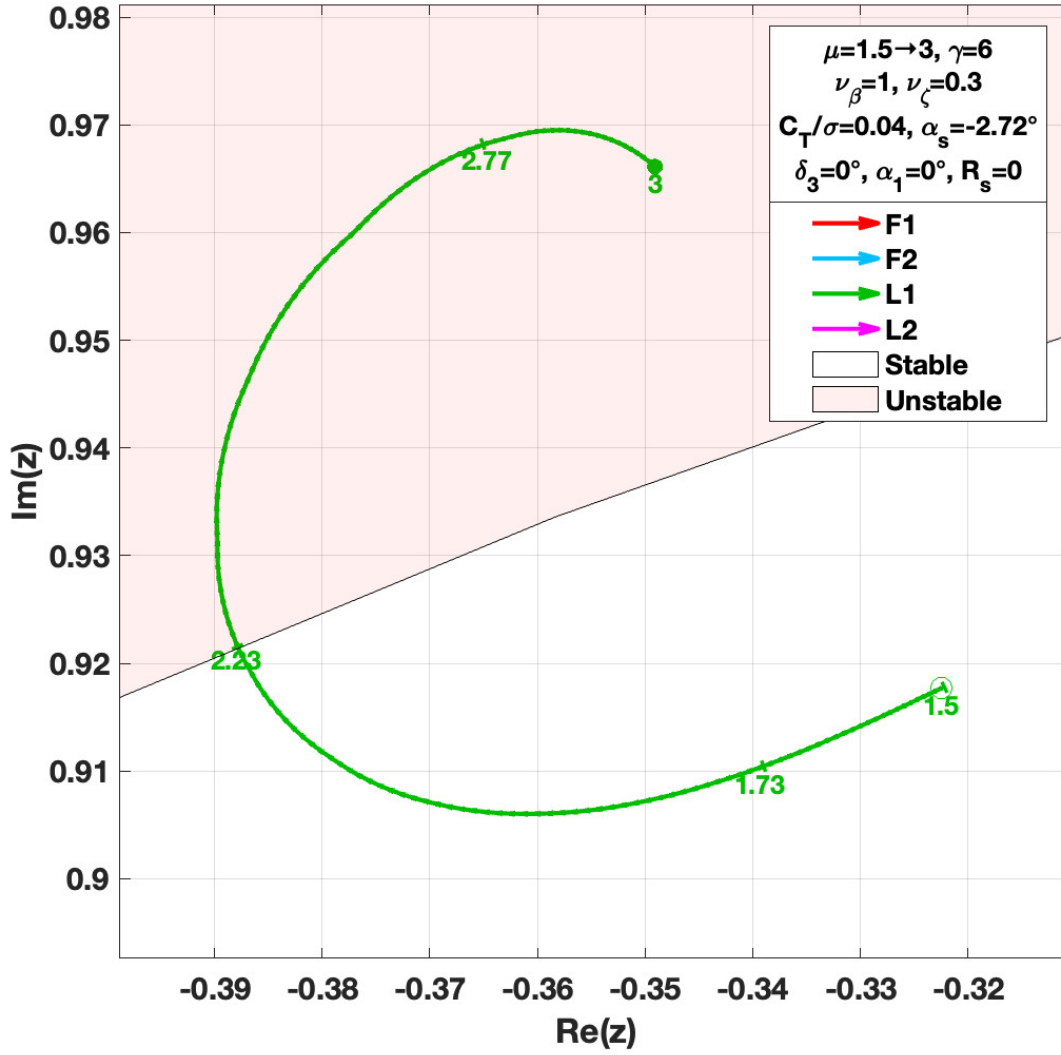


Figure 4.20: Flap-lag characteristic multipliers, L1 root, very soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.3/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $\mu = 1.5$ to 3 .

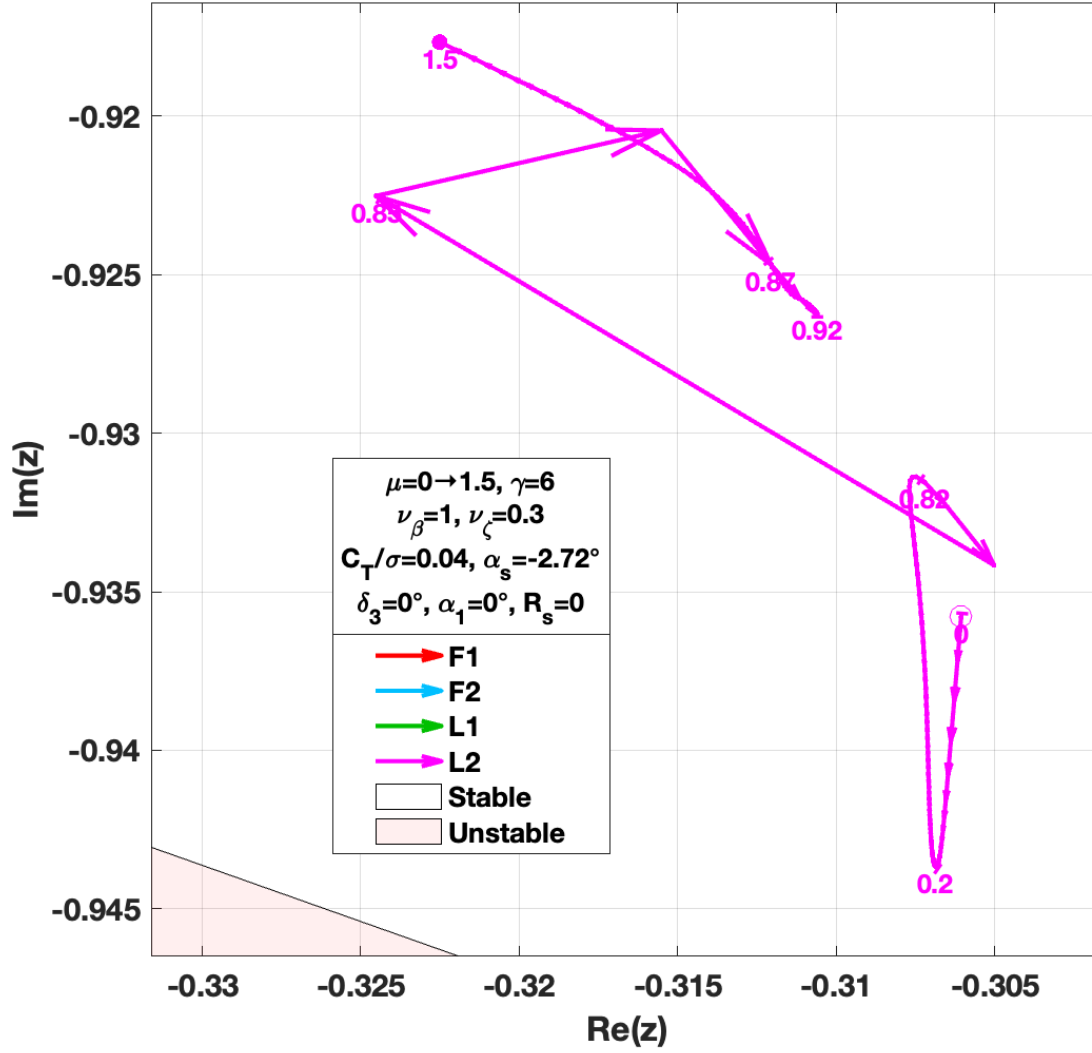


Figure 4.21: Flap-lag characteristic multipliers, L2 root, very soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.3/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $\mu = 0$ to 1.5.

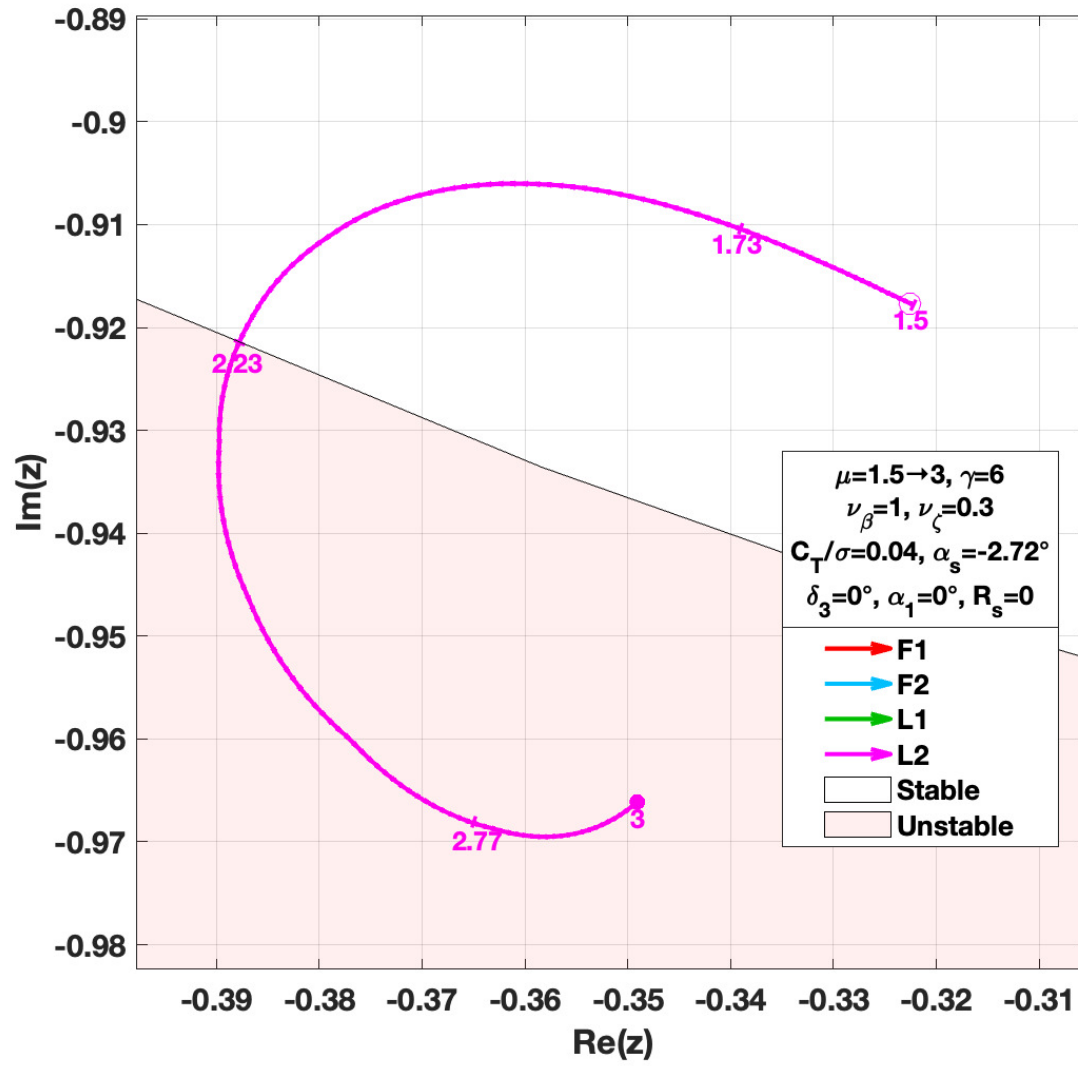


Figure 4.22: Flap-lag characteristic multipliers, L2 root, very soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.3/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $\mu = 1.5$ to 3 .

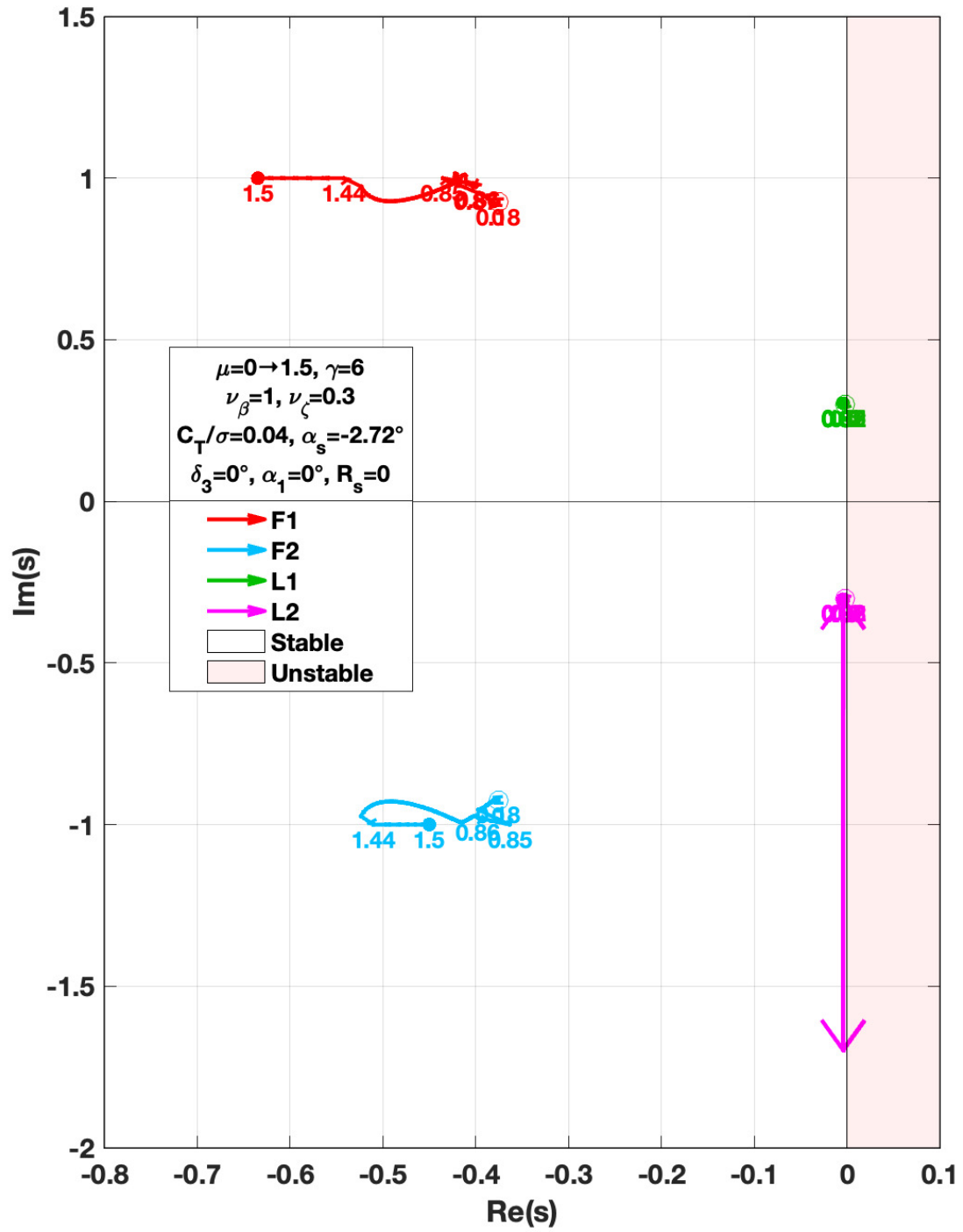


Figure 4.23: Flap-lag root locus, very soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.3/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $\mu = 0$ to 1.5 .

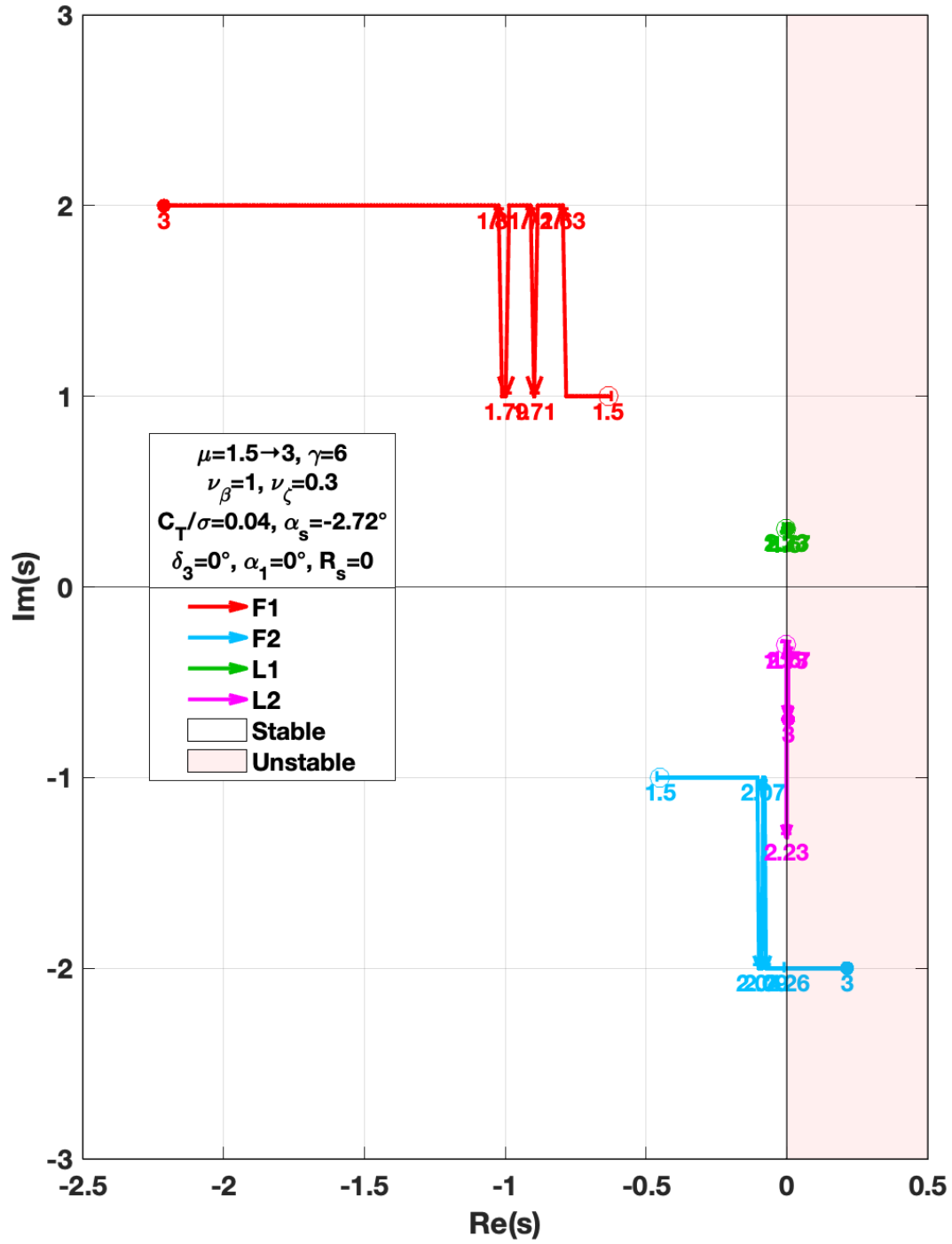


Figure 4.24: Flap-lag root locus, very soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.3/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $\mu = 1.5$ to 3.

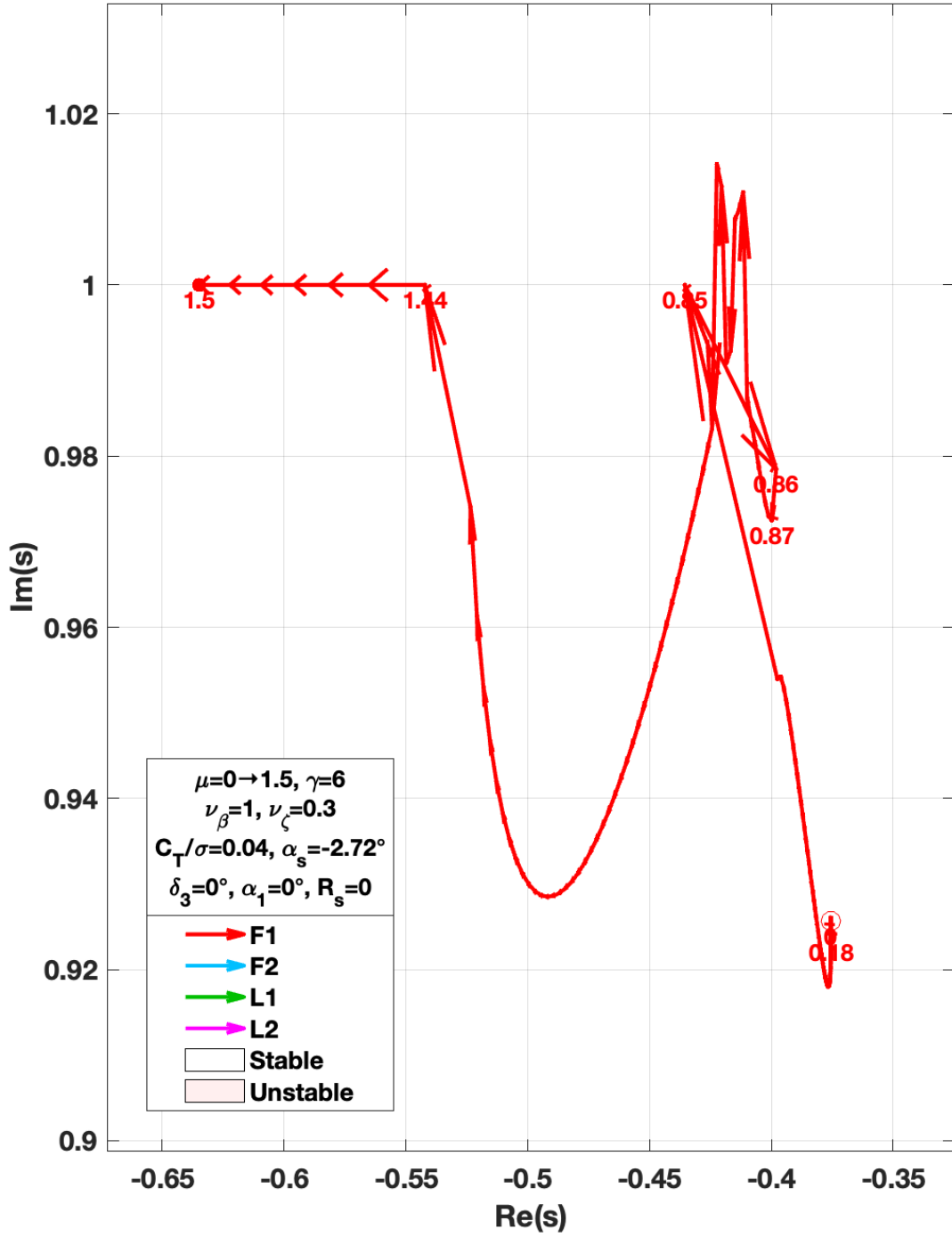


Figure 4.25: Flap-lag root locus, F1 root, very soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.3/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $\mu = 0$ to 1.5 .

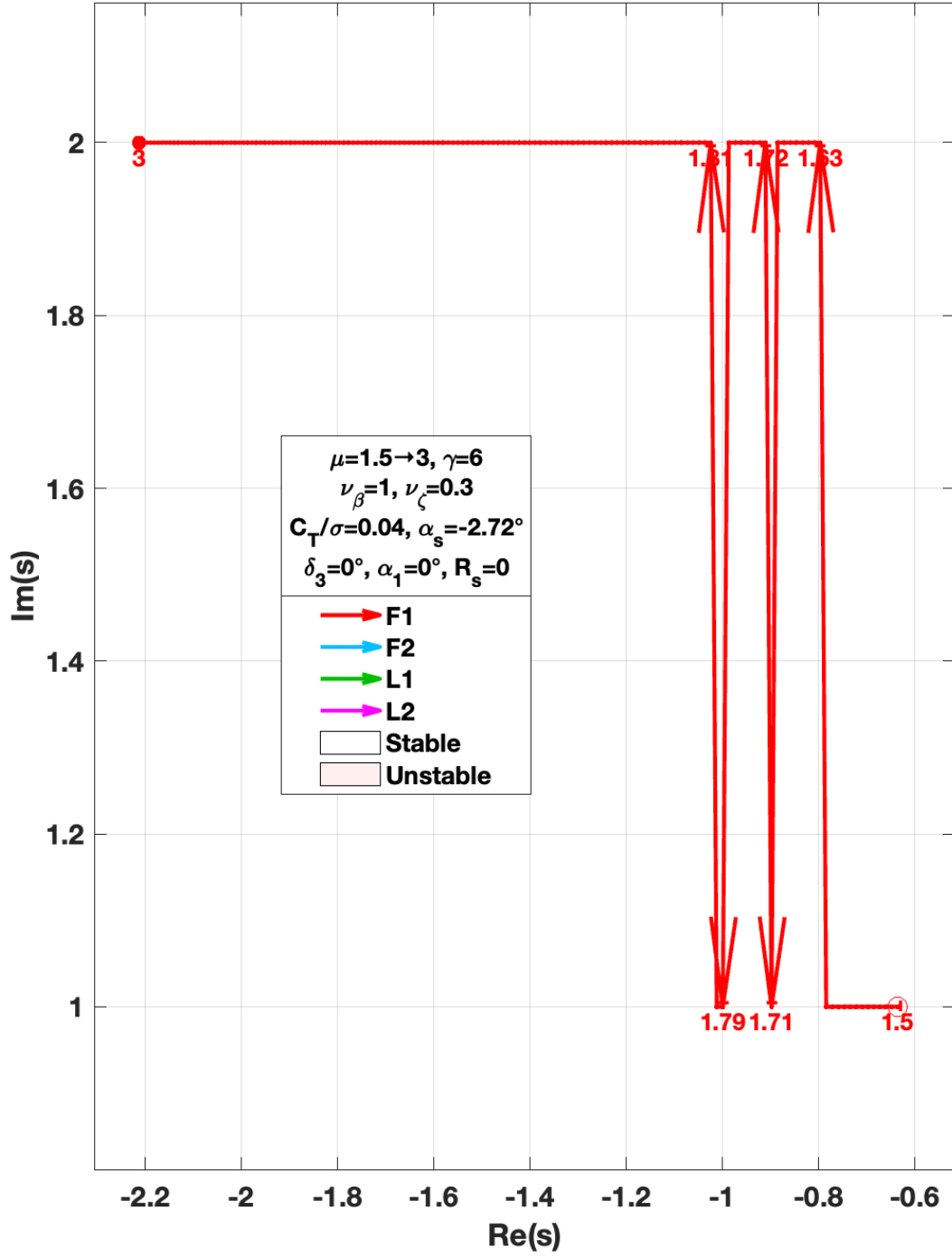


Figure 4.26: Flap-lag root locus, F1 root, very soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.3/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $\mu = 1.5$ to 3 .

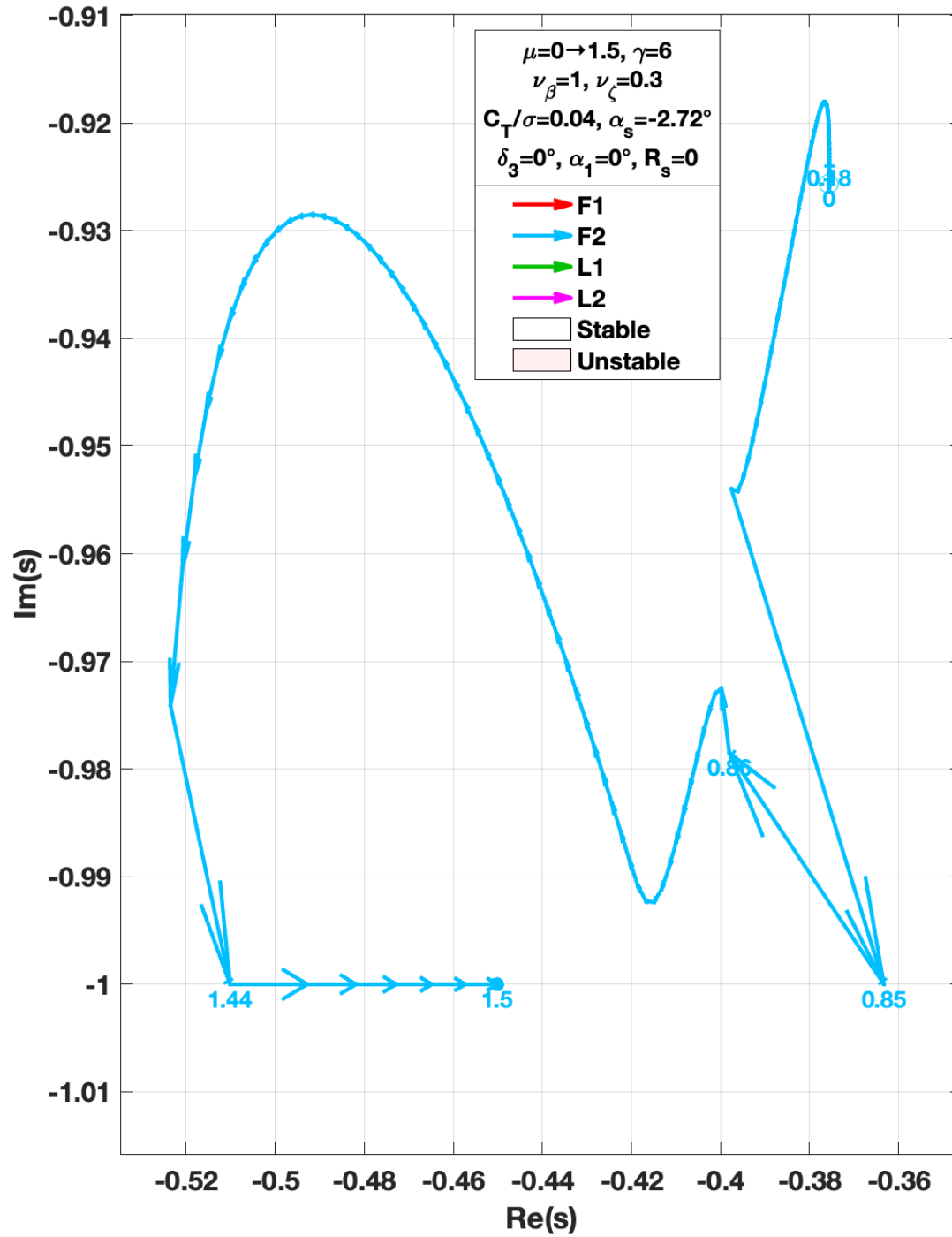


Figure 4.27: Flap-lag root locus, F2 root, very soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.3/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $\mu = 0$ to 1.5 .

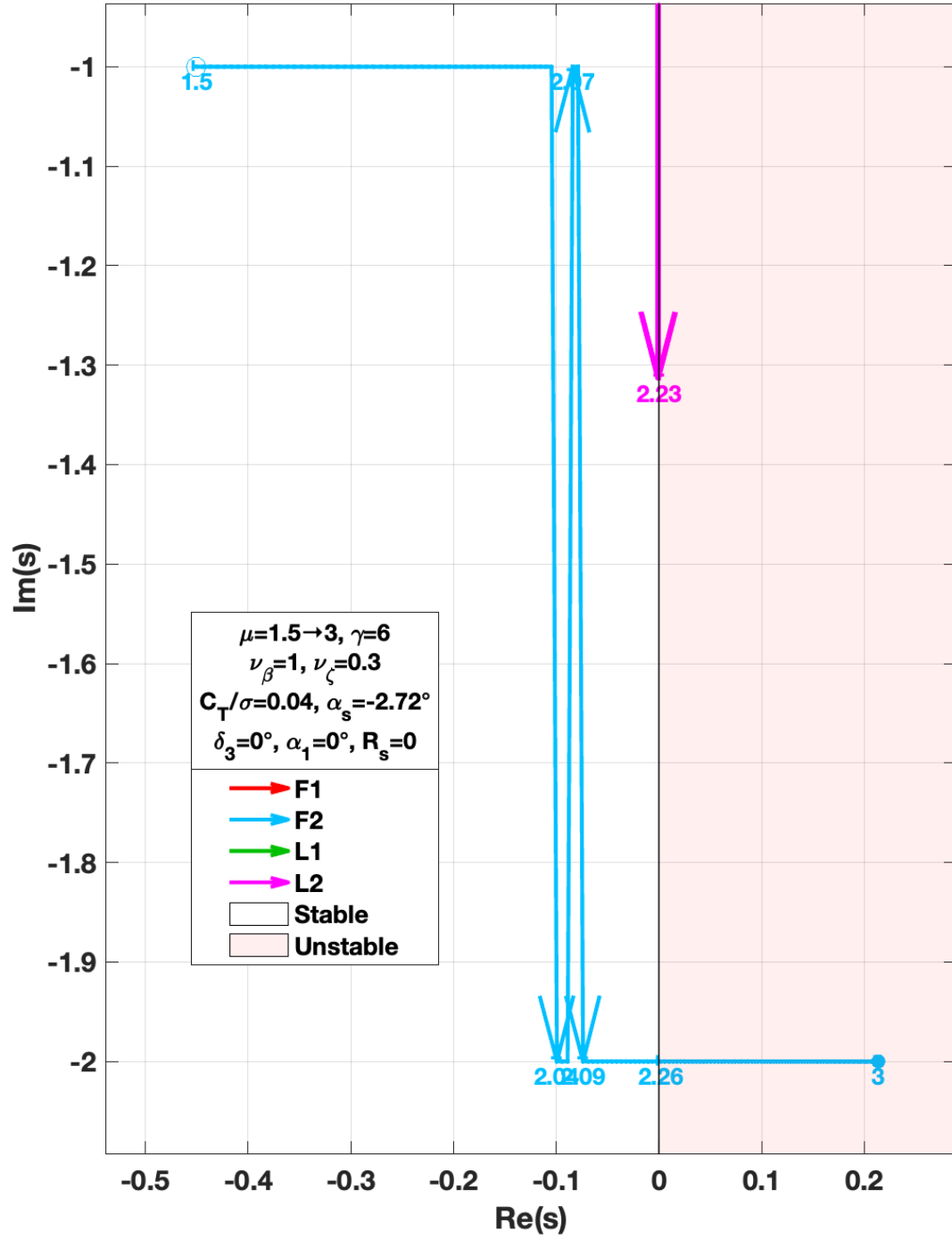


Figure 4.28: Flap-lag root locus, F2 root, very soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.3/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $\mu = 1.5$ to 3 .

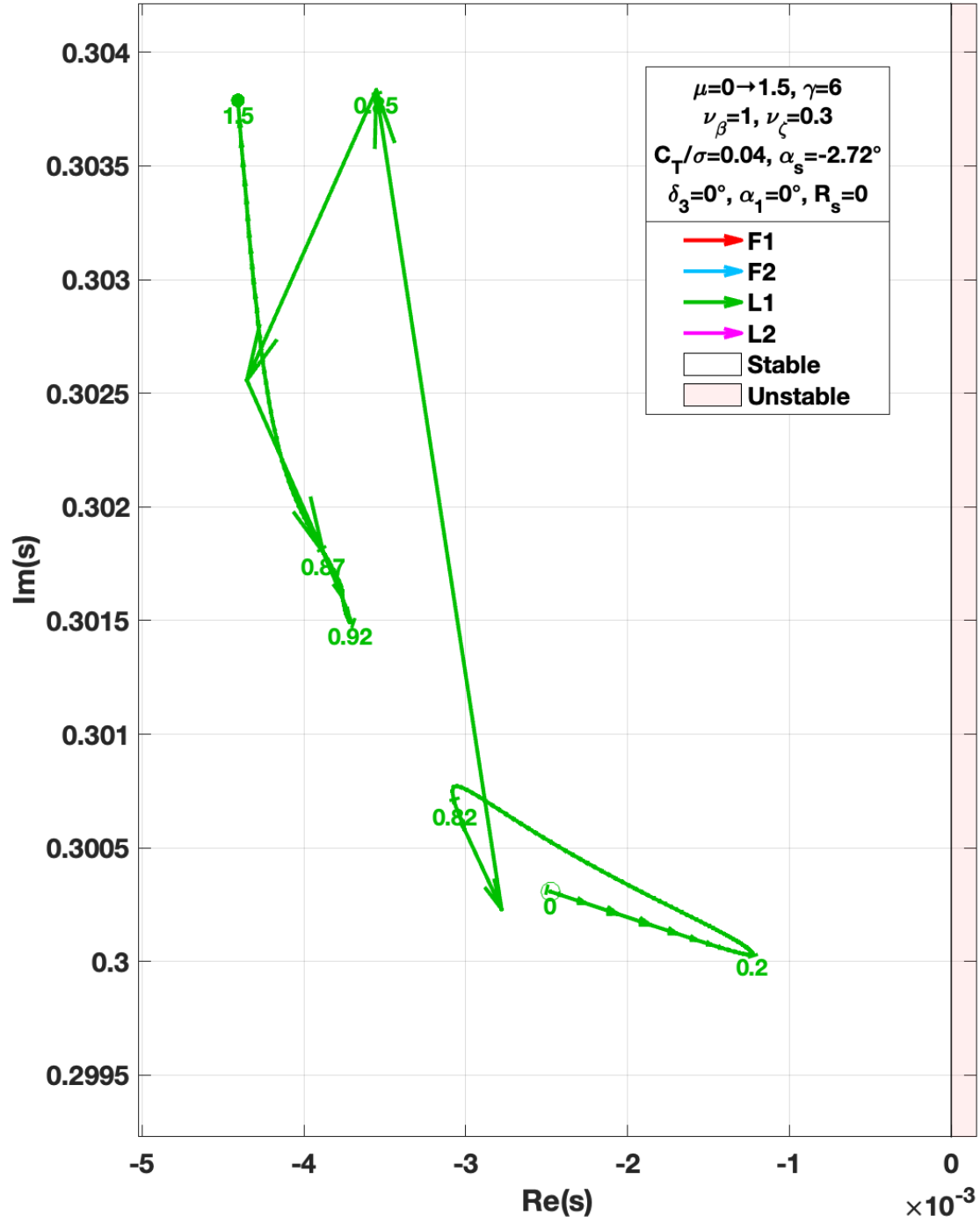


Figure 4.29: Flap-lag root locus, L1 root, very soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.3/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $\mu = 0$ to 1.5 .

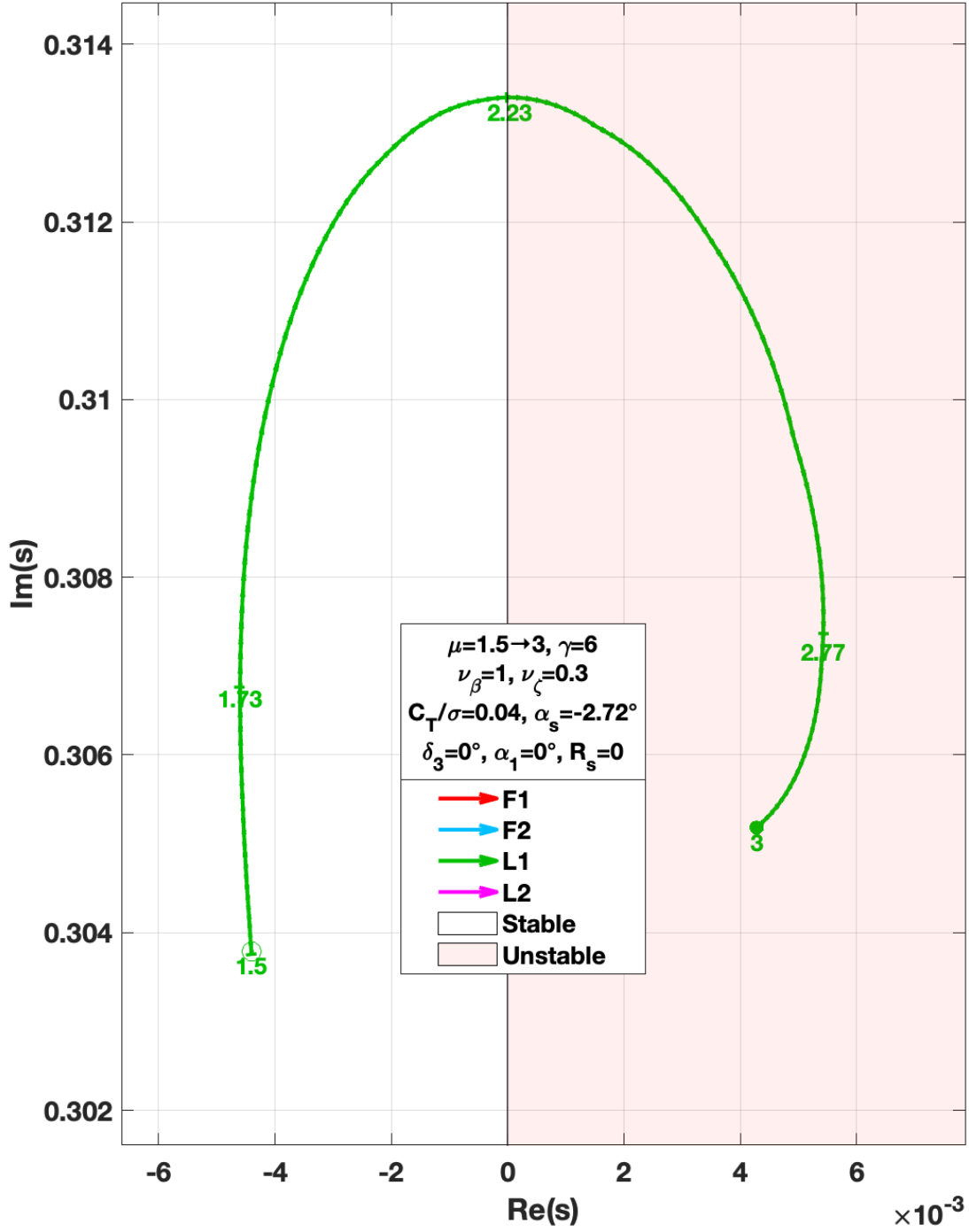


Figure 4.30: Flap-lag root locus, L1 root, very soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.3/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $\mu = 1.5$ to 3 .

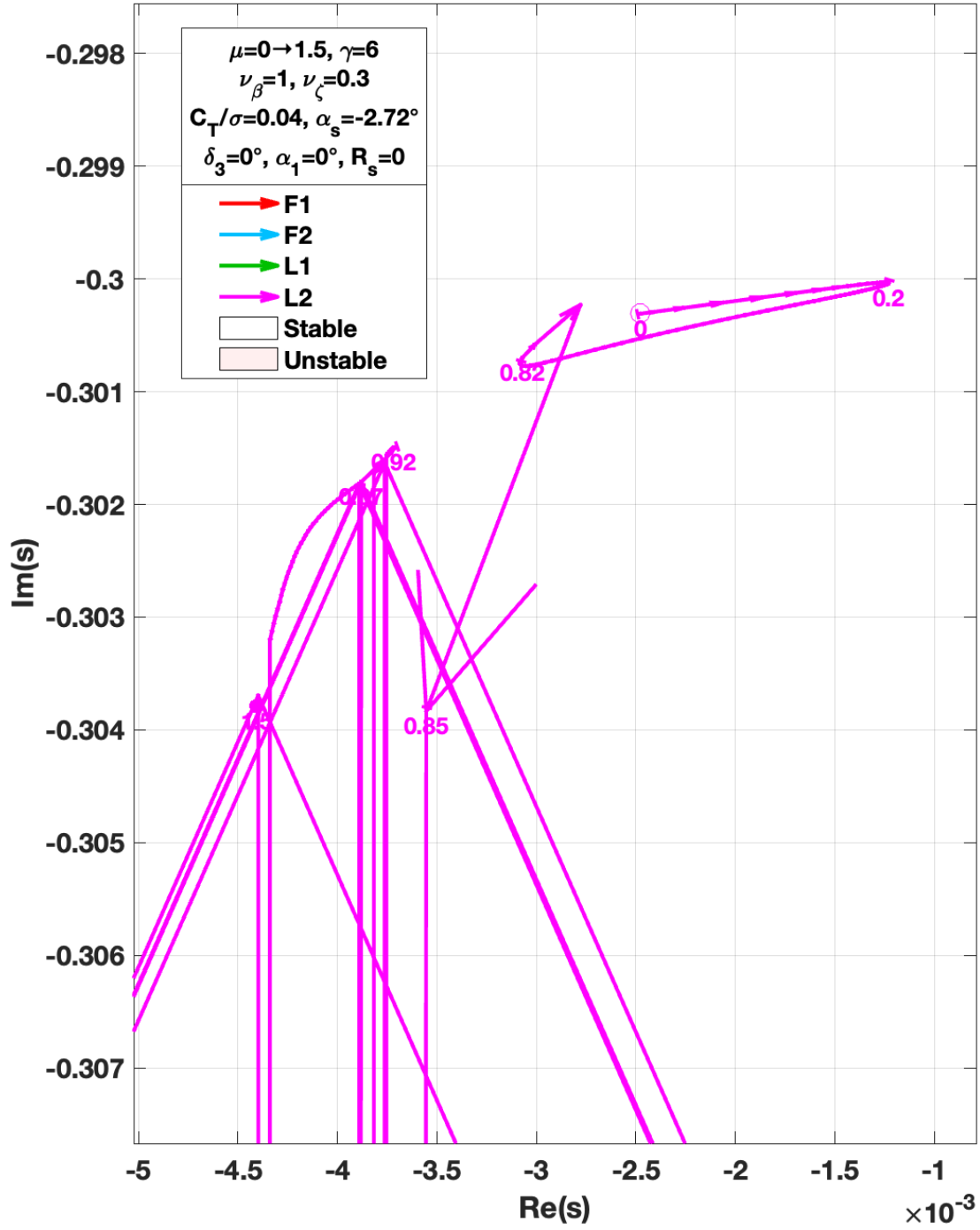


Figure 4.31: Flap-lag root locus, L2 root, very soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.3/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $\mu = 0$ to 1.5.

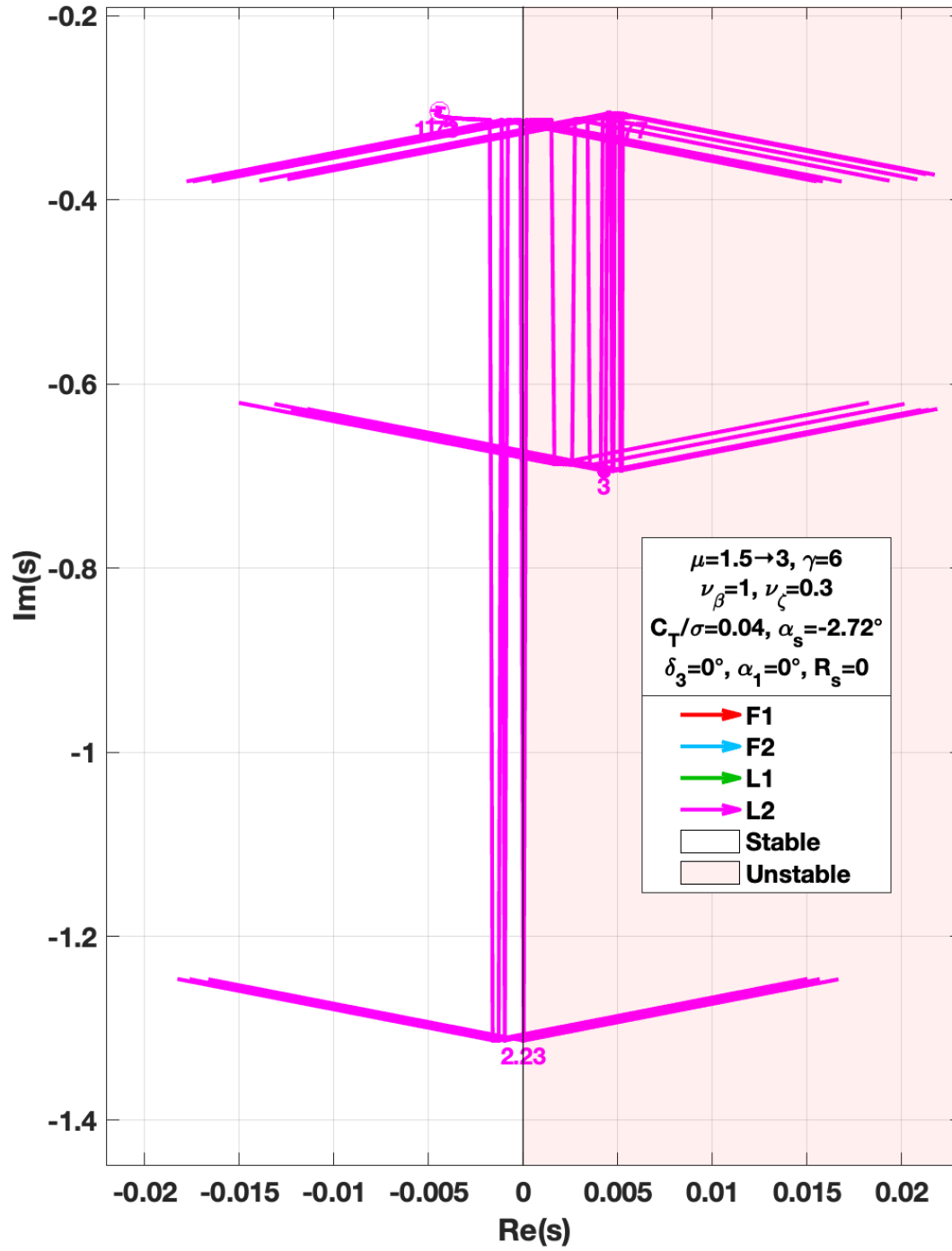


Figure 4.32: Flap-lag root locus, L2 root, very soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.3/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $\mu = 1.5$ to 3 .

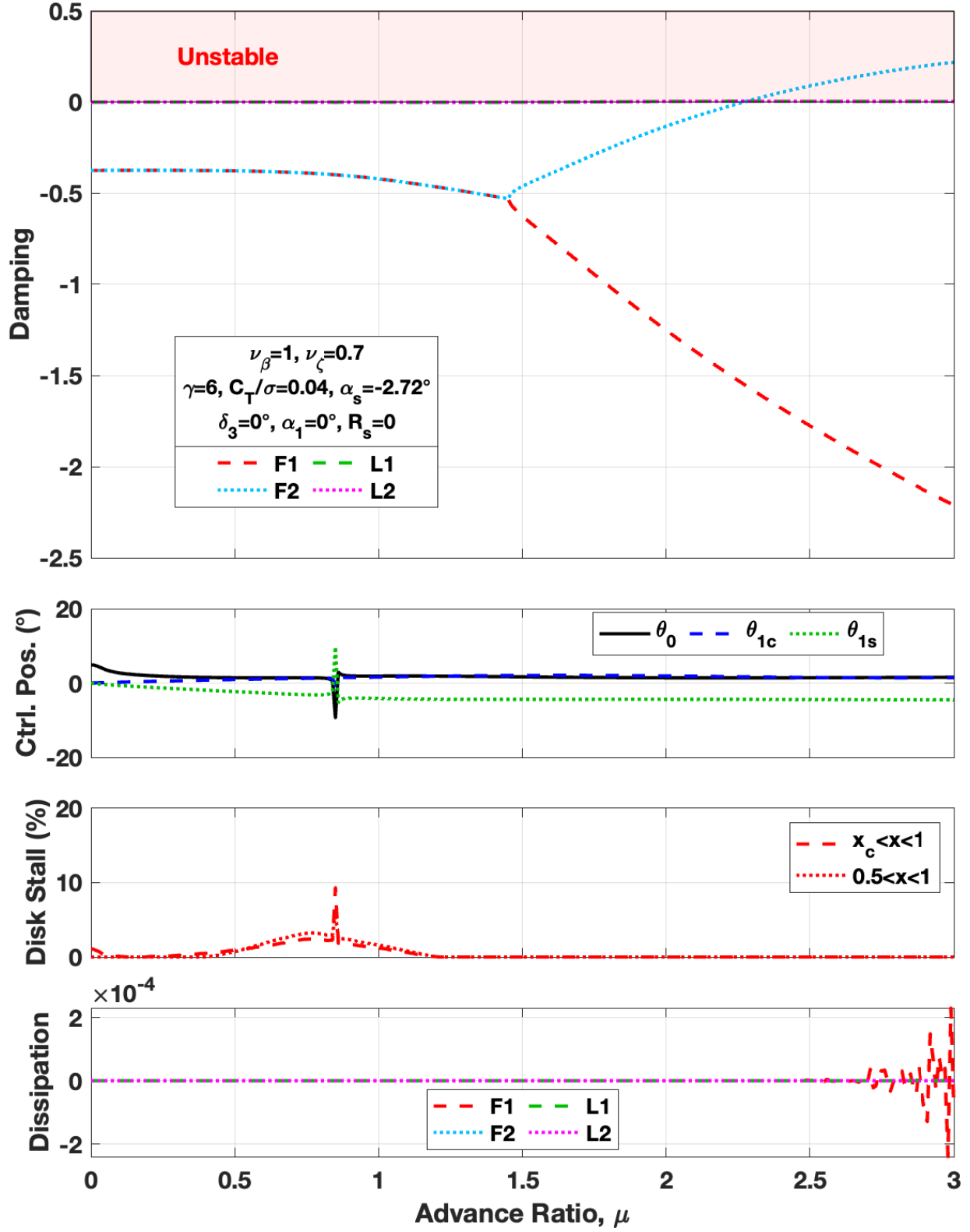


Figure 4.33: Flap-lag damping, trim, blade stall, and numerical stability, soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.7/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$.

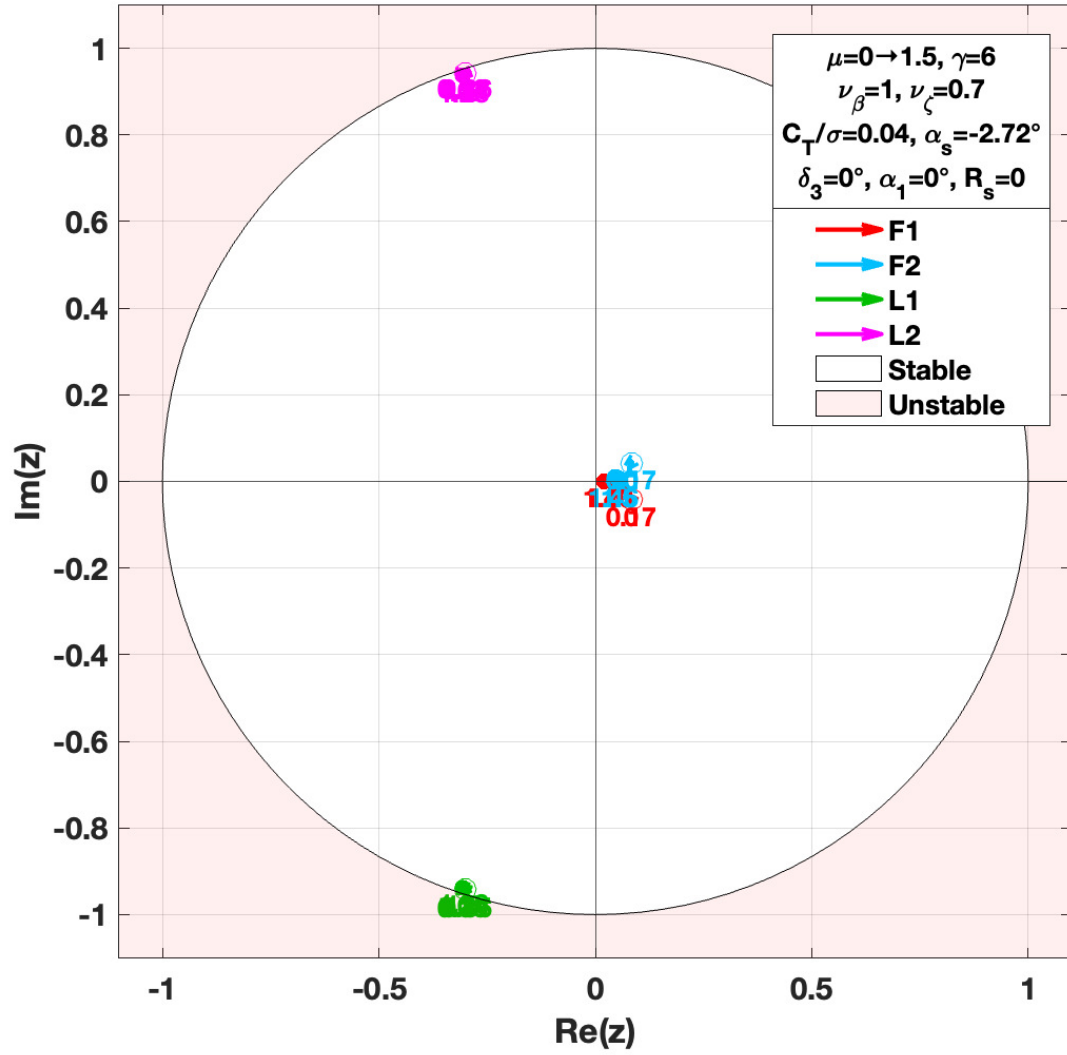


Figure 4.34: Flap-lag characteristic multipliers, soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.7/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $\mu = 0$ to 1.5 .

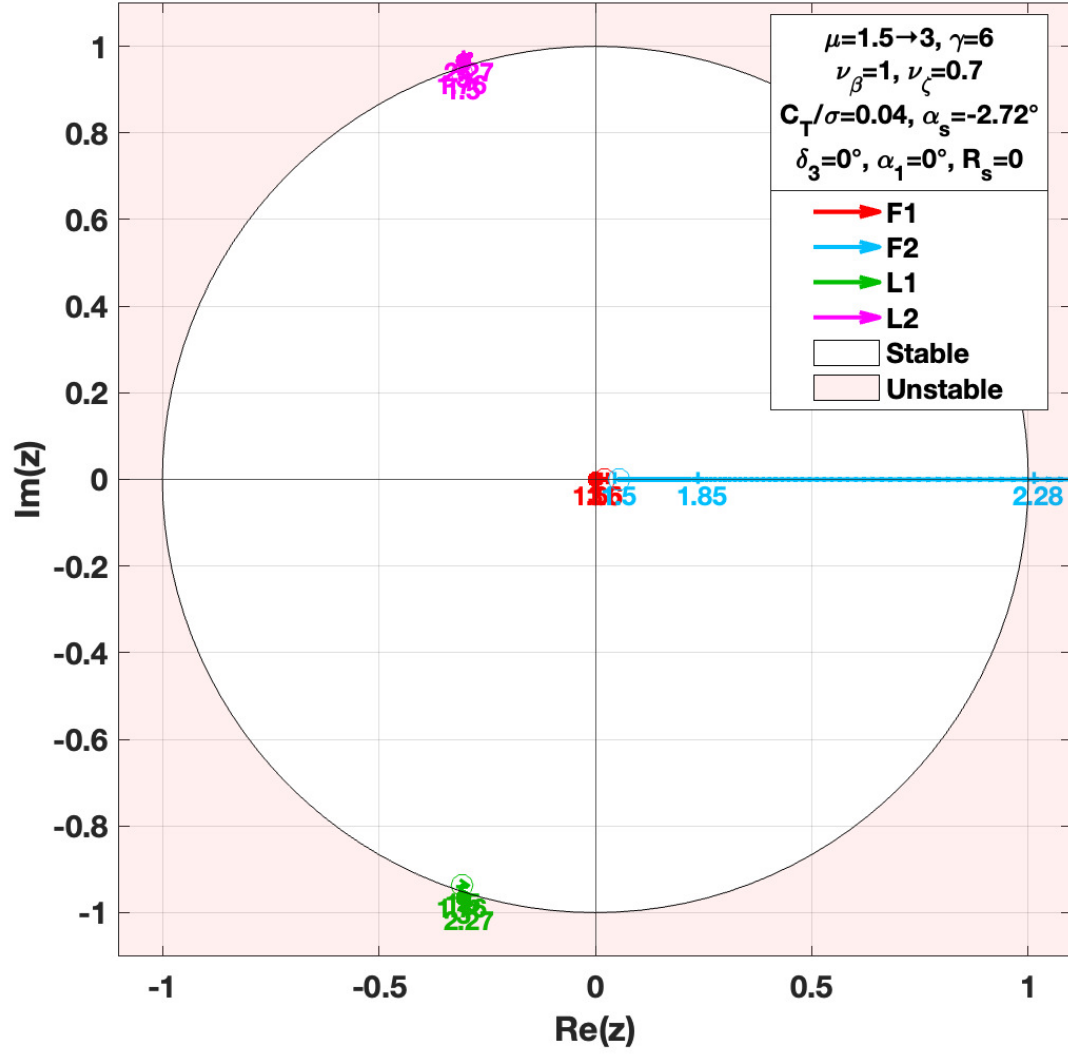


Figure 4.35: Flap-lag characteristic multipliers, soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.7/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $\mu = 1.5$ to 3 .

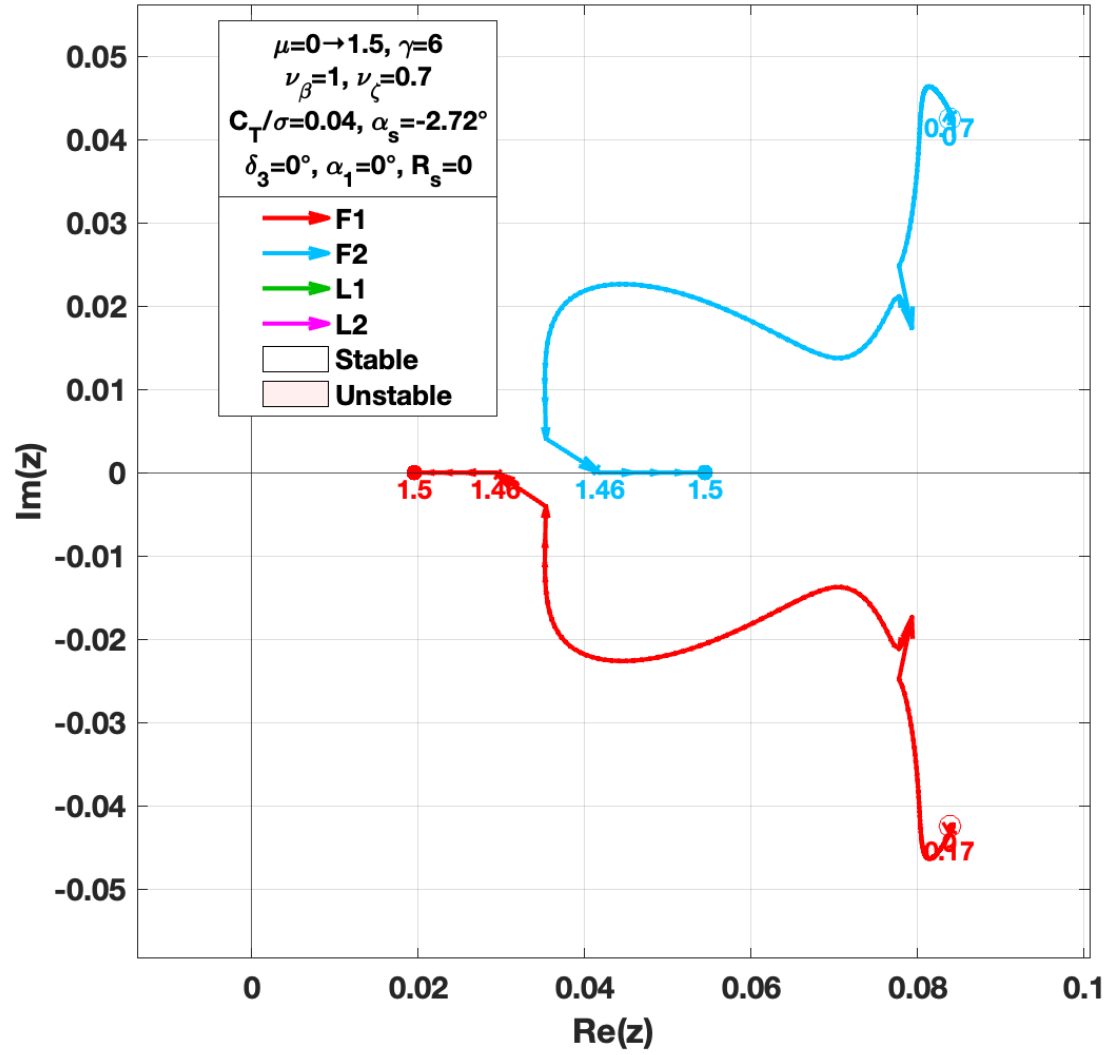


Figure 4.36: Flap-lag characteristic multipliers, F1 and F2 roots, soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.7/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $\mu = 0$ to 1.5 .

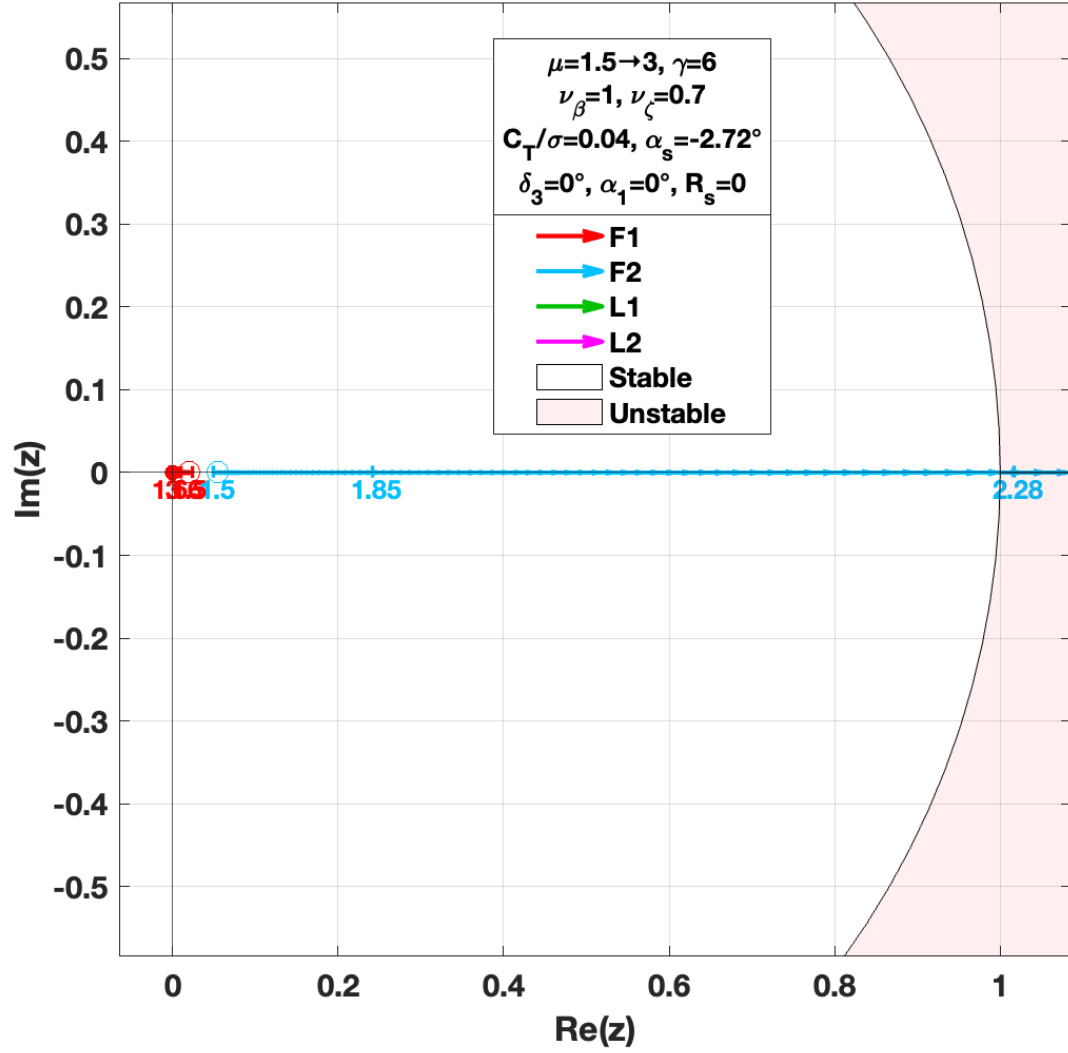


Figure 4.37: Flap-lag characteristic multipliers, F1 and F2 roots, soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.7/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $\mu = 1.5$ to 3 .

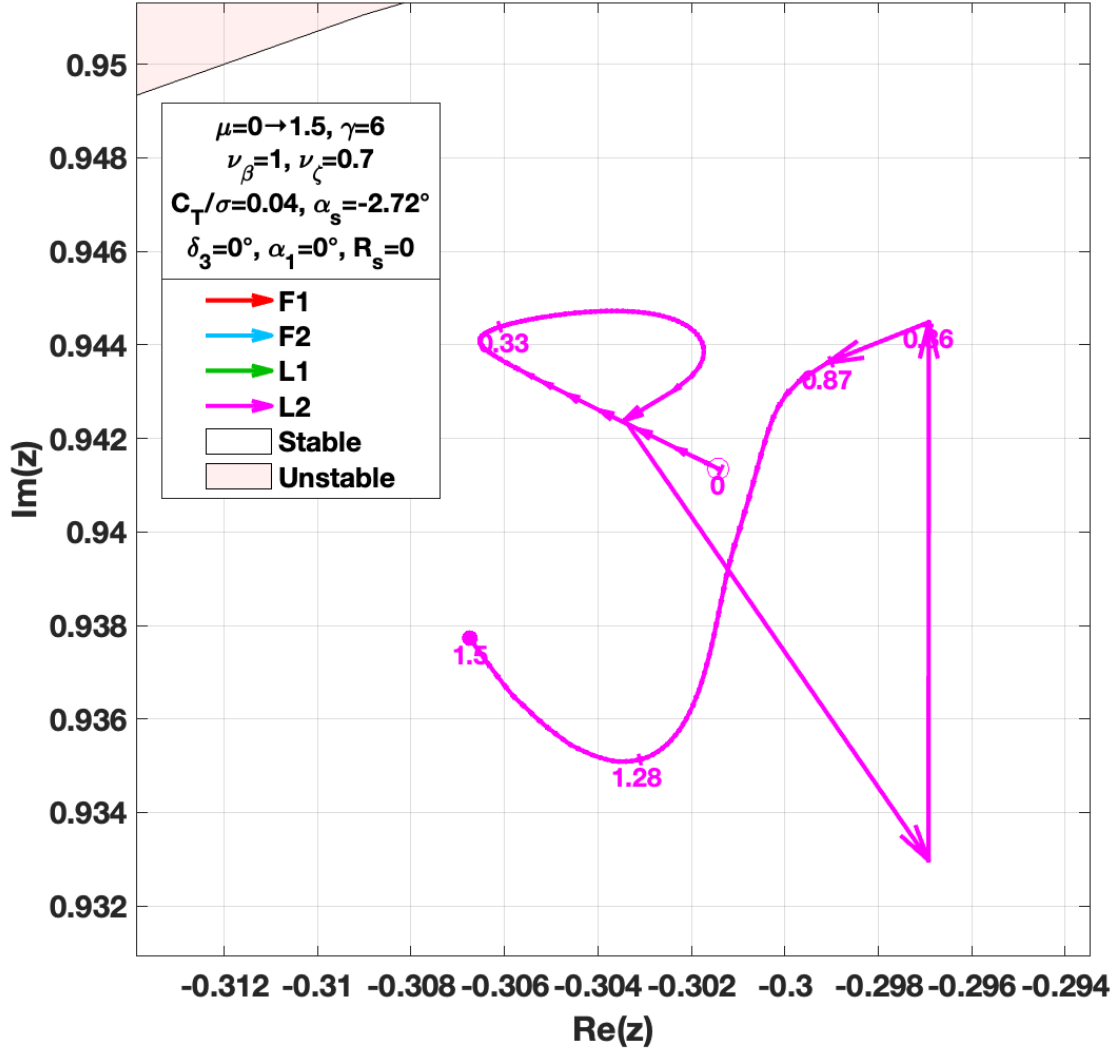


Figure 4.38: Flap-lag characteristic multipliers, L1 root, soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.7/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $\mu = 0$ to 1.5 .

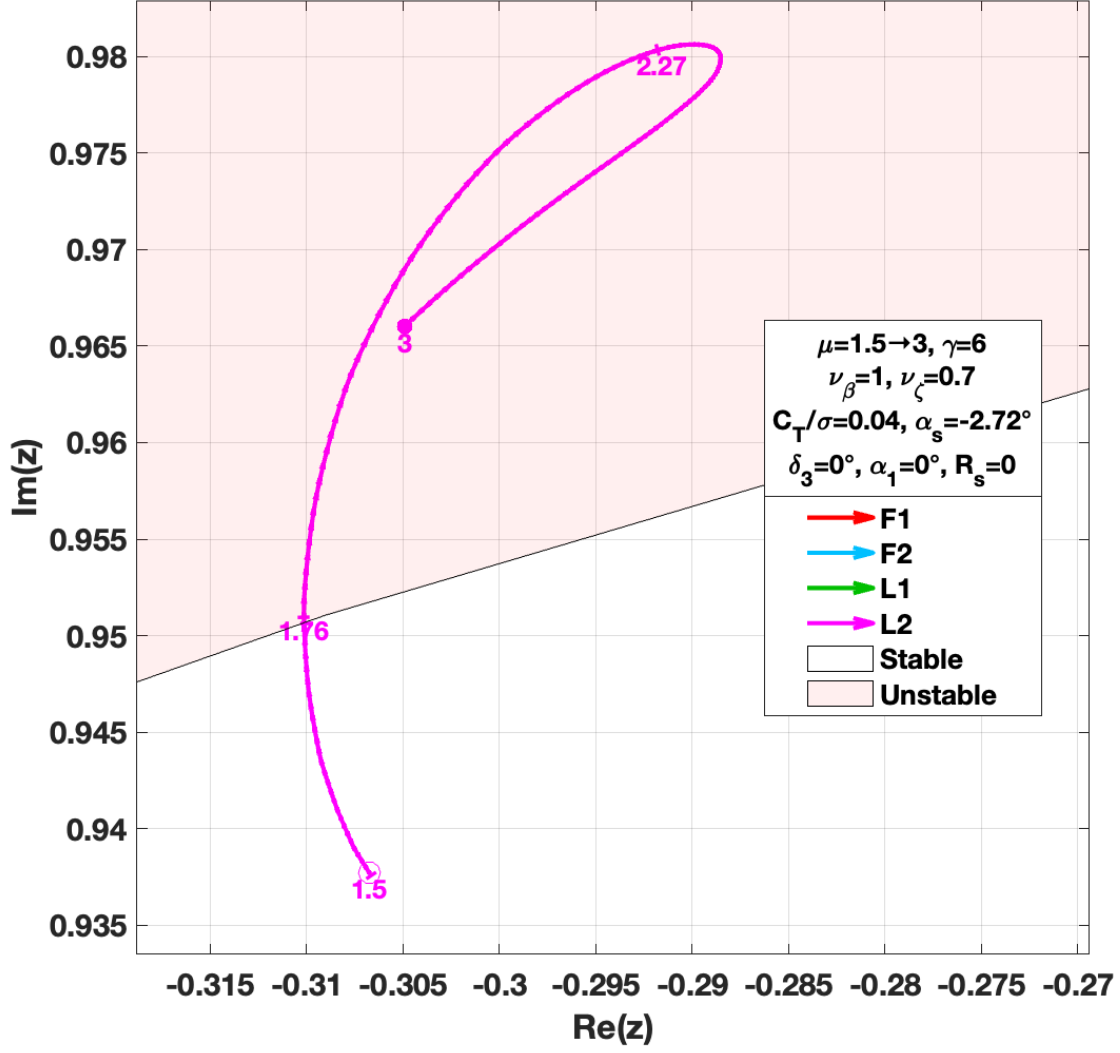


Figure 4.39: Flap-lag characteristic multipliers, L1 root, soft-in-plane articulated rotor, $\nu_\beta = 1, \nu_\zeta = 0.7/\text{rev}, \gamma = 6, C_T/\sigma = 0.04, \alpha_s = -2.72^\circ, \mu = 1.5$ to 3 .

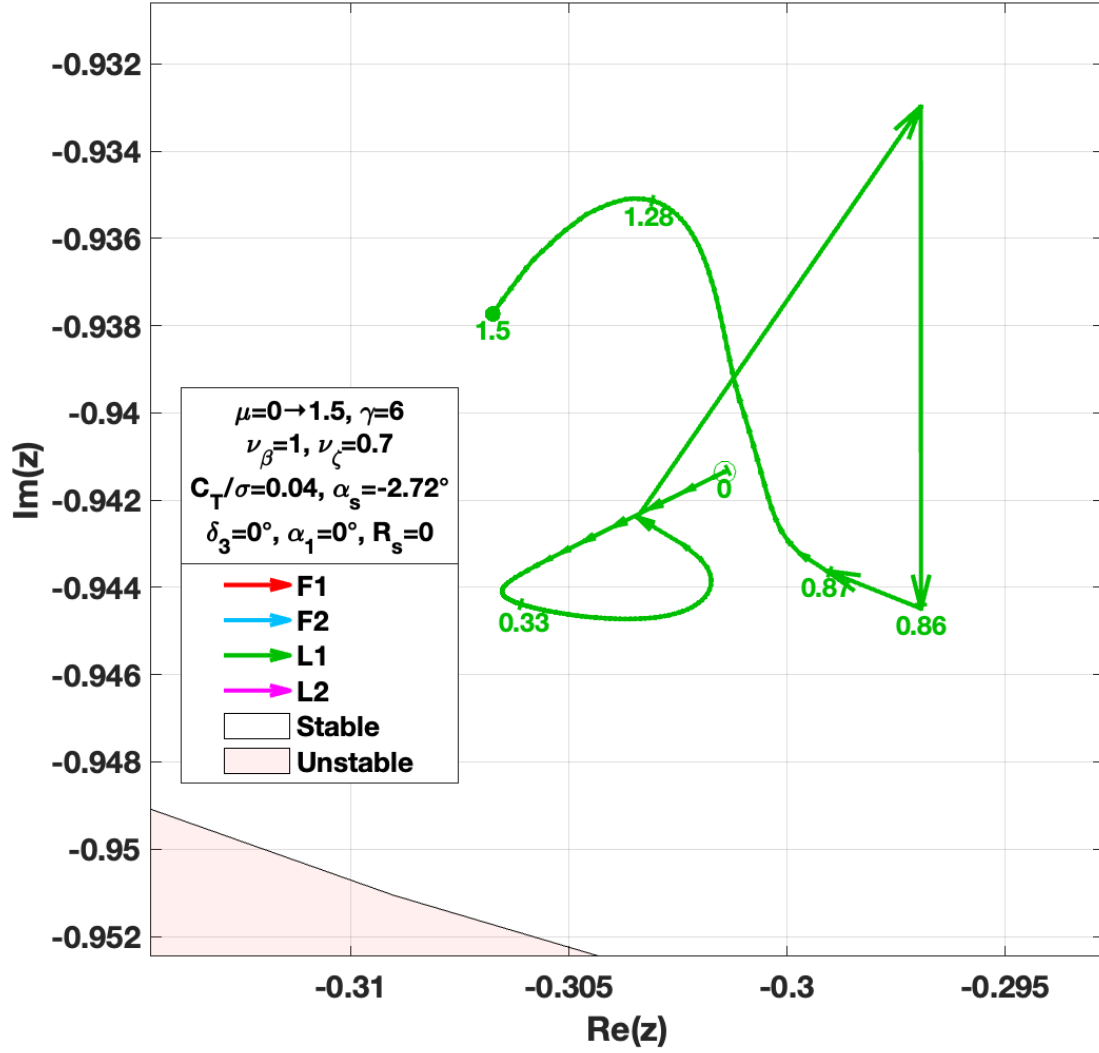


Figure 4.40: Flap-lag characteristic multipliers, L2 root, soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.7/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $\mu = 0$ to 1.5 .

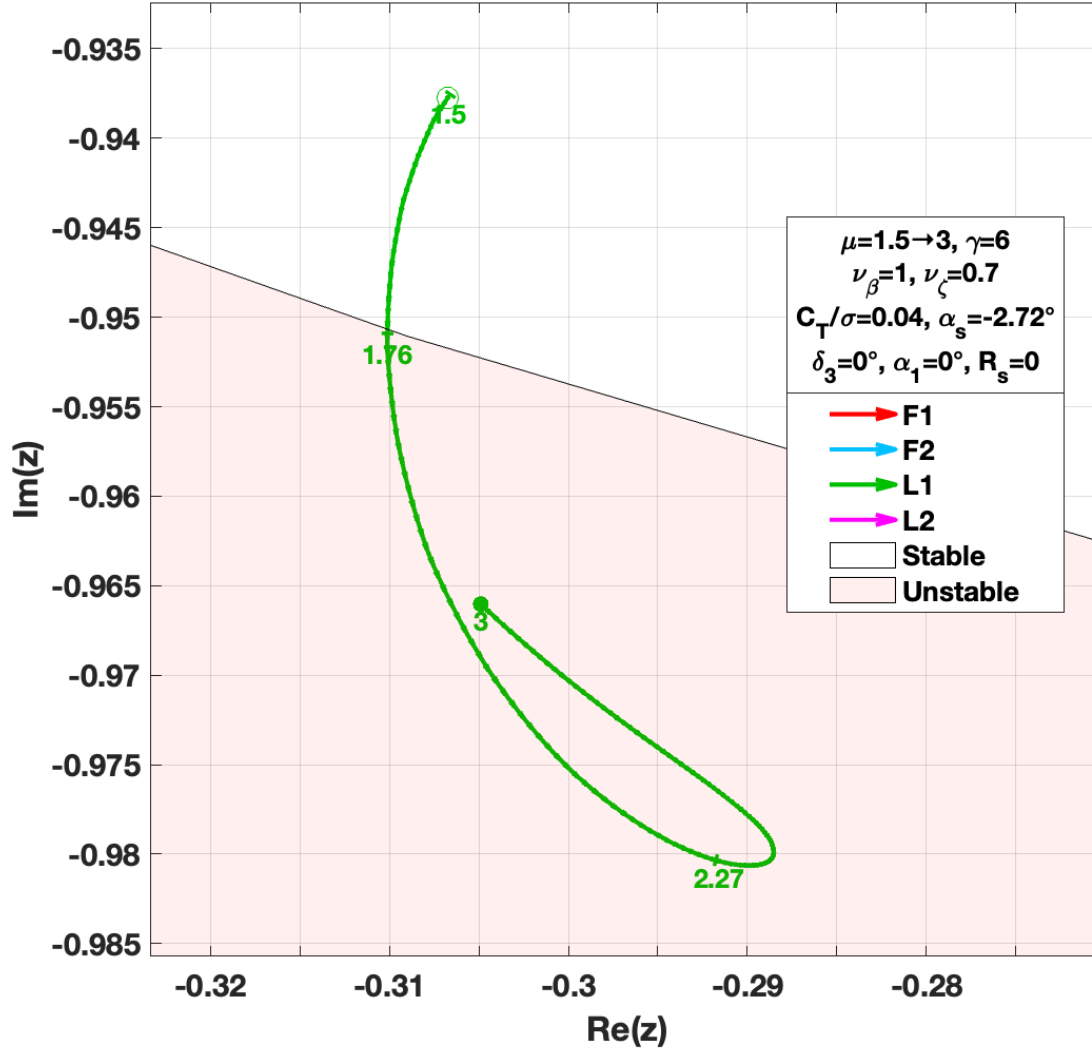


Figure 4.41: Flap-lag characteristic multipliers, L2 root, soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.7/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $\mu = 1.5$ to 3 .

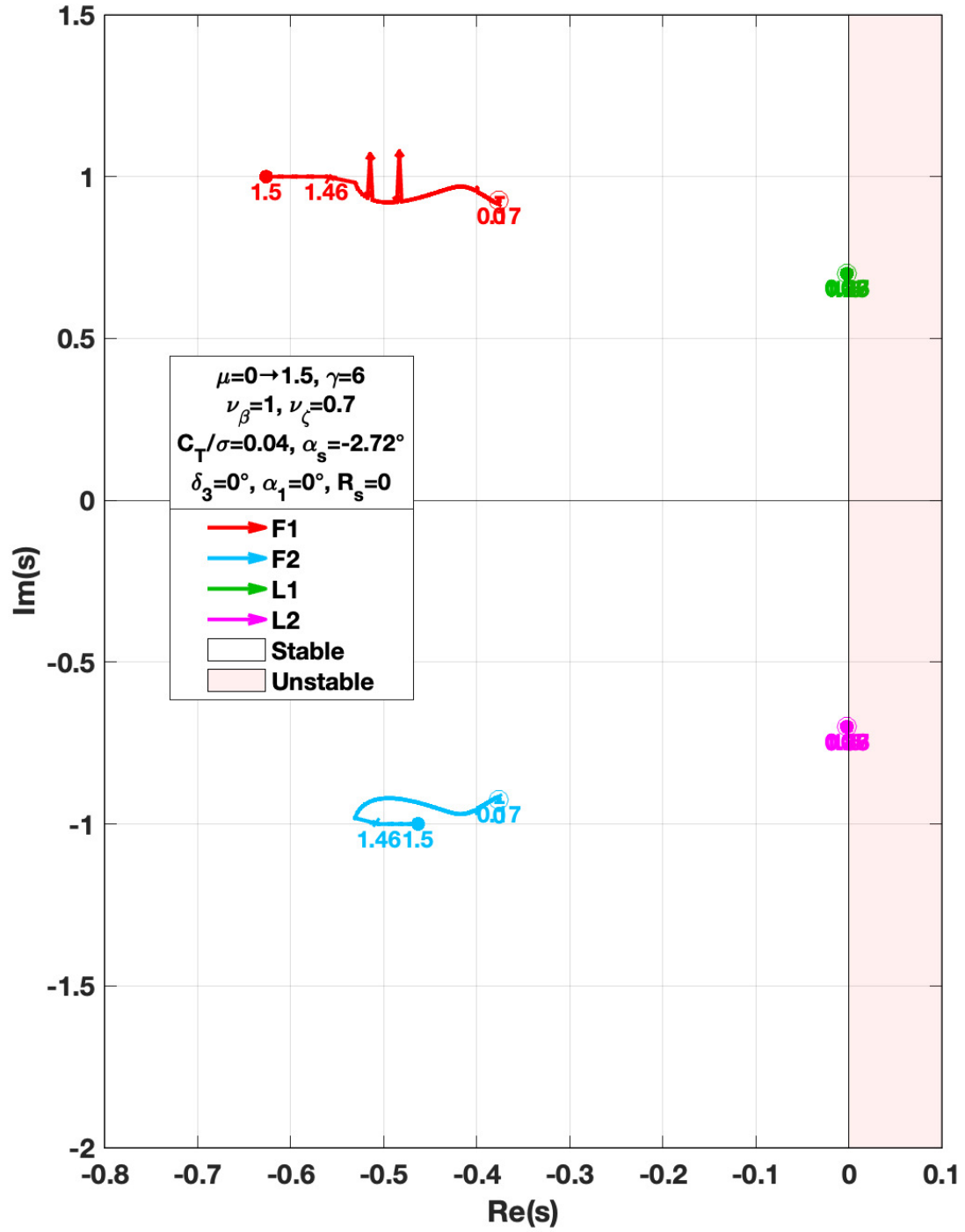


Figure 4.42: Flap-lag root locus, soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.7/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $\mu = 0$ to 1.5.

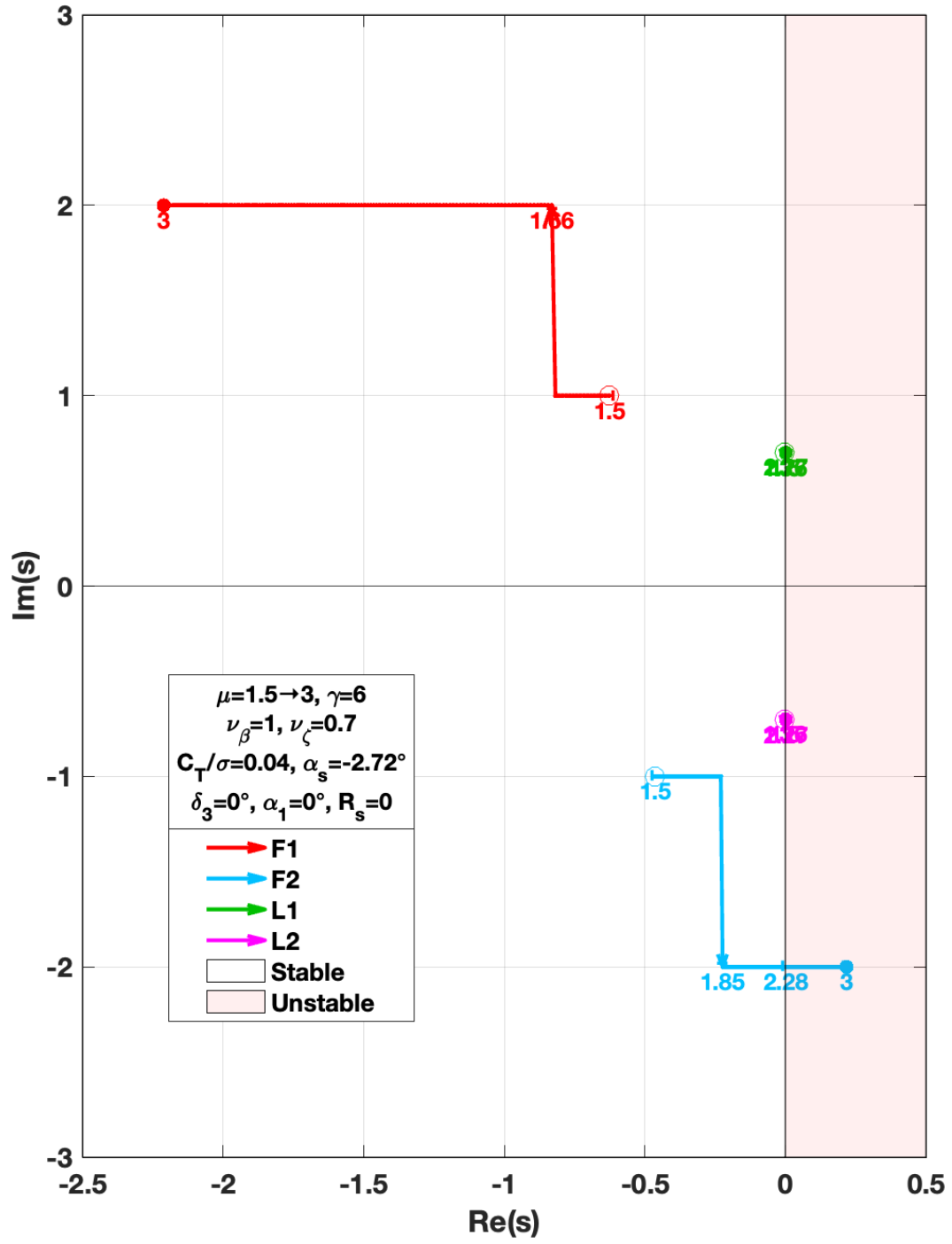


Figure 4.43: Flap-lag root locus, soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.7/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $\mu = 1.5$ to 3 .

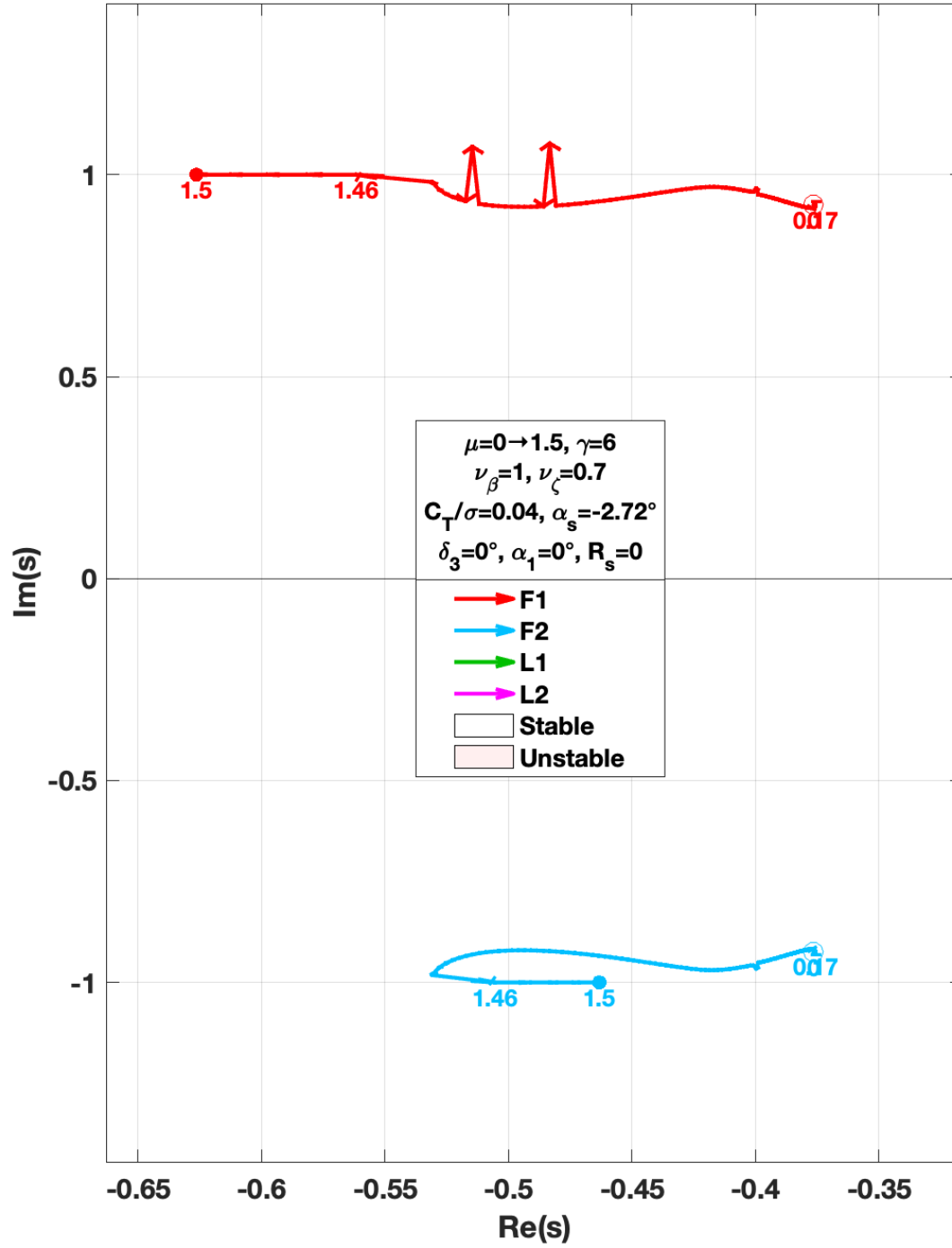


Figure 4.44: Flap-lag root locus, F1 and F2 roots, soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.7/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $\mu = 0$ to 1.5 .

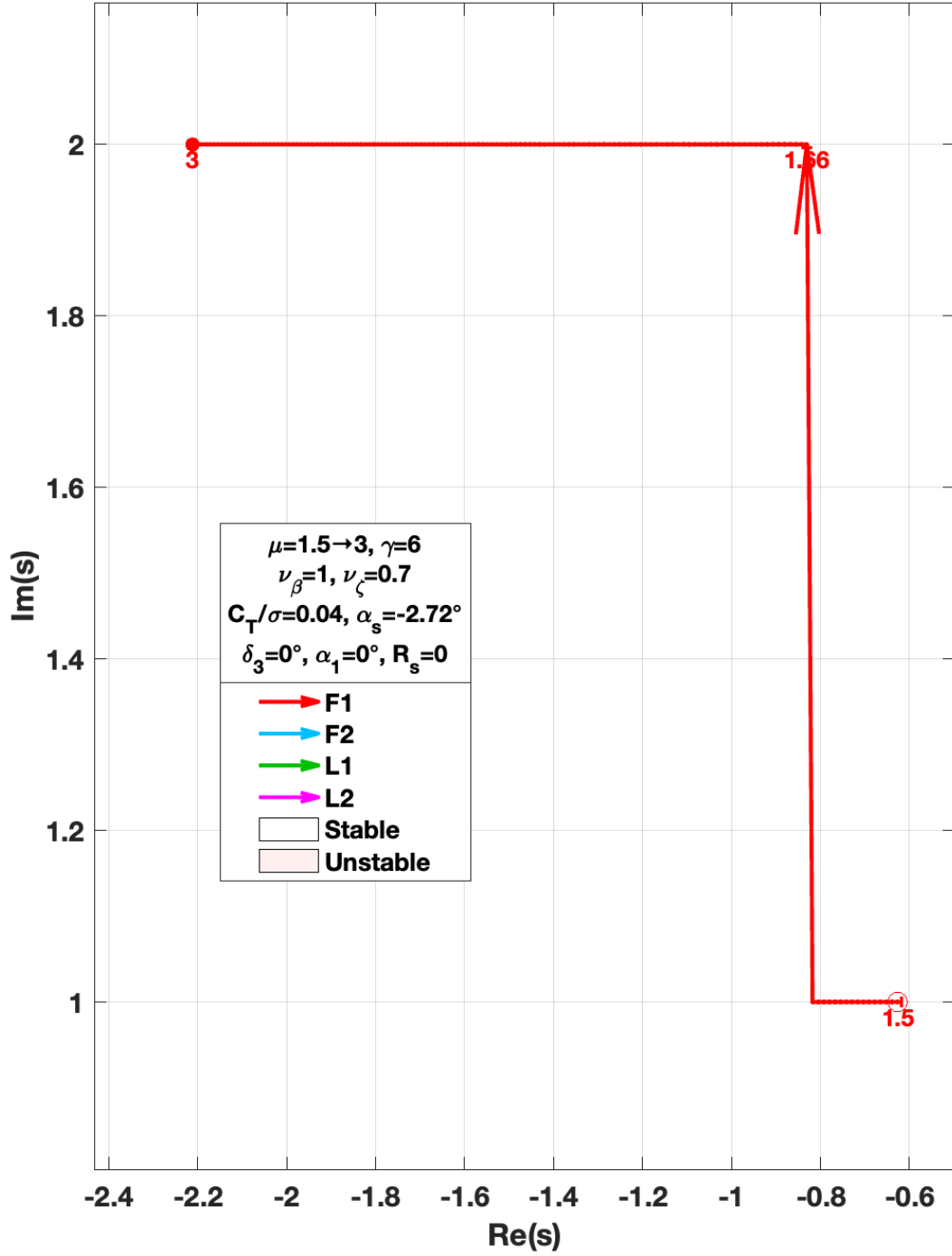


Figure 4.45: Flap-lag root locus, F1 root, soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.7/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $\mu = 1.5$ to 3 .

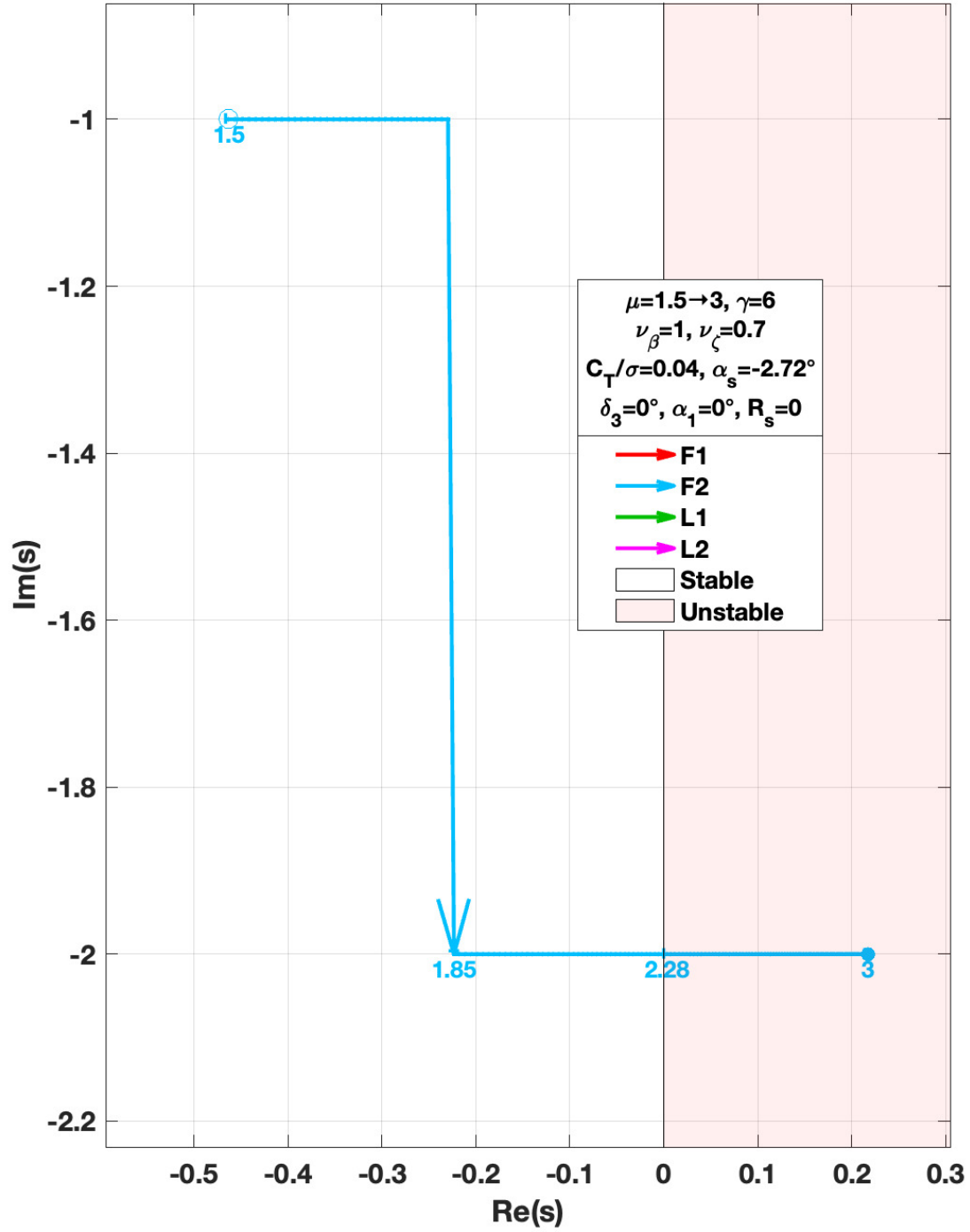


Figure 4.46: Flap-lag root locus, F2 root, soft-in-plane articulated rotor, $\nu_{\beta} = 1$, $\nu_{\zeta} = 0.7/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $\mu = 1.5$ to 3 .

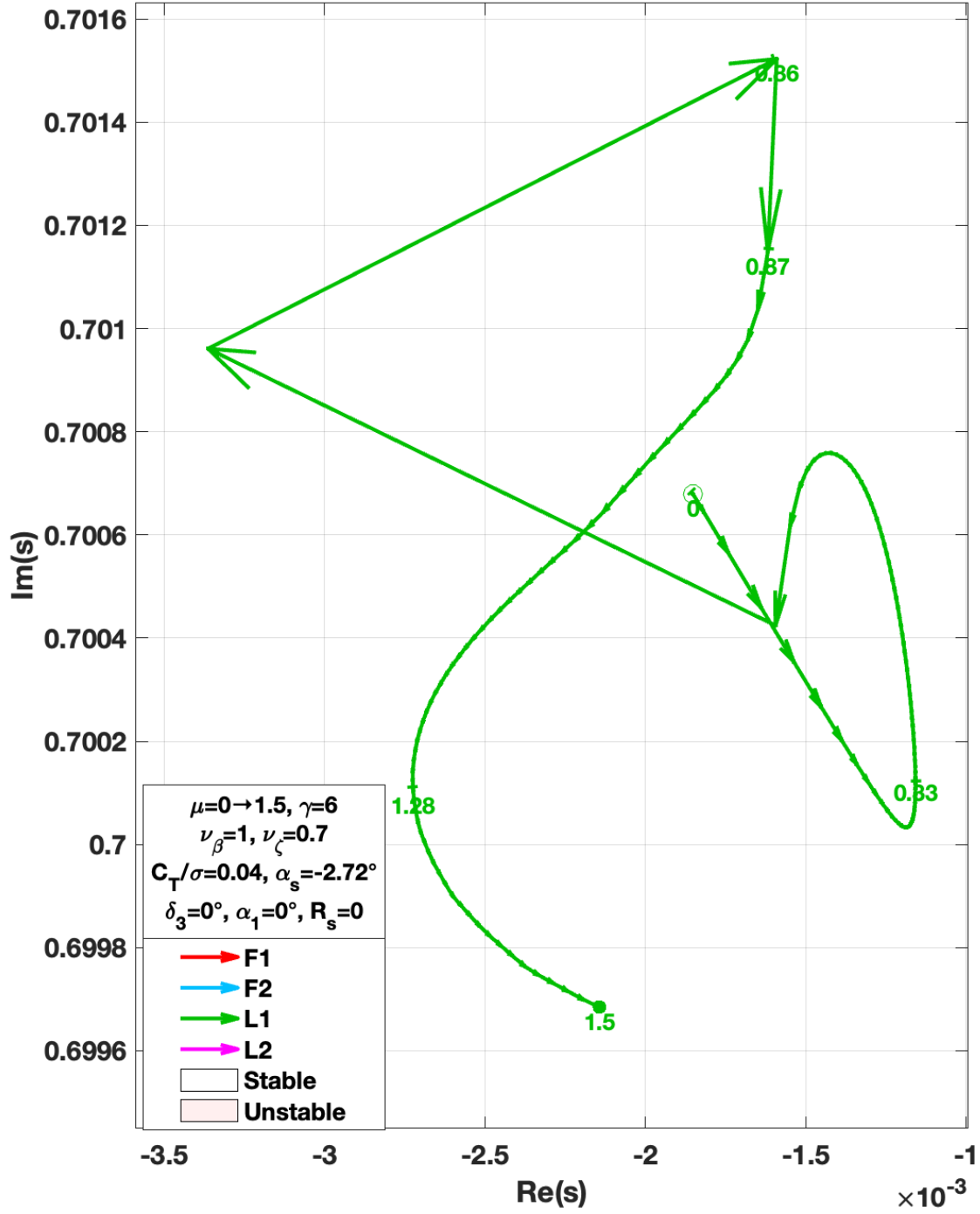


Figure 4.47: Flap-lag root locus, L1 root, soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.7/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $\mu = 0$ to 1.5.

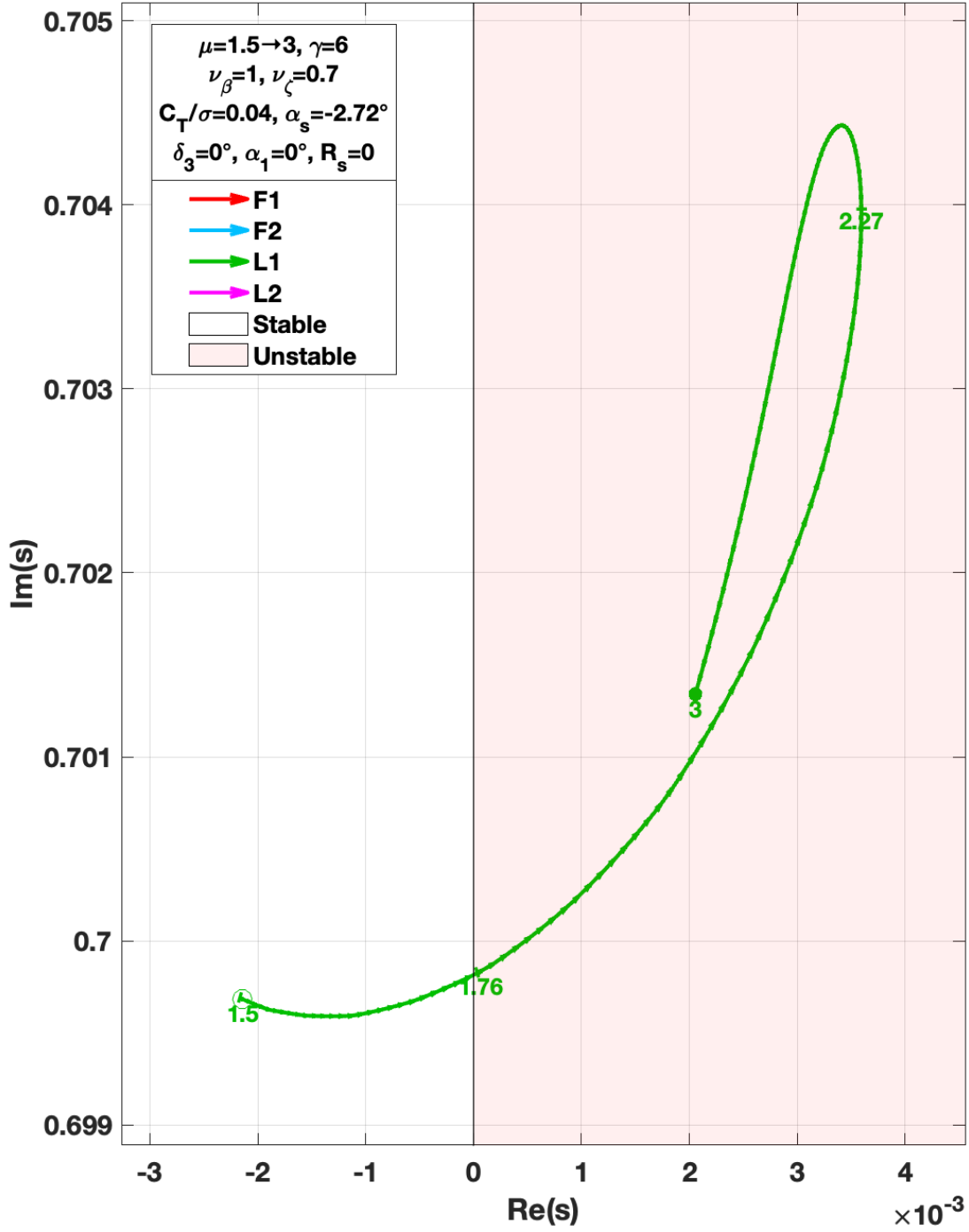


Figure 4.48: Flap-lag root locus, L1 root, soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.7/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $\mu = 1.5$ to 3.

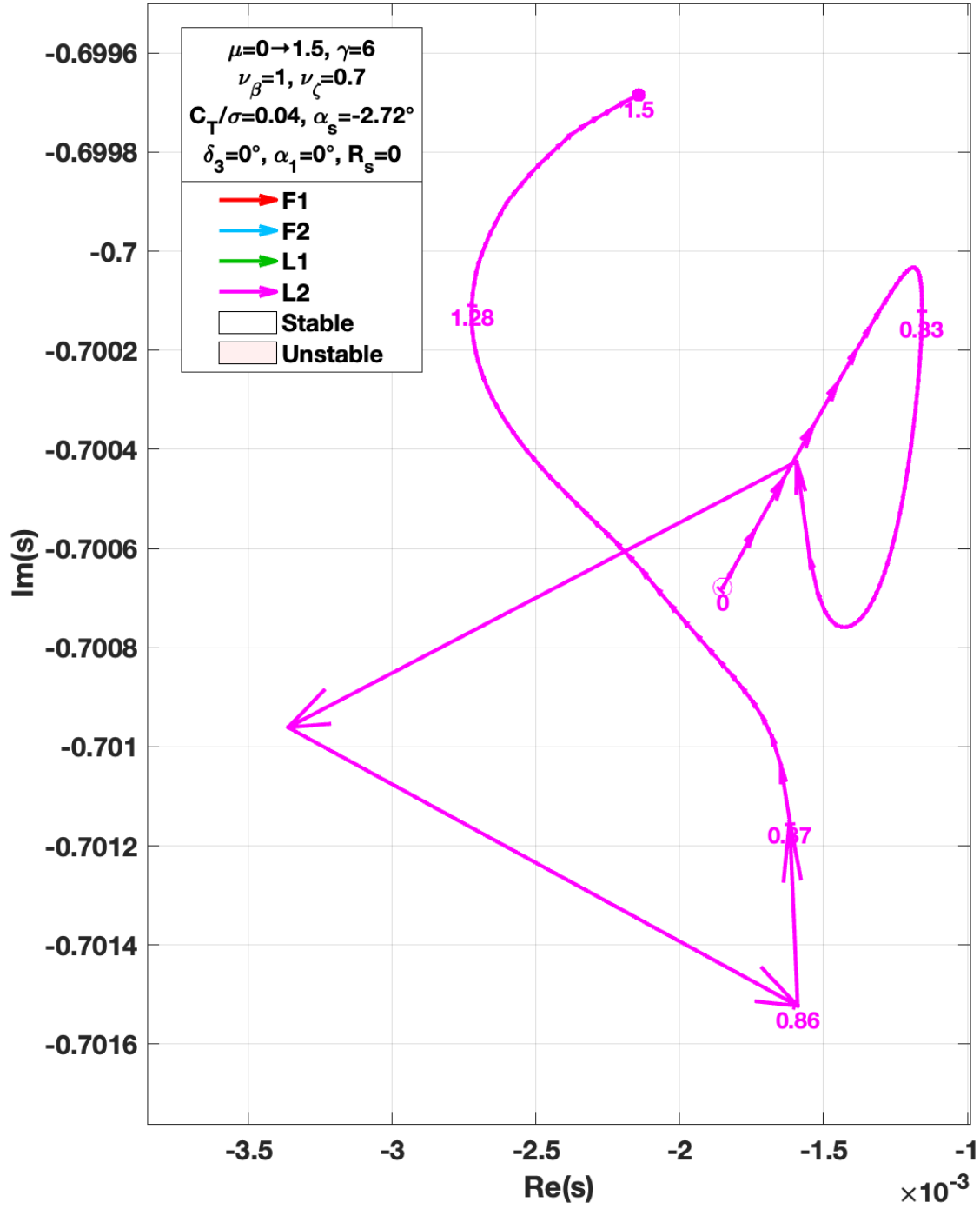


Figure 4.49: Flap-lag root locus, L2 root, soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.7/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $\mu = 0$ to 1.5.

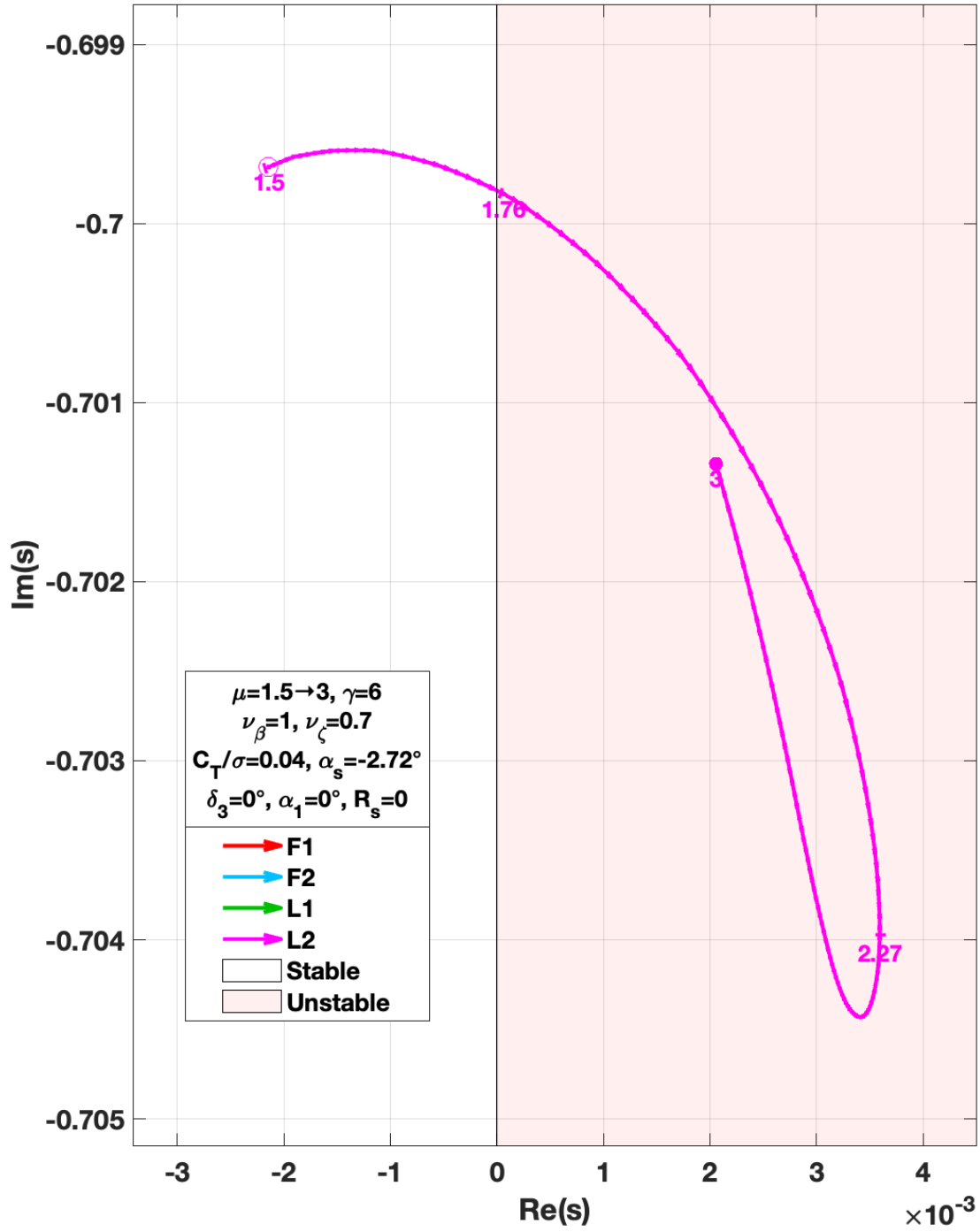


Figure 4.50: Flap-lag root locus, L2 root, soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.7/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $\mu = 1.5$ to 3 .

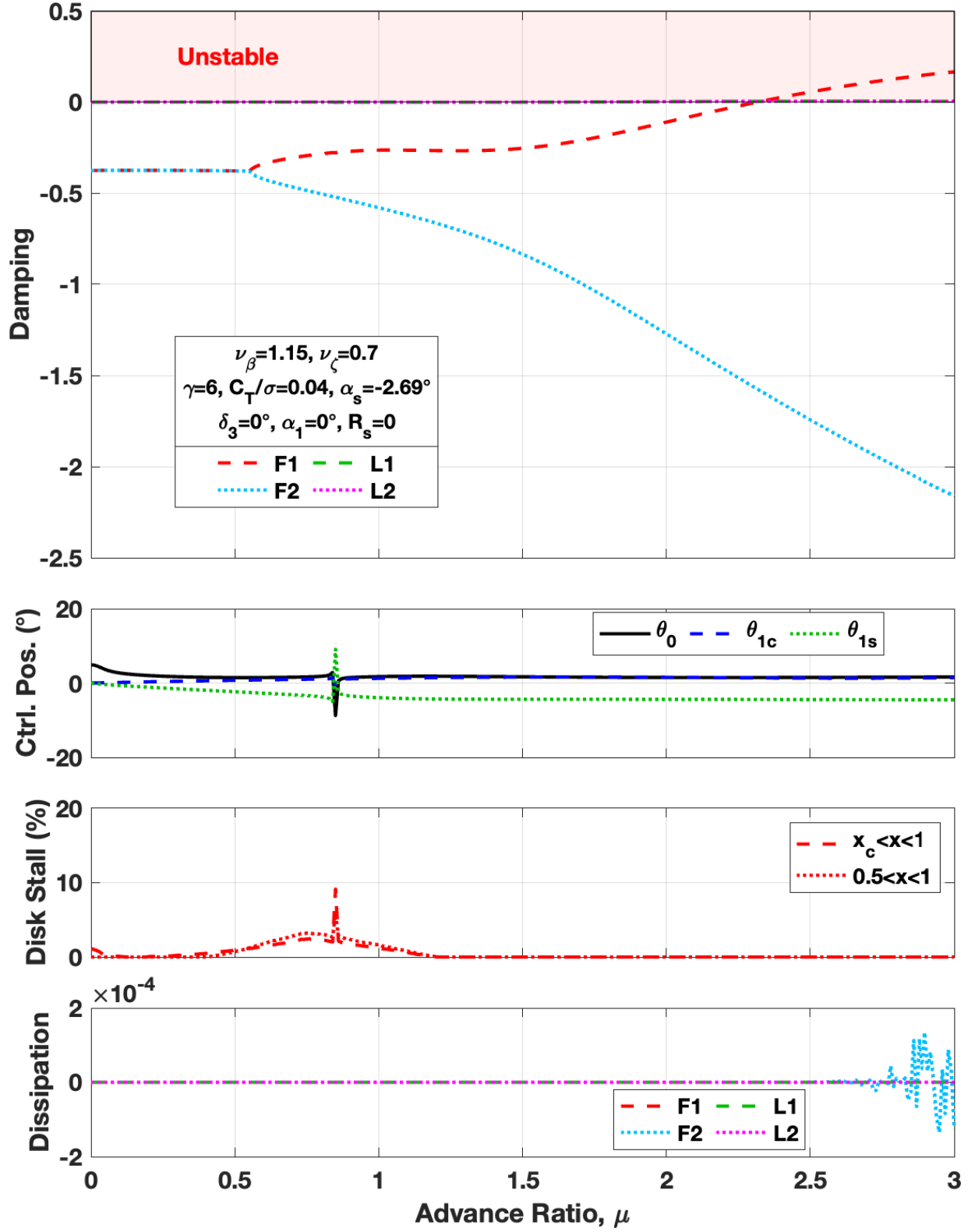


Figure 4.51: Flap-lag damping, trim, blade stall, and numerical stability, soft-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 0.7/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$.

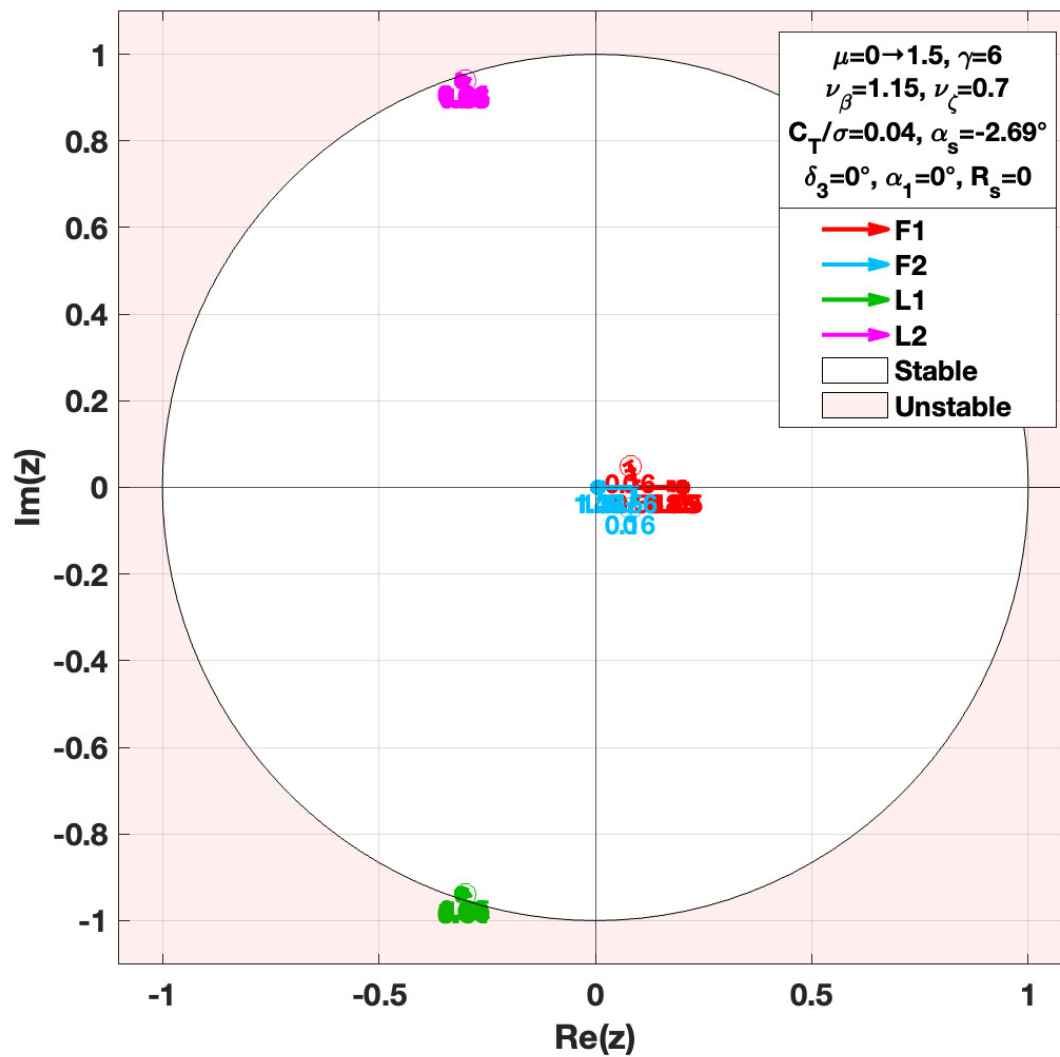


Figure 4.52: Flap-lag characteristic multipliers, soft-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 0.7/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $\mu = 0$ to 1.5 .

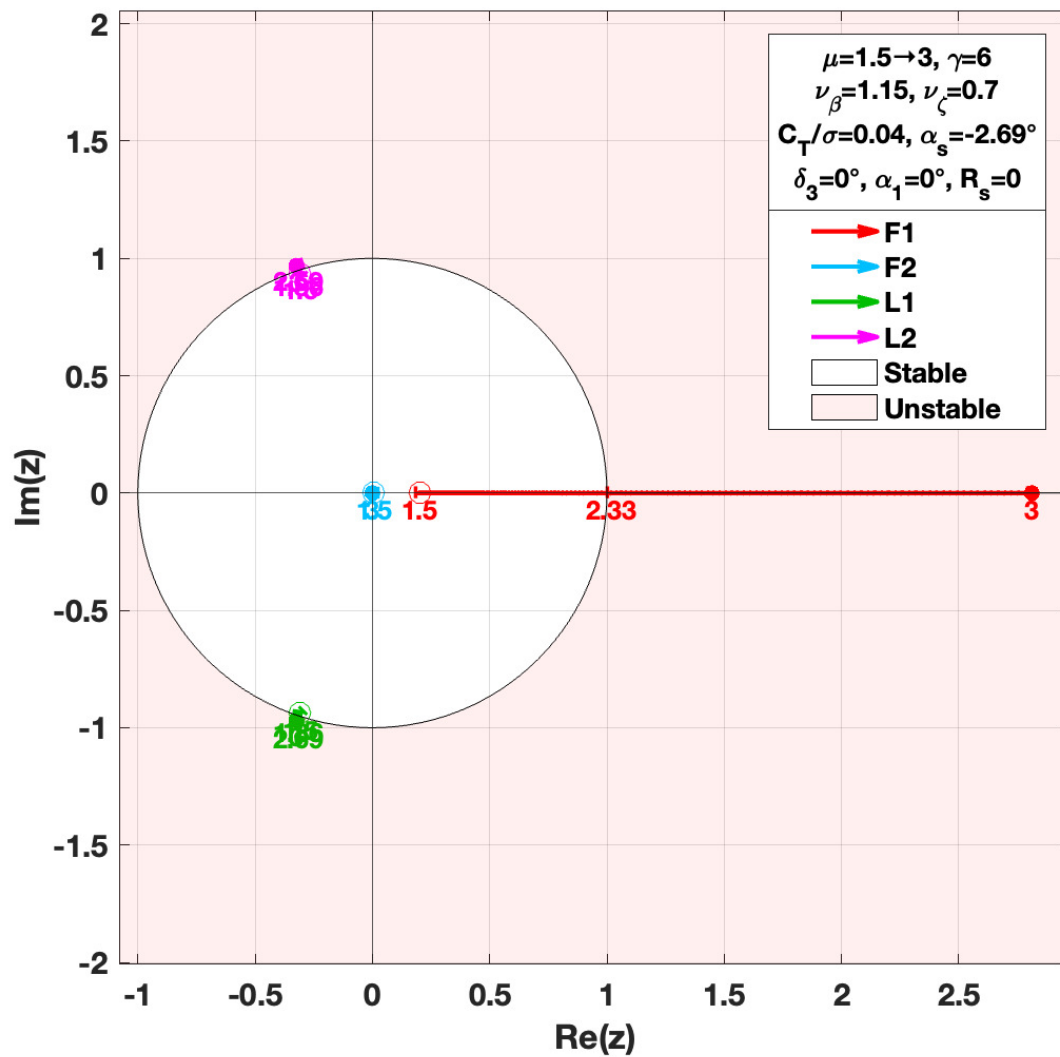


Figure 4.53: Flap-lag characteristic multipliers, soft-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 0.7/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $\mu = 1.5$ to 3 .

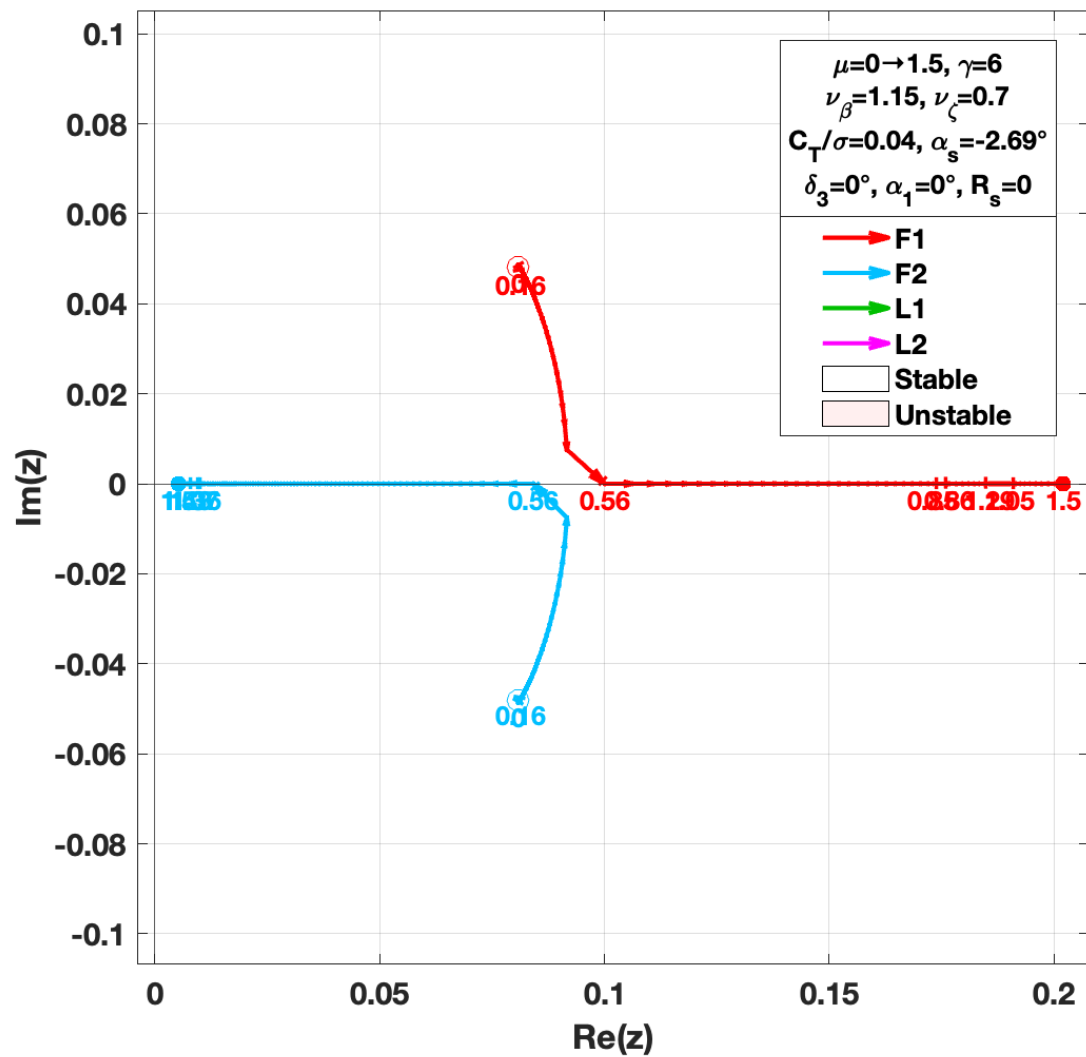


Figure 4.54: Flap-lag characteristic multipliers, F1 and F2 roots, soft-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 0.7/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $\mu = 0$ to 1.5 .

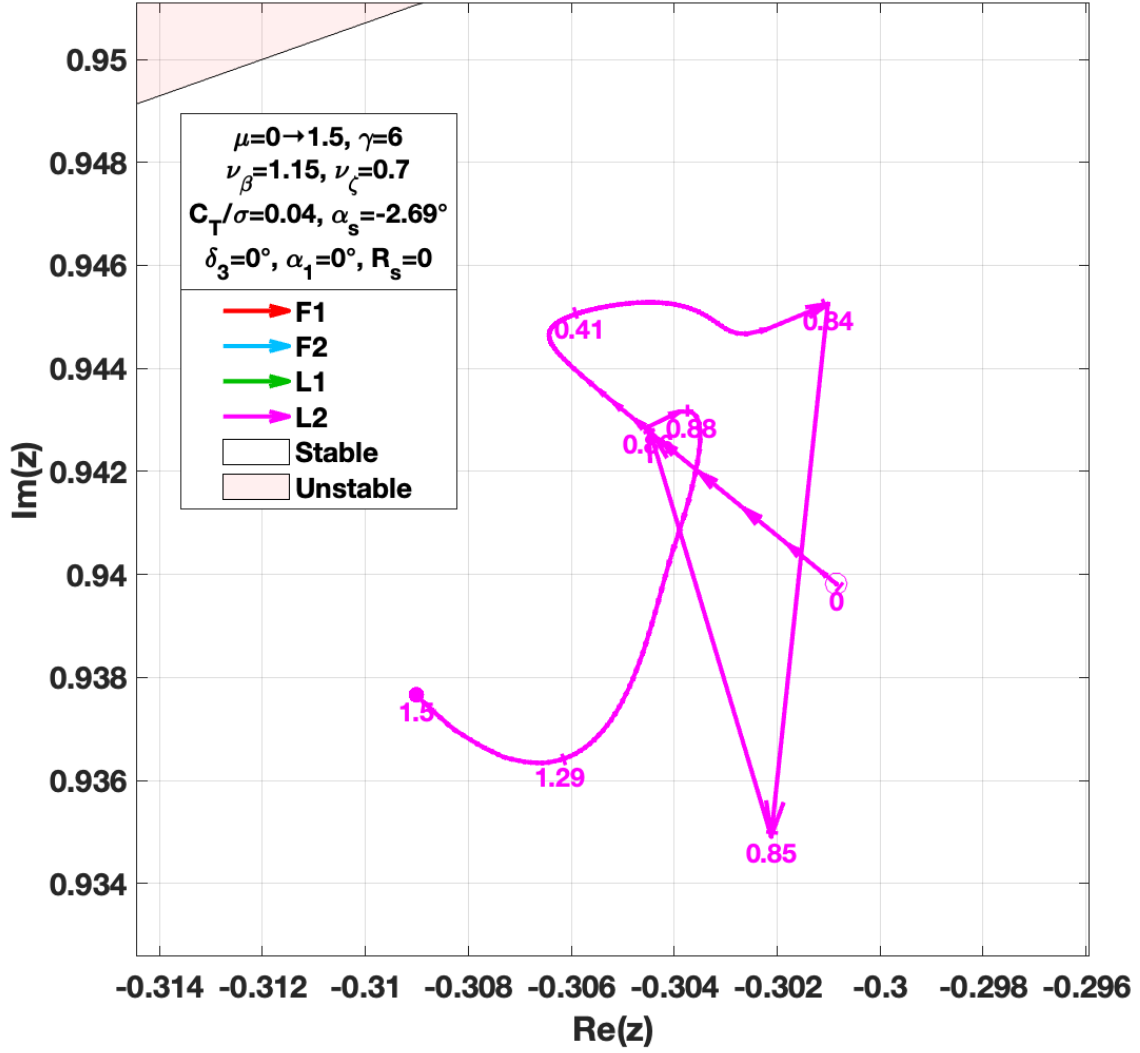


Figure 4.55: Flap-lag characteristic multipliers, L1 root, soft-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 0.7/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $\mu = 0$ to 1.5.

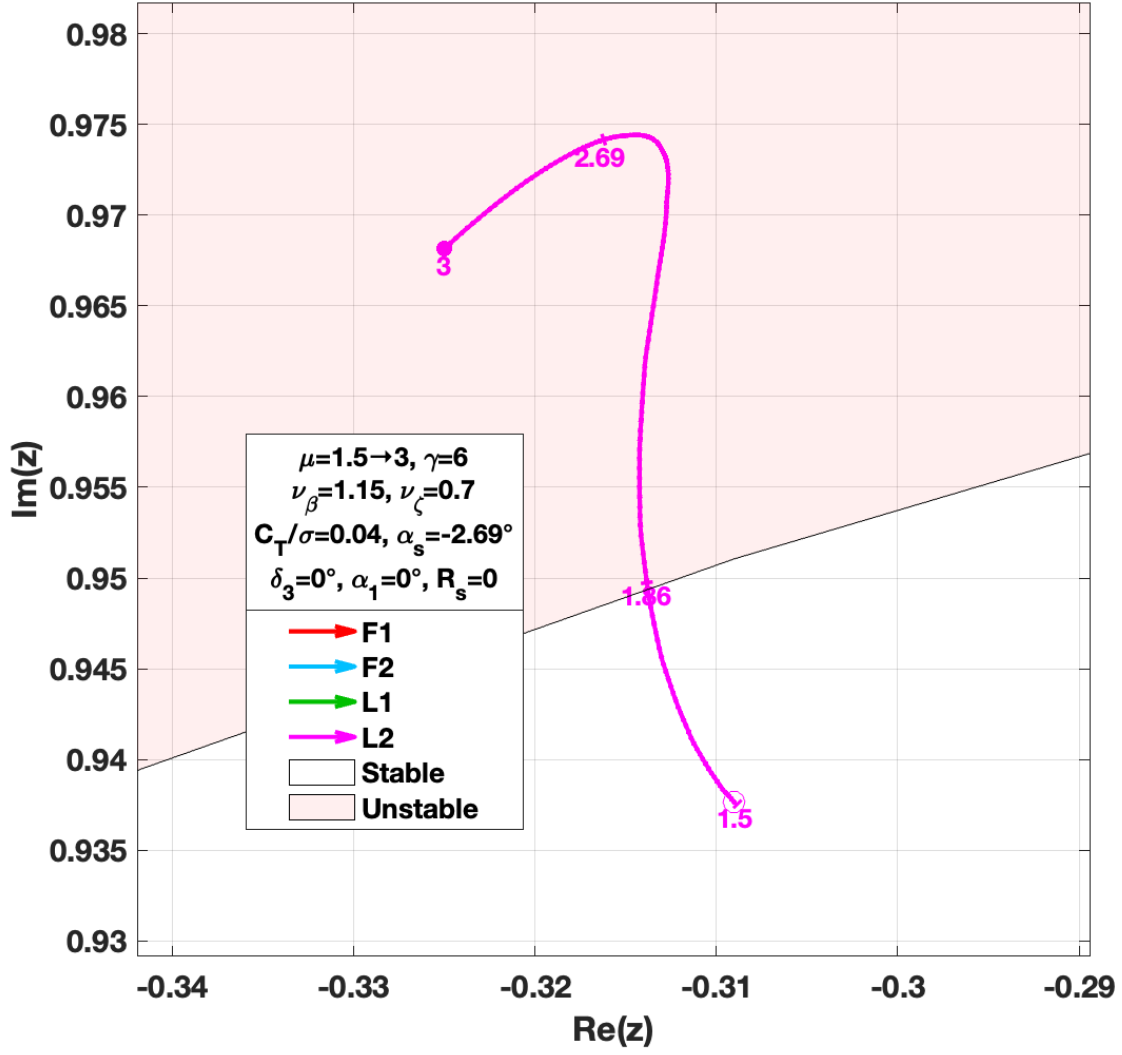


Figure 4.56: Flap-lag characteristic multipliers, L1 root, soft-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 0.7/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $\mu = 1.5$ to 3 .

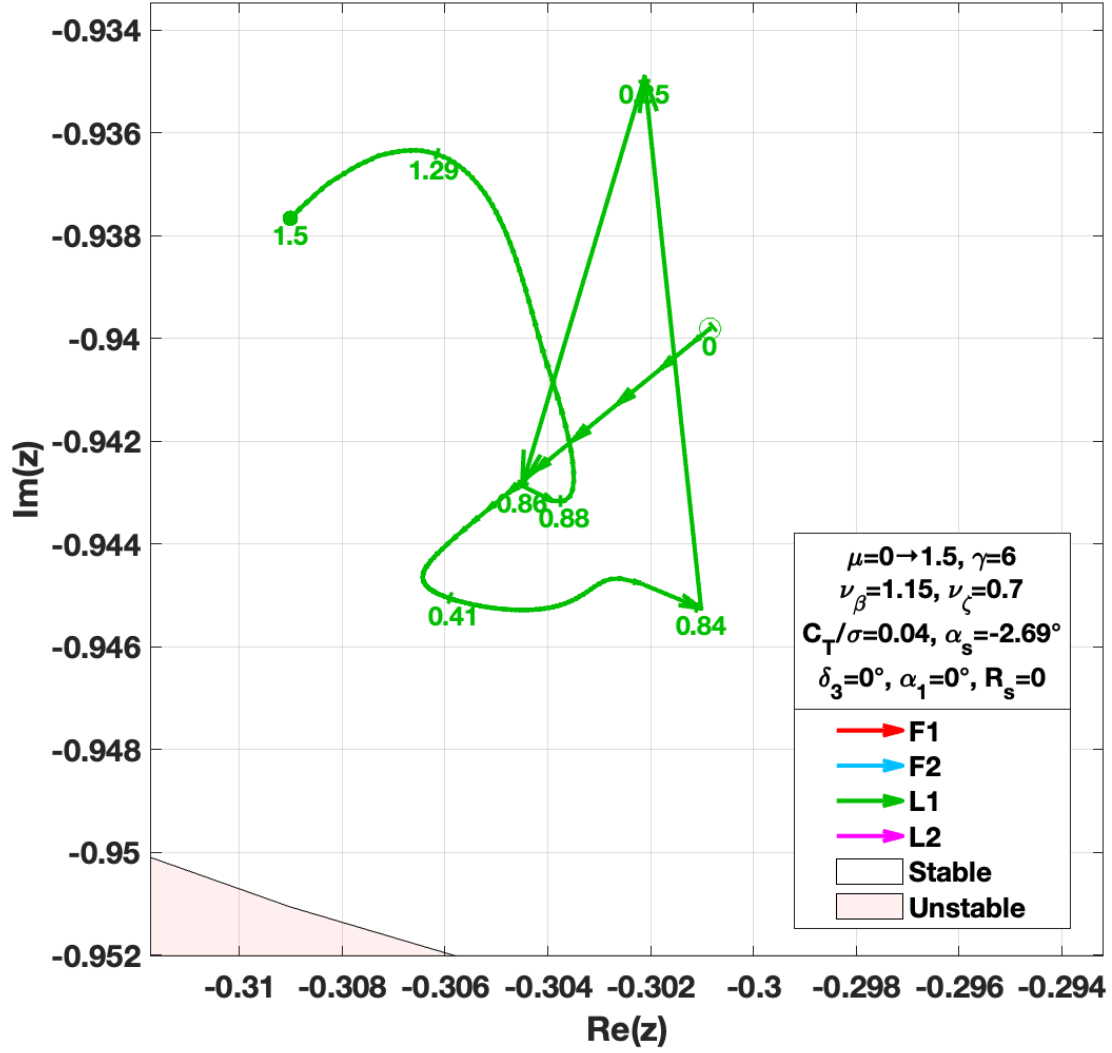


Figure 4.57: Flap-lag characteristic multipliers, L2 root, soft-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 0.7/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $\mu = 0$ to 1.5 .

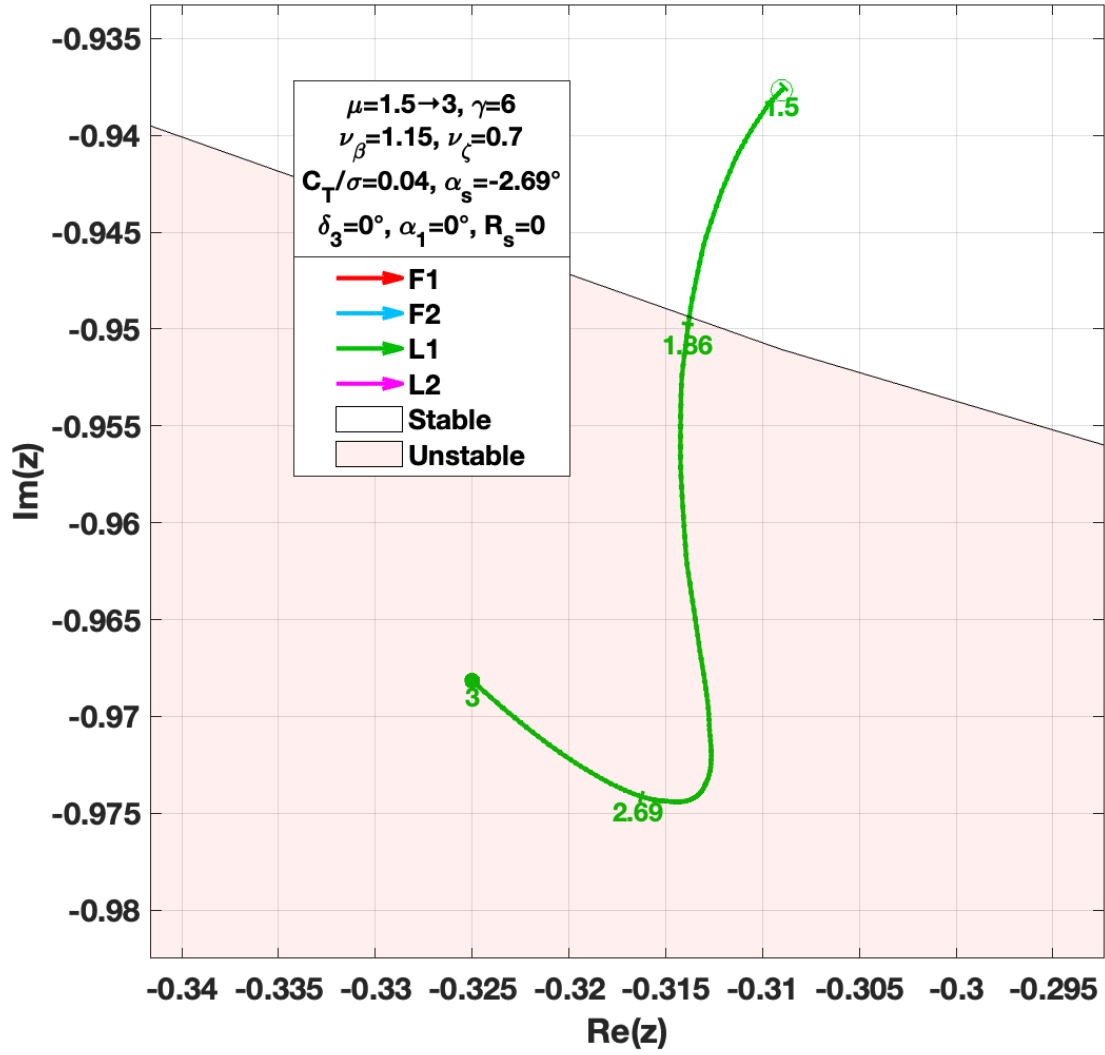


Figure 4.58: Flap-lag characteristic multipliers, L2 root, soft-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 0.7/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $\mu = 1.5$ to 3 .

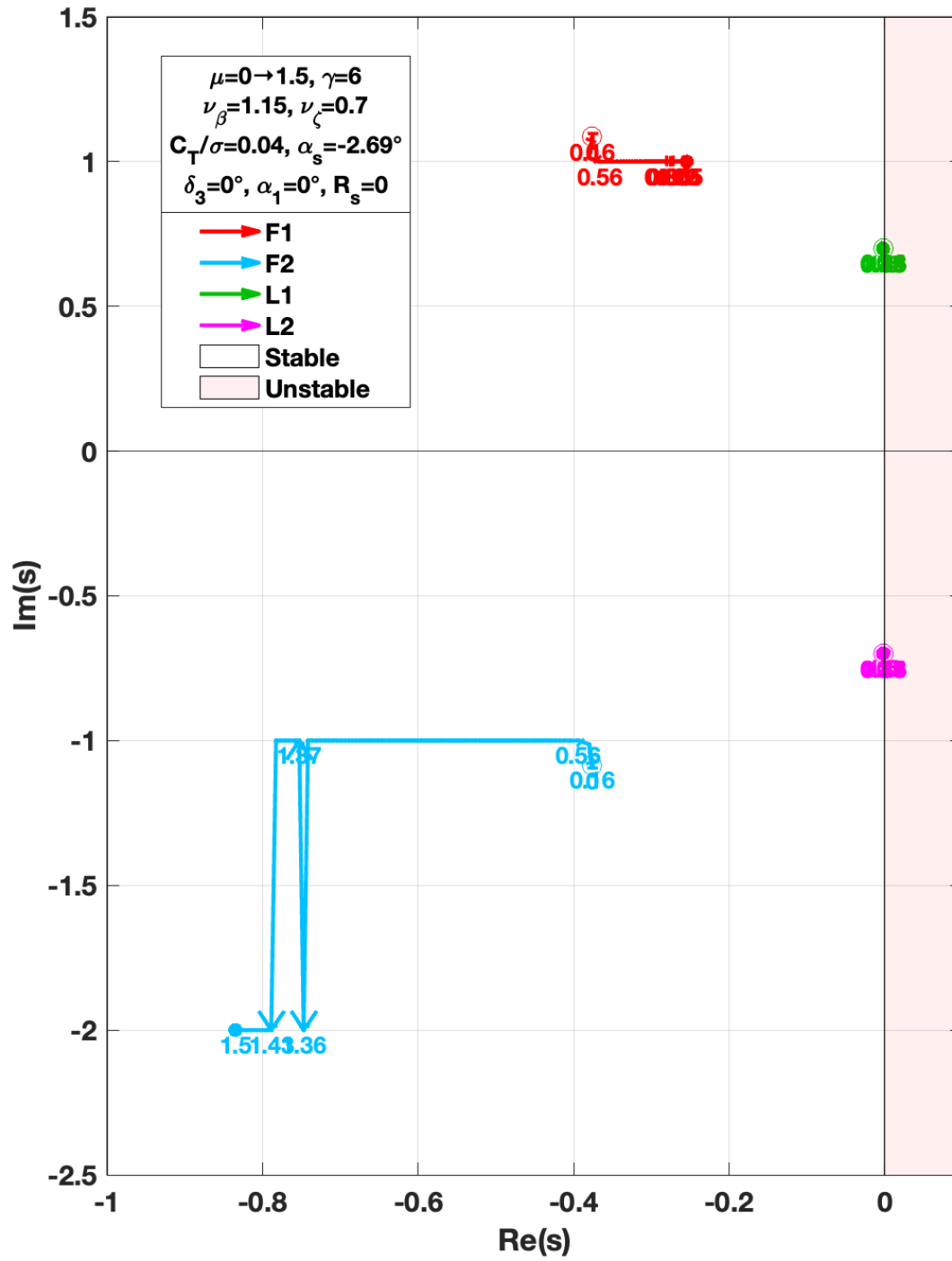


Figure 4.59: Flap-lag root locus, soft-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 0.7/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $\mu = 0$ to 1.5 .

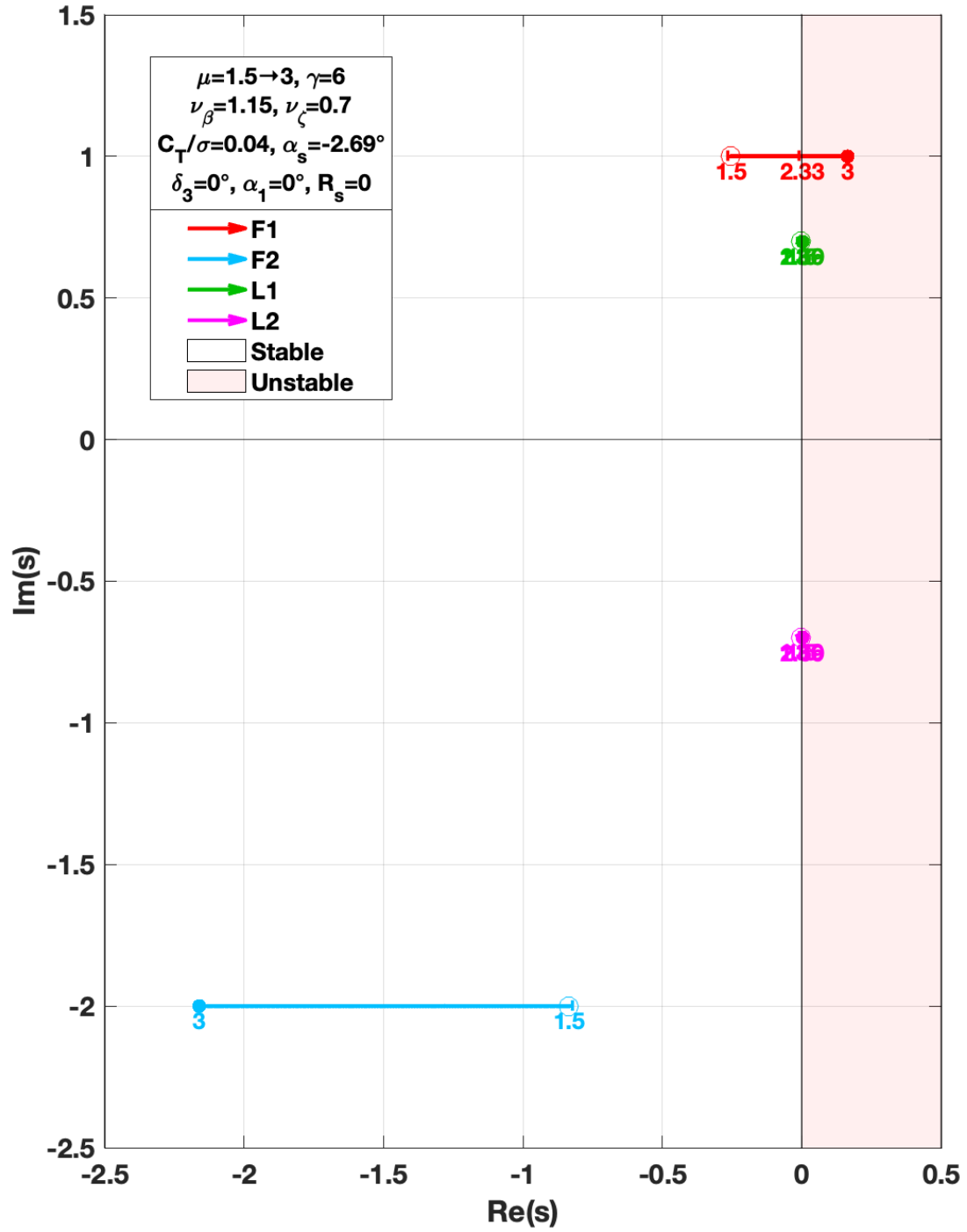


Figure 4.60: Flap-lag root locus, soft-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 0.7/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $\mu = 1.5$ to 3 .

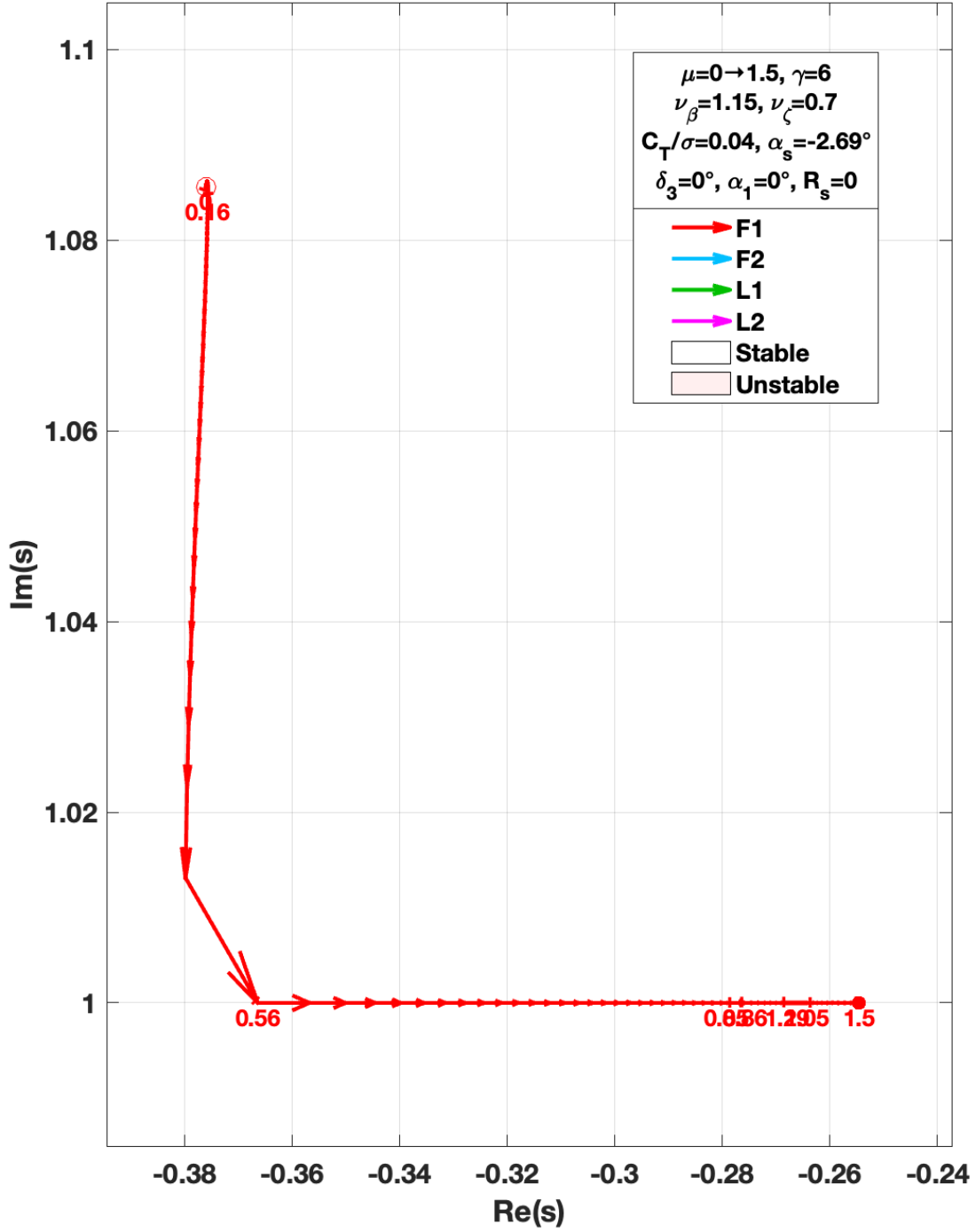


Figure 4.61: Flap-lag root locus, F1 root, soft-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 0.7/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $\mu = 0$ to 1.5 .

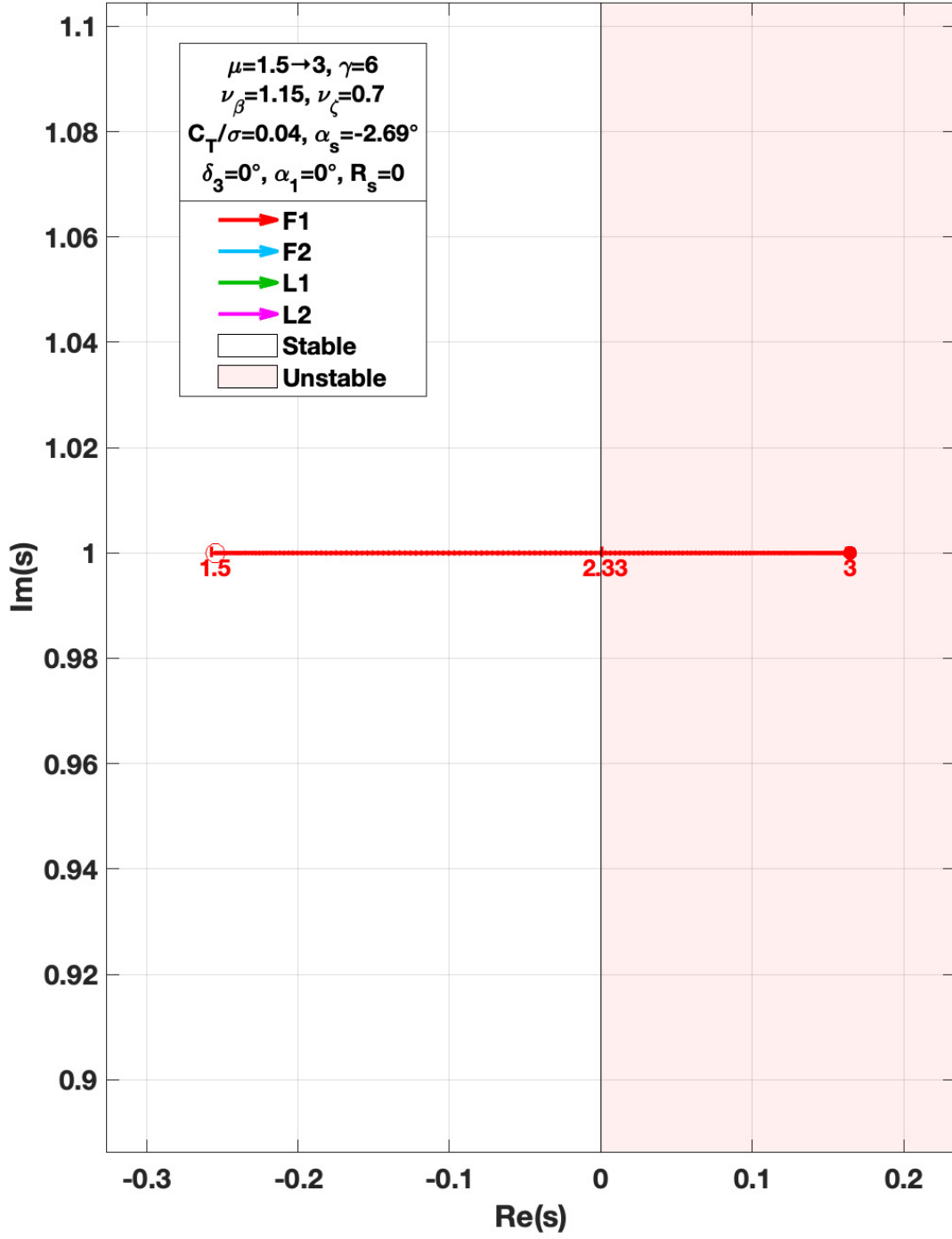


Figure 4.62: Flap-lag root locus, F1 root, soft-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 0.7/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $\mu = 1.5$ to 3 .

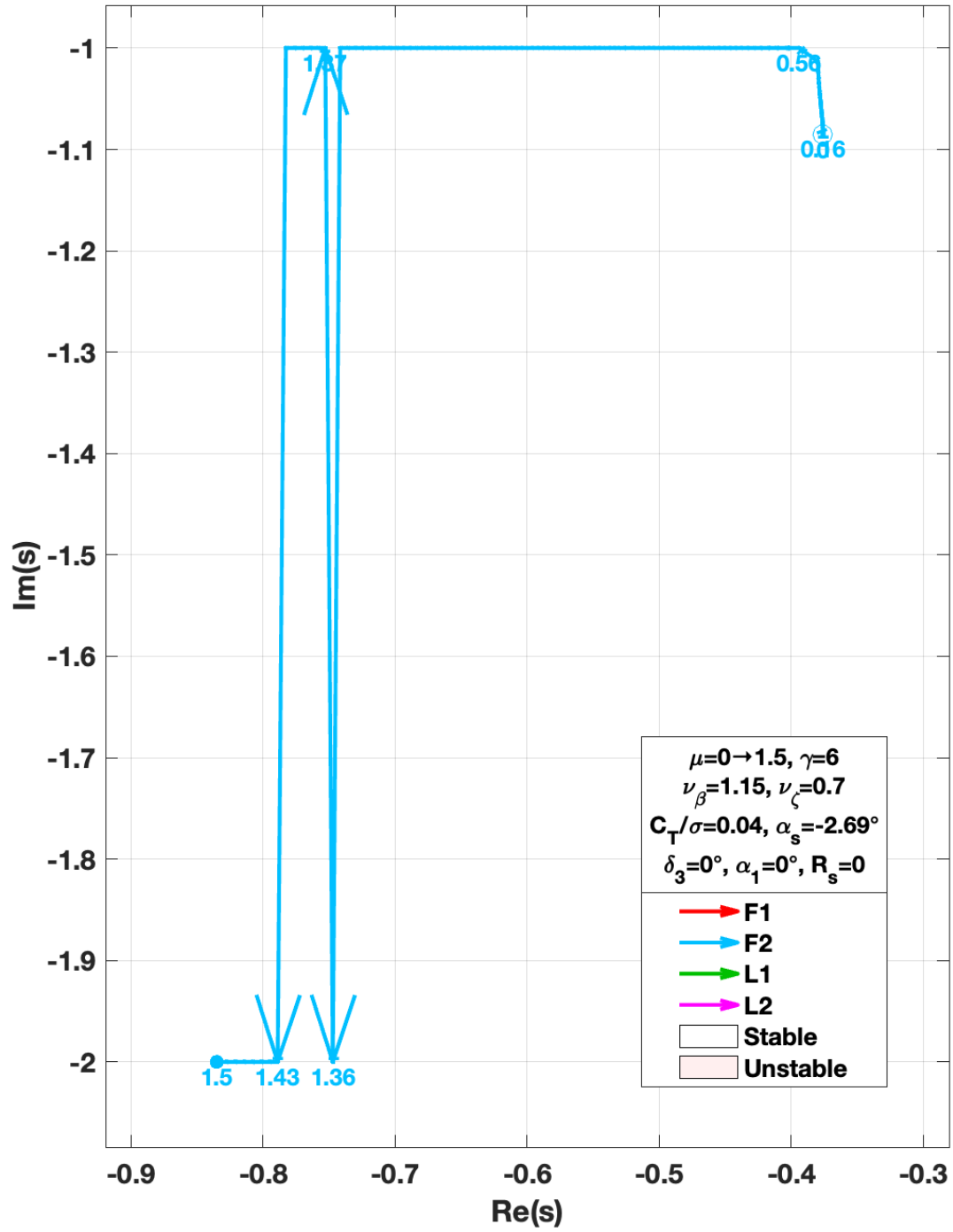


Figure 4.63: Flap-lag root locus, F2 root, soft-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 0.7/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $\mu = 0$ to 1.5 .

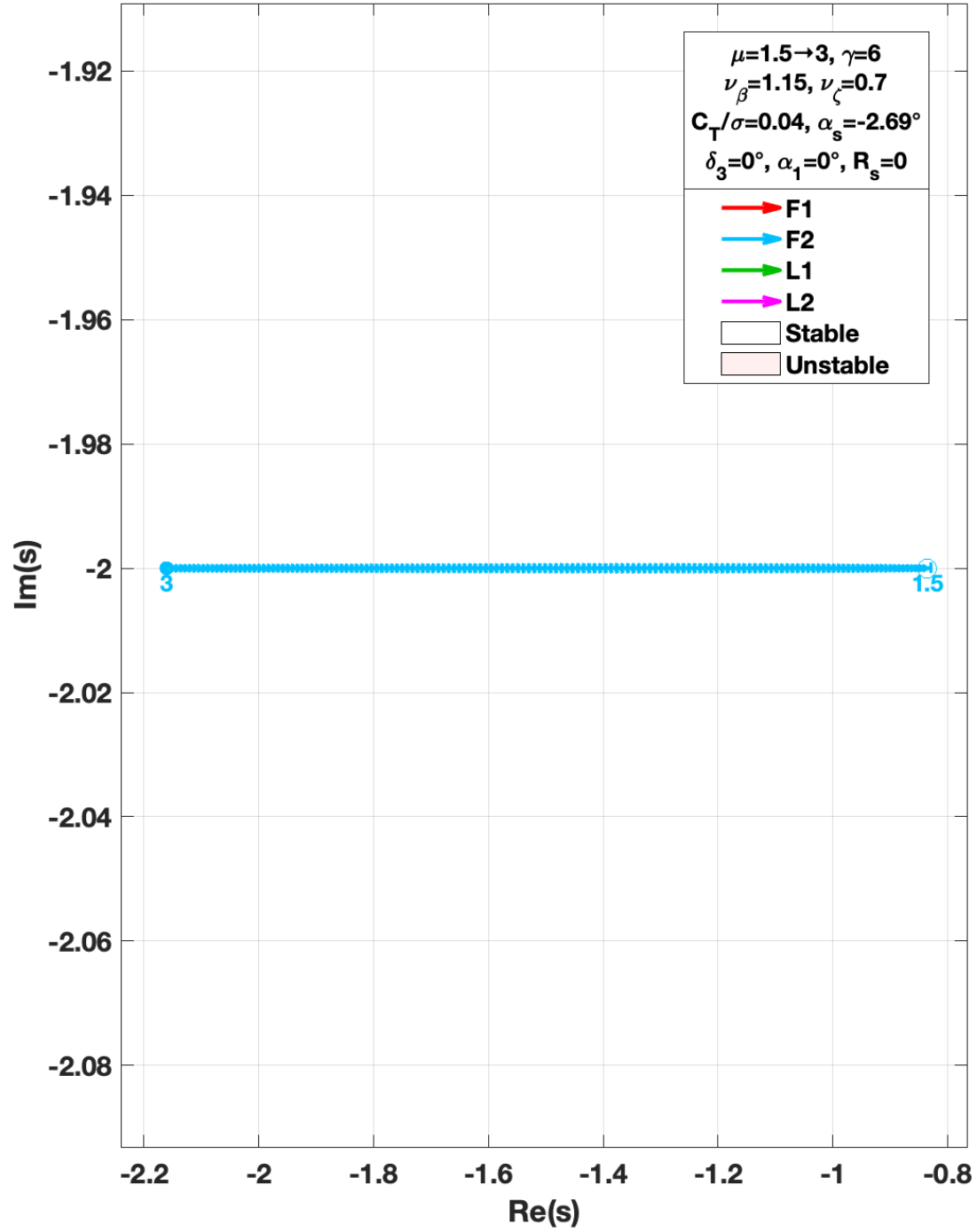


Figure 4.64: Flap-lag root locus, F2 root, soft-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 0.7/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $\mu = 1.5$ to 3.

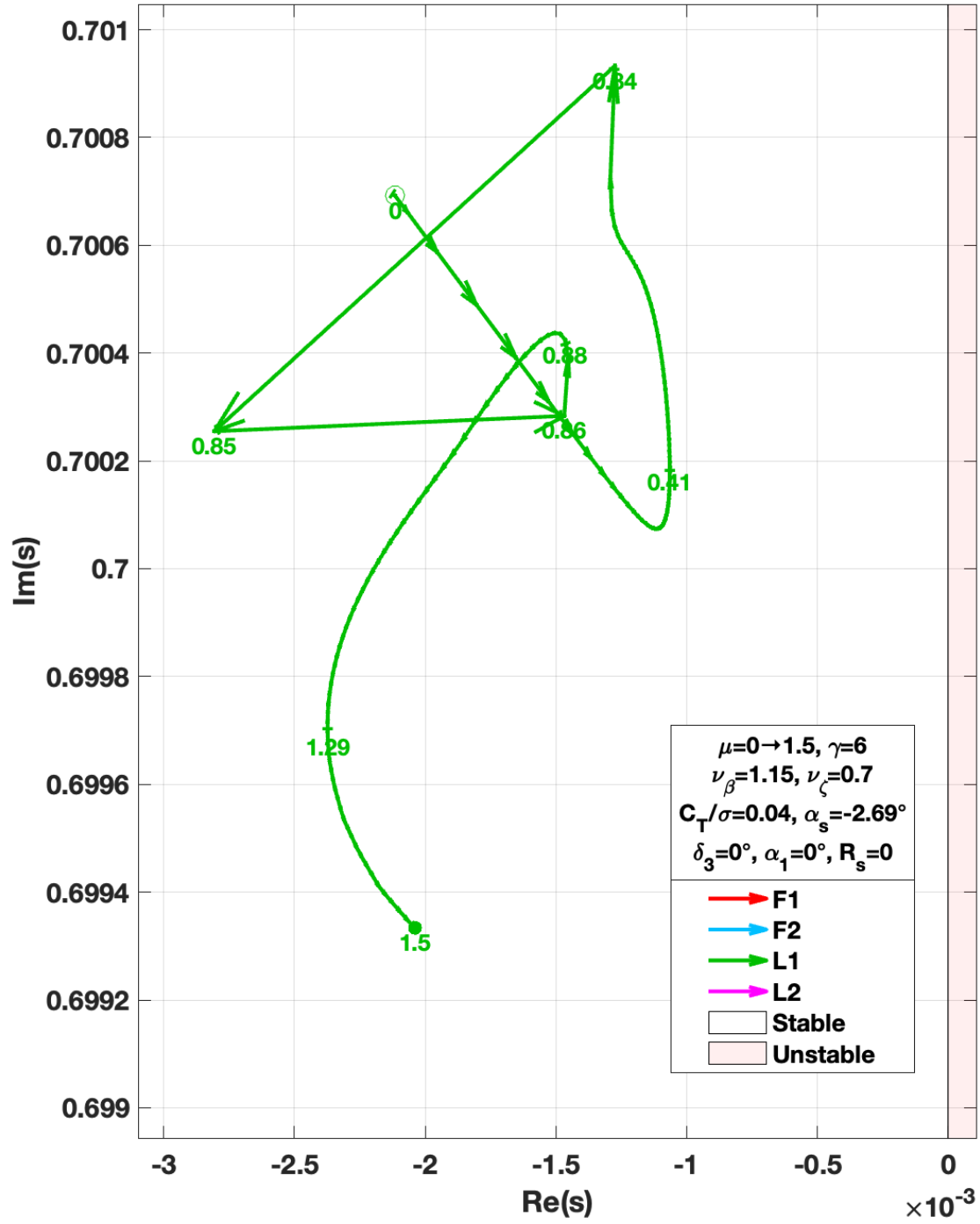


Figure 4.65: Flap-lag root locus, L1 root, soft-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 0.7/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $\mu = 0$ to 1.5 .

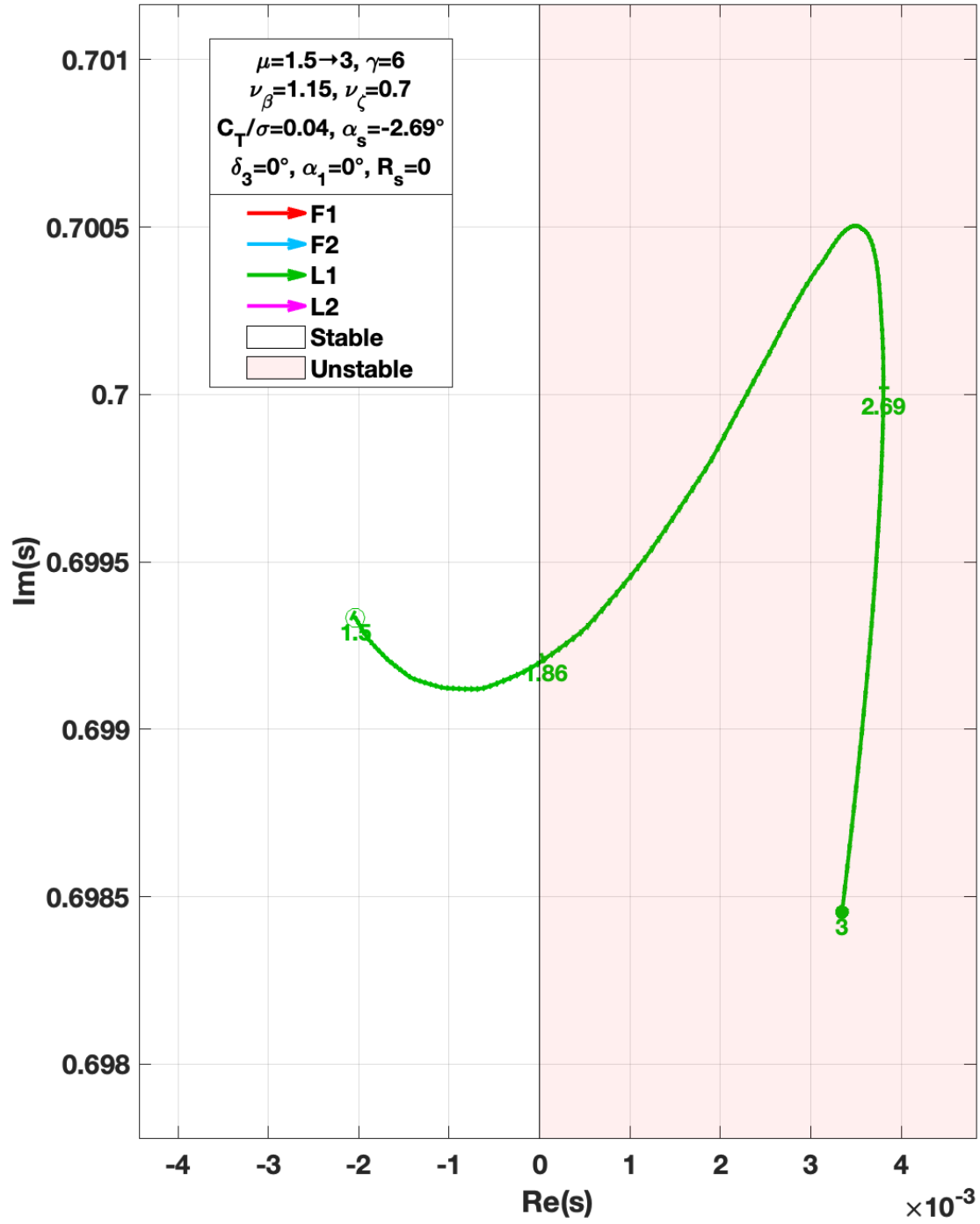


Figure 4.66: Flap-lag root locus, L1 root, soft-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 0.7/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $\mu = 1.5$ to 3 .

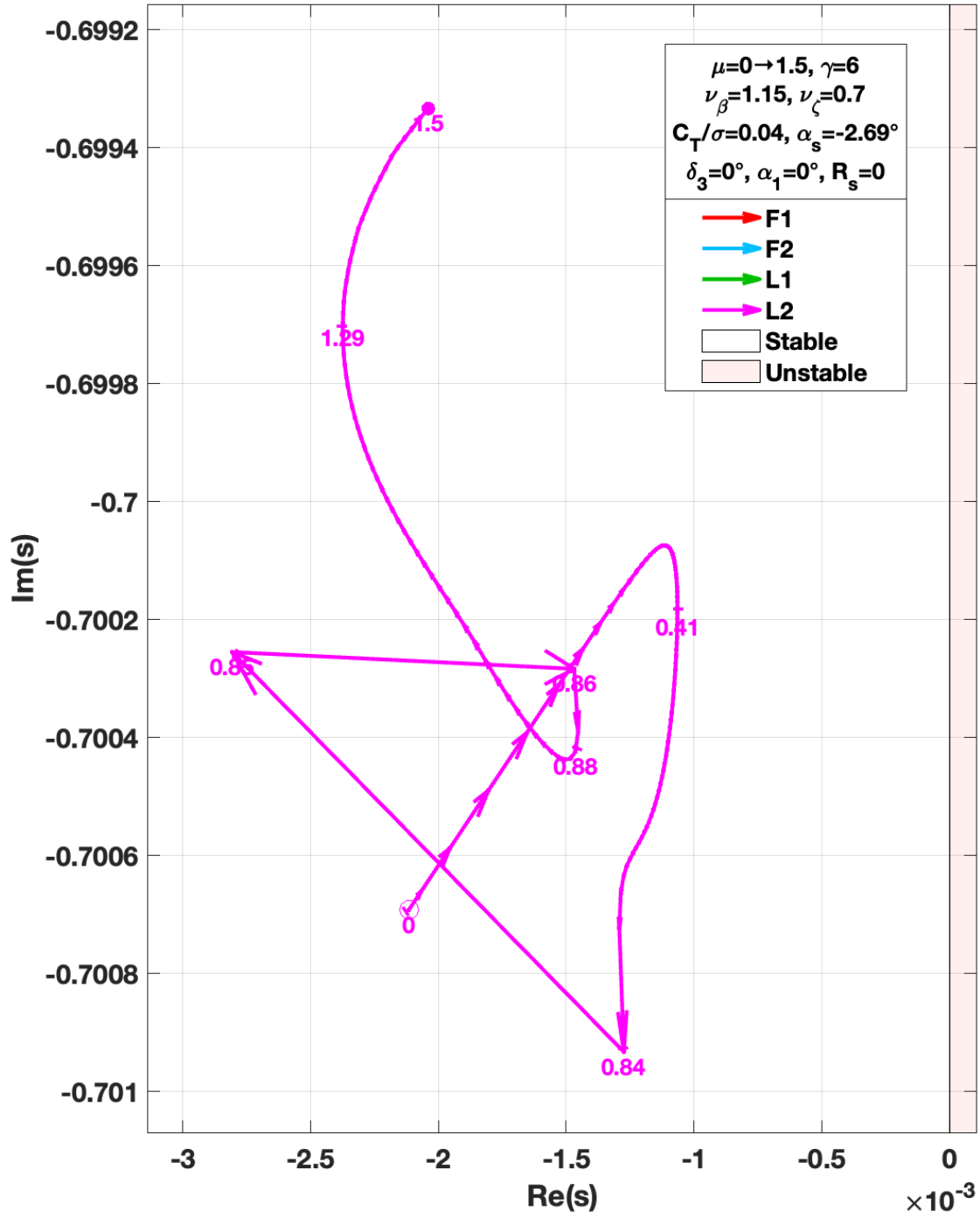


Figure 4.67: Flap-lag root locus, L2 root, soft-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 0.7/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $\mu = 0$ to 1.5.

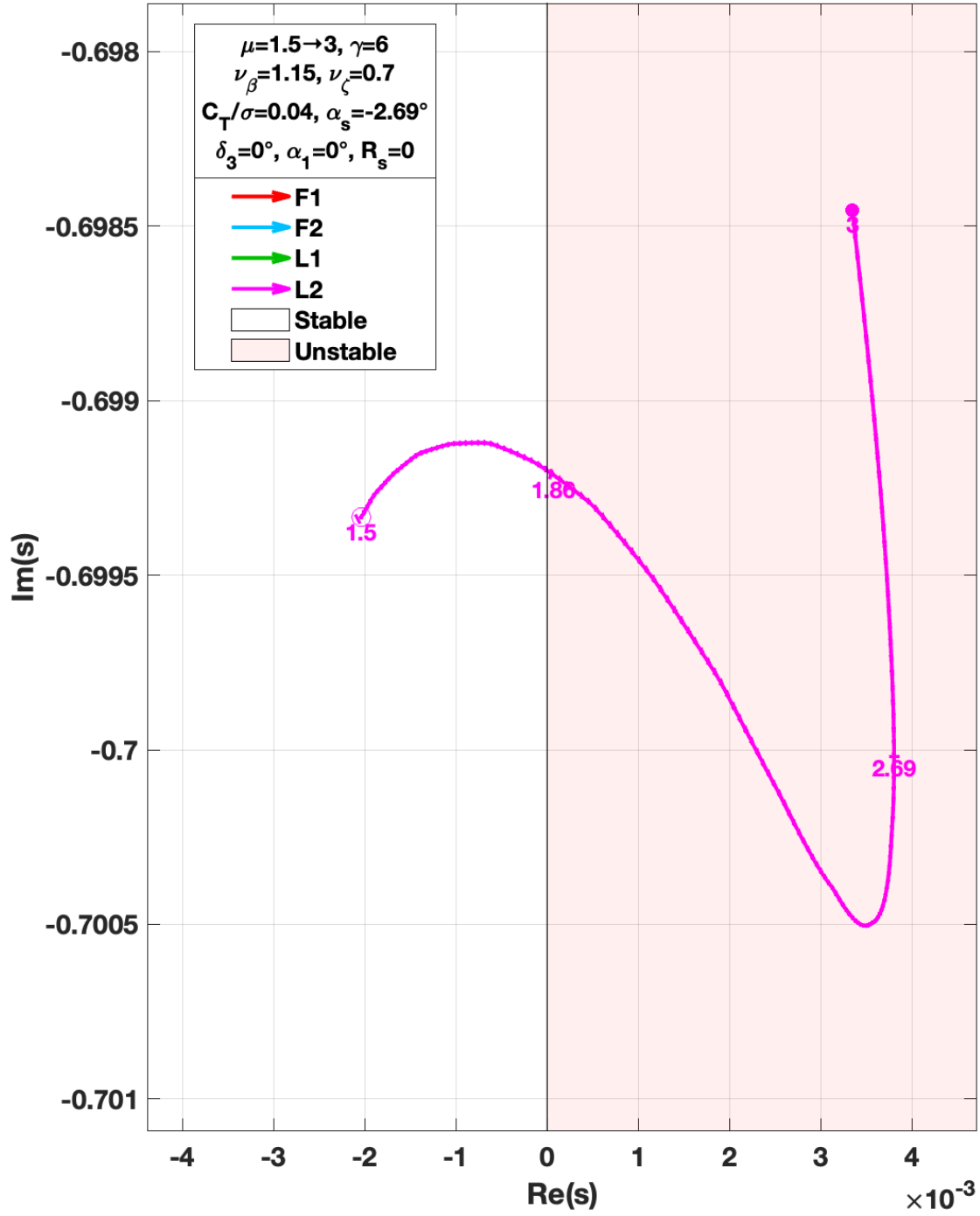


Figure 4.68: Flap-lag root locus, L2 root, soft-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 0.7/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $\mu = 1.5$ to 3.

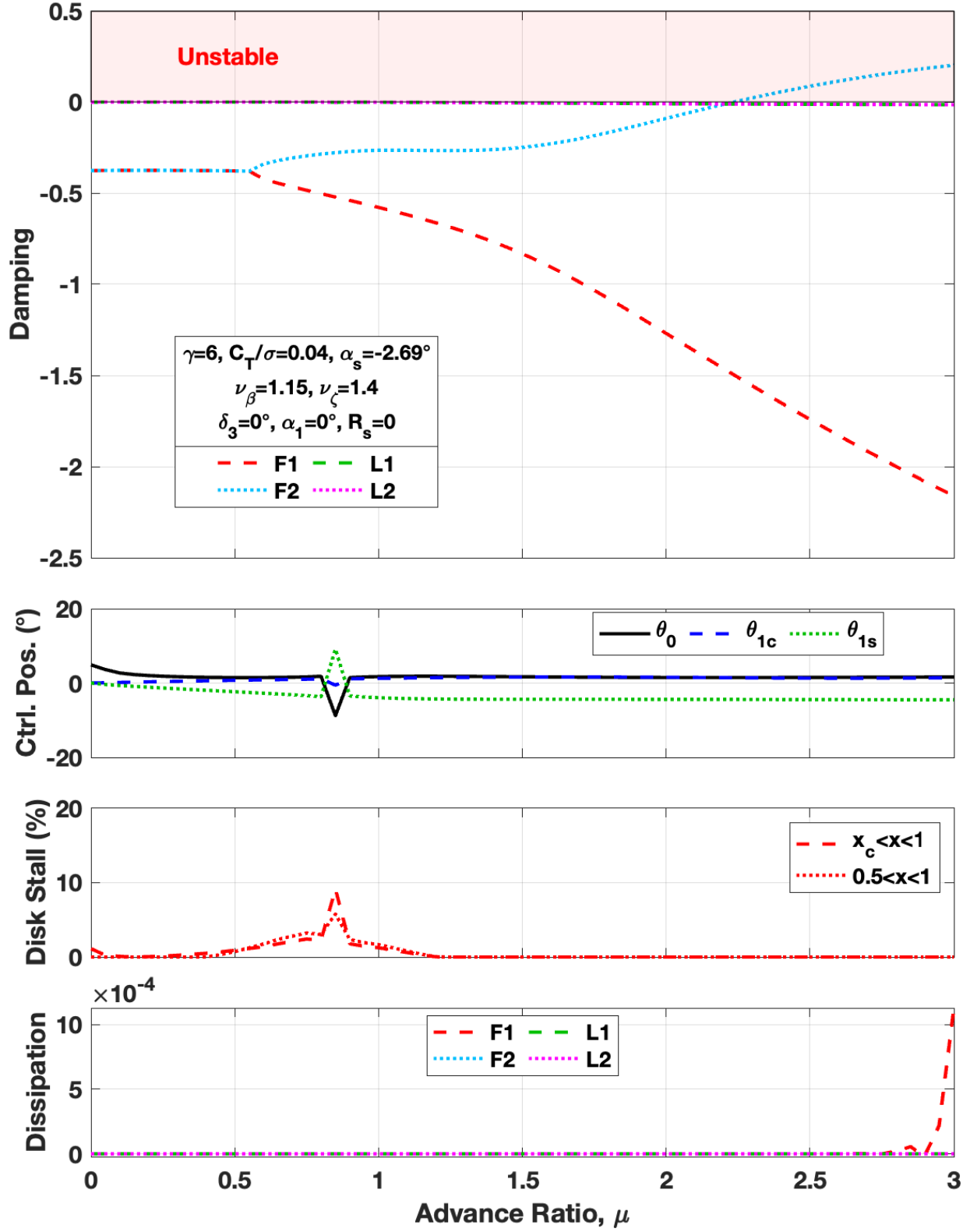


Figure 4.69: Flap-lag damping, trim, blade stall, and numerical stability, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$.

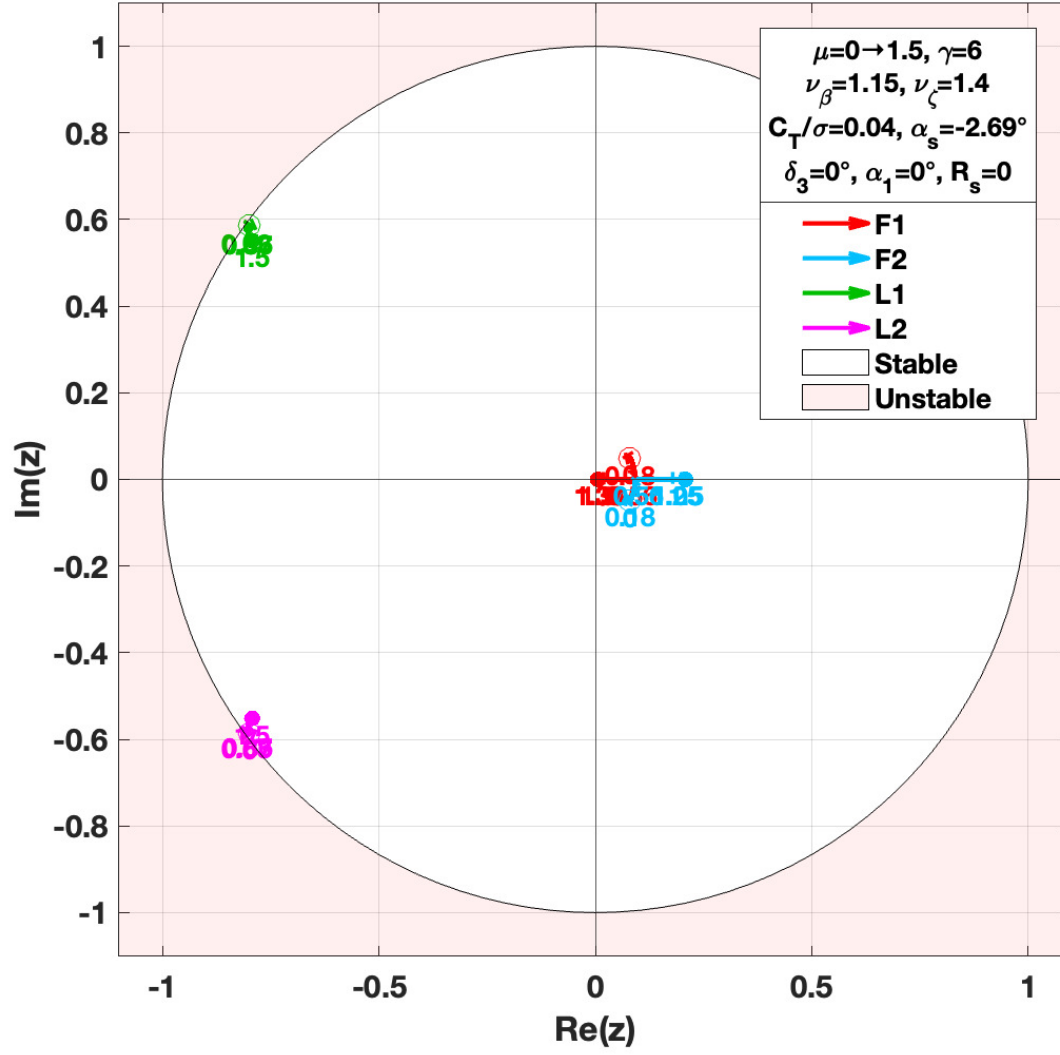


Figure 4.70: Flap-lag characteristic multipliers, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $\mu = 0$ to 1.5 .

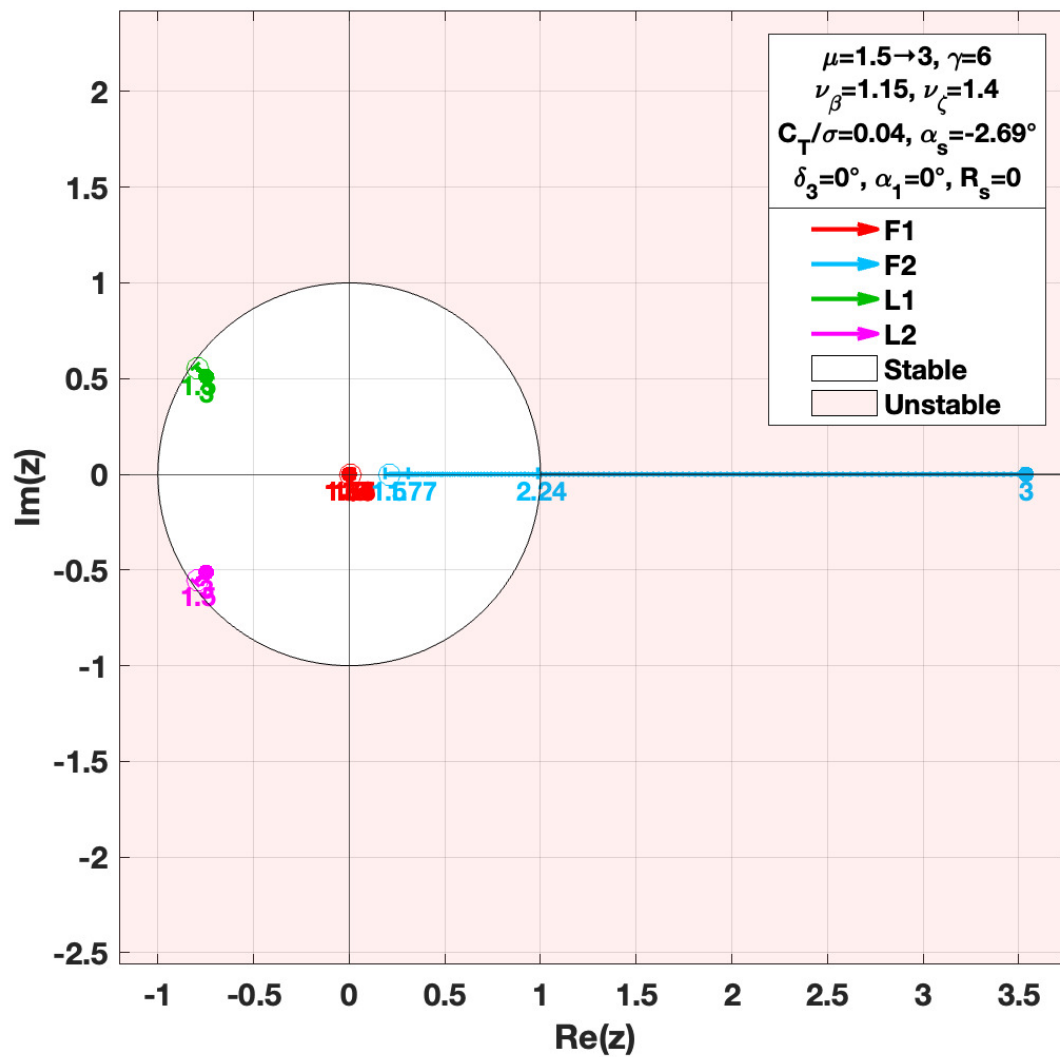


Figure 4.71: Flap-lag characteristic multipliers, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $\mu = 1.5$ to 3 .

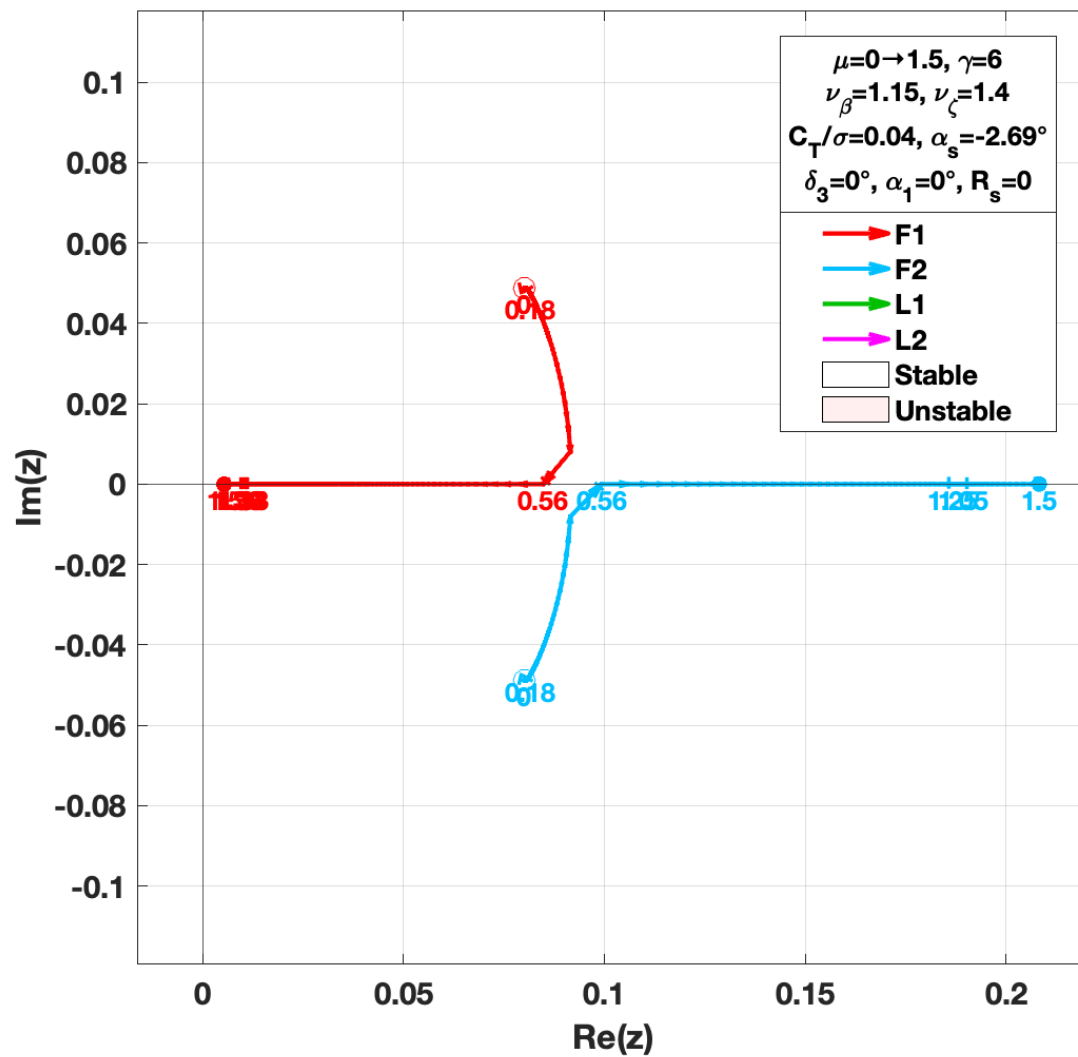


Figure 4.72: Flap-lag characteristic multipliers, F1 and F2 roots, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $\mu = 0$ to 1.5.

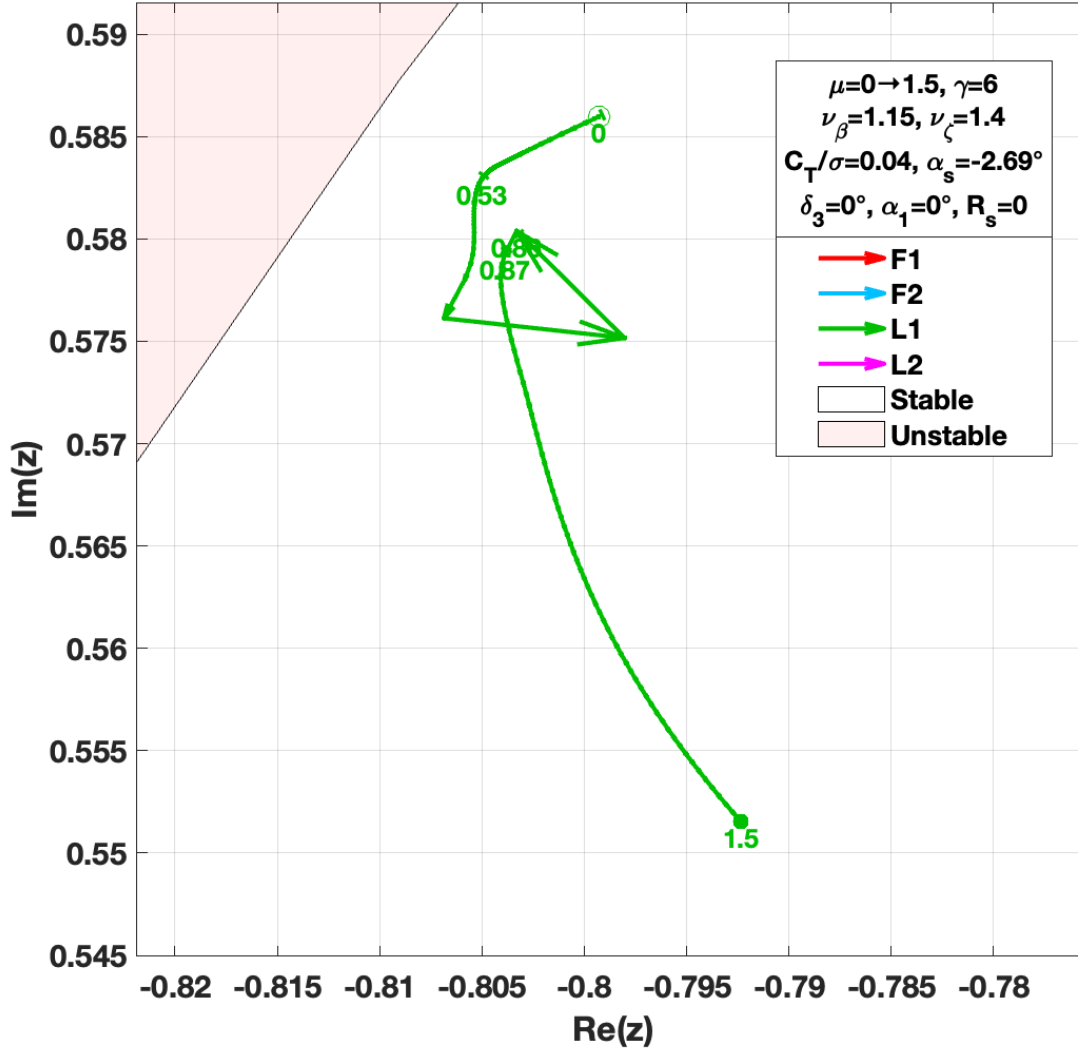


Figure 4.73: Flap-lag characteristic multipliers, L1 root, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $\mu = 0$ to 1.5 .

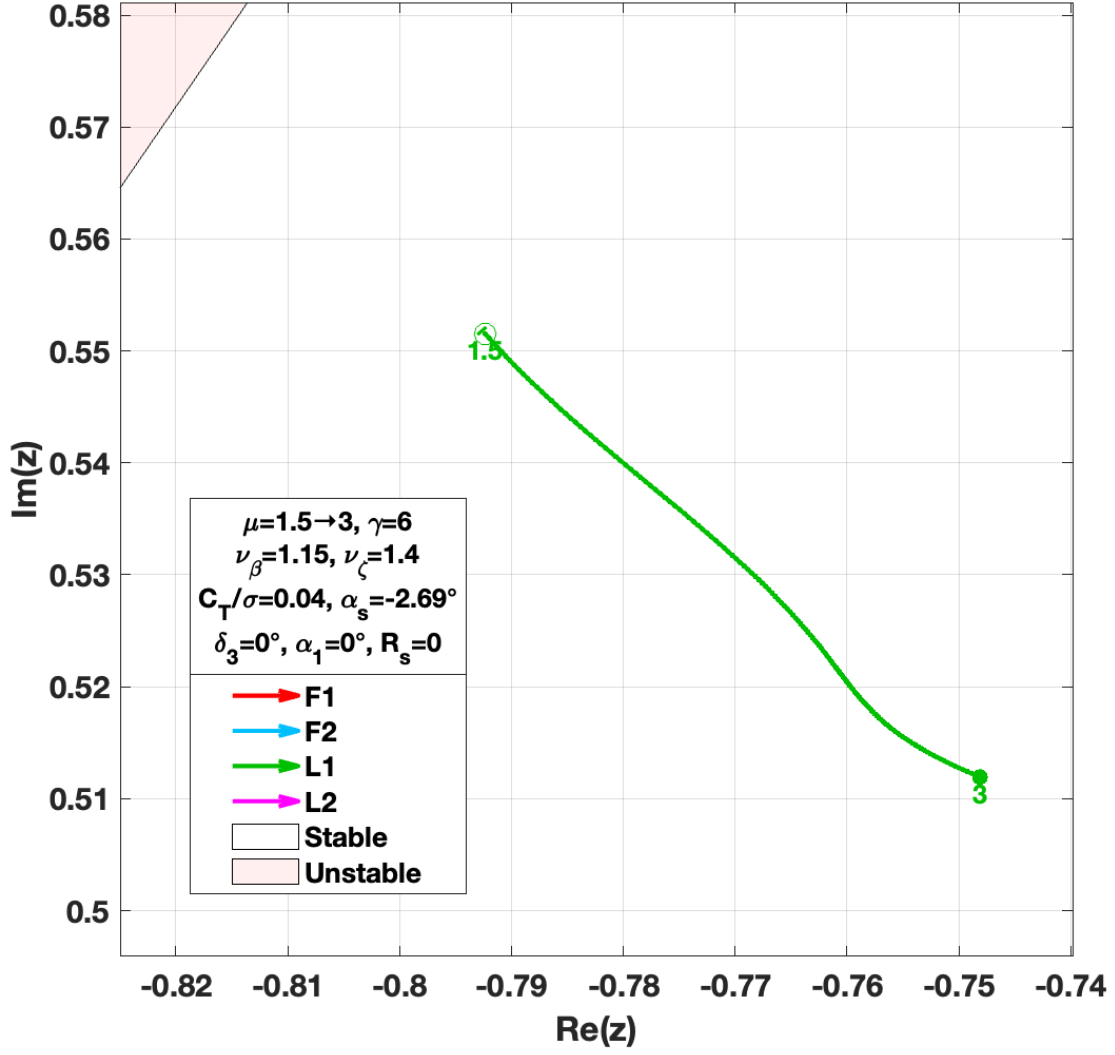


Figure 4.74: Flap-lag characteristic multipliers, L1 root, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $\mu = 1.5$ to 3 .

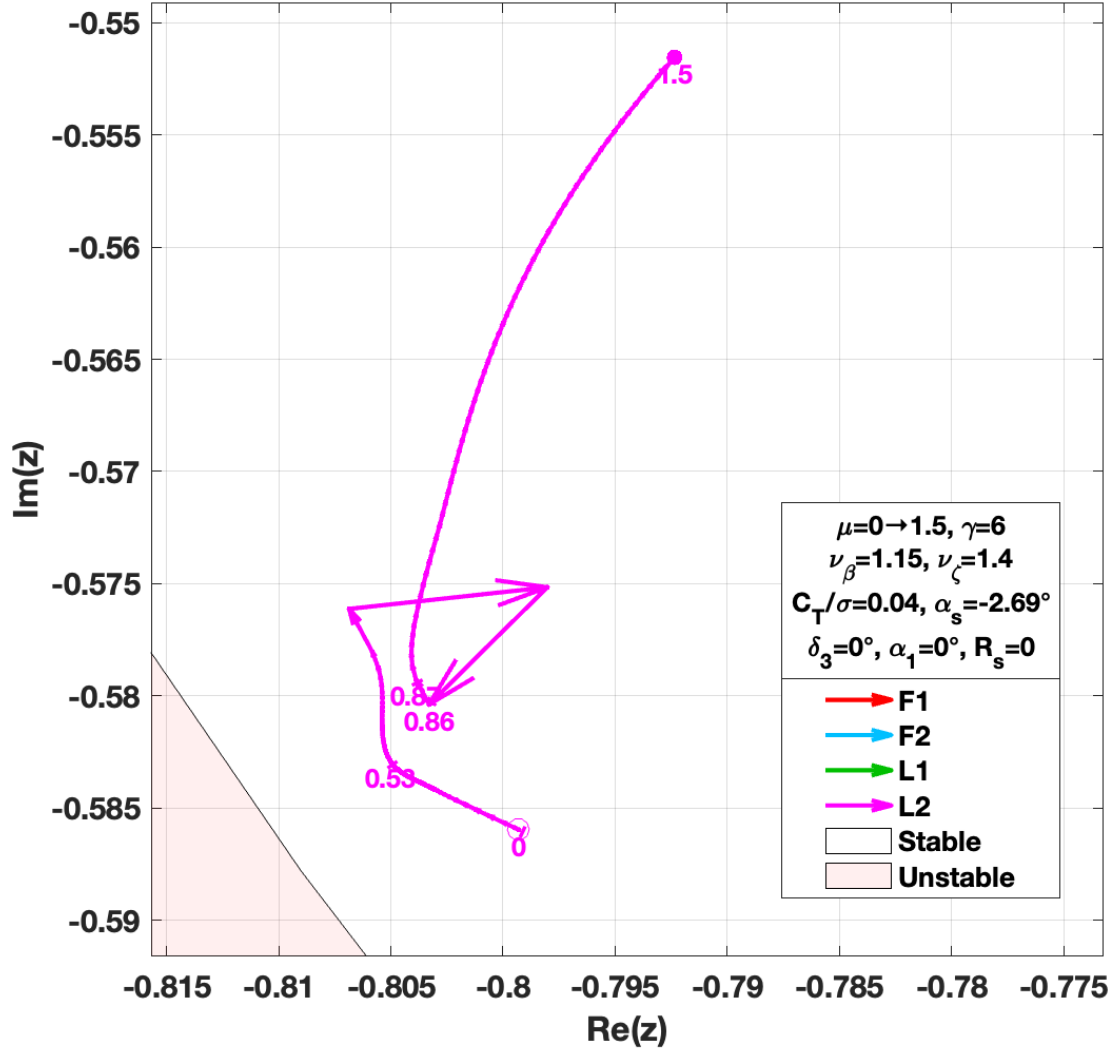


Figure 4.75: Flap-lag characteristic multipliers, L2 root, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $\mu = 0$ to 1.5 .

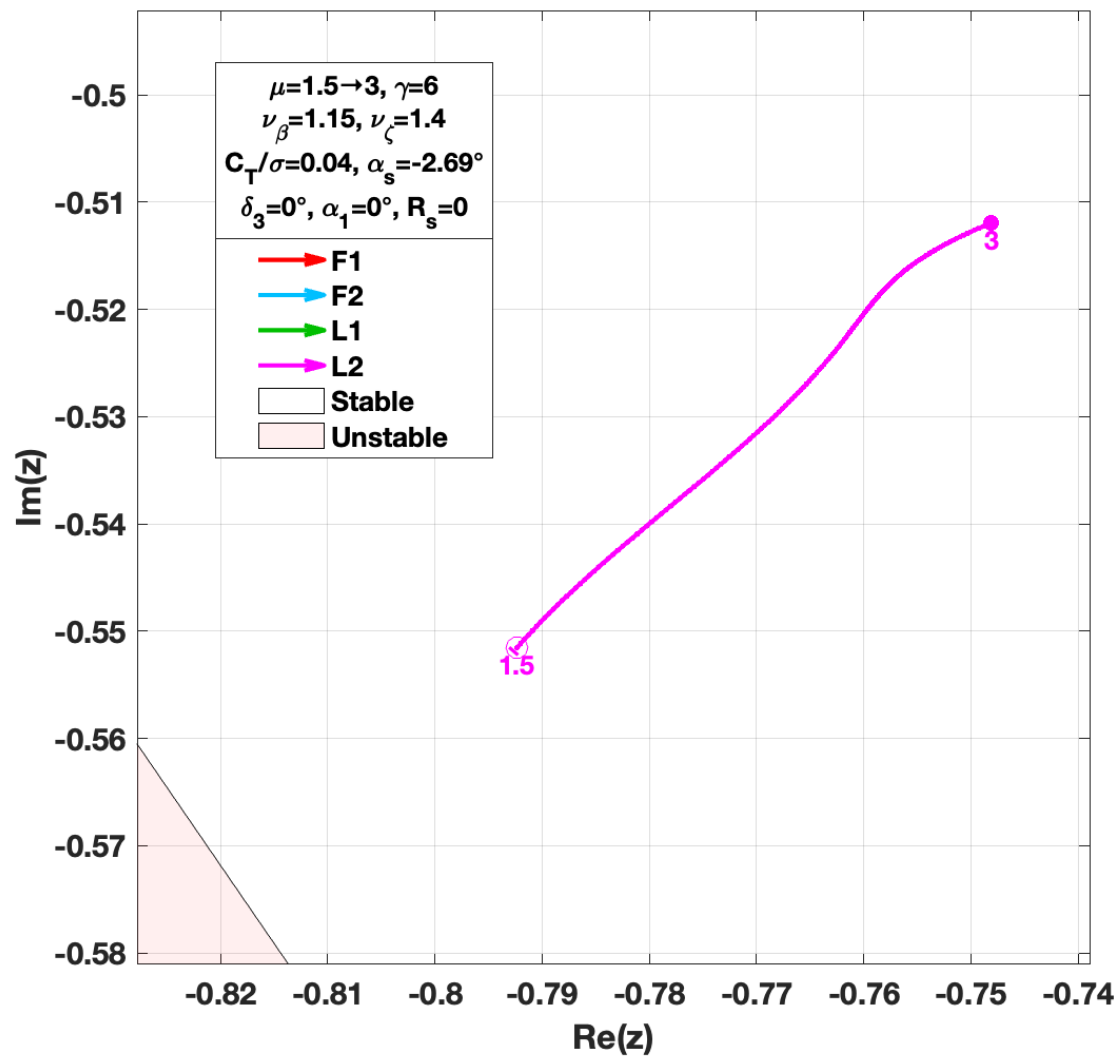


Figure 4.76: Flap-lag characteristic multipliers, L2 root, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $\mu = 1.5$ to 3 .

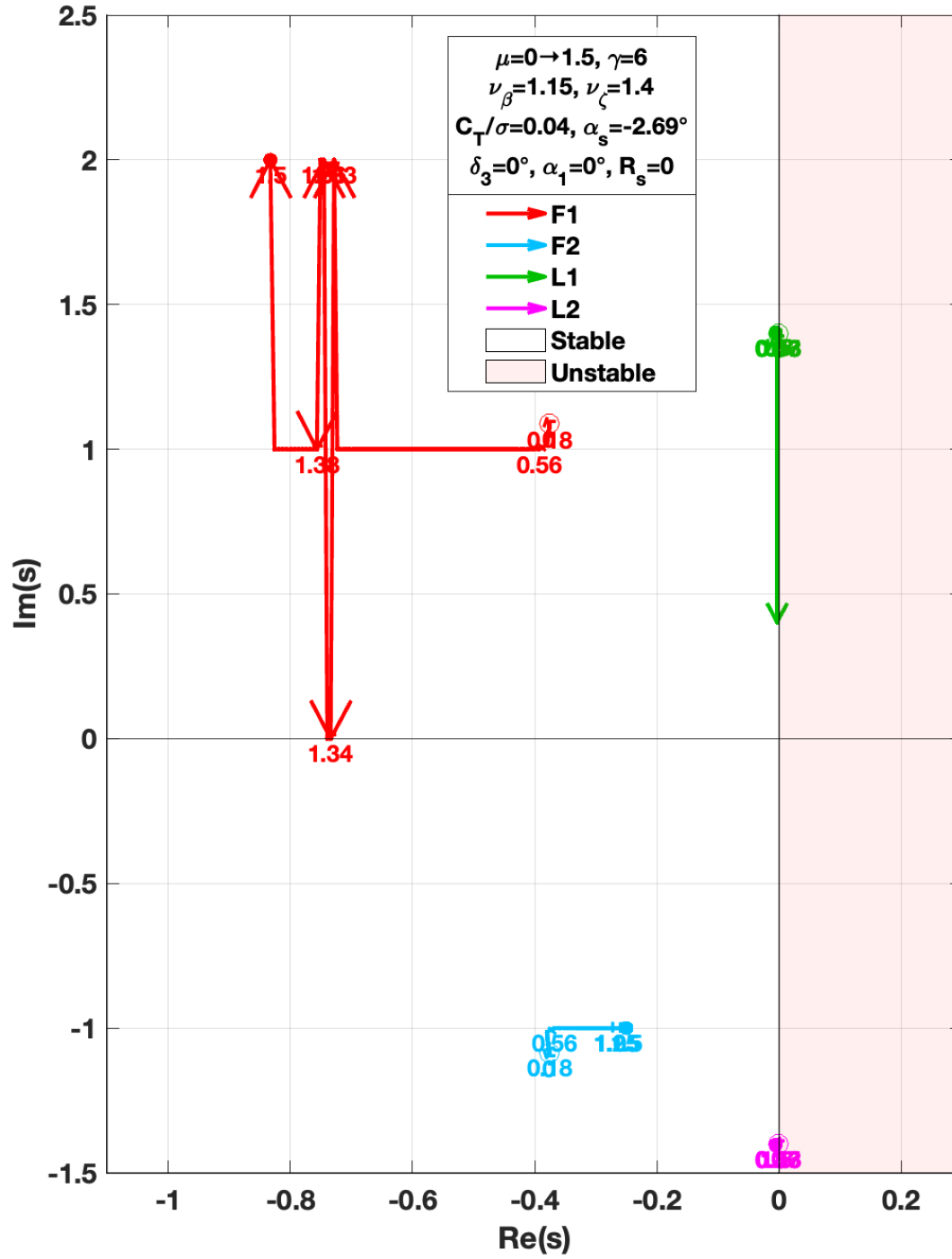


Figure 4.77: Flap-lag root locus, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $\mu = 0$ to 1.5 .

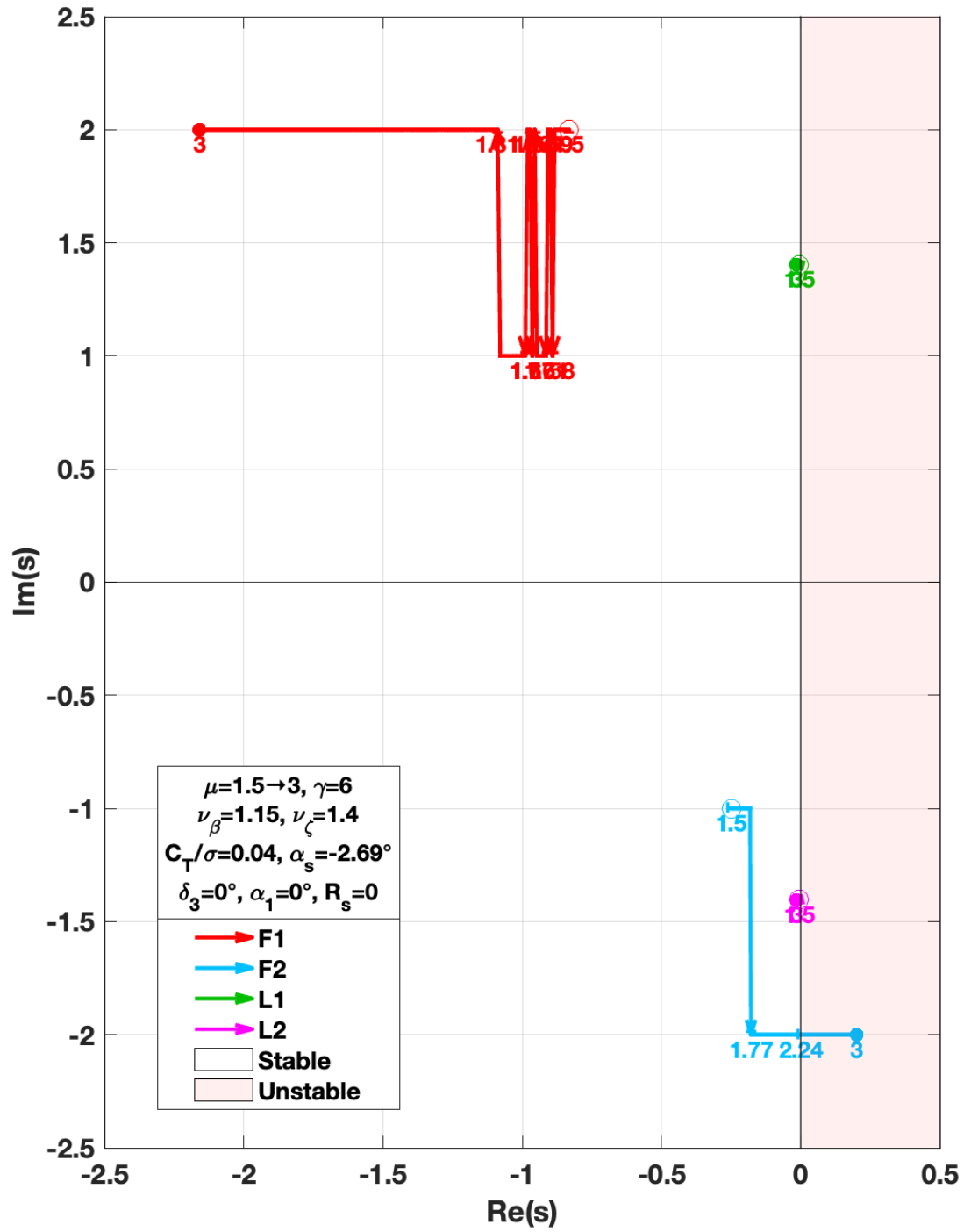


Figure 4.78: Flap-lag root locus, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $\mu = 1.5$ to 3 .

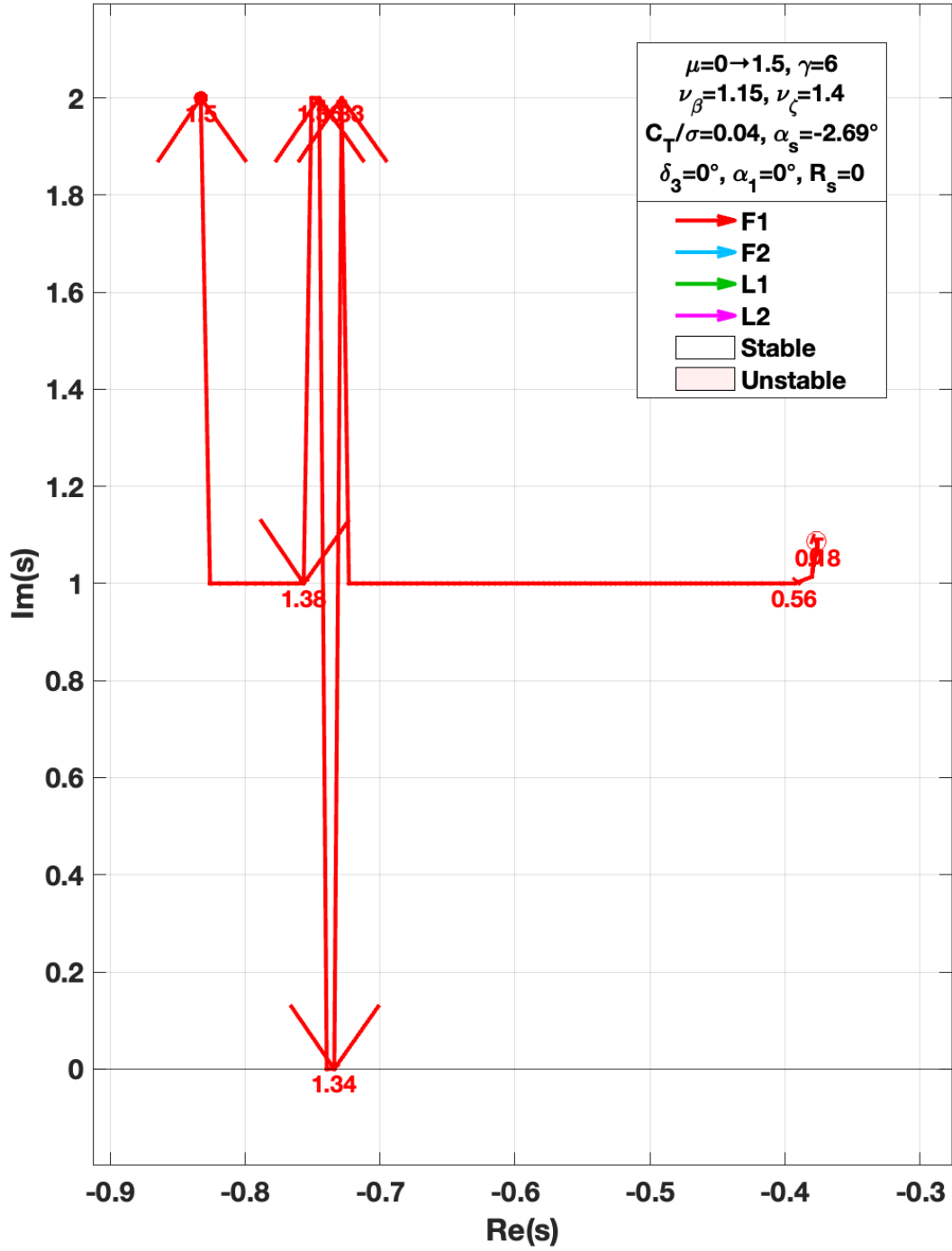


Figure 4.79: Flap-lag root locus, F1 root, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $\mu = 0$ to 1.5.

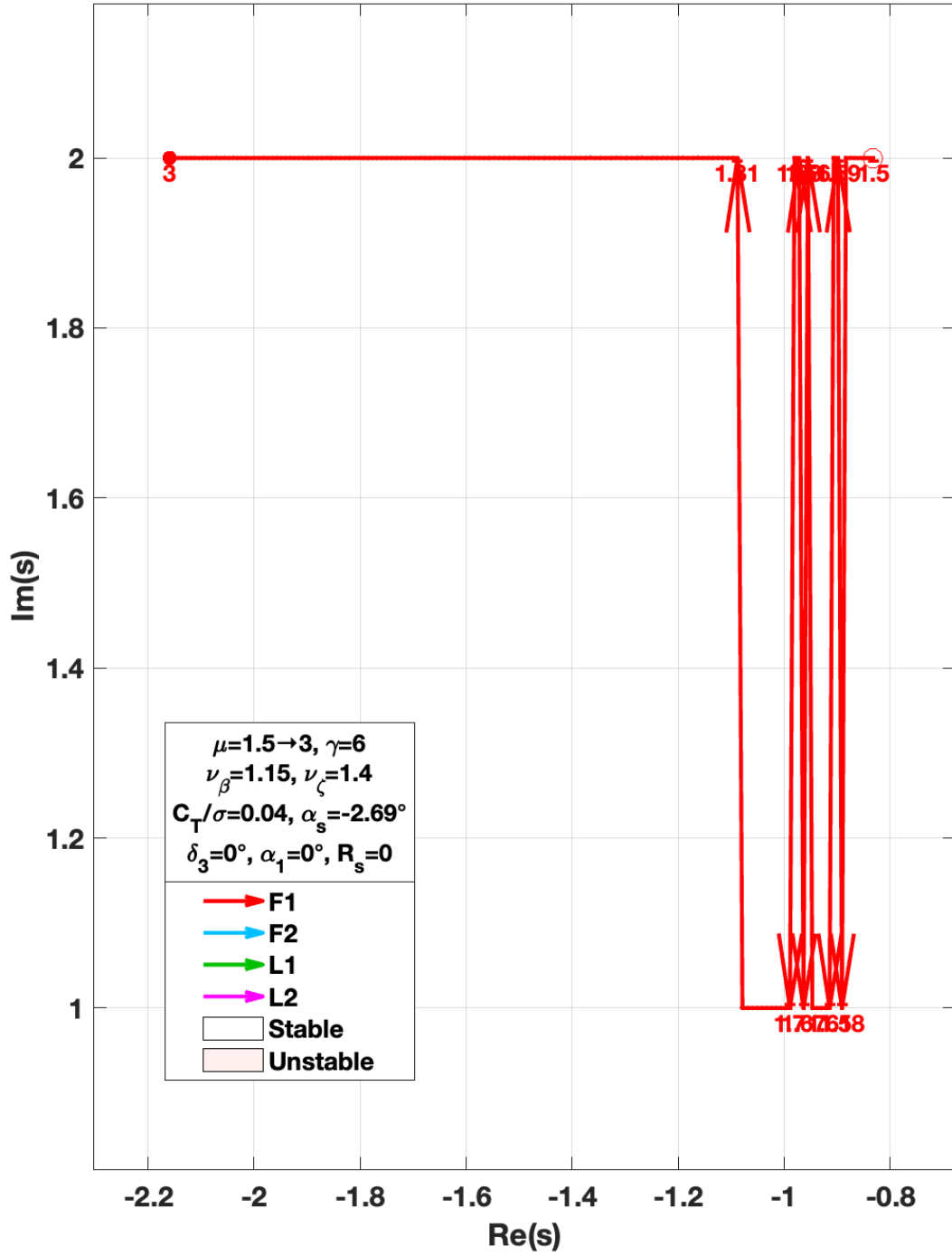


Figure 4.80: Flap-lag root locus, F1 root, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $\mu = 1.5$ to 3 .

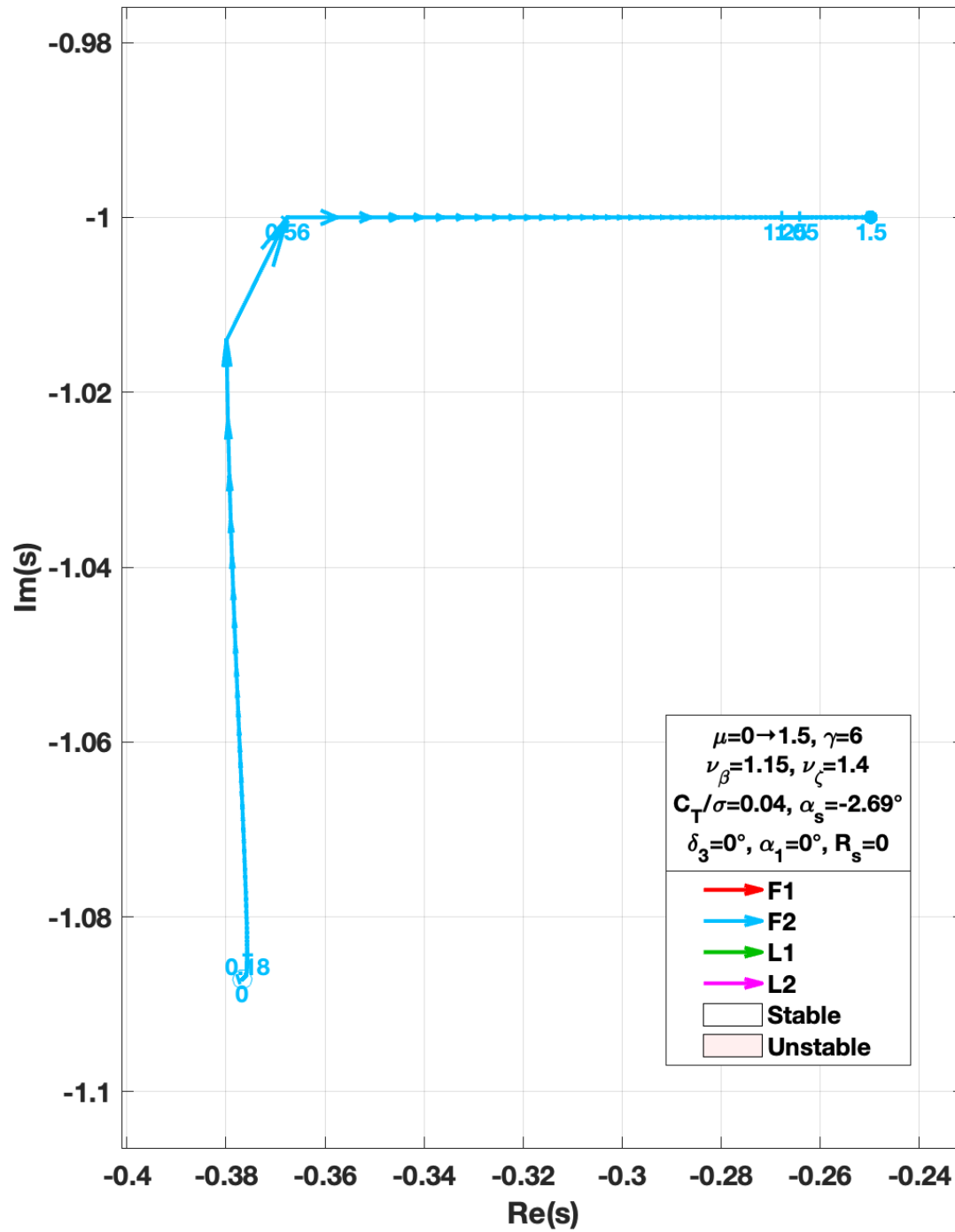


Figure 4.81: Flap-lag root locus, F2 root, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $\mu = 0$ to 1.5 .

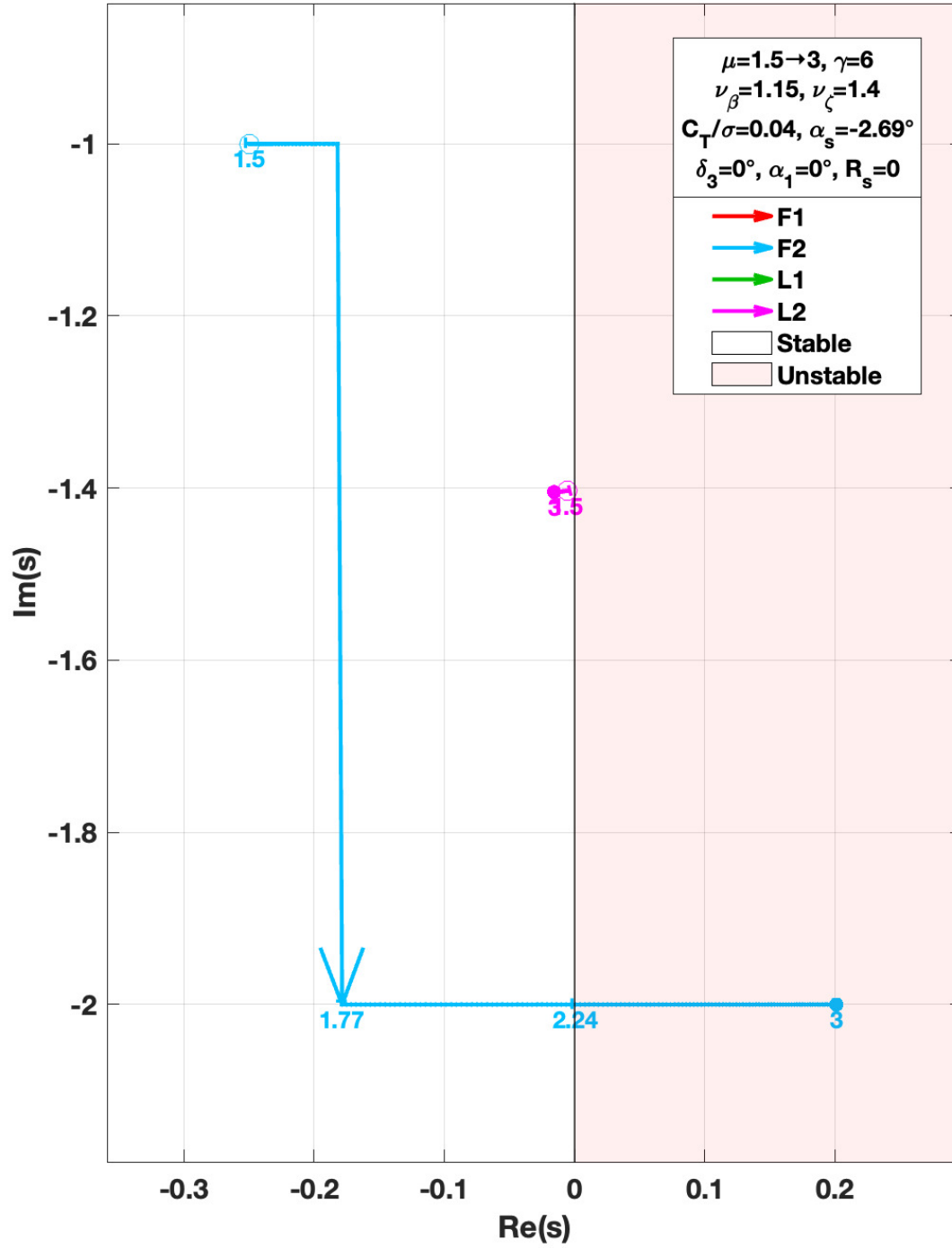


Figure 4.82: Flap-lag root locus, F2 root, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $\mu = 1.5$ to 3 .

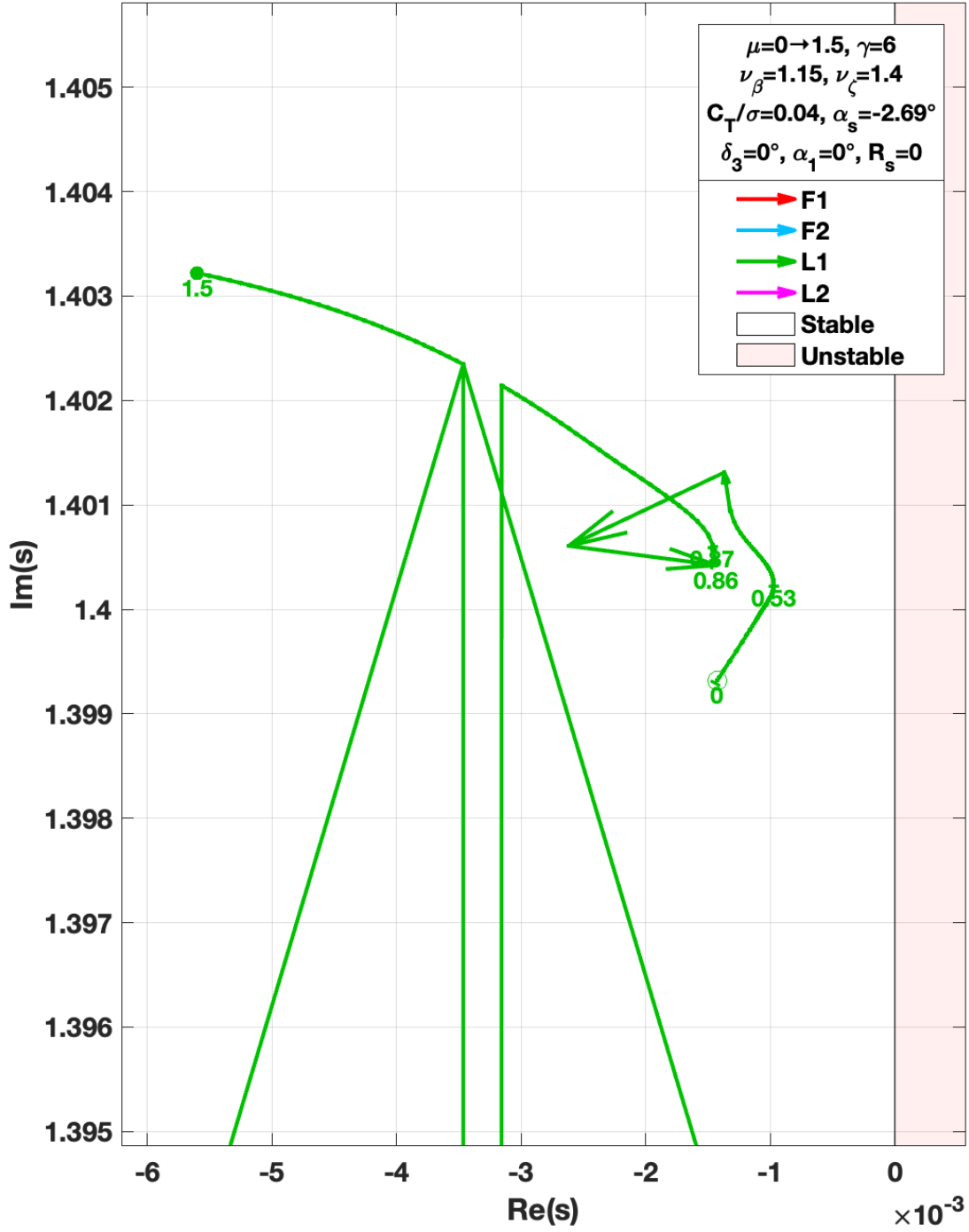


Figure 4.83: Flap-lag root locus, L1 root, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $\mu = 0$ to 1.5.

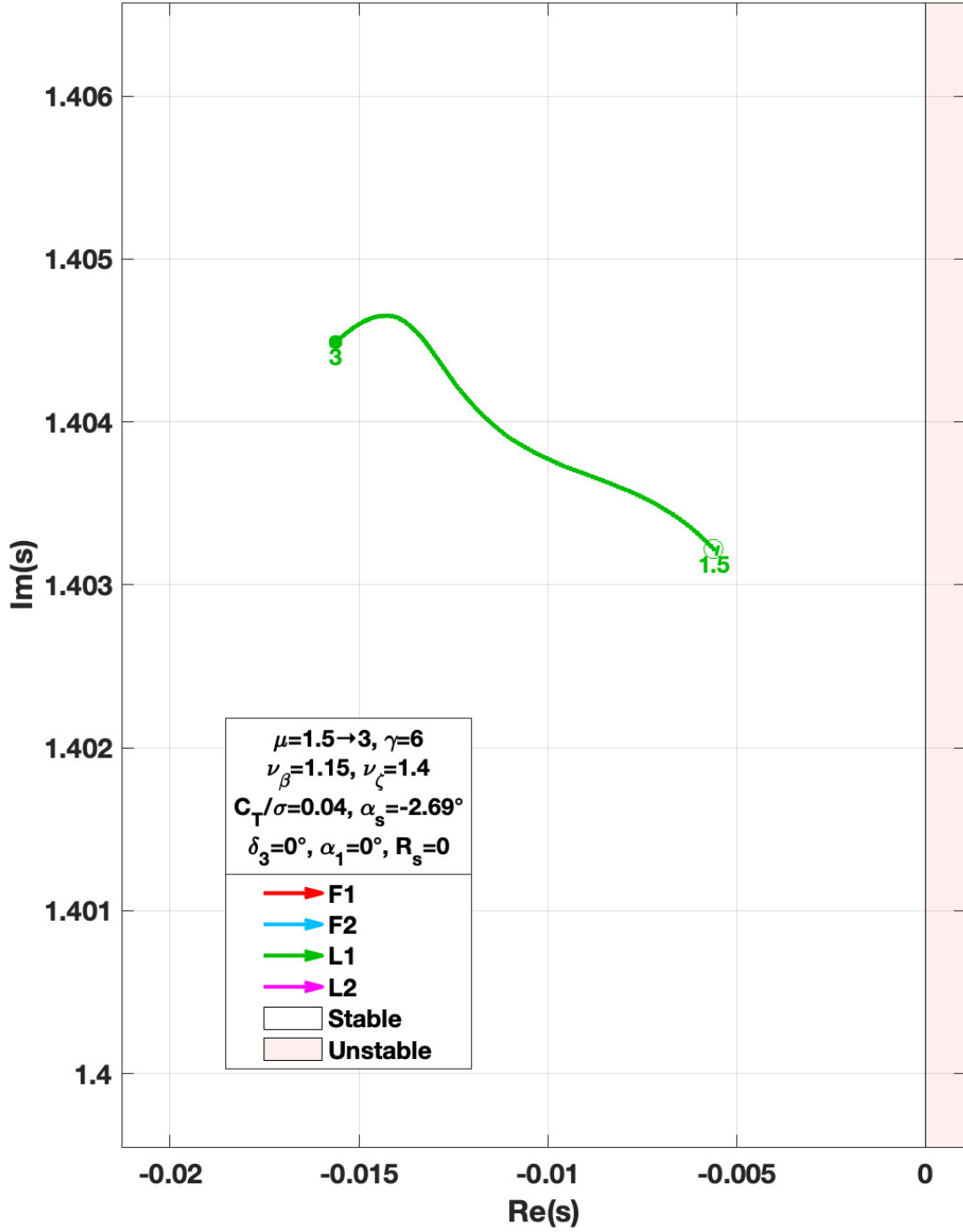


Figure 4.84: Flap-lag root locus, L1 root, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $\mu = 1.5$ to 3 .

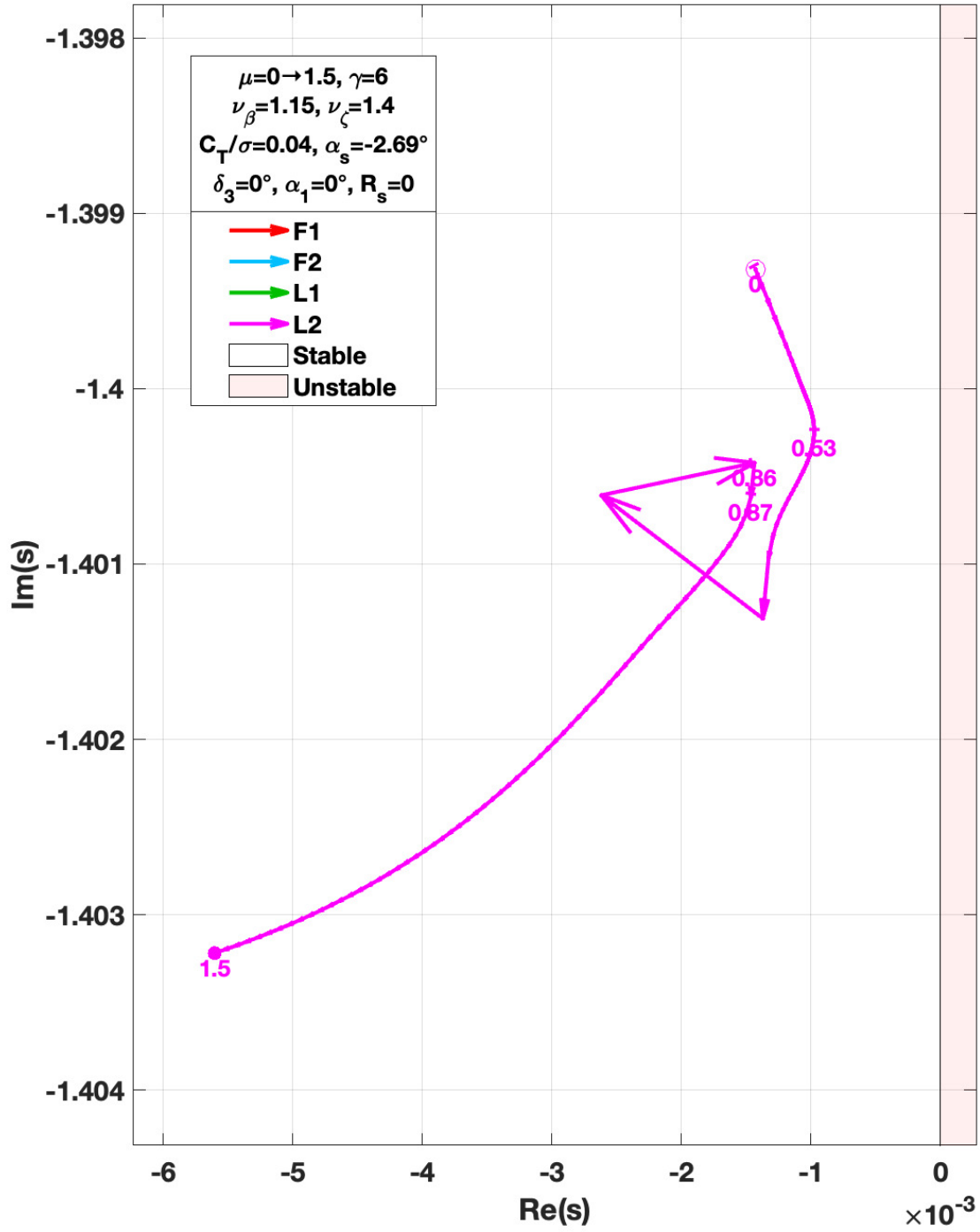


Figure 4.85: Flap-lag root locus, L2 root, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $\mu = 0$ to 1.5.

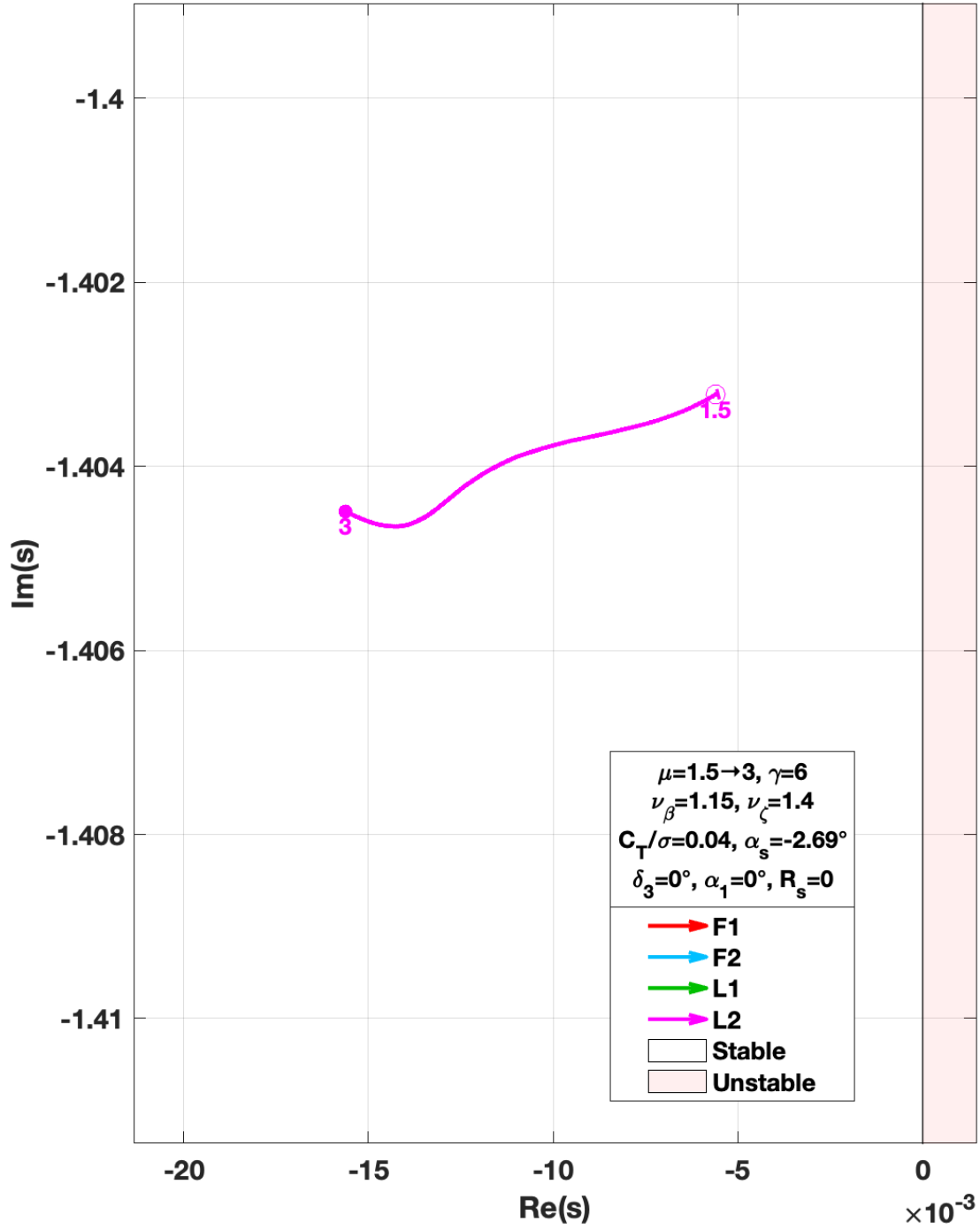


Figure 4.86: Flap-lag root locus, L2 root, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $\mu = 1.5$ to 3 .

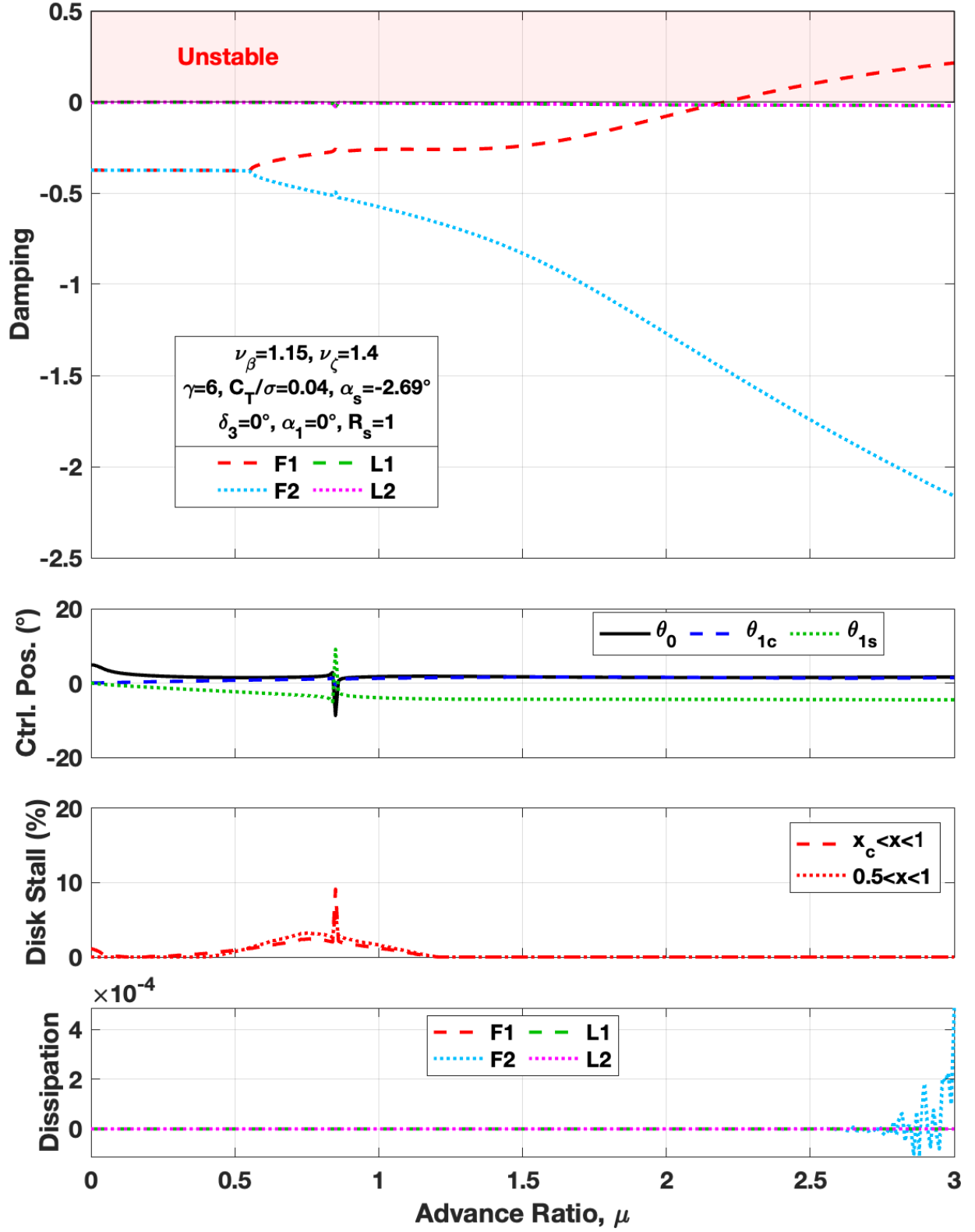


Figure 4.87: Flap-lag damping, trim, blade stall, and numerical stability, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $R_s = 1$.

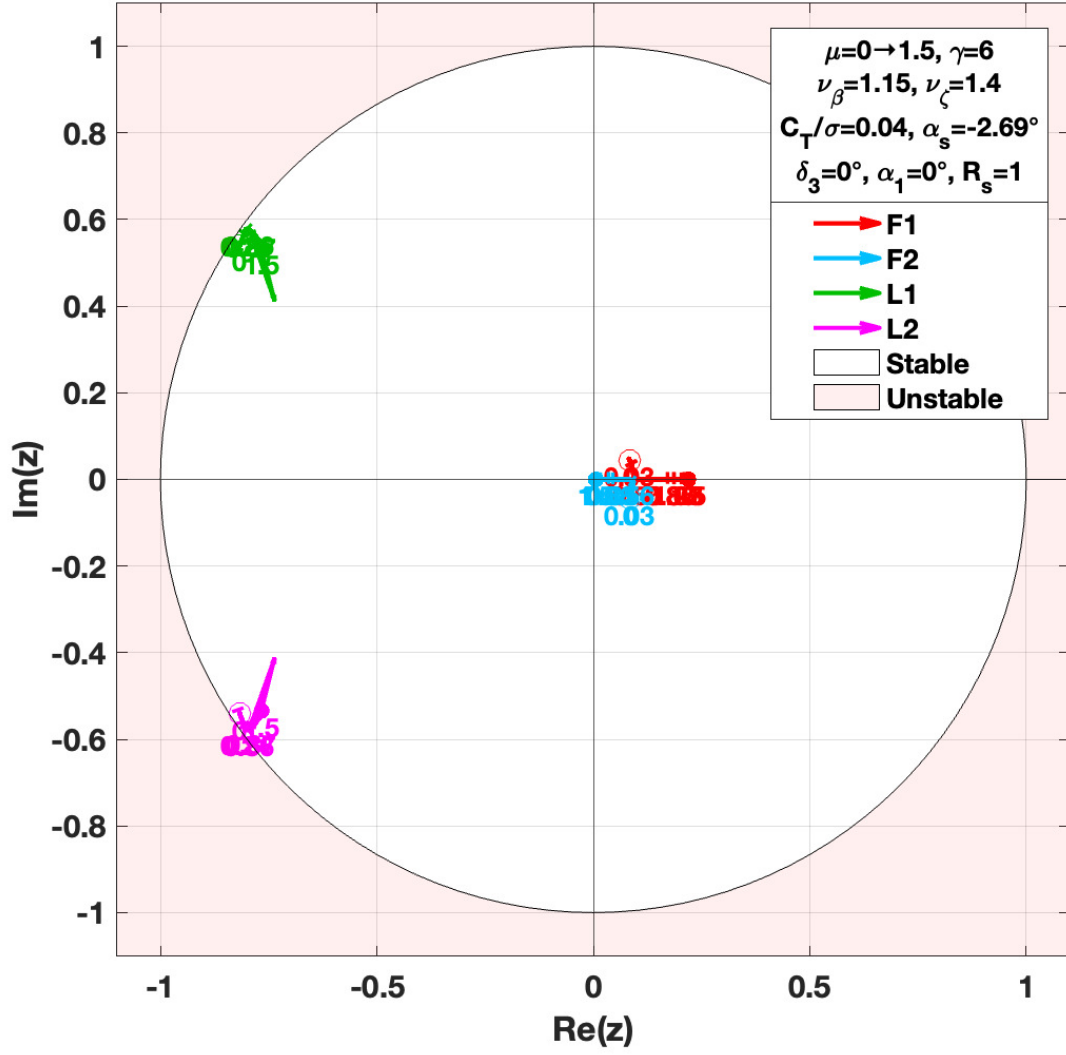


Figure 4.88: Flap-lag characteristic multipliers, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $R_s = 1$, $\mu = 0$ to 1.5 .

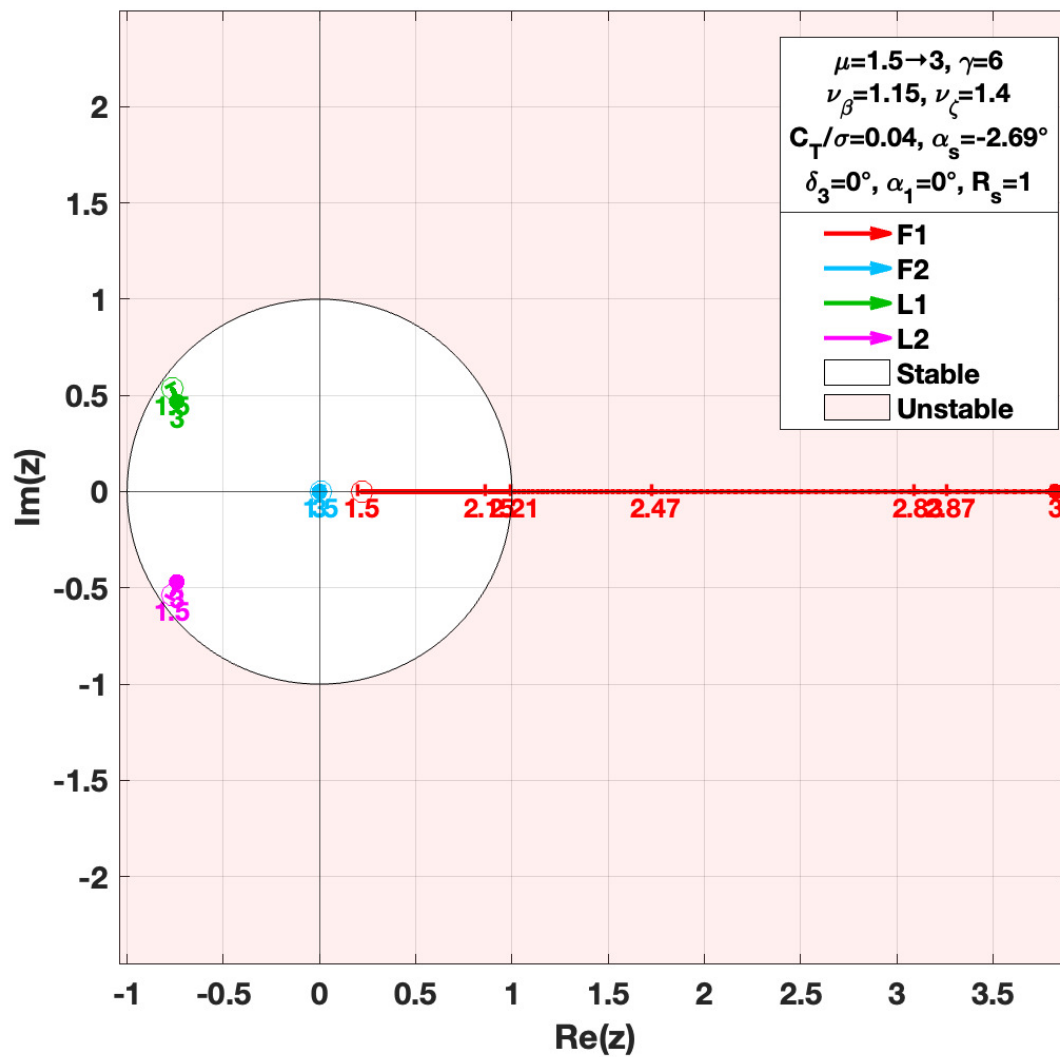


Figure 4.89: Flap-lag characteristic multipliers, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $R_s = 1$, $\mu = 1.5$ to 3 .

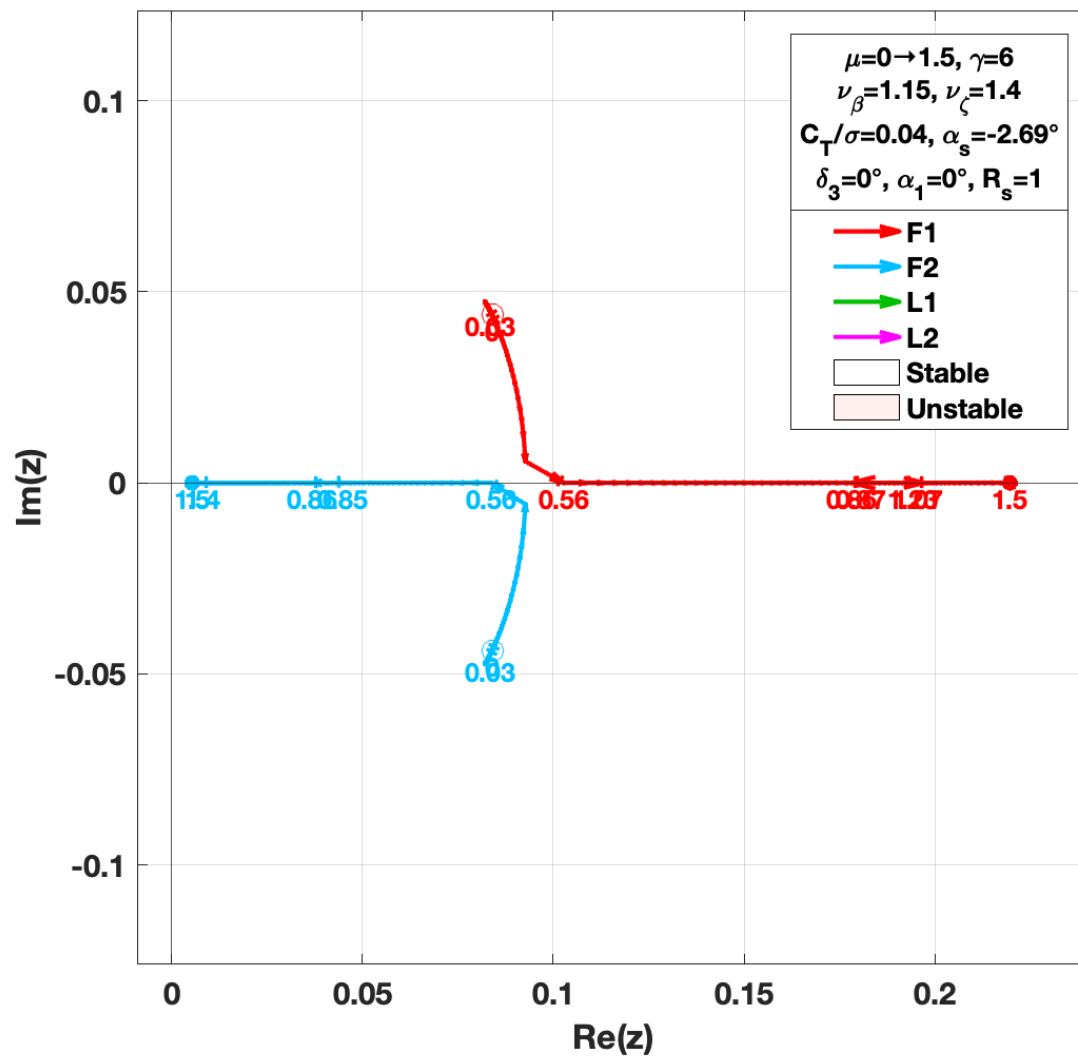


Figure 4.90: Flap-lag characteristic multipliers, F1 and F2 roots, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $R_s = 1$, $\mu = 0$ to 1.5 .

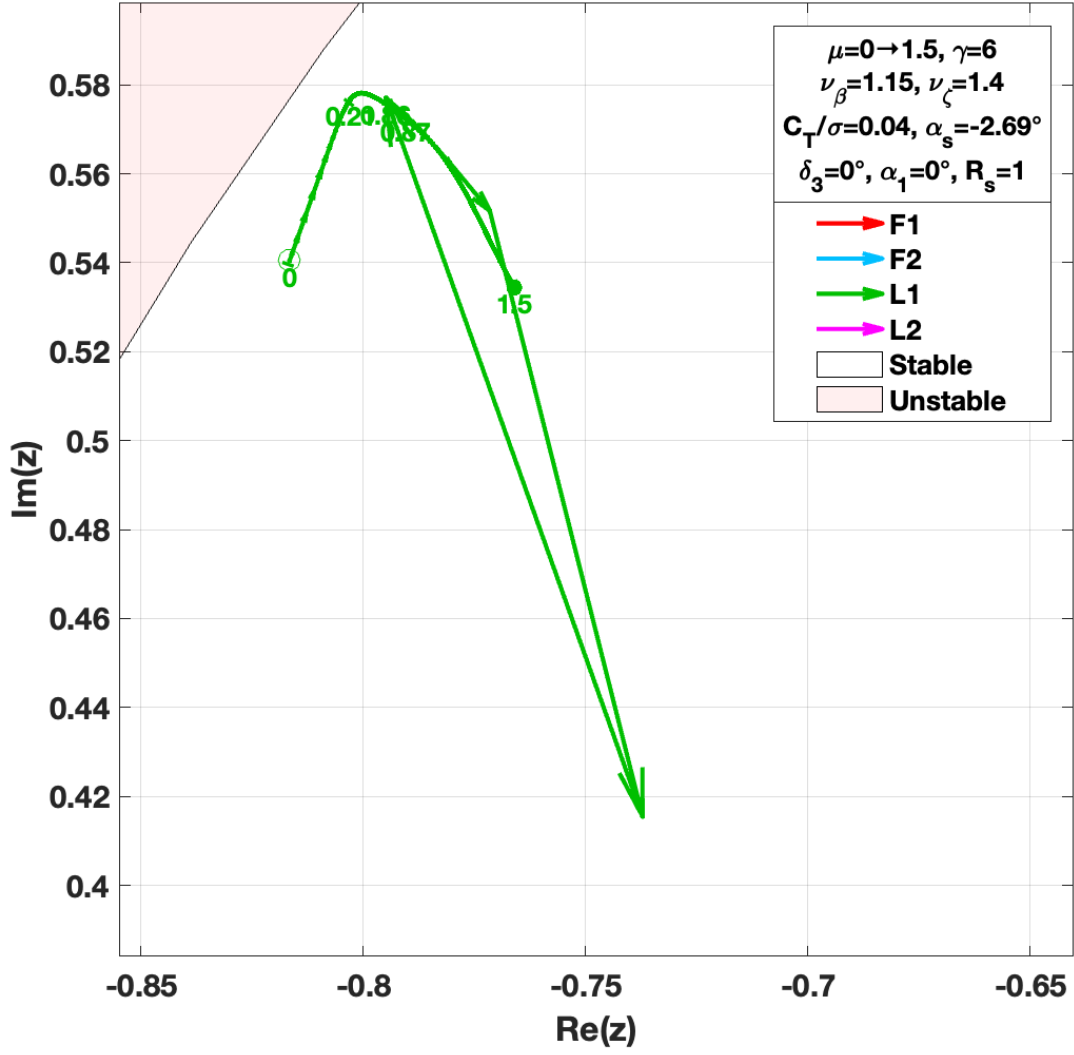


Figure 4.91: Flap-lag characteristic multipliers, L1 root, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $R_s = 1$, $\mu = 0$ to 1.5.

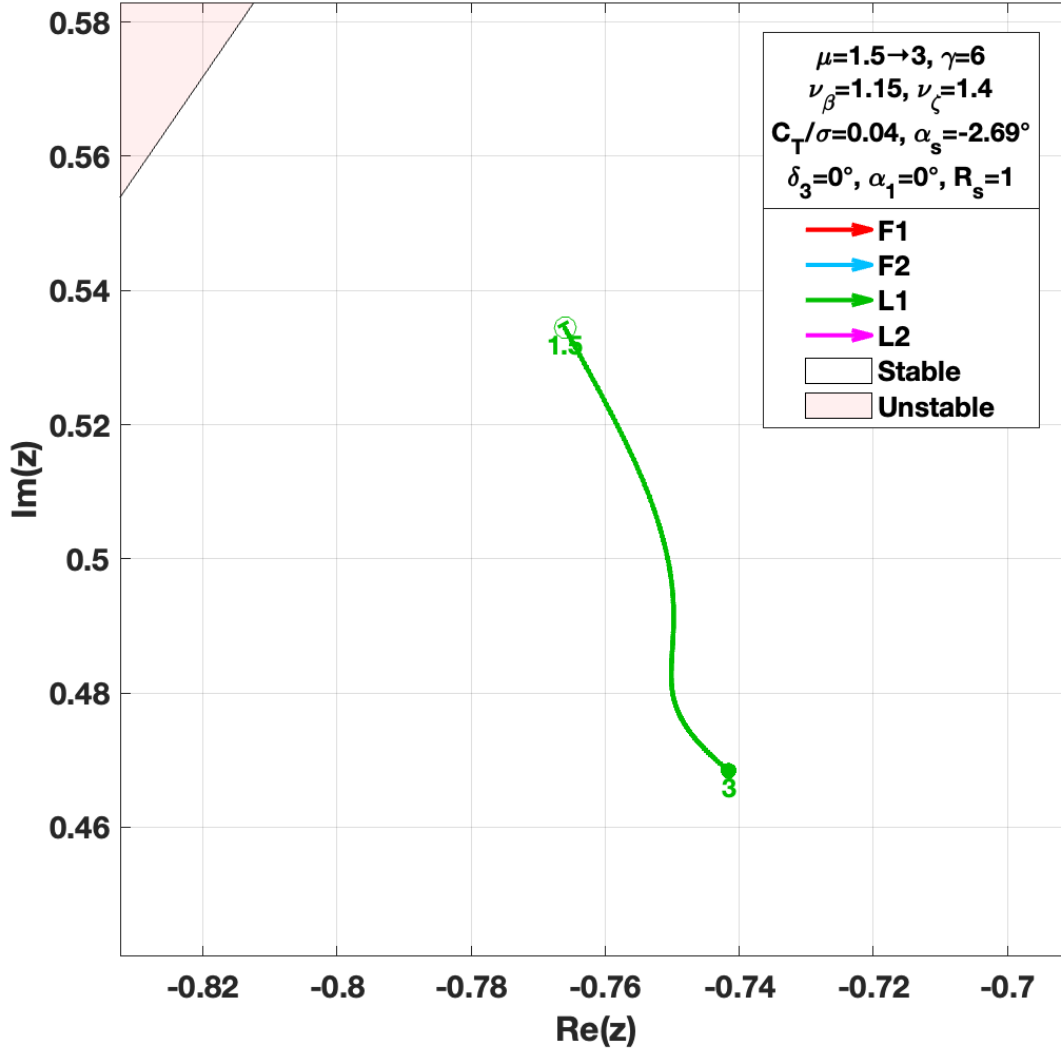


Figure 4.92: Flap-lag characteristic multipliers, L1 root, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $R_s = 1$, $\mu = 1.5$ to 3 .

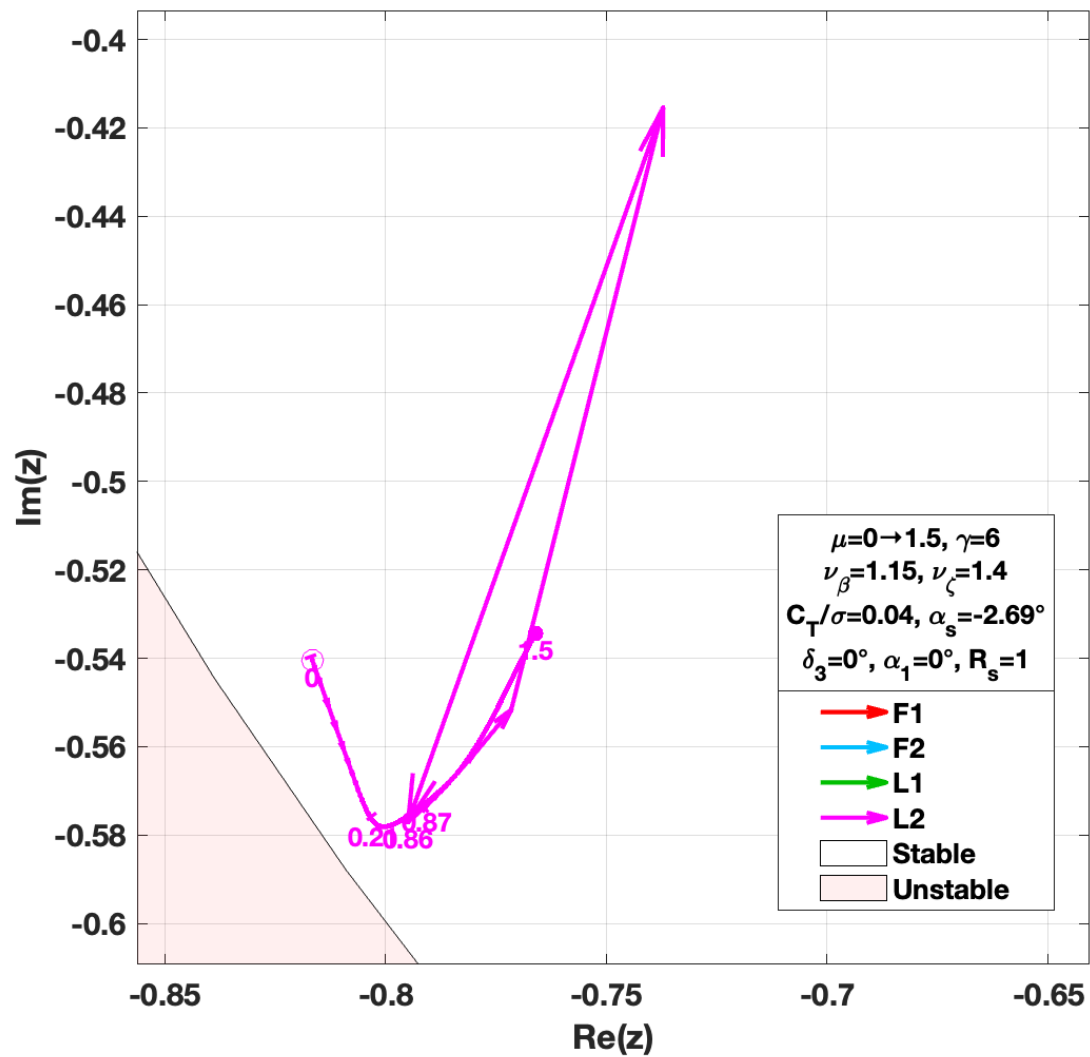


Figure 4.93: Flap-lag characteristic multipliers, L2 root, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $R_s = 1$, $\mu = 0$ to 1.5 .

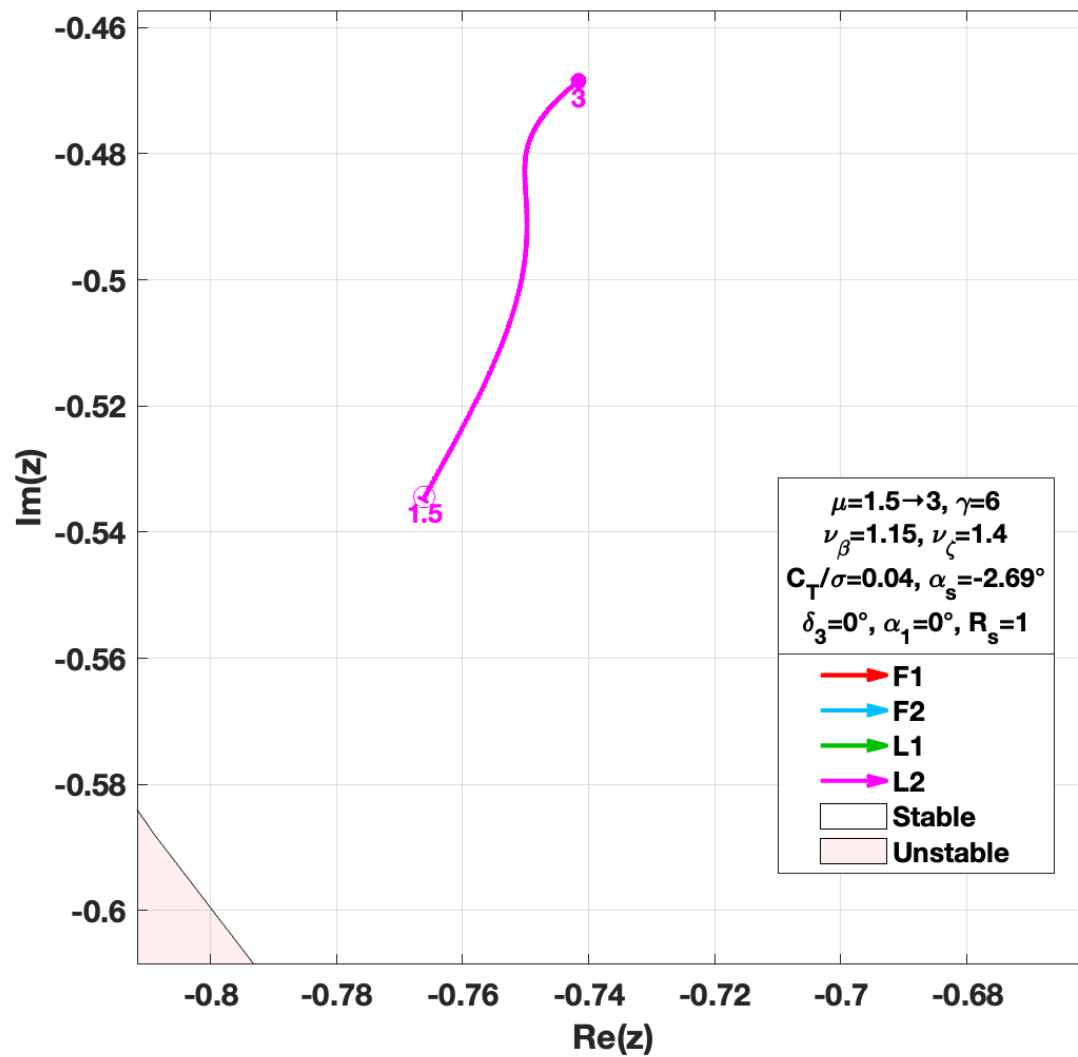


Figure 4.94: Flap-lag characteristic multipliers, L2 root, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $R_s = 1$, $\mu = 1.5$ to 3 .

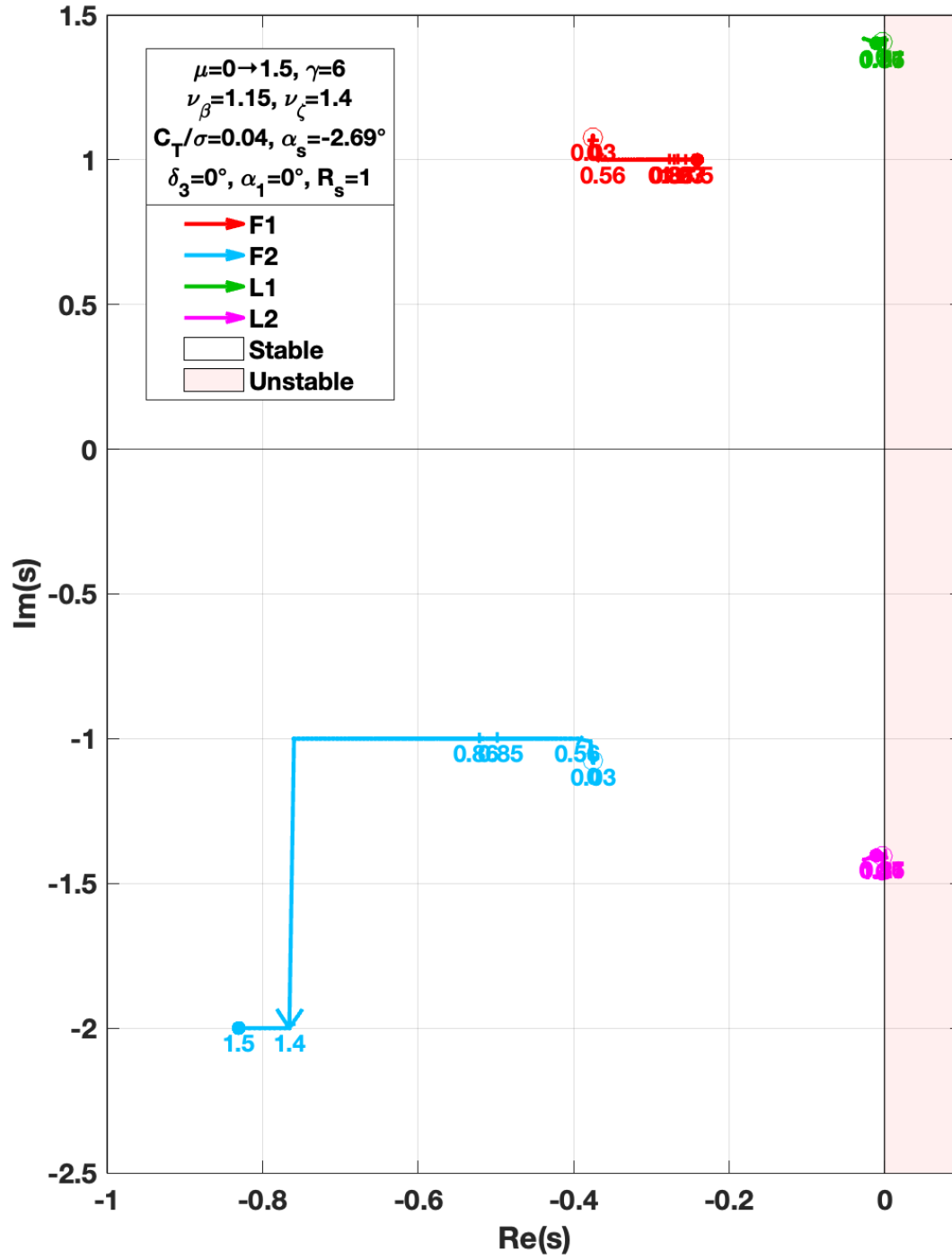


Figure 4.95: Flap-lag root locus, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $R_s = 1$, $\mu = 0$ to 1.5 .

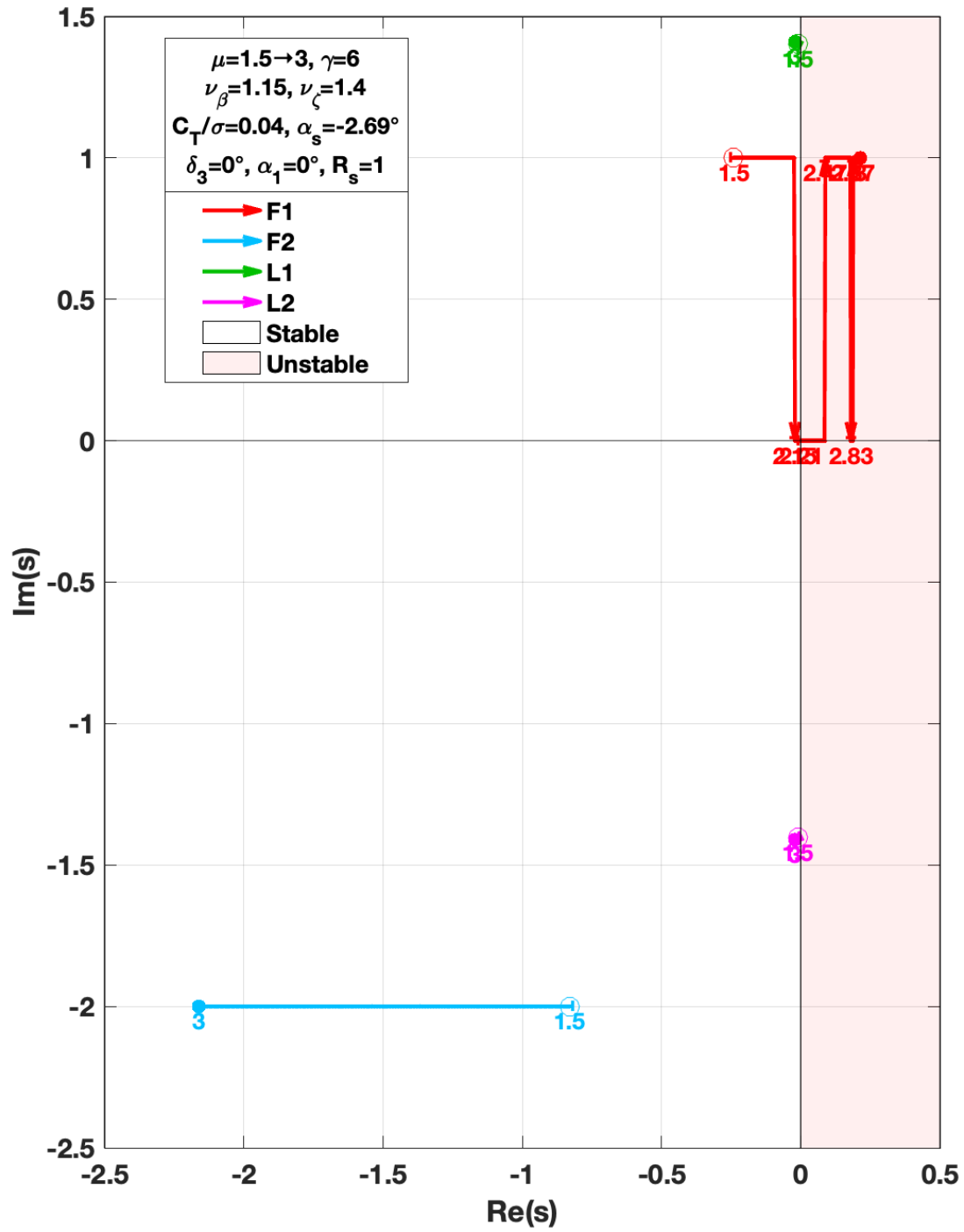


Figure 4.96: Flap-lag root locus, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $R_s = 1$, $\mu = 1.5$ to 3 .

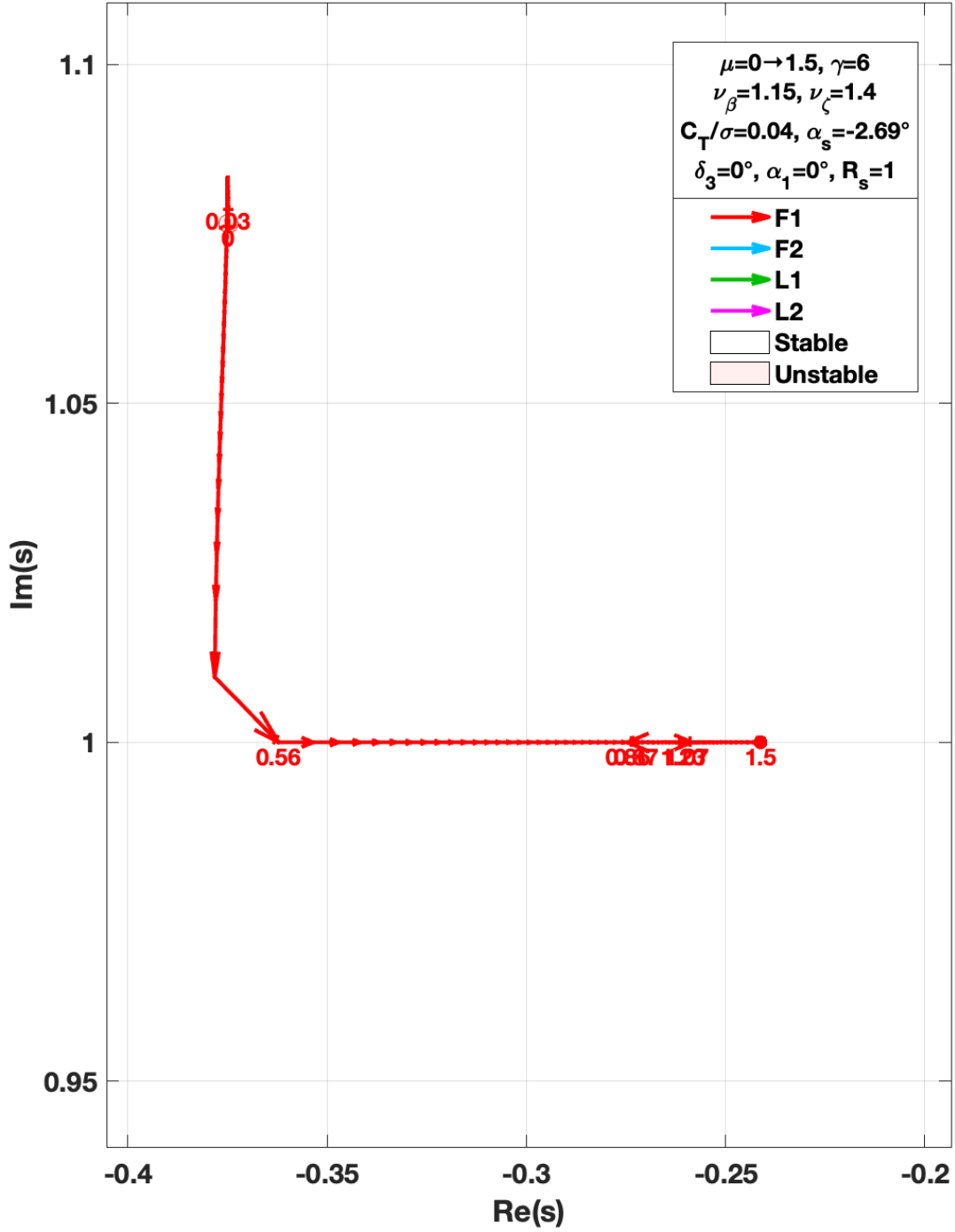


Figure 4.97: Flap-lag root locus, F1 root, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $R_s = 1$, $\mu = 0$ to 1.5 .

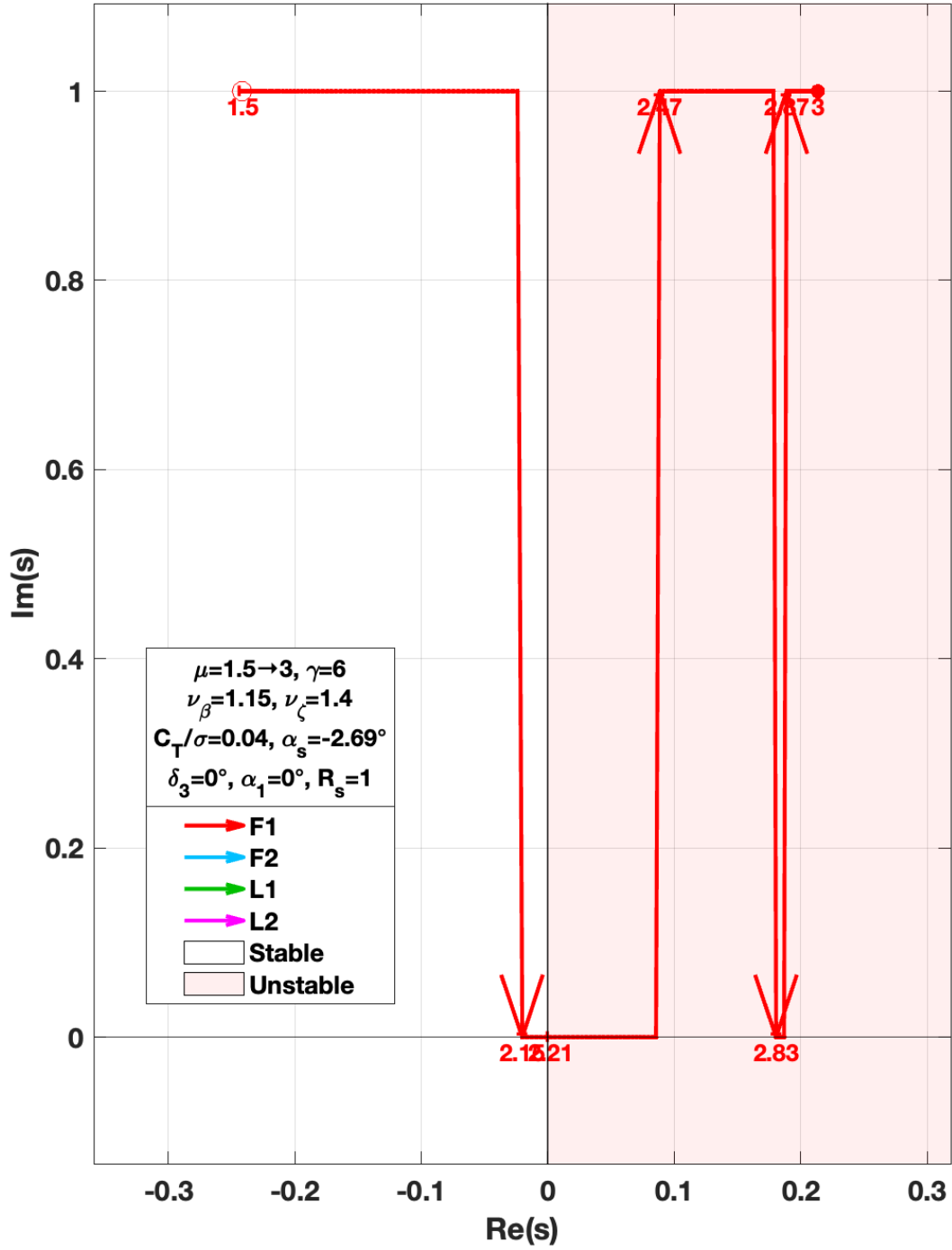


Figure 4.98: Flap-lag root locus, F1 root, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $R_s = 1$, $\mu = 1.5$ to 3 .

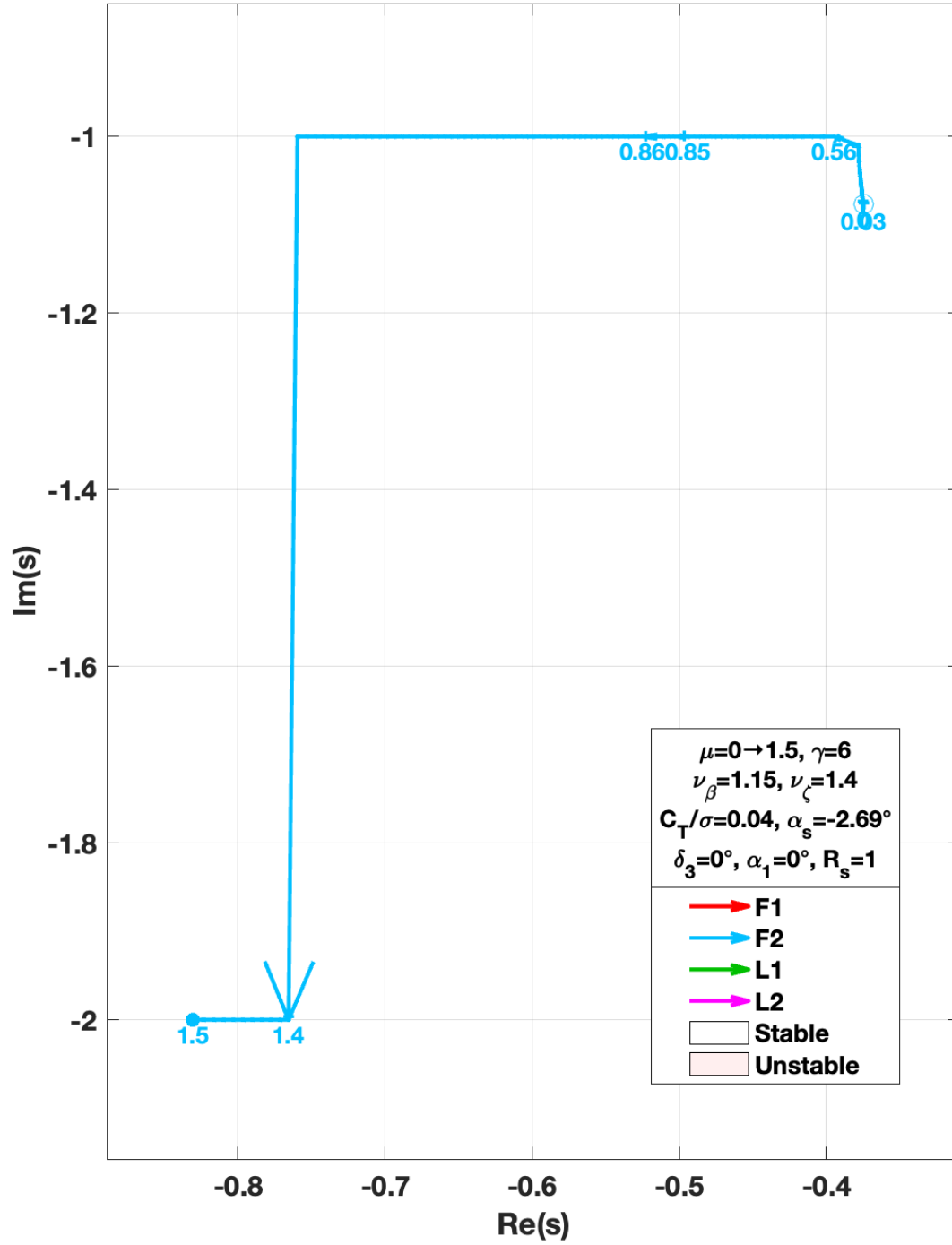


Figure 4.99: Flap-lag root locus, F2 root, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $R_s = 1$, $\mu = 0$ to 1.5 .

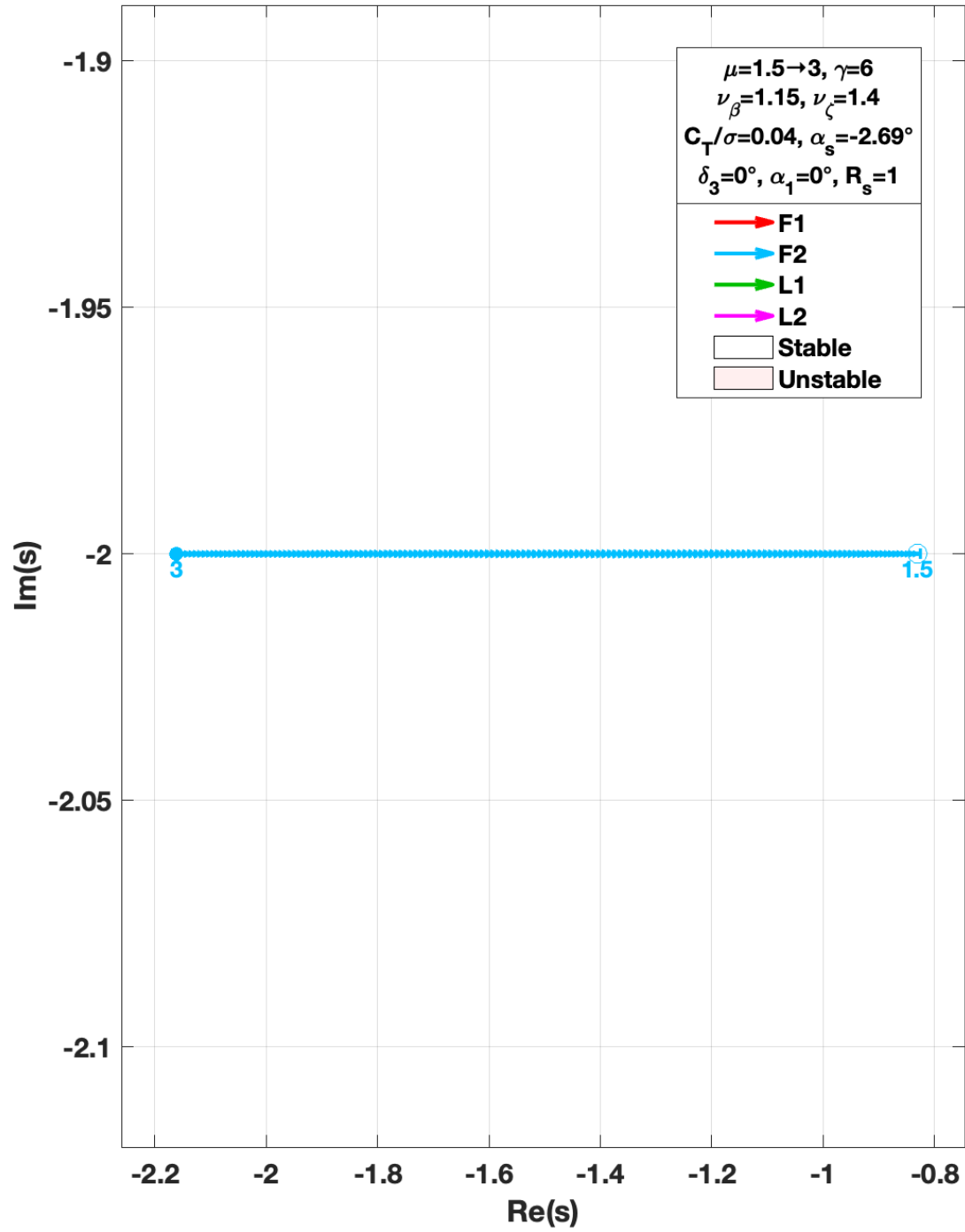


Figure 4.100: Flap-lag root locus, F2 root, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $R_s = 1$, $\mu = 1.5$ to 3 .

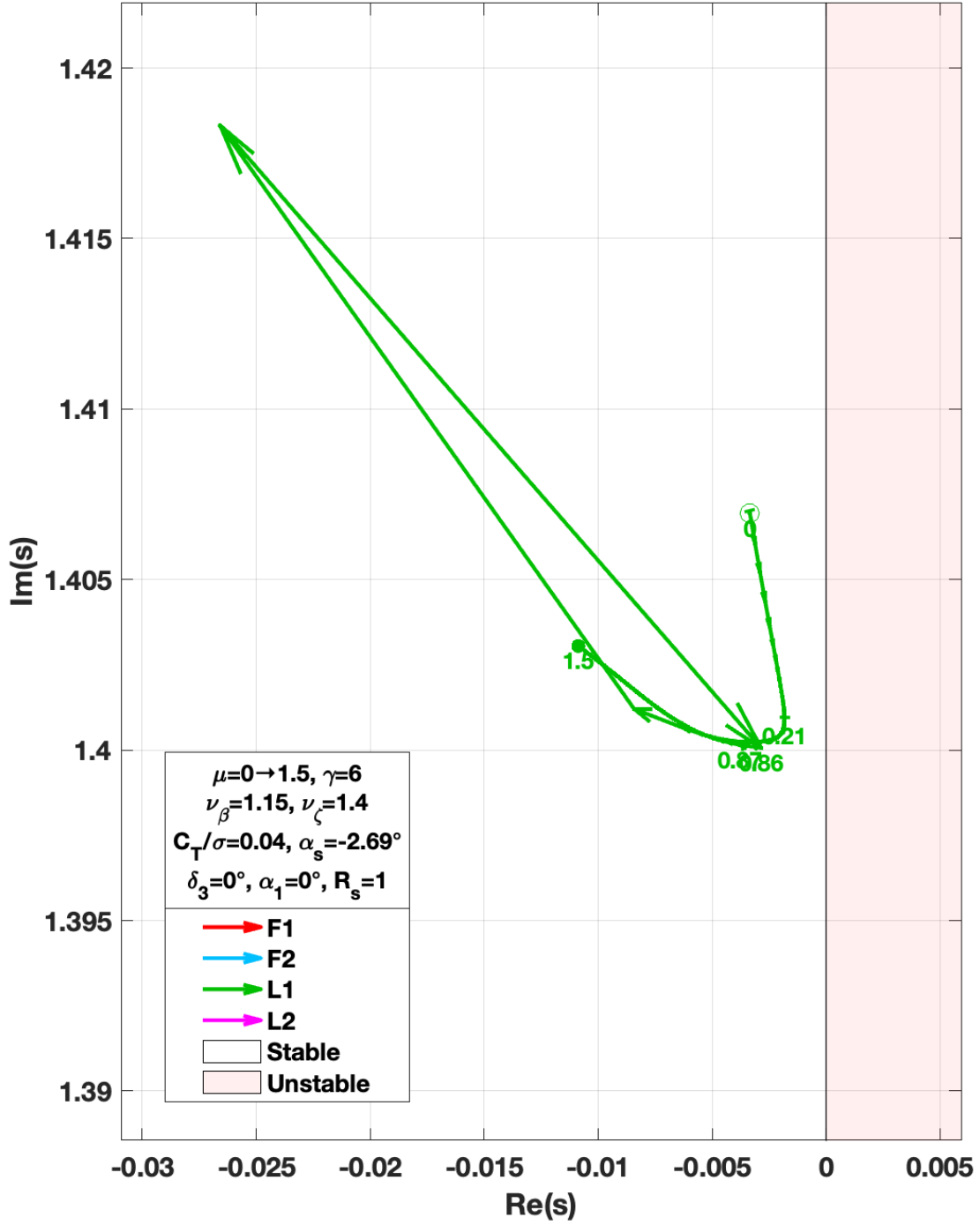


Figure 4.101: Flap-lag root locus, L1 root, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $R_s = 1$, $\mu = 0$ to 1.5 .

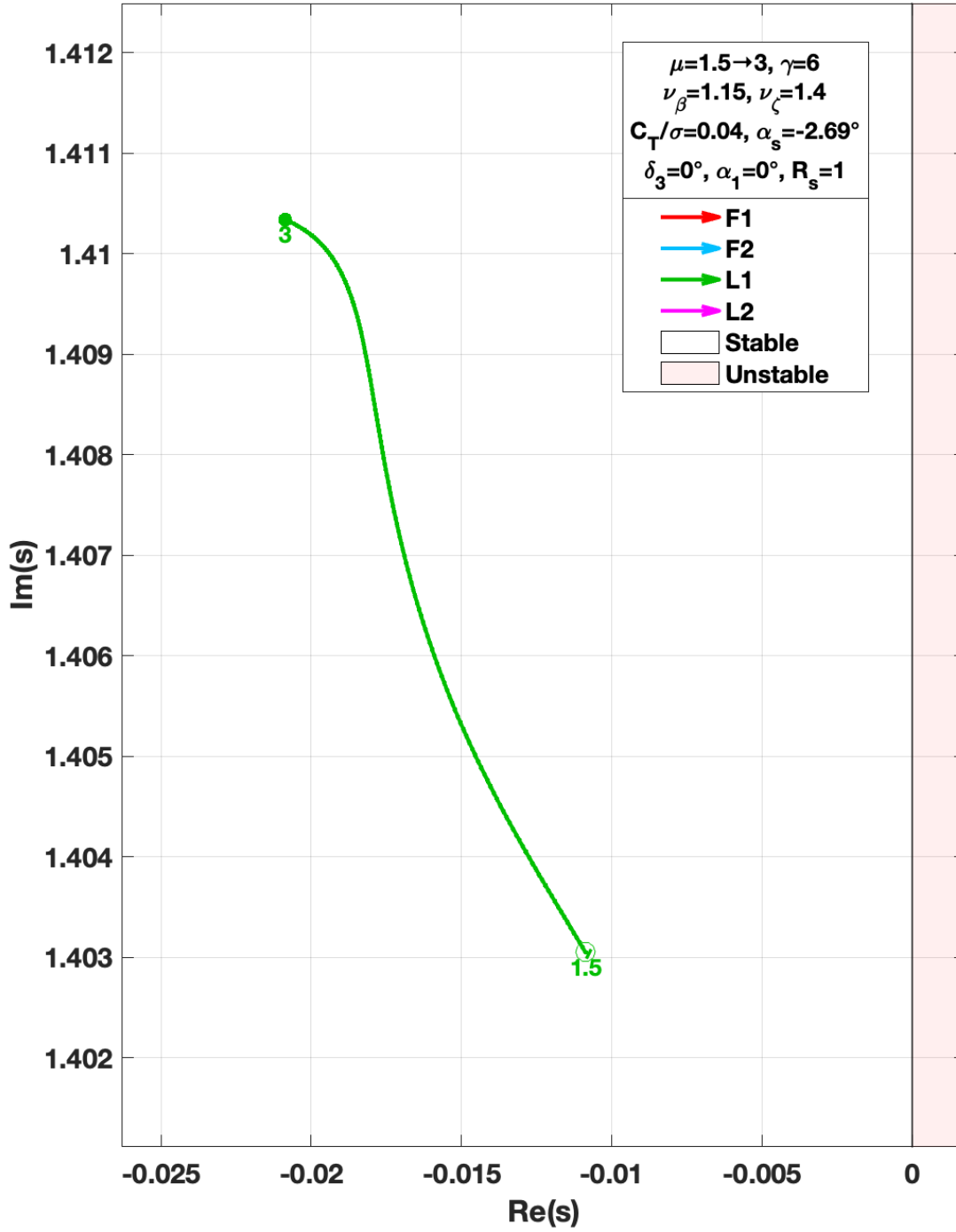


Figure 4.102: Flap-lag root locus, L1 root, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $R_s = 1$, $\mu = 1.5$ to 3 .

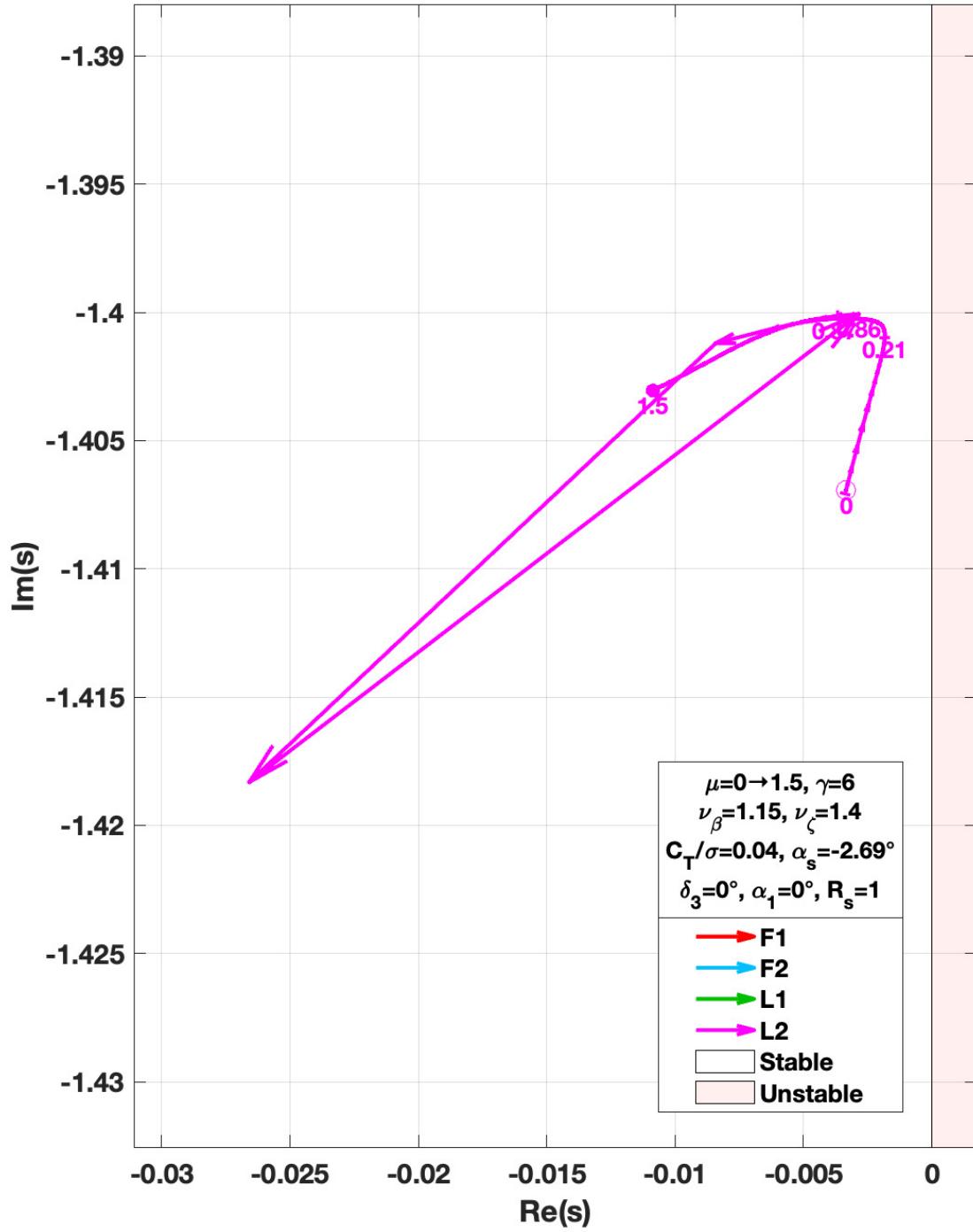


Figure 4.103: Flap-lag root locus, L2 root, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $R_s = 1$, $\mu = 0$ to 1.5.

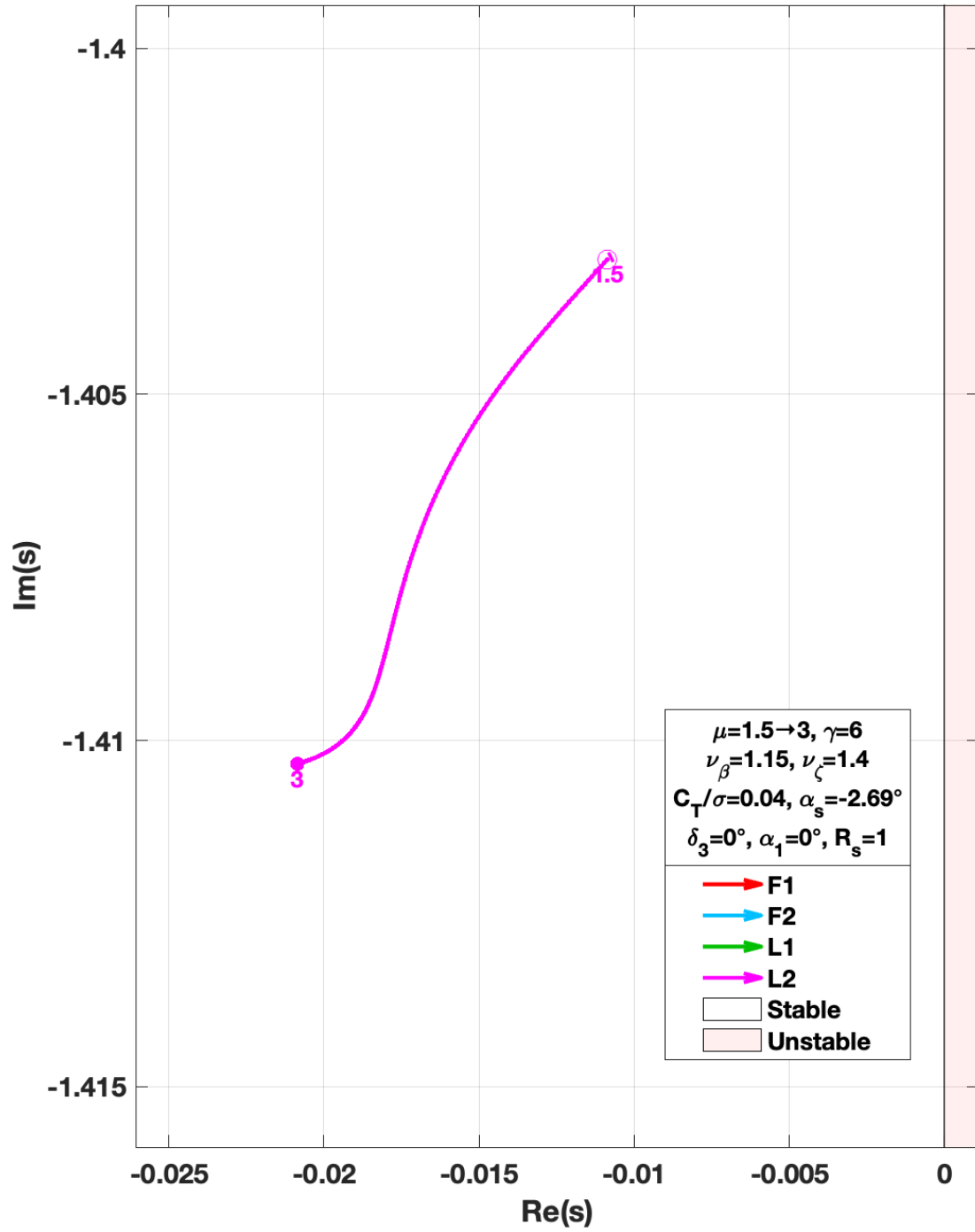


Figure 4.104: Flap-lag root locus, L2 root, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $R_s = 1$, $\mu = 1.5$ to 3 .

Correctly differentiating the flap and lag roots, especially when ν_β and ν_ζ are close and parametric resonance (frequency locking) is occurring, requires an eigenvalue shuffling algorithm that also a modal analysis of the eigenvector motion. Beginning with the time invariant case with zero thrust, then incrementally changing C_T/σ and μ , allows for the calculation of smooth root locus trajectories. Even when employing this scheme, it is challenging to differentiate between roots when flap-lag resonance and instability occur.

Chapter 5: Flap-Lag-Torsion Stability

5.1 Equations of Motion

Torsion is modeled with a rigid blade that has flexibility at the hub. The torsion moment is about the pitch axis, typically near the 1/4-chord position of the rotor blade. As before, flap and lag moments also act at the hub. The, flap, lag, and torsion aerodynamic forcing moments are

$$\begin{aligned}\bar{M}_\beta &= \int_{x_c}^1 x \frac{F_\beta}{\rho a c} dx \\ \bar{Q}_\zeta &= \int_{x_c}^1 x \frac{F_\zeta}{\rho a c} dx \\ \bar{T}_\theta &= \int_{x_c}^1 \frac{dM_\theta}{\rho a c}\end{aligned}\tag{5.1}$$

respectively.

Using thin airfoil theory, the non-dimensional forces and differential moment on a symmetrical airfoil are

$$\begin{aligned}\bar{F}_\beta &= \frac{1}{2} \left(u_T^2 \theta - \left(1 + \frac{c_d}{a} \right) u_P u_T \right) \\ \bar{F}_\zeta &= \frac{1}{2} \left(\frac{c_d}{a} u_T^2 + \theta u_P u_T - u_P^2 \right) \\ d\bar{M}_\theta &= -\frac{1}{2} \left(\frac{x_a}{c} \frac{c}{R} \right) \left(u_T^2 \theta - u_P u_T \right)\end{aligned}\tag{5.2}$$

The steady pitching moment is produced due to the chord-wise location of the quasisteady lift force. The arm x_a is the chordwise distance of the aerodynamic center (AC) distance aft of the pitch axis (PA) position. A positive x_a (AC forward of the PA) produces a negative dM_θ , which means a “nose-down” pitching moment. A collocated AC and PA ($x_a = 0$) eliminates the steady pitching moment. The blade radius to chord aspect ratio (or its inverse) is used in the nondimensionalization scheme. R/c is on the order of 15 to 20 for a typical rotor blade.

In reverse flow the aerodynamic center shifts to near the 3/4-chord measured from the leading edge, or 1/4-chord from the trailing edge. In this backwards orientation, the pitch axis is approximately at a 3/4-chord distance from the trailing edge. The periodically alternating x_a means an additional perturbation velocity is required; this is incorporated into the reverse flow force and moment equations where applicable.

In the reverse flow region, the notation for the forces and differential moment are

$$\begin{aligned}\bar{\tilde{F}}_\beta &= -\frac{1}{2}\frac{\hat{a}}{a}\left(u_T^2\theta - \left(1 + \frac{\hat{c}_d}{\hat{a}}\right)u_P u_T\right) \\ \bar{\tilde{F}}_\zeta &= -\frac{1}{2}\frac{\hat{a}}{a}\left(\frac{\hat{c}_d}{\hat{a}}u_T^2 + \theta u_P u_T - u_P^2\right) \\ d\bar{\tilde{M}}_\theta &= \frac{1}{2}\frac{\hat{a}}{a}\left(\frac{\hat{x}_a}{c}\frac{c}{R}\right)\left(u_T^2\theta - u_P u_T\right)\end{aligned}\tag{5.3}$$

As in the flap-lag case, stability is analyzed about the flap trim state. The flap, lag,

and pitch perturbations are

$$\begin{aligned}
\beta &= \beta_T + \delta\beta \\
\zeta &= \zeta_T + \delta\zeta \approx \delta\zeta \\
\theta &= \theta_T + \delta\theta
\end{aligned} \tag{5.4}$$

where the trimmed blade pitch is

$$\theta_T = \theta_C + x\theta_{tw} - k_{p\beta}\beta_T \tag{5.5}$$

The total pitch perturbation is the sum of the torsion and pitch-flap/ pitch-lag perturbations:

$$\delta\theta = \theta - k_{p\beta}\delta\beta - k_{p\zeta}\delta\zeta \tag{5.6}$$

5.2 Incorporation of Unsteady Aerodynamics

Due to the unsteady motion of the blade, the lift deficiency function $C(k)$ is appended to the quasisteady flap and lag perturbation forces:

$$\begin{aligned}
\delta\bar{F}_\beta &= \frac{1}{2}C(k)\left(\delta u_T\left(2u_T\theta_T - \left(1 + \frac{c_d}{a}\right)u_P\right) \right. \\
&\quad \left. - \left(1 + \frac{c_d}{a}\right)\delta u_P u_T + \delta\theta u_T^2\right) \\
\delta\bar{F}_\zeta &= \frac{1}{2}C(k)\left(\delta u_T\left(2u_T\frac{c_d}{a} + u_P\theta_T\right) \right. \\
&\quad \left. + \delta u_P\left(u_T\theta_T - 2u_P\right) + \delta\theta\left(u_P u_T\right)\right)
\end{aligned} \tag{5.7}$$

The pitching moment perturbation equation considers forces acting at both the 1/4-chord (L_1) and 3/4-chord (L_3). The former is simply the quasisteady lift with a moment arm x_a , and is affected by $C(k)$. The latter is a non-circulatory term caused by the rate of change of the angle of attack (pitch rate), $\dot{\theta}^*$.

$$\begin{aligned}\delta d\bar{M}_\theta = & -\frac{1}{2}C(k)\left(\frac{x_a}{c}\frac{c}{R}\right)\left(\delta u_T\left(2u_T\theta_T - u_P\right)\right. \\ & \left. - \delta u_P u_T + \delta\theta u_T^2\right) - \frac{1}{8}\left(\frac{c}{R}\right)^2\left(\frac{1}{2} + \frac{x_a}{c}\right)u_T\dot{\theta}^*\end{aligned}\quad (5.8)$$

In reverse flow,

$$\begin{aligned}\delta\bar{F}_\beta = & -\frac{1}{2}\frac{\hat{a}}{a}C(k)\left(\delta u_T\left(2u_T\theta_T - \left(1 + \frac{\hat{c}_d}{\hat{a}}\right)u_P\right)\right. \\ & \left. - \left(1 + \frac{\hat{c}_d}{\hat{a}}\right)\delta\hat{u}_P u_T + \delta\theta u_T^2\right) \\ \delta\bar{F}_\zeta = & -\frac{1}{2}\frac{\hat{a}}{a}C(k)\left(\delta u_T\left(2u_T\frac{\hat{c}_d}{\hat{a}} + u_P\theta_T\right)\right. \\ & \left. + \delta\hat{u}_P\left(u_T\theta_T - 2u_P\right) + \delta\theta\left(u_P u_T\right)\right) \\ \delta d\bar{M}_\theta = & -\frac{1}{2}\frac{\hat{a}}{a}C(k)\left(\frac{\hat{x}_a}{c}\frac{c}{R}\right)\left(\delta u_T\left(2u_T\theta_T - u_P\right)\right. \\ & \left. - \delta\hat{u}_P u_T + \delta\theta u_T^2\right) - \frac{1}{8}\frac{\hat{a}}{a}\left(\frac{c}{R}\right)^2\left(\frac{1}{2} + \frac{\hat{x}_a}{c}\right)u_T\dot{\theta}^*\end{aligned}\quad (5.9)$$

The perturbation velocities are

$$\begin{aligned}\delta u_T = & -x\zeta^* - \delta\zeta\mu\cos\psi \\ \delta u_P = & x\delta\beta^* + \delta\beta\mu\cos\psi - \frac{c}{R}\left(\frac{1}{2} + \frac{x_a}{c}\right)\delta\dot{\theta}^* \\ \delta\hat{u}_P = & x\delta\beta^* + \delta\beta\mu\cos\psi - \frac{c}{R}\left(\frac{1}{2} + \frac{\hat{x}_a}{c}\right)\delta\dot{\theta}^*\end{aligned}\quad (5.10)$$

The $\dot{\theta}^*$ term in the δu_P equation is another L_3 contribution from unsteady aerodynamics.

The aerodynamic center and pitch axis are both near the 1/4-chord position for normal flow, and x_a will be near zero. In this normal case, the overall sign of the $\dot{\theta}^*$ term will remain negative, meaning the blade pitch rate will decrease. This is the main source of torsional damping. In reverse flow, x_a moves to approximately $-1/2$, and the pitch damping term is either very small or vanishes completely.

It might be observed that neither the pitching moment coefficient c_m or its slope $c_{m\alpha}$ have been directly included in the equations of motion. Recalling Fig. 1.4, these seem especially important in reverse flow. However, the thin airfoil theory method essentially derives these values by accounting for the AC and PA offsets, while also providing a mechanism to model pitch damping.

Some unsteady terms have been neglected in this analysis. For flap, the unsteady forces L_2 (vertical acceleration of the mid-chord) and L_3 are considered small compared to L_1 . In torsion, the effects of L_2 (vertical acceleration of the mid-chord) and the non-circulatory moment due to pitch acceleration are not considered. These would yield small $\ddot{\beta}^*$ and $\ddot{\theta}^*$ terms, which would affect the mass matrix $\underline{\underline{M}}$.

5.3 Dynamic Inflow and the Lift Deficiency Function

There are many choices for the lift deficiency function, but a simple model is desired to match with the level of this analysis. The notation $C(k)$ used above usually refers to the Theodorsen lift deficiency function, the pioneering quantification of a pitching and plunging airfoil in general. Theodorsen's approach uses Bessel functions that causes a decrease in C below 1 depending on the reduced frequency $k = \frac{\omega c}{2U}$, along with an azimuth phase

(time) lag. Although Loewy and others have developed lift deficiency functions catered towards unsteady rotary wings, including the effects of μ have been challenging. This is especially true at advance ratios greater than 1 where many assumptions collapse. Utilizing the dynamic inflow approach introduced by Pitt and Peters in Refs. 61 and 62 results in an inflow perturbation $\delta\lambda$. The consequent reduction in thrust takes the form of a lift deficiency function:

$$C' = \frac{1}{1 + \frac{\sigma a}{8V_{eff}}} \quad (5.11)$$

where

$$V_{eff} = \frac{\mu^2 + \lambda(\lambda + \lambda_i)}{\sqrt{(\mu^2 + \lambda^2)}} \quad (5.12)$$

In hover, the present C' definition appears similar to Loewy's lift deficiency function (also commonly notated as C') simplified to order 1, neglecting k due to a small ω . The difference is that Pitt and Peters' V_{eff} reduces to 2λ when $\mu = 0$; Loewy's definition is just λ .

This version of C' is substituted for $C(k)$ in the previous equations when dynamic inflow is considered in the present work. This method effectively reduces the Lock number for the circulatory terms due to the wake. Although it is a simple approximation, the general wake spacing effects and low-frequency lift reduction are captured. There are also perturbations to the pitch and roll hub moments, but these terms are neglected in the present case because the corresponding longitudinal and lateral flapping are trimmed to zero (i.e. wind tunnel trim).

Except where noted in the subsequent analyses, $C(k) = 1$. As can be seen in Fig. 5.1 for a stiff-in-plane hingeless rotor, the Pitt-Peters dynamic inflow/ lift deficiency term decreases flap damping most noticeably when $\mu < 0.1$. At the high advance ratios of

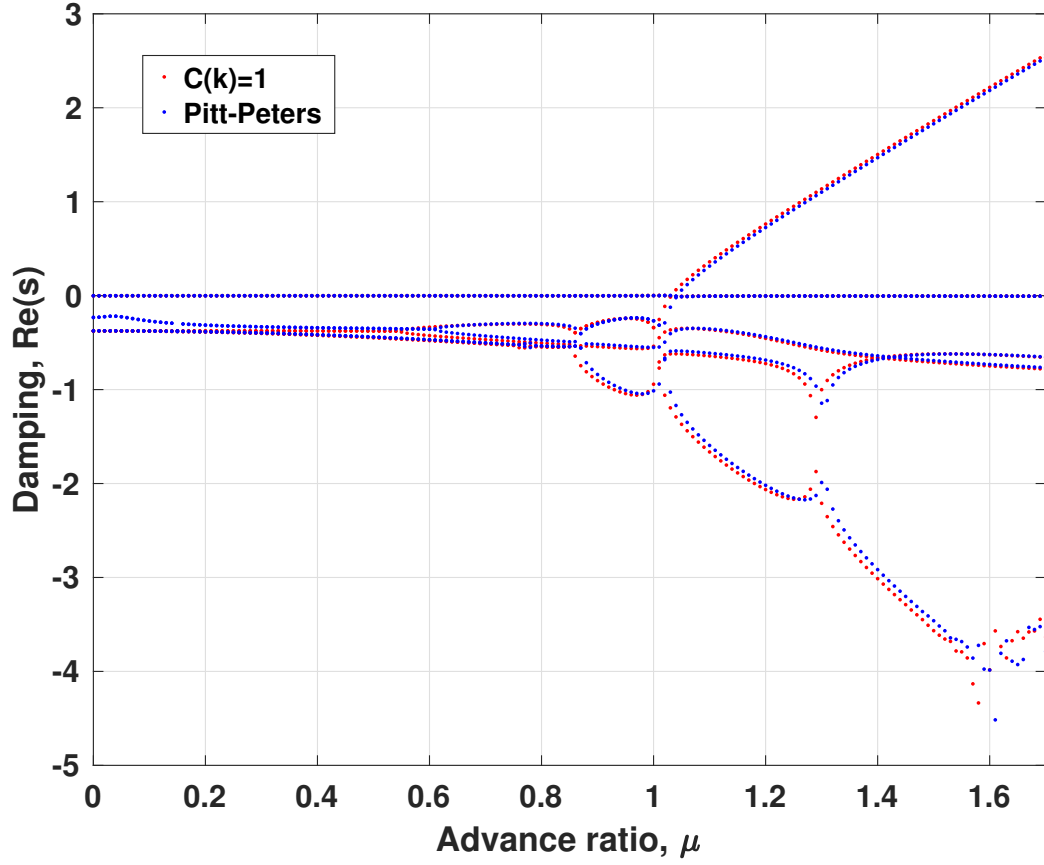


Figure 5.1: Effect of lift deficiency function and dynamic inflow on flap-lag-torsion damping, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$.

interest ($\mu > 0.5$), the inflow perturbation effects are small and even slightly stabilizing.

5.4 Assembly of the Coefficient Matrix

The reverse flow regions and integration limits are as defined in the flap and flap-lag cases. The aerodynamic forces and differential moment are integrated to find the moments

for each azimuth block, and collected as follows:

$$\begin{aligned}
\bar{M}_\beta &= M_\beta^* \beta + M_\zeta^* \zeta + M_\theta^* \theta + M_\beta \beta + M_\zeta \zeta + M_\theta \theta \\
\bar{Q}_\zeta &= Q_\beta^* \beta + Q_\zeta^* \zeta + Q_\theta^* \theta + Q_\beta \beta + Q_\zeta \zeta + Q_\theta \theta \\
\bar{T}_\theta &= T_\beta^* \beta + T_\zeta^* \zeta + T_\theta^* \theta + T_\beta \beta + T_\zeta \zeta + T_\theta \theta
\end{aligned} \tag{5.13}$$

Re-arranging the system of equations into its homogeneous perturbation form,

$$\underline{\underline{M}} \delta \underline{\underline{q}} + \underline{\underline{C}} \delta \underline{\underline{q}} + \underline{\underline{K}} \delta \underline{\underline{q}} = \underline{0} \tag{5.14}$$

where

$$\delta \underline{\underline{q}} = \begin{pmatrix} \delta \beta \\ \delta \zeta \\ \delta \theta \end{pmatrix}, \delta \underline{\underline{q}}^* = \begin{pmatrix} \delta \beta^* \\ \delta \zeta^* \\ \delta \theta^* \end{pmatrix} \tag{5.15}$$

The linearized coefficient matrix is

$$\underline{\underline{A}} = \begin{pmatrix} \underline{0} & \underline{I} \\ -\underline{\underline{M}}^{-1} \underline{\underline{K}} & -\underline{\underline{M}}^{-1} \underline{\underline{C}} \end{pmatrix} \tag{5.16}$$

The component matrices are all 3 by 3, combining to form a 6 by 6 coefficient matrix.

Like the flap frequency, in the rigid model the rotating torsion frequency is related to the non-rotating frequency by the relation

$$\nu_\theta^2 = 1 + \nu_{\theta_0}^2 \tag{5.17}$$

Modeling the blade with a thin rectangular cross section, the normalized feathering moment of inertia is

$$I_f^* = I_f/I_b = \frac{1}{4}(c/R)^2 \quad (5.18)$$

The moment of inertia term due to the chord-wise center of gravity location differing from feathering axis is

$$I_x^* = I_x/I_b = \frac{3}{2}x_I/R \quad (5.19)$$

If the blade c.g. is just forward of the pitch axis, e.g. $x_I/c = -0.01$, then stability is improved due to the “nose down” moment. Moving the c.g. aft of the pitch axis is destabilizing.

As before, the linearized flap-lag Coriolis perturbation terms are

$$\begin{aligned} \delta(\beta\dot{\zeta}^*) &= \beta_T\delta\dot{\zeta}^* \approx \beta_0\delta\dot{\zeta}^* \\ \delta(\beta\dot{\beta}^*) &= \beta_T\delta\dot{\beta}^* + \beta_T^*\delta\beta_T \end{aligned} \quad (5.20)$$

which depend upon the flap trim state.

The coupled mass, damping, and stiffness matrices are

$$\begin{aligned}
\underline{\underline{M}} &= \begin{pmatrix} 1 & 0 & -I_x^* \\ 0 & 1 & 0 \\ -I_x^* & 0 & I_f^* \end{pmatrix} \\
\underline{\underline{C}} &= \begin{pmatrix} -\gamma M_\beta^* & -2\beta_T - \gamma M_\zeta^* & -\gamma M_\theta^* \\ 2\beta_T - \gamma Q_\beta^* & 2\nu_{\zeta_0} \zeta_L - \gamma Q_\zeta^* & -2\beta_T I_x^* - \gamma Q_\theta^* \\ -\gamma T_\beta^* & 2\beta_T I_x^* - \gamma T_\zeta^* & 2\nu_{\theta_0} \zeta_T I_f^* - \gamma T_\theta^* \end{pmatrix} \\
\underline{\underline{K}} &= \begin{pmatrix} \nu_\beta^2 - \gamma M_\beta & -\gamma M_\zeta + Z & -I_x^* - \gamma M_\theta \\ 2\beta_T^* - \gamma Q_\beta + Z & \nu_\zeta^2 - \gamma Q_\zeta & -\gamma Q_\theta \\ -I_x^* - \gamma T_\beta & -\gamma T_\zeta & I_f^* \nu_\theta^2 - \gamma T_\theta \end{pmatrix}
\end{aligned} \tag{5.21}$$

The full equations for the periodic aerodynamic forcing moments (M_β^* , T_θ , etc.) are provided in Appendix B.

Consider the third column of the $\underline{\underline{C}}$ matrix. Without unsteady aerodynamics and the non-circulatory terms, $M_\theta^* = 0$, $Q_\theta^* = 0$, and $T_\theta^* = 0$. In that case, beneficial torsion damping would rely on the structural damping ratio ζ_T , which is small or zero. Although the trimmed flap motion β_T is also present in the lag-torsion damping term, this would only be stabilizing when the flap angle is negative. With positive coning and longitudinal and lateral flapping trimmed to zero, the flap motion is destabilizing. Thus the importance of including more than quasisteady aerodynamics is apparent when modeling torsion.

Flap-lag-torsion stability was derived directly in this chapter, but the simpler coupling of flap-torsion may also be of interest. For this latter two degree-of-freedom case, the lag

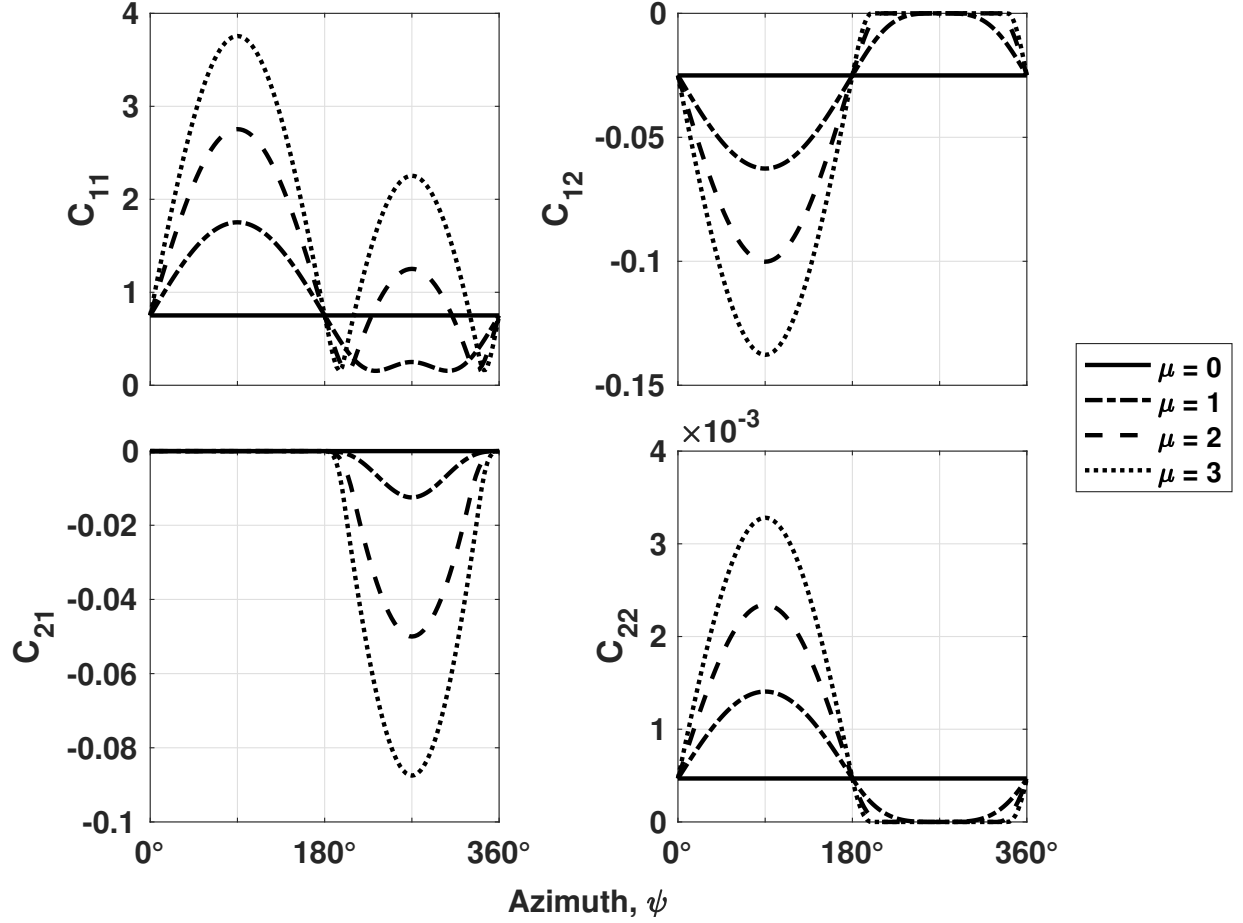


Figure 5.2: Flap-torsion damping matrix ($\underline{\underline{C}}$), $\nu_\beta = 1$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0$, $\alpha_s = 0^\circ$, $C(k) = 1$.

elements and the corresponding coupling terms can simply be removed, reducing the system to 2×2 periodic coefficient matrices.

Examples of $\underline{\underline{C}}$ and $\underline{\underline{K}}$ for flap-torsion are shown in in Figs. 5.2 through 5.5, where $\nu_\beta = 1$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, and $C(k) = 1$. For Figs. 5.2 and 5.3, $C_T/\sigma = 0$ and $\alpha_s = 0^\circ$. Unlike the flap-lag matrices, there is coupling for the time-invariant and zero thrust periodic coefficients.

The effect of adding thrust and aft shaft tilt to the flap-torsion damping and stiffness matrices is shown in Figs. 5.4 and 5.5, $C_T/\sigma = 0.04$ and $\alpha_s = -2.72^\circ$.

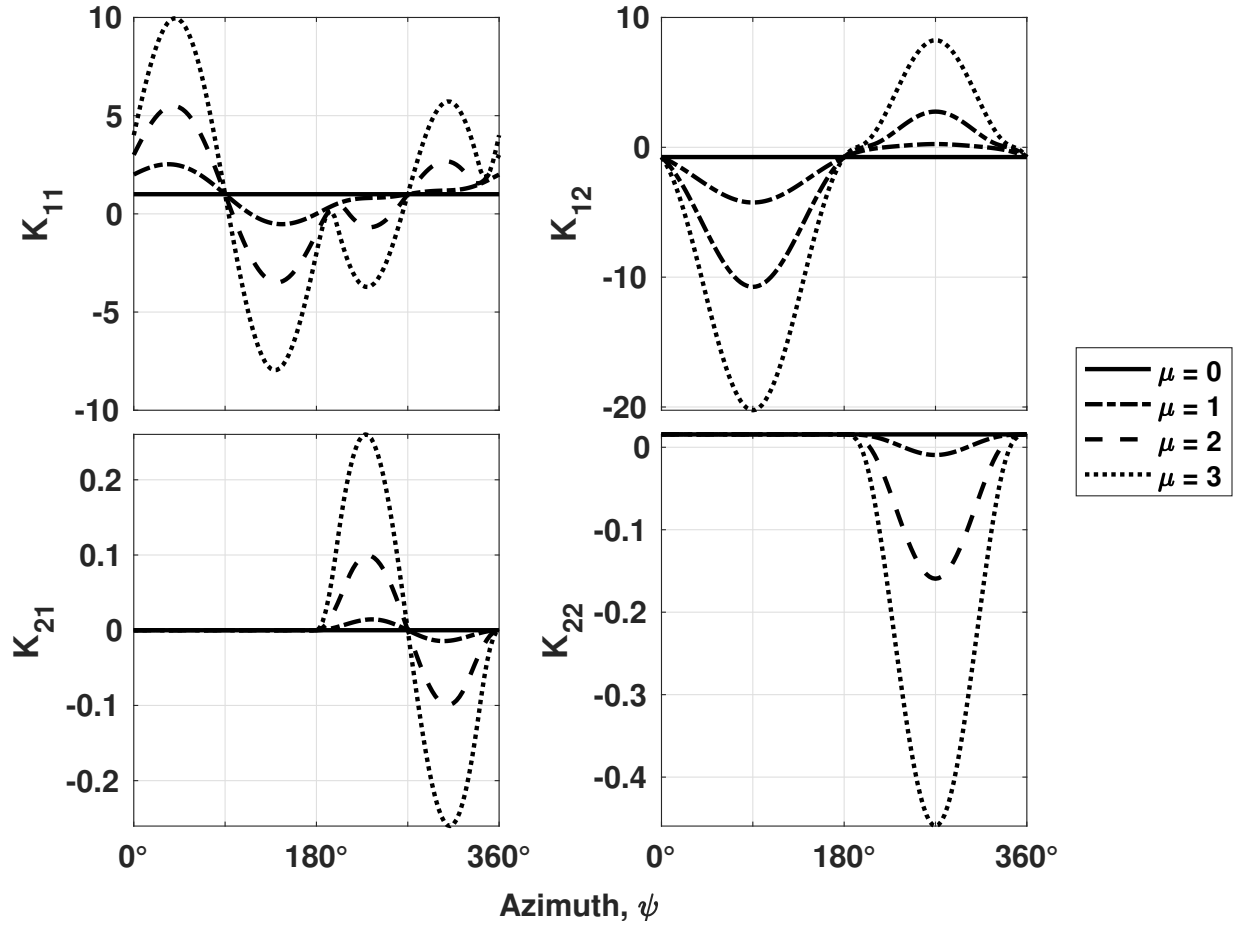


Figure 5.3: Flap-torsion stiffness matrix ($\underline{\underline{K}}$), $\nu_\beta = 1$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0$, $\alpha_s = 0^\circ$, $C(k) = 1$.

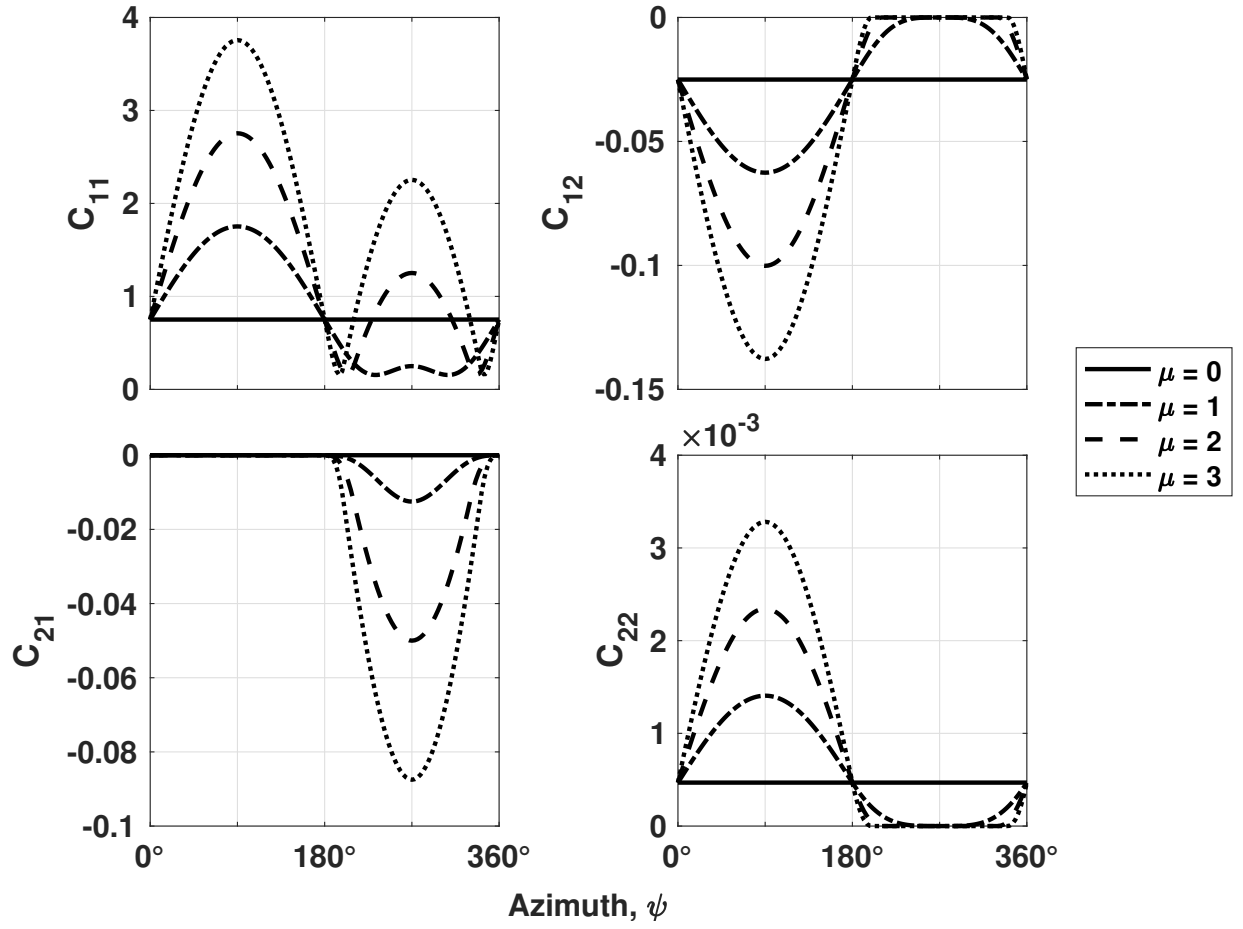


Figure 5.4: Flap-torsion damping matrix ($\underline{\underline{C}}$), $\nu_\beta = 1$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$.

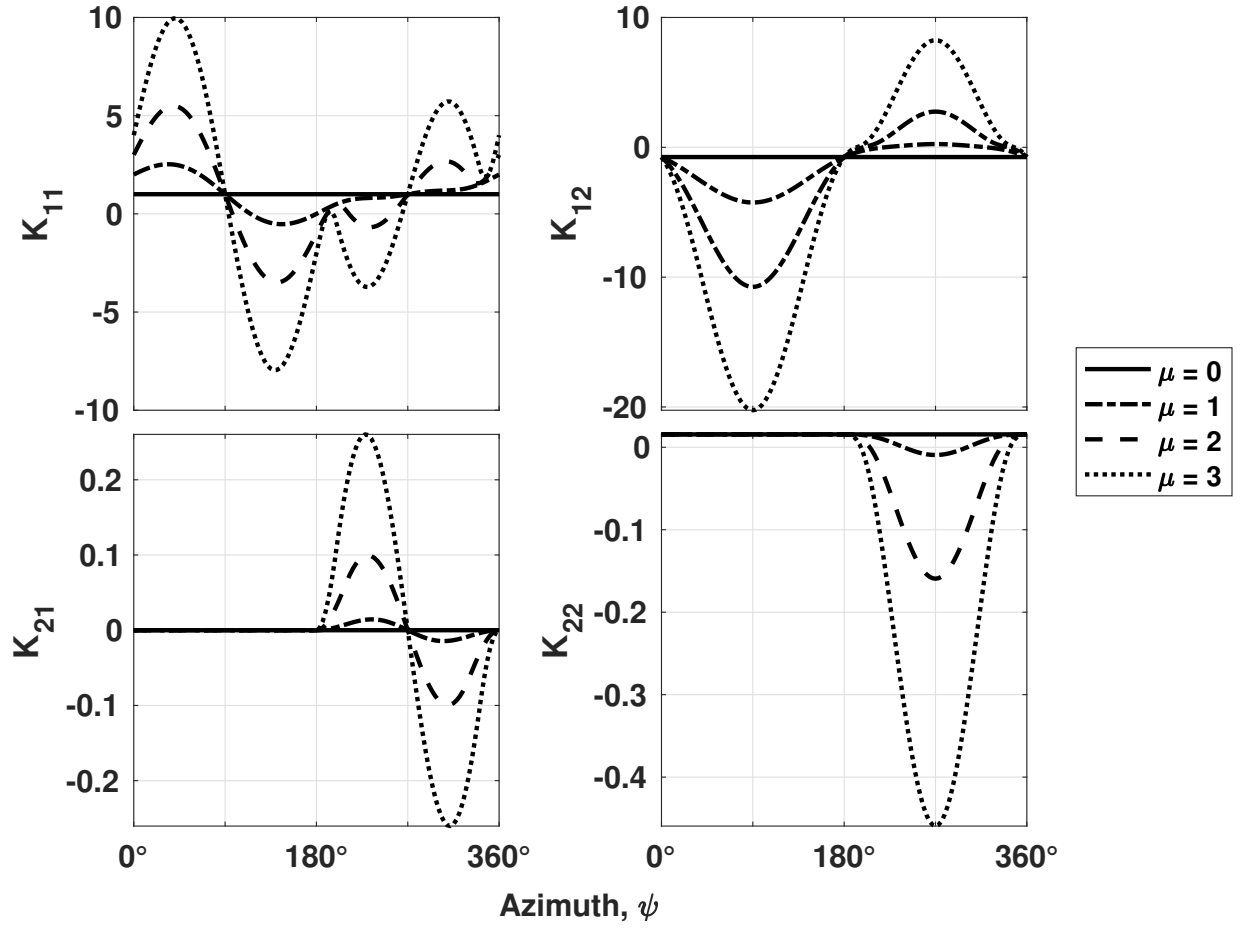


Figure 5.5: Flap-torsion stiffness matrix ($\underline{\underline{K}}$), $\nu_\beta = 1$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$.

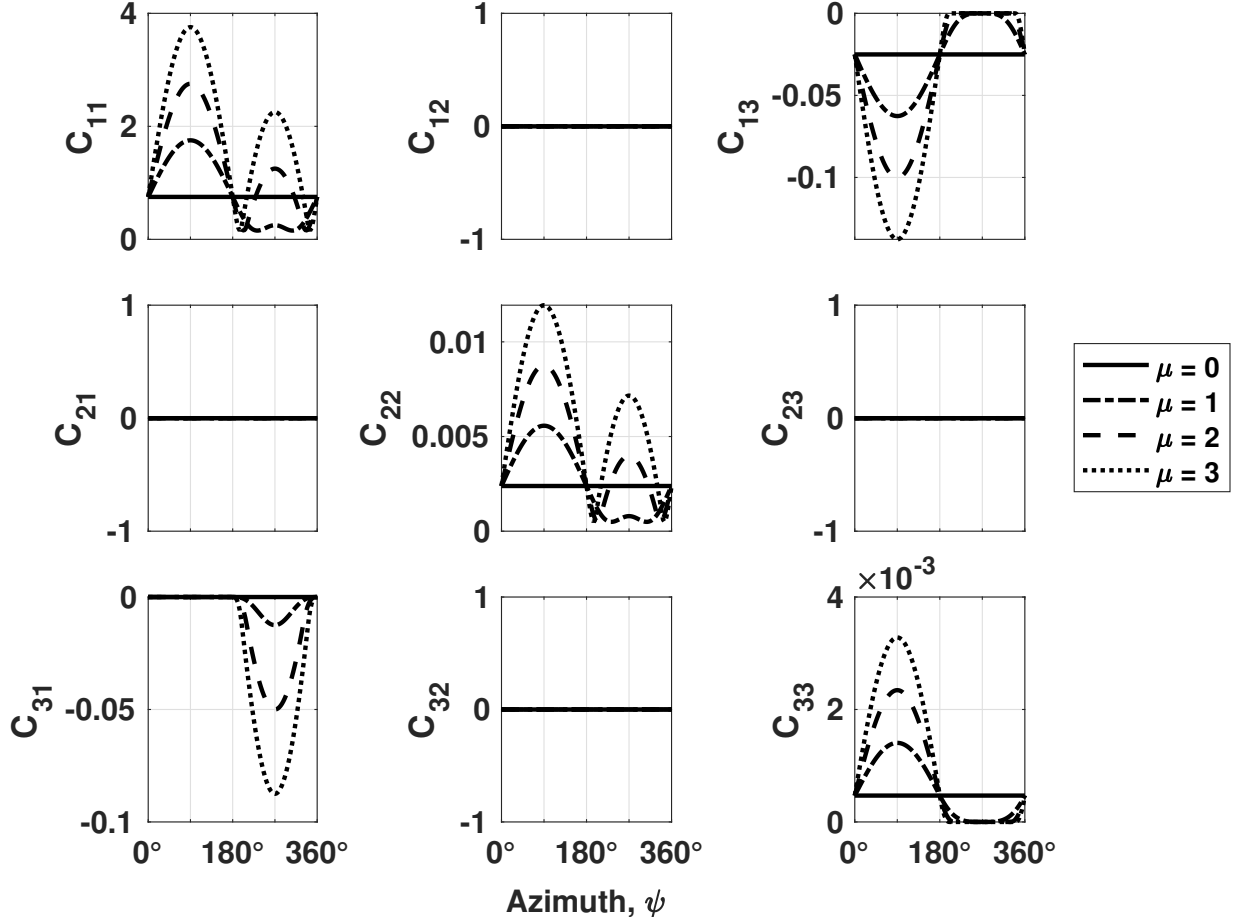


Figure 5.6: Flap-lag-torsion damping matrix ($\underline{\underline{C}}$) , $\nu_\beta = 1$, $\nu_\zeta = 0.3$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0$, $\alpha_s = 0^\circ$, $C(k) = 1$.

Applying these same parameters to a coupled flap-lag-torsion example, the 3×3 $\underline{\underline{C}}$ and $\underline{\underline{K}}$ matrices are calculated and plotted in Figs. 5.6 through 5.9, with $\nu_\zeta = 0.3/\text{rev}$.

5.5 Verification of the Coupled Torsion Stability Model

In Ref. 72, Friedmann and Kottapalli presented a coupled flap-lag-torsion analysis of a hingeless rotor in forward flight. Unlike the present work, they modeled moderate elastic blade deformations with a multimodal representation, which was fairly new at the time. Through an iterative quasi-linearization procedure, they determined the importance

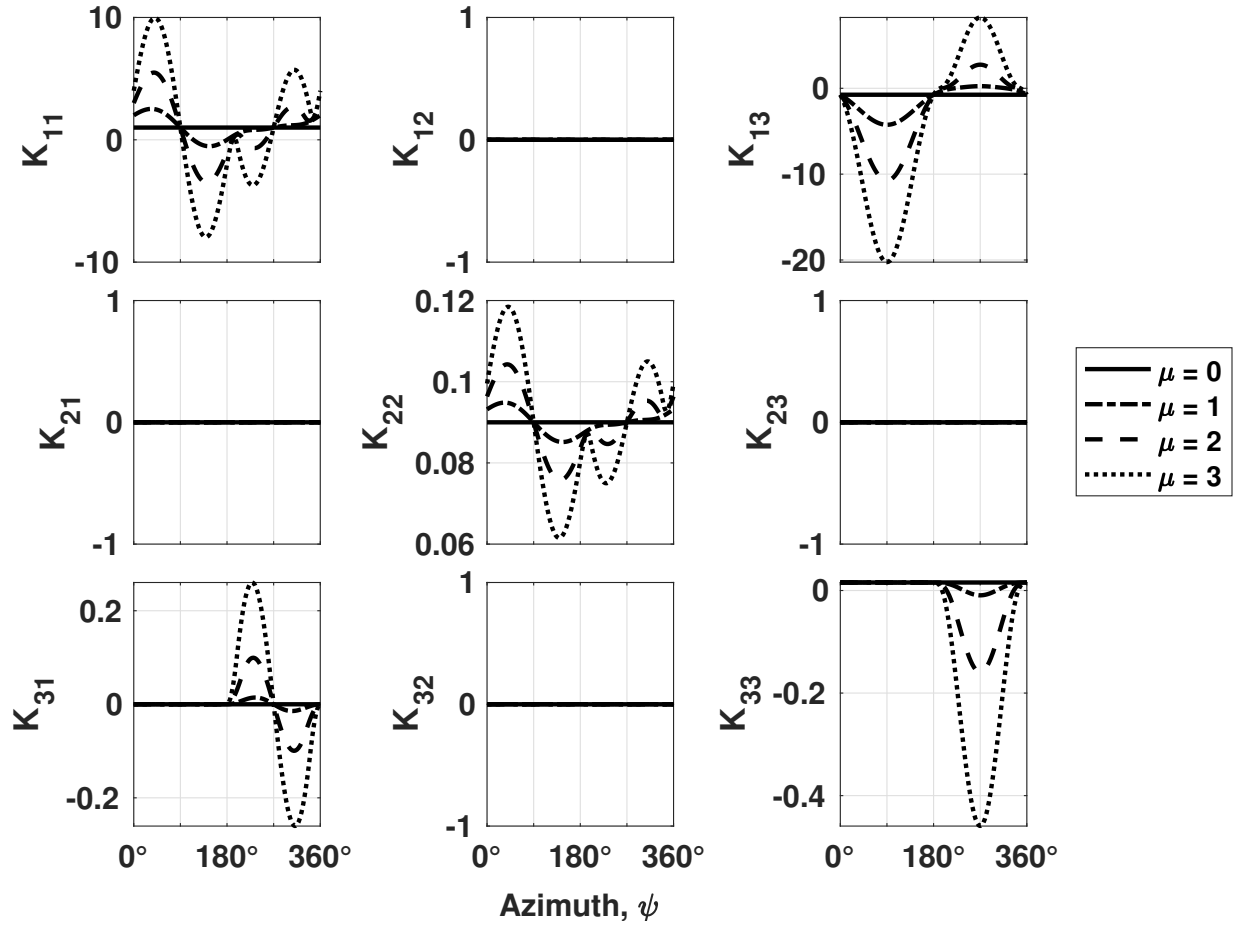


Figure 5.7: Flap-lag-torsion stiffness matrix ($\underline{\underline{K}}$), $\nu_\beta = 1$, $\nu_\zeta = 0.3$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0$, $\alpha_s = 0^\circ$, $C(k) = 1$.

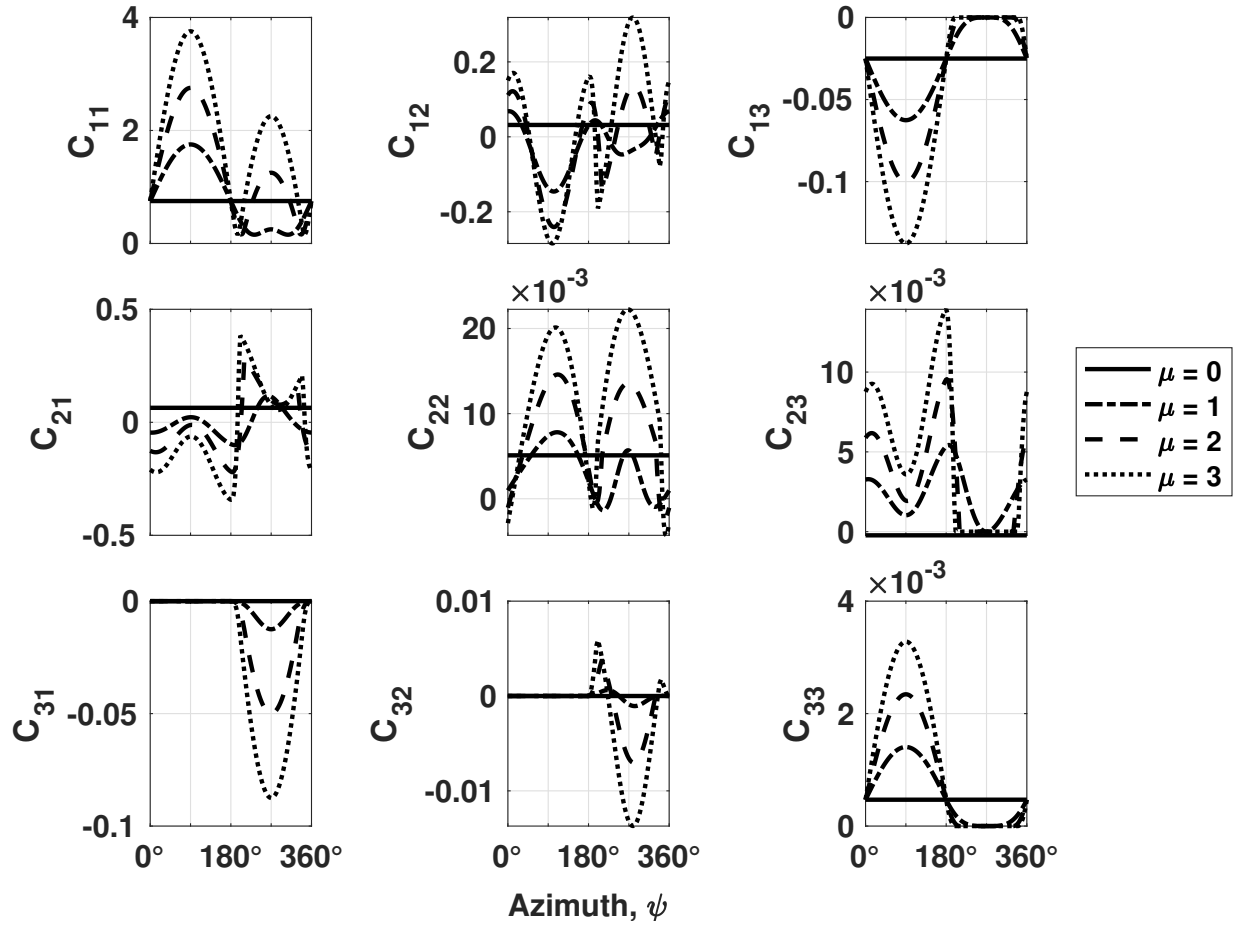


Figure 5.8: Flap-lag-torsion damping matrix ($\underline{\underline{C}}$), $\nu_\beta = 1$, $\nu_\zeta = 0.3$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$.

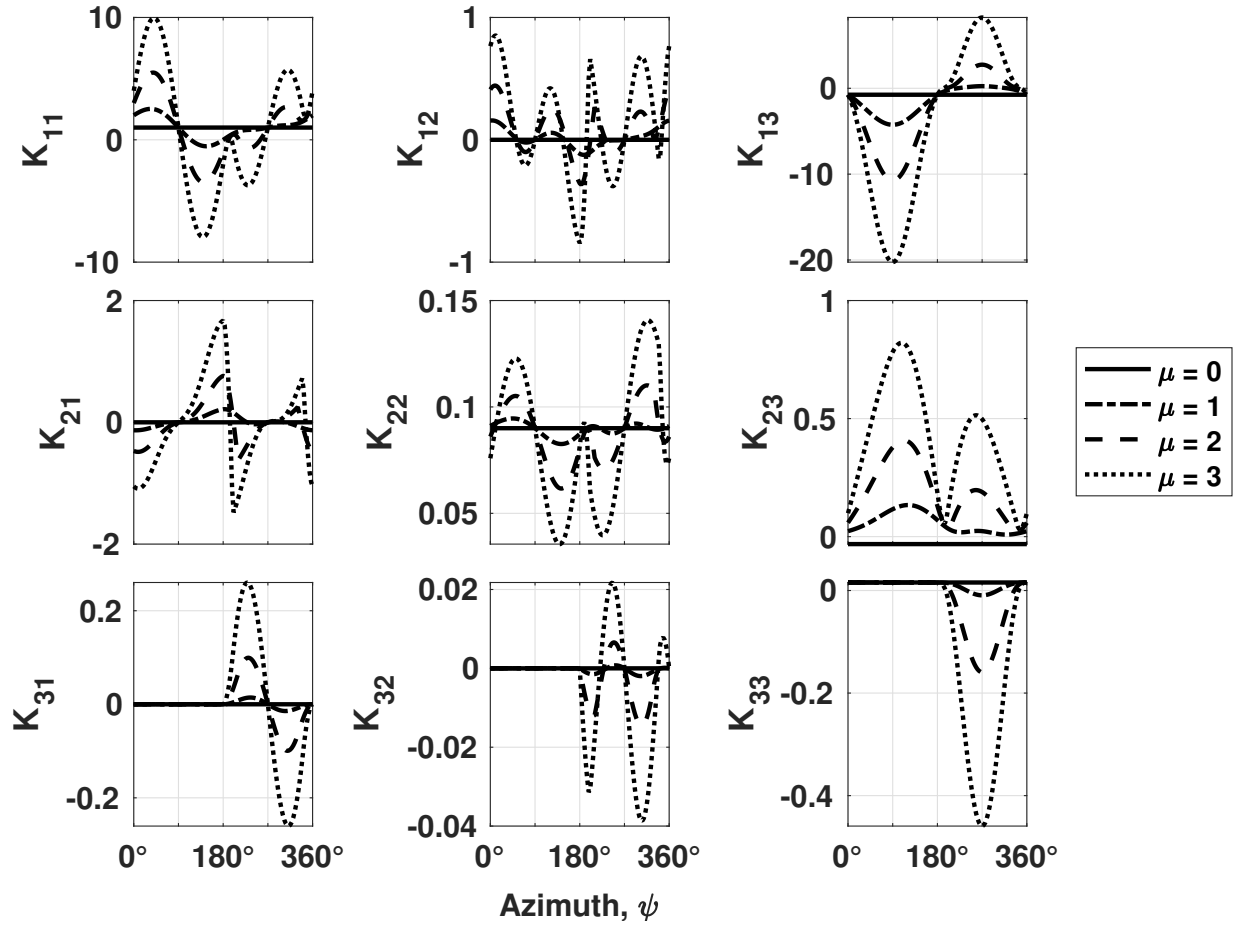


Figure 5.9: Flap-lag-torsion stiffness matrix ($\underline{\underline{K}}$), $\nu_\beta = 1$, $\nu_\zeta = 0.3$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$.

of nonlinear terms. Although reverse flow and unsteady aerodynamics were neglected, their work provides some verification of the current model at low- μ and moderate thrust.

The present model's flap, lag, and torsion damping values for a stiff-in-plane configuration are overlaid with their results in Figs. 5.10, 5.11, and 5.12. The three values of k (0, 1, and 2) correspond to neglecting the nonlinear terms, linearizing the response, and approximating the full nonlinear behavior, respectively.

The rigid flap and torsion results are fairly close to the linearized first modes of the elastic analysis; the second modes not captured by the current work are less damped, but only have a small or zero stability degradation with increasing μ . Note that the flap damping “split” above $\mu = 0.3$ corresponds to frequency locking.

The first lag mode also follows their results for $k = 0$. However, lag was found to be very sensitive to nonlinear terms, as was previously acknowledged in Ref. 76. The subsequent lag damping results for stiff-in-plane blades might be even more unstable at high- μ ; the influence of different trim settings and reverse flow could temper or aggravate this concern. It should be noted that the soft-in-plane results did not have the same nonlinear lag instability trend.

A similar verification with Panda and Chopra's FEM flap-lag-torsion stability analysis (Ref. 75) is shown in Fig. 5.13. The coupled lag stability results are obviously very similar for the rigid cases due to similar assumptions, and the present model is also close to the stability of the elastic the soft-in-plane blade. The decreasing lag damping trend of the stiff-in-plane blade is again only captured by the elastic model.

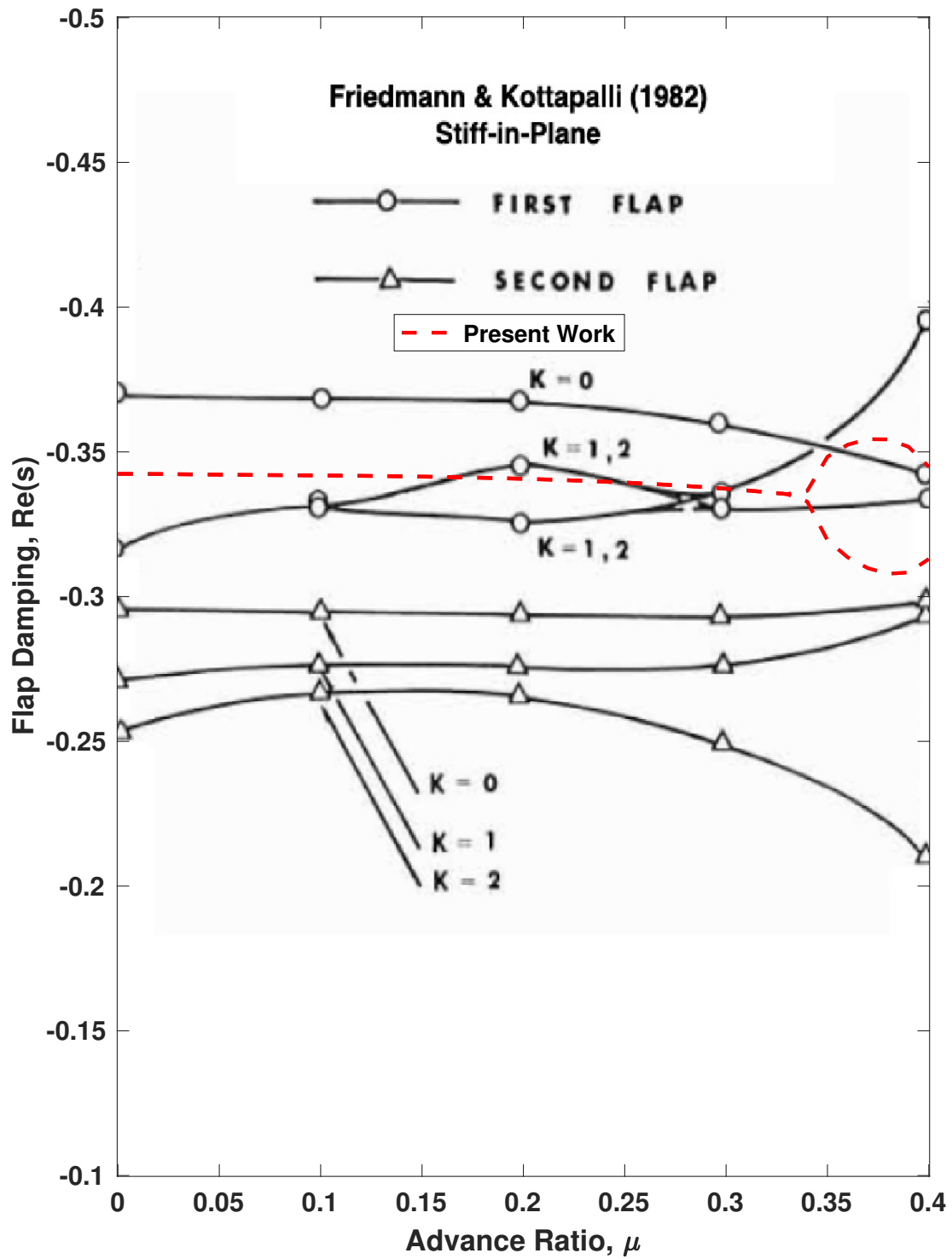


Figure 5.10: Comparison of coupled flap stability model with multimodal nonlinear forward flight analysis of stiff-in-plane hingeless rotor, quasisteady aero., propulsive trim.

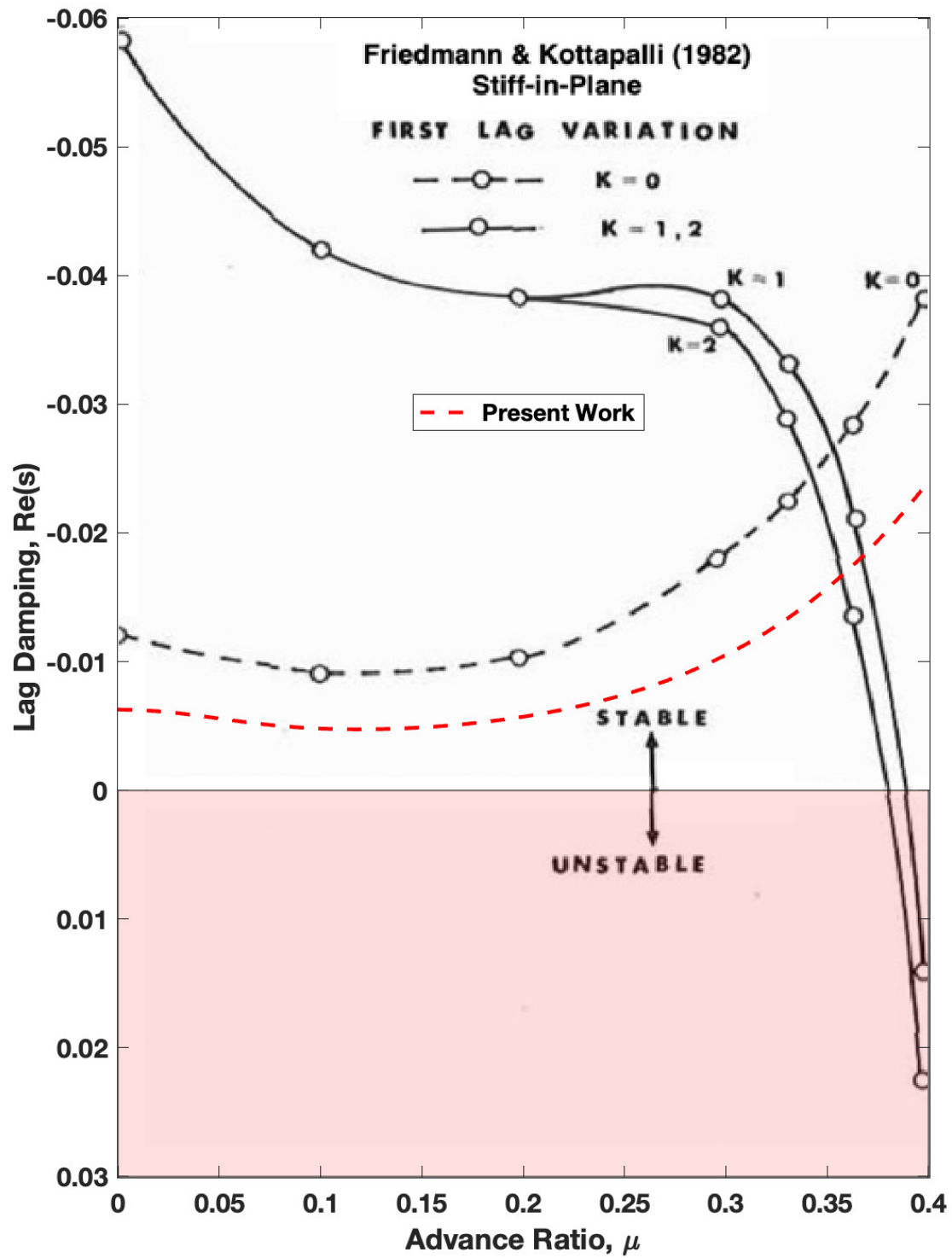


Figure 5.11: Comparison of coupled lag stability model with multimodal nonlinear forward flight analysis of stiff-in-plane hingeless rotor, quasisteady aero., propulsive trim.

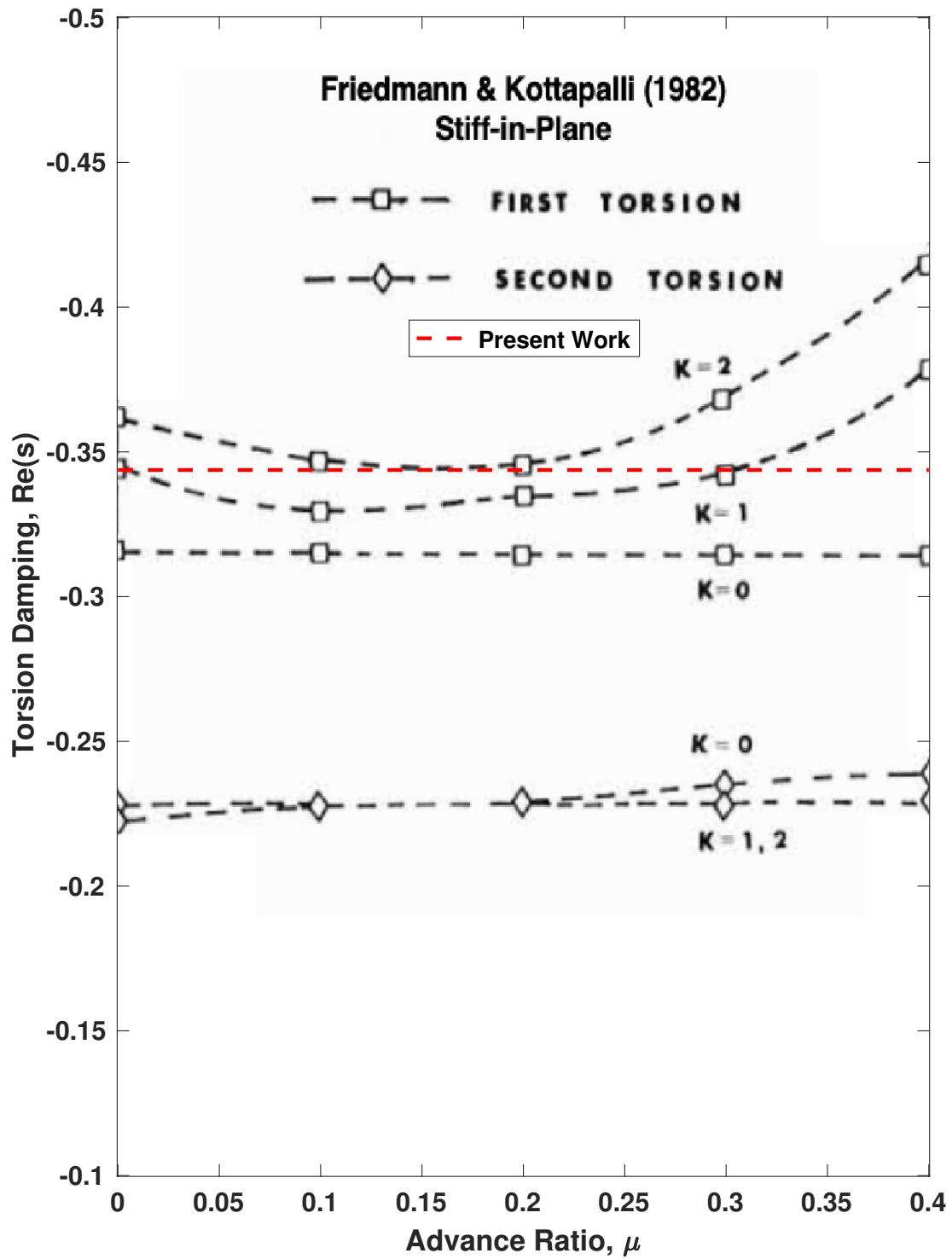


Figure 5.12: Comparison of coupled torsion stability model with multimodal nonlinear forward flight analysis of stiff-in-plane hingeless rotor, quasisteady aero., propulsive trim.

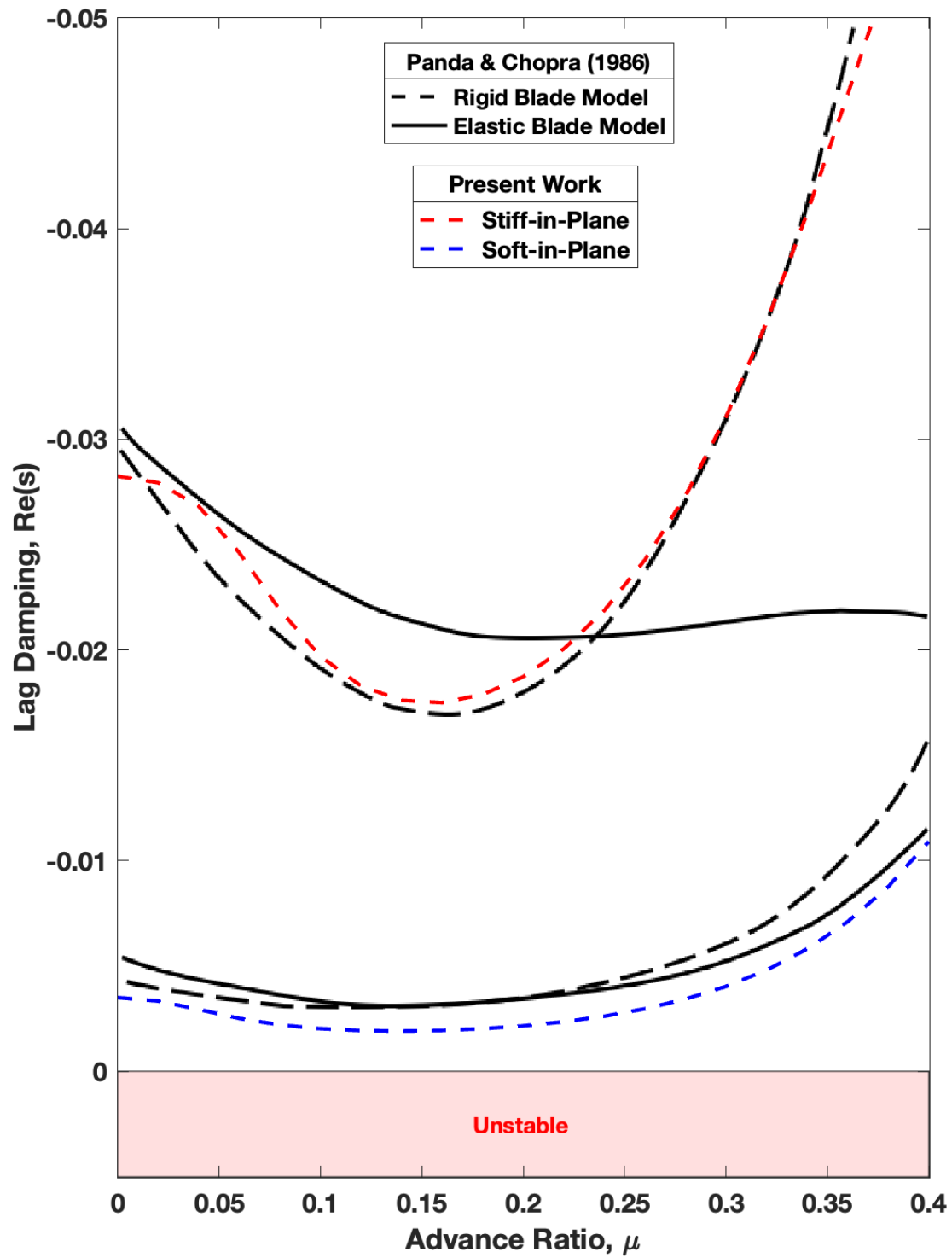


Figure 5.13: Comparison of coupled lag stability model with elastic FEM forward flight analysis of soft and stiff-in-plane hingeless rotors, dynamic inflow, propulsive trim.

5.6 Parametric Evaluation of Coupled Flap-Lag-Torsion Damping

Next, flap-lag-torsion stability will be analyzed with respect to a wide range of parameters for advance ratios between 0 and 3. As in Chapter 4, flap and lag frequencies are varied between 1 to 1.5 and 0.1 to 2, respectively, to produce ν_ζ vs. ν_β stability maps. Torsion frequencies are fixed at 5, 10, and 15 per rev. The latter stiff-in-torsion case provides similar results to the flap-lag analysis. Blade loading is set to 0.04 and aft shaft tilt is trimmed to minimize coning at thrust reversal. Figs. 5.14 through 5.19 show the unstable regions for $\mu = 0.6$ and above.

Many conclusions about rotor configuration can be gleaned from these damping contour plots. In general, soft in plane lag frequencies close to 0.5 per rev are unstable, and a torsion frequency around 5 per rev is not stiff enough for high advance ratios. Attempting to increase blade loading in the onset of the thrust reversal region, especially between $\mu = 0.8$ and 1, can lead to instability for many rotor configurations, in addition to the blade stall issues noted in the earlier trim discussion. The multi-axis stiffness a hingeless rotor, which can be conservatively approximated with the upper right corner of the $\nu_\theta = 9.9$ per rev plots, appears to have a wide range of stable thrust and speed regimes.

5.7 Eigenvalue Trajectory Analysis

The automated sorting process to assign the eigenvalues to each degree of freedom as described in Chapters 2 and 4 is applied. Without eliminating any root combinations, there are a maximum of $6! = 720$ permutations at each point for the flap-lag-torsion problem,

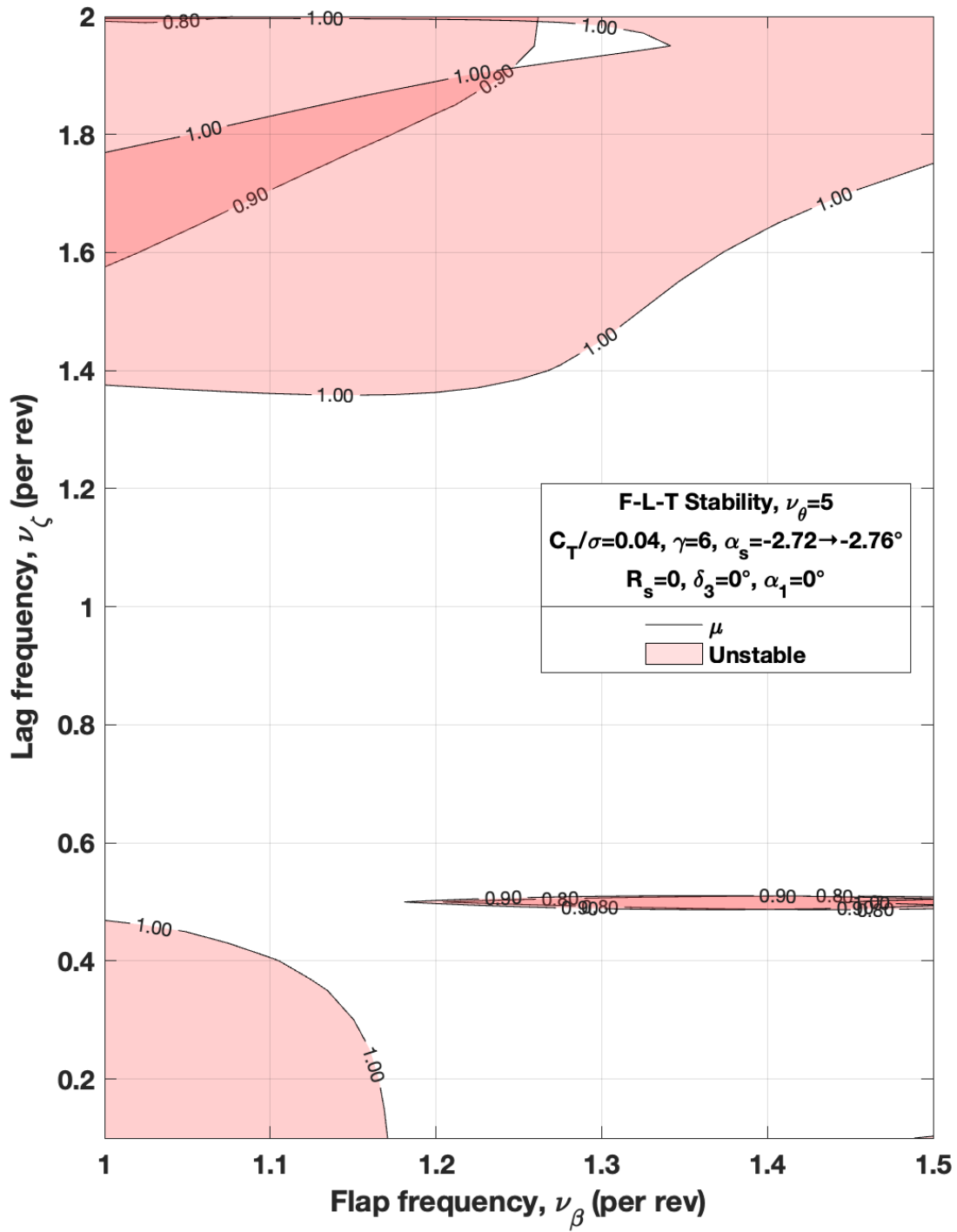


Figure 5.14: Flap-lag-torsion stability map (ν_ζ vs. ν_β), $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $C(k) = 1$, $\mu = 0.6$ to 1 .

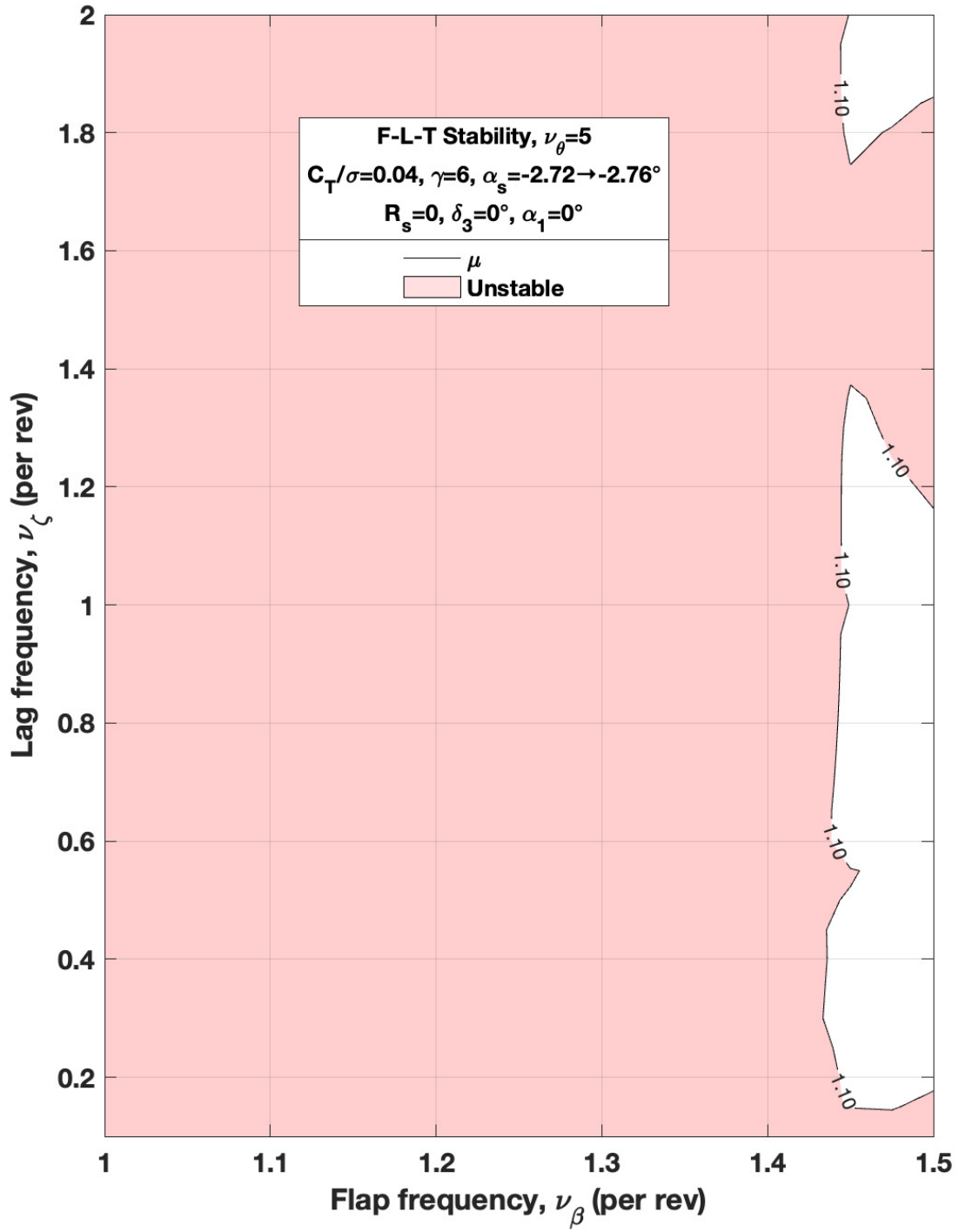


Figure 5.15: Flap-lag-torsion stability map (ν_ζ vs. ν_β), $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $C(k) = 1$, $\mu = 1.1$.

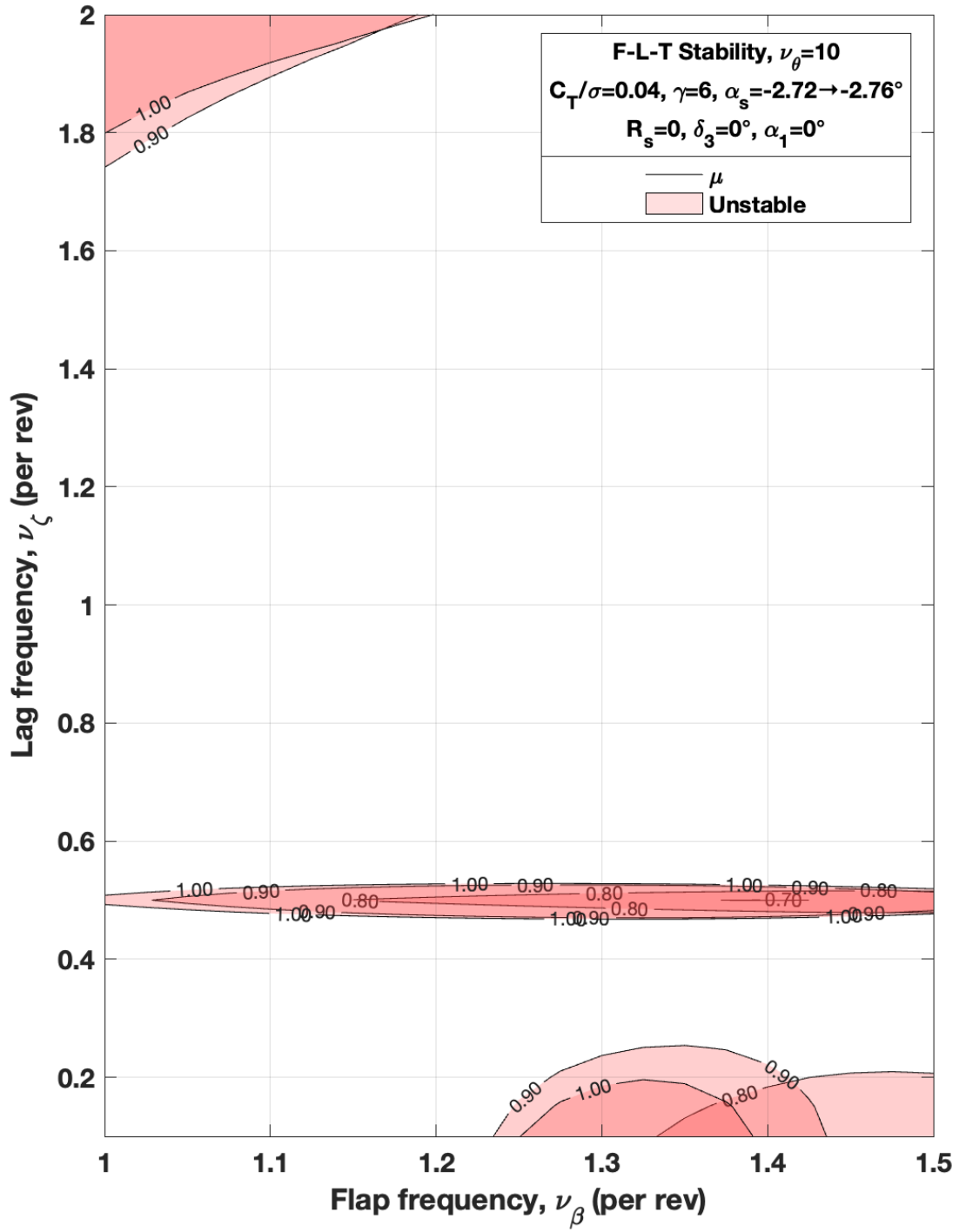


Figure 5.16: Flap-lag-torsion stability map (ν_ζ vs. ν_β), $\nu_\theta = 10/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $C(k) = 1$, $\mu = 0.6$ to 1.

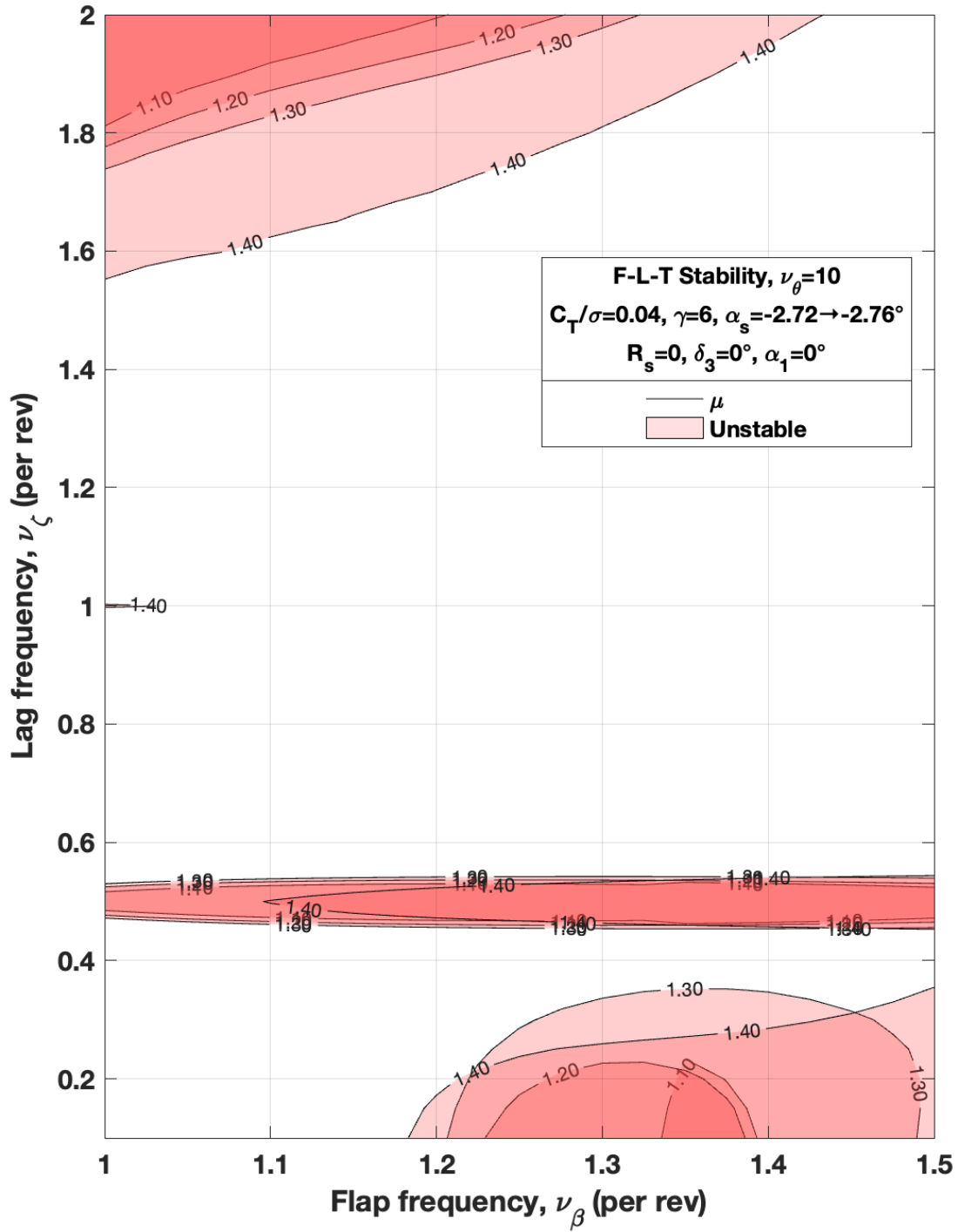


Figure 5.17: Flap-lag-torsion stability map (ν_ζ vs. ν_β), $\nu_\theta = 10/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $C(k) = 1$, $\mu = 1.1$ to 1.4 .

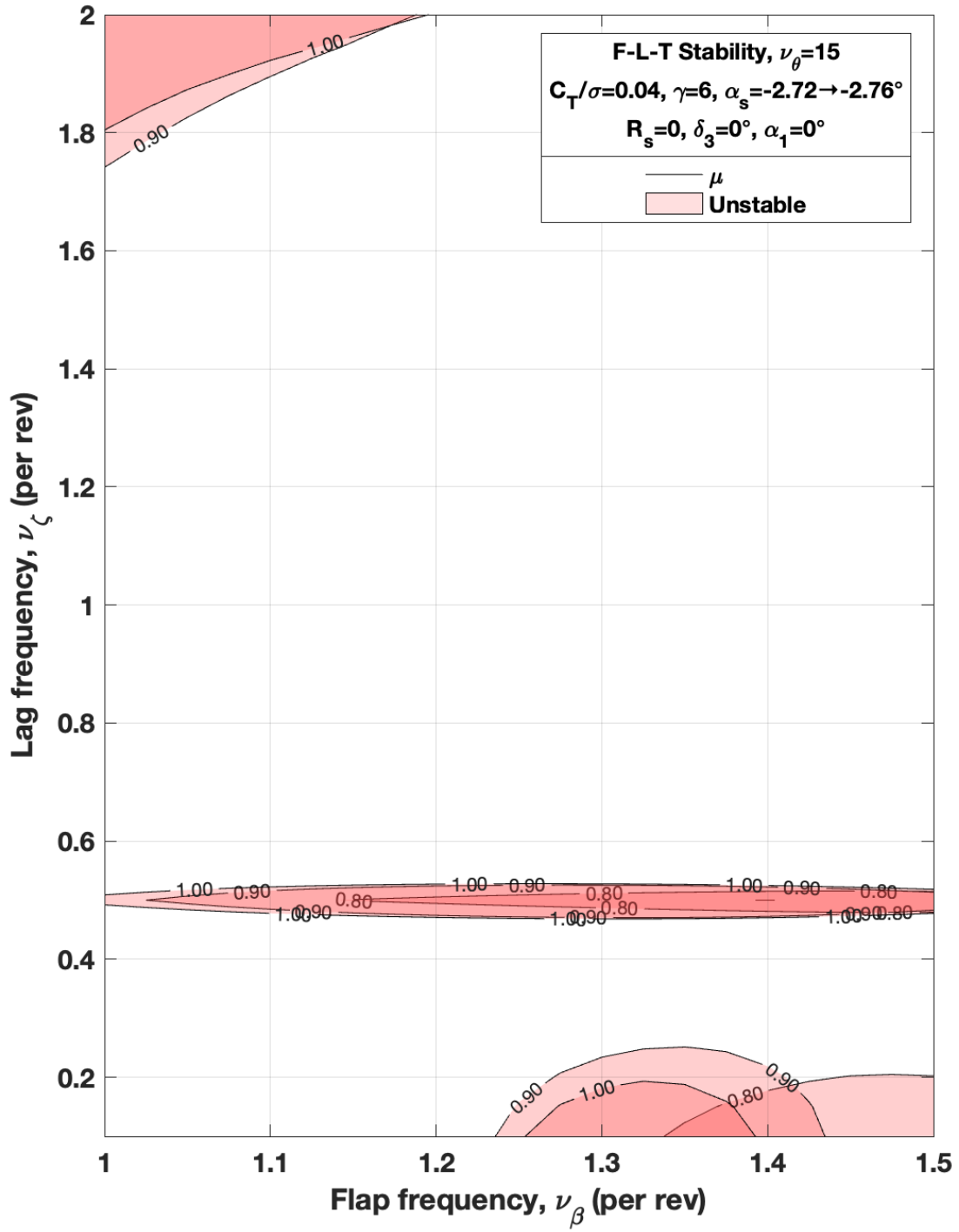


Figure 5.18: Flap-lag-torsion stability map (ν_ζ vs. ν_β), $\nu_\theta = 10/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $C(k) = 1$, $\mu = 0.6$ to 1.

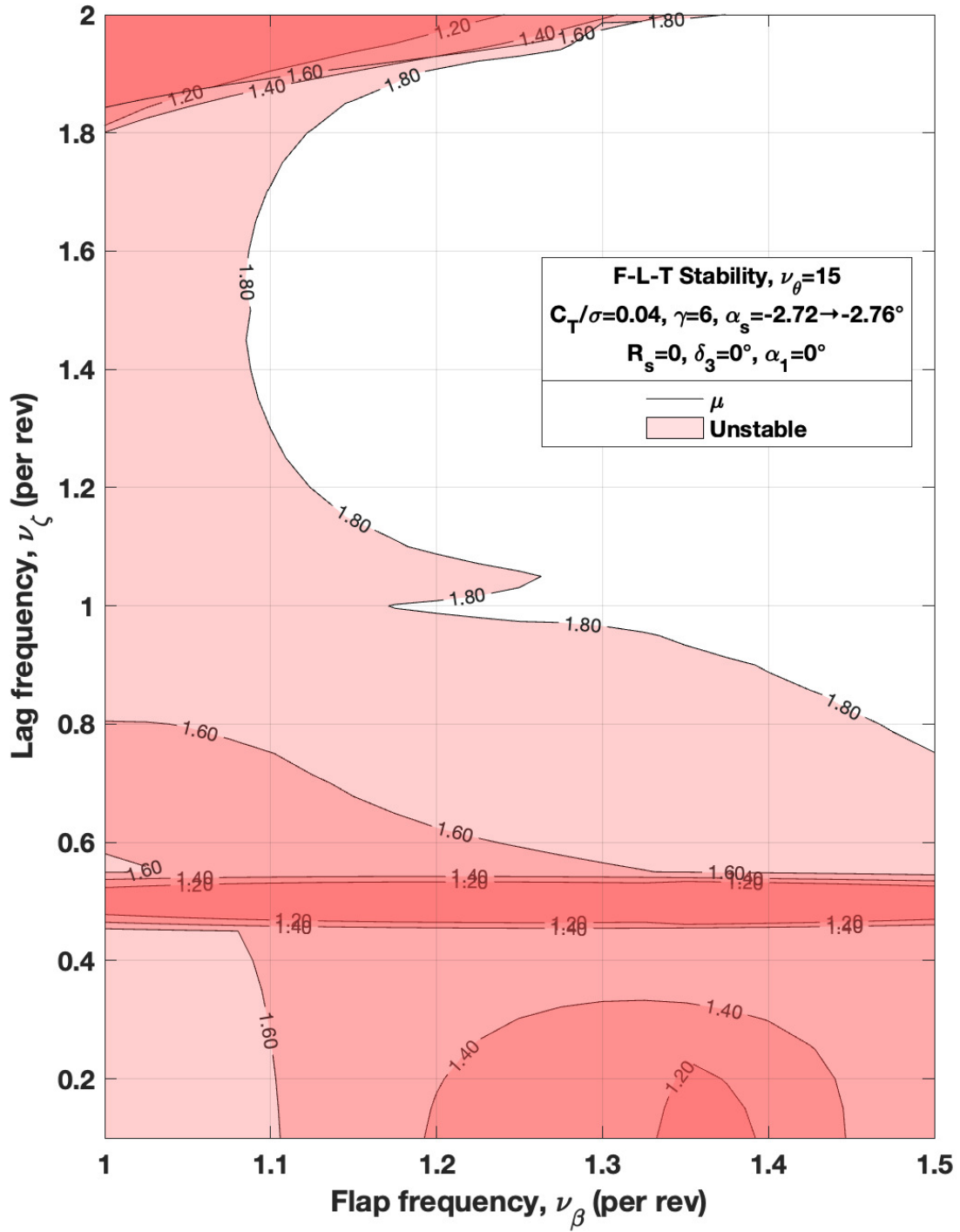


Figure 5.19: Flap-lag-torsion stability map (ν_ζ vs. ν_β), $\nu_\theta = 15/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $C(k) = 1$, $\mu = 1.2$ to 1.8 .

which increases computation time significantly compared to the $4! = 24$ combinations of F-T and F-L. Applying the eigenvalue pairing rules from Chapter 4 can reduce the number of necessary trial combinations. However, when multiple degrees of freedom become frequency locked, this approach becomes less effective because the roots may share principal frequencies.

Attempting to mitigate this issue by reducing the number of integration steps can lead to numerical instability, or poorly capturing frequency content when FFTs are applied. Increasing the parameter step size when projecting the trajectories from the time-invariant case can lead to incorrect root attribution when multiple parametric resonance branches occur in the same window. The following results utilize 120 time steps per revolution and $\Delta\mu = 0.01$.

A comparison of the root sorting algorithm with the default MATLAB eig ordering is shown in Fig. 5.20. In this stiff-in-plane hingeless rotor example, due to the numerical instability of the highly physically damped T2 root, the eigenvalues have their first incompatible solution at $\mu = 1.57$, and the sorting algorithm is only utilized until that point. Regardless, the system becomes unstable at much lower advance ratios: both L1 and L2 are briefly unstable for $\mu = 0.99$ to 1.02 , and F2 is unstable for $\mu > 1.03$ (this will be more apparent from the eigenvalue plots shown later).

For low torsion frequencies (soft-in-pitch), there are more cases of multi-axis parametric resonance, root locus crossings (s -plane overlaps), and numerical instability occurring at lower advance ratios. As will be shown, it can be difficult to ascertain whether flap or torsion becomes unstable at certain branching points without decreasing $\Delta\mu$ and increasing computational cost.

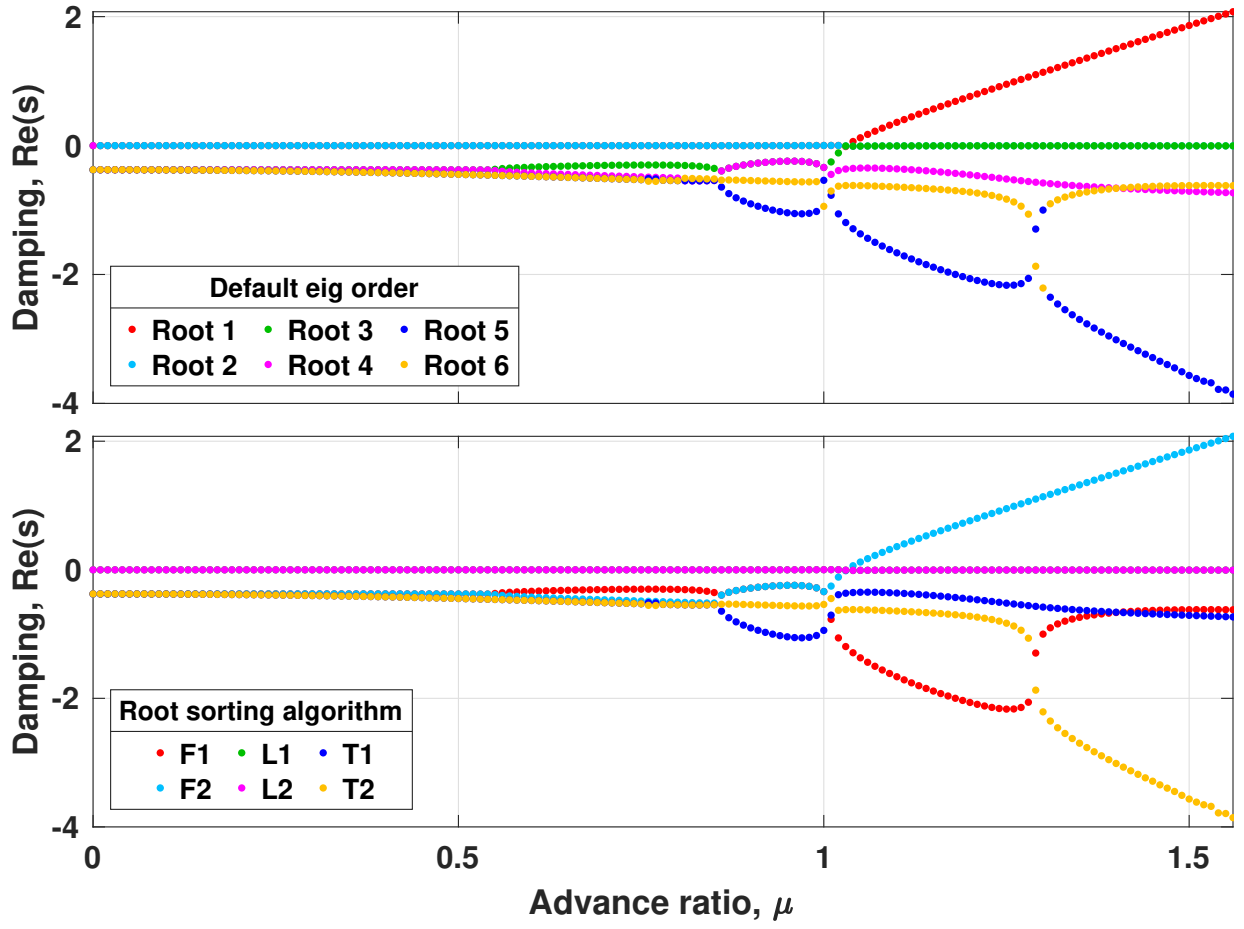


Figure 5.20: Comparison of default eig order with root sorting algorithm, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $C(k) = 1$.

As in in the flap-lag discussion, four rotor configuration cases will be explored for $\gamma = 6$, $C_T/\sigma = 0.04$, and α_s trimmed aft ($\approx -2.7^\circ$) to minimize flapping at thrust reversal. Now the torsion degree of freedom is added, with a rotating natural frequency of $\nu_\theta = 5/\text{rev}$.

1. Articulated rotor, very soft-in-plane: $\nu_\beta = 1$, $\nu_\zeta = 0.3/\text{rev}$ (Figs. 5.21 through 5.39).
2. Articulated rotor, soft-in-plane: $\nu_\beta = 1$, $\nu_\zeta = 0.7/\text{rev}$ (Figs. 5.40 through 5.58).
3. Hingeless rotor, soft-in-plane: $\nu_\beta = 1.15$, $\nu_\zeta = 0.7/\text{rev}$ (Figs. 5.59 through 5.74).
4. Hingeless rotor, stiff-in-plane: $\nu_\beta = 1.15$, $\nu_\zeta = 1.4/\text{rev}$ (Figs. 5.75 through 5.93).

5.8 Flap-Lag-Torsion Stability Conclusions

The torsion degree of freedom should not be neglected in the analysis of rotors operating at high advance ratios, even for those that are stiff in pitch. Including reverse flow and unsteady aerodynamics is essential to correctly model torsion and its coupling with flap and lag. Dynamic inflow does not have as significant of an impact on stability. Based on previous studies of elastic blades, the present rigid model may under-predict higher harmonic instabilities for stiff-in-plane blades.

The assessed soft-in-torsion articulated and hingeless rotor configurations entered into double parametric resonance (pitch-flap instability) at just above $\mu = 1$ and became unstable. Although a flap root appears to become the most unstable, it is difficult to numerically differentiate between the flap and torsion motion. All axes appear to resonate at a half-integer per revolution close to the torsion frequency during this parametric excitation. The

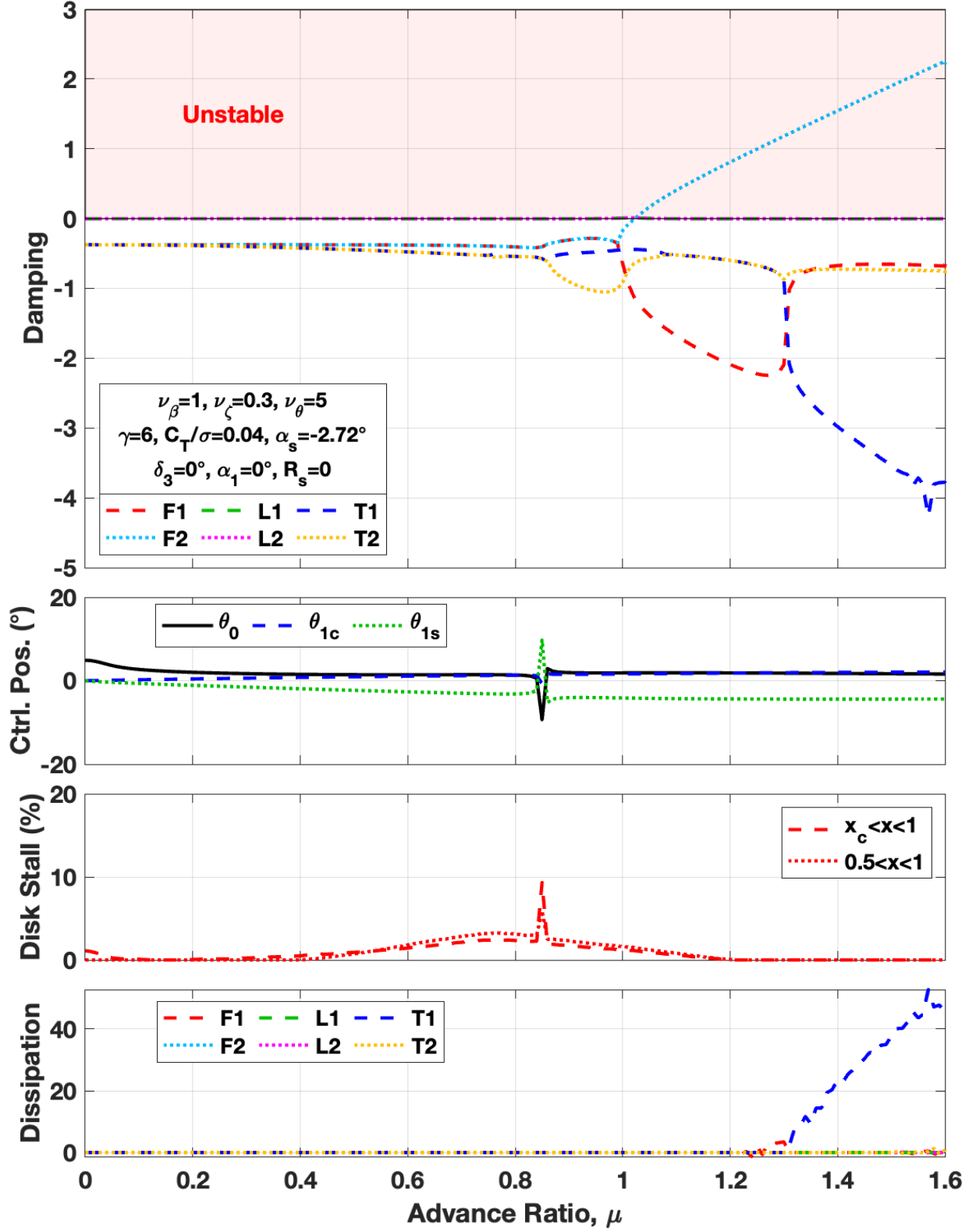


Figure 5.21: Flap-lag damping, trim, blade stall, and numerical stability, very soft-in-plane articulated rotor, $\nu_\beta = 1, \nu_\zeta = 0.3, \nu_\theta = 5/\text{rev}, \gamma = 6, C_T/\sigma = 0.04, \alpha_s = -2.72^\circ, C(k) = 1$.

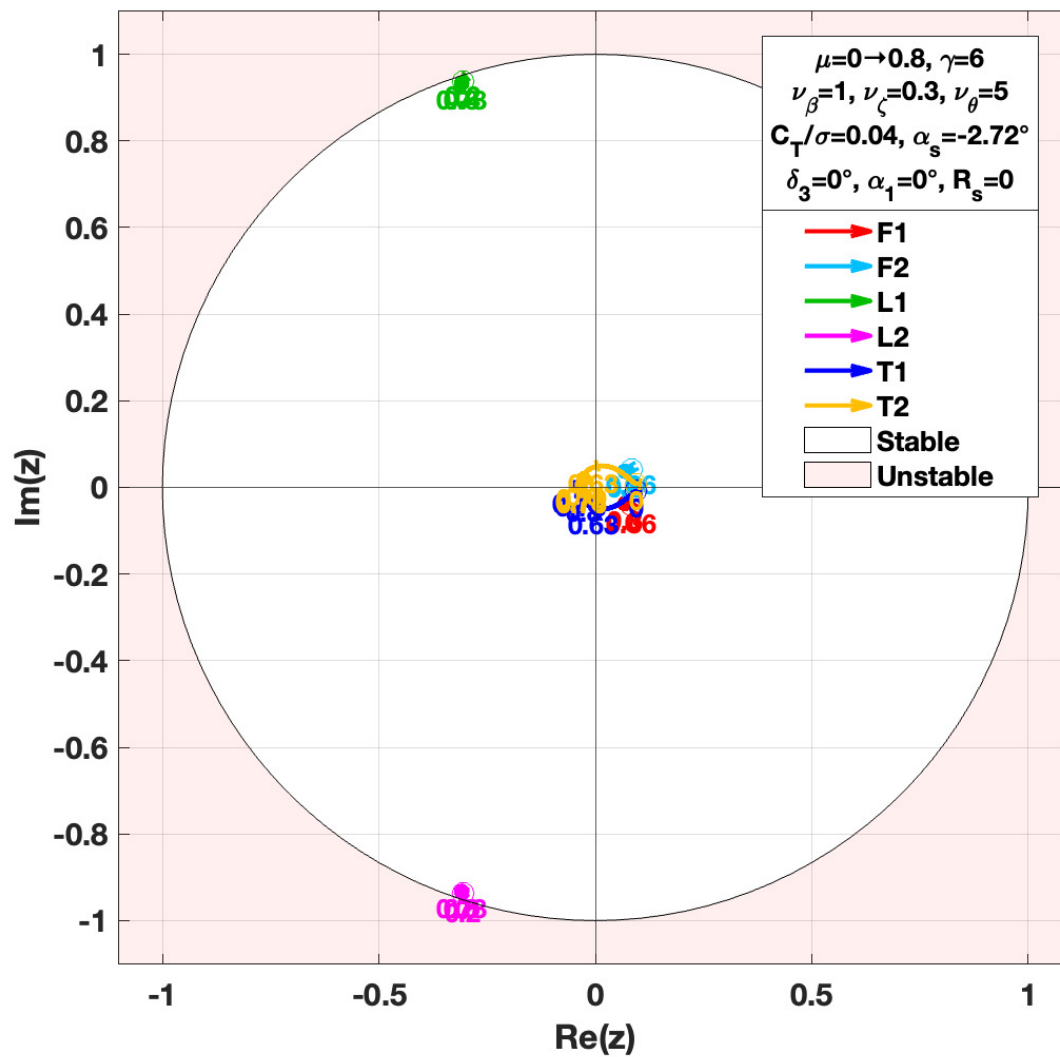


Figure 5.22: Flap-lag-torsion characteristic multipliers, very soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.3$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8 .

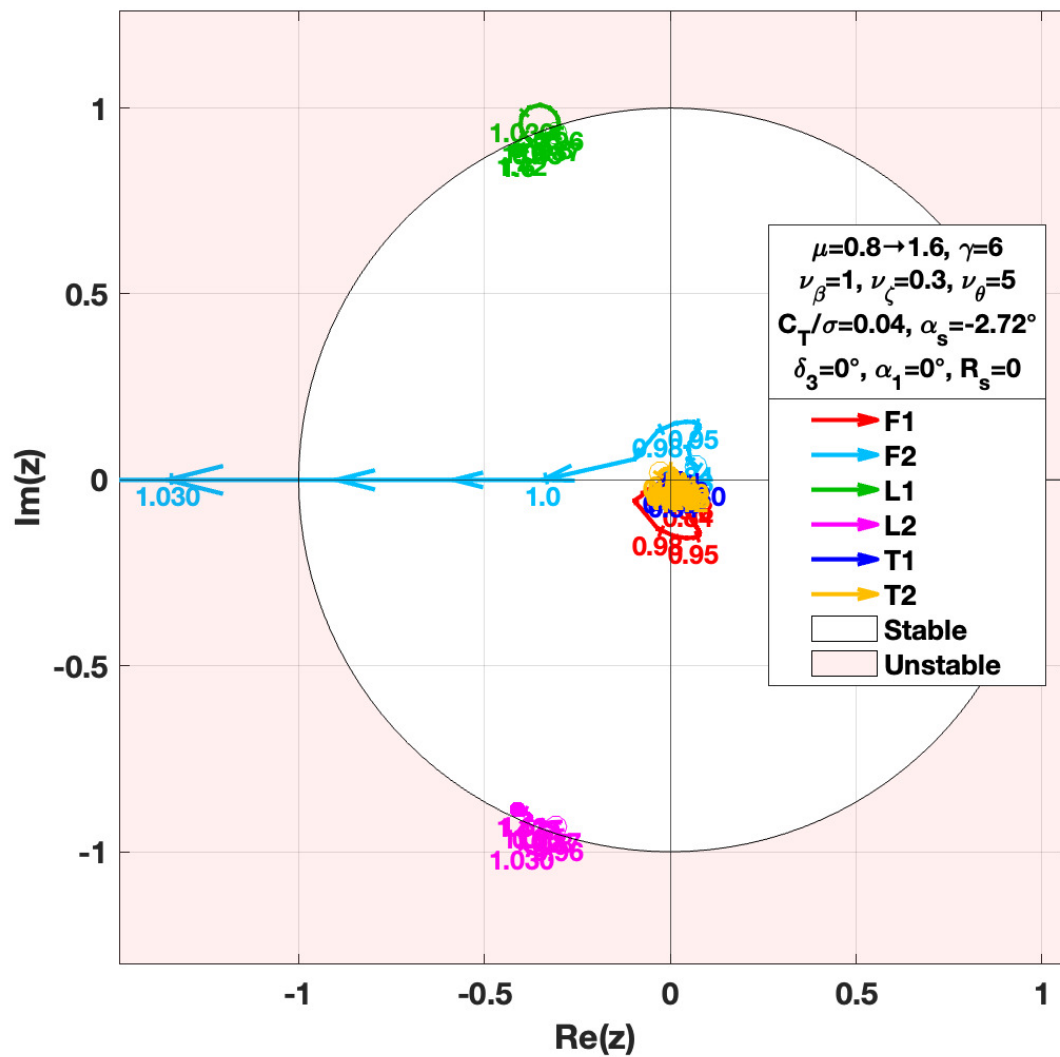


Figure 5.23: Flap-lag-torsion characteristic multipliers, very soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.3$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$, $\mu = 0.8$ to 1.6 .

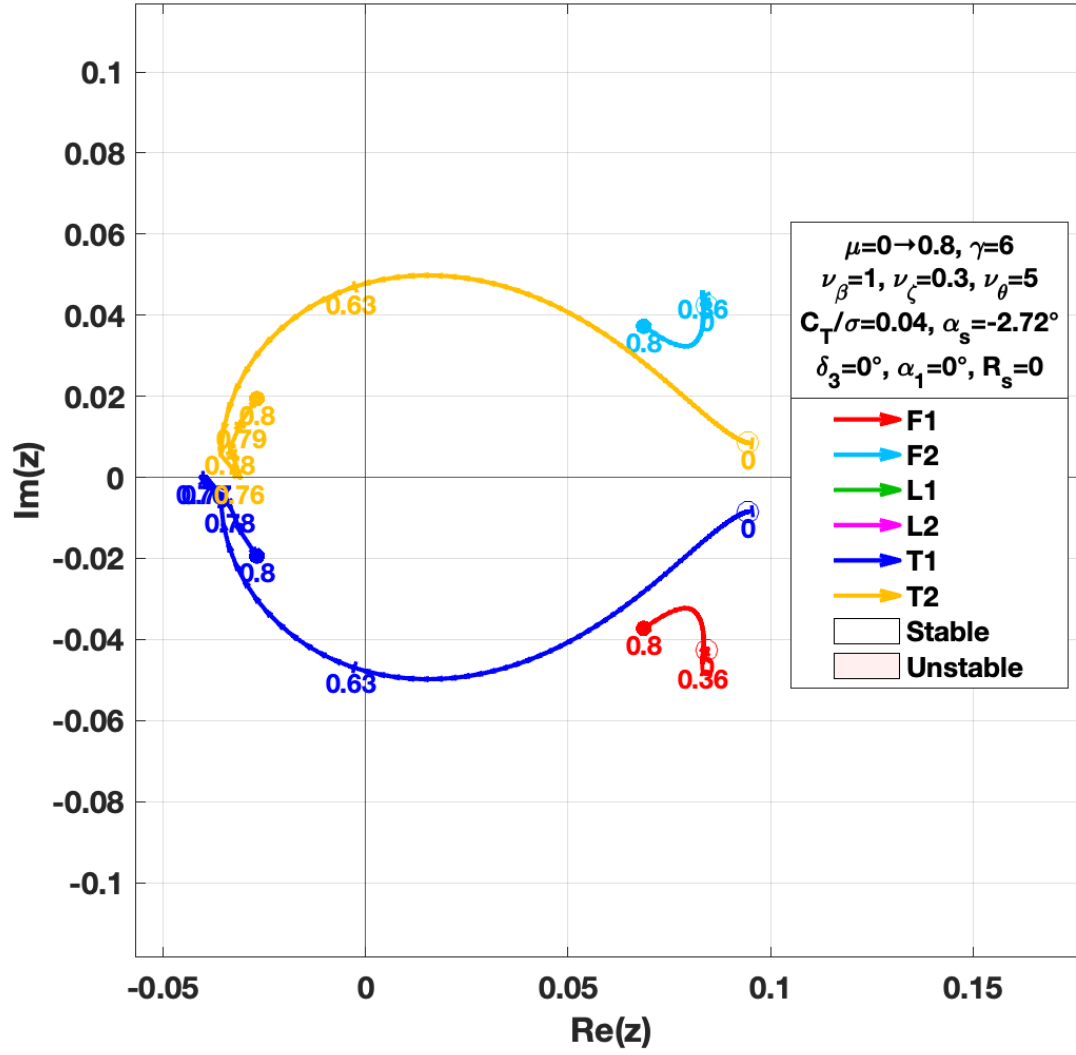


Figure 5.24: Flap-lag-torsion characteristic multipliers, F1, F2, T1, and T2 roots, very soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.3$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8 .

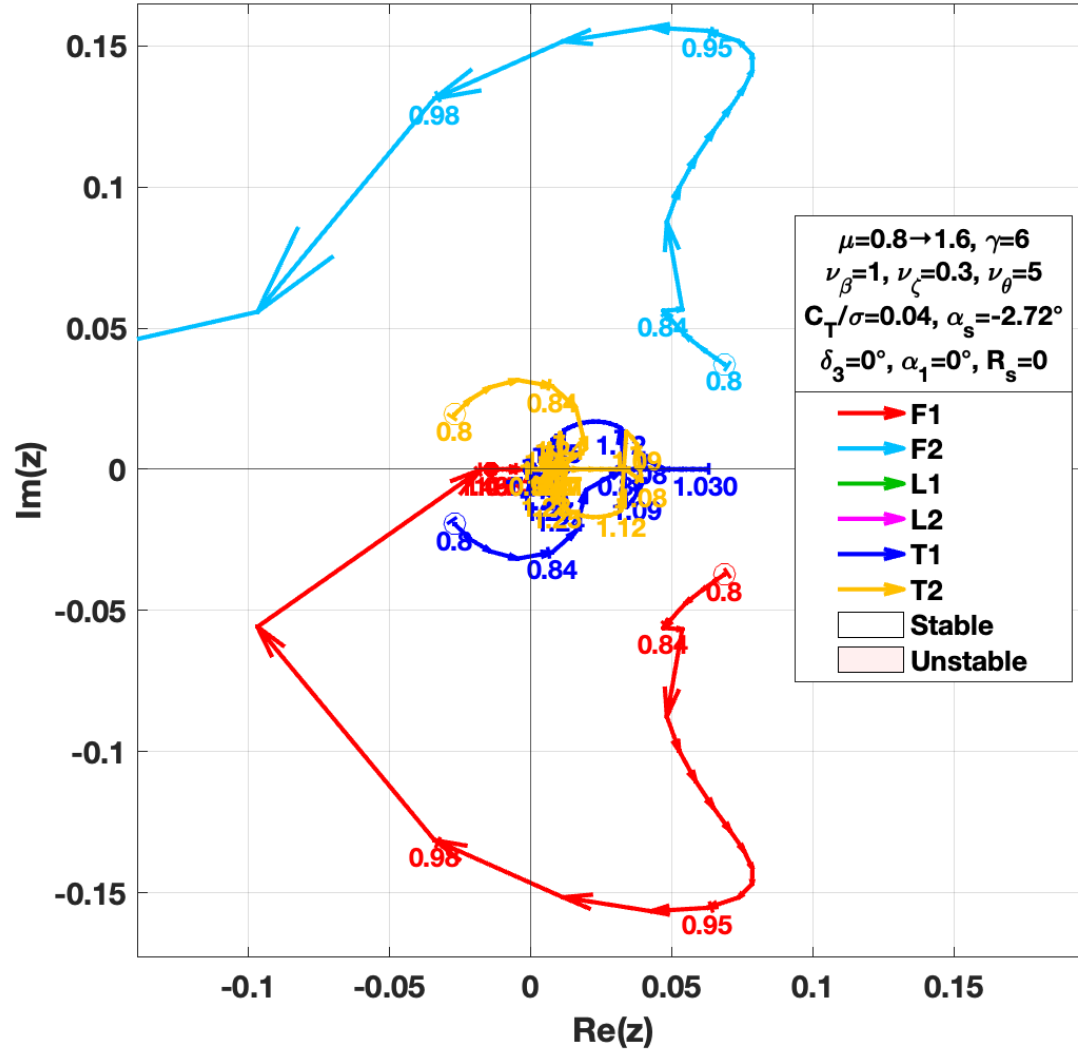


Figure 5.25: Flap-lag-torsion characteristic multipliers, F1, F2, T1, and T2 roots, very soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.3$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$, $\mu = 0.8$ to 1.6.

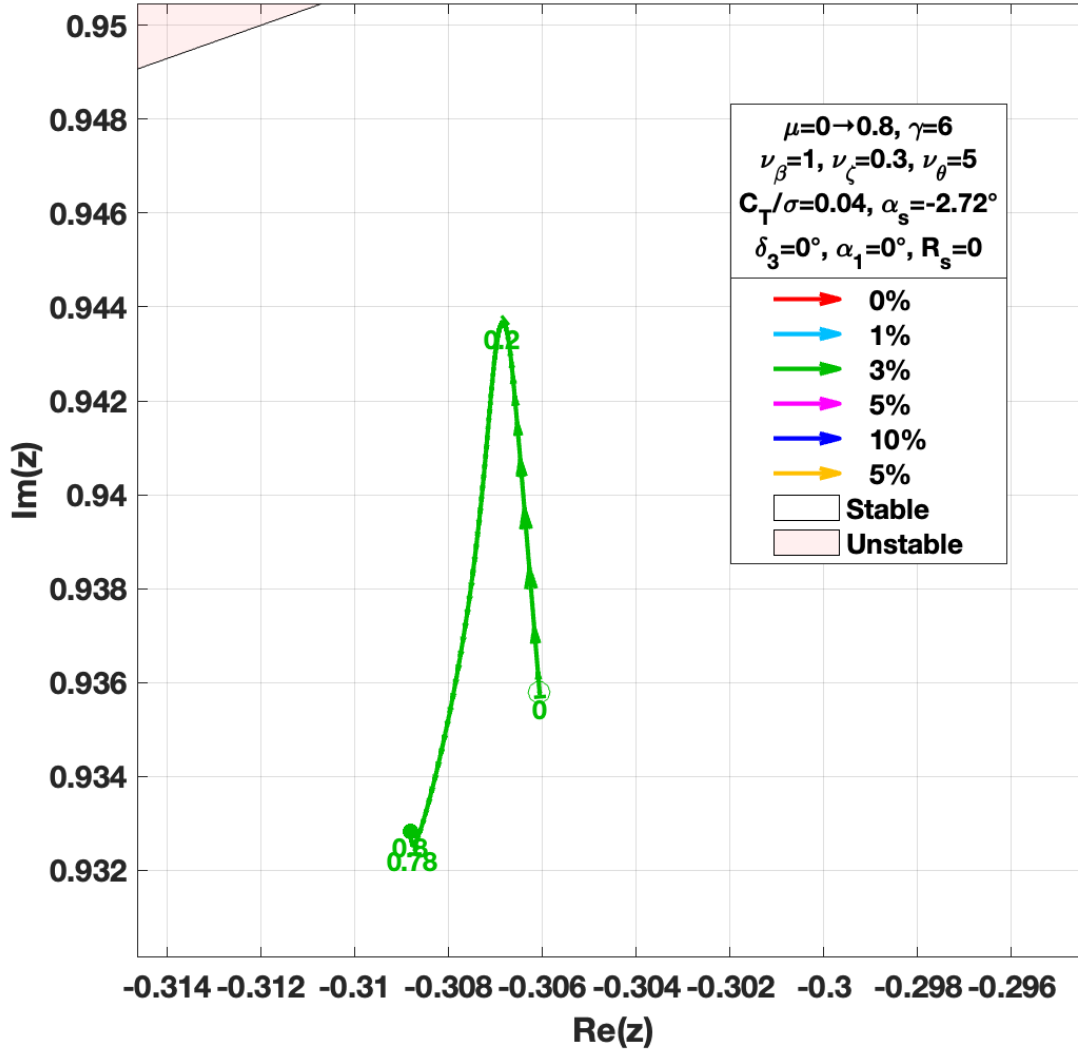


Figure 5.26: Flap-lag-torsion characteristic multipliers, L1 root, very soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.3$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8 .

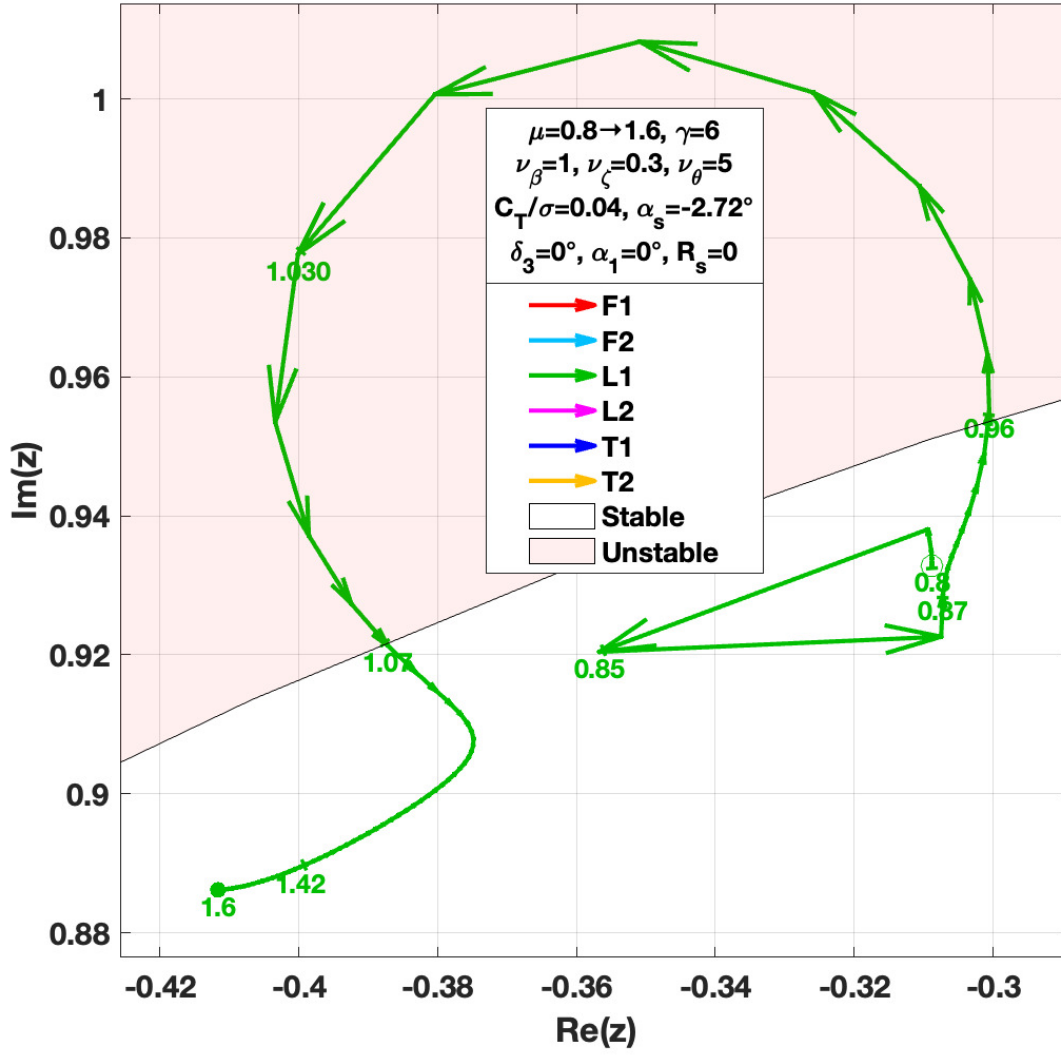


Figure 5.27: Flap-lag-torsion characteristic multipliers, L1 root, very soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.3$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$, $\mu = 0.8$ to 1.6 .

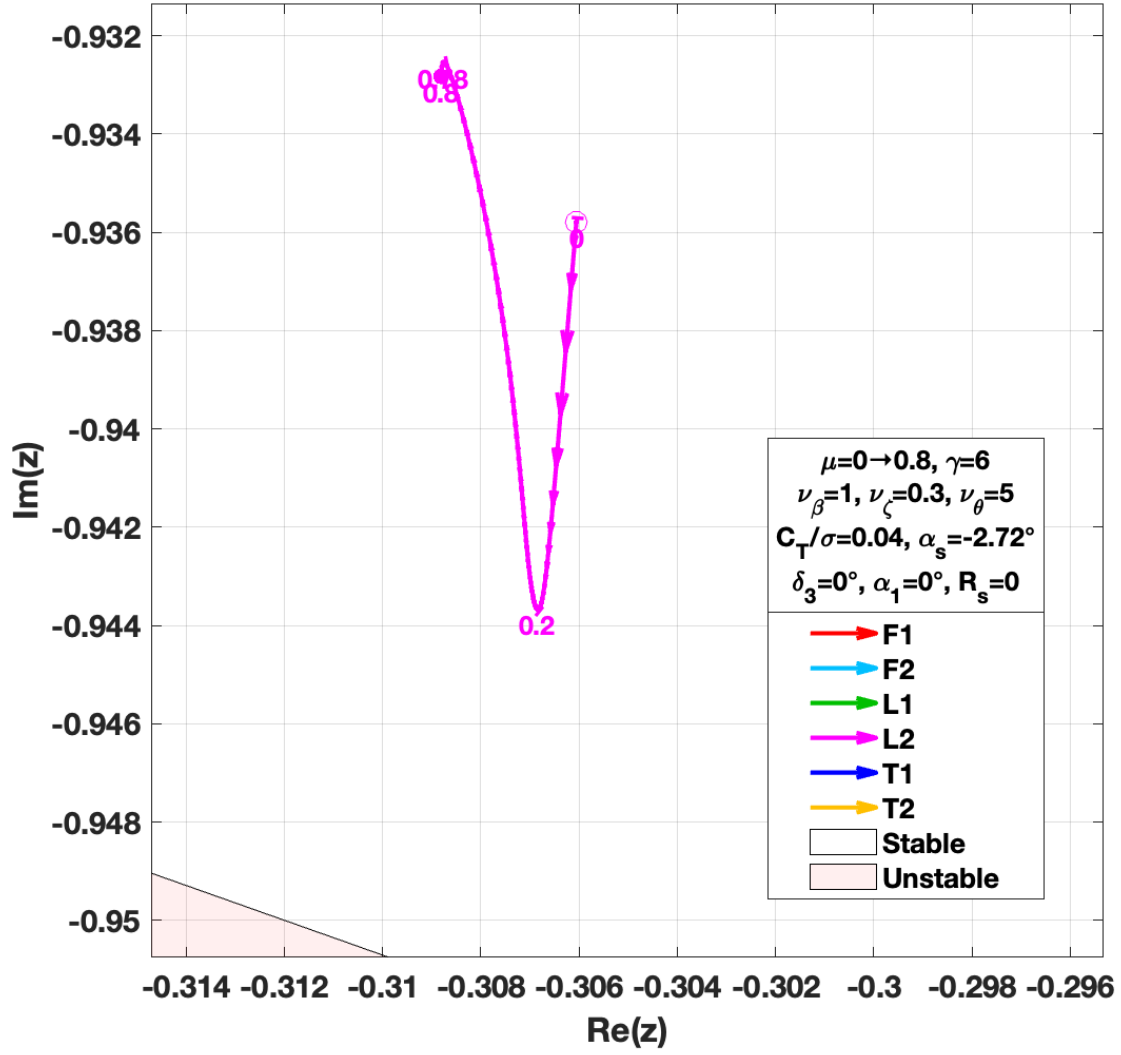


Figure 5.28: Flap-lag-torsion characteristic multipliers, L2 root, very soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.3$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8 .

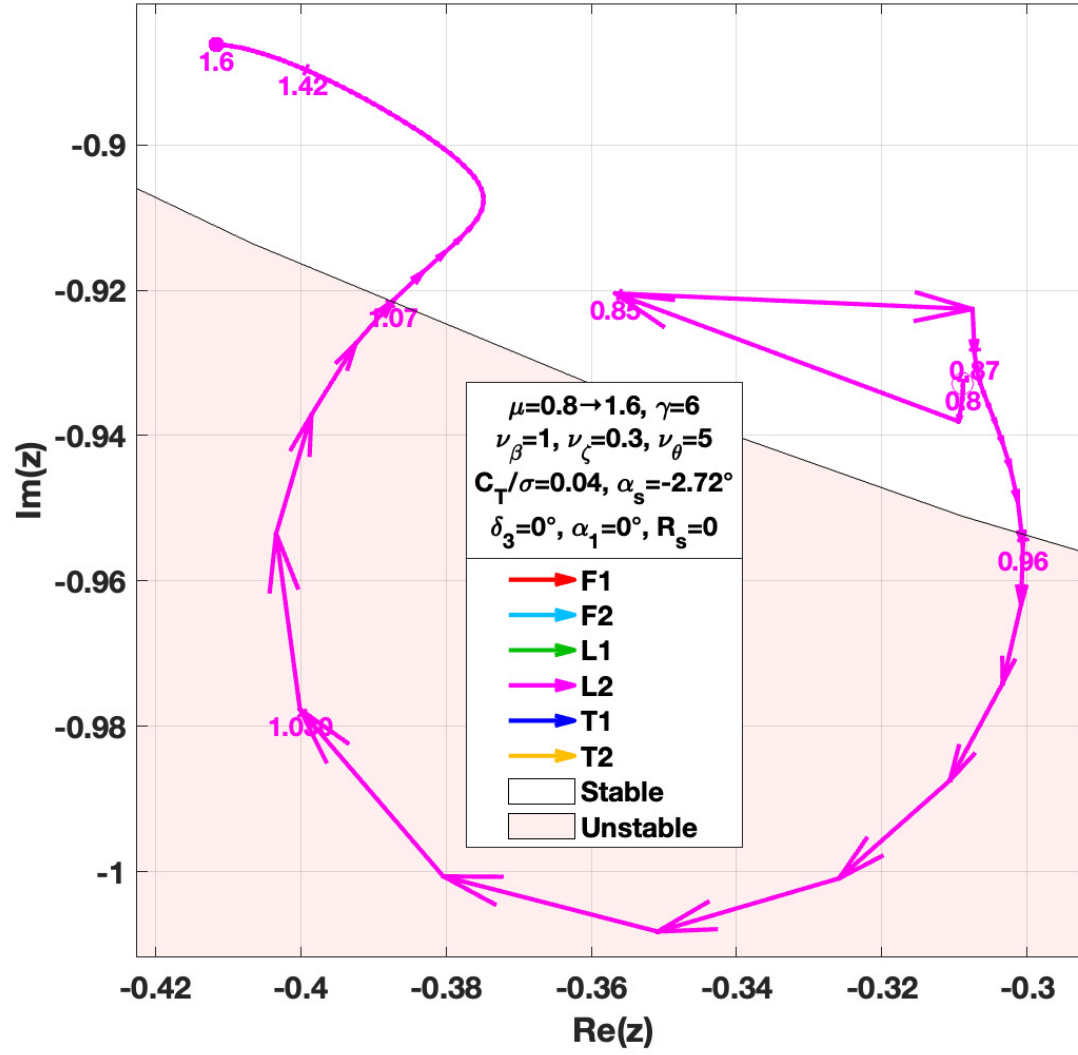


Figure 5.29: Flap-lag-torsion characteristic multipliers, L2 root, very soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.3$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$, $\mu = 0.8$ to 1.6 .

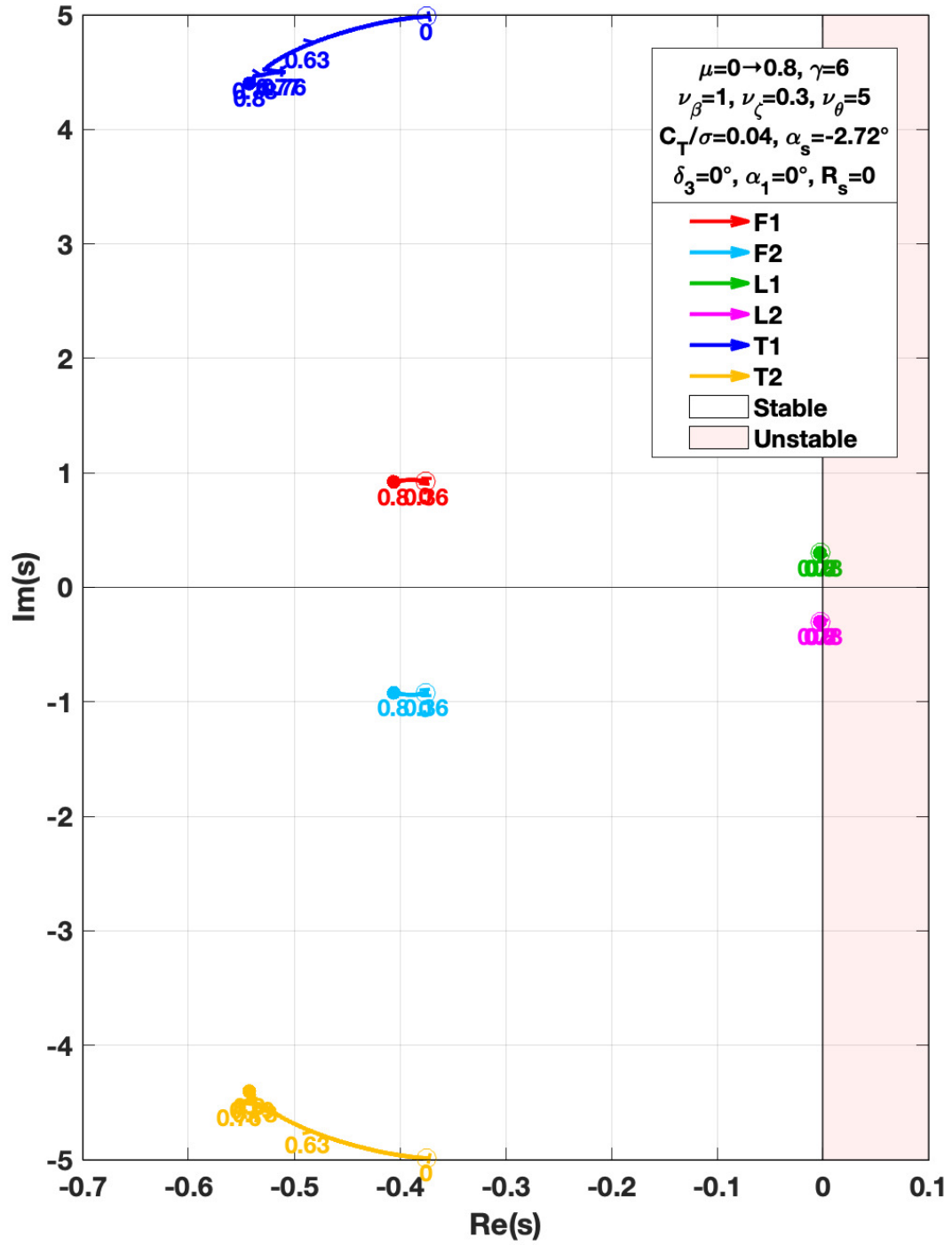


Figure 5.30: Flap-lag-torsion root locus, very soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.3$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8 .

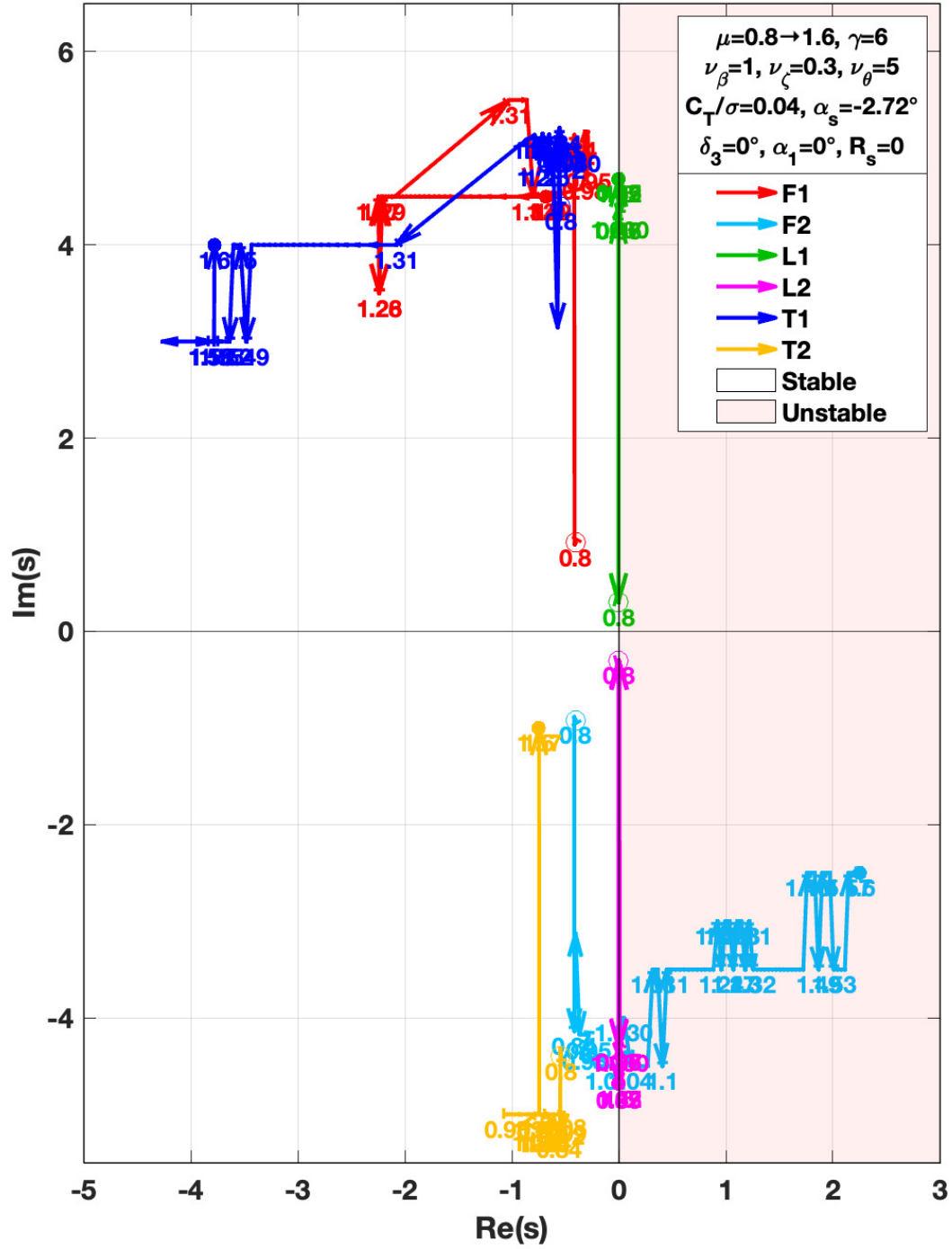


Figure 5.31: Flap-lag-torsion root locus, very soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.3$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$, $\mu = 0.8$ to 1.6 .

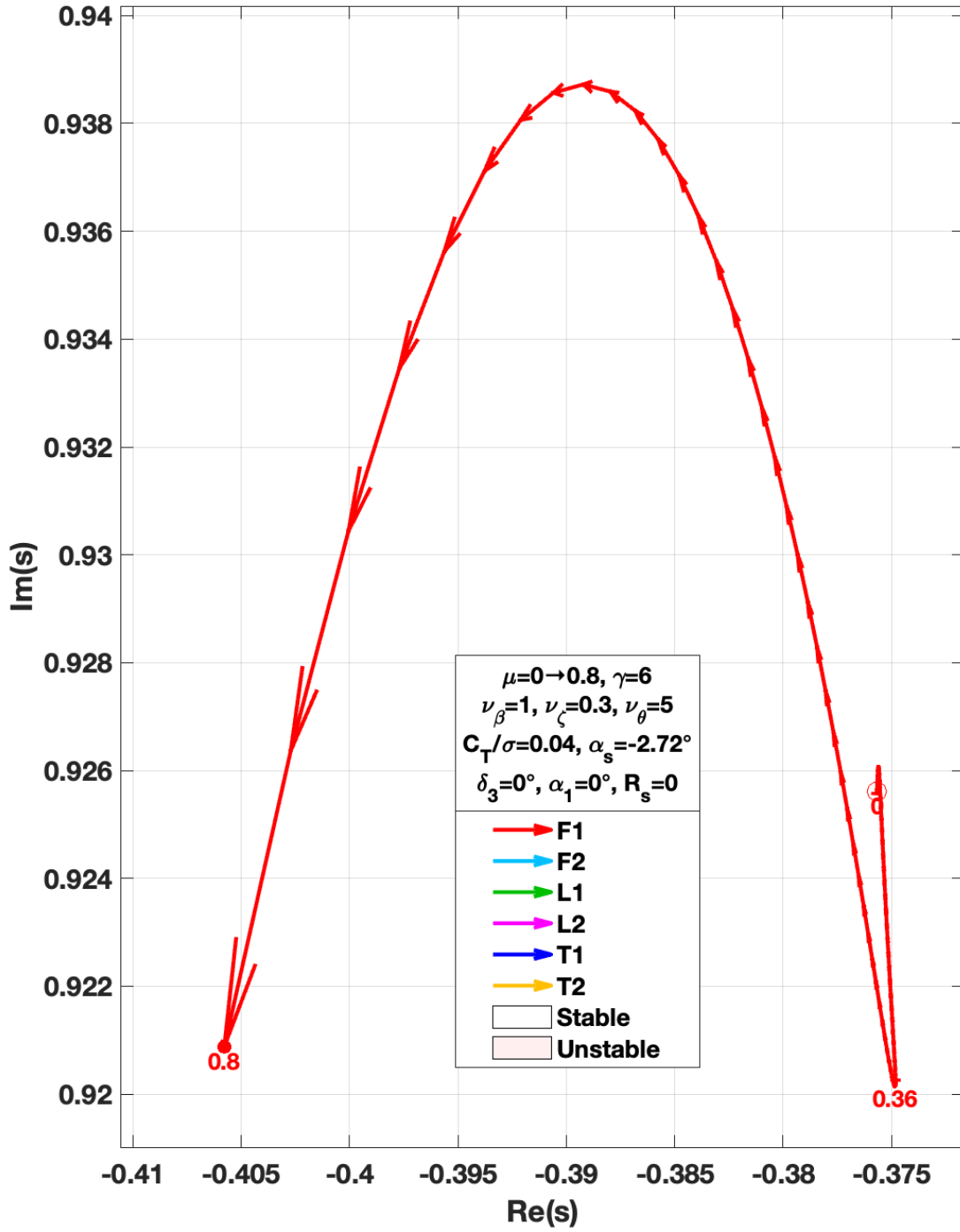


Figure 5.32: Flap-lag-torsion root locus, F1 root, very soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.3$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8 .

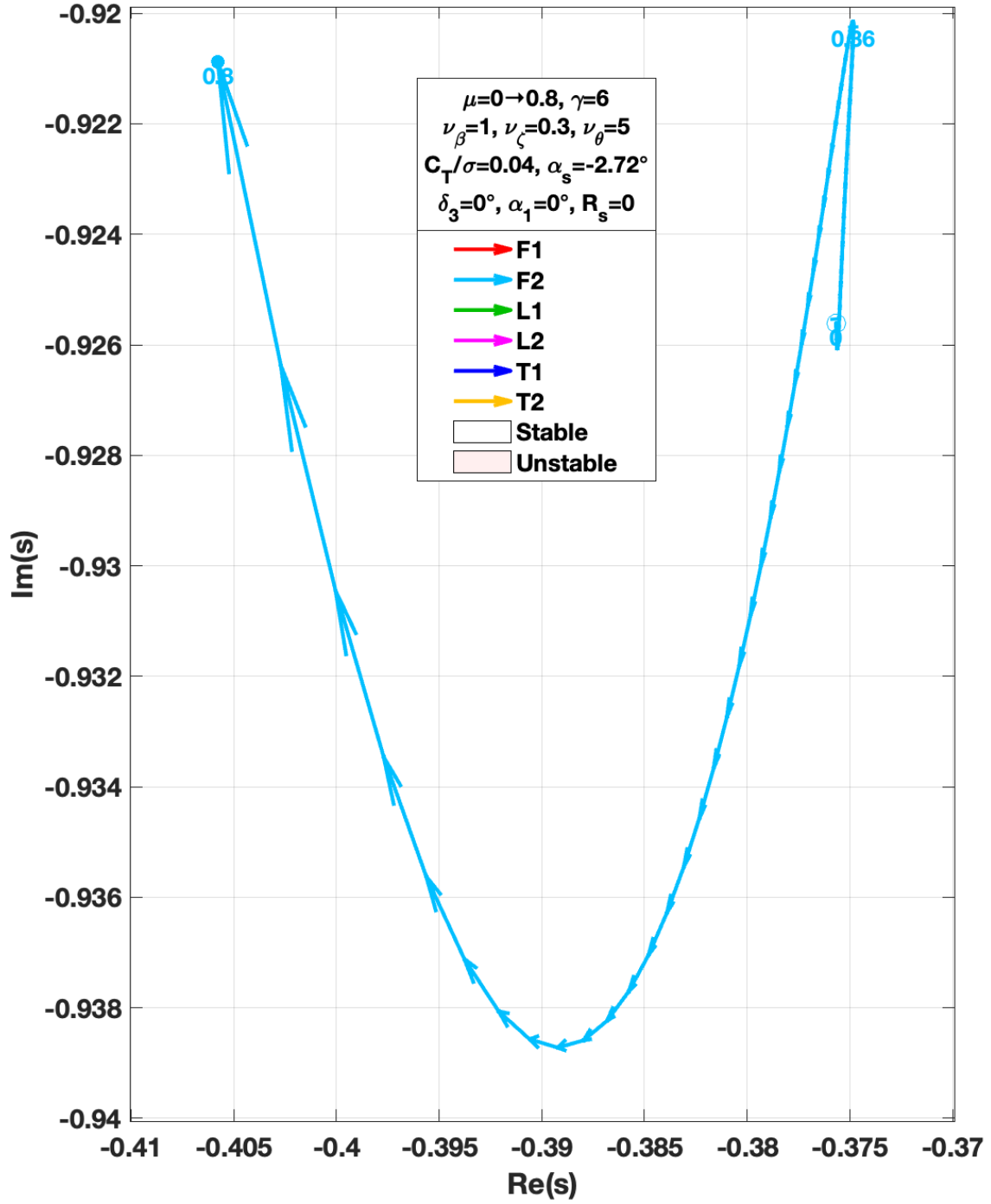


Figure 5.33: Flap-lag-torsion root locus, F2 root, very soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.3$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8 .

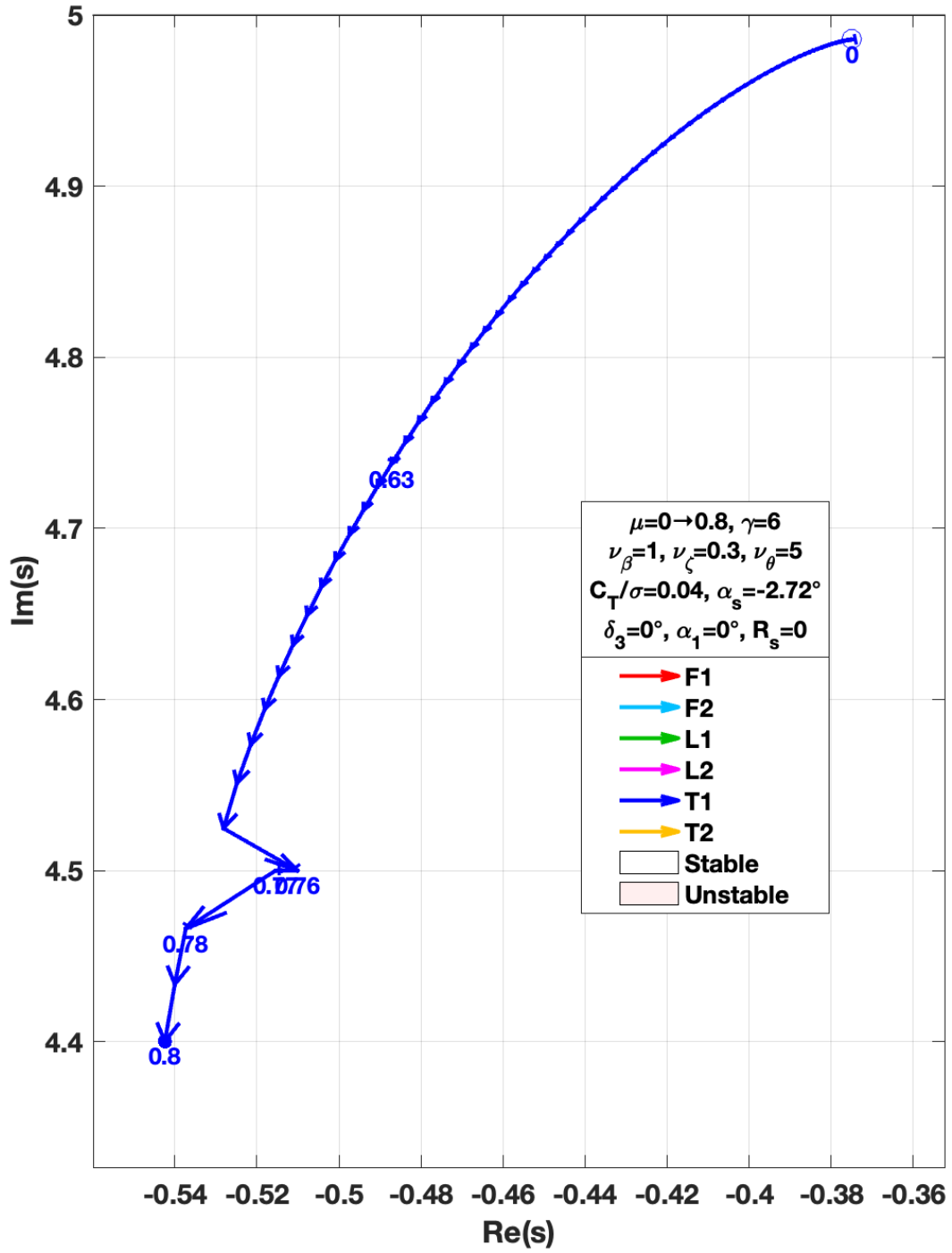


Figure 5.34: Flap-lag-torsion root locus, T1 root, very soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.3$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8 .

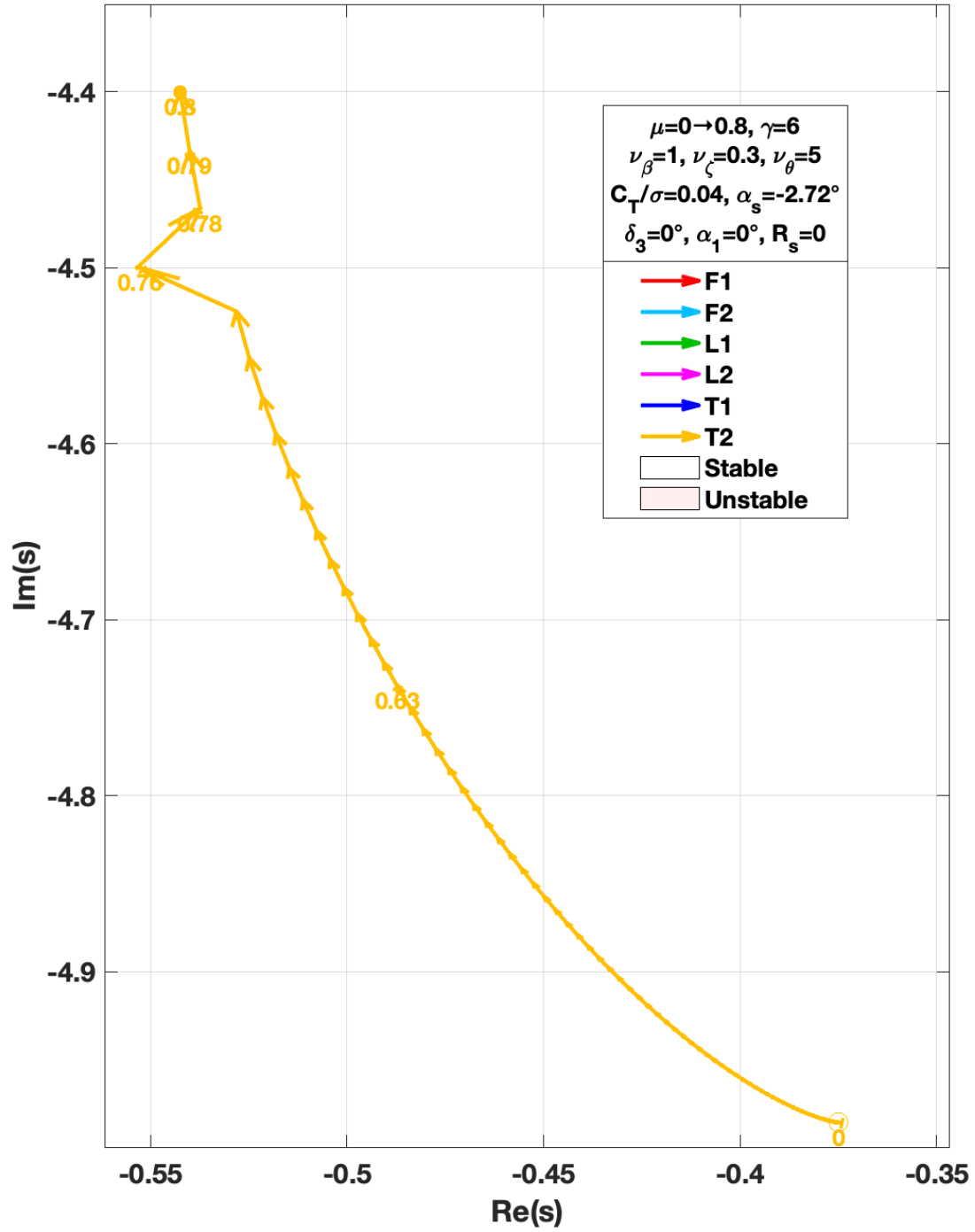


Figure 5.35: Flap-lag-torsion root locus, T2 root, very soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.3$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8 .

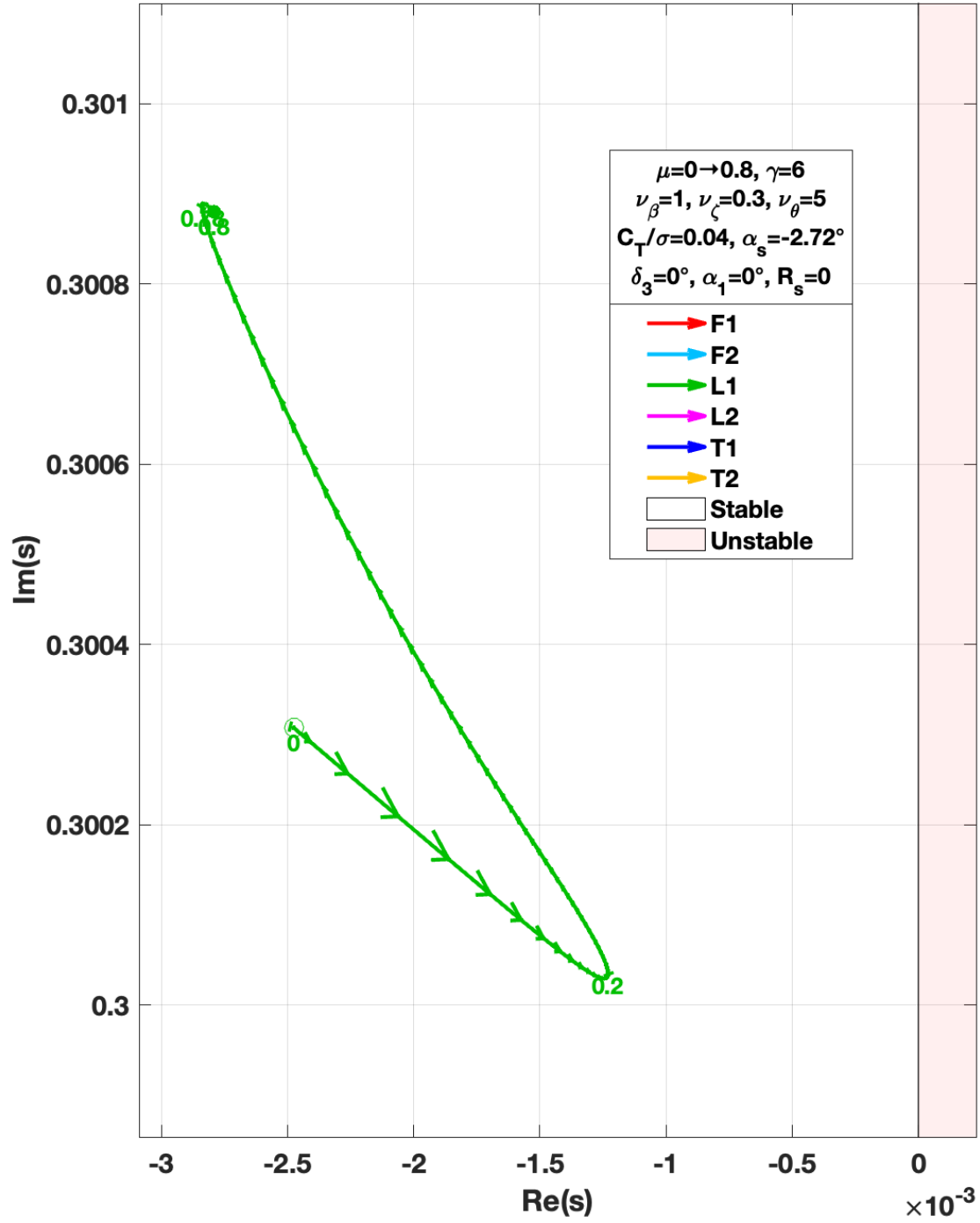


Figure 5.36: Flap-lag-torsion root locus, L1 root, very soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.3$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8 .

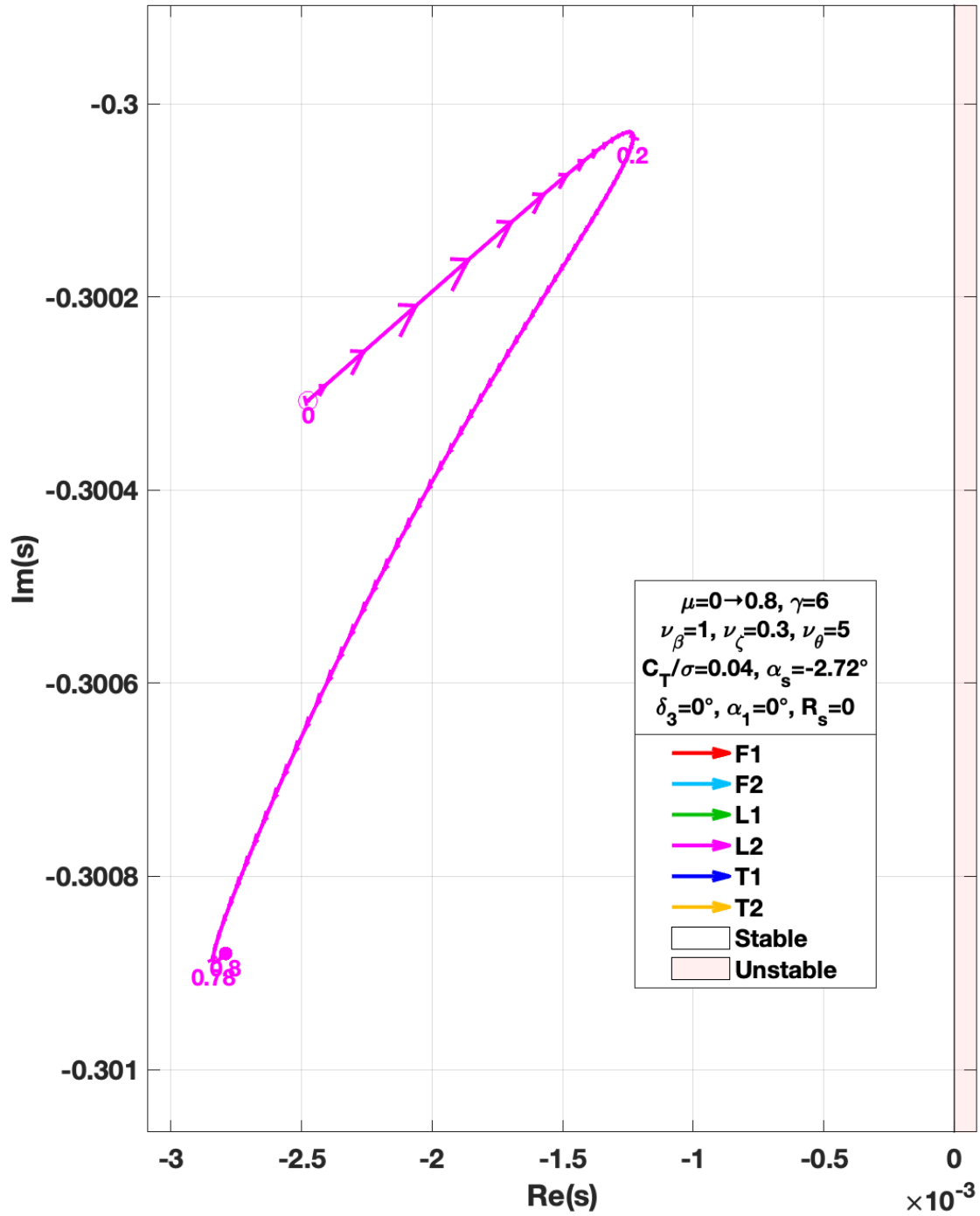


Figure 5.37: Flap-lag-torsion root locus, L2 root, very soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.3$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8 .

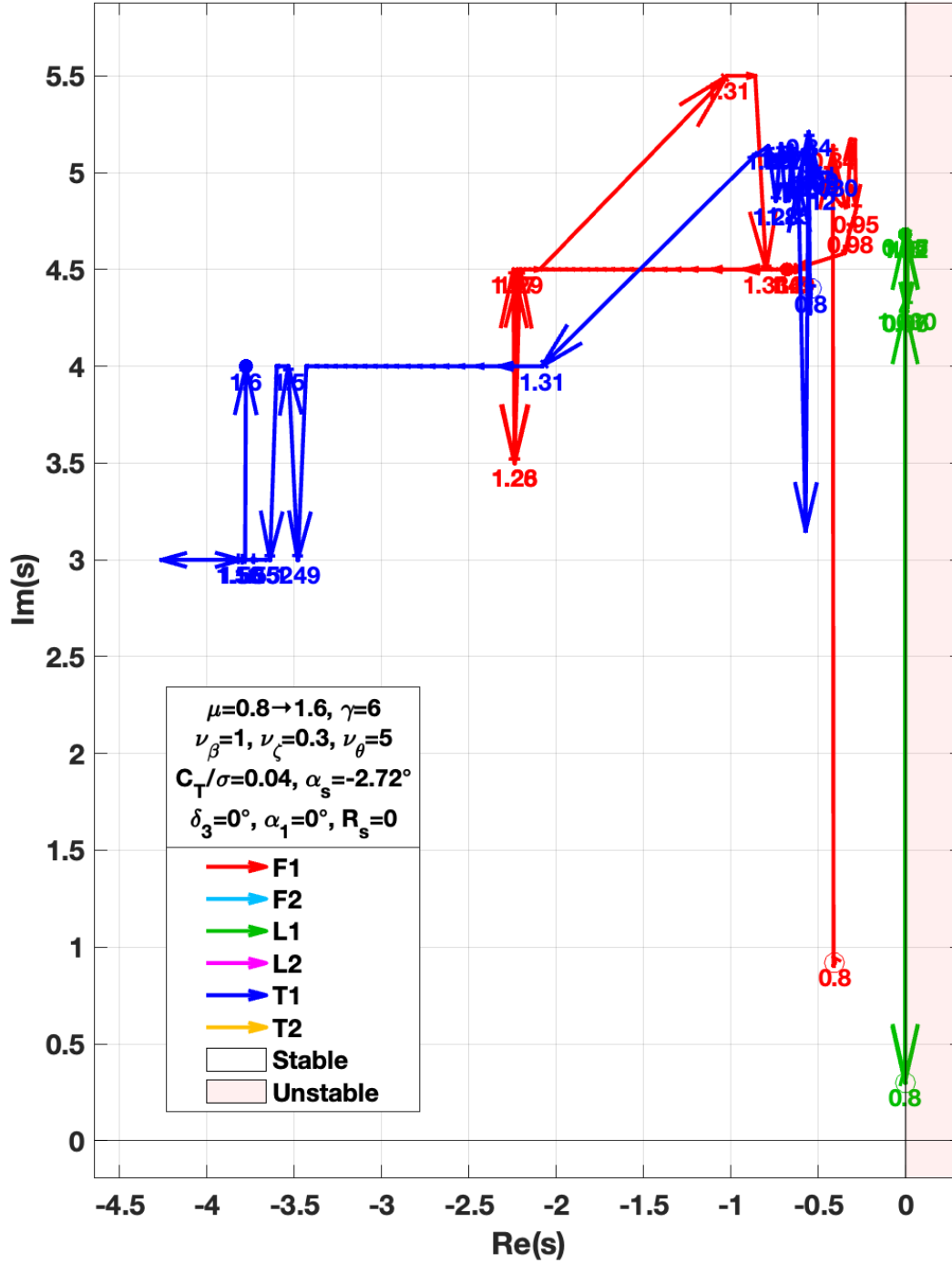


Figure 5.38: Flap-lag-torsion root locus, F1, L1, and T1 roots, very soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.3$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$, $C(k) = 1$, $\mu = 0.8$ to 1.6.

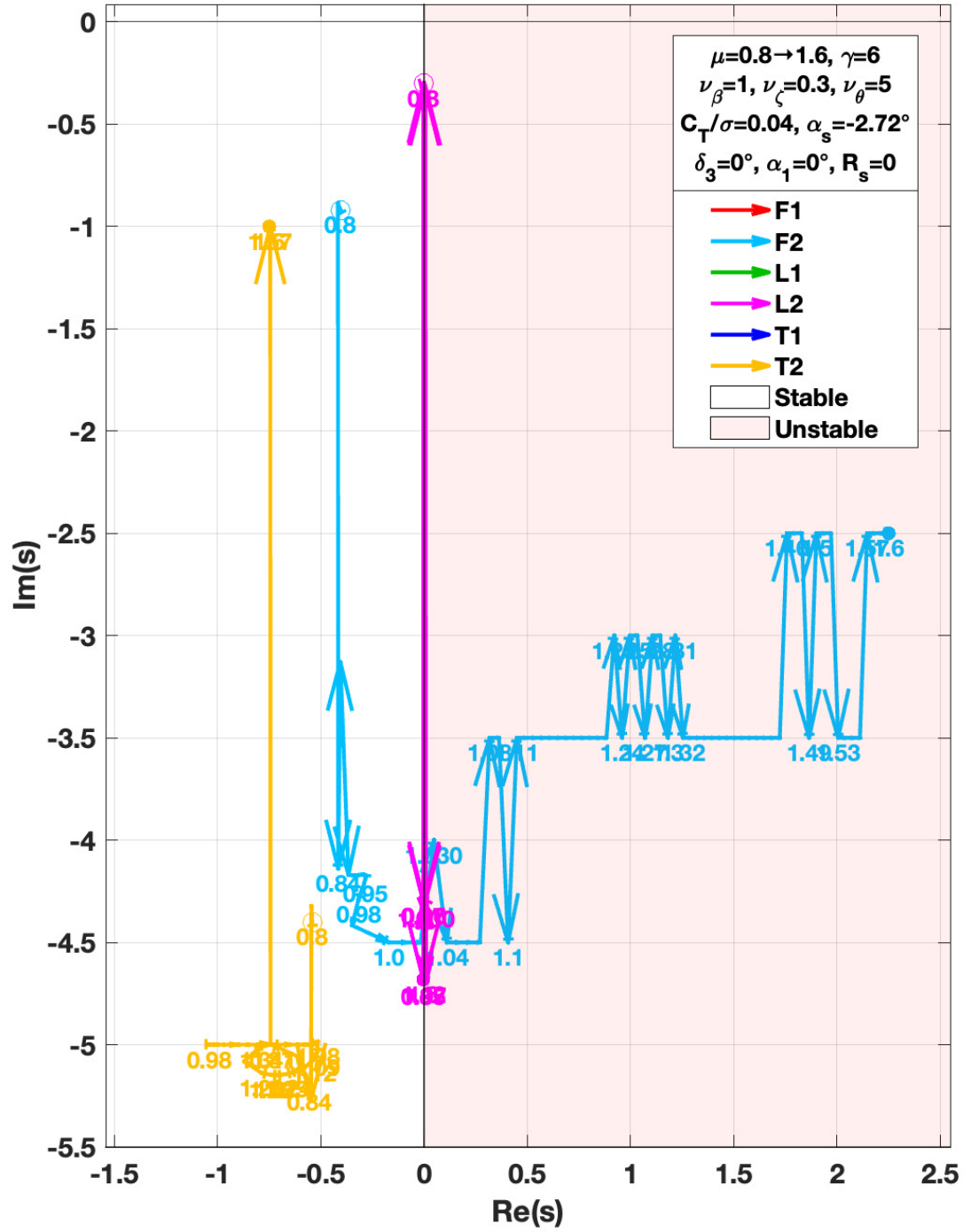


Figure 5.39: Flap-lag-torsion root locus, F2, L2, and T2 roots, very soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.3$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$, $\mu = 0.8$ to 1.6.

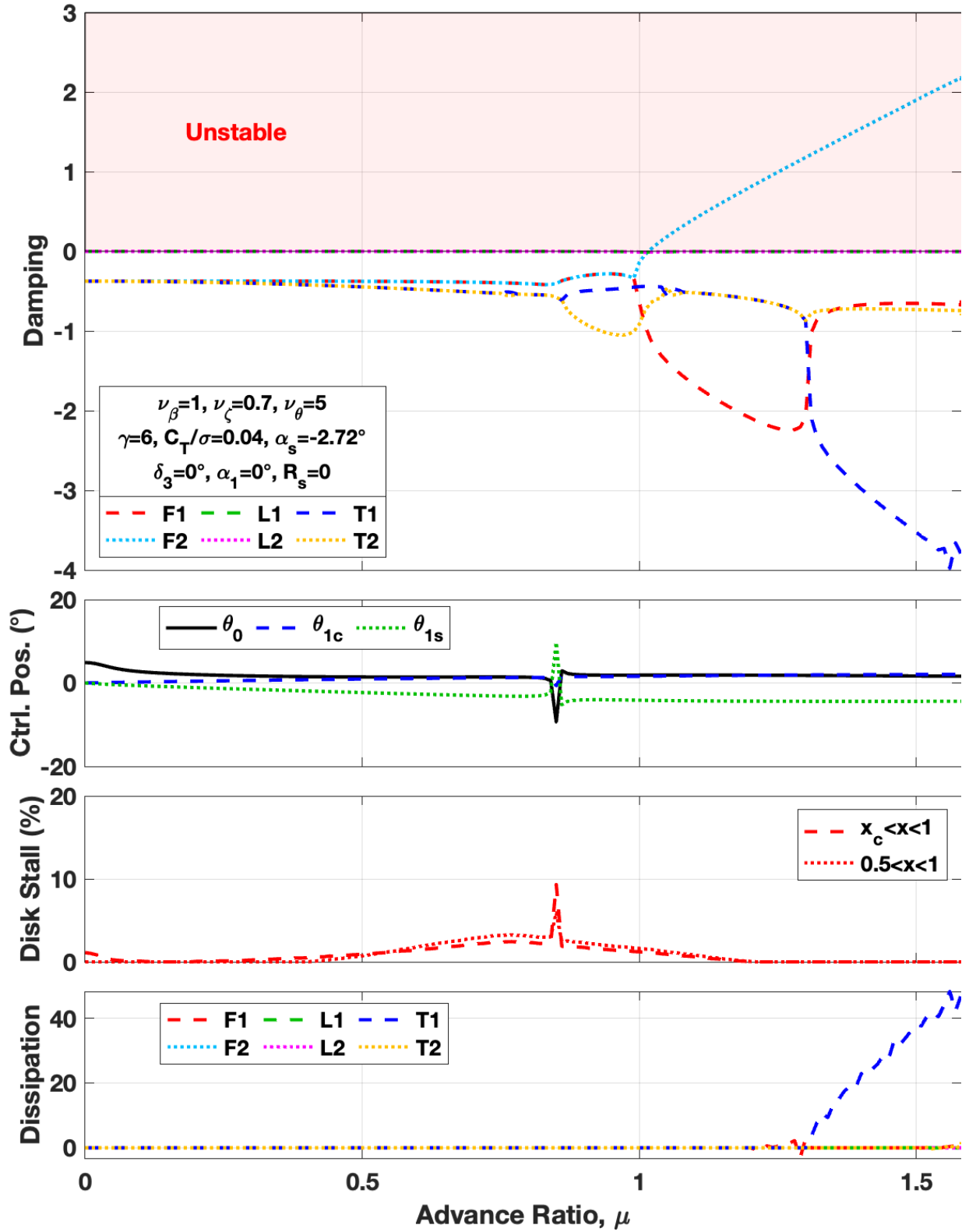
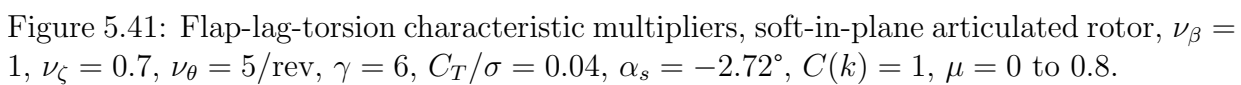


Figure 5.40: Flap-lag-torsion damping, trim, blade stall, and numerical stability, soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.7$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$.



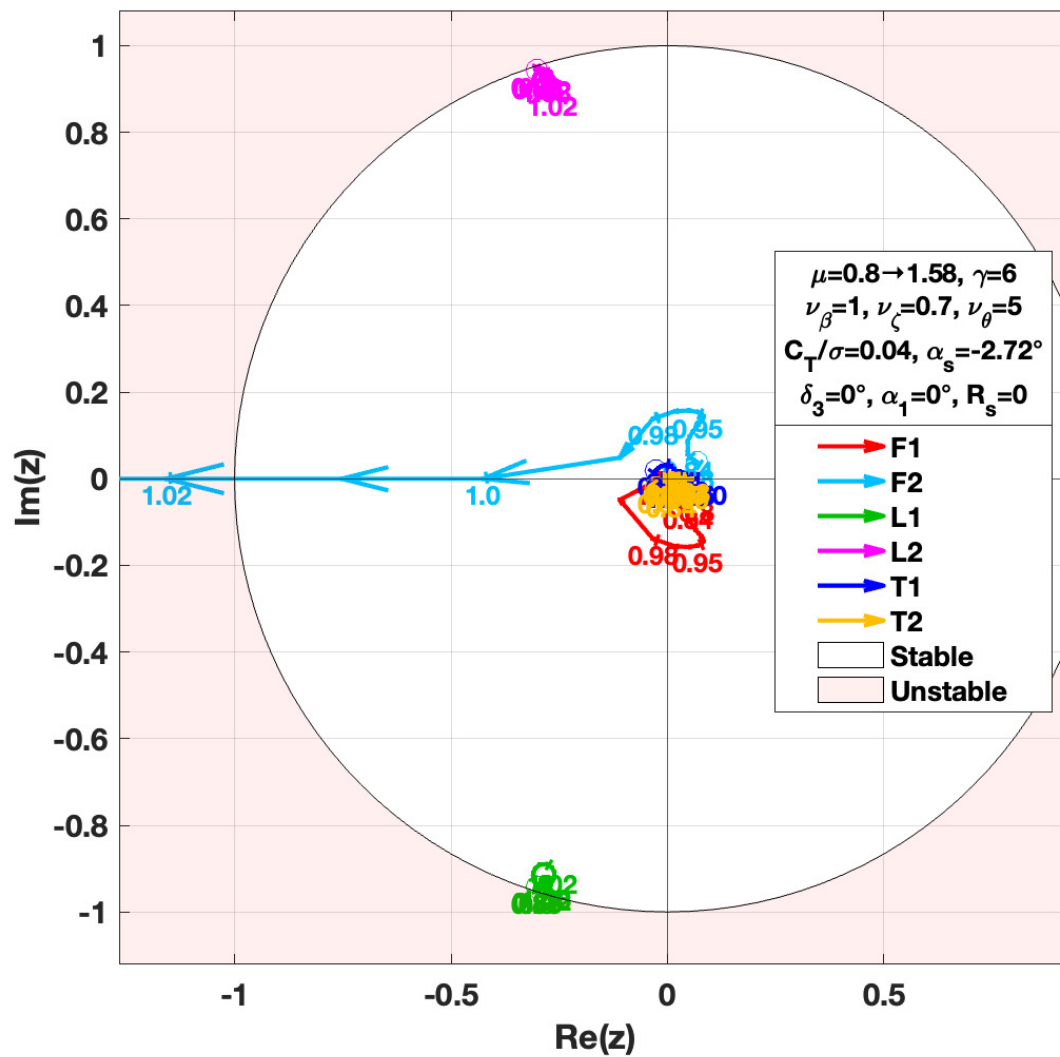


Figure 5.42: Flap-lag-torsion characteristic multipliers, soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.7$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$, $\mu = 0.8$ to 1.58 .

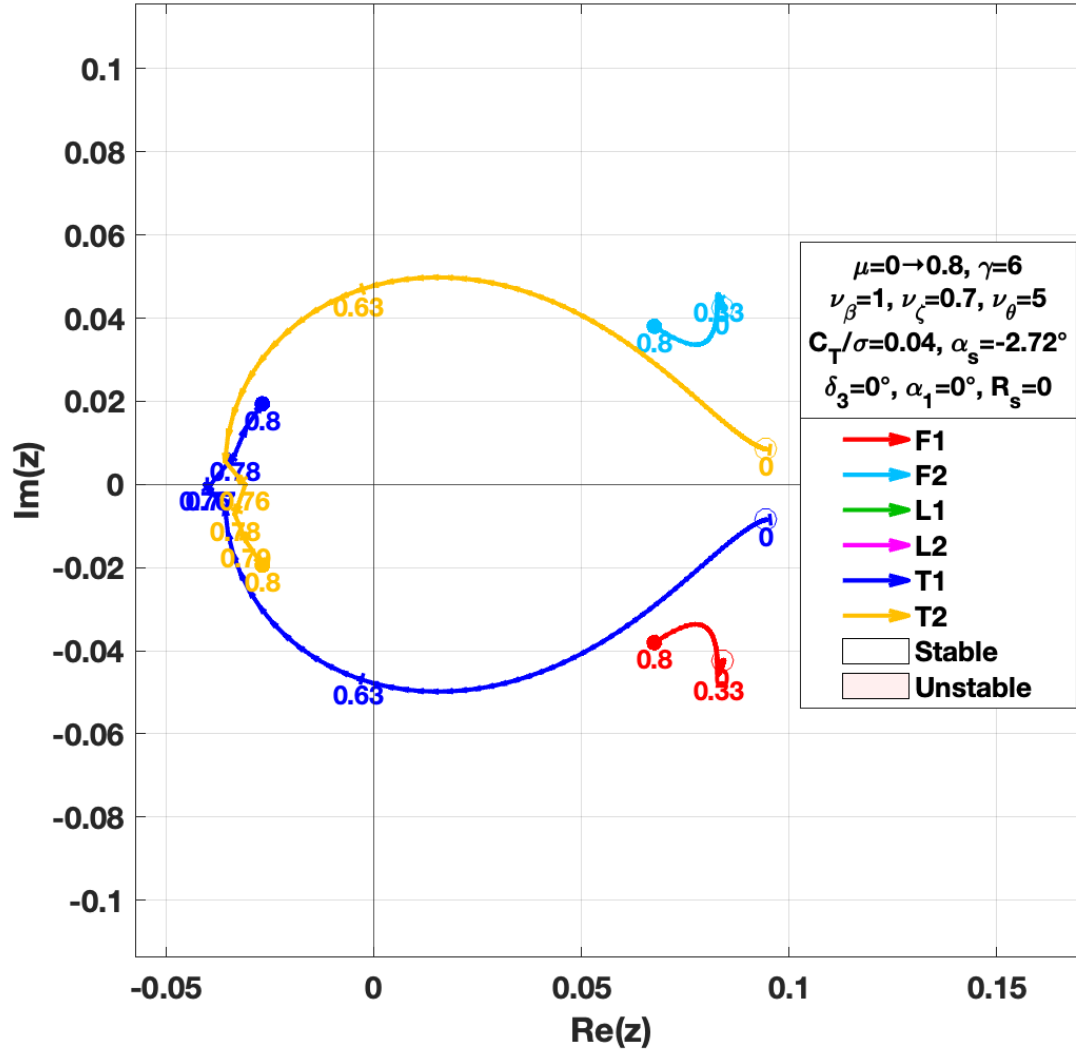


Figure 5.43: Flap-lag-torsion characteristic multipliers, $F1$, $F2$, $T1$, and $T2$ roots, soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.7$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8 .

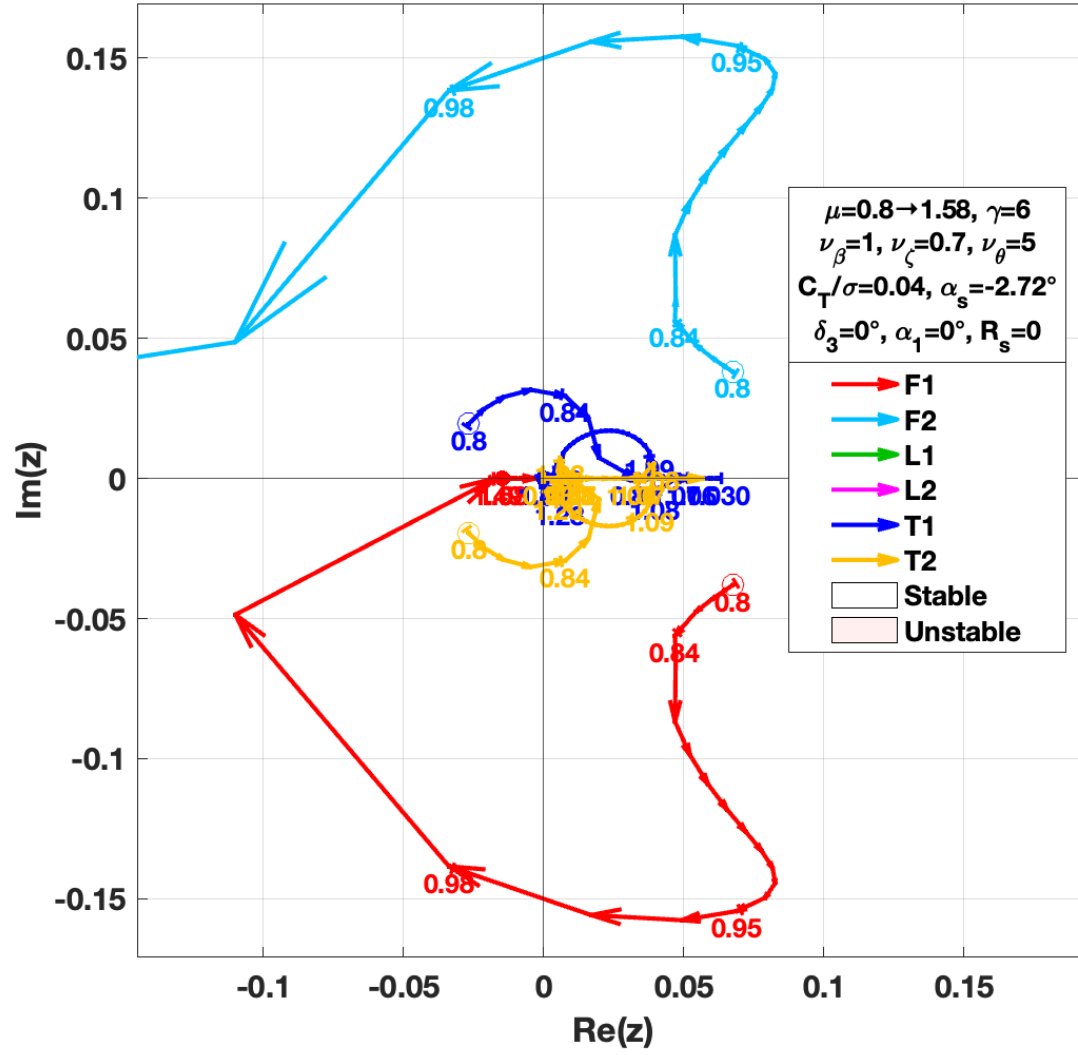


Figure 5.44: Flap-lag-torsion characteristic multipliers, F1, F2, T1, and T2 roots, soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.7$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$, $\mu = 0.8$ to 1.58 .

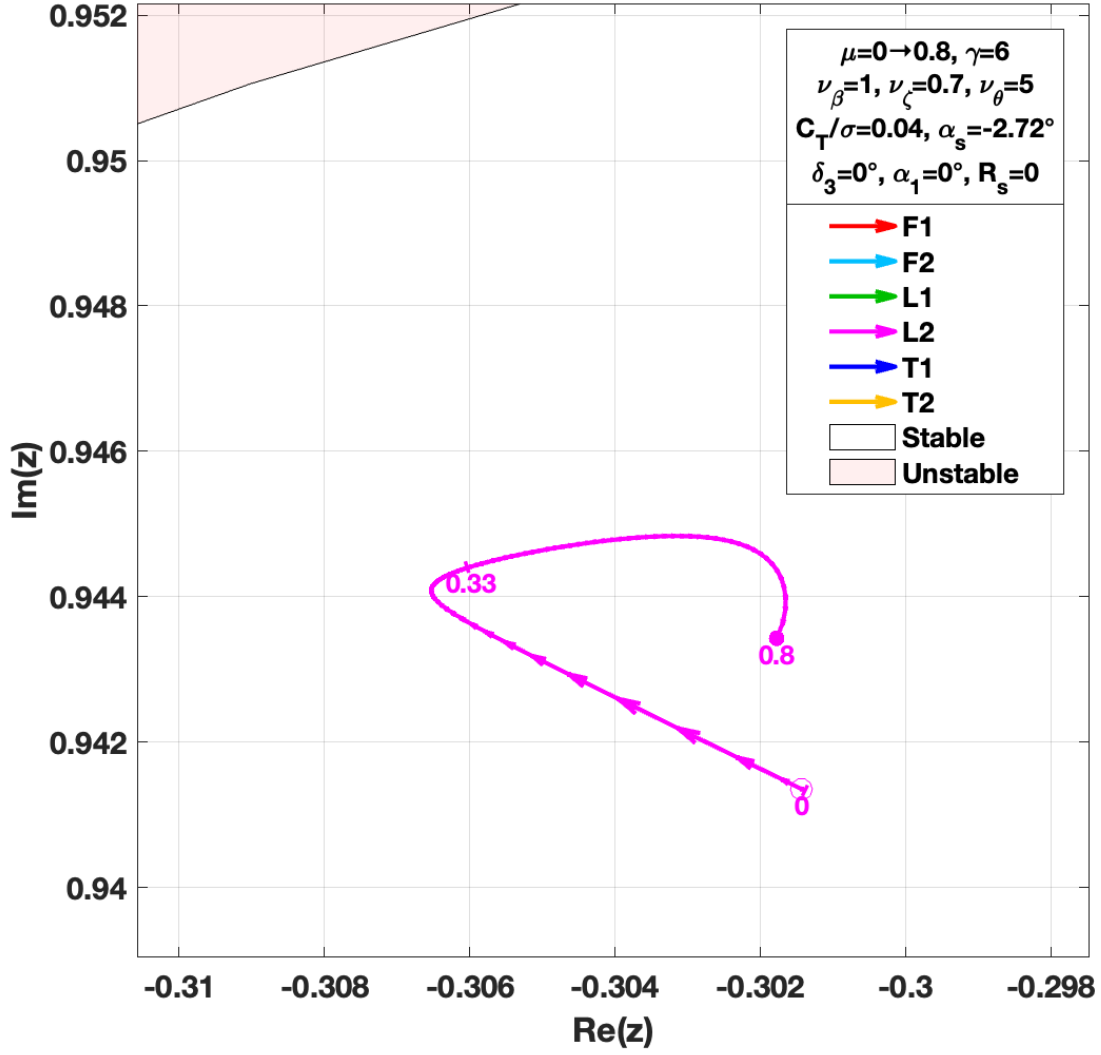


Figure 5.45: Flap-lag-torsion characteristic multipliers, L1 root, soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.7$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8 .

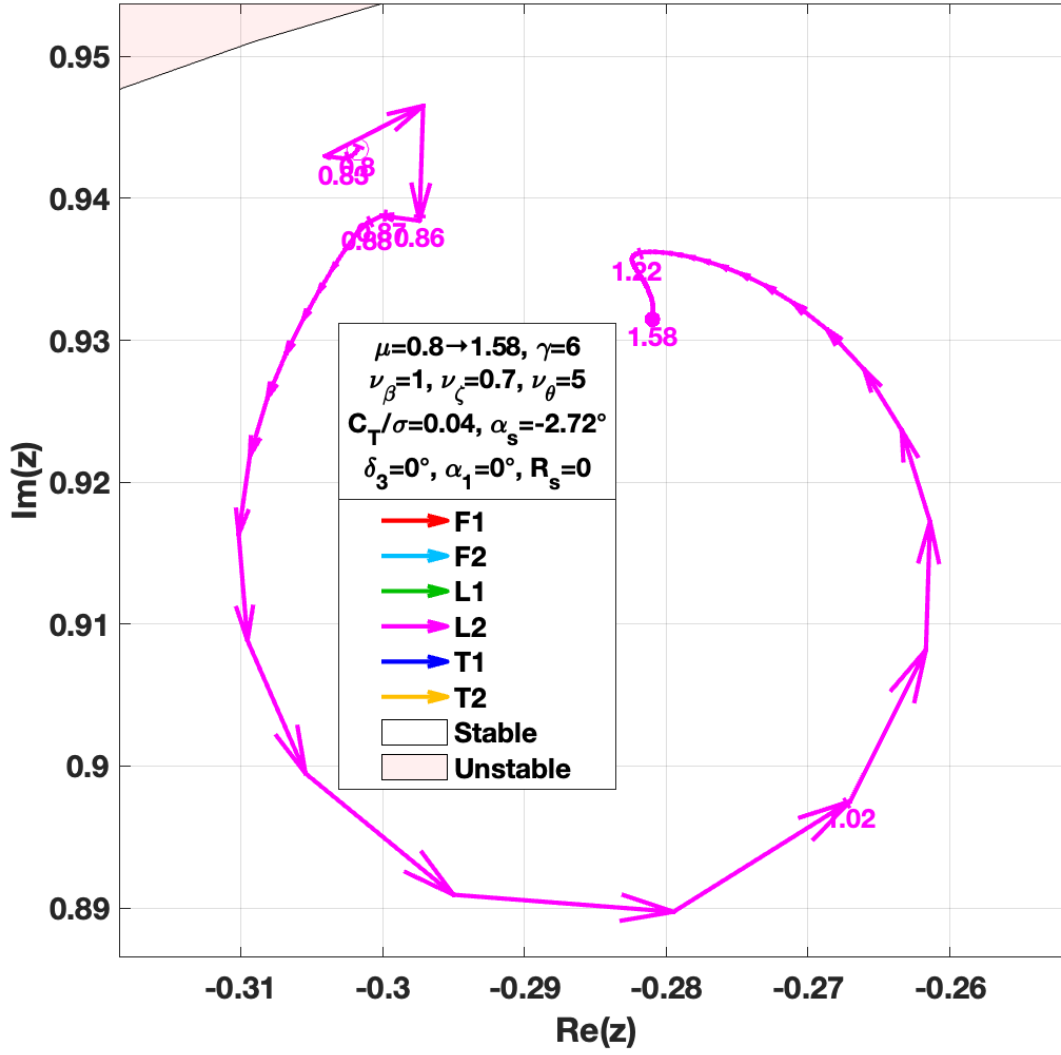


Figure 5.46: Flap-lag-torsion characteristic multipliers, L1 root, soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.7$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$, $\mu = 0.8$ to 1.58.

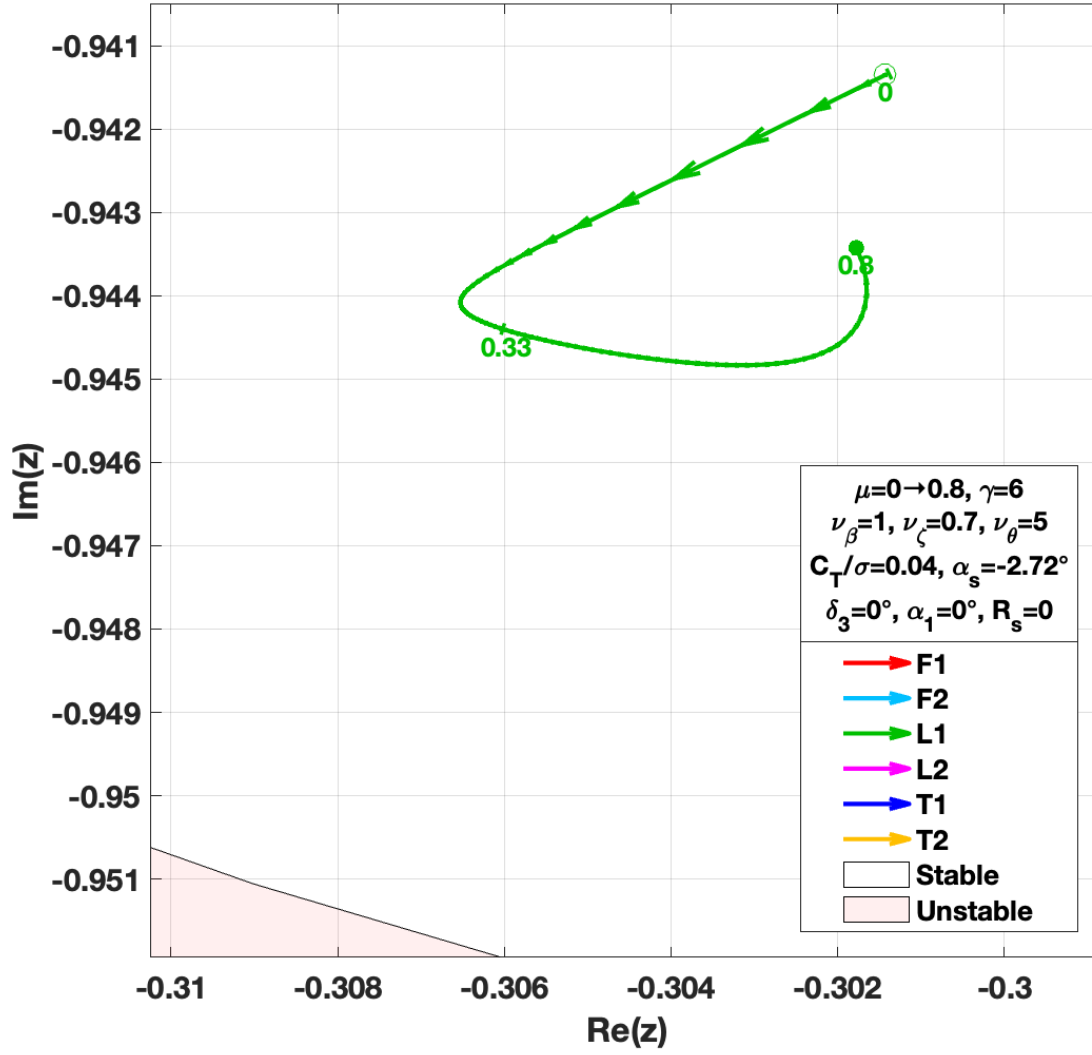


Figure 5.47: Flap-lag-torsion characteristic multipliers, L2 root, soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.7$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8 .

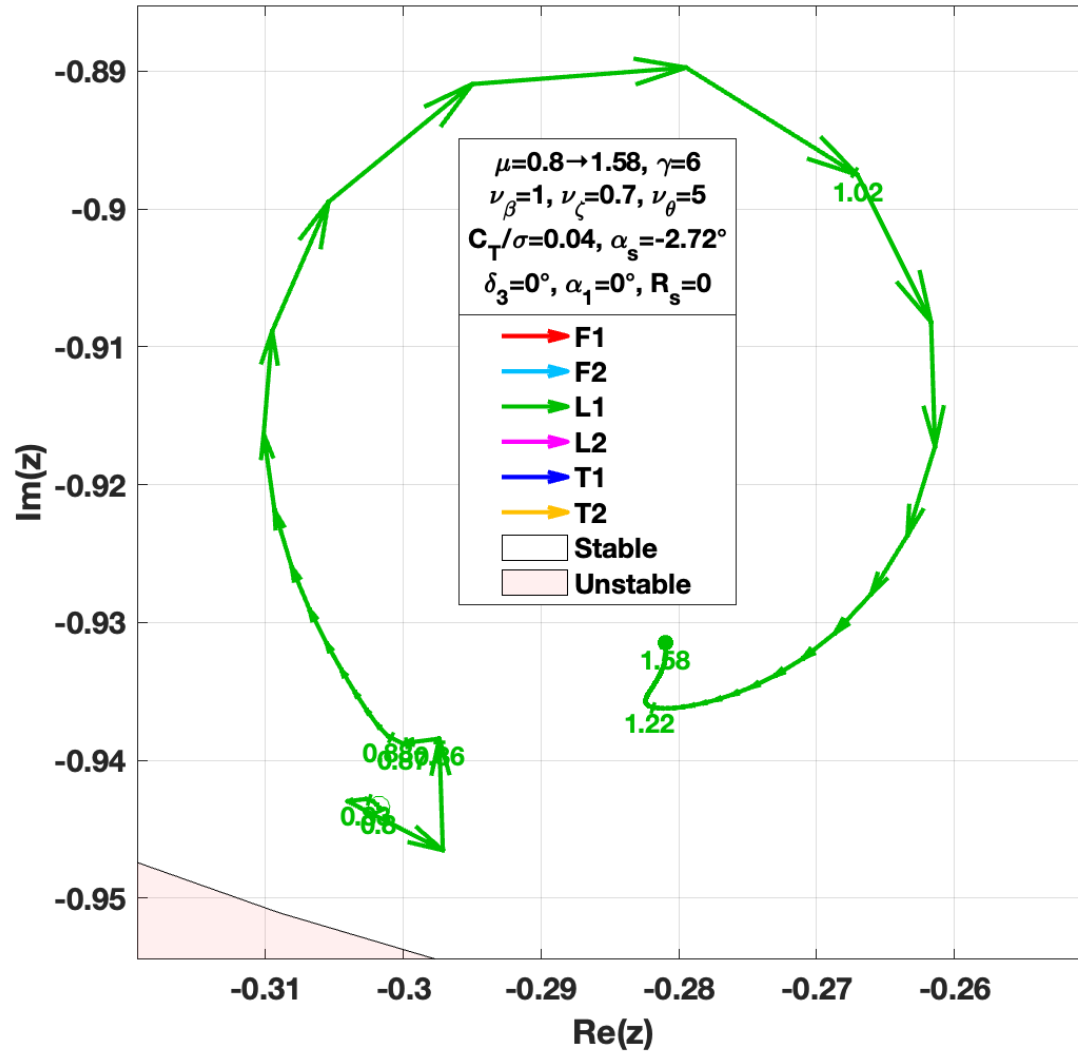


Figure 5.48: Flap-lag-torsion characteristic multipliers, L2 root, soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.7$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$, $\mu = 0.8$ to 1.58.

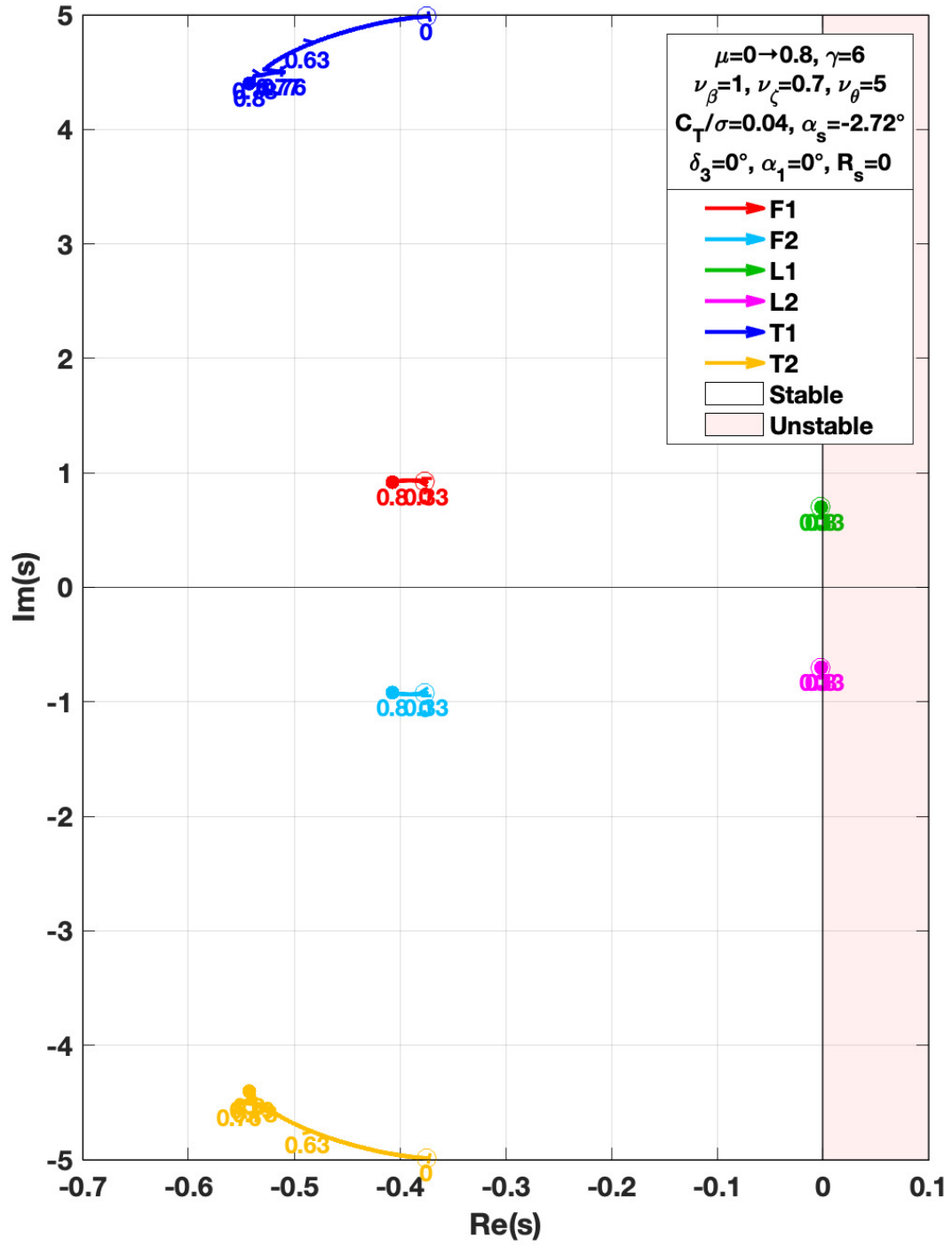


Figure 5.49: Flap-lag-torsion root locus, soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.7$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8 .

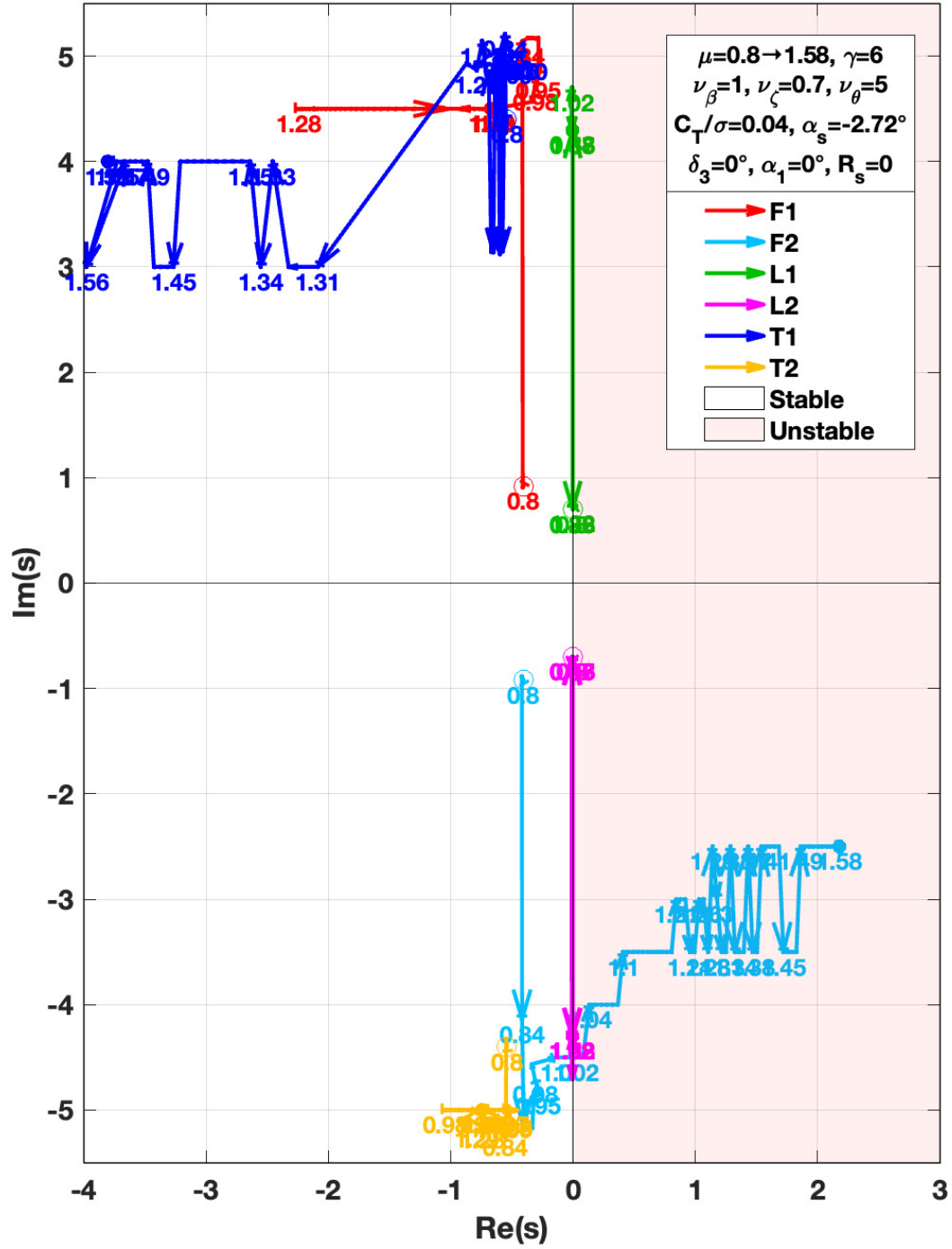


Figure 5.50: Flap-lag-torsion root locus, soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.7$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$, $\mu = 0.8$ to 1.58 .

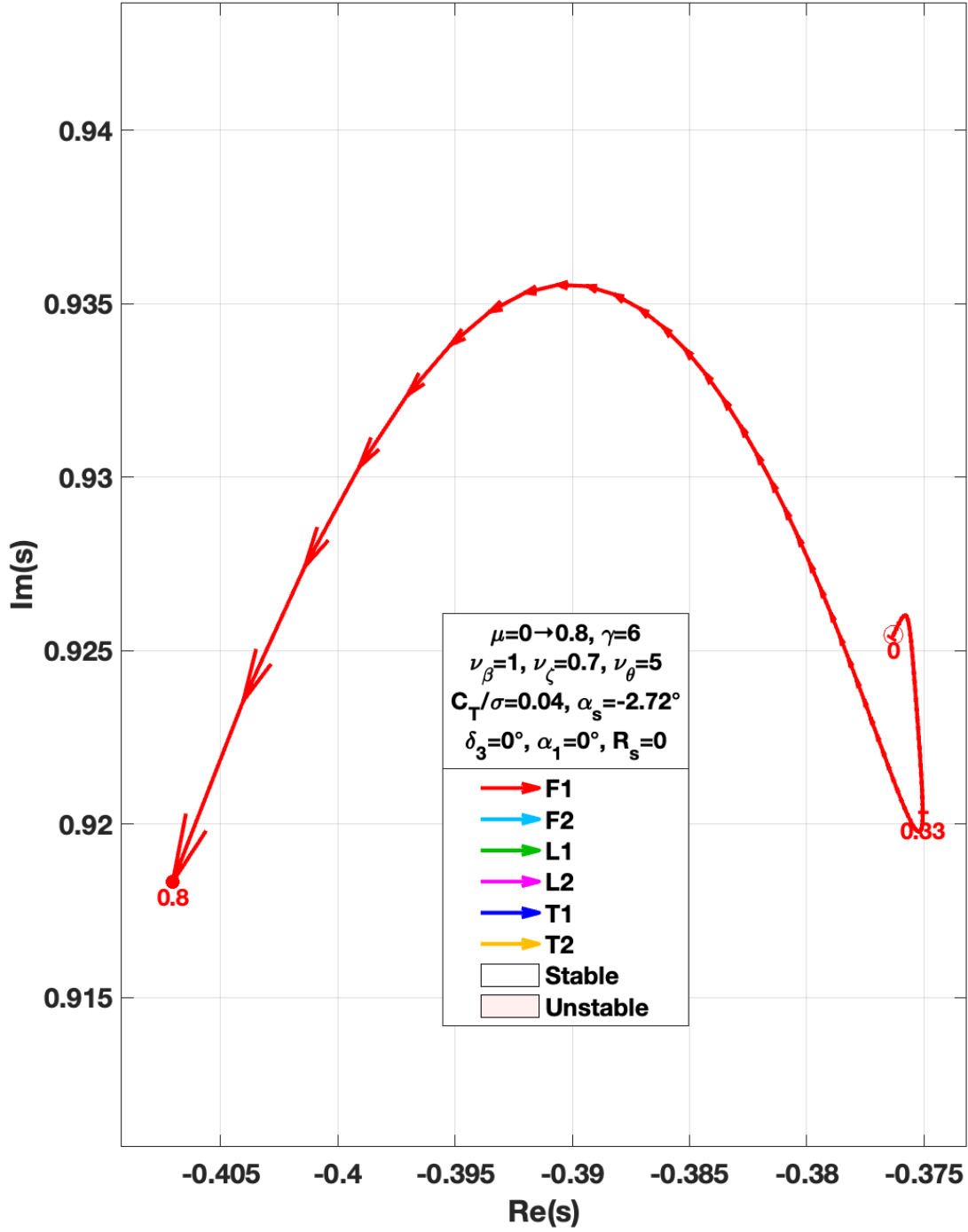


Figure 5.51: Flap-lag-torsion root locus, F1 root, soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.7$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8 .

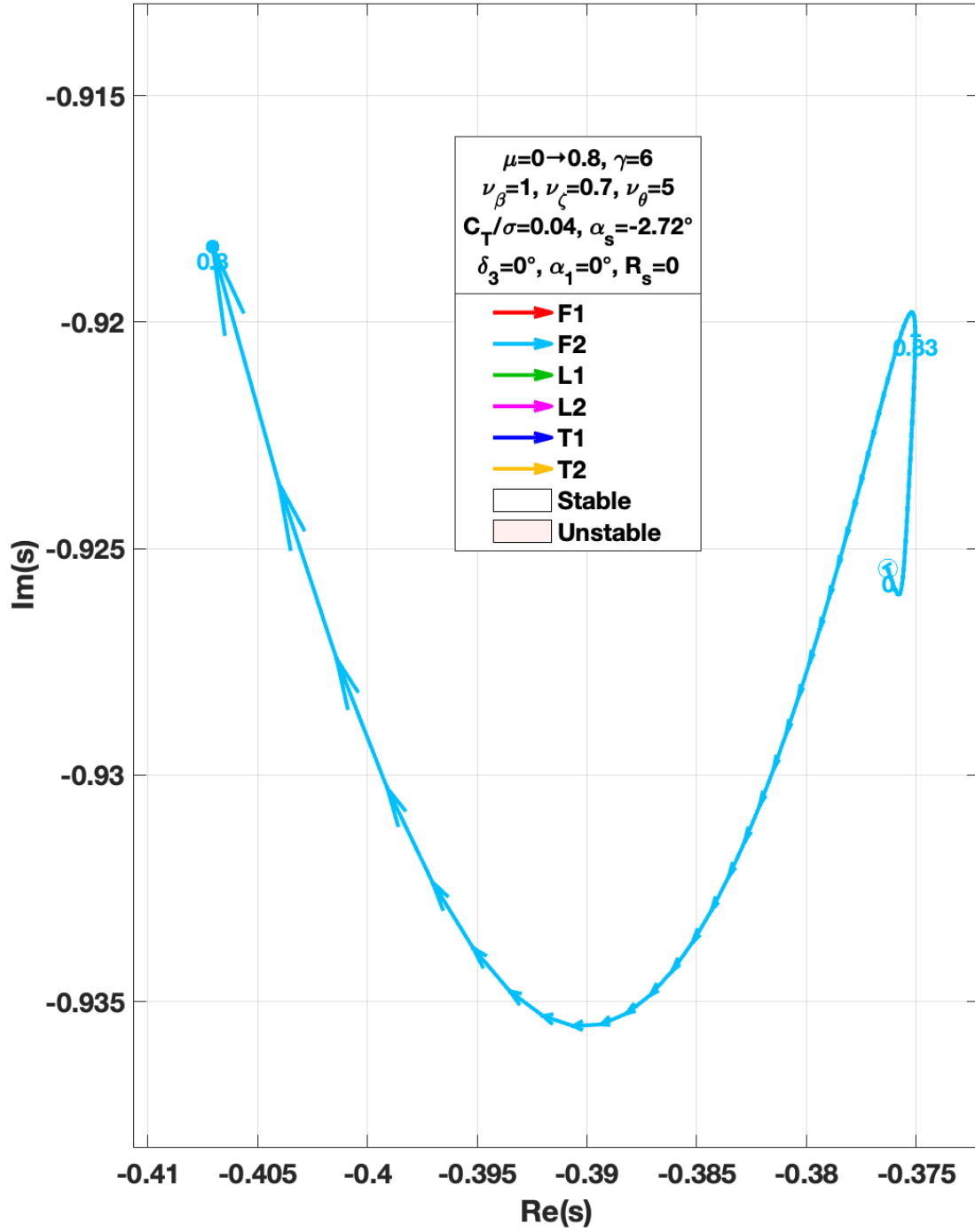


Figure 5.52: Flap-lag-torsion root locus, F2 root, soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.7$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8 .

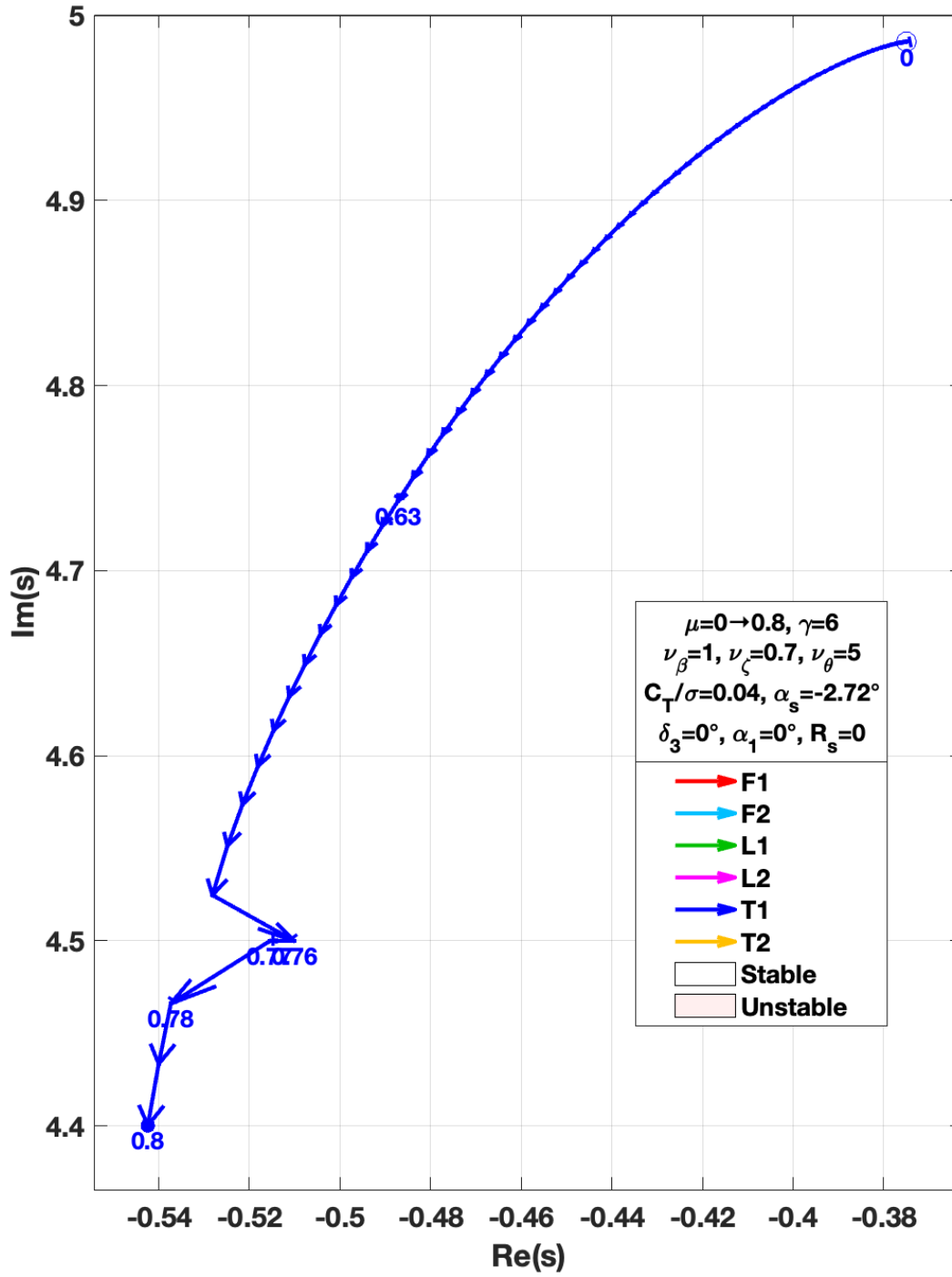


Figure 5.53: Flap-lag-torsion root locus, T1 root, soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.7$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8 .

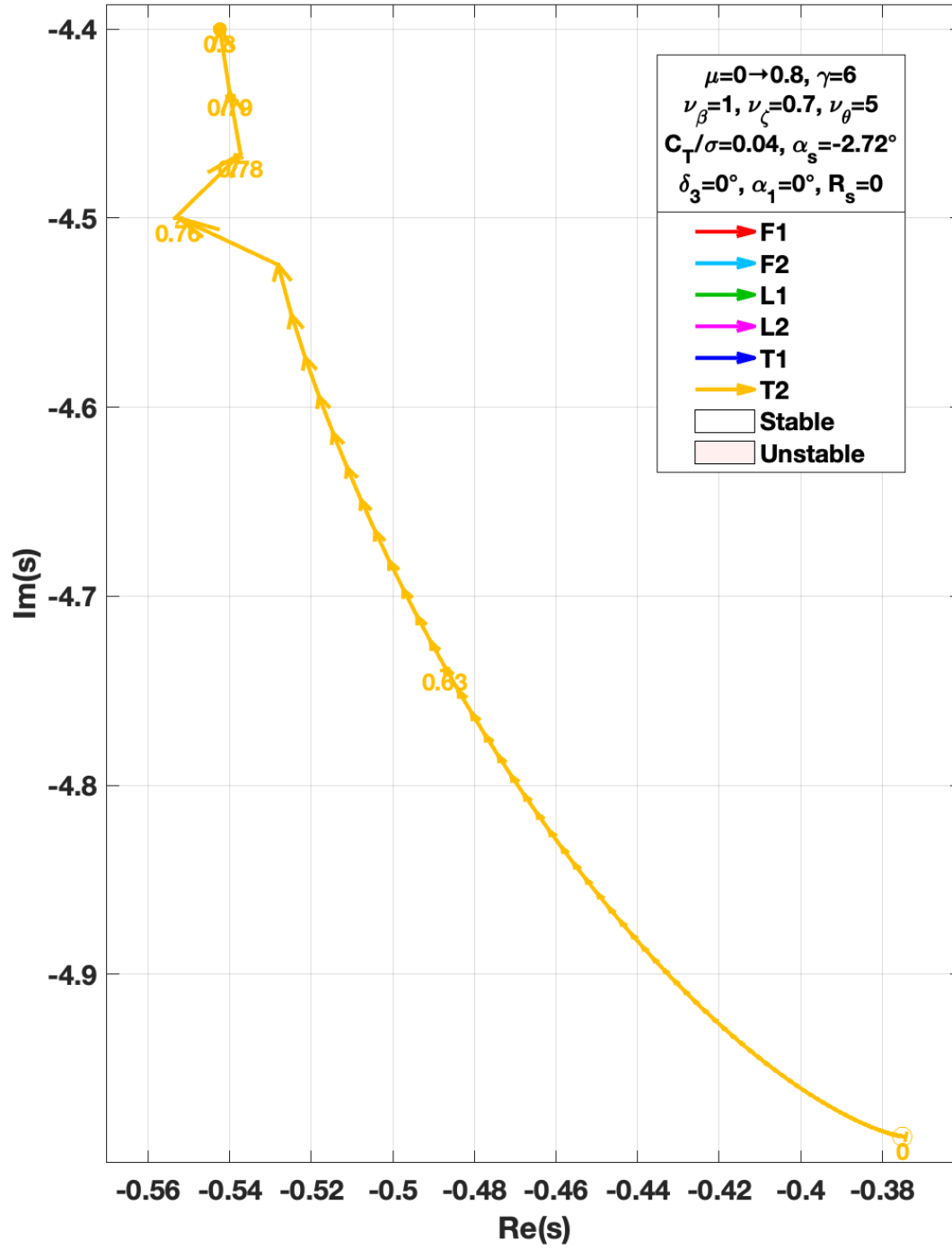


Figure 5.54: Flap-lag-torsion root locus, T2 root, soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.7$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8 .

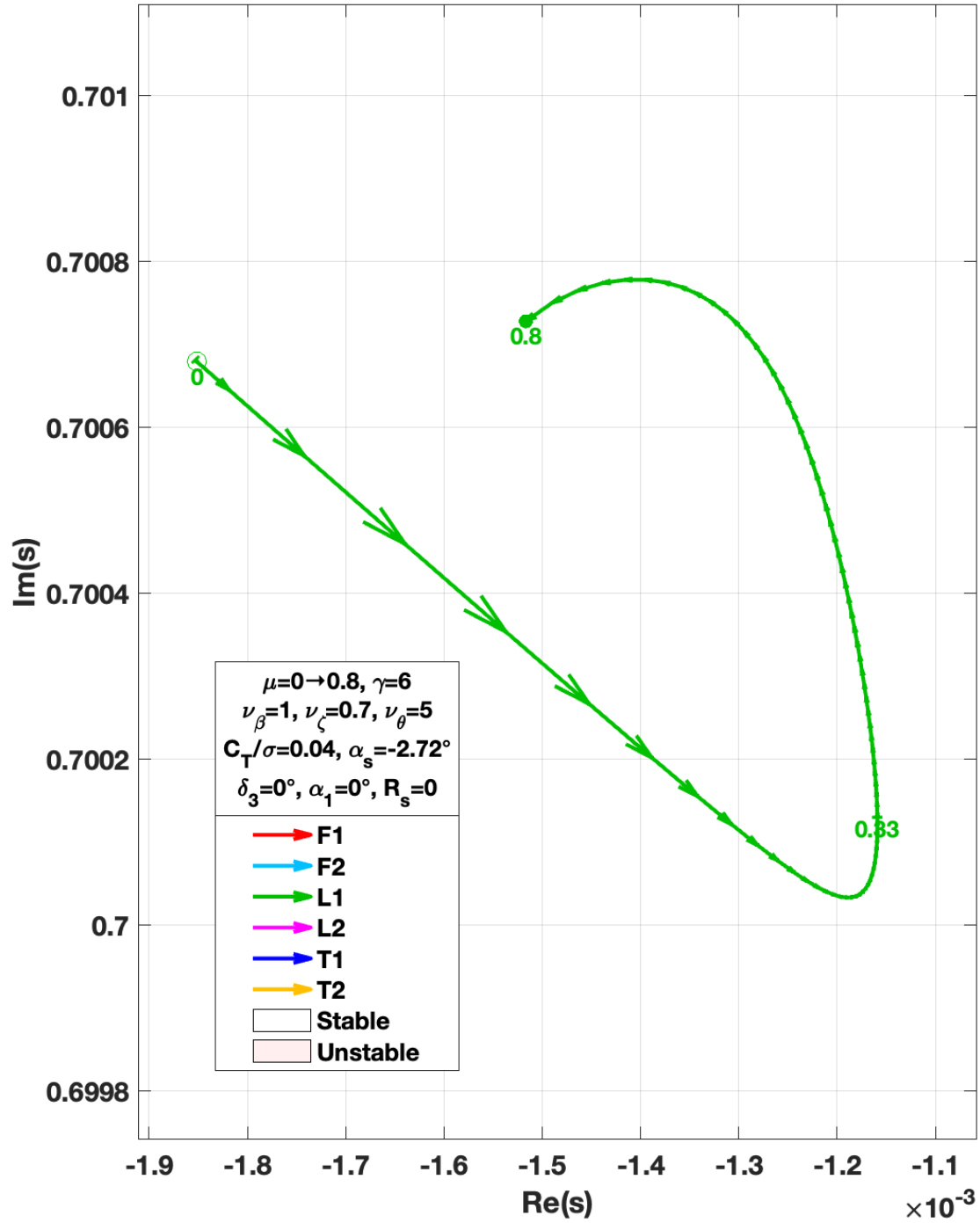


Figure 5.55: Flap-lag-torsion root locus, L1 root, soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.7$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8 .

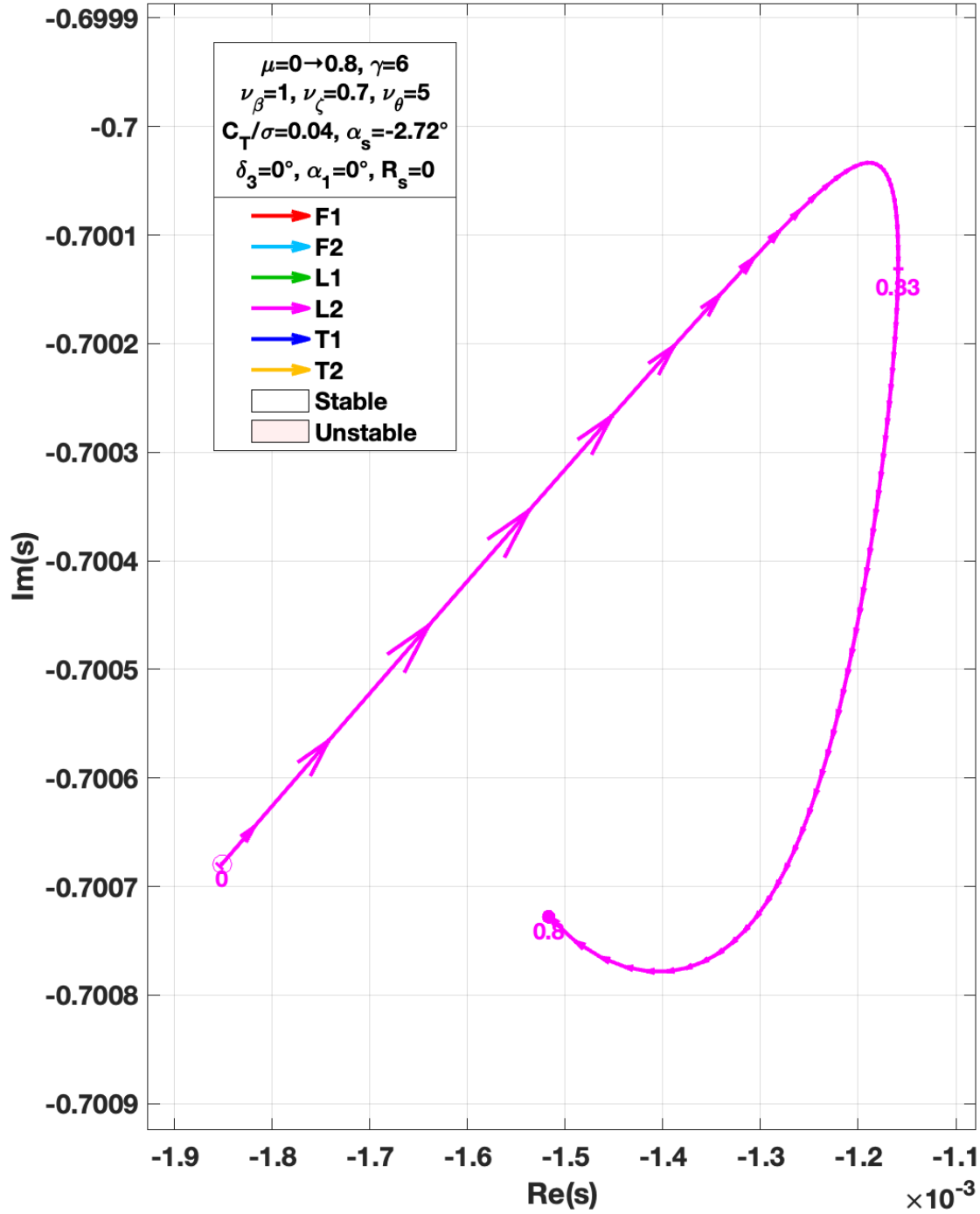


Figure 5.56: Flap-lag-torsion root locus, L2 root, soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.7$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8 .

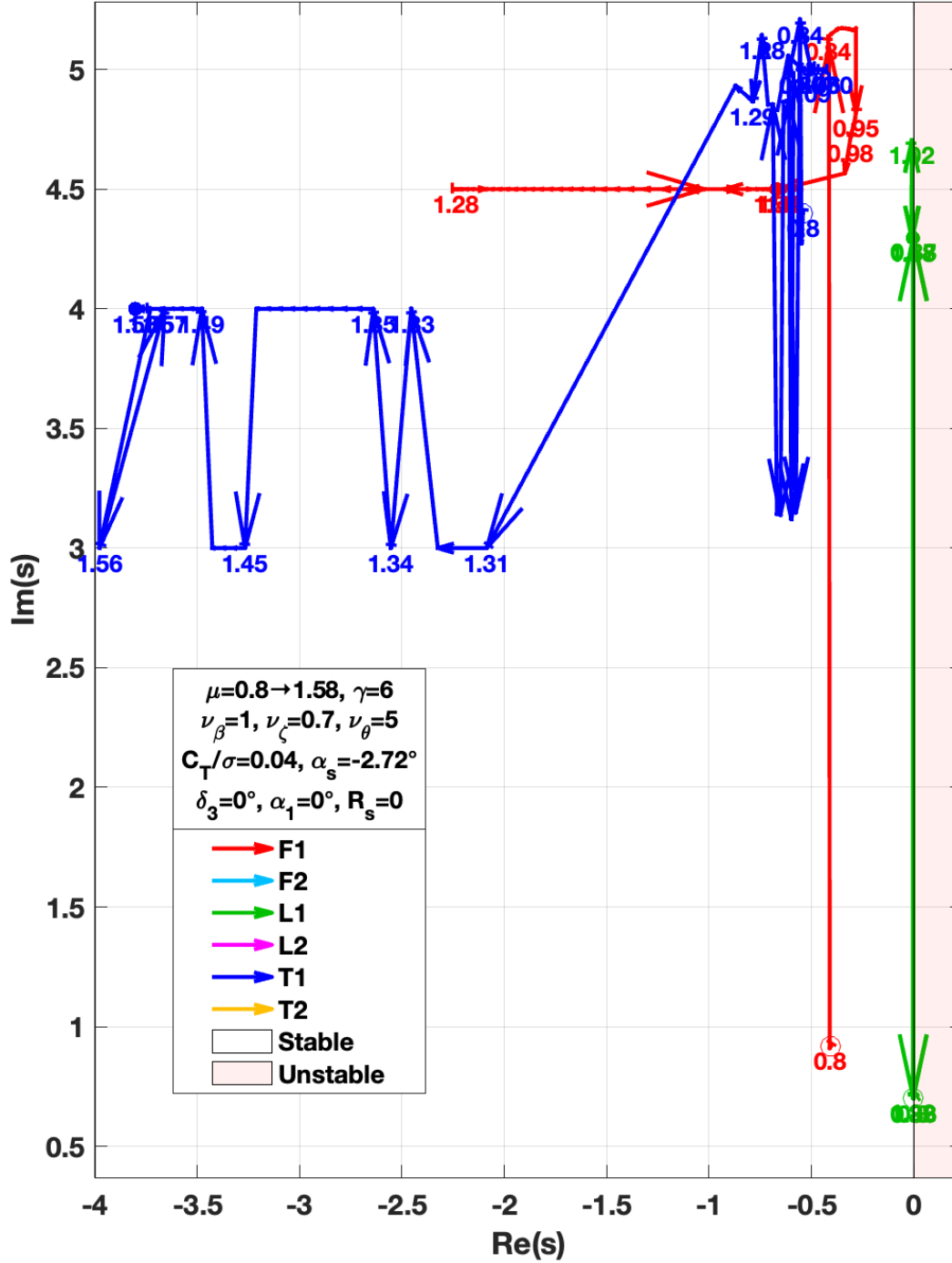


Figure 5.57: Flap-lag-torsion root locus, F1, L1, and T1 roots, soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.7$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$, $C(k) = 1$, $\mu = 0.8$ to 1.58 .

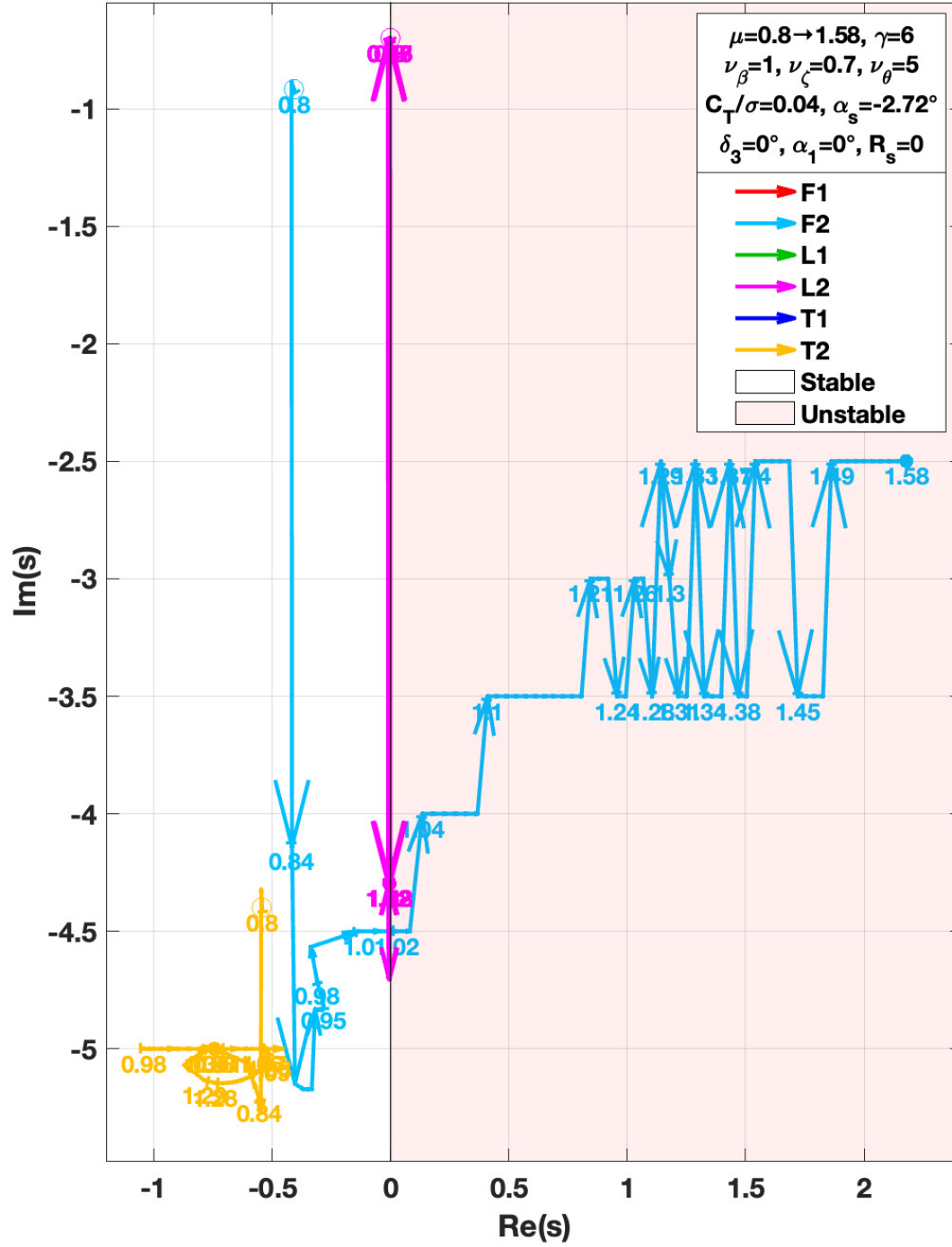


Figure 5.58: Flap-lag-torsion root locus, F2, L2, and T2 roots, soft-in-plane articulated rotor, $\nu_\beta = 1$, $\nu_\zeta = 0.7$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.72^\circ$, $C(k) = 1$, $\mu = 0.8$ to 1.58.

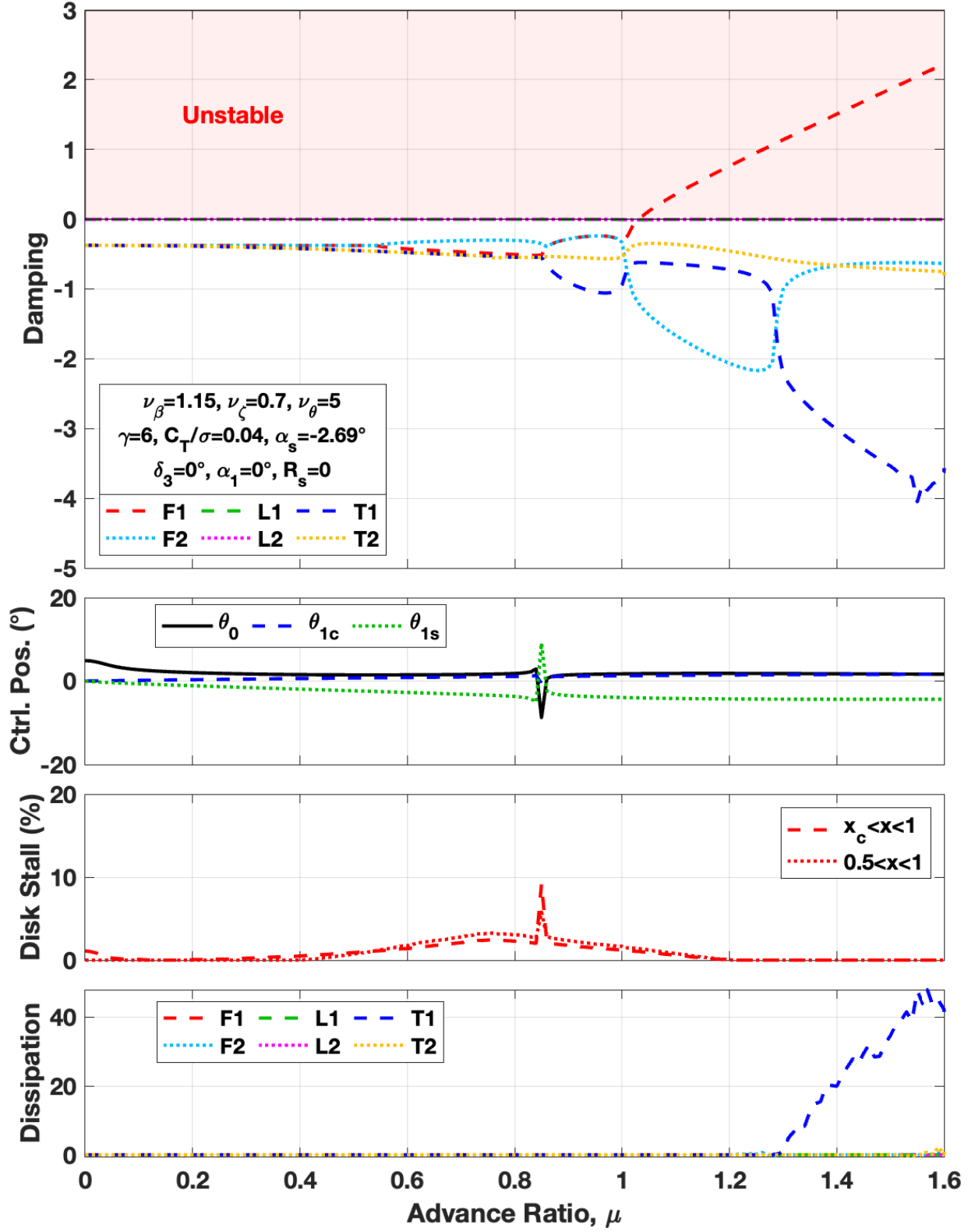
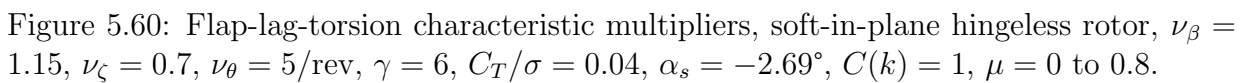


Figure 5.59: Flap-lag-torsion damping, trim, blade stall, and numerical stability, soft-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 0.7$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $C(k) = 1$.



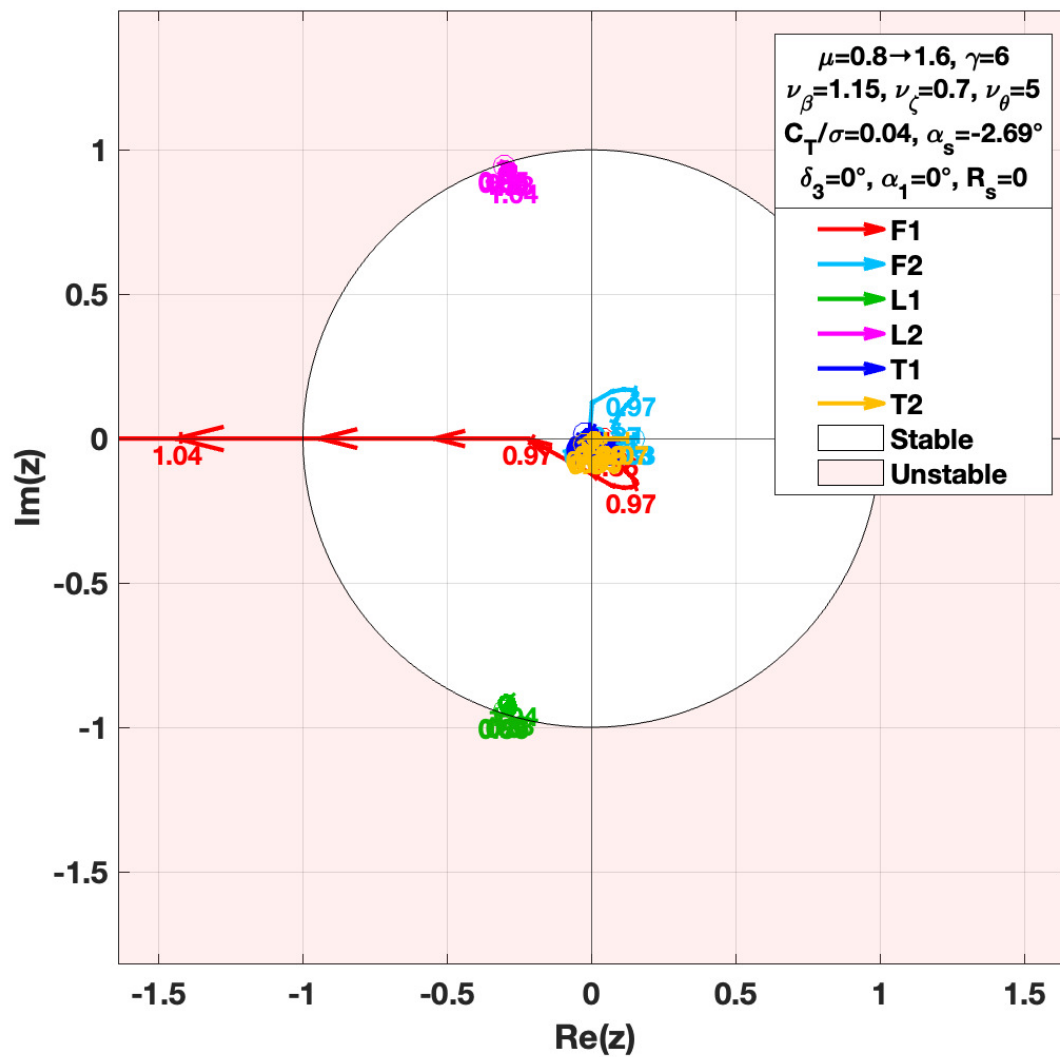


Figure 5.61: Flap-lag-torsion characteristic multipliers, soft-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 0.7$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $C(k) = 1$, $\mu = 0.8$ to 1.6 .

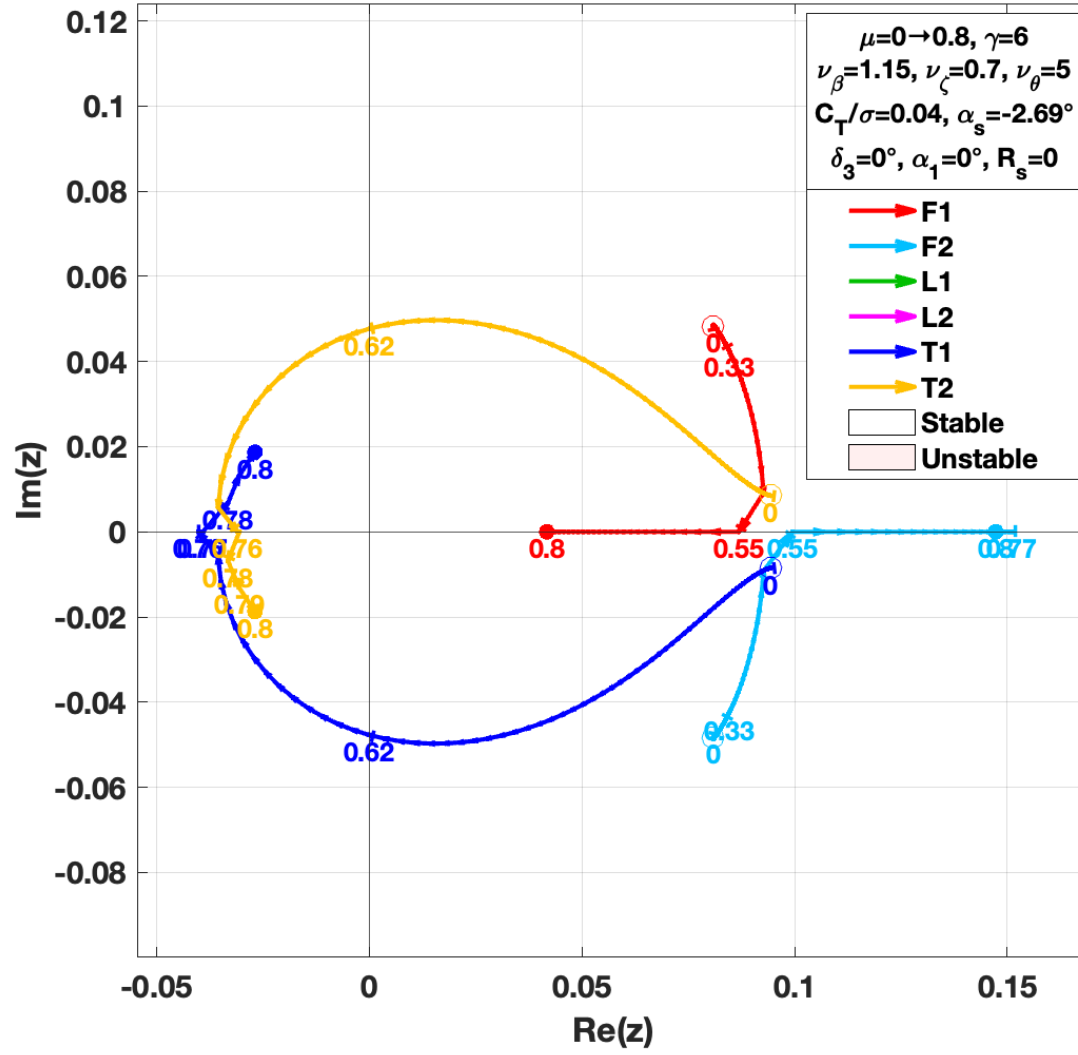


Figure 5.62: Flap-lag-torsion characteristic multipliers, $F1$, $F2$, $T1$, and $T2$ roots, soft-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 0.7$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8 .

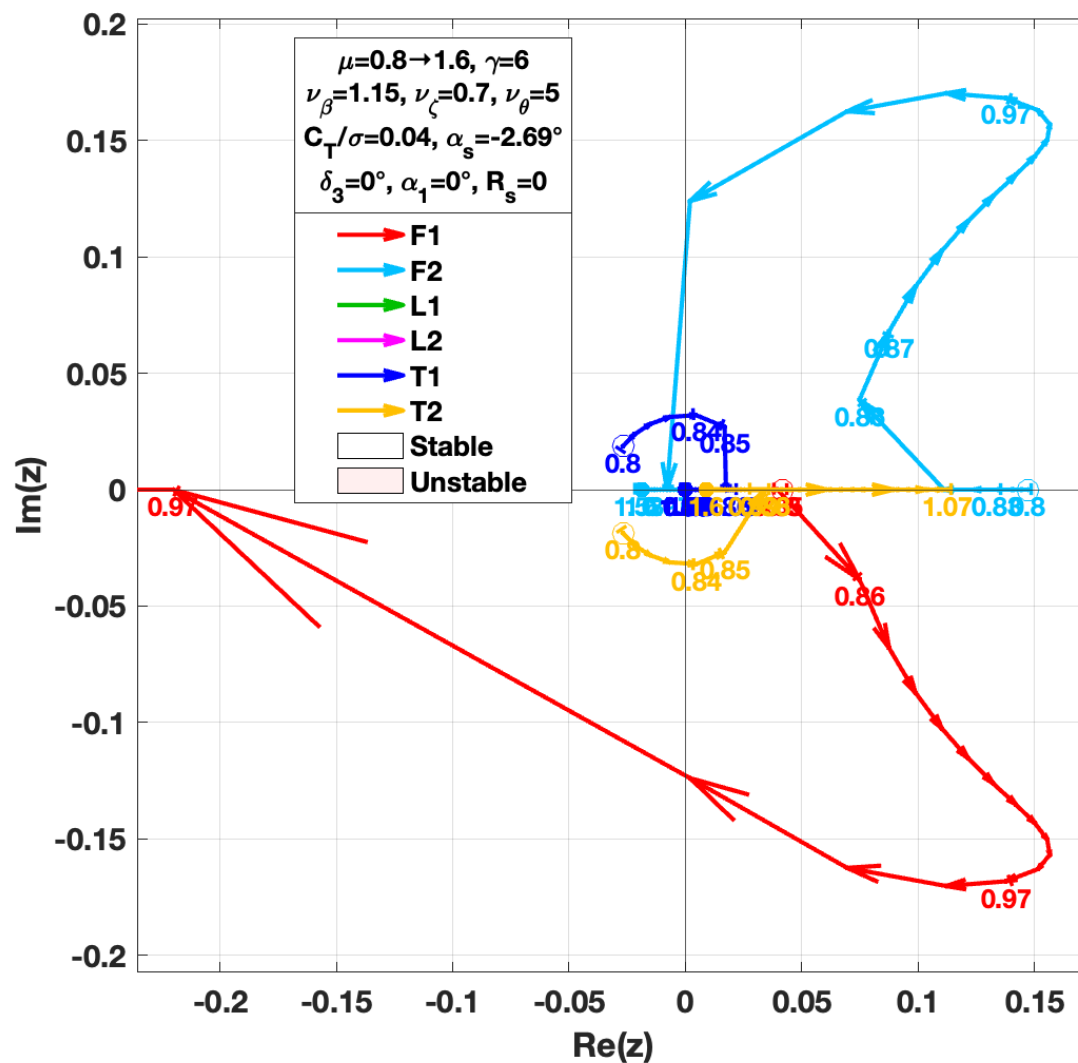


Figure 5.63: Flap-lag-torsion characteristic multipliers, F1, F2, T1, and T2 roots, soft-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 0.7$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $C(k) = 1$, $\mu = 0.8$ to 1.6 .

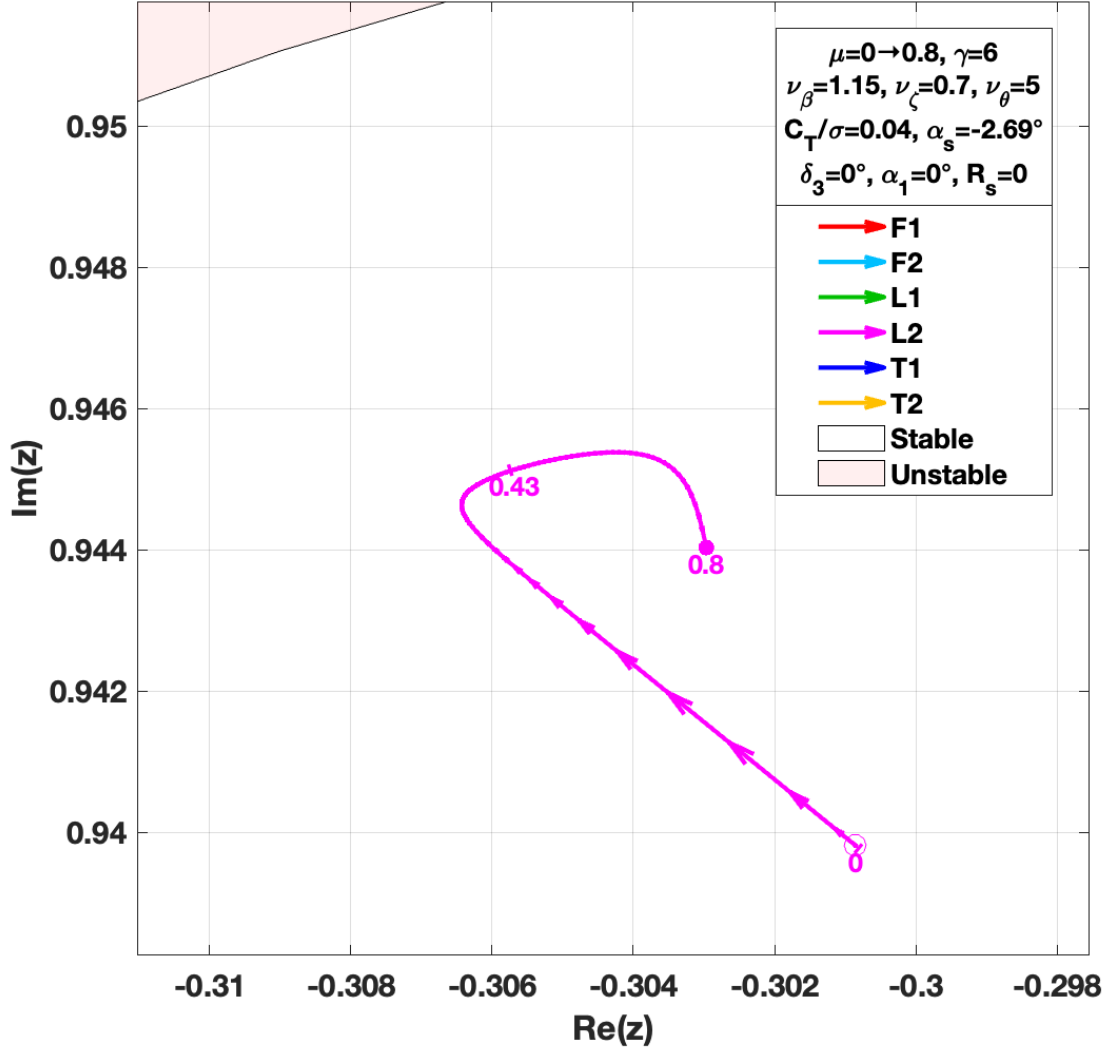


Figure 5.64: Flap-lag-torsion characteristic multipliers, L1 root, soft-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 0.7$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8.

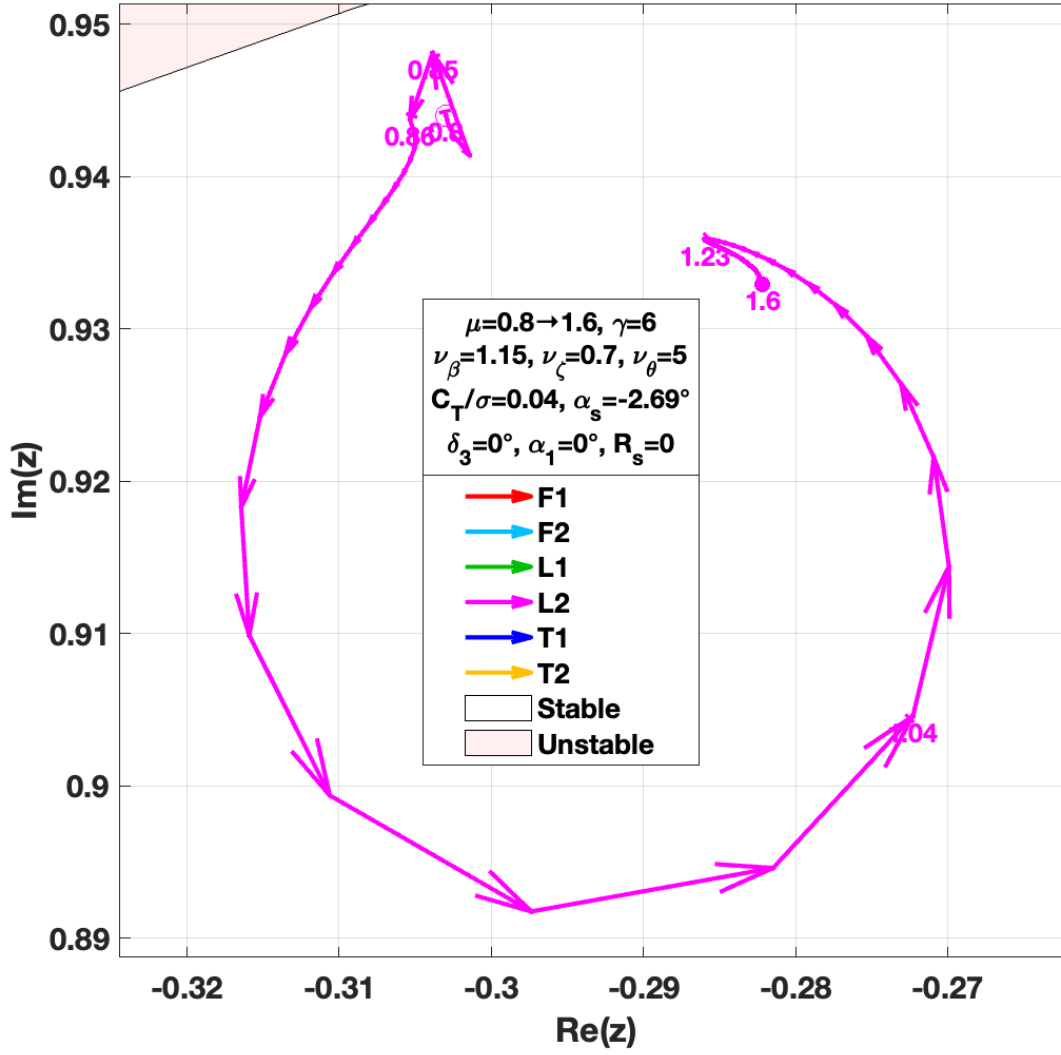


Figure 5.65: Flap-lag-torsion characteristic multipliers, L1 root, soft-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 0.7$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $C(k) = 1$, $\mu = 0.8$ to 1.6.

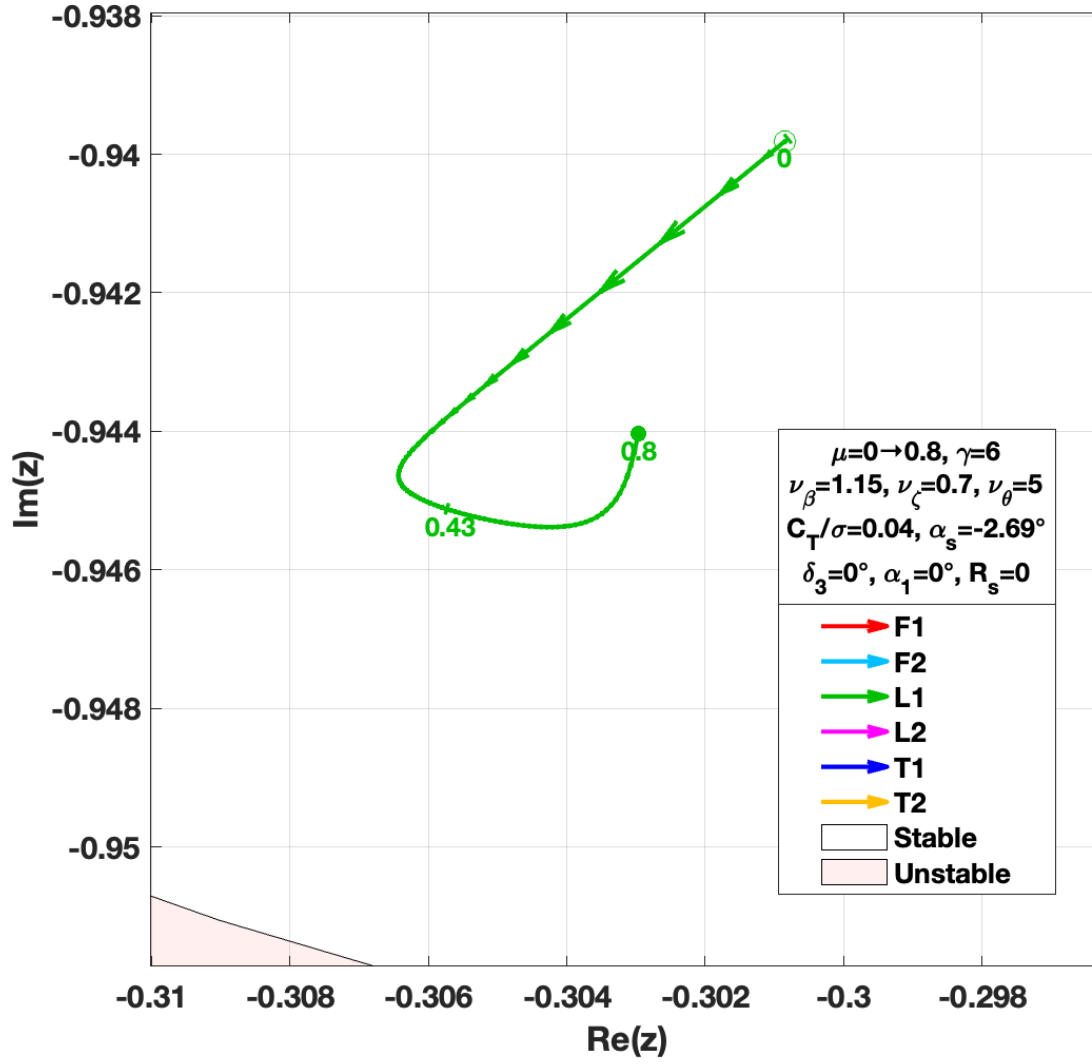


Figure 5.66: Flap-lag-torsion characteristic multipliers, L2 root, soft-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 0.7$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8.

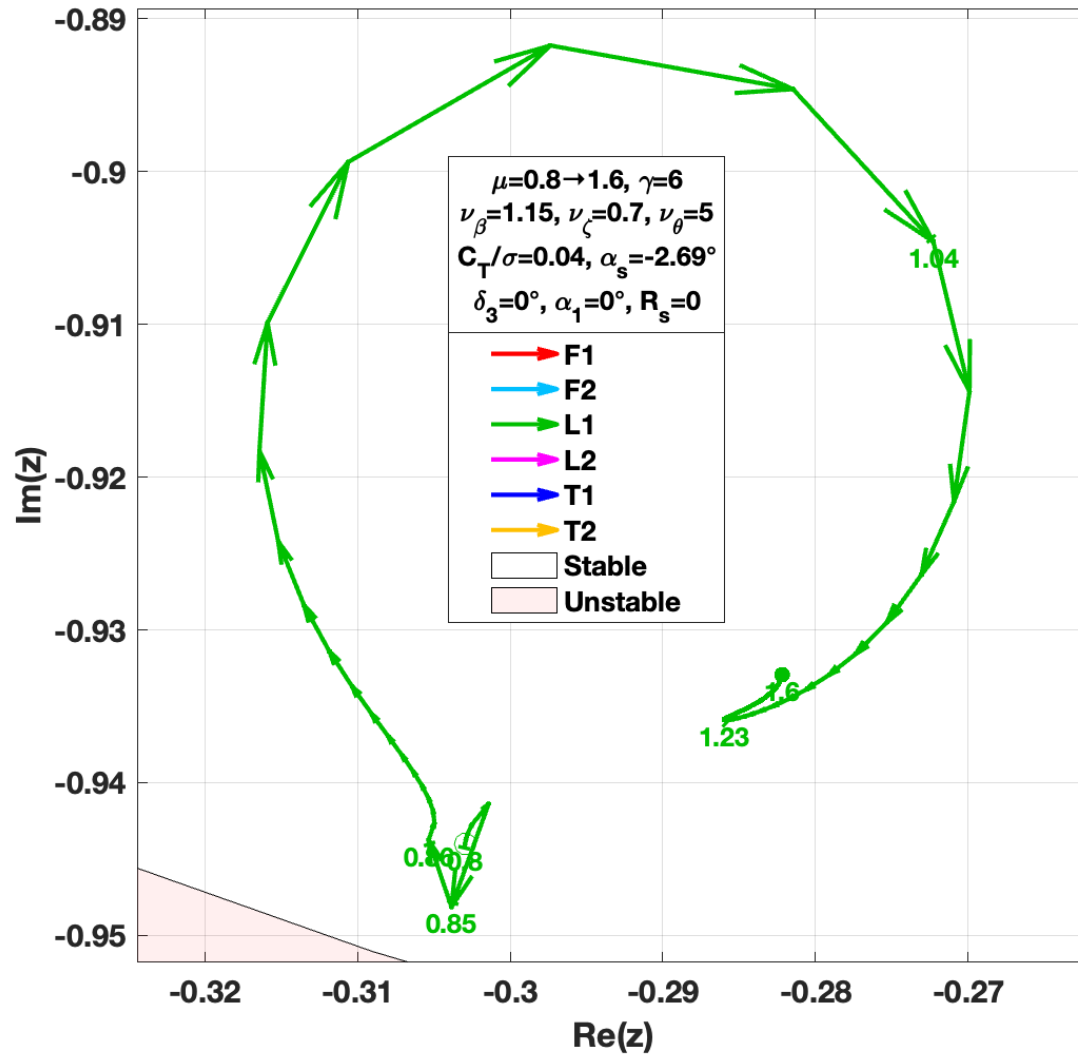


Figure 5.67: Flap-lag-torsion characteristic multipliers, L2 root, soft-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 0.7$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $C(k) = 1$, $\mu = 0.8$ to 1.6.

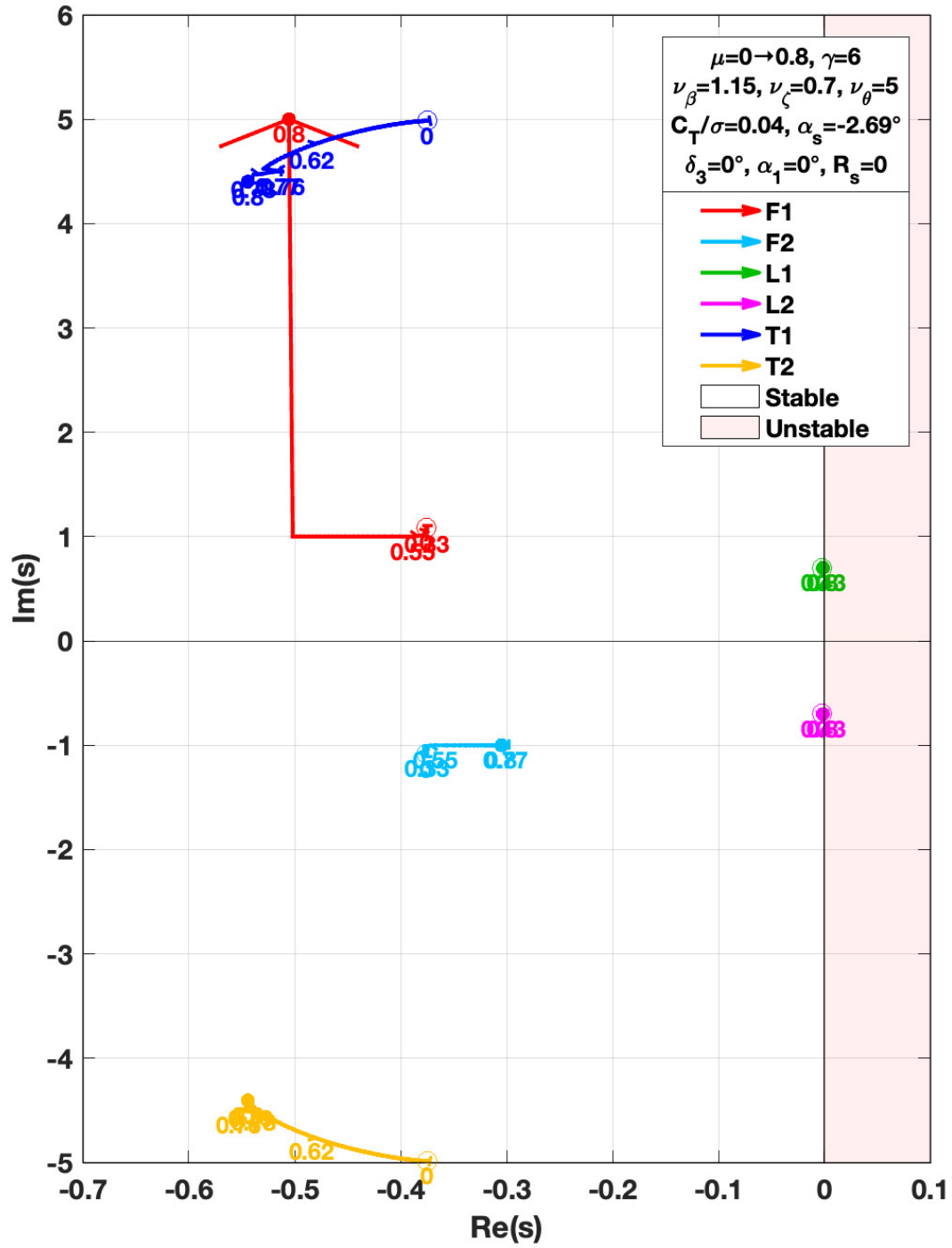


Figure 5.68: Flap-lag-torsion root locus, soft-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 0.7$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8 .

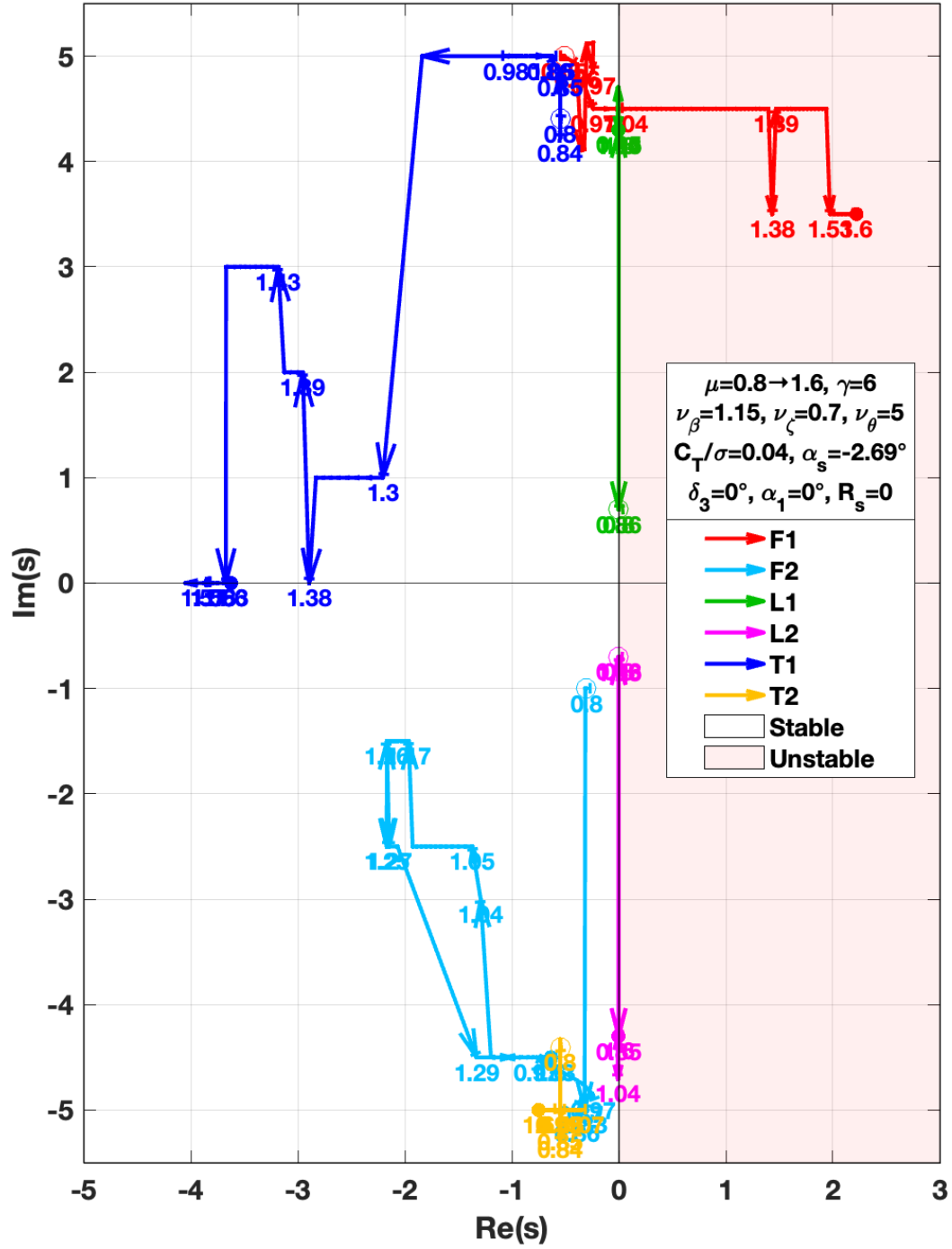


Figure 5.69: Flap-lag-torsion root locus, soft-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 0.7$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $C(k) = 1$, $\mu = 0.8$ to 1.6 .

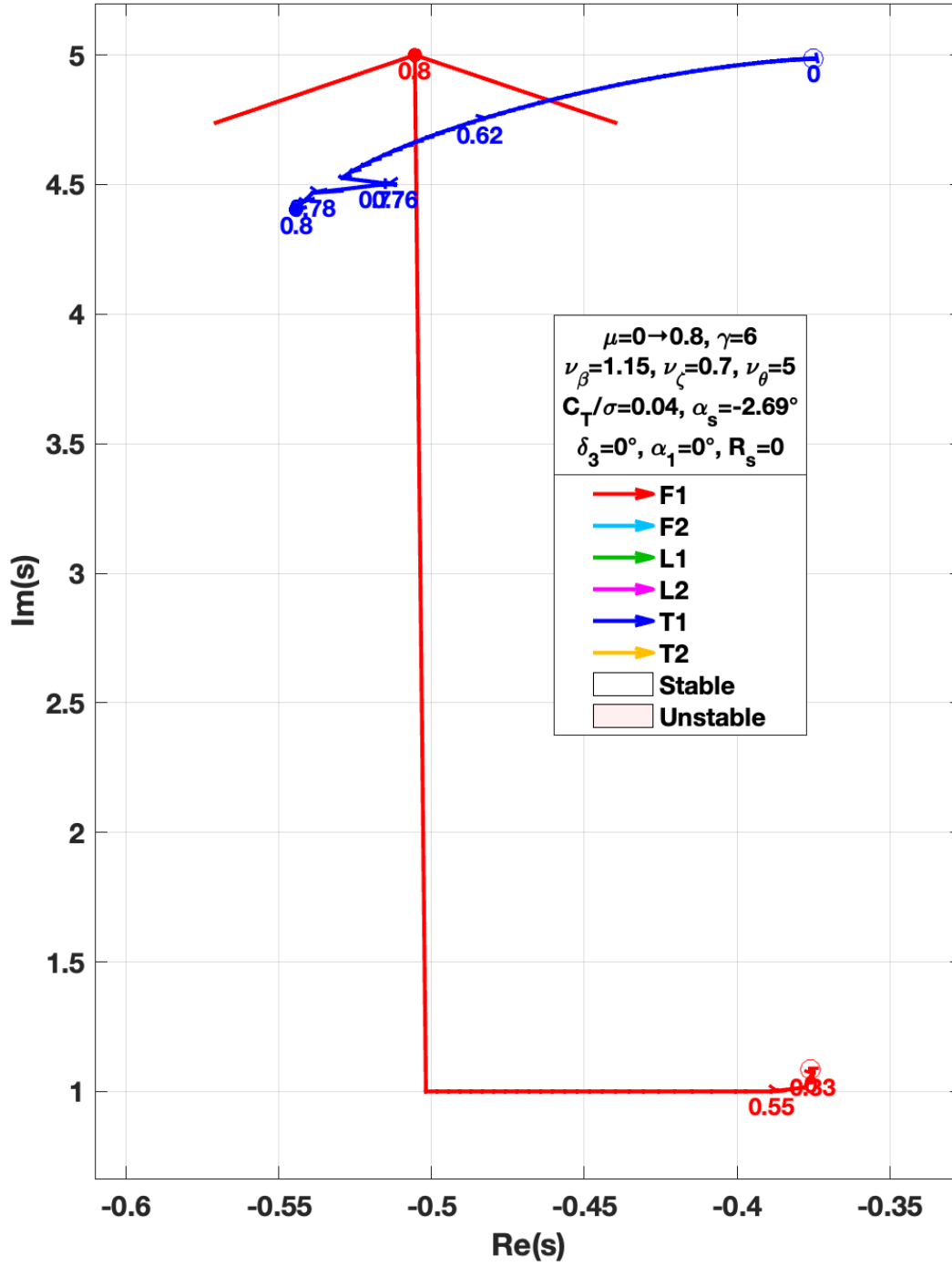


Figure 5.70: Flap-lag-torsion root locus, F1 and T1 roots, soft-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 0.7$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8 .

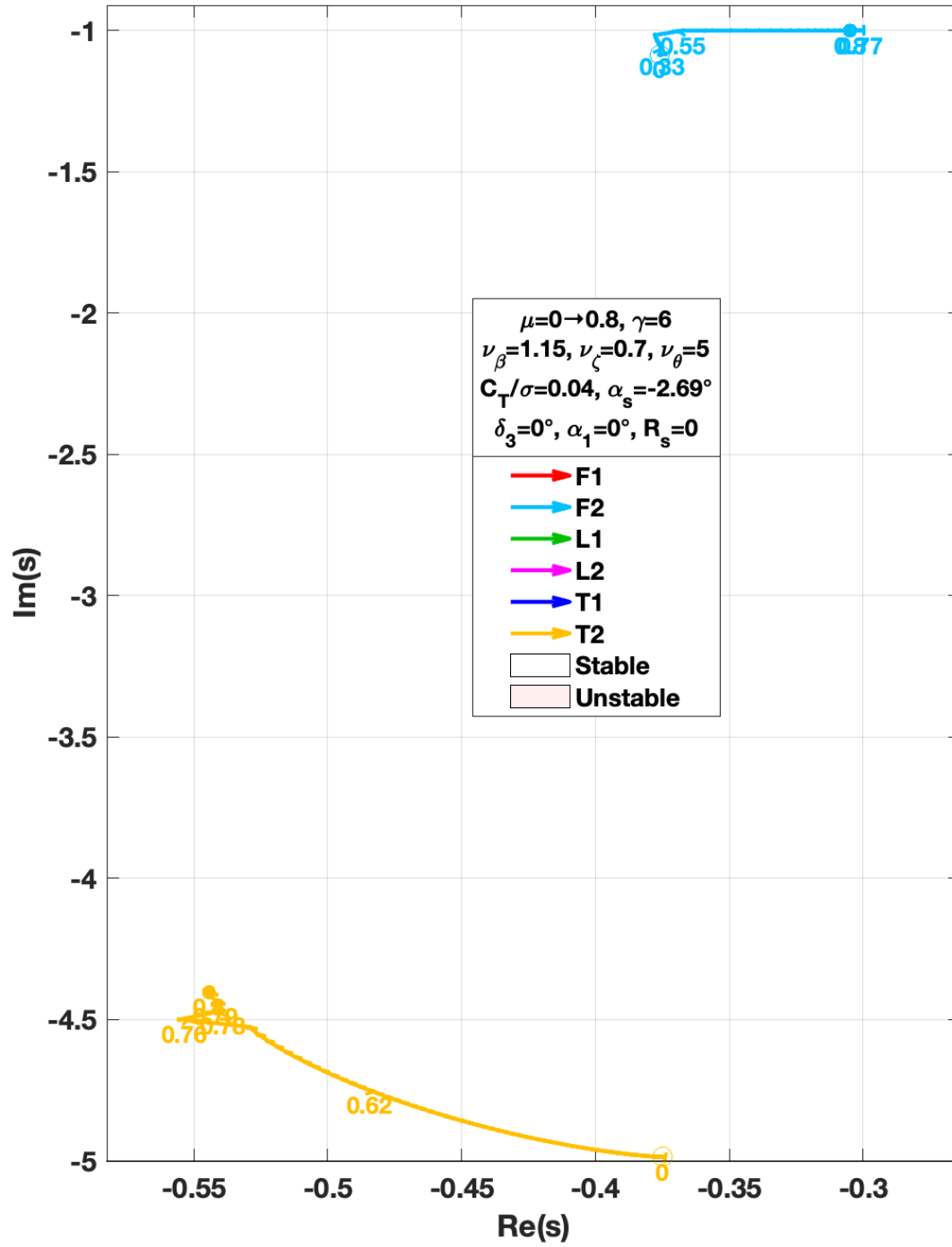


Figure 5.71: Flap-lag-torsion root locus, F2 and T2 roots, soft-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 0.7$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8 .

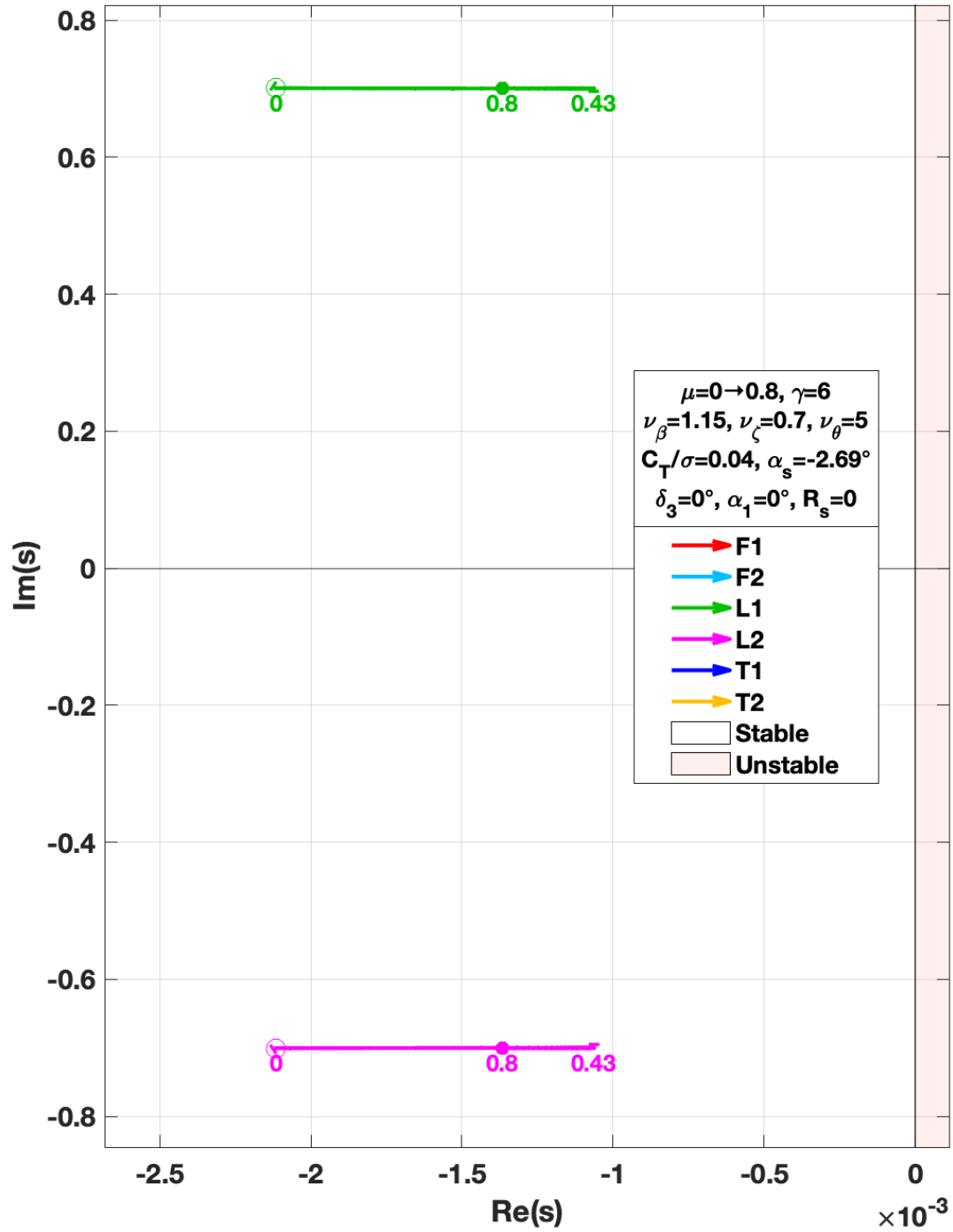


Figure 5.72: Flap-lag-torsion root locus, L1 and L2 roots, soft-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 0.7$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8 .

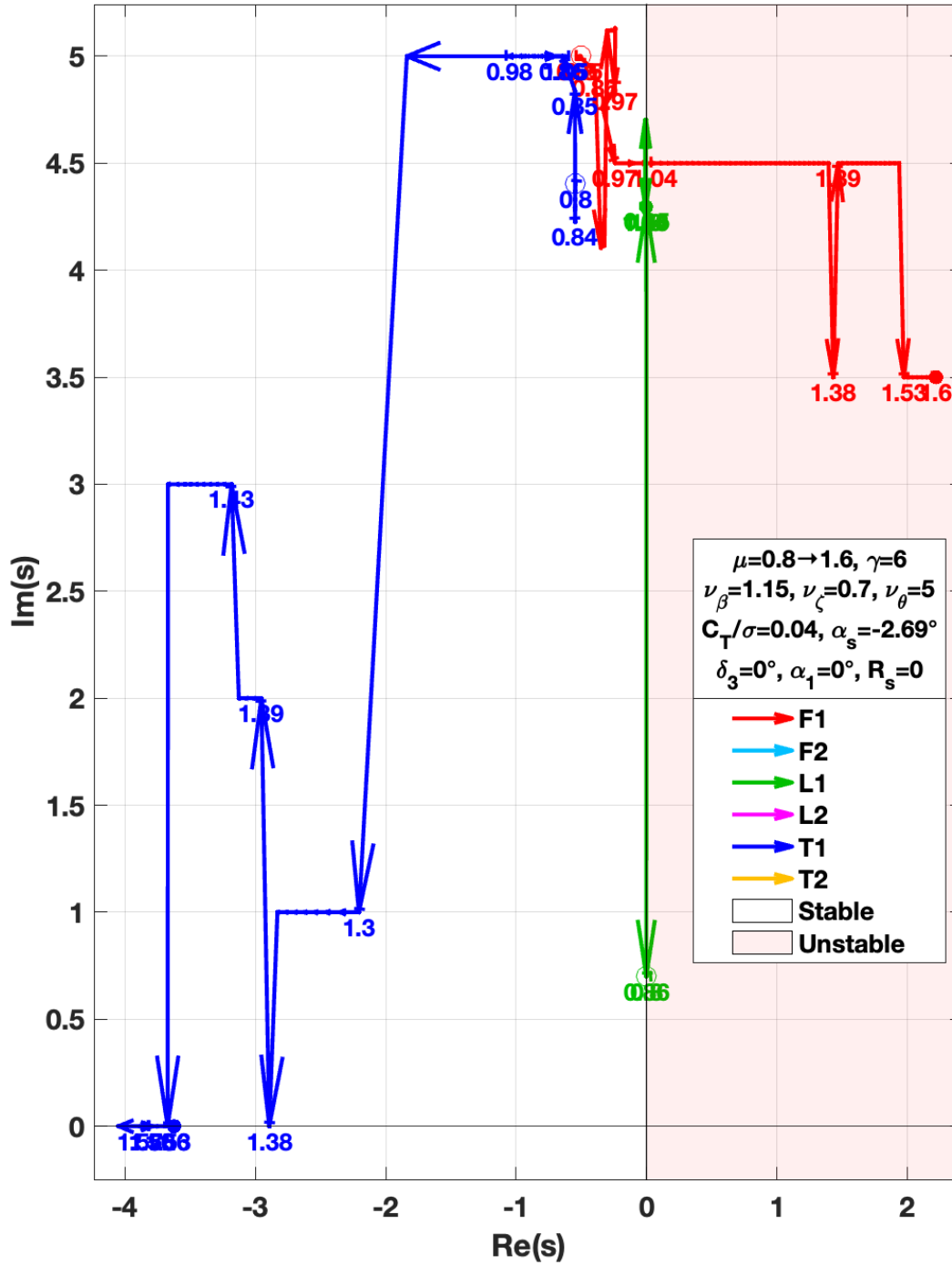


Figure 5.73: Flap-lag-torsion root locus, F1, L1, and T1 roots, soft-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 0.7$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $C(k) = 1$, $C(k) = 1$, $\mu = 0.8$ to 1.6 .

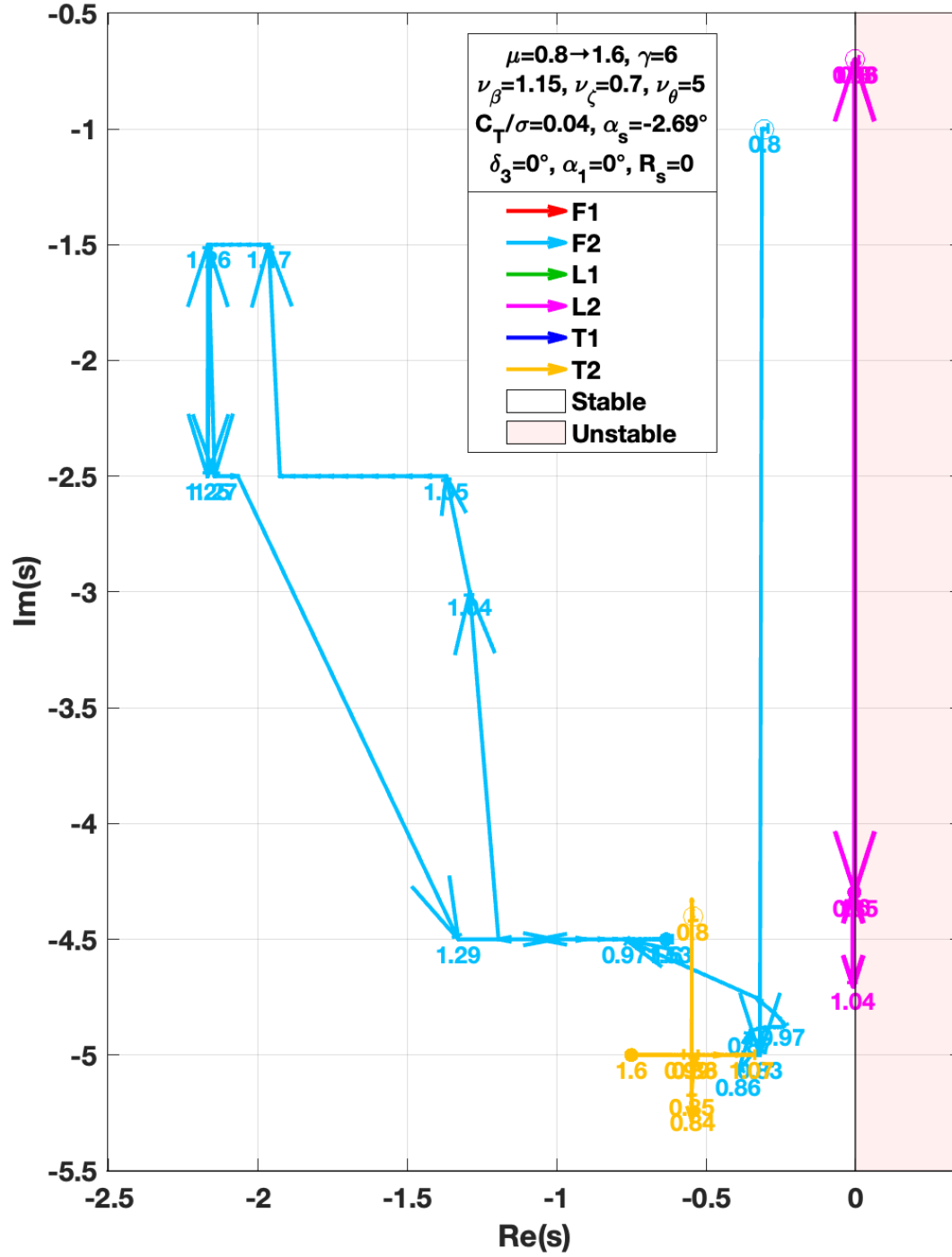


Figure 5.74: Flap-lag-torsion root locus, F2, L2, and T2 roots, soft-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 0.7$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $C(k) = 1$, $\mu = 0.8$ to 1.6 .

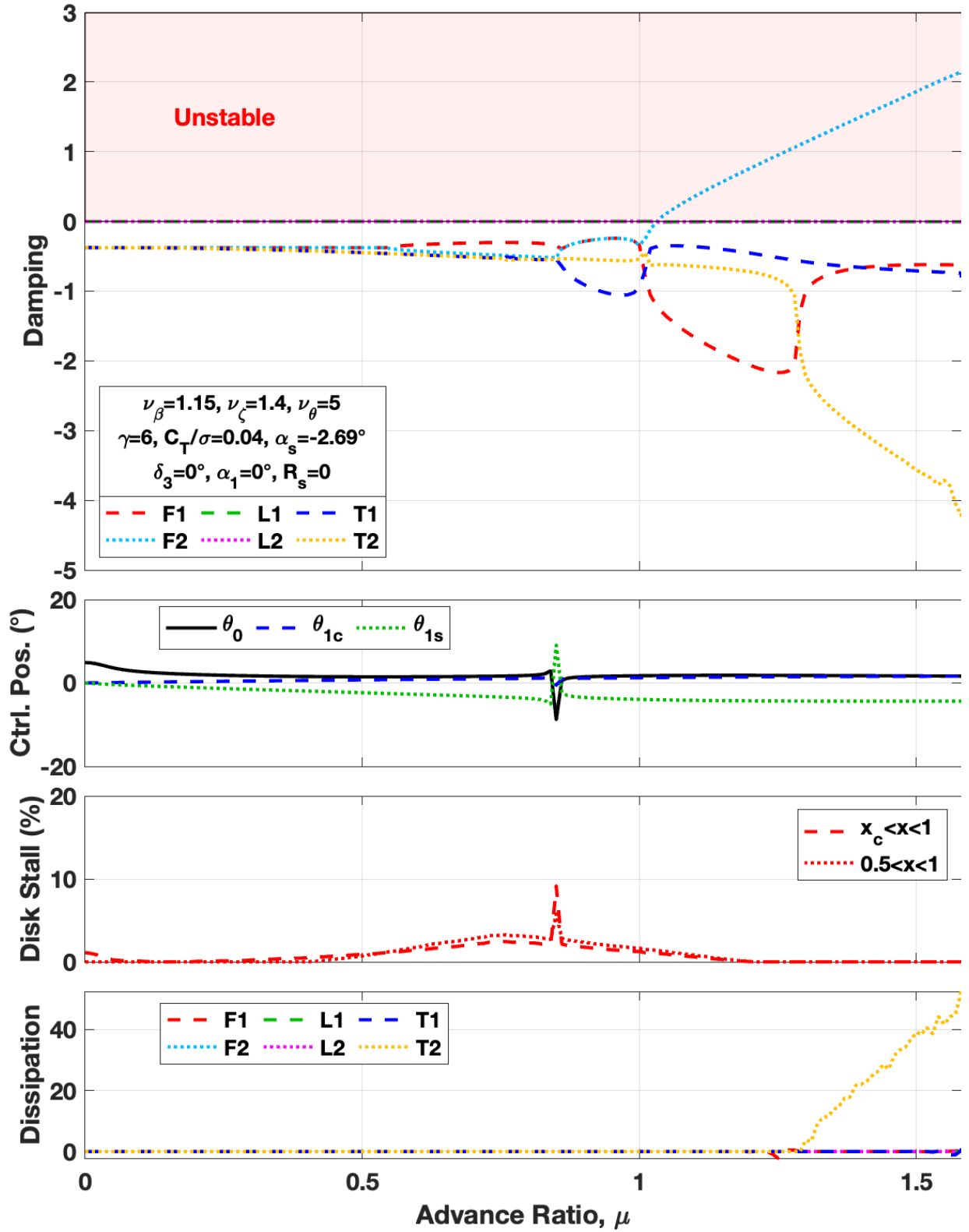
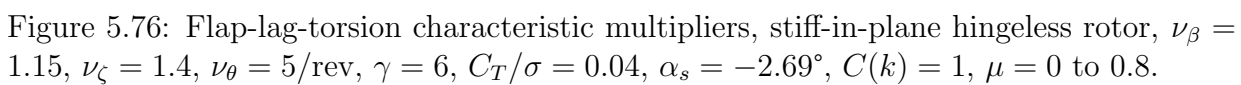
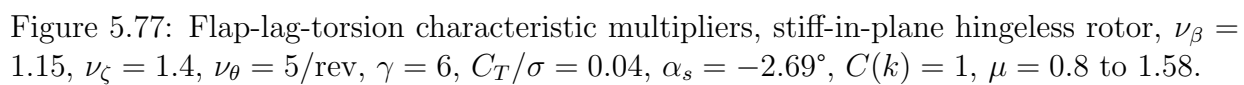


Figure 5.75: Flap-lag-torsion damping, trim, blade stall, and numerical stability, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $C(k) = 1$.





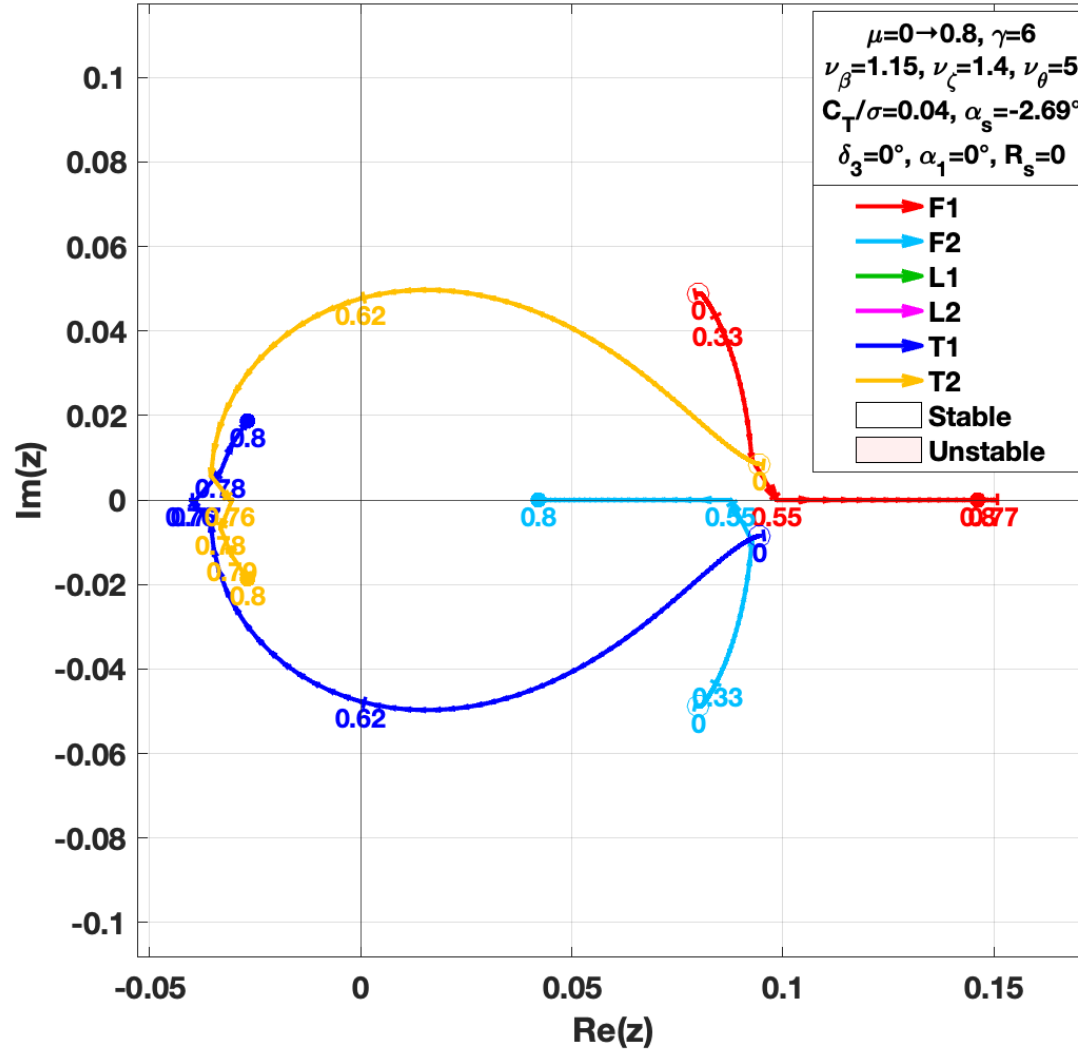


Figure 5.78: Flap-lag-torsion characteristic multipliers, F1, F2, T1, and T2 roots, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8 .

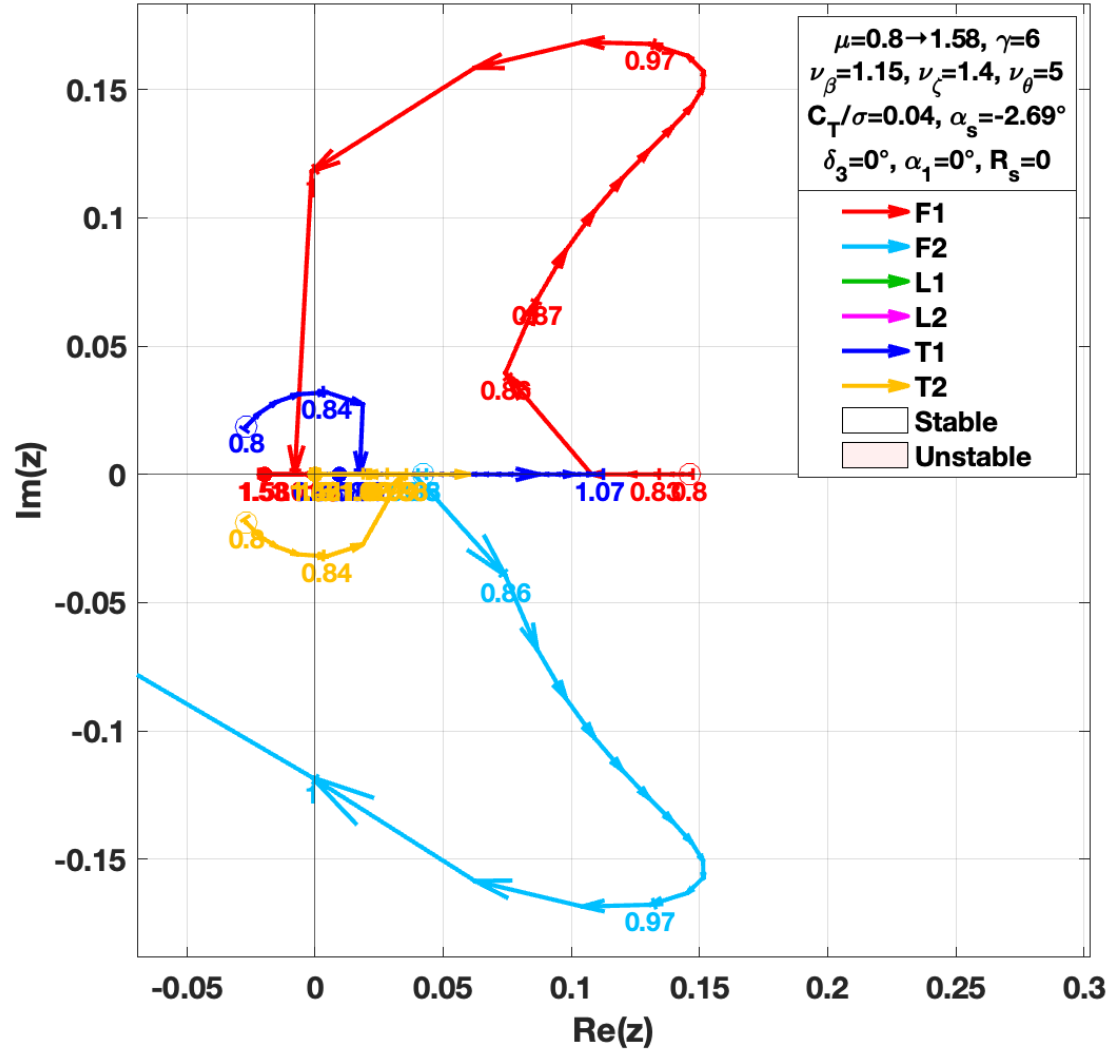


Figure 5.79: Flap-lag-torsion characteristic multipliers, F1, F2, T1, and T2 roots, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $C(k) = 1$, $\mu = 0.8$ to 1.58 .

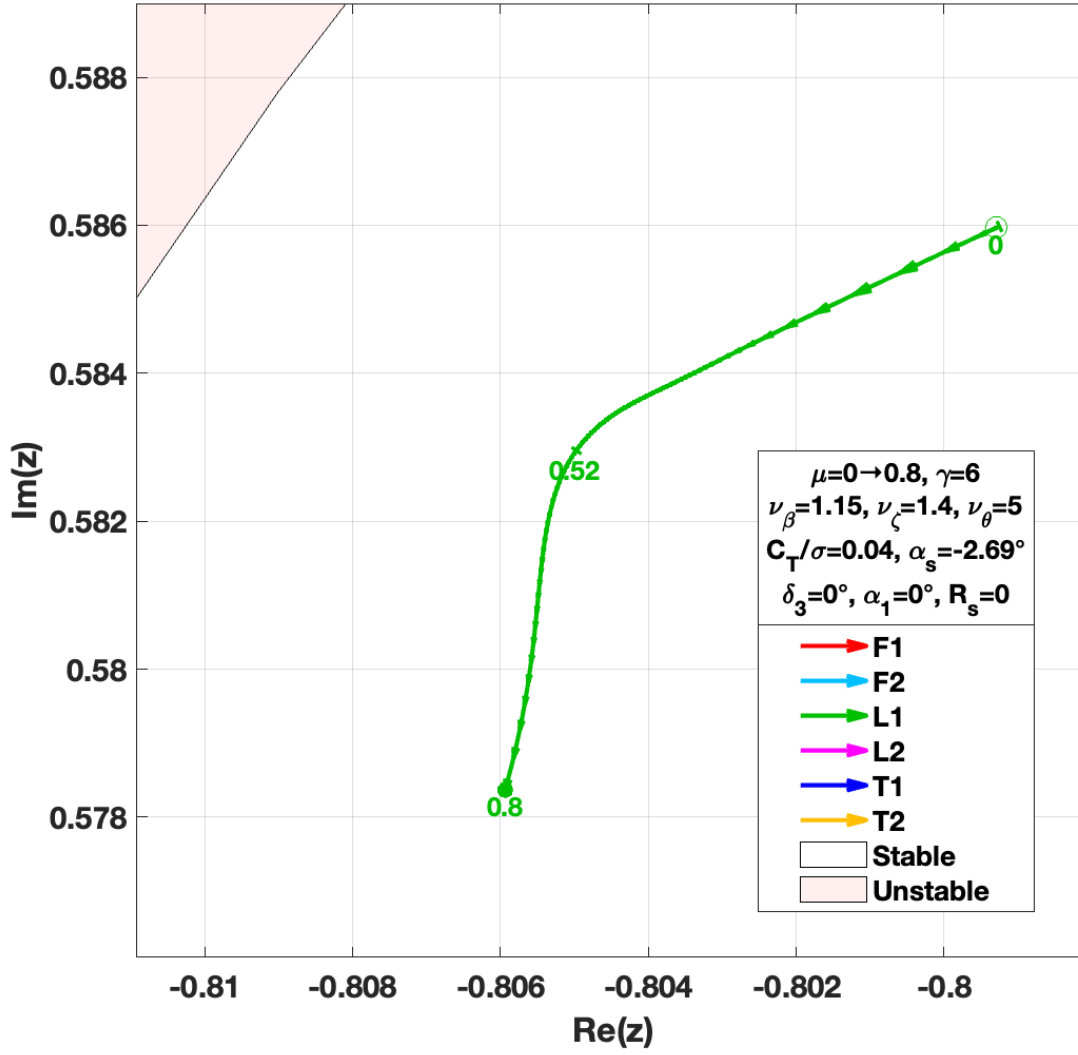


Figure 5.80: Flap-lag-torsion characteristic multipliers, L1 root, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8.

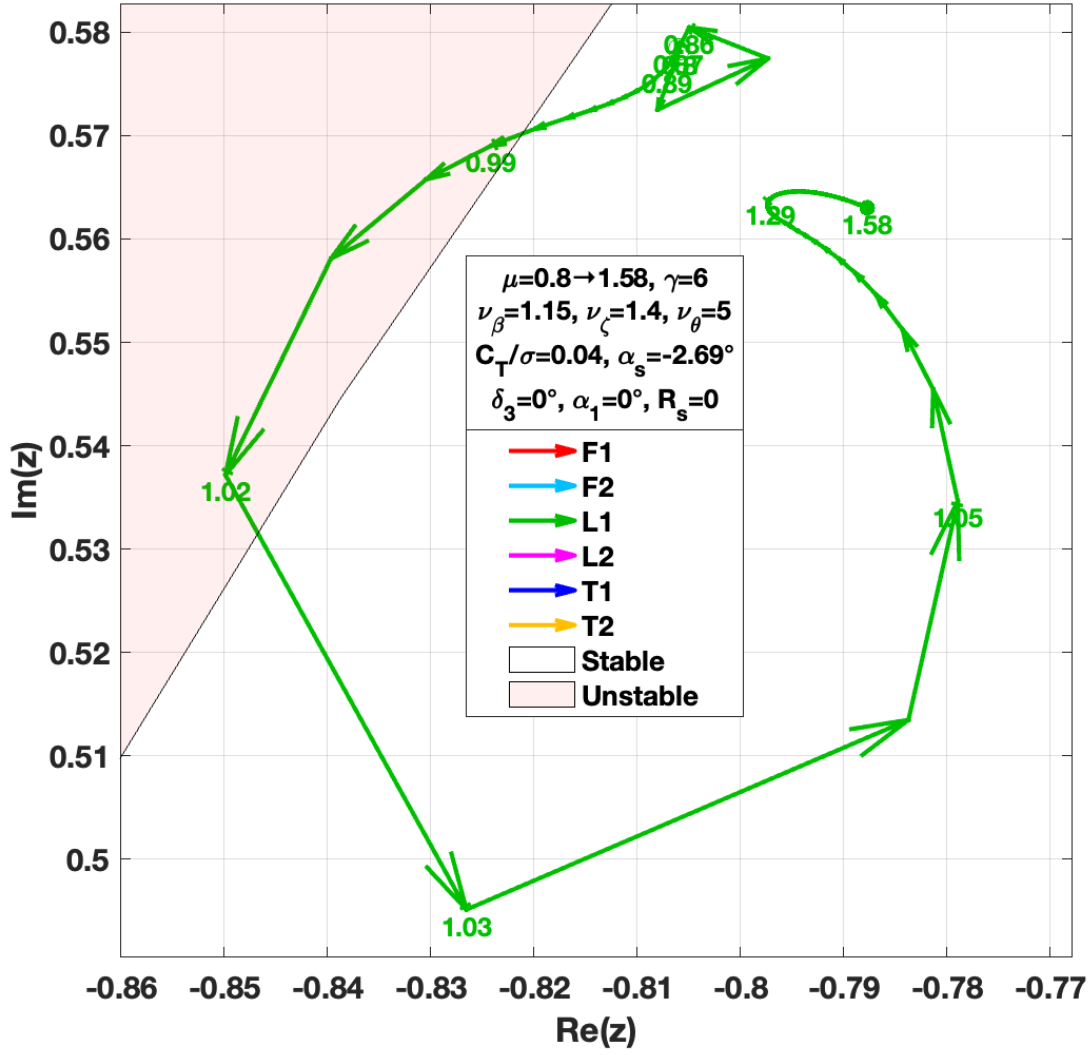


Figure 5.81: Flap-lag-torsion characteristic multipliers, L1 root, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $C(k) = 1$, $\mu = 0.8$ to 1.58.

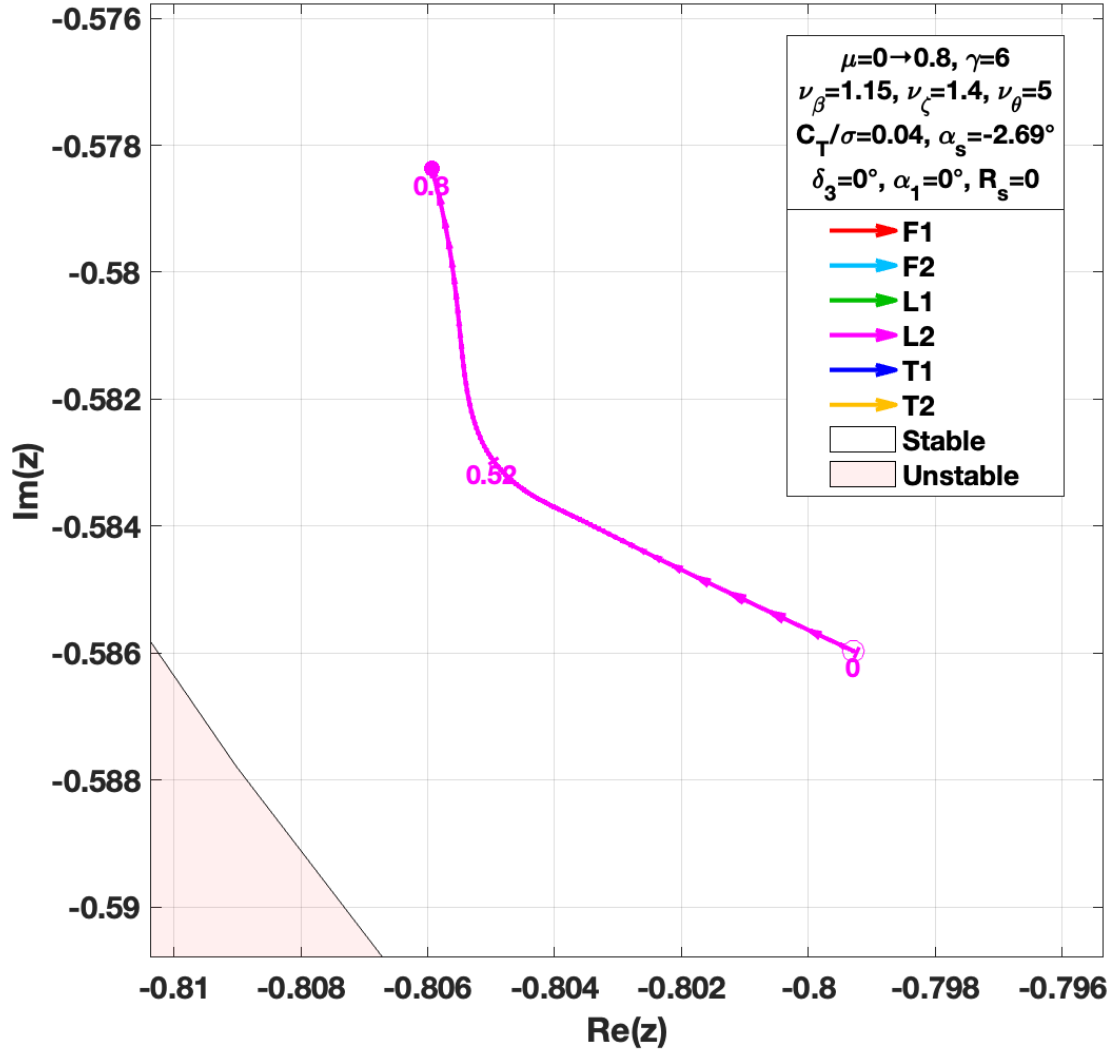


Figure 5.82: Flap-lag-torsion characteristic multipliers, L2 root, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8 .

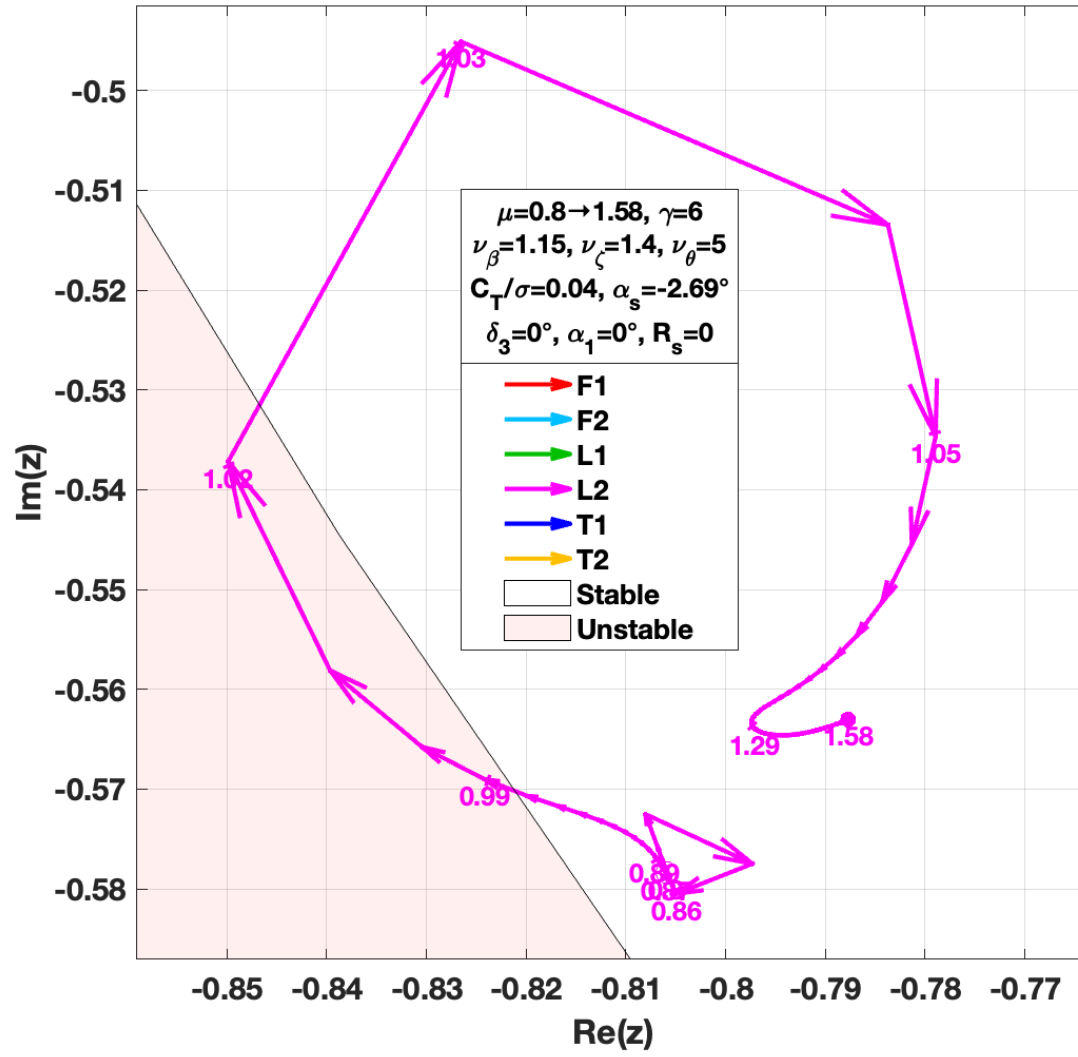


Figure 5.83: Flap-lag-torsion characteristic multipliers, L2 root, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $C(k) = 1$, $\mu = 0.8$ to 1.58.

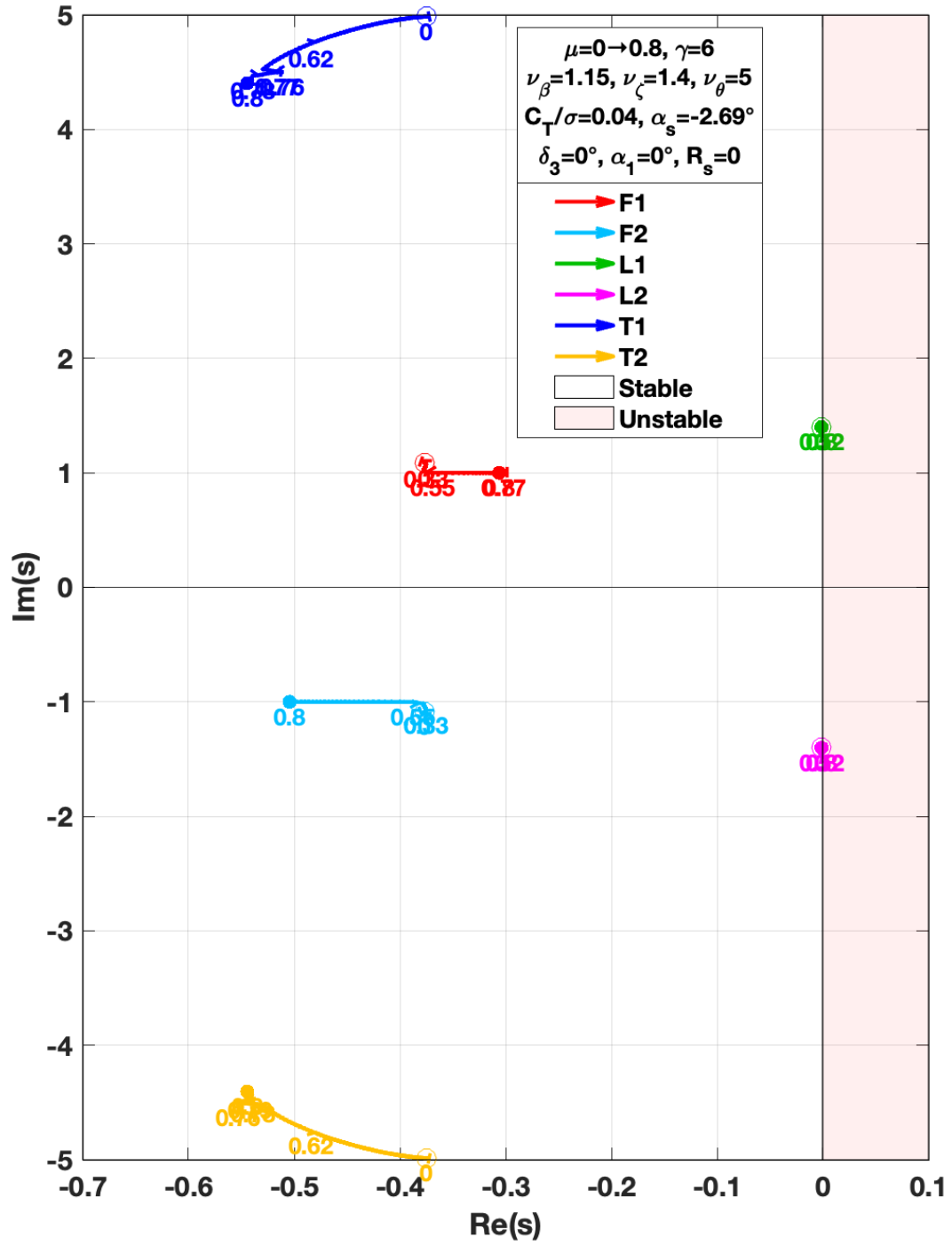


Figure 5.84: Flap-lag-torsion root locus, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8 .

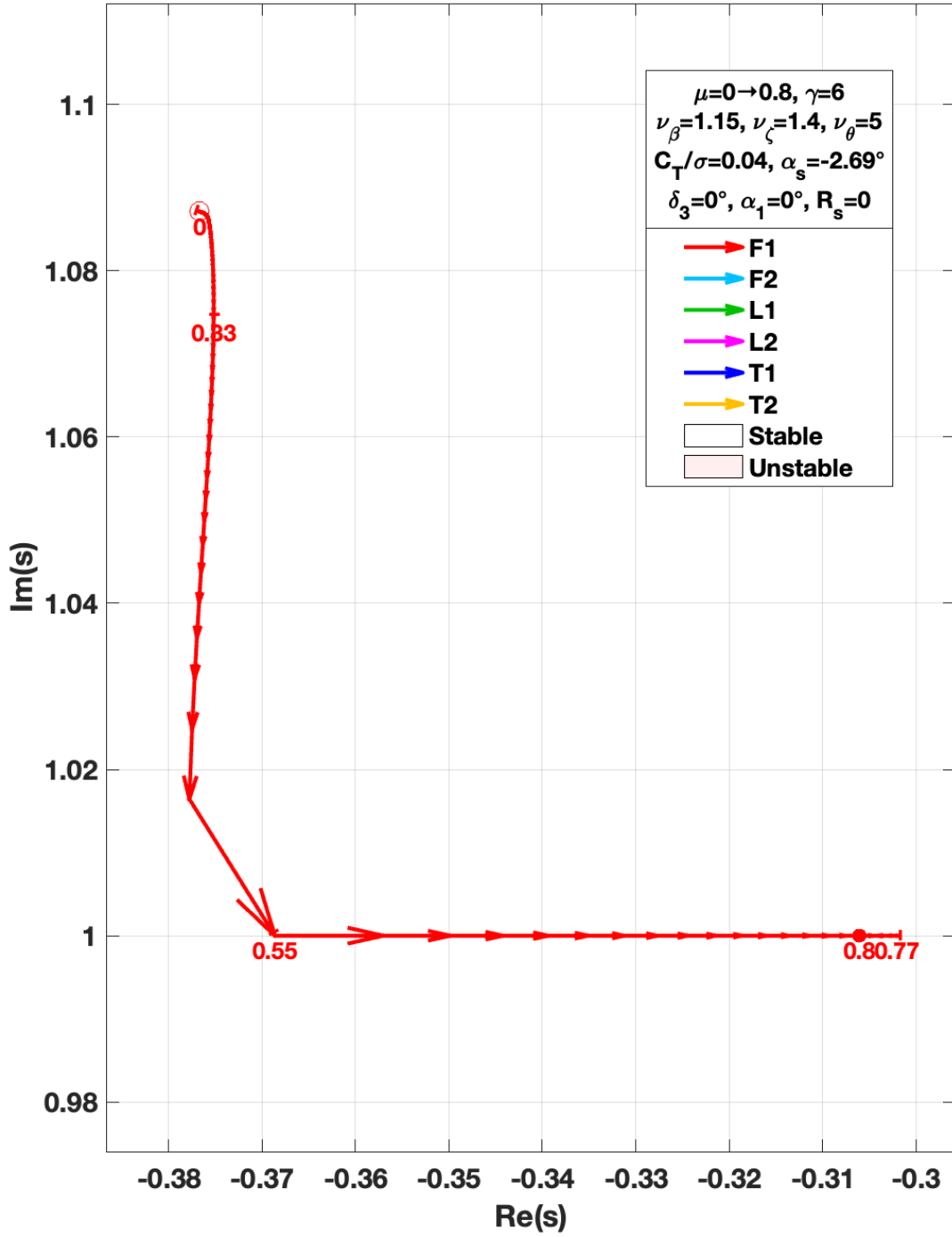


Figure 5.86: Flap-lag-torsion root locus, F1 root, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8 .

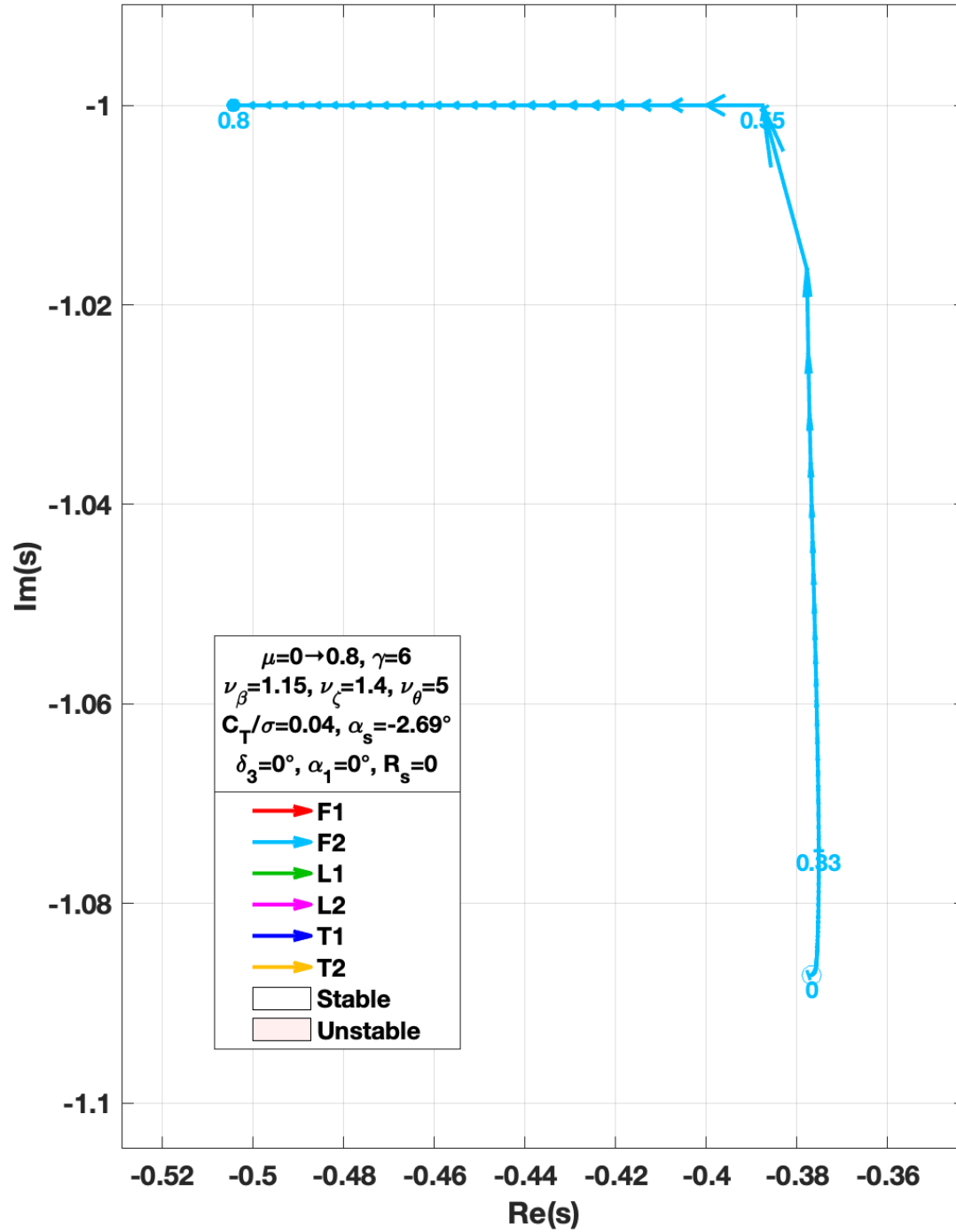


Figure 5.87: Flap-lag-torsion root locus, F2 root, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8 .

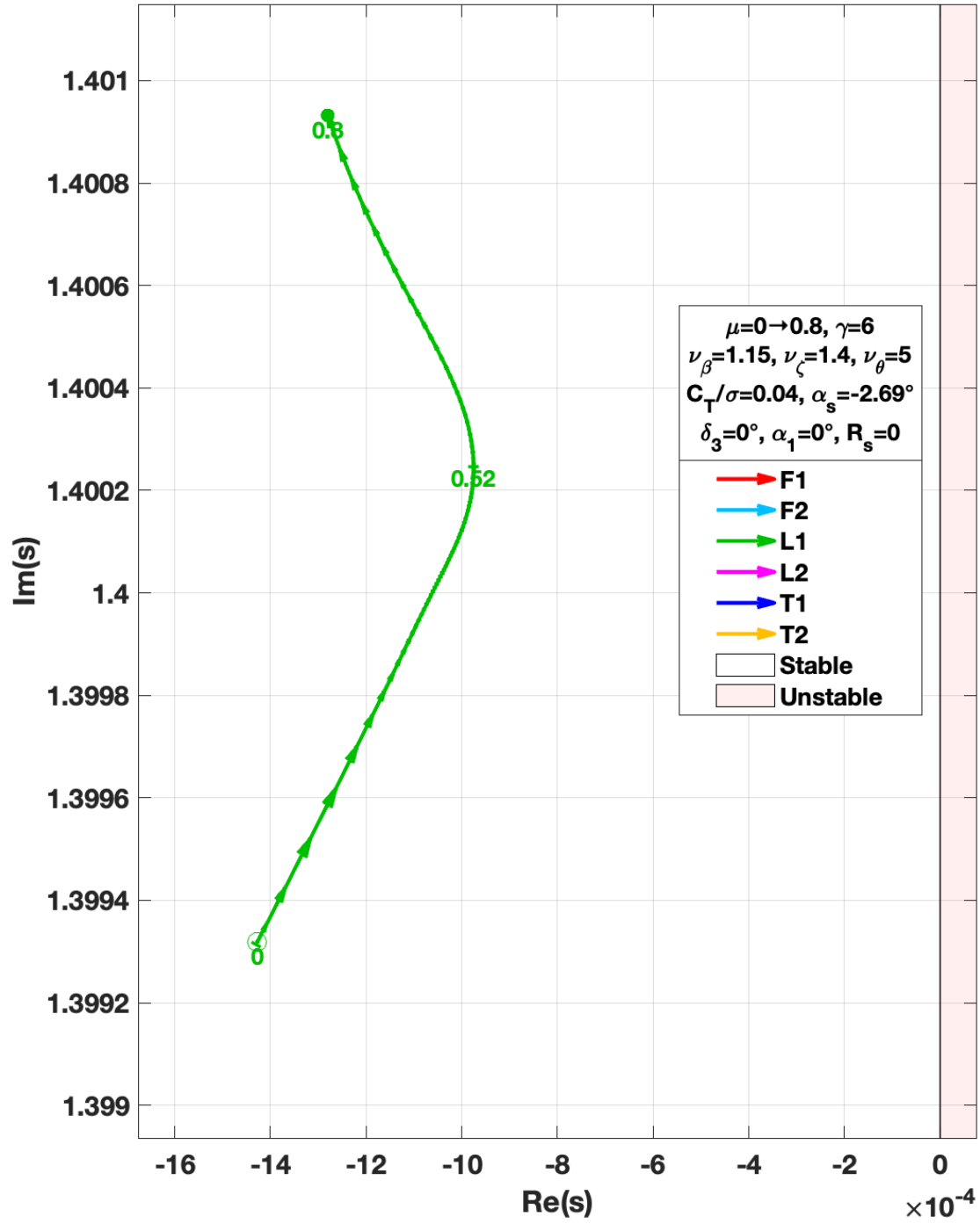


Figure 5.88: Flap-lag-torsion root locus, L1 root, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8 .

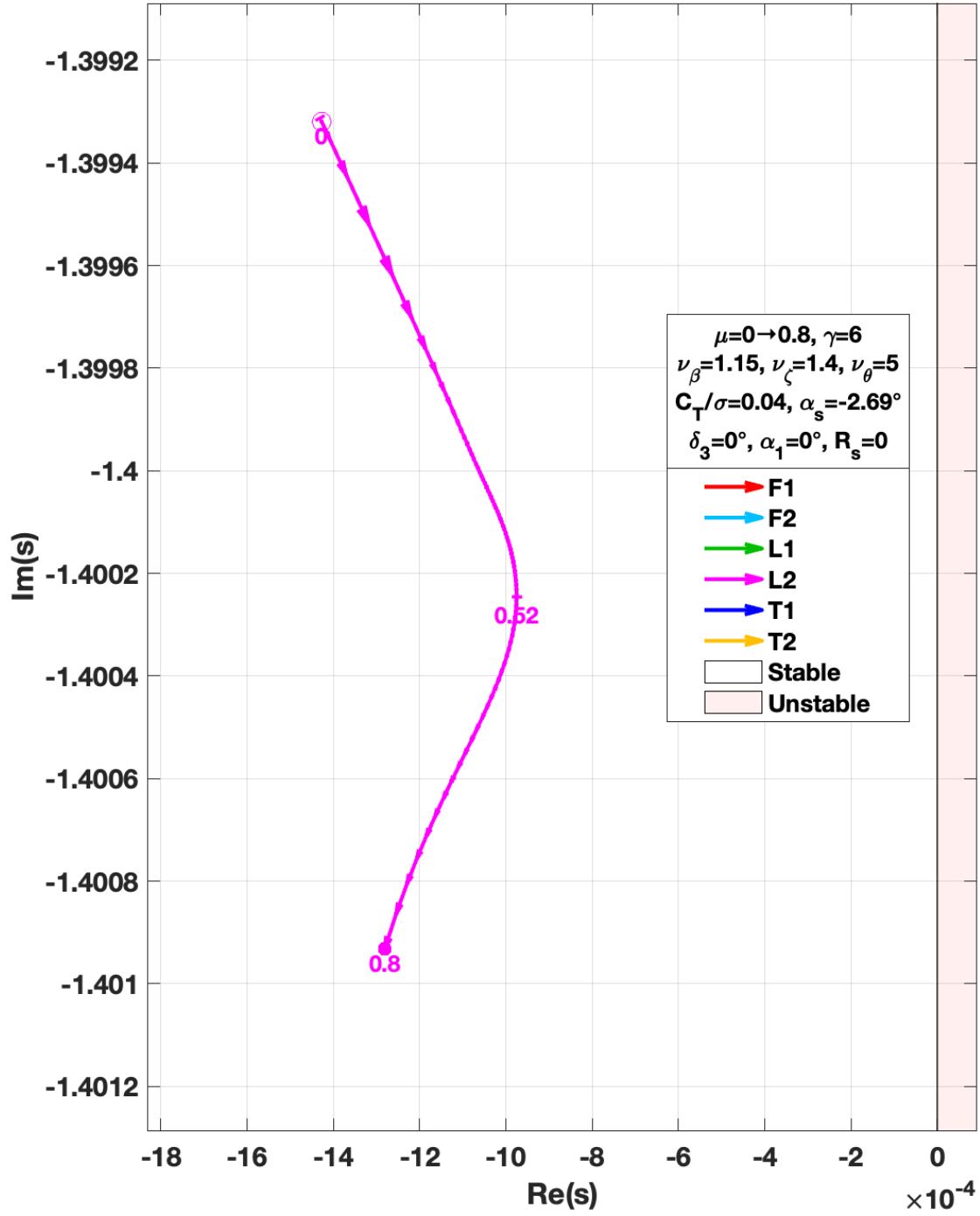


Figure 5.89: Flap-lag-torsion root locus, L2 root, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8 .

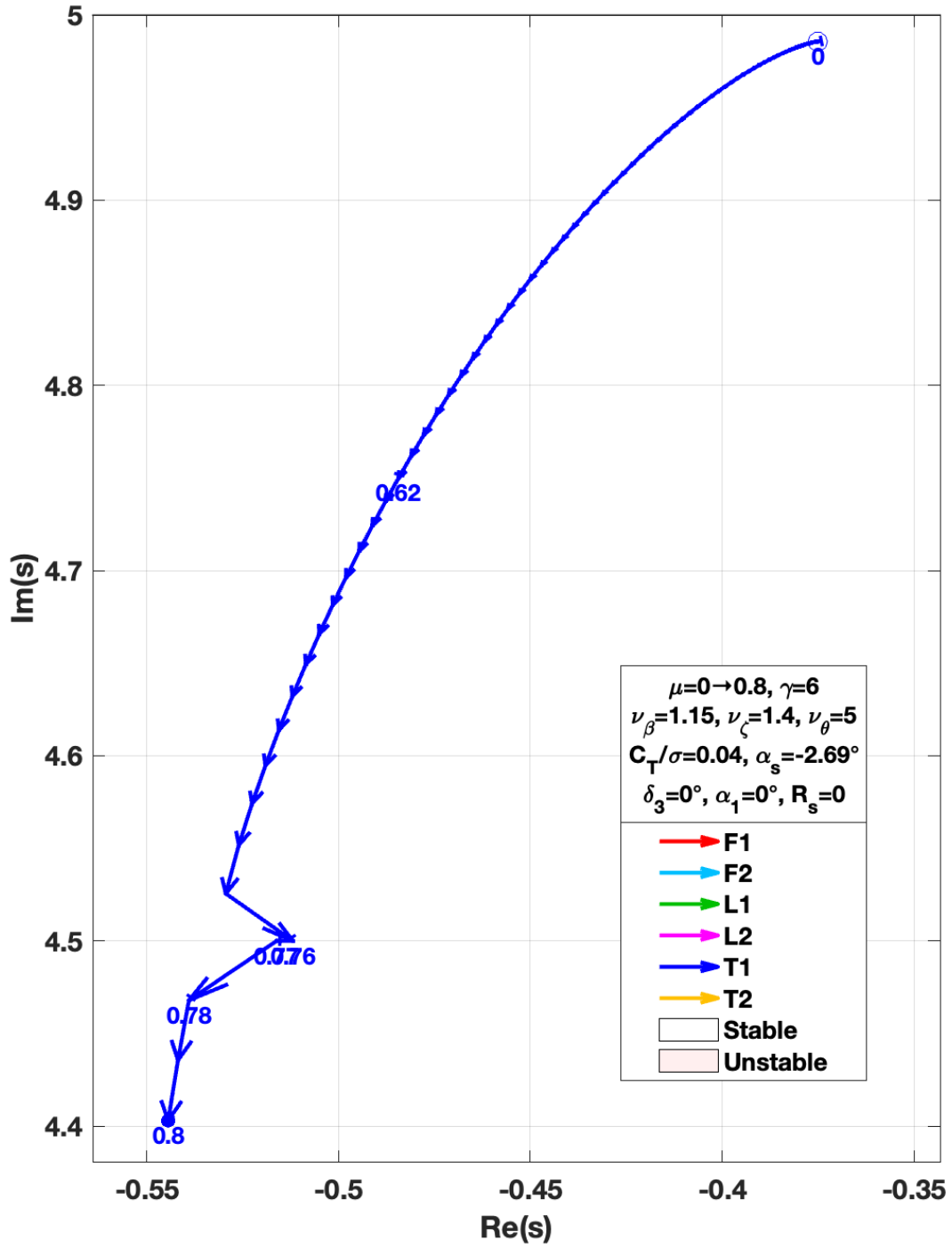


Figure 5.90: Flap-lag-torsion root locus, T1 root, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8 .

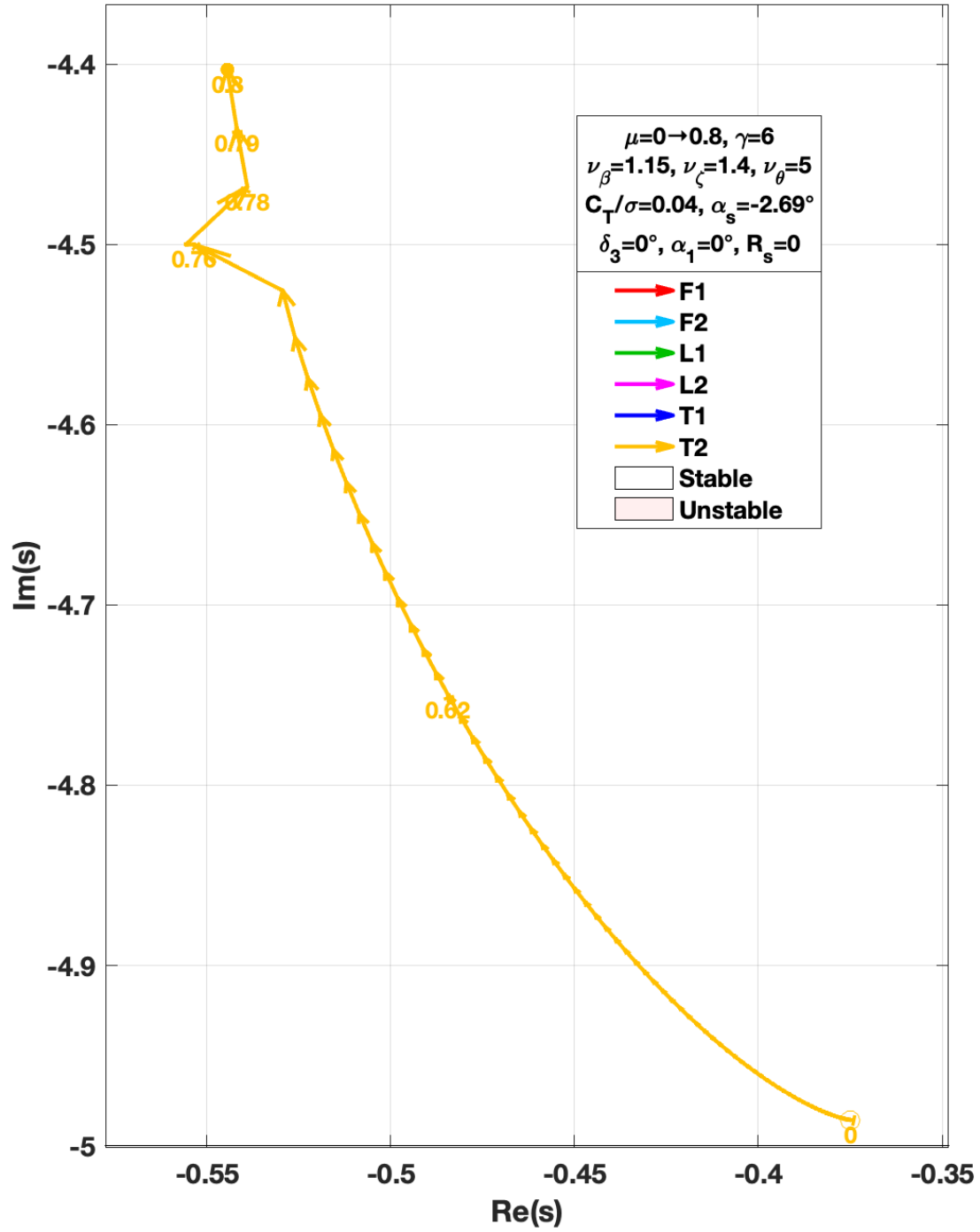
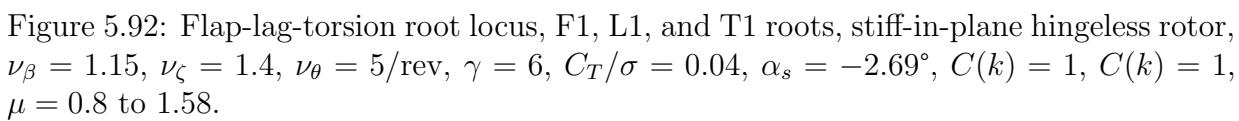


Figure 5.91: Flap-lag-torsion root locus, T2 root, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $C(k) = 1$, $\mu = 0$ to 0.8 .



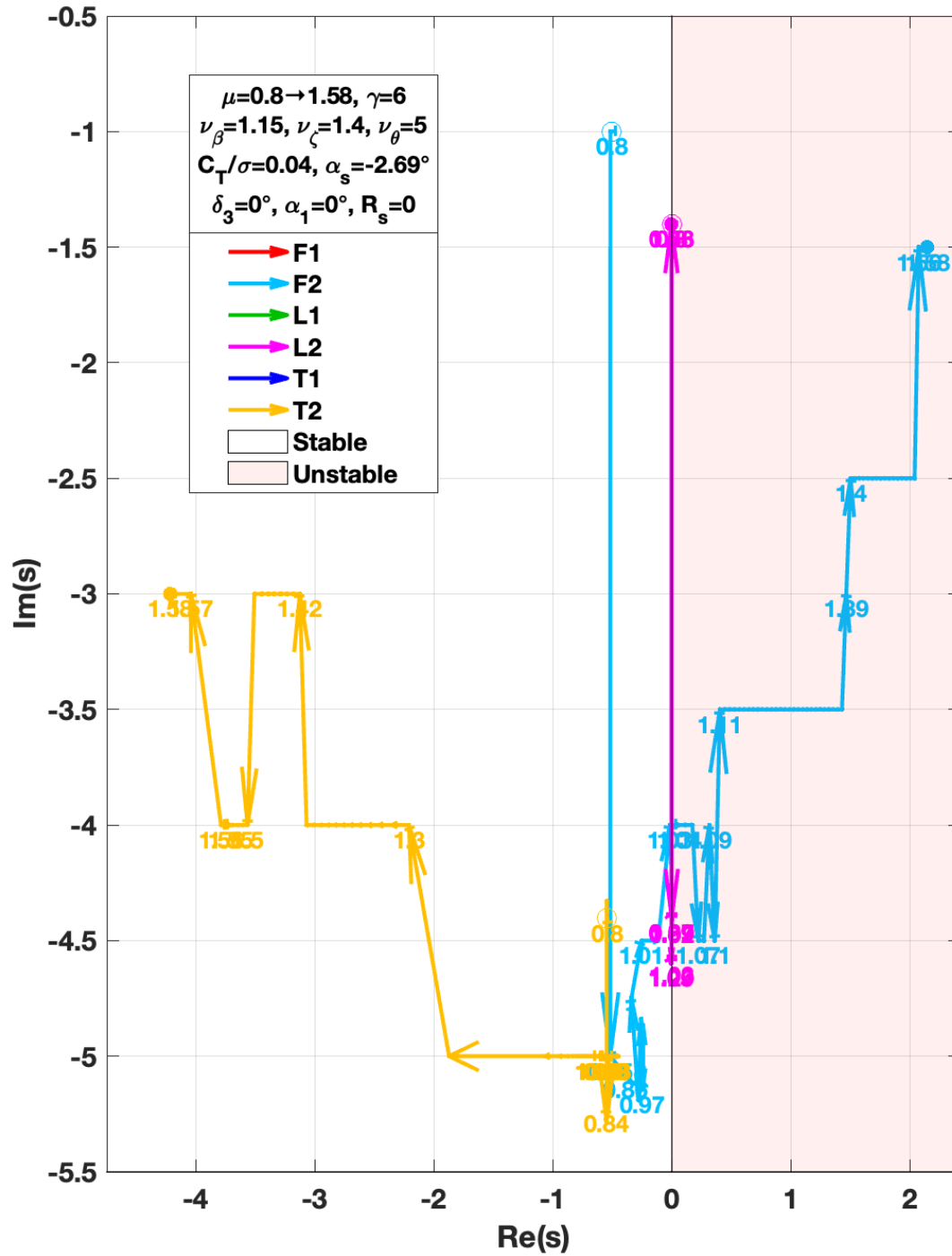


Figure 5.93: Flap-lag-torsion root locus, F2, L2, and T2 roots, stiff-in-plane hingeless rotor, $\nu_\beta = 1.15$, $\nu_\zeta = 1.4$, $\nu_\theta = 5/\text{rev}$, $\gamma = 6$, $C_T/\sigma = 0.04$, $\alpha_s = -2.69^\circ$, $C(k) = 1$, $\mu = 0.8$ to 1.58.

very soft-in-plane articulated rotor also briefly encountered conventional (non-resonant) lag instability at this point.

The stability maps predict that only rotors that are very stiff in all axes can operate near $\mu = 2$ with light blade loadings and an aft shaft tilt. An edgewise rotor operating at a zero-thrust condition could also theoretically approach the higher μ stability limit of the flap-only case. As was seen in the flap-lag stability investigation, this also assumes that the rotor is properly trimmed when transitioning through the thrust reversal region.

Although the derivations were provided for center of gravity and aerodynamic center offsets, no studies were performed regarding the influence of these terms besides accounting for the shifting center of lift during reverse flow. Nevertheless, the capability for evaluating these effects exists in the present model and computer code, and is one of many other parametric studies that can be performed.

Chapter 6: Conclusions and Future Work

6.1 Summary of Stability Studies

This work has demonstrated new a approach to modeling and evaluating blade stability in the rotating frame at high advance ratios. While blades that have low flap, lag, and torsion frequencies may not be able to exceed $\mu = 1$, stiffer configurations could approach $\mu = 2$ and beyond. The trim and stability results compared favorably to past experiments and analyses, with modeling extended to $\mu = 3$ and numerous combinations of parameters.

The flap stability maps matched previous results from the literature, showing that positive pitch-flap coupling and teetering rotor configurations were the most stable generally. Normal articulated and hingeless rotor configurations would lose flap stability around $\mu = 2.3$. There exist configurations that have theoretical flap stability past $\mu = 3$, but in all cases it is difficult to avoid 1/2 or 1 per/rev parametric resonance regions. During the root locus analysis, it was found that when this resonance is encountered, the frequency locking phenomenon not only drives one of the roots towards instability, but due to the distribution of modal participation factors over positive and negative frequencies, the associated eigenvalues are not truly progressive or regressive.

A major issue with flap trim at high speed is how to maintain control through thrust reversal, which peaks near $\mu = 0.85$. To minimize flight control divergence, blade stall, and

aeroelastic instability, the best solution is either a carefully trimmed aft shaft tilt based on the desired blade loading, or to have a level rotor with zero thrust. For most of the analyses in this work, the former approach is used.

Due to the Coriolis terms, lag stability is very sensitive to the trim solution. As expected, avoiding 0.5/rev and 1/rev multiples for the lag natural frequency was beneficial for stability. In general, blades that are stiff in all axes seem to have better stability characteristics as μ -increases. For soft-in-plane articulated and hingeless rotors in the low thrust, aft shaft tilt configuration, lag instability occurs prior to flap instability, near $\mu = 1.8$. The very soft-in-plane articulated rotor can maintain lag stability until flap becomes unstable, and the stiff-in-plane hingeless blade is stable in this configuration through $\mu = 3$. It was observed that lag became conventionally unstable (without frequency locking) in these cases.

The influence of the torsion degree of freedom at high advance ratios was significant, even for blades that are stiff in pitch. The reverse flow and unsteady aerodynamics effects were important. Soft-in-pitch rotor blades were observed to encounter an all-axis parametric resonance near the torsion frequency when approaching $\mu = 1$. It appears that torsion drives the flap motion towards instability, though the root trajectory analysis is difficult during this parametric excitation which causes the eigenvalues to coalesce in both damping and frequency. Blades with torsional frequencies above 10/rev can maintain stability near $\mu = 1.45$. The stability maps showed that very torsionally stiff blades ($> 15/\text{rev}$) could approach $\mu = 2$, but both flap and lag would need correspondingly high rotating natural frequencies as well.

6.2 Novel Developments

This work has expanded coupled flap-lag-torsion stability analysis to much higher advance ratios than previous analyses. This was accomplished using a hybrid symbolic-numerical approach to derive the linearized flap-lag-torsion perturbation equations and solve for stability using Floquet theory. Essential terms were considered, such as reverse flow and unsteady aerodynamics. Although the aerodynamics were limited to linear modeling, the blade stall solver provided a constraint to limit analyses to suitable conditions.

Root tracking was provided by a novel algorithm incorporating eigenvector motion analysis. In addition to providing a mechanism to automatically attribute eigenvalues to a degree of freedom, this provided insights into root behavior during frequency locking and parametric resonance. The types of high- μ root loci in this work have never been presented before.

A new method to determine the numerical stability of the numerical integration was also found. By analyzing the motion of the modal waveforms over two revolutions, the resultant dissipation and phase error were determined, which allowed for better conclusions regarding the physical stability of the coupled system.

6.3 Future Research

There are many ways to improve the existing analysis and computer code. The first is probably elastic blade (mode shape) analysis through FEM or another approach. It is noted that the present rigid blade model has a demonstrated limitation when modeling lag

stability for stiff-in-plane blades. Ironically, such configurations were shown to have the best stability at high advance ratios.

Other improvements could include airfoil tables to include Mach and Reynolds number corrections, wake modeling, hub motion, and blade differences. The present method can provide initial conditions for comprehensive code analyses. Some of the routines and code used this work may be used to improve the stability functions and libraries within UMARC.

The eigenvalue trajectory analysis and sorting algorithm developed in this work is promising, but more investigation is needed to assess its efficacy and improve its performance during multi-axis parametric resonance.

The parametric analyses presented in this work can be used as predictions of the damping ratios and frequency responses of model rotors operating at high advance ratios. It is hoped that these results will provide risk reduction and a theoretical foundation for high- μ wind tunnel stability testing and the eventual design of revolutionary high speed rotorcraft.

Appendix A: High- μ Flap Trim Validation

A.1 Articulated Rotor Trim Validation

The linear trim model used in this work is compared to past high advance ratio experiments of articulated rotors. The first is Jenkins' 1965 tests up $\mu = 1.45$ with varying shaft tilts (Ref. 35), shown in Figs. A.1, Figs. A.2, and Figs. A.3. There is an under-prediction of blade loading, but the trim results are otherwise fairly close to the present work.

The model in this work is also compared to the University of Maryland's high- μ articulated rotor wind tunnel experiments conducted by Bowen-Davies and Chopra in Ref. 91. Trimmed collective and cyclic control positions are presented in Figs. A.4 through A.13 up to an advance ratio of 1.20. The present linear model's trends compare favorably with both the test data and the UMARC comprehensive code. There is a trend of slightly under-predicting blade loading and cyclic values for given collective settings, a deviation shared with the more sophisticated UMARC.

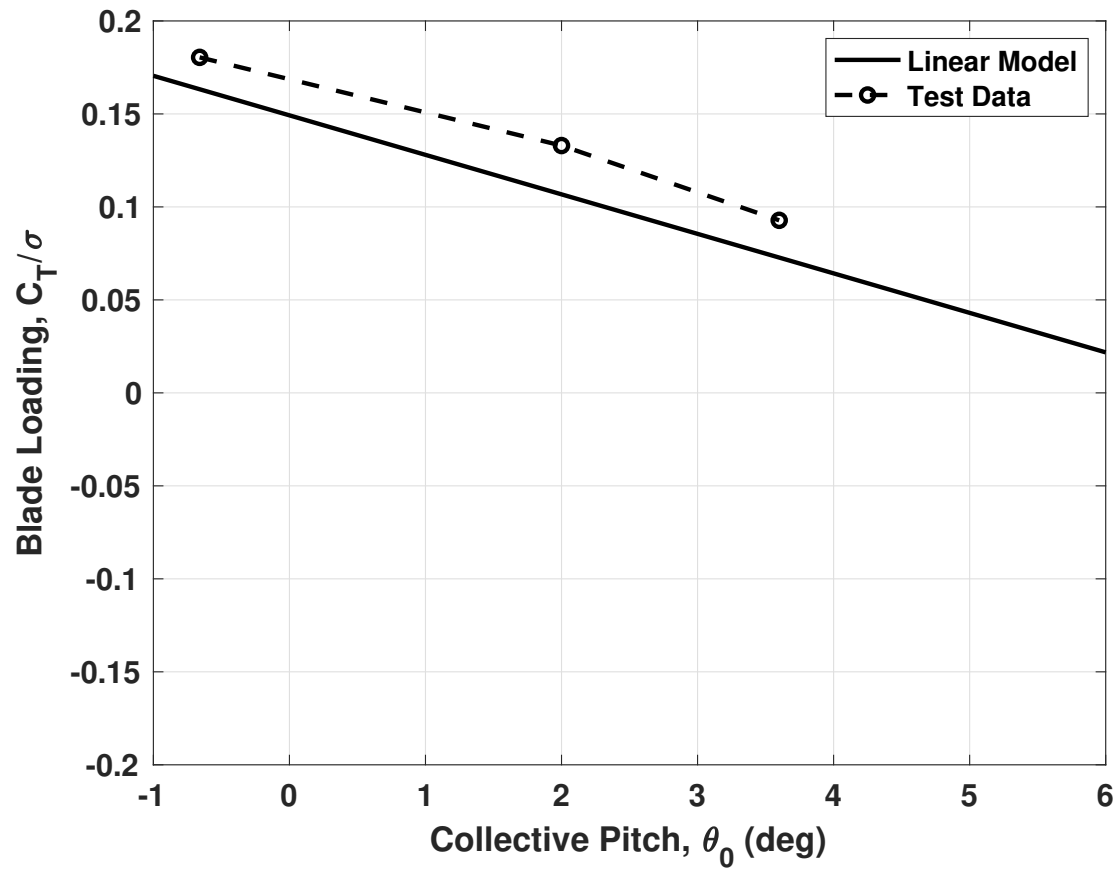


Figure A.1: Trim comparison with NASA high- μ articulated rotor tests, blade loading vs. collective, $\mu = 1.45$, $\alpha_s = 0$.

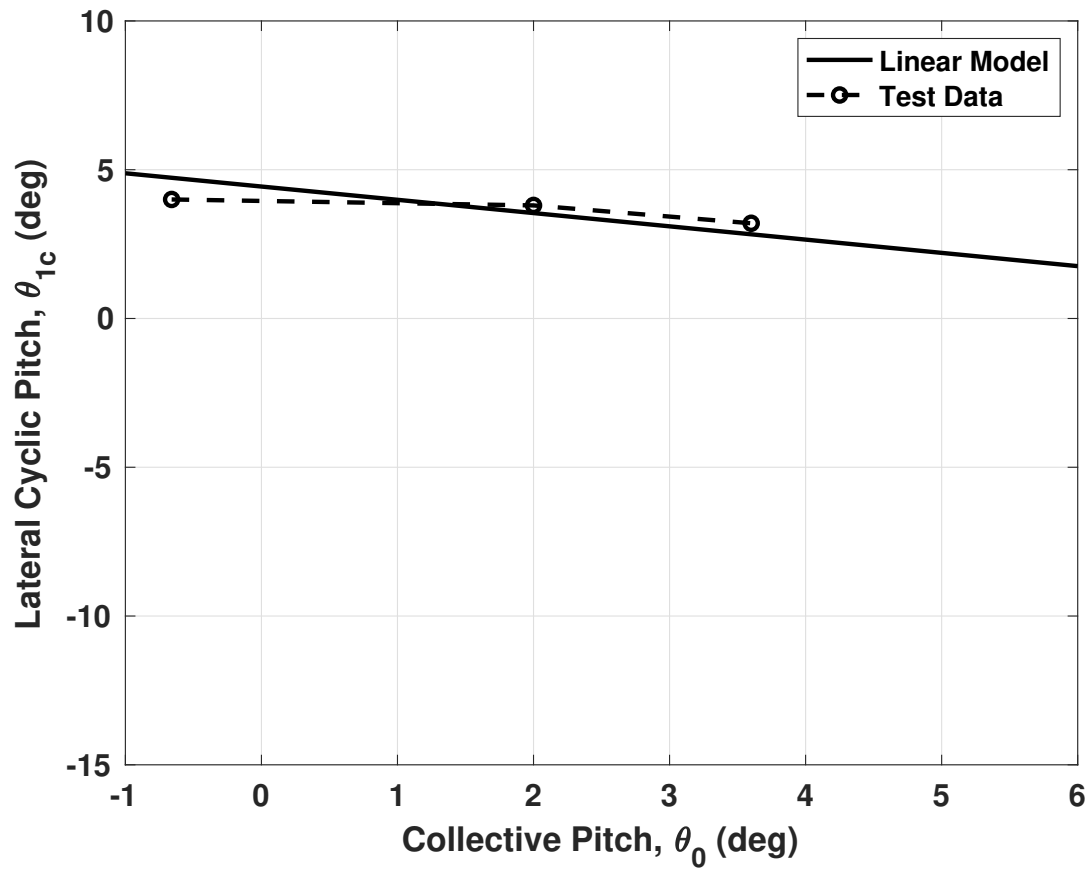


Figure A.2: Trim comparison with NASA high- μ articulated rotor tests, lateral cyclic vs. collective, $\mu = 1.45$, $\alpha_s = 0$.

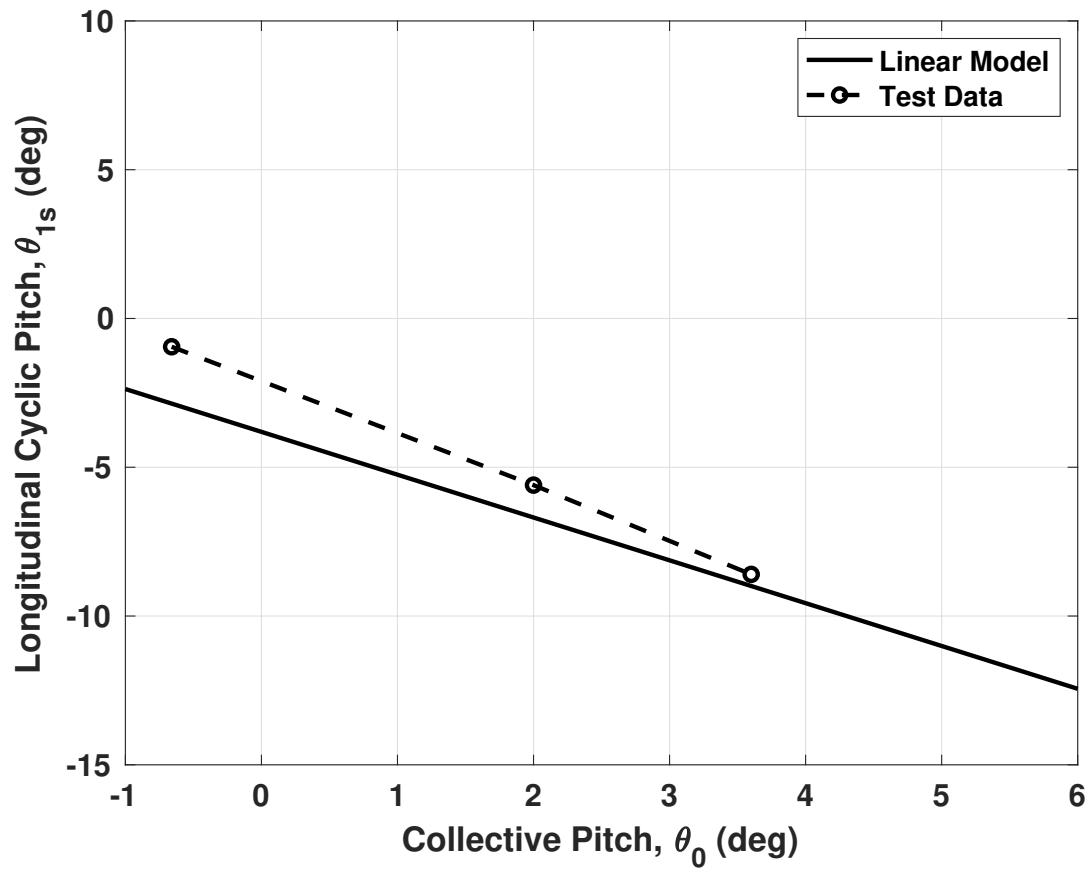


Figure A.3: Trim comparison with NASA high- μ articulated rotor tests, longitudinal cyclic vs. collective, $\mu = 1.45$, $\alpha_s = 0$.

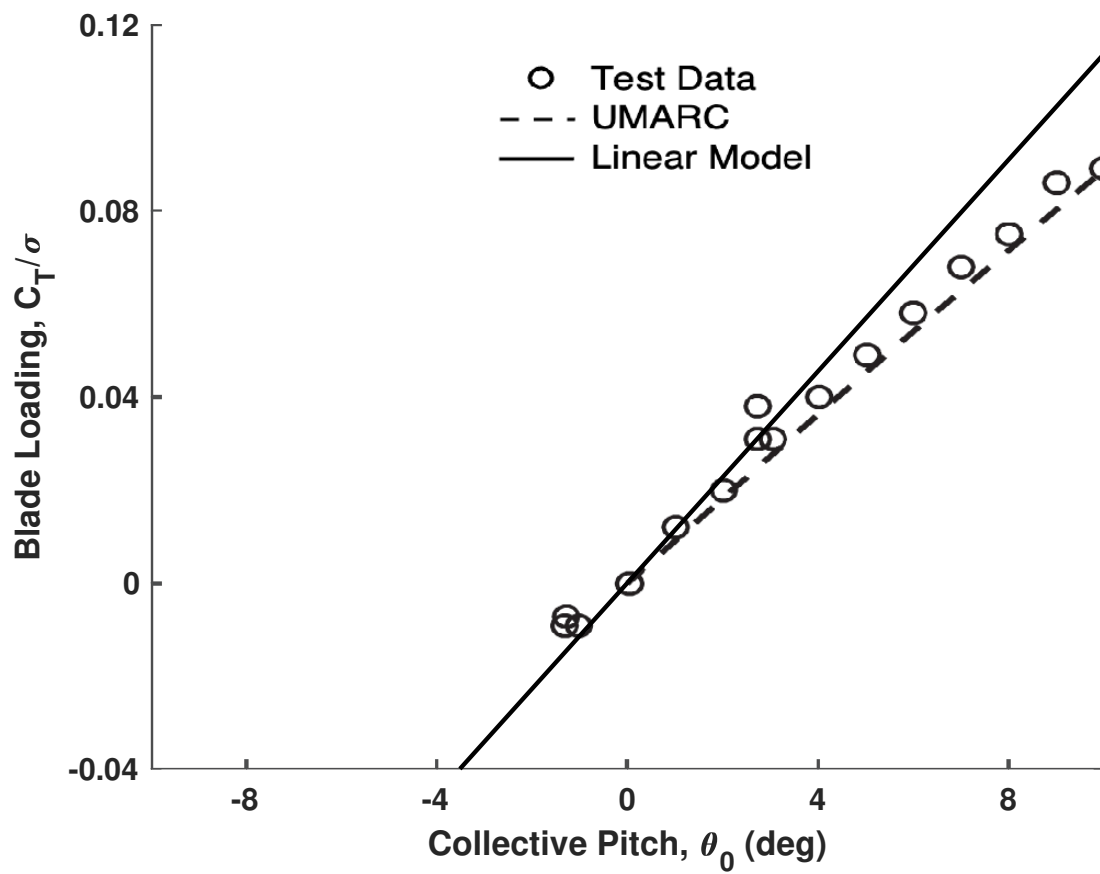


Figure A.4: Trim omparison with UMD high- μ articulated rotor tests, blade loading vs. collective, $\mu = 0.25$.

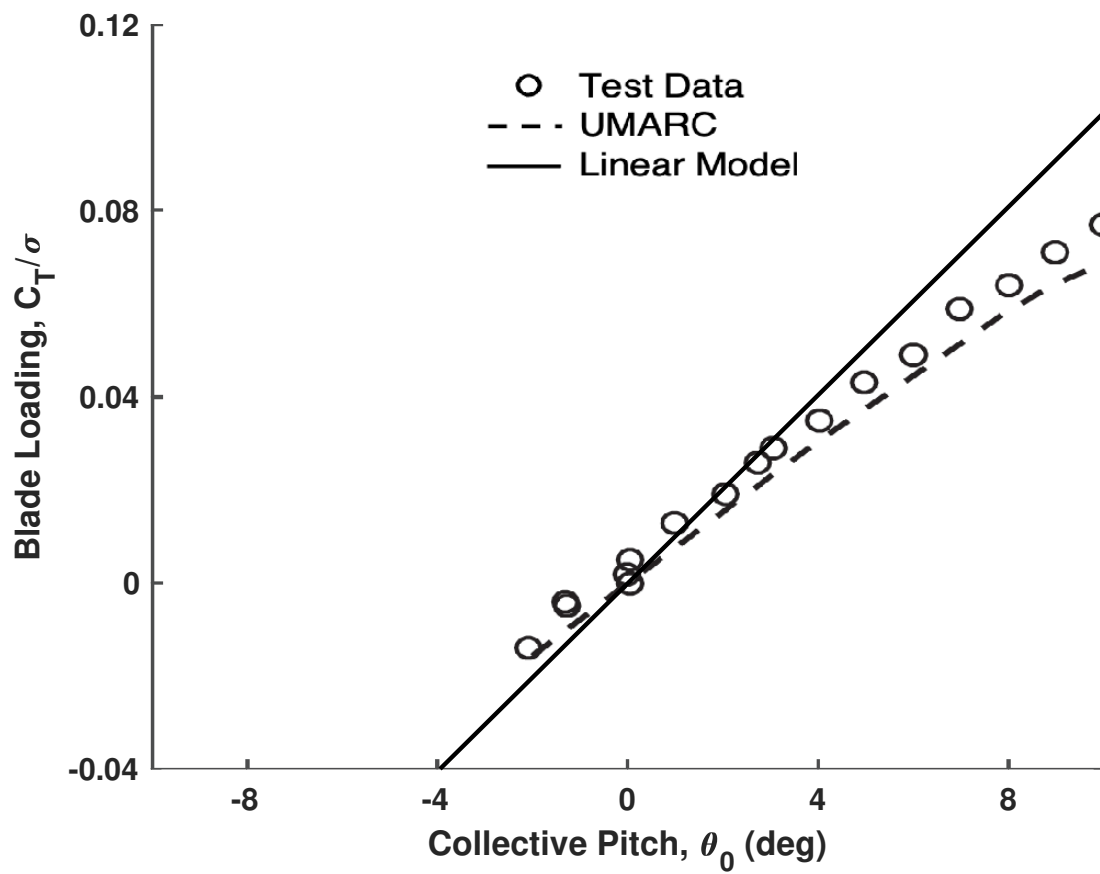


Figure A.5: Trim comparison with UMD high- μ articulated rotor tests, blade loading vs. collective. $\mu = 0.41$.

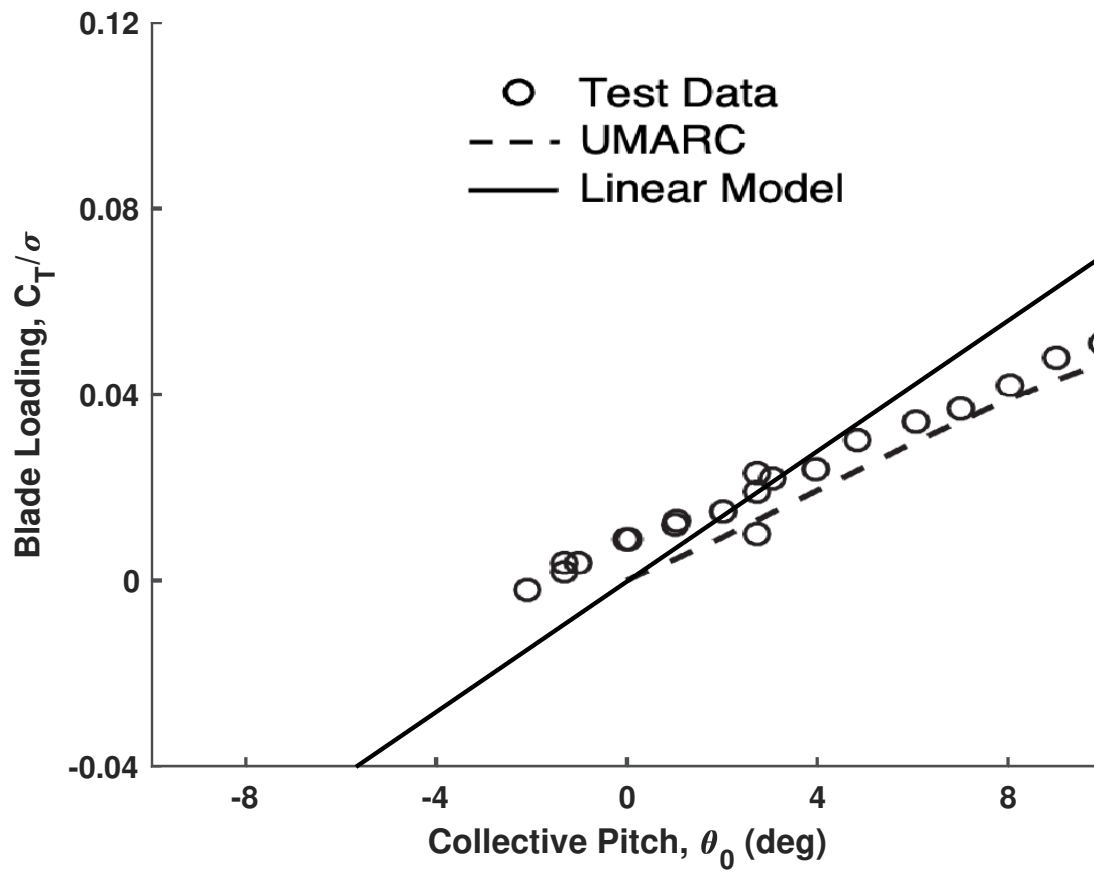


Figure A.6: Trim omparison with UMD high- μ articulated rotor tests, blade loading vs. collective, $\mu = 0.62$.

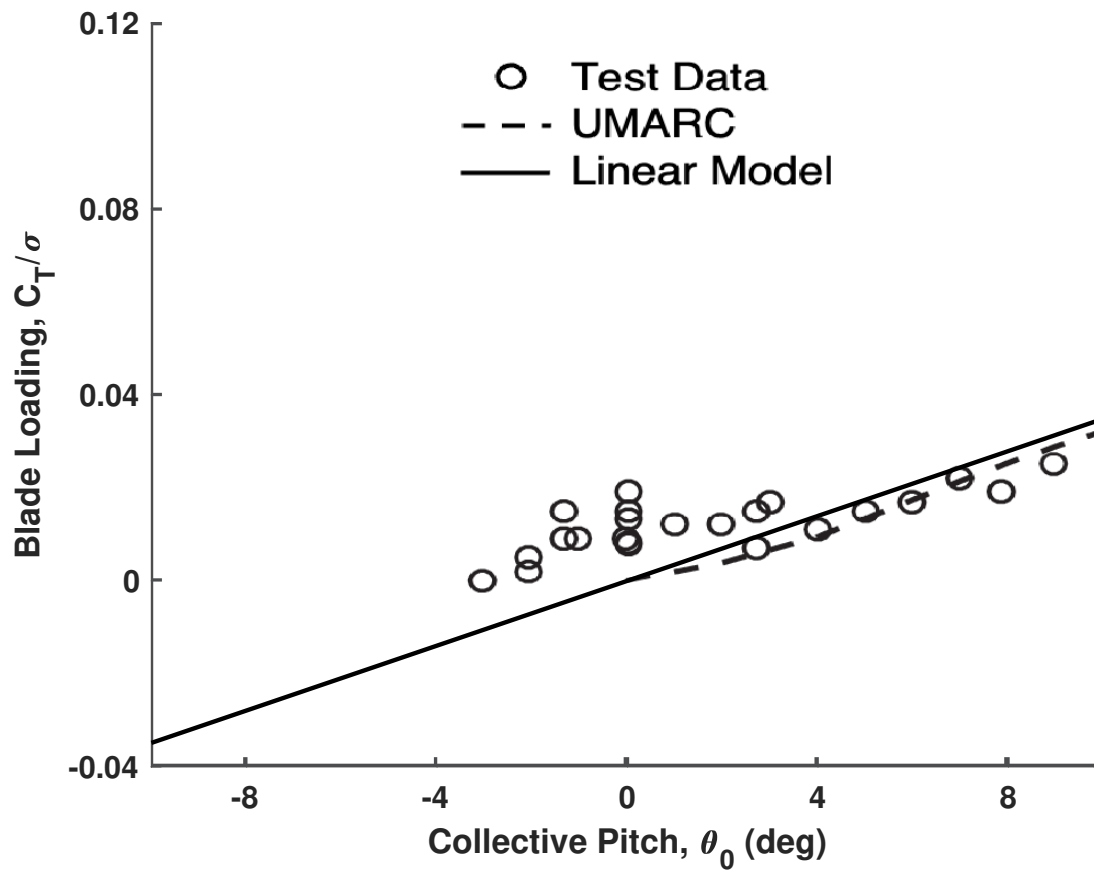


Figure A.7: Trim omparison with UMD high- μ articulated rotor tests, blade loading vs. collective, $\mu = 0.83$.

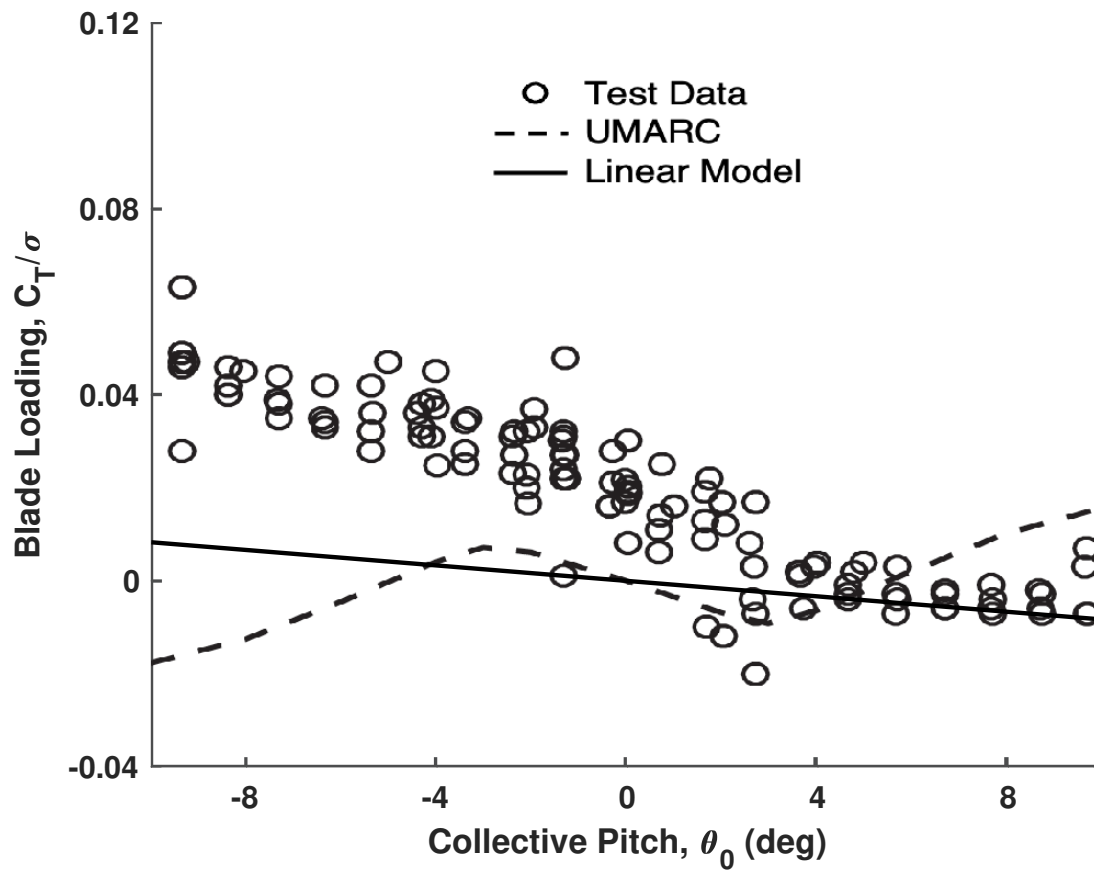


Figure A.8: Trim comparison with UMD high- μ articulated rotor tests, blade loading vs. collective, $\mu = 1.03$.

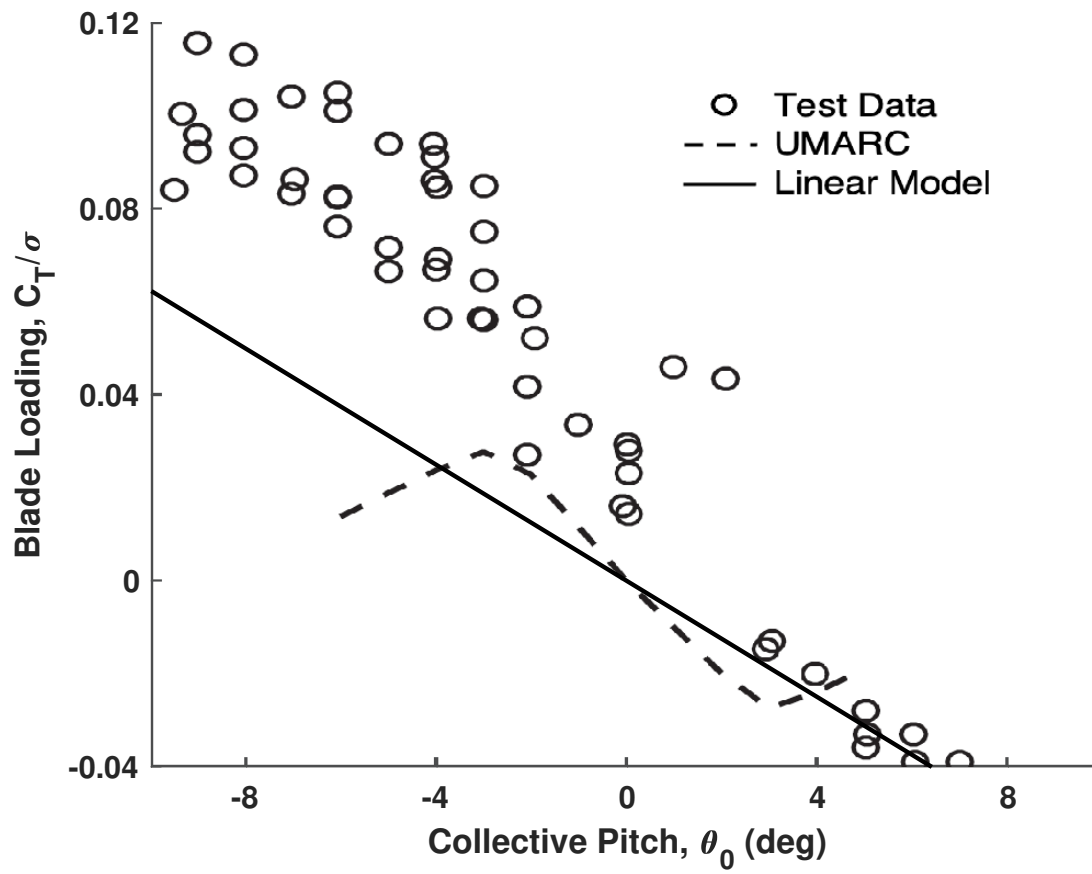


Figure A.9: Trim comparison with UMD high- μ articulated rotor tests, blade loading vs. collective, $\mu = 1.20$.

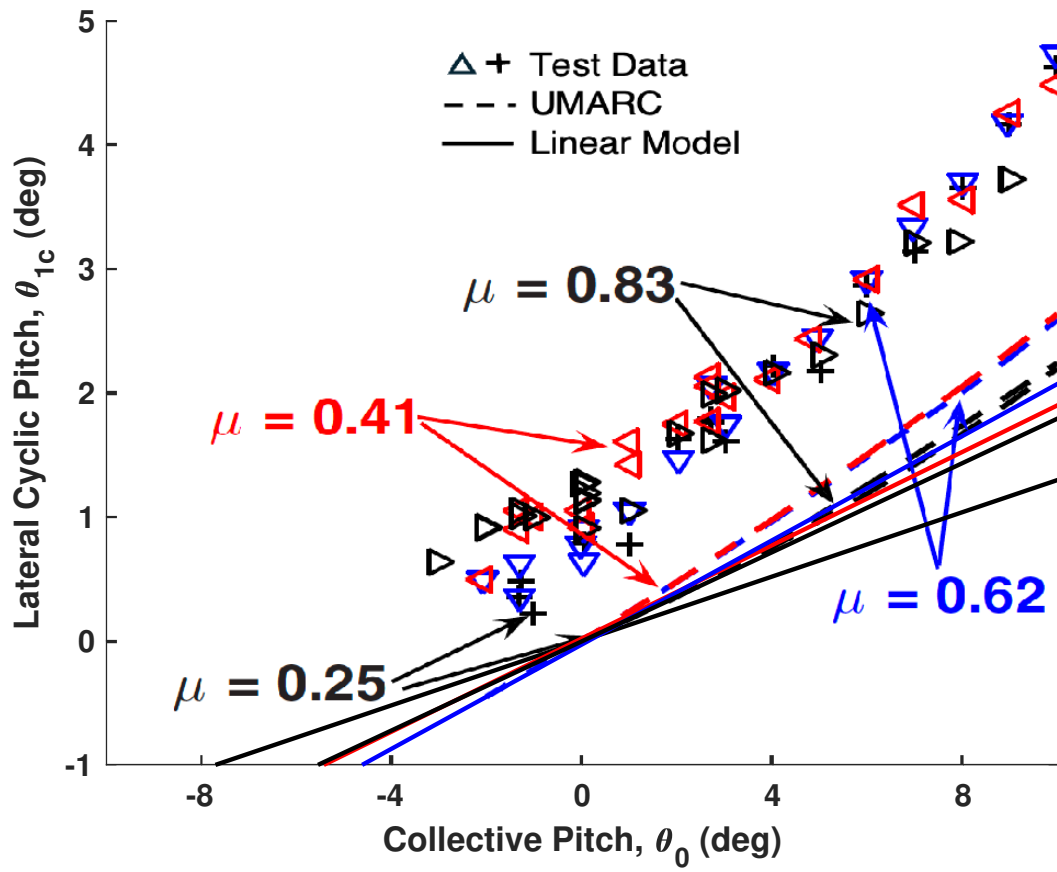


Figure A.10: Trim comparison with UMD high- μ articulated rotor tests, lateral cyclic vs. collective, low μ .

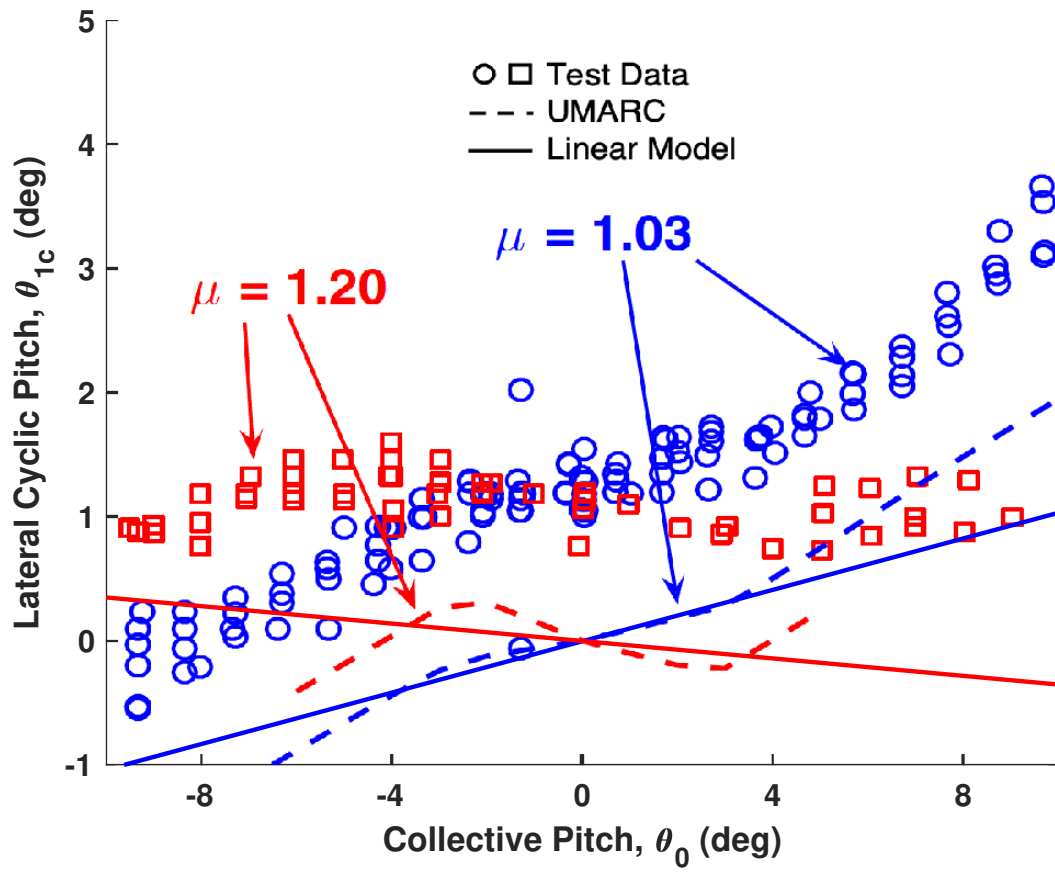


Figure A.11: Trim comparison with UMD high- μ articulated rotor tests, lateral cyclic vs. collective, high μ .

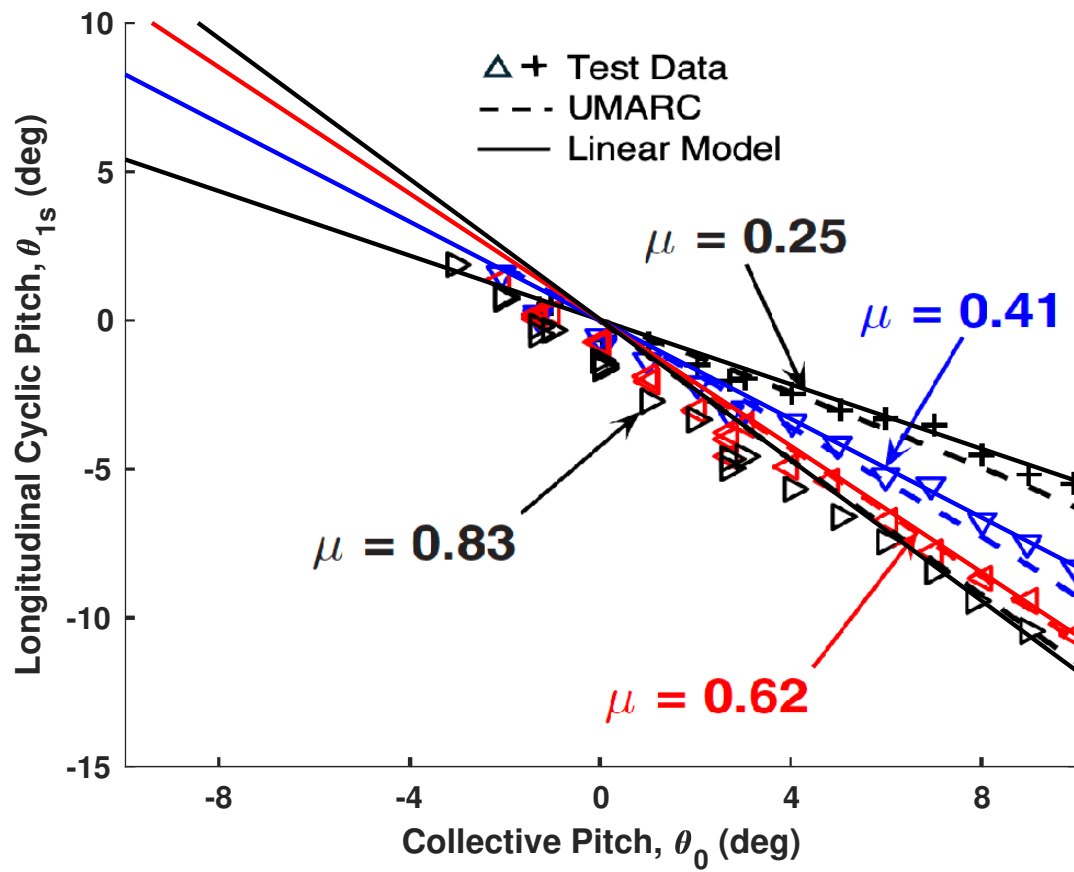


Figure A.12: Trim comparison with UMD high- μ articulated rotor tests, longitudinal cyclic vs. collective, low μ .

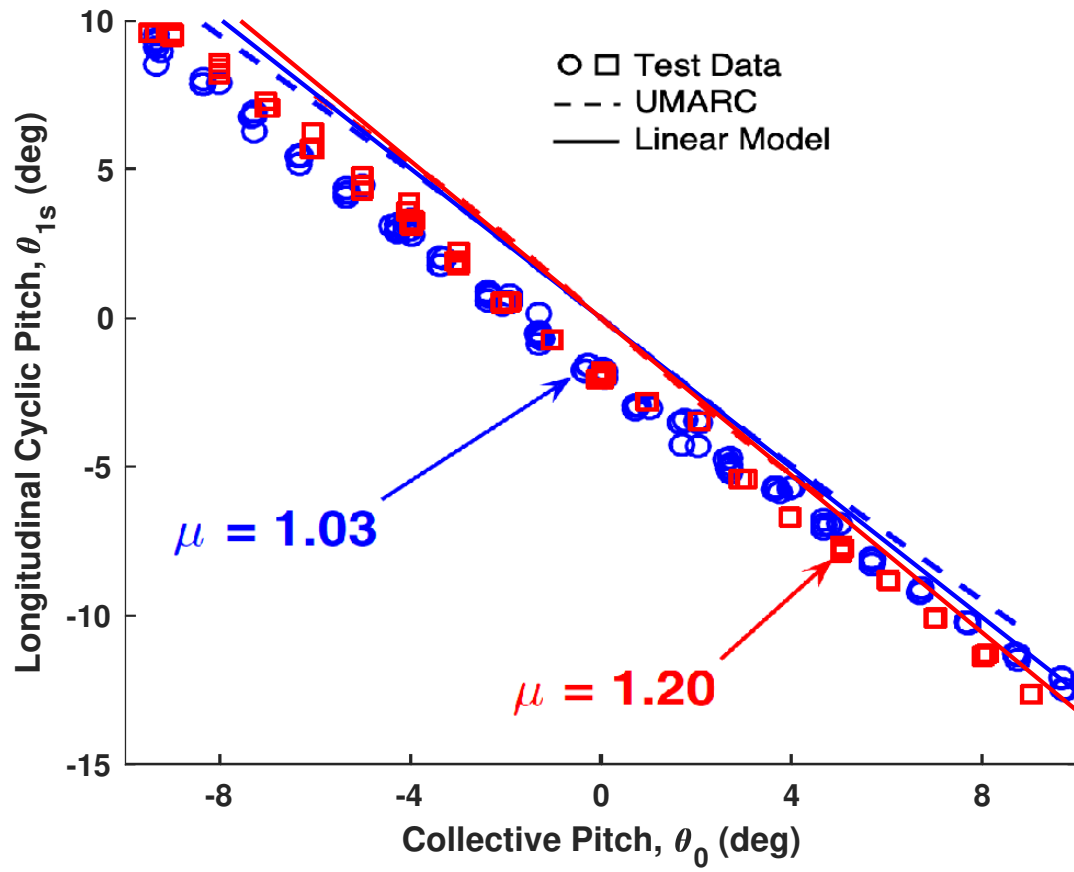


Figure A.13: Trim comparison with UMD high- μ articulated rotor tests, longitudinal cyclic vs. collective, high μ .

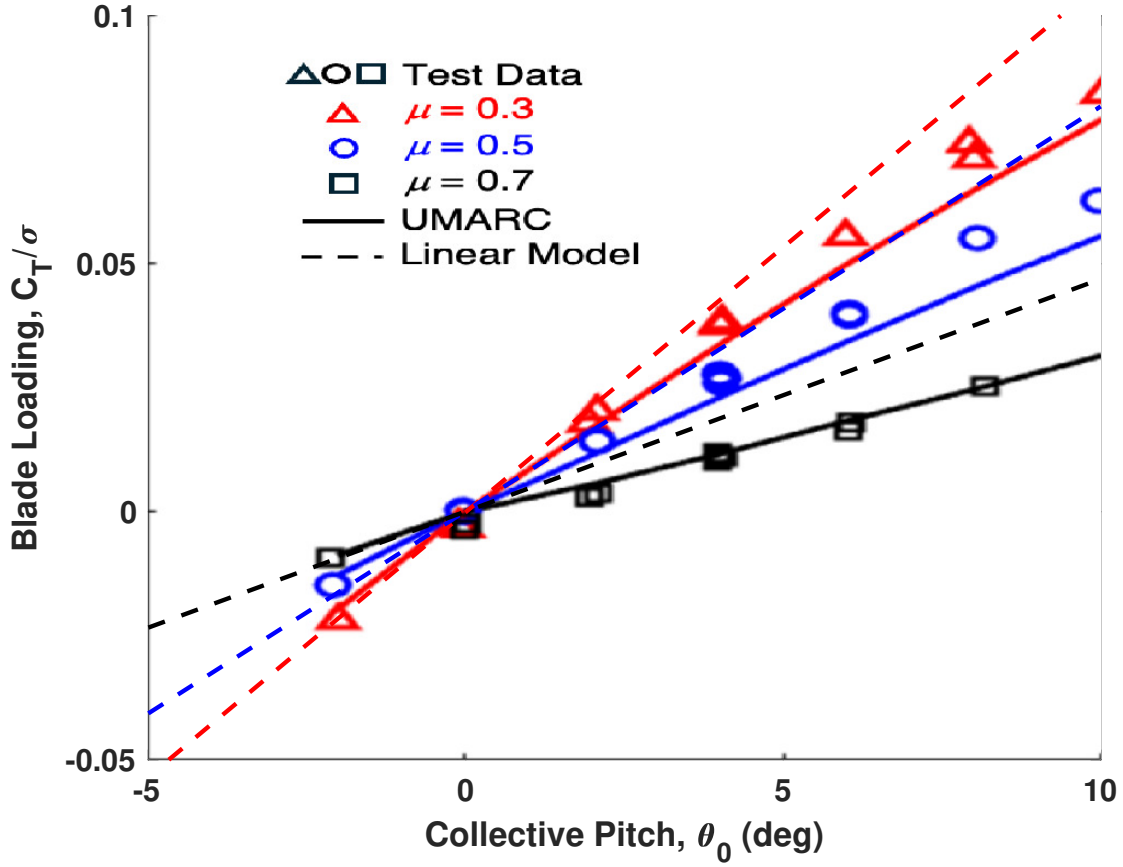


Figure A.14: Trim comparison with UMD high- μ hingeless rotor tests, blade loading vs. collective

A.2 Hingeless Rotor Trim Validation

The present model is compared with the UMD hingeless rotor wind tunnel tests performed by Wang et al. Ref. 92 Trimmed flight control positions are also shown in Figs. A.14 through valid A.16 up to $\mu = 0.7$. The linear model tends to slightly over-predict thrust and under-predict lateral cyclic for a given collective pitch.

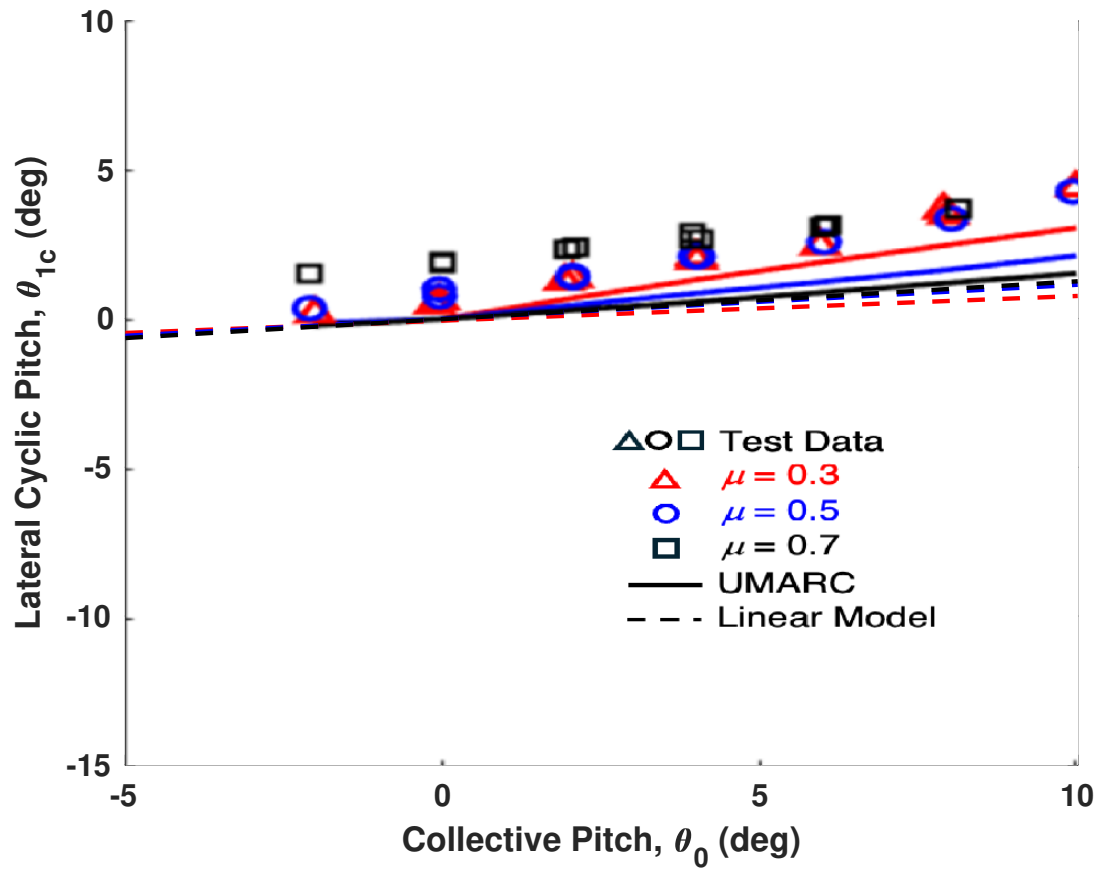


Figure A.15: Trim comparison with UMD high- μ hingeless rotor tests, lateral cyclic vs. collective.

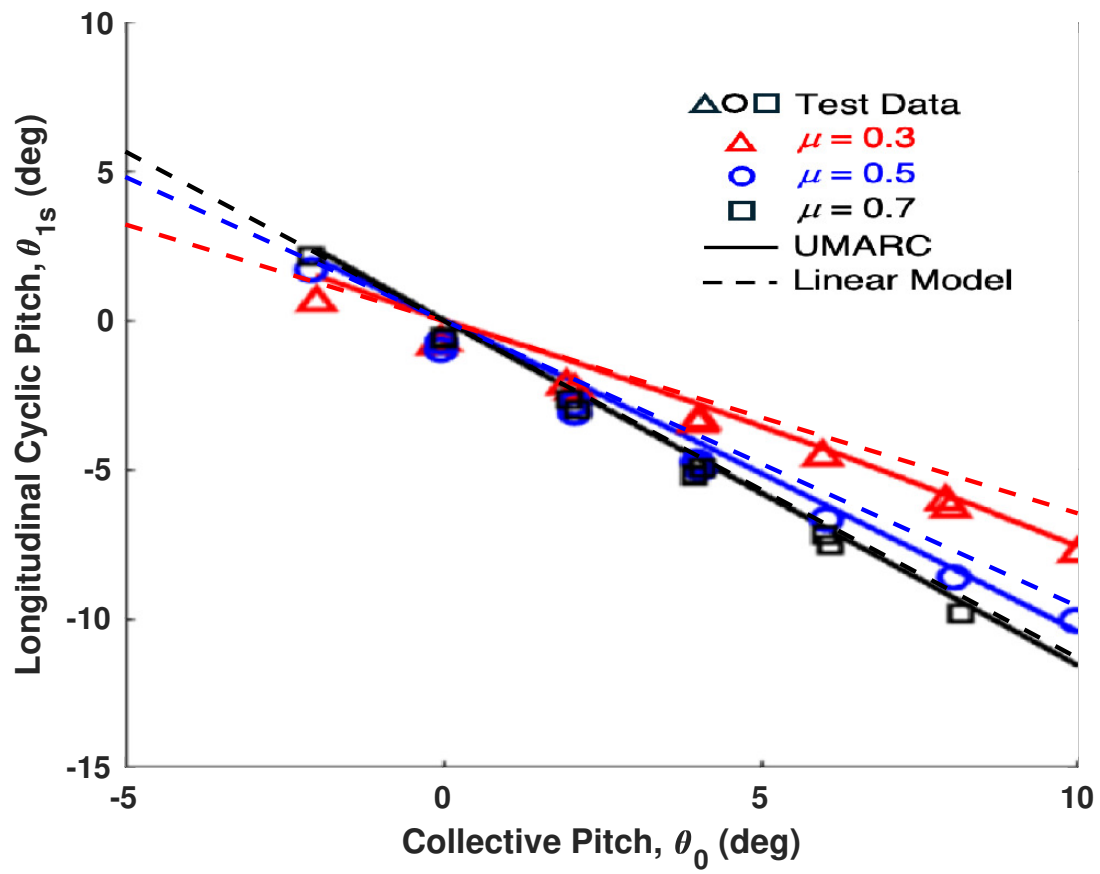


Figure A.16: Trim comparison with UMD high- μ hingeless rotor tests, longitudinal cyclic vs. collective.

Appendix B: Flap-Lag-Torsion Aerodynamic Forcing Moment Equations

The coupled damping and stiffness matrices are

$$\begin{aligned} \underline{\underline{C}} &= \begin{pmatrix} -\gamma M_{\beta}^* & -2\beta_T - \gamma M_{\zeta}^* & -\gamma M_{\theta}^* \\ 2\beta_T - \gamma Q_{\beta}^* & 2\nu_{\zeta_0}\zeta_L - \gamma Q_{\zeta}^* & -2\beta_T I_x^* - \gamma Q_{\theta}^* \\ -\gamma T_{\beta}^* & 2\beta_T I_x^* - \gamma T_{\zeta}^* & 2\nu_{\theta_0}\zeta_T I_f^* - \gamma T_{\theta}^* \end{pmatrix} \\ \underline{\underline{K}} &= \begin{pmatrix} \nu_{\beta}^2 - \gamma M_{\beta} & -\gamma M_{\zeta} + Z & -I_x^* - \gamma M_{\theta} \\ 2\beta_T^* - \gamma Q_{\beta} + Z & \nu_{\zeta}^2 - \gamma Q_{\zeta} & -\gamma Q_{\theta} \\ -I_x^* - \gamma T_{\beta} & -\gamma T_{\zeta} & I_f \nu_{\theta}^2 - \gamma T_{\theta} \end{pmatrix} \end{aligned} \quad (\text{B.1})$$

The periodic aerodynamic forcing moments contained in these matrices (M_{β}^* , T_{θ} , etc.) are derived for each flow region:

1. Region I, normal flow: The advancing side of the rotor, $\psi \leq 180^\circ$.
2. Region II, partial reverse flow: When $\mu < 1$, the entire retreating side of the rotor ($\psi > 180^\circ$) is in partial reverse flow. When $\mu \geq 1$, there are two sub-regions: $180^\circ < \psi < 270^\circ - \cos^{-1}(1/\mu)$ and $\psi > 270^\circ + \cos^{-1}(1/\mu)$.
3. Region III, complete reverse flow: When $\mu > 1$, there is a section of the retreating side where the entire blade is in reverse flow: $270^\circ - \cos^{-1}(1/\mu) \geq \psi \geq 270^\circ + \cos^{-1}(1/\mu)$.

B.1 Region I, Normal Flow

$$M_{\beta}^* = \frac{C(k) \left(\frac{c_d}{a} + 1\right) x_c^4}{8} + \frac{C(k) \mu \sin(\psi) \left(\frac{c_d}{a} + 1\right) x_c^3}{6} - \frac{C(k) \left(\frac{c_d}{a} + 1\right)}{8} - \frac{C(k) \mu \sin(\psi) \left(\frac{c_d}{a} + 1\right)}{6} \quad (\text{B.2})$$

$$\begin{aligned} M_{\zeta}^* = & \frac{C(k) \theta_{tw} x_c^5}{5} + \frac{C(k) \left(2\theta_C - 2\beta_T k_{p\beta} - \dot{\beta}_T \left(\frac{c_d}{a} + 1\right) + 2\mu \theta_{tw} \sin(\psi)\right) x_c^4}{8} \\ & - \frac{C(k) \left((\lambda + \beta_T \mu \cos(\psi)) \left(\frac{c_d}{a} + 1\right) - 2\mu \sin(\psi) (\theta_C - \beta_T k_{p\beta})\right) x_c^3}{6} - \frac{C(k) \theta_{tw}}{5} \\ & + \frac{C(k) \left((\lambda + \beta_T \mu \cos(\psi)) \left(\frac{c_d}{a} + 1\right) - 2\mu \sin(\psi) (\theta_C - \beta_T k_{p\beta})\right)}{6} \\ & - \frac{C(k) \left(2\theta_C - 2\beta_T k_{p\beta} - \dot{\beta}_T \left(\frac{c_d}{a} + 1\right) + 2\mu \theta_{tw} \sin(\psi)\right)}{8} \end{aligned} \quad (\text{B.3})$$

$$\begin{aligned} M_{\theta}^* = & \frac{C(k) \left(\frac{c}{R}\right) \left(\left(\frac{x_a}{c}\right) - \frac{1}{2}\right) \left(\frac{c_d}{a} + 1\right) x_c^3}{6} + \frac{C(k) \left(\frac{c}{R}\right) \mu \sin(\psi) \left(\left(\frac{x_a}{c}\right) - \frac{1}{2}\right) \left(\frac{c_d}{a} + 1\right) x_c^2}{4} \\ & - \frac{C(k) \left(\frac{c}{R}\right) \left(\left(\frac{x_a}{c}\right) - \frac{1}{2}\right) \left(\frac{c_d}{a} + 1\right)}{6} - \frac{C(k) \left(\frac{c}{R}\right) \mu \sin(\psi) \left(\left(\frac{x_a}{c}\right) - \frac{1}{2}\right) \left(\frac{c_d}{a} + 1\right)}{4} \end{aligned} \quad (\text{B.4})$$

$$\begin{aligned}
M_\beta = & \frac{C(k) k_{p\beta} x_c^4}{8} + \frac{C(k) (2 k_{p\beta} \mu \sin(\psi) + \mu \cos(\psi) \left(\frac{c_d}{a} + 1\right)) x_c^3}{6} + \frac{C(k) (k_{p\beta} \mu^2 \sin(\psi)^2 + \cos(\psi) \left(\frac{c_d}{a} + 1\right) \mu^2 \sin(\psi)) x_c^2}{4} \\
& - \frac{C(k) k_{p\beta}}{8} - \frac{C(k) (2 k_{p\beta} \mu \sin(\psi) + \mu \cos(\psi) \left(\frac{c_d}{a} + 1\right))}{6} - \frac{C(k) (k_{p\beta} \mu^2 \sin(\psi)^2 + \cos(\psi) \left(\frac{c_d}{a} + 1\right) \mu^2 \sin(\psi))}{4}
\end{aligned} \tag{B.5}$$

$$\begin{aligned}
M_\zeta = & \frac{C(k) (k_{p\zeta} + 2 \mu \theta_{tw} \cos(\psi)) x_c^4}{8} + \frac{C(k) \left(2 k_{p\zeta} \mu \sin(\psi) + \mu \cos(\psi) \left(2 \theta_C - 2 \beta_T k_{p\beta} - \dot{\beta}_T \left(\frac{c_d}{a} + 1\right) + 2 \mu \theta_{tw} \sin(\psi)\right)\right) x_c^3}{6} \\
& - \frac{C(k) (\mu \cos(\psi) ((\lambda + \beta_T \mu \cos(\psi)) \left(\frac{c_d}{a} + 1\right) - 2 \mu \sin(\psi) (\theta_C - \beta_T k_{p\beta})) - k_{p\zeta} \mu^2 \sin(\psi)^2) x_c^2}{4} \\
& - \frac{C(k) (k_{p\zeta} + 2 \mu \theta_{tw} \cos(\psi))}{8} + \frac{C(k) (\mu \cos(\psi) ((\lambda + \beta_T \mu \cos(\psi)) \left(\frac{c_d}{a} + 1\right) - 2 \mu \sin(\psi) (\theta_C - \beta_T k_{p\beta})) - k_{p\zeta} \mu^2 \sin(\psi)^2)}{4} \\
& - \frac{C(k) \left(2 k_{p\zeta} \mu \sin(\psi) + \mu \cos(\psi) \left(2 \theta_C - 2 \beta_T k_{p\beta} - \dot{\beta}_T \left(\frac{c_d}{a} + 1\right) + 2 \mu \theta_{tw} \sin(\psi)\right)\right)}{6}
\end{aligned} \tag{B.6}$$

$$M_\theta = -\frac{C(k) \mu^2 x_c^2 \sin(\psi)^2}{4} + \frac{C(k) \mu^2 \sin(\psi)^2}{4} - \frac{C(k) \mu x_c^3 \sin(\psi)}{3} + \frac{C(k) \mu \sin(\psi)}{3} - \frac{C(k) x_c^4}{8} + \frac{C(k)}{8} \tag{B.7}$$

$$\begin{aligned}
Q_\beta^* = & -\frac{C(k) \theta_{tw} x_c^5}{10} + \frac{C(k) \left(2 \dot{\beta}_T - \theta_C + \beta_T k_{p\beta} - \mu \theta_{tw} \sin(\psi)\right) x_c^4}{8} + \frac{C(k) (2 \lambda + 2 \beta_T \mu \cos(\psi) - \mu \sin(\psi) (\theta_C - \beta_T k_{p\beta})) x_c^3}{6} \\
& + \frac{C(k) \theta_{tw}}{10} - \frac{C(k) (2 \lambda + 2 \beta_T \mu \cos(\psi) - \mu \sin(\psi) (\theta_C - \beta_T k_{p\beta}))}{6} - \frac{C(k) \left(2 \dot{\beta}_T - \theta_C + \beta_T k_{p\beta} - \mu \theta_{tw} \sin(\psi)\right)}{8}
\end{aligned} \tag{B.8}$$

$$\begin{aligned}
Q_\zeta^* = & \frac{C(k) \dot{\beta}_T \theta_{tw} x_c^5}{10} + \frac{C(k) \left(2 \frac{c_d}{a} + \theta_{tw} (\lambda + \beta_T \mu \cos(\psi)) + \dot{\beta}_T (\theta_C - \beta_T k_{p\beta}) \right) x_c^4}{8} \\
& + \frac{C(k) ((\theta_C - \beta_T k_{p\beta}) (\lambda + \beta_T \mu \cos(\psi)) + 2 \frac{c_d}{a} \mu \sin(\psi)) x_c^3}{6} - \frac{C(k) ((\theta_C - \beta_T k_{p\beta}) (\lambda + \beta_T \mu \cos(\psi)) + 2 \frac{c_d}{a} \mu \sin(\psi))}{6} \\
& - \frac{C(k) \left(2 \frac{c_d}{a} + \theta_{tw} (\lambda + \beta_T \mu \cos(\psi)) + \dot{\beta}_T (\theta_C - \beta_T k_{p\beta}) \right)}{8} - \frac{C(k) \dot{\beta}_T \theta_{tw}}{10}
\end{aligned} \tag{B.9}$$

$$\begin{aligned}
Q_\theta^* = & - \frac{C(k) (\frac{c}{R}) \theta_{tw} ((\frac{x_a}{c}) - \frac{1}{2}) x_c^4}{8} + \frac{C(k) (\frac{c}{R}) ((\frac{x_a}{c}) - \frac{1}{2}) \left(2 \dot{\beta}_T - \theta_C + \beta_T k_{p\beta} - \mu \theta_{tw} \sin(\psi) \right) x_c^3}{6} \\
& + \frac{C(k) (\frac{c}{R}) ((\frac{x_a}{c}) - \frac{1}{2}) (2\lambda + 2\beta_T \mu \cos(\psi) - \mu \sin(\psi) (\theta_C - \beta_T k_{p\beta})) x_c^2}{4} \\
& - \frac{C(k) (\frac{c}{R}) ((\frac{x_a}{c}) - \frac{1}{2}) \left(2 \dot{\beta}_T - \theta_C + \beta_T k_{p\beta} - \mu \theta_{tw} \sin(\psi) \right)}{6} + \frac{C(k) (\frac{c}{R}) \theta_{tw} ((\frac{x_a}{c}) - \frac{1}{2})}{8} \\
& - \frac{C(k) (\frac{c}{R}) ((\frac{x_a}{c}) - \frac{1}{2}) (2\lambda + 2\beta_T \mu \cos(\psi) - \mu \sin(\psi) (\theta_C - \beta_T k_{p\beta}))}{4}
\end{aligned} \tag{B.10}$$

$$\begin{aligned}
Q_\beta = & \frac{C(k) \left(\dot{\beta}_T k_{p\beta} - \mu \theta_{tw} \cos(\psi) \right) x_c^4}{8} - \frac{C(k) \left(\dot{\beta}_T k_{p\beta} - \mu \theta_{tw} \cos(\psi) \right)}{8} \\
& + \frac{C(k) \left(k_{p\beta} \left(\lambda + \beta_T \mu \cos(\psi) + \dot{\beta}_T \mu \sin(\psi) \right) + \mu \cos(\psi) \left(2 \dot{\beta}_T - \theta_C + \beta_T k_{p\beta} - \mu \theta_{tw} \sin(\psi) \right) \right) x_c^3}{6} \\
& + \frac{C(k) \left(\mu \cos(\psi) \left(2\lambda + 2\beta_T \mu \cos(\psi) - \mu \sin(\psi) (\theta_C - \beta_T k_{p\beta}) \right) + k_{p\beta} \mu \sin(\psi) (\lambda + \beta_T \mu \cos(\psi)) \right) x_c^2}{4} \\
& - \frac{C(k) \left(\mu \cos(\psi) \left(2\lambda + 2\beta_T \mu \cos(\psi) - \mu \sin(\psi) (\theta_C - \beta_T k_{p\beta}) \right) + k_{p\beta} \mu \sin(\psi) (\lambda + \beta_T \mu \cos(\psi)) \right)}{4} \\
& - \frac{C(k) \left(k_{p\beta} \left(\lambda + \beta_T \mu \cos(\psi) + \dot{\beta}_T \mu \sin(\psi) \right) + \mu \cos(\psi) \left(2 \dot{\beta}_T - \theta_C + \beta_T k_{p\beta} - \mu \theta_{tw} \sin(\psi) \right) \right)}{6}
\end{aligned} \tag{B.11}$$

$$\begin{aligned}
Q_\zeta = & \frac{C(k) \left(\dot{\beta}_T k_{p\zeta} + \dot{\beta}_T \mu \theta_{tw} \cos(\psi) \right) x_c^4}{8} - \frac{C(k) \left(\dot{\beta}_T k_{p\zeta} + \dot{\beta}_T \mu \theta_{tw} \cos(\psi) \right)}{8} \\
& + \frac{C(k) \left(k_{p\zeta} \left(\lambda + \beta_T \mu \cos(\psi) + \dot{\beta}_T \mu \sin(\psi) \right) + \mu \cos(\psi) \left(2 \frac{c_d}{a} + \theta_{tw} (\lambda + \beta_T \mu \cos(\psi)) + \dot{\beta}_T (\theta_C - \beta_T k_{p\beta}) \right) \right) x_c^3}{6} \\
& + \frac{C(k) \left(\mu \cos(\psi) \left((\theta_C - \beta_T k_{p\beta}) (\lambda + \beta_T \mu \cos(\psi)) + \frac{2c_d \mu \sin(\psi)}{a} \right) + k_{p\zeta} \mu \sin(\psi) (\lambda + \beta_T \mu \cos(\psi)) \right) x_c^2}{4} \\
& + \frac{C(k) \left(k_{p\zeta} \left(\lambda + \beta_T \mu \cos(\psi) + \dot{\beta}_T \mu \sin(\psi) \right) + \mu \cos(\psi) \left(2 \frac{c_d}{a} + \theta_{tw} (\lambda + \beta_T \mu \cos(\psi)) + \dot{\beta}_T (\theta_C - \beta_T k_{p\beta}) \right) \right)}{4} \\
& - \frac{C(k) \left(\mu \cos(\psi) \left((\theta_C - \beta_T k_{p\beta}) (\lambda + \beta_T \mu \cos(\psi)) + \frac{2c_d \mu \sin(\psi)}{a} \right) + k_{p\zeta} \mu \sin(\psi) (\lambda + \beta_T \mu \cos(\psi)) \right)}{4}
\end{aligned} \tag{B.12}$$

$$Q_\theta = -\frac{C(k)\dot{\beta}_T x_c^4}{8} - \frac{C(k)\left(\lambda + \beta_T \mu \cos(\psi) + \dot{\beta}_T \mu \sin(\psi)\right) x_c^3}{6} - \frac{C(k) \mu \sin(\psi) (\lambda + \beta_T \mu \cos(\psi)) x_c^2}{4} + \frac{C(k) \dot{\beta}_T}{8} + \frac{C(k)\left(\lambda + \beta_T \mu \cos(\psi) + \dot{\beta}_T \mu \sin(\psi)\right)}{6} + \frac{C(k) \mu \sin(\psi) (\lambda + \beta_T \mu \cos(\psi))}{4} \quad (\text{B.13})$$

$$T_\beta^* = -\frac{C(k)\left(\frac{c}{R}\right)\left(\frac{x_a}{c}\right) x_c^3}{6} - \frac{C(k)\left(\frac{c}{R}\right) \mu \left(\frac{x_a}{c}\right) \sin(\psi) x_c^2}{4} + \frac{C(k)\left(\frac{c}{R}\right) \mu \left(\frac{x_a}{c}\right) \sin(\psi)}{6} + \frac{C(k)\left(\frac{c}{R}\right) \mu \left(\frac{x_a}{c}\right) \sin(\psi)}{4} \quad (\text{B.14})$$

$$\begin{aligned} T_\zeta^* &= x_c^2 \left(\frac{C(k)\left(\frac{c}{R}\right)\left(\frac{x_a}{c}\right) (\lambda + \beta_T \mu \cos(\psi) - 2 \mu \sin(\psi) (\theta_C - \beta_T k_{p\beta}))}{4} + \frac{C(k)\left(\frac{c}{R}\right) c_m \mu \sin(\psi)}{2a} \right) \\ &\quad + x_c^3 \left(\frac{C(k)\left(\frac{c}{R}\right) c_m}{3a} + \frac{\dot{\beta}_T - 2\theta_C + 2\beta_T k_{p\beta} - 2\mu \theta_{tw} \sin(\psi)}{6} \right) - \frac{C(k)\left(\frac{c}{R}\right) c_m}{3a} \\ &\quad - \frac{C(k)\left(\frac{c}{R}\right) \left(\dot{\beta}_T - 2\theta_C + 2\beta_T k_{p\beta} - 2\mu \theta_{tw} \sin(\psi)\right)}{6} - \frac{C(k)\left(\frac{c}{R}\right) (\lambda + \beta_T \mu \cos(\psi) - 2\mu \sin(\psi) (\theta_C - \beta_T k_{p\beta}))}{4} \\ &\quad + \frac{C(k)\left(\frac{c}{R}\right) \theta_{tw} \left(\frac{x_a}{c}\right)}{4} - \frac{C(k)\left(\frac{c}{R}\right) \theta_{tw} \left(\frac{x_a}{c}\right) x_c^4}{4} - \frac{C(k)\left(\frac{c}{R}\right) c_m \mu \sin(\psi)}{2a} \end{aligned} \quad (\text{B.15})$$

$$\begin{aligned} T_\theta^* &= x_c^2 \left(\frac{\left(\frac{c}{R}\right)^2 \left(\left(\frac{x_a}{c}\right) + \frac{1}{2}\right)}{16} - \frac{C(k)\left(\frac{c}{R}\right)^2 \left(\frac{x_a}{c}\right) \left(\left(\frac{x_a}{c}\right) - \frac{1}{2}\right)}{4} \right) + x_c \left(\frac{\left(\frac{c}{R}\right)^2 \mu \sin(\psi) \left(\left(\frac{x_a}{c}\right) + \frac{1}{2}\right)}{8} - \frac{C(k)\left(\frac{c}{R}\right)^2 \mu \left(\frac{x_a}{c}\right) \sin(\psi) \left(\left(\frac{x_a}{c}\right) - \frac{1}{2}\right)}{2} \right) \\ &\quad - \frac{\left(\frac{c}{R}\right)^2 \left(\left(\frac{x_a}{c}\right) + \frac{1}{2}\right)}{16} - \frac{\left(\frac{c}{R}\right)^2 \mu \sin(\psi) \left(\left(\frac{x_a}{c}\right) + \frac{1}{2}\right)}{8} + \frac{C(k)\left(\frac{c}{R}\right)^2 \left(\left(\frac{x_a}{c}\right) - \frac{1}{2}\right)}{4} + \frac{C(k)\left(\frac{c}{R}\right)^2 \mu \left(\frac{x_a}{c}\right) \sin(\psi) \left(\left(\frac{x_a}{c}\right) - \frac{1}{2}\right)}{2} \end{aligned} \quad (\text{B.16})$$

$$\begin{aligned}
T_\beta = & -\frac{C(k)(\frac{c}{R})(\frac{x_a}{c})(\mu \cos(\psi) + 2k_{p\beta} \mu \sin(\psi))x_c^2}{4} - \frac{C(k)(\frac{c}{R})(\frac{x_a}{c})(k_{p\beta} \mu^2 \sin(\psi)^2 + \cos(\psi) \mu^2 \sin(\psi))x_c}{2} \\
& + \frac{C(k)(\frac{c}{R})(\frac{x_a}{c})(\mu \cos(\psi) + 2k_{p\beta} \mu \sin(\psi))}{4} + \frac{C(k)(\frac{c}{R})(\frac{x_a}{c})(k_{p\beta} \mu^2 \sin(\psi)^2 + \cos(\psi) \mu^2 \sin(\psi))}{2} \\
& + \frac{C(k)(\frac{c}{R})k_{p\beta}(\frac{x_a}{c})}{6} - \frac{C(k)(\frac{c}{R})k_{p\beta}(\frac{x_a}{c})x_c^3}{6}
\end{aligned} \tag{B.17}$$

$$\begin{aligned}
T_\zeta = & x_c \left(\frac{C(k)(\frac{c}{R})(\frac{x_a}{c})(\mu \cos(\psi)(\lambda + \beta_T \mu \cos(\psi) - 2\mu \sin(\psi)(\theta_C - \beta_T k_{p\beta})) - k_{p\zeta} \mu^2 \sin(\psi)^2)}{2} + \frac{C(k)(\frac{c}{R})c_m \mu^2 \cos(\psi) \sin(\psi)}{a} \right) \\
& - x_c^2 \left(\frac{C(k)(\frac{c}{R})(\frac{x_a}{c})(2k_{p\zeta} \mu \sin(\psi) - \mu \cos(\psi)(\dot{\beta}_T - 2\theta_C + 2\beta_T k_{p\beta} - 2\mu \theta_{tw} \sin(\psi)))}{4} - \frac{C(k)(\frac{c}{R})c_m \mu \cos(\psi)}{2a} \right) \\
& + \frac{C(k)(\frac{c}{R})(\frac{x_a}{c})(k_{p\zeta} + 2\mu \theta_{tw} \cos(\psi))}{6} + \frac{C(k)(\frac{c}{R})(\frac{x_a}{c})(2k_{p\zeta} \mu \sin(\psi) - \mu \cos(\psi)(\dot{\beta}_T - 2\theta_C + 2\beta_T k_{p\beta} - 2\mu \theta_{tw} \sin(\psi)))}{4} \\
& - \frac{C(k)(\frac{c}{R})(\frac{x_a}{c})(\mu \cos(\psi)(\lambda + \beta_T \mu \cos(\psi) - 2\mu \sin(\psi)(\theta_C - \beta_T k_{p\beta})) - k_{p\zeta} \mu^2 \sin(\psi)^2)}{6} \\
& - \frac{C(k)(\frac{c}{R})(\frac{x_a}{c})x_c^3(k_{p\zeta} + 2\mu \theta_{tw} \cos(\psi))}{6} - \frac{C(k)(\frac{c}{R})c_m \mu \cos(\psi)}{2a} - \frac{C(k)(\frac{c}{R})c_m \mu^2 \cos(\psi) \sin(\psi)}{a}
\end{aligned} \tag{B.18}$$

$$\begin{aligned}
T_\theta = & \frac{C(k)(\frac{c}{R})(\frac{x_a}{c})\mu^2 x_c \sin(\psi)^2}{2} - \frac{C(k)(\frac{c}{R})(\frac{x_a}{c})\mu^2 \sin(\psi)^2}{2} + \frac{C(k)(\frac{c}{R})(\frac{x_a}{c})\mu x_c^2 \sin(\psi)}{2} \\
& - \frac{C(k)(\frac{c}{R})(\frac{x_a}{c})\mu \sin(\psi)}{2} + \frac{C(k)(\frac{c}{R})(\frac{x_a}{c})x_c^3}{6} - \frac{C(k)(\frac{c}{R})(\frac{x_a}{c})}{6}
\end{aligned} \tag{B.19}$$

B.2 Region II, Partial Reverse Flow

$$M_{\beta}^* = -\frac{C(k) \left(\frac{c_d}{a} + 1\right)}{8} - \frac{C(k) \mu^4 \sin(\psi)^4 \left(\frac{c_d}{a} + 1\right)}{24} - \frac{C(k) \mu \sin(\psi) \left(\frac{c_d}{a} + 1\right)}{6} - \frac{\hat{C}(k) \hat{a} x_c^4 \left(\frac{c_d}{a} + 1\right)}{8a} - \frac{\hat{C}(k) \hat{a} \mu^4 \sin(\psi)^4 \left(\frac{c_d}{a} + 1\right)}{24a} - \frac{\hat{C}(k) \hat{a} \mu x_c^3 \sin(\psi) \left(\frac{c_d}{a} + 1\right)}{6a} \quad (\text{B.20})$$

$$M_{\zeta}^* = \frac{C(k) \left((\lambda + \beta_T \mu \cos(\psi)) \left(\frac{c_d}{a} + 1\right) - 2 \mu \sin(\psi) (\theta_C - \beta_T k_{p\beta})\right)}{6} - \frac{C(k) \left(2\theta_C - 2\beta_T k_{p\beta} - \dot{\beta}_T \left(\frac{c_d}{a} + 1\right) + 2 \mu \theta_{tw} \sin(\psi)\right)}{8} - \frac{C(k) \theta_{tw}}{5} - \frac{C(k) \mu^5 \theta_{tw} \sin(\psi)^5}{5} + \frac{C(k) \mu^3 \sin(\psi)^3 \left((\lambda + \beta_T \mu \cos(\psi)) \left(\frac{c_d}{a} + 1\right) - 2 \mu \sin(\psi) (\theta_C - \beta_T k_{p\beta})\right)}{6} + \frac{C(k) \mu^4 \sin(\psi)^4 \left(2\theta_C - 2\beta_T k_{p\beta} - \dot{\beta}_T \left(\frac{c_d}{a} + 1\right) + 2 \mu \theta_{tw} \sin(\psi)\right)}{8} - \frac{\hat{C}(k) \hat{a} x_c^3 \left((\lambda + \beta_T \mu \cos(\psi)) \left(\frac{c_d}{a} + 1\right) - 2 \mu \sin(\psi) (\theta_C - \beta_T k_{p\beta})\right)}{6a} + \frac{\hat{C}(k) \hat{a} x_c^4 \left(2\theta_C - 2\beta_T k_{p\beta} - \dot{\beta}_T \left(\frac{c_d}{a} + 1\right) + 2 \mu \theta_{tw} \sin(\psi)\right)}{8a} - \frac{\hat{C}(k) \hat{a} \mu^3 \sin(\psi)^3 \left((\lambda + \beta_T \mu \cos(\psi)) \left(\frac{c_d}{a} + 1\right) - 2 \mu \sin(\psi) (\theta_C - \beta_T k_{p\beta})\right)}{6a} + \frac{\hat{C}(k) \hat{a} \mu^4 \sin(\psi)^4 \left(2\theta_C - 2\beta_T k_{p\beta} - \dot{\beta}_T \left(\frac{c_d}{a} + 1\right) + 2 \mu \theta_{tw} \sin(\psi)\right)}{8a} \quad (\text{B.21})$$

$$\begin{aligned}
M_{\theta}^* = & \frac{C(k) \left(\frac{c}{R}\right) \mu^3 \sin(\psi)^3 \left(\left(\frac{x_a}{c}\right) - \frac{1}{2}\right) \left(\frac{c_d}{a} + 1\right)}{12} - \frac{C(k) \left(\frac{c}{R}\right) \mu \sin(\psi) \left(\left(\frac{x_a}{c}\right) - \frac{1}{2}\right) \left(\frac{c_d}{a} + 1\right)}{4} \\
& - \frac{C(k) \left(\frac{c}{R}\right) \left(\left(\frac{x_a}{c}\right) - \frac{1}{2}\right) \left(\frac{c_d}{a} + 1\right)}{6} + \frac{\hat{C}(k) \hat{a} \left(\frac{c}{R}\right) x_c^3 \left(\left(\frac{x_a}{c}\right) + \frac{1}{2}\right) \left(\frac{c_d}{a} + 1\right)}{6a} \\
& - \frac{\hat{C}(k) \hat{a} \left(\frac{c}{R}\right) \mu^3 \sin(\psi)^3 \left(\left(\frac{x_a}{c}\right) + \frac{1}{2}\right) \left(\frac{c_d}{a} + 1\right)}{12a} + \frac{\hat{C}(k) \hat{a} \left(\frac{c}{R}\right) \mu x_c^2 \sin(\psi) \left(\left(\frac{x_a}{c}\right) + \frac{1}{2}\right) \left(\frac{c_d}{a} + 1\right)}{4a}
\end{aligned} \tag{B.22}$$

$$\begin{aligned}
M_{\beta} = & \frac{C(k) k_{p\beta} \mu^4 \sin(\psi)^4}{8} - \frac{C(k) (2 k_{p\beta} \mu \sin(\psi) + \mu \cos(\psi) \left(\frac{c_d}{a} + 1\right))}{6} \\
& - \frac{C(k) (k_{p\beta} \mu^2 \sin(\psi)^2 + \cos(\psi) \left(\frac{c_d}{a} + 1\right) \mu^2 \sin(\psi))}{4} - \frac{C(k) \mu^3 \sin(\psi)^3 (2 k_{p\beta} \mu \sin(\psi) + \mu \cos(\psi) \left(\frac{c_d}{a} + 1\right))}{6} \\
& + \frac{C(k) \mu^2 \sin(\psi)^2 (k_{p\beta} \mu^2 \sin(\psi)^2 + \cos(\psi) \left(\frac{c_d}{a} + 1\right) \mu^2 \sin(\psi))}{4} \\
& - \frac{\hat{C}(k) \hat{a} x_c^3 (2 k_{p\beta} \mu \sin(\psi) + \mu \cos(\psi) \left(\frac{c_d}{a} + 1\right))}{6a} - \frac{\hat{C}(k) \hat{a} x_c^2 (k_{p\beta} \mu^2 \sin(\psi)^2 + \cos(\psi) \left(\frac{c_d}{a} + 1\right) \mu^2 \sin(\psi))}{4a} \\
& + \frac{\hat{C}(k) \hat{a} k_{p\beta} \mu^4 \sin(\psi)^4}{8a} - \frac{\hat{C}(k) \hat{a} \mu^3 \sin(\psi)^3 (2 k_{p\beta} \mu \sin(\psi) + \mu \cos(\psi) \left(\frac{c_d}{a} + 1\right))}{6a} - \frac{C(k) k_{p\beta}}{8} \\
& - \frac{\hat{C}(k) \hat{a} k_{p\beta} x_c^4}{8a} + \frac{\hat{C}(k) \hat{a} \mu^2 \sin(\psi)^2 (k_{p\beta} \mu^2 \sin(\psi)^2 + \cos(\psi) \left(\frac{c_d}{a} + 1\right) \mu^2 \sin(\psi))}{4a}
\end{aligned} \tag{B.23}$$

$$\begin{aligned}
M_\zeta = & \frac{C(k) (\mu \cos(\psi) ((\lambda + \beta_T \mu \cos(\psi)) (\frac{c_d}{a} + 1) - 2 \mu \sin(\psi) (\theta_C - \beta_T k_{p\beta})) - k_{p\zeta} \mu^2 \sin(\psi)^2)}{4} \\
& - \frac{C(k) (2 k_{p\zeta} \mu \sin(\psi) + \mu \cos(\psi) (2 \theta_C - 2 \beta_T k_{p\beta} - \dot{\beta}_T (\frac{c_d}{a} + 1) + 2 \mu \theta_{tw} \sin(\psi)))}{6} \\
& - \frac{C(k) \mu^2 \sin(\psi)^2 (\mu \cos(\psi) ((\lambda + \beta_T \mu \cos(\psi)) (\frac{c_d}{a} + 1) - 2 \mu \sin(\psi) (\theta_C - \beta_T k_{p\beta})) - k_{p\zeta} \mu^2 \sin(\psi)^2)}{4} \\
& - \frac{C(k) \mu^3 \sin(\psi)^3 (2 k_{p\zeta} \mu \sin(\psi) + \mu \cos(\psi) (2 \theta_C - 2 \beta_T k_{p\beta} - \dot{\beta}_T (\frac{c_d}{a} + 1) + 2 \mu \theta_{tw} \sin(\psi)))}{6} \\
& + \frac{\hat{C}(k) \hat{a} x_c^2 (\mu \cos(\psi) ((\lambda + \beta_T \mu \cos(\psi)) (\frac{c_d}{a} + 1) - 2 \mu \sin(\psi) (\theta_C - \beta_T k_{p\beta})) - k_{p\zeta} \mu^2 \sin(\psi)^2)}{4a} \\
& - \frac{\hat{C}(k) \hat{a} x_c^3 (2 k_{p\zeta} \mu \sin(\psi) + \mu \cos(\psi) (2 \theta_C - 2 \beta_T k_{p\beta} - \dot{\beta}_T (\frac{c_d}{a} + 1) + 2 \mu \theta_{tw} \sin(\psi)))}{6a} \\
& - \frac{\hat{C}(k) \hat{a} \mu^3 \sin(\psi)^3 (2 k_{p\zeta} \mu \sin(\psi) + \mu \cos(\psi) (2 \theta_C - 2 \beta_T k_{p\beta} - \dot{\beta}_T (\frac{c_d}{a} + 1) + 2 \mu \theta_{tw} \sin(\psi)))}{6a} \\
& + \frac{\hat{C}(k) \hat{a} \mu^4 \sin(\psi)^4 (k_{p\zeta} + 2 \mu \theta_{tw} \cos(\psi))}{8a} - \frac{C(k) (k_{p\zeta} + 2 \mu \theta_{tw} \cos(\psi))}{8} \\
& - \frac{\hat{C}(k) \hat{a} \mu^2 \sin(\psi)^2 (\mu \cos(\psi) ((\lambda + \beta_T \mu \cos(\psi)) (\frac{c_d}{a} + 1) - 2 \mu \sin(\psi) (\theta_C - \beta_T k_{p\beta})) - k_{p\zeta} \mu^2 \sin(\psi)^2)}{4a} \\
& - \frac{\hat{C}(k) \hat{a} x_c^4 (k_{p\zeta} + 2 \mu \theta_{tw} \cos(\psi))}{8a} + \frac{C(k) \mu^4 \sin(\psi)^4 (k_{p\zeta} + 2 \mu \theta_{tw} \cos(\psi))}{8}
\end{aligned}
\tag{B.24}$$

$$M_\theta = \frac{C(k)}{8} + \frac{C(k) \mu^2 \sin(\psi)^2}{4} - \frac{C(k) \mu^4 \sin(\psi)^4}{24} + \frac{C(k) \mu \sin(\psi)}{3} + \frac{\hat{C}(k) \hat{a} x_c^4}{8a} - \frac{\hat{C}(k) \hat{a} \mu^4 \sin(\psi)^4}{24a} + \frac{\hat{C}(k) \hat{a} \mu^2 x_c^2 \sin(\psi)^2}{4a} + \frac{\hat{C}(k) \hat{a} \mu x_c^3 \sin(\psi)}{3a} \quad (\text{B.25})$$

$$Q_\beta^* = \frac{C(k) \theta_{tw}}{10} - \frac{C(k) (2\lambda + 2\beta_T \mu \cos(\psi) - \mu \sin(\psi) (\theta_C - \beta_T k_{p\beta}))}{6} - \frac{C(k) (2\dot{\beta}_T - \theta_C + \beta_T k_{p\beta} - \mu \theta_{tw} \sin(\psi))}{8} - \frac{\hat{C}(k) \hat{a} x_c^4 (2\dot{\beta}_T - \theta_C + \beta_T k_{p\beta} - \mu \theta_{tw} \sin(\psi))}{8a} + \frac{C(k) \mu^5 \theta_{tw} \sin(\psi)^5}{10} + \frac{\hat{C}(k) \hat{a} \theta_{tw} x_c^5}{10a} - \frac{C(k) \mu^3 \sin(\psi)^3 (2\lambda + 2\beta_T \mu \cos(\psi) - \mu \sin(\psi) (\theta_C - \beta_T k_{p\beta}))}{6} + \frac{C(k) \mu^4 \sin(\psi)^4 (2\dot{\beta}_T - \theta_C + \beta_T k_{p\beta} - \mu \theta_{tw} \sin(\psi))}{8} - \frac{\hat{C}(k) \hat{a} \mu^4 \sin(\psi)^4 (2\dot{\beta}_T - \theta_C + \beta_T k_{p\beta} - \mu \theta_{tw} \sin(\psi))}{8a} - \frac{\hat{C}(k) \hat{a} x_c^3 (2\lambda + 2\beta_T \mu \cos(\psi) - \mu \sin(\psi) (\theta_C - \beta_T k_{p\beta}))}{6a} + \frac{\hat{C}(k) \hat{a} \mu^5 \theta_{tw} \sin(\psi)^5}{10a} - \frac{\hat{C}(k) \hat{a} \mu^3 \sin(\psi)^3 (2\lambda + 2\beta_T \mu \cos(\psi) - \mu \sin(\psi) (\theta_C - \beta_T k_{p\beta}))}{6a} \quad (\text{B.26})$$

$$\begin{aligned}
Q_\zeta^* = & \frac{C(k) \mu^4 \sin(\psi)^4 \left(2 \frac{c_d}{a} + \theta_{tw} (\lambda + \beta_T \mu \cos(\psi)) + \dot{\beta}_T (\theta_C - \beta_T k_{p\beta}) \right)}{8} \\
& - \frac{C(k) \left(2 \frac{c_d}{a} + \theta_{tw} (\lambda + \beta_T \mu \cos(\psi)) + \dot{\beta}_T (\theta_C - \beta_T k_{p\beta}) \right)}{8} \\
& - \frac{C(k) \dot{\beta}_T \theta_{tw}}{10} - \frac{C(k) \mu^3 \sin(\psi)^3 \left((\theta_C - \beta_T k_{p\beta}) (\lambda + \beta_T \mu \cos(\psi)) + 2 \frac{c_d}{a} \mu \sin(\psi) \right)}{6} \\
& - \frac{C(k) \left((\theta_C - \beta_T k_{p\beta}) (\lambda + \beta_T \mu \cos(\psi)) + 2 \frac{c_d}{a} \mu \sin(\psi) \right)}{6} \\
& - \frac{\hat{C}(k) \hat{a} x_c^3 \left((\theta_C - \beta_T k_{p\beta}) (\lambda + \beta_T \mu \cos(\psi)) + \frac{2 c_d \mu \sin(\psi)}{\hat{a}} \right)}{6 a} \\
& - \frac{C(k) \dot{\beta}_T \mu^5 \theta_{tw} \sin(\psi)^5}{10} - \frac{\hat{C}(k) \hat{a} x_c^4 \left(\frac{2 c_d}{\hat{a}} + \theta_{tw} (\lambda + \beta_T \mu \cos(\psi)) + \dot{\beta}_T (\theta_C - \beta_T k_{p\beta}) \right)}{8 a} \\
& + \frac{\hat{C}(k) \hat{a} \mu^4 \sin(\psi)^4 \left(\frac{2 c_d}{\hat{a}} + \theta_{tw} (\lambda + \beta_T \mu \cos(\psi)) + \dot{\beta}_T (\theta_C - \beta_T k_{p\beta}) \right)}{8 a} \\
& - \frac{\hat{C}(k) \hat{a} \mu^3 \sin(\psi)^3 \left((\theta_C - \beta_T k_{p\beta}) (\lambda + \beta_T \mu \cos(\psi)) + \frac{2 c_d \mu \sin(\psi)}{\hat{a}} \right)}{8 a} \\
& - \frac{\hat{C}(k) \hat{a} \dot{\beta}_T \theta_{tw} x_c^5}{10 a} - \frac{\hat{C}(k) \hat{a} \dot{\beta}_T \mu^5 \theta_{tw} \sin(\psi)^5}{10 a}
\end{aligned}
\tag{B.27}$$

$$\begin{aligned}
Q_\theta^* = & \frac{C(k) (\frac{c}{R}) \theta_{tw} ((\frac{x_a}{c}) - \frac{1}{2})}{8} - \frac{C(k) (\frac{c}{R}) ((\frac{x_a}{c}) - \frac{1}{2}) \left(2\dot{\beta}_T - \theta_C + \beta_T k_{p\beta} - \mu \theta_{tw} \sin(\psi) \right)}{6} \\
& - \frac{C(k) (\frac{c}{R}) ((\frac{x_a}{c}) - \frac{1}{2}) (2\lambda + 2\beta_T \mu \cos(\psi) - \mu \sin(\psi) (\theta_C - \beta_T k_{p\beta}))}{4} \\
& - \frac{C(k) (\frac{c}{R}) \mu^3 \sin(\psi)^3 ((\frac{x_a}{c}) - \frac{1}{2}) \left(2\dot{\beta}_T - \theta_C + \beta_T k_{p\beta} - \mu \theta_{tw} \sin(\psi) \right)}{6} - \frac{C(k) (\frac{c}{R}) \mu^4 \theta_{tw} \sin(\psi)^4 ((\frac{x_a}{c}) - \frac{1}{2})}{8} \\
& + \frac{C(k) (\frac{c}{R}) \mu^2 \sin(\psi)^2 ((\frac{x_a}{c}) - \frac{1}{2}) (2\lambda + 2\beta_T \mu \cos(\psi) - \mu \sin(\psi) (\theta_C - \beta_T k_{p\beta}))}{4} \\
& + \frac{\hat{C}(k) \hat{a} (\frac{c}{R}) x_c^3 ((\frac{\hat{x}_a}{c}) + \frac{1}{2}) \left(2\dot{\beta}_T - \theta_C + \beta_T k_{p\beta} - \mu \theta_{tw} \sin(\psi) \right)}{6a} - \frac{\hat{C}(k) \hat{a} (\frac{c}{R}) \theta_{tw} x_c^4 ((\frac{\hat{x}_a}{c}) + \frac{1}{2})}{8a} \\
& + \frac{\hat{C}(k) \hat{a} (\frac{c}{R}) x_c^2 ((\frac{\hat{x}_a}{c}) + \frac{1}{2}) (2\lambda + 2\beta_T \mu \cos(\psi) - \mu \sin(\psi) (\theta_C - \beta_T k_{p\beta}))}{4a} \\
& + \frac{\hat{C}(k) \hat{a} (\frac{c}{R}) \mu^3 \sin(\psi)^3 ((\frac{\hat{x}_a}{c}) + \frac{1}{2}) \left(2\dot{\beta}_T - \theta_C + \beta_T k_{p\beta} - \mu \theta_{tw} \sin(\psi) \right)}{6a} + \frac{\hat{C}(k) \hat{a} (\frac{c}{R}) \mu^4 \theta_{tw} \sin(\psi)^4 ((\frac{\hat{x}_a}{c}) + \frac{1}{2})}{8a} \\
& - \frac{\hat{C}(k) \hat{a} (\frac{c}{R}) \mu^2 \sin(\psi)^2 ((\frac{\hat{x}_a}{c}) + \frac{1}{2}) (2\lambda + 2\beta_T \mu \cos(\psi) - \mu \sin(\psi) (\theta_C - \beta_T k_{p\beta}))}{4a}
\end{aligned}
\tag{B.28}$$

$$\begin{aligned}
Q_\beta = & \frac{C(k) \mu^4 \sin(\psi)^4 \left(\dot{\beta}_T k_{p\beta} - \mu \theta_{tw} \cos(\psi) \right)}{8} \\
& - \frac{C(k) (\mu \cos(\psi) (2\lambda + 2\beta_T \mu \cos(\psi) - \mu \sin(\psi) (\theta_C - \beta_T k_{p\beta})) + k_{p\beta} \mu \sin(\psi) (\lambda + \beta_T \mu \cos(\psi)))}{4} \\
& - \frac{C(k) \left(k_{p\beta} \left(\lambda + \beta_T \mu \cos(\psi) + \dot{\beta}_T \mu \sin(\psi) \right) + \mu \cos(\psi) \left(2\dot{\beta}_T - \theta_C + \beta_T k_{p\beta} - \mu \theta_{tw} \sin(\psi) \right) \right)}{6} \\
& - \frac{C(k) \mu^2 \sin(\psi)^2 (\mu \cos(\psi) (2\lambda + 2\beta_T \mu \cos(\psi) - \mu \sin(\psi) (\theta_C - \beta_T k_{p\beta})) + k_{p\beta} \mu \sin(\psi) (\lambda + \beta_T \mu \cos(\psi)))}{4} \\
& - \frac{C(k) \mu^3 \sin(\psi)^3 \left(k_{p\beta} \left(\lambda + \beta_T \mu \cos(\psi) + \dot{\beta}_T \mu \sin(\psi) \right) + \mu \cos(\psi) \left(2\dot{\beta}_T - \theta_C + \beta_T k_{p\beta} - \mu \theta_{tw} \sin(\psi) \right) \right)}{6} \\
& - \frac{\hat{C}(k) \hat{a} x_c^4 \left(\dot{\beta}_T k_{p\beta} - \mu \theta_{tw} \cos(\psi) \right)}{8a} - \frac{C(k) \left(\dot{\beta}_T k_{p\beta} - \mu \theta_{tw} \cos(\psi) \right)}{8} + \frac{\hat{C}(k) \hat{a} \mu^4 \sin(\psi)^4 \left(\dot{\beta}_T k_{p\beta} - \mu \theta_{tw} \cos(\psi) \right)}{8a} \\
& - \frac{\hat{C}(k) \hat{a} x_c^2 (\mu \cos(\psi) (2\lambda + 2\beta_T \mu \cos(\psi) - \mu \sin(\psi) (\theta_C - \beta_T k_{p\beta})) + k_{p\beta} \mu \sin(\psi) (\lambda + \beta_T \mu \cos(\psi)))}{4a} \\
& - \frac{\hat{C}(k) \hat{a} x_c^3 \left(k_{p\beta} \left(\lambda + \beta_T \mu \cos(\psi) + \dot{\beta}_T \mu \sin(\psi) \right) + \mu \cos(\psi) \left(2\dot{\beta}_T - \theta_C + \beta_T k_{p\beta} - \mu \theta_{tw} \sin(\psi) \right) \right)}{6a} \\
& + \frac{\hat{C}(k) \hat{a} \mu^2 \sin(\psi)^2 (\mu \cos(\psi) (2\lambda + 2\beta_T \mu \cos(\psi) - \mu \sin(\psi) (\theta_C - \beta_T k_{p\beta})) + k_{p\beta} \mu \sin(\psi) (\lambda + \beta_T \mu \cos(\psi)))}{4a} \\
& - \frac{\hat{C}(k) \hat{a} \mu^3 \sin(\psi)^3 \left(k_{p\beta} \left(\lambda + \beta_T \mu \cos(\psi) + \dot{\beta}_T \mu \sin(\psi) \right) + \mu \cos(\psi) \left(2\dot{\beta}_T - \theta_C + \beta_T k_{p\beta} - \mu \theta_{tw} \sin(\psi) \right) \right)}{6a}
\end{aligned}
\tag{B.29}$$

$$\begin{aligned}
Q_\zeta = & \frac{C(k) \mu^2 \sin(\psi)^2 \left(\mu \cos(\psi) \left((\theta_C - \beta_T k_{p\beta}) (\lambda + \beta_T \mu \cos(\psi)) + 2 \frac{c_d}{a} \mu \sin(\psi) \right) + k_{p\zeta} \mu \sin(\psi) (\lambda + \beta_T \mu \cos(\psi)) \right)}{4} \\
& - \frac{C(k) \left(\mu \cos(\psi) \left((\theta_C - \beta_T k_{p\beta}) (\lambda + \beta_T \mu \cos(\psi)) + 2 \frac{c_d}{a} \mu \sin(\psi) \right) + k_{p\zeta} \mu \sin(\psi) (\lambda + \beta_T \mu \cos(\psi)) \right)}{4} \\
& - \frac{C(k) \left(k_{p\zeta} \left(\lambda + \beta_T \mu \cos(\psi) + \dot{\beta}_T \mu \sin(\psi) \right) + \mu \cos(\psi) \left(2 \frac{c_d}{a} + \theta_{tw} (\lambda + \beta_T \mu \cos(\psi)) + \dot{\beta}_T (\theta_C - \beta_T k_{p\beta}) \right) \right)}{6} \\
& - \frac{C(k) \mu^3 \sin(\psi)^3 \left(k_{p\zeta} \left(\lambda + \beta_T \mu \cos(\psi) + \dot{\beta}_T \mu \sin(\psi) \right) + \mu \cos(\psi) \left(2 \frac{c_d}{a} + \theta_{tw} (\lambda + \beta_T \mu \cos(\psi)) + \dot{\beta}_T (\theta_C - \beta_T k_{p\beta}) \right) \right)}{6} \\
& + \frac{C(k) \mu^4 \sin(\psi)^4 \left(\dot{\beta}_T k_{p\zeta} + \dot{\beta}_T \mu \theta_{tw} \cos(\psi) \right)}{8} - \frac{\hat{C}(k) \hat{a} x_c^4 \left(\dot{\beta}_T k_{p\zeta} + \dot{\beta}_T \mu \theta_{tw} \cos(\psi) \right)}{8a} \\
& - \frac{\hat{C}(k) \hat{a} x_c^3 \left(k_{p\zeta} \left(\lambda + \beta_T \mu \cos(\psi) + \dot{\beta}_T \mu \sin(\psi) \right) + \mu \cos(\psi) \left(2 \frac{c_d}{a} + \theta_{tw} (\lambda + \beta_T \mu \cos(\psi)) + \dot{\beta}_T (\theta_C - \beta_T k_{p\beta}) \right) \right)}{8a} \\
& - \frac{\hat{C}(k) \hat{a} x_c^2 \left(\mu \cos(\psi) \left((\theta_C - \beta_T k_{p\beta}) (\lambda + \beta_T \mu \cos(\psi)) + 2 \frac{c_d}{a} \mu \sin(\psi) \right) + k_{p\zeta} \mu \sin(\psi) (\lambda + \beta_T \mu \cos(\psi)) \right)}{6a} \\
& + \frac{\hat{C}(k) \hat{a} \mu^2 \sin(\psi)^2 \left(\mu \cos(\psi) \left((\theta_C - \beta_T k_{p\beta}) (\lambda + \beta_T \mu \cos(\psi)) + 2 \frac{c_d}{a} \mu \sin(\psi) \right) + k_{p\zeta} \mu \sin(\psi) (\lambda + \beta_T \mu \cos(\psi)) \right)}{4a} \\
& - \frac{\hat{C}(k) \hat{a} \mu^3 \sin(\psi)^3 \left(k_{p\zeta} \left(\lambda + \beta_T \mu \cos(\psi) + \dot{\beta}_T \mu \sin(\psi) \right) + \mu \cos(\psi) \left(2 \frac{c_d}{a} + \theta_{tw} (\lambda + \beta_T \mu \cos(\psi)) + \dot{\beta}_T (\theta_C - \beta_T k_{p\beta}) \right) \right)}{6a} \\
& + \frac{\hat{C}(k) \hat{a} \mu^4 \sin(\psi)^4 \left(\dot{\beta}_T k_{p\zeta} + \dot{\beta}_T \mu \theta_{tw} \cos(\psi) \right)}{8a}
\end{aligned}
\tag{B.30}$$

$$\begin{aligned}
Q_\theta = & \frac{C(k) \dot{\beta}_T}{8} + \frac{C(k) \left(\lambda + \beta_T \mu \cos(\psi) + \dot{\beta}_T \mu \sin(\psi) \right)}{6} - \frac{C(k) \dot{\beta}_T \mu^4 \sin(\psi)^4}{8} \\
& - \frac{C(k) \mu^3 \sin(\psi)^3 (\lambda + \beta_T \mu \cos(\psi))}{4} + \frac{C(k) \mu \sin(\psi) (\lambda + \beta_T \mu \cos(\psi))}{4} \\
& + \frac{C(k) \mu^3 \sin(\psi)^3 \left(\lambda + \beta_T \mu \cos(\psi) + \dot{\beta}_T \mu \sin(\psi) \right)}{6} + \frac{\hat{C}(k) \hat{a} \dot{\beta}_T x_c^4}{8a} \\
& + \frac{\hat{C}(k) \hat{a} x_c^3 \left(\lambda + \beta_T \mu \cos(\psi) + \dot{\beta}_T \mu \sin(\psi) \right)}{6a} - \frac{\hat{C}(k) \hat{a} \dot{\beta}_T \mu^4 \sin(\psi)^4}{8a} \\
& + \frac{\hat{C}(k) \hat{a} \mu x_c^2 \sin(\psi) (\lambda + \beta_T \mu \cos(\psi))}{4a} - \frac{\hat{a} \left(\frac{c}{R} \right)^2 \mu \sin(\psi) \left(\left(\frac{x_a}{c} \right) + \frac{1}{2} \right)}{8a} \\
& - \frac{\hat{C}(k) \hat{a} \mu^3 \sin(\psi)^3 (\lambda + \beta_T \mu \cos(\psi))}{4a} + \frac{\hat{C}(k) \hat{a} \mu^3 \sin(\psi)^3 \left(\lambda + \beta_T \mu \cos(\psi) + \dot{\beta}_T \mu \sin(\psi) \right)}{6a}
\end{aligned} \tag{B.31}$$

$$\begin{aligned}
T_\beta^* = & \frac{C(k) \left(\frac{c}{R} \right) \left(\frac{x_a}{c} \right)}{6} - \frac{C(k) \left(\frac{c}{R} \right) \mu^3 \left(\frac{x_a}{c} \right) \sin(\psi)^3}{12} + \frac{C(k) \left(\frac{c}{R} \right) \mu \left(\frac{x_a}{c} \right) \sin(\psi)}{4} + \frac{\hat{C}(k) \hat{a} \left(\frac{c}{R} \right) x_c^3 \left(\frac{x_a}{c} \right)}{6a} \\
& - \frac{\hat{C}(k) \hat{a} \left(\frac{c}{R} \right) \mu^3 \left(\frac{x_a}{c} \right) \sin(\psi)^3}{12a} + \frac{\hat{C}(k) \hat{a} \left(\frac{c}{R} \right) \mu x_c^2 \left(\frac{x_a}{c} \right) \sin(\psi)}{4a}
\end{aligned} \tag{B.32}$$

$$\begin{aligned}
T_{\zeta}^* = & \frac{C(k) \left(\frac{c}{R}\right) \theta_C \left(\frac{x_a}{c}\right)}{3} - \frac{C(k) \dot{\beta}_T \left(\frac{c}{R}\right) \left(\frac{x_a}{c}\right)}{6} - \frac{C(k) \left(\frac{c}{R}\right) \lambda \left(\frac{x_a}{c}\right)}{4} - \frac{C(k) \left(\frac{c}{R}\right) c_m}{3a} + \frac{C(k) \left(\frac{c}{R}\right) \theta_{tw} \left(\frac{x_a}{c}\right)}{4} \\
& - \frac{C(k) \beta_T \left(\frac{c}{R}\right) k_{p\beta} \left(\frac{x_a}{c}\right)}{3} - \frac{C(k) \left(\frac{c}{R}\right) \mu^3 \theta_C \left(\frac{x_a}{c}\right) \sin(\psi)^3}{6} + \frac{C(k) \left(\frac{c}{R}\right) \mu^4 \theta_{tw} \left(\frac{x_a}{c}\right) \sin(\psi)^4}{12} \\
& - \frac{C(k) \beta_T \left(\frac{c}{R}\right) \mu \left(\frac{x_a}{c}\right) \cos(\psi)}{4} + \frac{C(k) \left(\frac{c}{R}\right) c_m \mu^3 \sin(\psi)^3}{6a} + \frac{C(k) \left(\frac{c}{R}\right) \mu \theta_C \left(\frac{x_a}{c}\right) \sin(\psi)}{2} \\
& + \frac{C(k) \left(\frac{c}{R}\right) \mu \theta_{tw} \left(\frac{x_a}{c}\right) \sin(\psi)}{3} - \frac{C(k) \left(\frac{c}{R}\right) c_m \mu \sin(\psi)}{2a} - \frac{C(k) \dot{\beta}_T \left(\frac{c}{R}\right) \mu^3 \left(\frac{x_a}{c}\right) \sin(\psi)^3}{6} \\
& + \frac{C(k) \left(\frac{c}{R}\right) \lambda \mu^2 \left(\frac{x_a}{c}\right) \sin(\psi)^2}{4} + \frac{C(k) \beta_T \left(\frac{c}{R}\right) k_{p\beta} \mu^3 \left(\frac{x_a}{c}\right) \sin(\psi)^3}{6} \\
& - \frac{\hat{C}(k) \hat{a} \left(\frac{c}{R}\right) x_c^3 \left(\frac{x_a}{c}\right) \left(\dot{\beta}_T - 2\theta_C + 2\beta_T k_{p\beta} - 2\mu \theta_{tw} \sin(\psi)\right)}{6a} - \frac{C(k) \beta_T \left(\frac{c}{R}\right) k_{p\beta} \mu \left(\frac{x_a}{c}\right) \sin(\psi)}{2} \\
& + \frac{C(k) \beta_T \left(\frac{c}{R}\right) \mu^3 \left(\frac{x_a}{c}\right) \cos(\psi) \sin(\psi)^2}{4} + \frac{\hat{C}(k) \hat{a} \left(\frac{c}{R}\right) \theta_{tw} x_c^4 \left(\frac{x_a}{c}\right)}{4a} \\
& - \frac{\hat{C}(k) \hat{a} \left(\frac{c}{R}\right) x_c^2 \left(\frac{x_a}{c}\right) (\lambda + \beta_T \mu \cos(\psi) - 2\mu \sin(\psi) (\theta_C - \beta_T k_{p\beta}))}{4} - \frac{\hat{C}(k) \hat{a} \left(\frac{c}{R}\right) \mu^4 \theta_{tw} \left(\frac{x_a}{c}\right) \sin(\psi)^4}{4a} \\
& + \frac{\hat{C}(k) \hat{a} \left(\frac{c}{R}\right) \mu^2 \left(\frac{x_a}{c}\right) \sin(\psi)^2 (\lambda + \beta_T \mu \cos(\psi) - 2\mu \sin(\psi) (\theta_C - \beta_T k_{p\beta}))}{4a} \\
& - \frac{\hat{C}(k) \hat{a} \left(\frac{c}{R}\right) \mu^3 \left(\frac{x_a}{c}\right) \sin(\psi)^3 \left(\dot{\beta}_T - 2\theta_C + 2\beta_T k_{p\beta} - 2\mu \theta_{tw} \sin(\psi)\right)}{6a}
\end{aligned}
\tag{B.33}$$

$$\begin{aligned}
T_{\theta}^* = & x_c^2 \left(\frac{\hat{a}(\frac{c}{R})^2 ((\frac{\hat{x}_a}{c})^2 + \frac{1}{2})}{16a} - \frac{\hat{C}(k) \hat{a}(\frac{c}{R})^2 ((\frac{\hat{x}_a}{c})^2 + \frac{1}{2})}{4a} \right) - \frac{C(k) (\frac{c}{R})^2 (\frac{x_a}{c})}{8} - \frac{(\frac{c}{R})^2 (\frac{x_a}{c})}{16} - \frac{(\frac{c}{R})^2}{32} \\
& + x_c \left(\frac{\hat{a}(\frac{c}{R})^2 \mu \sin(\psi) ((\frac{\hat{x}_a}{c})^2 + \frac{1}{2})}{8a} - \frac{\hat{C}(k) \hat{a}(\frac{c}{R})^2 \mu (\frac{\hat{x}_a}{c}) \sin(\psi) ((\frac{\hat{x}_a}{c})^2 + \frac{1}{2})}{2a} \right) - \frac{(\frac{c}{R})^2 \mu^2 \sin(\psi)^2}{32} \\
& - \frac{(\frac{c}{R})^2 \mu \sin(\psi)}{16} + \mu \sin(\psi) \left(\frac{\hat{a}(\frac{c}{R})^2 \mu \sin(\psi) ((\frac{\hat{x}_a}{c})^2 + \frac{1}{2})}{8a} - \frac{\hat{C}(k) \hat{a}(\frac{c}{R})^2 \mu (\frac{\hat{x}_a}{c}) \sin(\psi) ((\frac{\hat{x}_a}{c})^2 + \frac{1}{2})}{2a} \right) \\
& - \mu^2 \sin(\psi)^2 \left(\frac{\hat{a}(\frac{c}{R})^2 ((\frac{\hat{x}_a}{c})^2 + \frac{1}{2})}{16a} - \frac{\hat{C}(k) \hat{a}(\frac{c}{R})^2 ((\frac{\hat{x}_a}{c})^2 + \frac{1}{2})}{4a} \right) - \frac{(\frac{c}{R})^2 \mu^2 (\frac{x_a}{c}) \sin(\psi)^2}{16} - \frac{(\frac{c}{R})^2 \mu (\frac{x_a}{c}) \sin(\psi)}{8} \\
& - \frac{C(k) (\frac{c}{R})^2 \mu (\frac{x_a}{c}) \sin(\psi)}{4} + \frac{C(k) (\frac{c}{R})^2 \mu^2 (\frac{x_a}{c})^2 \sin(\psi)^2}{4} + \frac{C(k) (\frac{c}{R})^2 \mu (\frac{x_a}{c})^2 \sin(\psi)}{2} \\
& - \frac{C(k) (\frac{c}{R})^2 \mu^2 (\frac{x_a}{c}) \sin(\psi)^2}{8} + \frac{C(k) (\frac{c}{R})^2 (\frac{x_a}{c})^2}{4} \\
\end{aligned} \tag{B.34}$$

$$\begin{aligned}
T_{\beta} = & \frac{C(k) (\frac{c}{R}) k_{p\beta} (\frac{x_a}{c})}{6} + \frac{C(k) (\frac{c}{R}) \mu (\frac{x_a}{c}) \cos(\psi)}{4} + \frac{C(k) (\frac{c}{R}) \mu^2 (\frac{x_a}{c}) \cos(\psi) \sin(\psi)}{2} + \frac{C(k) (\frac{c}{R}) k_{p\beta} \mu (\frac{x_a}{c}) \sin(\psi)}{2} \\
& + \frac{C(k) (\frac{c}{R}) \mu^3 (\frac{x_a}{c}) \cos(\psi) \sin(\psi)^2}{4} + \frac{C(k) (\frac{c}{R}) k_{p\beta} \mu^2 (\frac{x_a}{c}) \sin(\psi)^2}{2} + \frac{C(k) (\frac{c}{R}) k_{p\beta} \mu^3 (\frac{x_a}{c}) \sin(\psi)^3}{6} \\
& + \frac{\hat{C}(k) \hat{a}(\frac{c}{R}) x_c (\frac{\hat{x}_a}{c}) (k_{p\beta} \mu^2 \sin(\psi)^2 + \cos(\psi) \mu^2 \sin(\psi))}{2a} + \frac{\hat{C}(k) \hat{a}(\frac{c}{R}) x_c^2 (\frac{\hat{x}_a}{c}) (\mu \cos(\psi) + 2 k_{p\beta} \mu \sin(\psi))}{4a} \\
& + \frac{\hat{C}(k) \hat{a}(\frac{c}{R}) \mu (\frac{\hat{x}_a}{c}) \sin(\psi) (k_{p\beta} \mu^2 \sin(\psi)^2 + \cos(\psi) \mu^2 \sin(\psi))}{2a} + \frac{\hat{C}(k) \hat{a}(\frac{c}{R}) k_{p\beta} \mu^3 (\frac{\hat{x}_a}{c}) \sin(\psi)^3}{6a} \\
& - \frac{\hat{C}(k) \hat{a}(\frac{c}{R}) \mu^2 (\frac{\hat{x}_a}{c}) \sin(\psi)^2 (\mu \cos(\psi) + 2 k_{p\beta} \mu \sin(\psi))}{4a} + \frac{\hat{C}(k) \hat{a}(\frac{c}{R}) k_{p\beta} x_c^3 (\frac{\hat{x}_a}{c})}{6a} \\
\end{aligned} \tag{B.35}$$

$$\begin{aligned}
T_\zeta = & \frac{C(k)(\frac{c}{R})k_{p\zeta}(\frac{x_a}{c})}{6} - \frac{C(k)\dot{\beta}_T(\frac{c}{R})\mu(\frac{x_a}{c})\cos(\psi)}{4} - \frac{C(k)(\frac{c}{R})\lambda\mu(\frac{x_a}{c})\cos(\psi)}{2} + \frac{C(k)(\frac{c}{R})\mu\theta_C(\frac{x_a}{c})\cos(\psi)}{2} \\
+ & \frac{C(k)(\frac{c}{R})\mu\theta_{tw}(\frac{x_a}{c})\cos(\psi)}{3} + \frac{C(k)(\frac{c}{R})k_{p\zeta}\mu(\frac{x_a}{c})\sin(\psi)}{2} - \frac{C(k)(\frac{c}{R})c_m\mu\cos(\psi)}{2a} - \frac{C(k)\beta_T(\frac{c}{R})\mu^2(\frac{x_a}{c})\cos(\psi)^2}{2} \\
+ & \frac{C(k)(\frac{c}{R})k_{p\zeta}\mu^2(\frac{x_a}{c})\sin(\psi)^2}{2} + \frac{C(k)(\frac{c}{R})k_{p\zeta}\mu^3(\frac{x_a}{c})\sin(\psi)^3}{6} - \frac{C(k)(\frac{c}{R})c_m\mu^3\cos(\psi)\sin(\psi)^2}{2} \\
+ & \frac{\hat{C}(k)\hat{a}(\frac{c}{R})x_c^3(\frac{x_a}{c})(k_{p\zeta}+2\mu\theta_{tw}\cos(\psi))}{6a} - \frac{C(k)(\frac{c}{R})\lambda\mu^2(\frac{x_a}{c})\cos(\psi)\sin(\psi)}{2} + \frac{C(k)(\frac{c}{R})\mu^2\theta_C(\frac{x_a}{c})\cos(\psi)\sin(\psi)}{2} \\
- & \frac{\hat{C}(k)\hat{a}(\frac{c}{R})x_c(\frac{x_a}{c})(\mu\cos(\psi)(\lambda+\beta_T\mu\cos(\psi)-2\mu\sin(\psi)(\theta_C-\beta_Tk_{p\beta})) - k_{p\zeta}\mu^2\sin(\psi)^2)}{2a} \\
+ & \frac{\hat{C}(k)\hat{a}(\frac{c}{R})x_c^2(\frac{x_a}{c})(2k_{p\zeta}\mu\sin(\psi)-\mu\cos(\psi)(\dot{\beta}_T-2\theta_C+2\beta_Tk_{p\beta}-2\mu\theta_{tw}\sin(\psi)))}{4a} \\
+ & \frac{C(k)(\frac{c}{R})\mu^2\theta_{tw}(\frac{x_a}{c})\cos(\psi)\sin(\psi)}{2} - \frac{C(k)\beta_T(\frac{c}{R})k_{p\beta}\mu(\frac{x_a}{c})\cos(\psi)}{2} - \frac{C(k)(\frac{c}{R})c_m\mu^2\cos(\psi)\sin(\psi)}{a} \\
- & \frac{C(k)\beta_T(\frac{c}{R})\mu^3(\frac{x_a}{c})\cos(\psi)^2\sin(\psi)}{2} + \frac{C(k)\dot{\beta}_T(\frac{c}{R})\mu^3(\frac{x_a}{c})\cos(\psi)\sin(\psi)^2}{4} + \frac{C(k)(\frac{c}{R})\mu^3\theta_C(\frac{x_a}{c})\cos(\psi)\sin(\psi)^2}{2} \\
- & \frac{C(k)(\frac{c}{R})\mu^4\theta_{tw}(\frac{x_a}{c})\cos(\psi)\sin(\psi)^3}{6} - \frac{C(k)\beta_T(\frac{c}{R})k_{p\beta}\mu^3(\frac{x_a}{c})\cos(\psi)\sin(\psi)^2}{2} \\
- & \frac{\hat{C}(k)\hat{a}(\frac{c}{R})\mu(\frac{x_a}{c})\sin(\psi)(\mu\cos(\psi)(\lambda+\beta_T\mu\cos(\psi)-2\mu\sin(\psi)(\theta_C-\beta_Tk_{p\beta})) - k_{p\zeta}\mu^2\sin(\psi)^2)}{2a} \\
- & \frac{\hat{C}(k)\hat{a}(\frac{c}{R})\mu^2(\frac{x_a}{c})\sin(\psi)^2(2k_{p\zeta}\mu\sin(\psi)-\mu\cos(\psi)(\dot{\beta}_T-2\theta_C+2\beta_Tk_{p\beta}-2\mu\theta_{tw}\sin(\psi)))}{2a} \\
+ & \frac{\hat{C}(k)\hat{a}(\frac{c}{R})\mu^3(\frac{x_a}{c})\sin(\psi)^3(k_{p\zeta}+2\mu\theta_{tw}\cos(\psi))}{6a} - \frac{C(k)\beta_T(\frac{c}{R})k_{p\beta}\mu^2(\frac{x_a}{c})\cos(\psi)\sin(\psi)}{4a}
\end{aligned}$$

(B.36)

$$\begin{aligned}
T_\theta = & -\frac{C(k)(\frac{c}{R})(\frac{x_a}{c})}{6} - \frac{C(k)(\frac{c}{R})\mu^2(\frac{x_a}{c})\sin(\psi)^2}{2} - \frac{C(k)(\frac{c}{R})\mu^3(\frac{x_a}{c})\sin(\psi)^3}{6} - \frac{C(k)(\frac{c}{R})\mu(\frac{x_a}{c})\sin(\psi)}{2} \\
& - \frac{\hat{C}(k)\hat{a}(\frac{c}{R})x_c^3(\frac{x_a}{c})}{6a} - \frac{\hat{C}(k)\hat{a}(\frac{c}{R})\mu^3(\frac{x_a}{c})\sin(\psi)^3}{6a} - \frac{\hat{C}(k)\hat{a}(\frac{c}{R})\mu^2x_c(\frac{x_a}{c})\sin(\psi)^2}{2a} - \frac{\hat{C}(k)\hat{a}(\frac{c}{R})\mu x_c^2(\frac{x_a}{c})\sin(\psi)}{2a}
\end{aligned} \tag{B.37}$$

B.3 Region III, Complete Reverse Flow

$$\begin{aligned}
M_\beta^* = & \frac{\hat{C}(k)\hat{a}(\frac{c_i}{a}+1)}{8a} - \frac{\hat{C}(k)\hat{a}x_c^4(\frac{c_i}{a}+1)}{8a} + \frac{\hat{C}(k)\hat{a}\mu\sin(\psi)(\frac{c_i}{a}+1)}{6a} \\
& - \frac{\hat{C}(k)\hat{a}\mu x_c^3\sin(\psi)(\frac{c_i}{a}+1)}{6a}
\end{aligned} \tag{B.38}$$

$$\begin{aligned}
M_\zeta^* = & \frac{\hat{C}(k)\hat{a}\theta_{tw}}{5a} - \frac{\hat{C}(k)\hat{a}((\lambda + \beta_T\mu\cos(\psi))(\frac{c_i}{a}+1) - 2\mu\sin(\psi)(\theta_C - \beta_Tk_{p\beta}))}{6a} \\
& + \frac{\hat{C}(k)\hat{a}(2\theta_C - 2\beta_Tk_{p\beta} - \dot{\beta}_T(\frac{c_i}{a}+1) + 2\mu\theta_{tw}\sin(\psi))}{8a} - \frac{\hat{C}(k)\hat{a}\theta_{tw}x_c^5}{5a} \\
& + \frac{\hat{C}(k)\hat{a}x_c^3((\lambda + \beta_T\mu\cos(\psi))(\frac{c_i}{a}+1) - 2\mu\sin(\psi)(\theta_C - \beta_Tk_{p\beta}))}{6a} \\
& - \frac{\hat{C}(k)\hat{a}x_c^4(2\theta_C - 2\beta_Tk_{p\beta} - \dot{\beta}_T(\frac{c_i}{a}+1) + 2\mu\theta_{tw}\sin(\psi))}{8a}
\end{aligned} \tag{B.39}$$

$$M_\theta^* = \frac{\hat{C}(k) \hat{a} (\frac{c}{R}) x_c^3 ((\frac{\hat{x}a}{c}) + \frac{1}{2}) (\frac{c\hat{a}}{a} + 1)}{6a} - \frac{\hat{C}(k) \hat{a} (\frac{c}{R}) ((\frac{\hat{x}a}{c}) + \frac{1}{2}) (\frac{c\hat{a}}{a} + 1)}{6a} - \frac{\hat{C}(k) \hat{a} (\frac{c}{R}) \mu \sin(\psi) ((\frac{\hat{x}a}{c}) + \frac{1}{2}) (\frac{c\hat{a}}{a} + 1)}{4a} + \frac{\hat{C}(k) \hat{a} (\frac{c}{R}) \mu x_c^2 \sin(\psi) ((\frac{\hat{x}a}{c}) + \frac{1}{2}) (\frac{c\hat{a}}{a} + 1)}{4a} \quad (\text{B.40})$$

$$M_\beta = \frac{\hat{C}(k) \hat{a} k_{p\beta}}{8a} + \frac{\hat{C}(k) \hat{a} (2k_{p\beta} \mu \sin(\psi) + \mu \cos(\psi) (\frac{c\hat{a}}{a} + 1))}{6a} + \frac{\hat{C}(k) \hat{a} (k_{p\beta} \mu^2 \sin(\psi)^2 + \cos(\psi) (\frac{c\hat{a}}{a} + 1) \mu^2 \sin(\psi))}{4a} - \frac{\hat{C}(k) \hat{a} k_{p\beta} x_c^4}{8a} - \frac{\hat{C}(k) \hat{a} x_c^3 (2k_{p\beta} \mu \sin(\psi) + \mu \cos(\psi) (\frac{c\hat{a}}{a} + 1))}{6a} - \frac{\hat{C}(k) \hat{a} x_c^2 (k_{p\beta} \mu^2 \sin(\psi)^2 + \cos(\psi) (\frac{c\hat{a}}{a} + 1) \mu^2 \sin(\psi))}{4a} \quad (\text{B.41})$$

$$M_\zeta = \frac{\hat{C}(k) \hat{a} (2k_{p\zeta} \mu \sin(\psi) + \mu \cos(\psi) (2\theta_C - 2\beta_T k_{p\beta} - \dot{\beta}_T (\frac{c\hat{a}}{a} + 1) + 2\mu \theta_{tw} \sin(\psi)))}{6a} - \frac{\hat{C}(k) \hat{a} (\mu \cos(\psi) ((\lambda + \beta_T \mu \cos(\psi)) (\frac{c\hat{a}}{a} + 1) - 2\mu \sin(\psi) (\theta_C - \beta_T k_{p\beta})) - k_{p\zeta} \mu^2 \sin(\psi)^2)}{4a} + \frac{\hat{C}(k) \hat{a} x_c^2 (\mu \cos(\psi) ((\lambda + \beta_T \mu \cos(\psi)) (\frac{c\hat{a}}{a} + 1) - 2\mu \sin(\psi) (\theta_C - \beta_T k_{p\beta})) - k_{p\zeta} \mu^2 \sin(\psi)^2)}{4a} + \frac{\hat{C}(k) \hat{a} x_c^3 (2k_{p\zeta} \mu \sin(\psi) + \mu \cos(\psi) (2\theta_C - 2\beta_T k_{p\beta} - \dot{\beta}_T (\frac{c\hat{a}}{a} + 1) + 2\mu \theta_{tw} \sin(\psi)))}{6a} + \frac{\hat{C}(k) \hat{a} (k_{p\zeta} + 2\mu \theta_{tw} \cos(\psi))}{8a} - \frac{\hat{C}(k) \hat{a} x_c^4 (k_{p\zeta} + 2\mu \theta_{tw} \cos(\psi))}{8a} \quad (\text{B.42})$$

$$M_\theta = \frac{\hat{C}(k) \hat{a} x_c^4}{8a} - \frac{\hat{C}(k) \hat{a}}{8a} - \frac{\hat{C}(k) \hat{a} \mu \sin(\psi)}{3a} - \frac{\hat{C}(k) \hat{a} \mu^2 \sin(\psi)^2}{4a} + \frac{\hat{C}(k) \hat{a} \mu x_c^3 \sin(\psi)}{3a} \quad (\text{B.43})$$

$$\begin{aligned}
Q_\beta^* = & \frac{\hat{C}(k) \hat{a} \left(2 \dot{\beta}_T - \theta_C + \beta_T k_{p\beta} - \mu \theta_{tw} \sin(\psi) \right)}{8a} - \frac{\hat{C}(k) \hat{a} \theta_{tw}}{10a} + \frac{\hat{C}(k) \hat{a} \theta_{tw} x_c^5}{10a} \\
& - \frac{\hat{C}(k) \hat{a} x_c^3 (2\lambda + 2\beta_T \mu \cos(\psi) - \mu \sin(\psi) (\theta_C - \beta_T k_{p\beta}))}{6a}
\end{aligned} \tag{B.44}$$

$$\begin{aligned}
& - \frac{\hat{C}(k) \hat{a} x_c^4 \left(2 \dot{\beta}_T - \theta_C + \beta_T k_{p\beta} - \mu \theta_{tw} \sin(\psi) \right)}{8a} + \frac{\hat{C}(k) \hat{a} (2\lambda + 2\beta_T \mu \cos(\psi) - \mu \sin(\psi) (\theta_C - \beta_T k_{p\beta}))}{6a}
\end{aligned}$$

$$\begin{aligned}
Q_\zeta^* = & \frac{\hat{C}(k) \hat{a} \left(\frac{2c_{\hat{a}}}{\hat{a}} + \theta_{tw} (\lambda + \beta_T \mu \cos(\psi)) + \dot{\beta}_T (\theta_C - \beta_T k_{p\beta}) \right)}{8a} \\
& + \frac{\hat{C}(k) \hat{a} ((\theta_C - \beta_T k_{p\beta}) (\lambda + \beta_T \mu \cos(\psi)) + 2 \frac{c_{\hat{a}}}{\hat{a}} \mu \sin(\psi) right)}{6a} \\
& - \frac{\hat{C}(k) \hat{a} x_c^3 \left((\theta_C - \beta_T k_{p\beta}) (\lambda + \beta_T \mu \cos(\psi)) + \frac{2c_{\hat{a}} \mu \sin(\psi)}{\hat{a}} \right)}{6a} + \frac{\hat{C}(k) \hat{a} \dot{\beta}_T \theta_{tw}}{10a} \\
& - \frac{\hat{C}(k) \hat{a} x_c^4 \left(2 \frac{c_{\hat{a}}}{\hat{a}} + \theta_{tw} (\lambda + \beta_T \mu \cos(\psi)) + \dot{\beta}_T (\theta_C - \beta_T k_{p\beta}) \right)}{8a} - \frac{\hat{C}(k) \hat{a} \dot{\beta}_T \theta_{tw} x_c^5}{10a}
\end{aligned} \tag{B.45}$$

$$\begin{aligned}
Q_\theta^* = & \frac{\hat{C}(k) \hat{a} (\frac{c}{R}) \theta_{tw} ((\frac{x_{\hat{a}}}{c}) + \frac{1}{2})}{8a} - \frac{\hat{C}(k) \hat{a} (\frac{c}{R}) ((\frac{x_{\hat{a}}}{c}) + \frac{1}{2}) \left(2 \dot{\beta}_T - \theta_C + \beta_T k_{p\beta} - \mu \theta_{tw} \sin(\psi) \right)}{6a} \\
& - \frac{\hat{C}(k) \hat{a} (\frac{c}{R}) ((\frac{x_{\hat{a}}}{c}) + \frac{1}{2}) (2\lambda + 2\beta_T \mu \cos(\psi) - \mu \sin(\psi) (\theta_C - \beta_T k_{p\beta}))}{4a} \\
& + \frac{\hat{C}(k) \hat{a} (\frac{c}{R}) x_c^3 ((\frac{x_{\hat{a}}}{c}) + \frac{1}{2}) \left(2 \dot{\beta}_T - \theta_C + \beta_T k_{p\beta} - \mu \theta_{tw} \sin(\psi) \right)}{6a} - \frac{\hat{C}(k) \hat{a} (\frac{c}{R}) \theta_{tw} x_c^4 ((\frac{x_{\hat{a}}}{c}) + \frac{1}{2})}{8a} \\
& + \frac{\hat{C}(k) \hat{a} (\frac{c}{R}) x_c^2 ((\frac{x_{\hat{a}}}{c}) + \frac{1}{2}) (2\lambda + 2\beta_T \mu \cos(\psi) - \mu \sin(\psi) (\theta_C - \beta_T k_{p\beta}))}{4a}
\end{aligned} \tag{B.46}$$

$$\begin{aligned}
Q_\beta &= \frac{\hat{C}(k) \hat{a} (\mu \cos(\psi) (2\lambda + 2\beta_T \mu \cos(\psi) - \mu \sin(\psi) (\theta_C - \beta_T k_{p\beta})) + k_{p\beta} \mu \sin(\psi) (\lambda + \beta_T \mu \cos(\psi)))}{4a} \\
&+ \frac{\hat{C}(k) \hat{a} (k_{p\beta} (\lambda + \beta_T \mu \cos(\psi) + \dot{\beta}_T \mu \sin(\psi)) + \mu \cos(\psi) (2\dot{\beta}_T - \theta_C + \beta_T k_{p\beta} - \mu \theta_{tw} \sin(\psi)))}{6a} \\
&- \frac{\hat{C}(k) \hat{a} (\dot{\beta}_T k_{p\beta} - \mu \theta_{tw} \cos(\psi))}{8a} - \frac{\hat{C}(k) \hat{a} x_c^4 (\dot{\beta}_T k_{p\beta} - \mu \theta_{tw} \cos(\psi))}{8a} \\
&- \frac{\hat{C}(k) \hat{a} x_c^2 (\mu \cos(\psi) (2\lambda + 2\beta_T \mu \cos(\psi) - \mu \sin(\psi) (\theta_C - \beta_T k_{p\beta})) + k_{p\beta} \mu \sin(\psi) (\lambda + \beta_T \mu \cos(\psi)))}{4a} \\
&- \frac{\hat{C}(k) \hat{a} x_c^3 (k_{p\beta} (\lambda + \beta_T \mu \cos(\psi) + \dot{\beta}_T \mu \sin(\psi)) + \mu \cos(\psi) (2\dot{\beta}_T - \theta_C + \beta_T k_{p\beta} - \mu \theta_{tw} \sin(\psi)))}{6a} \\
Q_\zeta &= \frac{\hat{C}(k) \hat{a} (\mu \cos(\psi) ((\theta_C - \beta_T k_{p\beta}) (\lambda + \beta_T \mu \cos(\psi)) + 2\frac{c_d}{a} \mu \sin(\psi)) + k_{p\zeta} \mu \sin(\psi) (\lambda + \beta_T \mu \cos(\psi)))}{4a} \\
&+ \frac{\hat{C}(k) \hat{a} (k_{p\zeta} (\lambda + \beta_T \mu \cos(\psi) + \dot{\beta}_T \mu \sin(\psi)) + \mu \cos(\psi) (\frac{2c_d}{a} + \theta_{tw} (\lambda + \beta_T \mu \cos(\psi)) + \dot{\beta}_T (\theta_C - \beta_T k_{p\beta})))}{6a} \\
&- \frac{\hat{C}(k) \hat{a} x_c^3 (k_{p\zeta} (\lambda + \beta_T \mu \cos(\psi) + \dot{\beta}_T \mu \sin(\psi)) + \mu \cos(\psi) (\frac{2c_d}{a} + \theta_{tw} (\lambda + \beta_T \mu \cos(\psi)) + \dot{\beta}_T (\theta_C - \beta_T k_{p\beta})))}{6a} \\
&- \frac{\hat{C}(k) \hat{a} (\dot{\beta}_T k_{p\zeta} + \dot{\beta}_T \mu \theta_{tw} \cos(\psi))}{8a} - \frac{\hat{C}(k) \hat{a} x_c^4 (\dot{\beta}_T k_{p\zeta} + \dot{\beta}_T \mu \theta_{tw} \cos(\psi))}{8a} \\
&- \frac{\hat{C}(k) \hat{a} x_c^2 (\mu \cos(\psi) ((\theta_C - \beta_T k_{p\beta}) (\lambda + \beta_T \mu \cos(\psi)) + \frac{2c_d}{a} \mu \sin(\psi)) + k_{p\zeta} \mu \sin(\psi) (\lambda + \beta_T \mu \cos(\psi)))}{4a} \\
Q_\theta &= \frac{\hat{C}(k) \hat{a} x_c^3 (\lambda + \beta_T \mu \cos(\psi) + \dot{\beta}_T \mu \sin(\psi))}{6a} - \frac{\hat{C}(k) \hat{a} (\lambda + \beta_T \mu \cos(\psi) + \dot{\beta}_T \mu \sin(\psi))}{6a} - \frac{\hat{C}(k) \hat{a} \dot{\beta}_T}{8a} \\
&+ \frac{\hat{C}(k) \hat{a} \dot{\beta}_T x_c^4}{8a} - \frac{\hat{C}(k) \hat{a} \mu \sin(\psi) (\lambda + \beta_T \mu \cos(\psi))}{4a} + \frac{\hat{C}(k) \hat{a} \mu x_c^2 \sin(\psi) (\lambda + \beta_T \mu \cos(\psi))}{4a}
\end{aligned} \tag{B.47}$$

$$\begin{aligned}
Q_\zeta &= \frac{\hat{C}(k) \hat{a} (\mu \cos(\psi) ((\theta_C - \beta_T k_{p\beta}) (\lambda + \beta_T \mu \cos(\psi)) + 2\frac{c_d}{a} \mu \sin(\psi)) + k_{p\zeta} \mu \sin(\psi) (\lambda + \beta_T \mu \cos(\psi)))}{4a} \\
&+ \frac{\hat{C}(k) \hat{a} (k_{p\zeta} (\lambda + \beta_T \mu \cos(\psi) + \dot{\beta}_T \mu \sin(\psi)) + \mu \cos(\psi) (\frac{2c_d}{a} + \theta_{tw} (\lambda + \beta_T \mu \cos(\psi)) + \dot{\beta}_T (\theta_C - \beta_T k_{p\beta})))}{6a} \\
&- \frac{\hat{C}(k) \hat{a} x_c^3 (k_{p\zeta} (\lambda + \beta_T \mu \cos(\psi) + \dot{\beta}_T \mu \sin(\psi)) + \mu \cos(\psi) (\frac{2c_d}{a} + \theta_{tw} (\lambda + \beta_T \mu \cos(\psi)) + \dot{\beta}_T (\theta_C - \beta_T k_{p\beta})))}{6a} \\
&- \frac{\hat{C}(k) \hat{a} (\dot{\beta}_T k_{p\zeta} + \dot{\beta}_T \mu \theta_{tw} \cos(\psi))}{8a} - \frac{\hat{C}(k) \hat{a} x_c^4 (\dot{\beta}_T k_{p\zeta} + \dot{\beta}_T \mu \theta_{tw} \cos(\psi))}{8a} \\
&- \frac{\hat{C}(k) \hat{a} x_c^2 (\mu \cos(\psi) ((\theta_C - \beta_T k_{p\beta}) (\lambda + \beta_T \mu \cos(\psi)) + \frac{2c_d}{a} \mu \sin(\psi)) + k_{p\zeta} \mu \sin(\psi) (\lambda + \beta_T \mu \cos(\psi)))}{4a}
\end{aligned} \tag{B.48}$$

$$\begin{aligned}
Q_\theta &= \frac{\hat{C}(k) \hat{a} x_c^3 (\lambda + \beta_T \mu \cos(\psi) + \dot{\beta}_T \mu \sin(\psi))}{6a} - \frac{\hat{C}(k) \hat{a} (\lambda + \beta_T \mu \cos(\psi) + \dot{\beta}_T \mu \sin(\psi))}{6a} - \frac{\hat{C}(k) \hat{a} \dot{\beta}_T}{8a} \\
&+ \frac{\hat{C}(k) \hat{a} \dot{\beta}_T x_c^4}{8a} - \frac{\hat{C}(k) \hat{a} \mu \sin(\psi) (\lambda + \beta_T \mu \cos(\psi))}{4a} + \frac{\hat{C}(k) \hat{a} \mu x_c^2 \sin(\psi) (\lambda + \beta_T \mu \cos(\psi))}{4a}
\end{aligned} \tag{B.49}$$

$$T_{\beta}^* = \frac{\hat{C}(k) \hat{a}(\frac{c}{R}) x_c^3 (\frac{x_a}{c})}{6a} - \frac{\hat{C}(k) \hat{a}(\frac{c}{R}) (\frac{x_a}{c})}{6a} - \frac{\hat{C}(k) \hat{a}(\frac{c}{R}) \mu (\frac{x_a}{c}) \sin(\psi)}{4a} + \frac{\hat{C}(k) \hat{a}(\frac{c}{R}) \mu x_c^2 (\frac{x_a}{c}) \sin(\psi)}{4a} \quad (\text{B.50})$$

$$\begin{aligned} T_{\zeta}^* = & \frac{\hat{C}(k) \hat{a}(\frac{c}{R}) (\frac{x_a}{c}) (\lambda + \beta_T \mu \cos(\psi) - 2 \mu \sin(\psi) (\theta_C - \beta_T k_{p\beta}))}{4a} - \frac{\hat{C}(k) \hat{a}(\frac{c}{R}) \theta_{tw} (\frac{x_a}{c})}{4a} \\ & + \frac{\hat{C}(k) \hat{a}(\frac{c}{R}) \theta_{tw} x_c^4 (\frac{x_a}{c})}{4a} + \frac{\hat{C}(k) \hat{a}(\frac{c}{R}) (\frac{x_a}{c}) (\dot{\beta}_T - 2 \theta_C + 2 \beta_T k_{p\beta} - 2 \mu \theta_{tw} \sin(\psi))}{6a} \\ & - \frac{\hat{C}(k) \hat{a}(\frac{c}{R}) x_c^3 (\frac{x_a}{c}) (\dot{\beta}_T - 2 \theta_C + 2 \beta_T k_{p\beta} - 2 \mu \theta_{tw} \sin(\psi))}{6a} \\ & - \frac{\hat{C}(k) \hat{a}(\frac{c}{R}) x_c^2 (\frac{x_a}{c}) (\lambda + \beta_T \mu \cos(\psi) - 2 \mu \sin(\psi) (\theta_C - \beta_T k_{p\beta}))}{4a} \end{aligned} \quad (\text{B.51})$$

$$\begin{aligned} T_{\theta}^* = & x_c^2 \left(\frac{\hat{a}(\frac{c}{R})^2 ((\frac{x_a}{c})^2 + \frac{1}{2})}{16a} - \frac{\hat{C}(k) \hat{a}(\frac{c}{R}) (\frac{x_a}{c}) ((\frac{x_a}{c})^2 + \frac{1}{2})}{4a} \right) - \frac{\hat{a}(\frac{c}{R})^2 \mu \sin(\psi) ((\frac{x_a}{c})^2 + \frac{1}{2})}{8a} \\ & + x_c \left(\frac{\hat{a}(\frac{c}{R})^2 \mu \sin(\psi) ((\frac{x_a}{c})^2 + \frac{1}{2})}{8a} - \frac{\hat{C}(k) \hat{a}(\frac{c}{R})^2 \mu (\frac{x_a}{c}) \sin(\psi) ((\frac{x_a}{c})^2 + \frac{1}{2})}{2a} \right) - \frac{\hat{a}(\frac{c}{R})^2 ((\frac{x_a}{c})^2 + \frac{1}{2})}{16a} \\ & + \frac{\hat{C}(k) \hat{a}(\frac{c}{R})^2 (\frac{x_a}{c}) ((\frac{x_a}{c})^2 + \frac{1}{2})}{4a} + \frac{\hat{C}(k) \hat{a}(\frac{c}{R})^2 \mu (\frac{x_a}{c}) \sin(\psi) ((\frac{x_a}{c})^2 + \frac{1}{2})}{2a} \end{aligned} \quad (\text{B.52})$$

$$\begin{aligned} T_{\beta} = & \frac{\hat{C}(k) \hat{a}(\frac{c}{R}) x_c (\frac{x_a}{c}) (k_{p\beta} \mu^2 \sin(\psi)^2 + \cos(\psi) \mu^2 \sin(\psi))}{2a} - \frac{\hat{C}(k) \hat{a}(\frac{c}{R}) (\frac{x_a}{c}) (\mu \cos(\psi) + 2 k_{p\beta} \mu \sin(\psi))}{4a} \\ & - \frac{\hat{C}(k) \hat{a}(\frac{c}{R}) (\frac{x_a}{c}) (k_{p\beta} \mu^2 \sin(\psi)^2 + \cos(\psi) \mu^2 \sin(\psi))}{2a} - \frac{\hat{C}(k) \hat{a}(\frac{c}{R}) k_{p\beta} (\frac{x_a}{c})}{6a} \\ & + \frac{\hat{C}(k) \hat{a}(\frac{c}{R}) x_c^2 (\frac{x_a}{c}) (\mu \cos(\psi) + 2 k_{p\beta} \mu \sin(\psi))}{4a} + \frac{\hat{C}(k) \hat{a}(\frac{c}{R}) k_{p\beta} x_c^3 (\frac{x_a}{c})}{6a} \end{aligned} \quad (\text{B.53})$$

$$\begin{aligned}
T_\zeta = & \frac{\hat{C}(k) \hat{a}(\frac{c}{R})(\frac{\hat{x}_a}{c}) (\mu \cos(\psi) (\lambda + \beta_T \mu \cos(\psi) - 2 \mu \sin(\psi) (\theta_C - \beta_T k_{p\beta})) - k_{p\zeta} \mu^2 \sin(\psi)^2)}{2a} \\
& - \frac{\hat{C}(k) \hat{a}(\frac{c}{R})(\frac{\hat{x}_a}{c}) \left(2 k_{p\zeta} \mu \sin(\psi) - \mu \cos(\psi) \left(\dot{\beta}_T - 2 \theta_C + 2 \beta_T k_{p\beta} - 2 \mu \theta_{tw} \sin(\psi) \right) \right)}{4a} \\
& - \frac{\hat{C}(k) \hat{a}(\frac{c}{R})(\frac{\hat{x}_a}{c}) (k_{p\zeta} + 2 \mu \theta_{tw} \cos(\psi))}{6a} + \frac{\hat{C}(k) \hat{a}(\frac{c}{R}) x_c^3 (\frac{\hat{x}_a}{c}) (k_{p\zeta} + 2 \mu \theta_{tw} \cos(\psi))}{6a} \\
& - \frac{\hat{C}(k) \hat{a}(\frac{c}{R}) x_c (\frac{\hat{x}_a}{c}) (\mu \cos(\psi) (\lambda + \beta_T \mu \cos(\psi) - 2 \mu \sin(\psi) (\theta_C - \beta_T k_{p\beta})) - k_{p\zeta} \mu^2 \sin(\psi)^2)}{2a} \\
& + \frac{\hat{C}(k) \hat{a}(\frac{c}{R}) x_c^2 (\frac{\hat{x}_a}{c}) \left(2 k_{p\zeta} \mu \sin(\psi) - \mu \cos(\psi) \left(\dot{\beta}_T - 2 \theta_C + 2 \beta_T k_{p\beta} - 2 \mu \theta_{tw} \sin(\psi) \right) \right)}{4a}
\end{aligned} \tag{B.54}$$

$$\begin{aligned}
T_\theta = & \frac{\hat{C}(k) \hat{a}(\frac{c}{R})(\frac{\hat{x}_a}{c})}{6a} - \frac{\hat{C}(k) \hat{a}(\frac{c}{R}) x_c^3 (\frac{\hat{x}_a}{c})}{6a} + \frac{\hat{C}(k) \hat{a}(\frac{c}{R}) \mu^2 (\frac{\hat{x}_a}{c}) \sin(\psi)^2}{2a} \\
& + \frac{\hat{C}(k) \hat{a}(\frac{c}{R}) \mu (\frac{\hat{x}_a}{c}) \sin(\psi)}{2a} - \frac{\hat{C}(k) \hat{a}(\frac{c}{R}) \mu^2 x_c (\frac{\hat{x}_a}{c}) \sin(\psi)^2}{2a} - \frac{\hat{C}(k) \hat{a}(\frac{c}{R}) \mu x_c^2 (\frac{\hat{x}_a}{c}) \sin(\psi)}{2a}
\end{aligned} \tag{B.55}$$

Bibliography

- [1] McDonnell XV-1 Convertiplane. San Diego Air and Space Museum Archive, Catalog #: 00034977. Uploaded to Flickr on 30 April 2010.
- [2] C.H. Perisho. Analysis of the Stability of a Flexible Rotor Blade at High Advance Ratio. *Journal of the American Helicopter Society*, 4(2):4–18, April 1959.
- [3] R. W. Prouty. The Lockheed Helicopter Experience. In *Proceedings of the American Helicopter Society's 65th Annual Forum*, Grapevine, TX, May 27-29 2009.
- [4] K. I. Swartz. Jaunt's Journey. *Vertiflite*, pages 54–59, September/October 2021.
- [5] R. Coisson, Y. Skladanek, and D. Ferullo. Flap-lag Stability Analysis for High Speed Flight. In *Proceedings of the Vertical Flight Society's 6th Decennial Aeromechanics Specialists' Conference*, Santa Clara, CA, February 6-8 2024.
- [6] F.D. Harris. An Overview of Autogyros and The McDonnell XV-1 Convertiplane. Technical Report CR-2003-212799, NASA, October 2003.
- [7] F.D. Harris. Rotor Performance at High Advance Ratio; Theory versus Test. Technical Report CR-2008-215370, NASA, October 2008.
- [8] É. Mathieu. Memoir on the Vibratory Movement of an Elliptical Membrane (*published in French*). *Journal de mathématiques pures et appliquées 2e série*, 13:137–203, 1868.
- [9] G. Floquet. On Linear Differential Equations Periodic Coefficients (*published in French*). *Ann. Sci. Éc. Norm. Supér.*, 12:47–88, 1883.
- [10] G. W. Hill. On the Part of Motion of the Lunar Perigee which is Function of the Mean Motions of the Sun and Moon. *Acta Math.*, 8:1–36, 1886.
- [11] A. M. Lyapunov. *The General Problem of the Stability of Motion (published in Russian)*. Doctoral thesis, University of Kharkov, 1892.
- [12] H. K. Wilson. *Ordinary Differential Equations: Introductory and Intermediate Courses Using Matrix Methods*. Addison-Wesley Publishing Company, 1971.

- [13] V. A. Yakubovich and V. M. Starzhinskii. *Linear Differential Equations with Periodic Coefficients*. John Wiley & Sons, 1975.
- [14] M. Farkas. *Periodic Motions*, volume 104. Springer Science & Business Media, 1994.
- [15] J. Cronin. *Ordinary Differential Equations: Introduction and Qualitative Theory*. Chapman & Hall CRC Pure and Applied Mathematics, 2008.
- [16] G. Teschl. *Ordinary Differential Equations and Dynamical Systems*, volume 140. American Mathematical Society, 2012.
- [17] P. Friedmann, C. E. Hammond, and T. H. Woo. Efficient Numerical Treatment of Periodic Systems with Application to Stability Problems. *International Journal for Numerical Methods in Engineering*, 11(7):1117–1136, 1988.
- [18] J. Dugundji and J. H. Wendell. Some Analysis Methods for Rotating Systems with Periodic Coefficients. *AIAA Journal*, 21(6):890–897, 1981.
- [19] R. Seydel. *From Equilibrium to Chaos: Practical Bifurcation and Stability Analysis*. Elsevier Science Publishing Co., 1988.
- [20] W. Johnson. *Rotorcraft Aeromechanics*. Princeton University Press, Princeton, NJ, 2013.
- [21] J. Ledoux, S. Rizzo, and J. Salomon. Analysis of the Blade element momentum theory. *SIAM Journal on Applied Mathematics*, 81(6):2596–2621, 2021.
- [22] W.J.M. Rankine. On the Mechanical Principles of the Action of Propellers. *Transactions of the Institution of Naval Architects*, 6:13–30, 1865.
- [23] W. Froude. On the Elementary Relation between Pitch, Slip and Propulsive Efficiency.
- [24] C.N.H. Lock, H. Bateman, and H.C.H. Townend. *An Extension of the Vortex Theory of Airscrews with Applications to Airscrews of Small Pitch, Including Experimental Results*. A.R.C. Research Reports and Memoranda 1014, September 1925.
- [25] C. Seidel and D.A. Peters. How Big is a Lock Number? *Journal of the American Helicopter Society*, 66(1):1–3, January 2021.
- [26] H. Glauert. *The Elements of Aerofoil and Airscrew Theory*. Cambridge University Press, 1926.
- [27] H. Glauert and G. Shone. The Disturbed Motion of the Blades of a Gyroplane. *Aeronautical Research Council Paper*, 993, 1933.
- [28] G. Horvay. Rotor Blade Flapping Motion. *Quarterly of Applied Mathematics*, 5(2), Jul 1947.

- [29] H. Glauert. Airplane Propellers. *Aerodynamic Theory: A General Review of Progress Under a Grant of the Guggenheim Fund for the Promotion of Aeronautics*, pages 169–360, 1935.
- [30] L. Prandtl. *Four Treatises on Hydrodynamics and Aerodynamics (published in German)*. Aerodynamic Research Institute (AVA), University of Göttingen, 1927.
- [31] K. W. Mangler. Fourier Coefficients for the Downwash at a Helicopter Rotor. RAE Tech Note Aero 1058, May 1948.
- [32] T. Theodorsen. General Theory of Aerodynamic Instability and the Mechanism of Flutter. Technical Report Report 496, NACA, 1935.
- [33] R.G. Loewy. A Two-Dimensional Approximation to the Unsteady Aerodynamics of Rotary Wings. *Journal of the Aeronautical Sciences*, 24(2):81–92, February 1957.
- [34] C.C. Critzos, H.H. Heyson, and R.W. Boswinkle Jr. Aerodynamic Characteristics of NACA 0012 Airfoil Section at Angles of Attack from 0 deg to 180 deg. Technical Report TN3361, NASA, 1955.
- [35] J.L. Jenkins. Wind-tunnel Investigation of a Lifting Rotor Operating at Tip-speed Ratios from 0.65 to 1.45. Technical Report TN D-2628, NASA, 1965.
- [36] R. P. White. Instabilities Associated with a Rotor Blade Stopped in Flight. Washington, D.C., May 8-10 1968. Proceedings of the 24th Annual Forum of the American Helicopter Society.
- [37] J.L. Jenkins. A Numerical Method for Studying the Transient Blade Motions of Rotor with Flapping and Lead-Lag Degrees of Freedom. Technical Report TN D-4195, NASA, 1967.
- [38] P. J. Arcidiacono. Prediction of Rotor Instability at High Forward Speeds, Volume I: Steady Flight Differential Equations of Motion for a Flexible Helicopter Blade with Chordwise Mass Imbalance. Technical Report Technical Report 68-18A, US- AAVLABS, 1969.
- [39] C. J. Astill and C. F. Niebanck. Prediction of Rotor Instability at High Forward Speeds, Volume II: Classical Flutter. Technical Report Technical Report 68-18B, USAAVLABS, 1969.
- [40] F. O. Carta and C. F. Niebanck. Prediction of Rotor Instability at High Forward Speeds, Volume III: Stall Flutter. Technical Report Technical Report 68-18C, US- AAVLABS, 1969.
- [41] C. F. Niebanck and H. L. Elman. Prediction of Rotor Instability at High Forward Speeds, Volume IV: Torsional Divergence. Technical Report Technical Report 68-18D, USAAVLABS, 1969.

- [42] H. L. Elman and C. F. Niebanck. Prediction of Rotor Instability at High Forward Speeds, Volume V: Flapping and Flap-Lag Instability. Technical Report Technical Report 68-18E, USAAVLABS, 1969.
- [43] C. F. Niebanck and L. J. Bain. Rotor Aeroelastic Instability and Transient Characteristics. Technical Report Technical Report 69-88, USAAVLABS, 1970.
- [44] G.J. Sissingh. Dynamics of Rotors Operating at High Advance Ratios. *Journal of the American Helicopter Society*, 13(3):56–63, 1968.
- [45] D. A. Peters and K. H. Hohenemser. Application of the Floquet Transition Matrix to Problems of Lifting Rotor Stability. *Journal of the American Helicopter Society*, 16(2):25–33, Apr 1971.
- [46] W. Johnson. A Perturbation Solution of Rotor Flapping Stability. In *AIAA Atmospheric Flight Mechanics Conference*, Moffett Field, CA, September 11-14 1972.
- [47] J. C. Biggers. Some Approximations to the Flapping Stability of Helicopter Rotors. *Journal of the American Helicopter Society*, 19(4):23–33, Oct 1975.
- [48] K. H. Hohenemser and S. Yin. Some Applications of the Method of Multiblade Coordinates. *Journal of the American Helicopter Society*, 17(3):3–12, 1972.
- [49] M. Joglekar and R. Loewy. An Actuator-Disc Analysis of Helicopter Wake Geometry and the Corresponding Blade Response. Usaavlabs tr 69-66, 1970.
- [50] E. D. Bellinger. Analytical Investigation of the Effects of Blade Flexibility, Unsteady Aerodynamics, and Variable Inflow on Helicopter Rotor Stall Characteristics. *Journal of the American Helicopter Society*, 17(3):35–44, 1972.
- [51] R. A. Ormiston and D. H. Hodges. Linear Flap-Lag Dynamics of Hingeless Helicopter Rotor Blades in Hover. *Journal of the American Helicopter Society*, 17(2):2–14, April 1972.
- [52] R. A. Ormiston and W. G. Bousman. A Study of Stall-Induced Flap-Lag Instability of Hingeless Rotors. *Journal of the American Helicopter Society*, 20(1):20–30, January 1975.
- [53] D. A. Peters. Flap-Lag Stability of Helicopter Rotor Blades in Forward Flight. *Journal of the American Helicopter Society*, 20(4):2–13, October 1975.
- [54] K. R. V. Kaza and R. G. Kvaternick. Examination of the Flap-Lag Stability of Rigid Articulated Rotor Blades. *Journal of Aircraft*, 16(12):876–884, December 1979.
- [55] K. H. Hohenemser and R. A. Ormiston. Hingeless Rotorcraft Flight Dynamics. Technical Report AG-197, AGARD, 1974.
- [56] D.H. Hodges. A Theoretical Technique for Analyzing Aeroelastic Stability of Bearingless Rotors. *AIAA Journal*, 17(4):400–407, 1979.

- [57] D. A. Peters and A. P. Izadpanah. Helicopter Trim by Periodic Shooting with Newton-Raphson Iteration. In *Proceedings of the American Helicopter Society's 37th Annual Forum*, New Orleans, LA, May 1981.
- [58] R.A. Calico and W.E. Wiesel. Stabilization of helicopter blade flapping. *Journal of the American Helicopter Society*, 31(4):59–64, October 1986.
- [59] W. Johnson. Application of Unsteady Airfoil Theory to Rotary Wings. *Journal of Aircraft*, 17(4):285–286, 1980.
- [60] D. M. Pitt. Rotor Dynamic Inflow Derivatives and Time Constants from Various Inflow Models. USATSARCOM TR 81-2, December 1980.
- [61] D.M. Pitt and D.A. Peters. Theoretical Prediction of Dynamic-Inflow Derivatives. In *Paper No. 47, 6th European Rotorcraft Forum*, Bristol, England, U.K., September 1980.
- [62] D.M. Pitt and D.A. Peters. Theoretical Prediction of Dynamic-Inflow Derivatives. *Vertica*, 5(1), 1981.
- [63] G. H. Gaonkar and D. A. Peters. Effectiveness of Current Dynamic-Inflow Models in Hover and Forward Flight. *Journal of the American Helicopter Society*, 31(2):47–57, 1986.
- [64] G. H. Gaonkar, M. J. McNulty, and J. Nagabhushanam. An Experimental and Analytical Investigation of Isolated Rotor Flap-Lag Stability in Forward Flight. In *Paper No. 66, 11th European Rotorcraft Forum*, London, England, U.K., September 1985.
- [65] M. J. McNulty. Flap-Lag Stability Data for a Small-Scale Isolated Hingeless Rotor in Forward Flight. Technical Report TM-102189, NASA, 1989.
- [66] G. H. Gaonkar, M. J. McNulty, and J. Nagabhushanam. An Experimental and Analytical Investigation of Isolated Rotor Flap-Lag Stability in Forward Flight. *Journal of the American Helicopter Society*, 35(2):25–34, May 1990.
- [67] D. L. Sharpe. An Experimental Investigation of the Flap-Lag-Torsion Aeroelastic Stability of a Small-Scale Hingeless Helicopter Rotor in Hover. Technical Report TP-2546, NASA, 1986.
- [68] R. L. Peterson and W. Warmbrodt. Hover Test of a Full-Scale Hingeless Helicopter Rotor: Aeroelastic Stability, Performance and Loads Data. Technical Report TM-85892, NASA, 1984.
- [69] R.L. Peterson and W. Johnson. Aeroelastic Loads and Stability Investigation of a Full-Scale Hingeless Rotor. Technical Report TM-103867, NASA, 1991.
- [70] W. Johnson. Flap/ Lag/ Torsion Dynamics of a Uniform, Cantilever Rotor Blade in Hover. Technical Report TM-73248, NASA, 1977.

- [71] I. Chopra. Flap-Lag-Torsion Flutter Analysis of a Constant Lift Rotor. Technical Report CR-152244, NASA, 1979.
- [72] P. P. Friedmann and S. B. R. Kottapalli. Coupled Flap-Lag-Torsional Dynamics of Hingeless Rotor Blades in Forward Flight. *Journal of the American Helicopter Society*, 27(4):28–36, October 1982.
- [73] N. T. Sivaneri and I. Chopra. Dynamic Stability of a Rotor Blade using Finite Element Analysis. *AIAA Journal*, 20(5):716–723, 1983.
- [74] B. Panda and I. Chopra. Flap-Lag-Torsion Stability in Forward Flight. *Journal of the American Helicopter Society*, 30(10):30–39, October 1985.
- [75] B. Panda and I. Chopra. Dynamic Stability of Hingeless and Bearingless Rotors in Forward Flight. *Computers & Mathematics with Applications*, 12(1):111–130, 1986.
- [76] P. P. Friedmann. Recent Developments in Rotary-Wing Aeroelasticity. *Journal of Aircraft*, 14(11):1027–1041, 1977.
- [77] P. P. Friedmann. Rotary-Wing Aeroelasticity: Current Status and Future Trends. *AIAA Journal*, 42(10):1953–1972, 2004.
- [78] R. A. Ormiston, W. G. Warmbrodt, D. H. Hodges, and D. A. Peters. Survey of Army/NASA Rotorcraft Aeroelastic Stability Research. NASA TM 101026/USAAVSCOM-TR-88-A-005, October 1988.
- [79] I. Chopra. Perspectives in Aeromechanical Stability of Helicopter Rotors. *Vertica*, 14(4):457–508, 1990.
- [80] T.S.R. Reddy. Symbolic Generation of Elastic Rotor Blade Equations Using a FORTRAN Processor and Numerical Study on Dynamic Inflow Effects on the Stability of Helicopter Rotors. Technical Report TM-86750, NASA, 1986.
- [81] W. Johnson. A History of Rotorcraft Comprehensive Analyses. Technical Report TP-2012-216012, NASA, 2012.
- [82] J.G. Leishman and T.S. Beddoes. A Semi-Empirical Model for Dynamic Stall. *Journal of the American Helicopter Society*, 34(3):3–17, July 1989.
- [83] A. Bagai. Rotor Free-Wake Modeling Using a Relaxation Technique—Including Comparisons with Experimental Data. Washington, DC, May 11-13 1994. Proceedings of the 50th Annual Forum of the American Helicopter Society.
- [84] M.J. Bhagwat and J.G. Leishman. Stability, Consistency and Convergence of Time-Marching Free-Vortex Rotor Wake Algorithms. *Journal of the American Helicopter Society*, 46(1):59–71, 2001.
- [85] G. Bir, I. Chopra, and K. Nguyen. Development of UMARC (University of Maryland Advanced Rotorcraft Code). Washington, DC, May 1990. Proceedings of the 46th Annual Forum of the American Helicopter Society.

- [86] F.A. Tasker and I. Chopra. Assessment of Transient Analysis Techniques for Rotor Stability Testing. *Journal of the American Helicopter Society*, 35(1):39–50, January 1990.
- [87] D. A. Peters, S. M. Lieb, and L. A. Ahaus. Interpretation of Floquet Eigenvalues and Eigenvectors for Periodic Systems. *Journal of the American Helicopter Society*, 56(3):1–11, 2011.
- [88] J. H. Milgram and I. Chopra. Air Resonance of Hingeless Helicopter Rotor Helicopters in Trimmed Forward Flight. *Journal of the American Helicopter Society*, 29(4):46–58, 1994.
- [89] M. W. Floros and W. Johnson. Stability and Control Analysis of the Slowed-Rotor Compound Helicopter Configuration. *Journal of the American Helicopter Society*, 52(3):239–253, July 2007.
- [90] A. Datta, H. Yeo, and T. Norman. Experimental Investigation and Fundamental Understanding of a Full-Scale Slowed Rotor at High Advance Ratios. *Journal of the American Helicopter Society*, 58(2):1–17, 2013.
- [91] G. M. Bowen-Davies and I. Chopra. Performance and Loads Predictions of a Mach-Scaled Rotor at High Advance Ratios. *Journal of the American Helicopter Society*, 61(3):1–11, 2016.
- [92] X. Wang, A. Bauknecht, S. Maurya, and I. Chopra. Slowed Hingeless Rotor Wind Tunnel Tests and Validation at High Advance Ratios. *Journal of Aircraft*, 58(1):153–166, 2021.
- [93] U. Saetti and M. Lovera. Time-Periodic and High-Order Time-Invariant Linearized Models of Rotorcraft: A Survey. *Journal of the American Helicopter Society*, 67(1):1–19, 2022.
- [94] M.J.S. Lopez and J.V.R. Prasad. Linear Time Invariant Approximations of Linear Time Periodic Systems. *Journal of the American Helicopter Society*, 62(1):1–10, January 2017.
- [95] U. Saetti and J.F. Horn. Linear Time-Invariant Approximations of Nonlinear Time-Periodic Systems. *Journal of the American Helicopter Society*, 68(1):1–10, 2023.
- [96] R. Mohan and G. H. Gaonkar. A Unified Assessment of Fast Floquet, Generalized Floquet, and Periodic Eigenvector Methods for Rotorcraft Stability Predictions. *Journal of the American Helicopter Society*, 58(4):1–12, 2013.
- [97] S.M. Fishman and I. Chopra. Analysis of Flap-Lag Aeroelastic Stability at High Advance Ratios. West Palm Beach, FL, May 16-18 2023. Proceedings of the Vertical Flight Society’s 79th Annual Forum & Technology Display.

- [98] S.M. Fishman and I. Chopra. Analysis of Flap-Lag-Torsion Aeroelastic Stability at High Advance Ratios. Montreal, Quebec, Canada, May 7-9 2024. Proceedings of the Vertical Flight Society's 80th Annual Forum & Technology Display.
- [99] P. P. Friedmann, G. A. Lesieutre, and D. Huang. *Structural Dynamics*, volume 50. Cambridge University Press, 2023.
- [100] L. F. Shampine and M. W. Reichelt. The MATLAB ODE Suite. *SIAM Journal on Scientific Computing*, 18(1):1–22, 1997.
- [101] Mike Croucher. The MATLAB Blog - The new solution framework for Ordinary Differential Equations (ODEs) in MATLAB R2023b. <https://blogs.mathworks.com/matlab/2023/10/03/the-new-solution-framework-for-ordinary-differential-equations-odes-in-matlab-r2023b>. Published October 3 2023, accessed via web.archive.org.
- [102] MathWorks MATLAB Documentation. ode45: Solve nonstiff differential equations - medium order method. <https://www.mathworks.com/help/matlab/ref/ode45.html>. Archived 25 September 2012, accessed via web.archive.org).
- [103] C. Moler. Cleve's Corner: Cleve Moler on Mathematics and Computing - Ordinary Differential Equation Suite. <https://blogs.mathworks.com/cleve/2014/05/12/ordinary-differential-equation-suite>. Published May 12 2014, archived May 18 2014, accessed via web.archive.org.
- [104] MathWorks Support Team. Is there a fixed-step Ordinary Differential Equation (ODE) solver in MATLAB 8.0 (R2012b)? <https://www.mathworks.com/matlabcentral/answers/98293-is-there-a-fixed-step-ordinary-differential-equation-ode-solver-in-matlab-8-0-r2012b>. Published August 29 2019, accessed July 24 2023.
- [105] J. D'Errico. Eigenshuffle Version 2.0.0.0. <https://www.mathworks.com/matlabcentral/fileexchange/22885-eigenshuffle>. Published June 5 2020, accessed: October 24 2022.
- [106] Gilbert G. Strang and G. Fix. *An Analysis of the Finite Element Method, Second Edition*. Wellesley-Cambridge Press, 2008.
- [107] G. B. Folland. *Fourier Analysis and its Applications*, volume 4. American Mathematical Society, 2009.
- [108] C. E. Shannon. Communication in the Presence of Noise. *Proceedings of the Institute of Radio Engineers*, 37(1):10–21, January 1949. Reprinted in *Proceedings of the IEEE*, 86(2):447–457, February 1998.