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ABSTRACT

This paper is concerned with strong approximation in queueing networks. A model of a circuit-switched network with fixed routes is considered in the limiting regime where the link capacities and the offered traffic are increased at the same rate. The process of normalized queue lengths is shown to converge almost surely to a sliding mode solution of an ordinary differential equation. The solution is shown to posess a unique stable point. It is reached exponentially fast or in finite time, depending on the values of the parameters. This has implications on the settling time of the network. The technique is applicable to closed Jackson networks and their settling times. In contrast with other asymptotic results on queueing networks it does not make use of product form distributions and extends easily to non-Markovian models.

Key Words: Asymptotic methods; circuit switching; Jackson networks; sliding mode; strong approximation.

1. Introduction

The difficulty of computing the normalizing constant of product form distributions of networks that are moderately large has recently lead to the investigation of asymptotic methods. Pittel [6] and McKenna and Mitra [5] consider classes of closed Jackson networks, and Kelly [3] investigates an asymptotic regime for a product form model of a circuit switched network. The methods used in [5], [6] and [3] rely heavily on the product form of the stationary distribution of these networks. The aim of this paper is to show that standard strong approximation methods can be used to study the asymptotic regime in [3]. The same methods can be applied to other networks such as the ones in [5] and [6]. Our methods lead to approximations for the transient and the stationary behavior of these networks and do not rely on the product form of the stationary distribution.

We describe a special case of the model in [3]. The general case will be discussed elsewhere. The network consists of a graph whose edges represent communication links between the nodes. Link $j \in \{1,\ldots,J\}$ comprises C_j circuits. A route in the network is a simple path in the graph that establishes a connection between its endpoints. Assume that there are M possible routes and denote their set by R. Calls requesting route $r \in R$ form a Poisson process with rate ν_r and each such call uses $A_{jr} \in \{0,1\}$ circuits from link j. Let A denote the matrix (A_{jr}). If there are less than A_{jr} calls available on any link $j \in r$ the call is lost. Call arrivals for different routes are independent. Each successful call holds its circuits for a period of time which is exponentially distributed with mean μ_r and is independent of earlier arrival and holding times. Denote by $y_r(t)$ the number of calls in progress on route $r \in R$ at time $t \geq 0$. We will consider the asymptotic regime where the arrival rates of the calls and the link capacities increase to infinity at the same rate, i.e., the parameters $\nu = (\nu_r,\ r \in R)^T$, $\mathbf{C} = (C_j,\ j = 1,\ldots,J)^T$ are replaced by $\nu^N = (\nu_r^N,\ r \in R)^T$, $\mathbf{C}^N = (C_j^N,\ j = 1,\ldots,J)^T$, where

$$\begin{aligned} \lim_{N\to\infty} \frac{1}{N}\nu_r^N &= \nu_r,\ r \in R \\ \lim_{N\to\infty} \frac{1}{N}C_j^N &= C_j,\ j \in \{1,\ldots,J\} \end{aligned} \tag{1.1}$$

For a link j define the set of routes sharing that link,

$$R_j = \{r \in R \mid j \in r\}.$$

The vector process $\mathbf{y^N}(t) = (y_r^N(t),\ r \in R)^T$ satisfies the following stochastic integral equations, for each $N = 1,2,\ldots$.

$$\begin{aligned} y_r^N(t) = y_r^N(0) &+ Y_r^a\left(\nu_r^N \int_0^t \prod_{j\in r} 1\{A_{jr} + \sum_{q\in R_j} A_{jq}y_q^N(s) \leq C_j^N\}ds\right) \\ &- Y_r^d\left(\mu_r \int_0^t y_r^N(s)ds\right), \end{aligned} \tag{1.2}$$

for all $r \in R$, $t \geq 0$, where $Y_r^a(\cdot)$, $Y_r^d(\cdot)_{r\in R}$ are independent Poisson processes with unit rate.

As in Kurtz [4], by setting

$$\mathbf{z}^N(t) = \frac{1}{N}\mathbf{y^N}(t),$$

(1.2) becomes

$$\begin{aligned} z_r^N(t) = z_r^N(0) &+ \frac{1}{N}Y_r^a\left(\nu_r^N \int_0^t \prod_{j\in r} 1\{\frac{A_{jr}}{N} + \sum_{q\in R_j} A_{jq}z_q^N(s) \leq \frac{C_j^N}{N}\}ds\right) \\ &- \frac{1}{N}Y_r^d\left(N\mu_r \int_0^t z_r^N(s)ds\right). \end{aligned} \tag{1.3}$$

To proceed as in [4] one considers the limit equation (see Theorem 2.1),

$$z_r(t) = z_r(0) + \nu_r \int_0^t \prod_{j \in r} 1\{\sum_{q \in R_j} A_{jq} z_q(s) \le C_j\} ds - \mu_r \int_0^t z_r(s) ds, \; r \in R. \tag{1.4}$$

Note that the integrands on the right hand side of (1.4) are discontinuous and a solution in the usual sense may fail to exist. We will consider solutions of (1.4) in a wider sense, allowing $(\mathbf{z}(t))$ to be non-differentiable at a finite number of times but retaining continuity. This gives rise to so-called "sliding mode" solutions. It is shown that the solution of (1.3) converges to such a solution of (1.4) almost surely on finite time intervals.

The results in Kurtz [4] are not directly applicable because the right hand side of (1.4) fails to be continuous. Instead, in Section 2.1, the results in [4] are applied to simple perturbations of (1.4) which are given in (2.3) and (2.5). The limit of the perturbed processes, given in (2.4) and (2.6) is seen to approximate a "sliding mode" solution of (1.3), given by (2.2), (2.7) and (2.8) in Section 2.2. The corresponding deterministic limit is simpler in the closed Jackson networks considered in [5] and [6]. In Section 3 it is seen that this "sliding mode" solution has a unique stable point. Weak convergence of the stationary measure of $(\mathbf{z}^N(t))$ to this stable point follows from standard results in Ethier and Kurtz [2]. The method provides information on the settling times of networks. The corresponding results imply some of the results in Anantharam [1]. Furthermore, our results can be extended to non-Markovian models.

Of particular interest are non-product form networks where the "sliding mode" solution of the deterministic limit equation has multiple stable points. Such a situation will be considered in a future paper.

2. Limits

2.1 Approximation

We will henceforth set $\mu_r = 1$ in (1.3) and (1.4). This leads to solutions of (1.4) that are piecewise straight lines. The case of arbitrary μ's and of an arbitrary vector field can be handled by modifying the argument below. However, the description of the "sliding mode" solution of (1.4) becomes more complicated.

Equation (1.3) is rewritten as

$$\begin{aligned} z_r^N(t) = z_r^N(0) &+ \frac{1}{N} Y_r^a \left(\nu_r^N \int_0^t f_r^N(\mathbf{z}^N(s)) ds \right) \\ &- \frac{1}{N} Y_r^a \left(N \int_0^t z_r^N(s) ds \right). \end{aligned} \tag{2.1}$$

where we have set ,

$$f_r^N(\mathbf{z}) = \prod_{j \in r} 1\{A_{jr} + \sum_{q \in R_j} A_{jq} z_q(s) \le \frac{C_j^N}{N}\}, \quad r = 1, \ldots, M.$$

We now proceed to describe component-wise lower bounds for the process $(\mathbf{z}^N(t))_{t \ge 0}$.

Notation

(a) Denote the admissible set by $\mathcal{A} = \{\mathbf{z} \ge 0 \mid \mathrm{A}\mathbf{z} \le \mathrm{C}\}$.

(b) The relative boundary and the relative interior of a set K are denoted by ∂K and $IntK$ respectively. Set $\tau_K = \inf\{t > 0 \mid \mathbf{z}(t) \in K\}$.

(c) A set of indices $\{i_1, \ldots, i_l\}$ is denoted by $\mathbf{i}_l$ and $\mathbf{i}_l \mathbf{j}_m = \mathbf{i}_l \cup \mathbf{j}_l$.

(d) For $i_1, \ldots, i_l$ in $\{1, \ldots, J\}$, we will find useful to consider the subsets of intersections of hyperplanes

$$H_{\mathbf{i}_l} = \{\mathbf{z} \in \partial\mathcal{A} \mid \mathbf{a}_k^T \cdot \mathbf{z} = C_k, \; k \in \mathbf{i}_l; \; \mathbf{a}_k^T \cdot \mathbf{z} < C_k, \; k \notin \mathbf{i}_l\}$$

(e) For $j, \in \{1, \dots, J\}$, the sets of routes

$$R_{\mathbf{i}_l} = \bigcup_{k=1}^{l} R_{i_k}$$

$$R_{j \backslash \mathbf{i}_l} = R_j \backslash R_{\mathbf{i}_l}$$

will be repeatedly considered. The quantities $\mathcal{A}^N$, τ_K^N and $H_{\mathbf{i}_l}^N$, are defined in the obvious way.

Lower bounds will be developed so that the limit of the bounding process can be identified from standard strong approximation results. For simplicity, fix $\mathbf{z}^N(0) = \mathbf{z}^0$, a constant. We need to distinguish between three cases.

(i) $\mathbf{z}^0 \in Int\mathcal{A}^N$

The limit of $(\mathbf{z}^N(t))$ as $N \to \infty$ is directly identified as the solution of the differential equation

$$\mathbf{z}(t) = \nu - \mathbf{z}(t); \quad t \leq \tau_{\partial\mathcal{A}}; \quad \mathbf{z}(0) = \mathbf{z}^0 \qquad (2.2).$$

One has from [4], Theorem 2.1, and for $T \geq 0$,

$$\sup\{\|\mathbf{z}^N(t) - \mathbf{z}(t)\| \mid t \leq \tau_{\partial\mathcal{A}} \wedge T\} \xrightarrow{N\to\infty} 0, \text{ a.s.}$$

(ii) $\mathbf{z}^0 \in H_i^N$

For $\epsilon > 0$, set $I_i^\epsilon(\mathbf{z}(t)) = \operatorname{diag}\left(1\{\mathbf{z}(u) \in \partial\mathcal{A}^N,\ u \in [t-\epsilon, t],\ r \in R_i\},\quad r = 1, \dots, M\right)$. A componentwise lower bound can be obtained as the solution of equations

$$\begin{aligned} z_r^{N,\epsilon}(t) = z_r^0 &+ \frac{1}{N}\tilde{Y}_r^a\left(\nu_r^N \int_0^t f_r^N(\mathbf{z}^{N,\epsilon}(s)) I_{i,r}^\epsilon(\mathbf{z}^{N,\epsilon}(s))ds\right) \\ &- \frac{1}{N}\tilde{Y}_r^d\left(N \int_0^t z_r^{N,\epsilon}(s)ds\right), \quad r = 1, \dots, M, \end{aligned} \qquad (2.3)$$

where $\tilde{Y}_r^a(\cdot)$ and $\tilde{Y}_r^d(\cdot)$ are independent Poisson processes with unit rate and are chosen so that, almost surely, as $N \to \infty$, $\mathbf{z}^N(t) \geq \mathbf{z}^{N,\epsilon}(t)$, a.s. for $t \leq \tau_{H_i^N}^\epsilon$.

From the strong law of large numbers one has $Y(Nu)/N \approx u$, almost surely, and the limiting process can be seen to be the sawtooth-like solution of the differential equation

$$\dot{\mathbf{z}}^\epsilon(t) = \nu I_i^\epsilon(\mathbf{z}^\epsilon(t)) - \mathbf{z}^\epsilon(t), \quad \mathbf{z}^\epsilon(0) = \mathbf{z}^0, \quad t \leq \tau_{H_i}^\epsilon, \qquad (2.4)$$

where $\tau_{H_i}^\epsilon$ is defined in the obvious way. Recall that we allow a finite number of non-differentiable points in the solution of a differential equation. Indeed, one has, for $T \geq 0$,

$$\sup\left\{\|\mathbf{z}^{N,\epsilon}(t) - \mathbf{z}^\epsilon(t)\| \mid t \leq \tau_{H_i}^\epsilon \wedge T\right\} \xrightarrow{N\to\infty} 0, \quad \text{a.s.}$$

(iii) $\mathbf{z}^0 \in H_{\mathbf{i}_l}^N$

For $\epsilon > 0$, set $I_{\mathbf{i}_l}^\epsilon(\mathbf{z}(t)) = \operatorname{diag}(1\{\mathbf{z}(u) \in [t-\epsilon, t],\ r \in R_{i_l \backslash \mathbf{i}_{l-1}}\},\quad r = 1, \dots, M)$. As in the previous case, an asumptotic lower bound can be obtained that satisfies the equation

$$\begin{aligned} z_r^{N,\epsilon}(t) = z_r^0 &+ \frac{1}{N}\tilde{Y}_a^r\left(\nu_r^N \int_0^t f_r^N(\mathbf{z}^{N,\epsilon}(s)) I_{i,r}^\epsilon(\mathbf{z}^{N,\epsilon}(s))ds\right) \\ &- \frac{1}{N}\tilde{Y}_d^r\left(N \int_0^t z_r^{N,\epsilon}(s)ds\right), \quad r = 1, \dots, M, \end{aligned} \qquad (2.5)$$

where, again, $\tilde{Y}_r^a(\cdot)$ and $\tilde{Y}_r^d(\cdot)$ are independent Poisson processes with unit rate and are chosen so that $\mathbf{z}^N(t) \geq \mathbf{z}^{N,\epsilon}(t)$, a.s., for $t \leq \tau_{H_{\mathrm{I}_\mathrm{t}}}^{N,\epsilon}$. The limit process is the solution to the equation

$$\dot{\mathbf{z}}^\epsilon(t) = \nu I_{\mathrm{I}_\mathrm{t}}^\epsilon(\mathbf{z}^\epsilon(t)) - \mathbf{z}^\epsilon(t), \quad \mathbf{z}^\epsilon(0) = \mathbf{z}^0, \quad t \leq \tau_{H_{\mathrm{I}_\mathrm{t}}}^\epsilon, \tag{2.6}$$

and one obtains

$$\sup\left\{ \|\mathbf{z}^{N,\epsilon}(t) - \mathbf{z}^\epsilon(t)\| \mid t \leq \tau_{H_{\mathrm{I}_\mathrm{t}}}^\epsilon \wedge T \right\} \xrightarrow{N\to\infty} 0, \quad \text{a.s.}$$

2.2. The limit equation

We next examine the behavior of $(\mathbf{z}^\epsilon(t))$, defined by (2.4) and (2.6), as $\epsilon \to 0$. The case where $\nu \in Int\mathcal{A}$ is of no interest because then one obtains $\tau_{\partial\mathcal{A}} = \infty$ in (2.2). Again, we need to consider the same cases as above. Case (ii) is included below for clarity. In the sequel, certain assumptions are progressively imposed. They ensure that $(\mathbf{z}(t))$ will evolve in the corresponding intersection of hyperplanes for a positive time. No loss of generality results if the stated assumptions do not hold. In that case, $(\mathbf{z}(t))$ evolves in the intersection of fewer hyperplanes. See also the procedure of Section 3.

(i) $\mathbf{z}^0 \in H_i$

For $t \leq \epsilon$ one has

$$\dot{\mathbf{z}}^\epsilon(t) = -\mathbf{z}^\epsilon(t) + (0,\ r \in R_i,\ \nu_r,\ r \in R_i^c)^T,$$

and assuming that $\mathbf{a}_i^T \cdot \nu - C_i > 0$, one sets

$$B_i \stackrel{\text{def}}{=} \frac{\mathbf{a}_i^T \cdot \nu}{\mathbf{a}_i^T \cdot \nu - C_i} > 1.$$

Quantities $\tau_{\partial\mathcal{A}}^\epsilon = \tau_{H_i}^\epsilon$ and $\mathbf{z}^\epsilon(\tau_{H_i}^\epsilon)$ are calculated as

$$\tau_{\partial\mathcal{A}}^\epsilon = \tau_{H_i}^\epsilon = B_i\epsilon + o(\epsilon) \text{ and}$$

$$\mathbf{z}^\epsilon(\tau_{H_i}^\epsilon) = \mathbf{z}^0 + \tau_{H_i}^\epsilon \left\{\nu^i - \mathbf{z}^0\right\} + o(\epsilon), \text{ where,}$$

$$\nu^i \stackrel{\text{def}}{=} \nu - \frac{1}{B_i}(\nu_r,\ r \in R_i; 0,\ r \in R_i^c)^T. \tag{2.7}$$

From this, one concludes that $(\mathbf{z}^\epsilon(t))$ converges uniformly to $(\mathbf{z}(t))$ where

$$\dot{\mathbf{z}}(t) = \nu^i - \mathbf{z}(t), \quad \mathbf{z}(0) = \mathbf{z}^0, \quad 0 \leq t \leq \tau_{H_i^c}.$$

If $\mathbf{a}_i^T \cdot \nu - C_i \leq 0$, then $(\mathbf{z}^\epsilon(t))$ will never return to H_i and $(\mathbf{z}(t))$ will obey equation (2.2) for $t \leq \tau_{\partial\mathcal{A}}$.

(ii) $\mathbf{z}^0 \in H_{ij}$

It is assumed that $\mathbf{a}_i^T \cdot \nu \geq C_i$, $\mathbf{a}_j^T \cdot \nu \geq C_j$, $B_i < B_j$, and that $\mathbf{a}_j^T \cdot \nu^i - C_j > 0$. This last assumption implies that $\mathbf{a}_j^T \cdot \mathbf{z}^\epsilon(t) < C_j$ for $0 < t \leq \tau_{H_{ij}}^\epsilon$. Some algebra shows that

$$\tau_{H_i^c}^\epsilon = \tau_{H_{ij}}^\epsilon = B_{ij}\epsilon + o(\epsilon), \quad \text{where}$$

$$B_{ij} \stackrel{\text{def}}{=} \frac{\mathbf{a}_j^T \cdot \left(\nu_r,\ r \in R_{j\setminus i};\ 0,\ r \in R_{j\setminus i}^c\right)^T}{\mathbf{a}_j^T \cdot \nu^i - C_j}.$$

Similarly to the above case, one verifies that

$$\mathbf{z}^\epsilon(\tau_{H_{ij}}^\epsilon) = \mathbf{z}^0 + \tau_{H_{ij}}^\epsilon \left\{\nu^{ij} - \mathbf{z}^0\right\} + o(\epsilon), \quad \text{where}$$

$$\nu^{ij} \stackrel{\text{def}}{=} \nu^i - \frac{1}{B_{ij}} \left(\nu_r,\ r \in R_{j\backslash i}; 0,\ r \in R^c_{j\backslash i}\right)^T.$$

From this, one concludes that $(\mathbf{z}^\epsilon(t))$ converges uniformly to $(\mathbf{z}(t))$ where

$$\dot{\mathbf{z}}(t) = \nu^{ij} - \mathbf{z}(t), \qquad \mathbf{z}(0) = \mathbf{z}^0, \quad 0 \le t \le \tau_{H^c_{ij}}.$$

(iii) $\mathbf{z}^0 \in H_{\mathbf{i}_l}$

Motivated by the above considerations one defines inductively, for $l = 2, \ldots, J$,

$$B_{\mathbf{i}_l} \stackrel{\text{def}}{=} \frac{\mathbf{a}^T_{i_l} \cdot \left(\nu_r,\ r \in R_{i_l \backslash \mathbf{i}_{l-1}};\ 0,\ r \in R^c_{i_l \backslash \mathbf{i}_{l-1}}\right)^T}{\mathbf{a}^T_{i_l} \cdot \nu^{\mathbf{i}_{l-1}} - C_{i_l}}.$$

$$\nu^{\mathbf{i}_l} \stackrel{\text{def}}{=} \nu^{\mathbf{i}_{l-1}} - \frac{1}{B_{\mathbf{i}_l}} \left(\nu_r,\ r \in R_{i_l \backslash \mathbf{i}_{l-1}}; 0,\ r \in R^c_{i_l \backslash \mathbf{i}_{l-1}}\right)^T.$$

To ensure that the limit solution will evolve in $H_{\mathbf{i}_l}$ for some positive time one assumes,

$$\mathbf{a}^T_{i_k} \cdot \nu > C_{i_k}\ k = 1, \ldots, l; \quad B_{i_1} = \min\{B_{i_k} \mid k = 1, \ldots, l\}$$

$$\mathbf{a}^T_{i_k} \cdot \nu^{i_1} > C_{i_k}\ k = 2, \ldots, l; \quad B_{i_1, i_2} = \min\{B_{i_1, i_k} \mid k = 2, \ldots, l\}$$

$$\vdots$$

$$\mathbf{a}^T_{i_l} \cdot \nu^{\mathbf{i}_{l-1}} > C_{i_l}.$$

Then, it can be verified as in the previous cases that $(\mathbf{z}^\epsilon(t))$ converges uniformly to $(\mathbf{z}(t))$ where

$$\dot{\mathbf{z}}(t) = \nu^{\mathbf{i}_l} - \mathbf{z}(t), \qquad \mathbf{z}(0) = \mathbf{z}^0, \quad 0 \le t \le \tau_{H^c_{\mathbf{i}_l}} \tag{2.8}$$

It remains to show that the process $(\mathbf{z}^N(t))$ converges to $(\mathbf{z}(t))$ as $N \to \infty$. To this end note that for some $K > 0$ and for $t \ge 0$,

$$z_r(t) \ge z^\epsilon_r(t) \ge z_r(t) - K\epsilon + o(\epsilon), \quad r = 1, \ldots, M.$$

Then, since $z^N_r(t) \ge z^{N,\epsilon}_r(t)$ for all $r = 1, \ldots, M$, the limit results of Section 2.1 imply that

$$\limsup_{N \to \infty}\ \sup_{t \le T} \|\mathbf{z}(t) - \mathbf{z}^N(t)\| \le K\epsilon + o(\epsilon), \quad \text{a.s.}$$

By piecing together the above cases the following is established.

Theorem 2.1: For $T \ge 0$, one has

$$\lim_{N \to \infty} \sup_{t \le T} \|\mathbf{z}(t) - \mathbf{z}^N(t)\| = 0,$$

where $(\mathbf{z}(t))$ is determined by equations (2.2), (2.7) and (2.8).

3. Stability of the limit trajectory

In this section we study the behavior of $(\mathbf{z}(t))$, defined in (2.2), (2.7) and (2.8), as $t \to \infty$. It will be proved that if $\nu \notin Int\mathcal{A}$ then there exists a unique stable point on $\partial\mathcal{A}$. Depending on the location of the stable point, convergence is exponential or can be achieved in finite time. This is not the case for the closed Jackson network considered in Anantharam [1]. There, convergence is always achieved in finite time.

Given an intersection of hyperplanes we now present a procedure which determines the intersection of hyperplanes in which $(\mathbf{z}(t))$ will evolve. Specifically, for $\mathbf{z}^0 \in H_{\mathbf{j}_m} \neq \emptyset$, this procedure determines the set of indices $I(\mathbf{j}_m) \stackrel{\text{def}}{=} \mathbf{i}_l \subset \mathbf{j}_m$. Thus, by computing $\nu^{\mathbf{i}_l}$, one determines the intersection of hyperplanes on which $(\mathbf{z}(t))$ will evolve for some time interval of positive length. It is assumed that the indices $\{i_1, \ldots, i_l\}$ below are defined uniquely. Simple modifications of the procedure are needed for the general case.

Procedure

Step 0: Set $J_0 = \mathbf{j}_m \backslash \{j \in \mathbf{j}_m \mid \mathbf{a}_j^T \cdot \nu \leq C_j\}$. If $J_0 = \emptyset$, then $I(\mathbf{j}_m) = \emptyset$ and $(\mathbf{z}(t))$ satisfies (2.2); else proceed to Step 1.

Step 1: Set $i_1 = \operatorname{argmin}\{B_j \mid j \in J_0\}$ and $J_1 = J_0 \backslash \{j \in J_0 \mid \mathbf{a}_j^T \cdot \nu^{i_1} \leq C_j\}$. If $J_1 = \emptyset$, then $I(\mathbf{j}_m) = \{i_1\}$ and $(\mathbf{z}(t))$ satisfies (2.7) with $i = i_1$; else proceed to Step 2.

Step l: Set $i_n = \operatorname{argmin}\{B_j \mid j \in J_{n-1}\}$ and $J_n = J_{n-1} \backslash \{j \in J_{n-1} \mid \mathbf{a}_j^T \cdot \nu^{\mathbf{i}_n} \leq C_j\}$. If $J_n = \emptyset$, then $I(\mathbf{j}_m) = \{i_1, \ldots, i_n\}$ and $(\mathbf{z}(t))$ satisfies (2.8); else proceed to Step $l+1$.

The procedure terminates in at most m steps.

From equations (2.2), (2.7) and (2.8) one sees that $\dot{\mathbf{z}}(t) = 0$ iff $\mathbf{z}(t) = \nu^{\mathbf{i}_n}$ for some set of indices $\mathbf{i}_n$ produced by the above procedure and such that $\nu^{\mathbf{i}_n} \in \mathcal{A}$. By following the trajectory of $(\mathbf{z}(t))$ for $\mathbf{z}^0 = 0$ one verifies that $\mathbf{z}(t) \xrightarrow{t \to \infty} \nu^{\mathbf{i}}$ where $\mathbf{i} \stackrel{\text{def}}{=} I(1, \ldots, M)$. It can be checked that $\nu^{\mathbf{i}} \in \mathcal{A}$. Furthermore, it is proven next that $\nu^{\mathbf{i}}$ is the unique such point in $\mathcal{A}$.

Theorem 3.1: For all index sets $\mathbf{i}_l$ produced by the Procedure,

$$\nu^{\mathbf{i}_l} \in \mathcal{A} \text{ iff } \mathbf{i}_l = I(1, \ldots, M).$$

The only if part of the theorem is a consequence of the following.

Lemma 3.1: Let $\mathbf{i}_l = \{i_1, \ldots, i_l\}$ and $\mathbf{j}_m = \{j_1, \ldots, j_m\}$ be sets of indices in $\{1, \ldots, M\}$ such that $\mathbf{i}_l\mathbf{j}_m$ is produced by the Procedure. If for some $k \notin \mathbf{i}_l\mathbf{j}_m$,

$$B_{\mathbf{i}_l k} < B_{\mathbf{i}_l j_1}, \text{ then } \mathbf{a}_k^T \cdot \nu^{\mathbf{i}_l\mathbf{j}_m} > C_k.$$

The proof is by induction using the definitions of Section 2.1.

4. References

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