GRAVITATIONAL RADIATION DETECTION

BY

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ABSTRACT

This dissertation studies resonant gravitational wave detectors and related data analysis.

Different forms (strain amplitude) of the equation of motion for a medium responding to a gravitational wave are discussed in relation to the detection of such waves.

Utilizing "Bayesian techniques" an optimal method for data analysis is developed. Noise and filter theory is reviewed. It is seen that the "Bayesian techniques" integrates filter theory and data analysis, providing both filter properties and optimal methods for integrating the data. (In particular the method leads to a non threshold type of analysis, and "looks for" correlation between two detectors without the use of time delay).

Expressions for optimal sensitivity (and filters) of detector systems are given, including the limit of perfect sensors and electronics. The signal to noise ratio in terms of the spectral power of the gravitational radiation is derived. Long baseline interferometry is discussed.

A computer program simulating a pair of Weber type detectors is developed to study different approaches to data analysis.

CHAPTER 1. THE ANTENNA

CHAPTER 2. CONCEPTS OF PROBABILITY

CHAPTER 3. NOISE AND FILTERS

CHAPTER 4. COMPUTER SIMULATION OF A WEBER TYPE DETECTOR SYSTEM

INTRODUCTION

When Joseph Weber in 1969-1970 announced that coincident exitations were observed in his two gravitational wave detectors--one at the University of Maryland and the other at Argonne National Laboratory-- it came as a surprise to most scientists. Probably no one except Weber had even contemplated the possibility of such experiments. The results generated enormous interest, and in retrospect one may state that a new field of research was opened. The birth of this new field has not been without pain or difficulties however. It was realized that the amount of energy radiated at the source (assumed to be at the center of the Galaxy) would have to be enormous, assuming that (1) the theory of General Relativity is correct, and (2) that the excitations were of gravitational origin.

In fact, unless the strength of the radiation could be explained by "beaming" or some other energy saving effect, under the above assumptions most astronomical theories would have to be discareded. In time the difficulties mounted when other groups started to announce negative results from similar experiments.

Even if it should turn out that the origin of Weber's coincidences is other than gravitational, it seems quite possible that continued research efforts will in the not too distant future lead to the detection of gravitational radiation. It has been realized that with the possibilities of todays techniques, an increase in sensitivity of many orders of magnitude can be achieved if one is willing to make the effort. William M. Fairbank has estimated that his $3\ 10^{-3}$ °K detector ultimately will be, 10^{6} times more sensitive than Weber's (1972) detector, and able to detect supernovae from the nearest 1000 galaxies, which means his antenna should be able to pick up an average of 30 events per year.

In the following discussion some conventions are useful. We shall refer to a "bare" detector as just the antenna (e.g. Webers aluminum cylinder) without sensors and amplifiers. In referring to the sensitivity etc. of a "bare" detector, we mean the sensitivity such a detector would have if equipped with perfect (noiseless) sensors and amplifiers. Further on, we shall refer to a "dressed" detector as the antenna plus realistic sensors and amplifiers. The sensors and amplifiers alone shall be described as the "dress".

If it should be possible to construct a massive (on the order of a ton or more) low temperature detector of a single crystal of quartz or sapphire, a considerable increase in sensitivity can be obtained because of the high mechanical Q of such crystals. Presently, the highest measured Q for sapphire at room temperature is 10^9, while it has been estimated that a Q as large as 10^{18} might be attainable. Furthermore, it is not unlikely (judging from reports on experiments with quartz and other materials) that the Q will increase markedly for low temperatures. The "bare" detector sensitivity is linear in Q^{-1} but the actual improvement depends critically on the "dress." Thus if the dress can be improved to a corresponding degree, it is not unrealistic to envision low temperature sapphire detector sensitivities 10^3, or more times that of Fairbank's millidegree detector. In spite of present difficulties, one may thus consider the future of gravitational radiation detection to be bright.

Indeed, if the envisioned improvements are even partially realized and long baseline interferometry techniques are employed, a new dimension will be added to observational astronomy. The ability to "look behind the scenes" of astronomical events with gravitational radiation detectors may well surpass that based on all other methods (except possibly neutrino detection). Gravitational radiation, once created, can travel practically without loss through almost any type of matter. The experimental importance of highly sensitive gravitational radiation detectors is not, of course, limited to astrophysical studies. Such a device could be an important tool in studying the gravitational field itself. A controlled emission-detection (Hertz) experiment would provide information most valuable in distinguishing among various gravitational theories.

Before we go into the topics of this dissertation, it is useful to introduce some notational conventions. The pair (x,y) will denote the raw filtered corotating output amplitudes (see e.g. equation (1.17), and $(\bar{x},\bar{y})$ will denote the corresponding (smoothly) filtered variables. If the filter is of the special type that is matched to a delta function signal input, we use a dot notation i.e. $(\dot{x},\dot{y})$, since this filter will in most cases approximate a differentiation. In the litterature P usually labels a quantity which is proportional to $x^2 + y^2$, and stands for power. This may be a bit confusing since $x^2 + y^2$ represents the energy of the detector; however, the output power of the amplifier is certainly proportional to $x^2 + y^2$. We shall usually use the notation E for $x^2 + y^2$. Because of tradition however, the quantity $[\frac{d}{dt}(x^2+y^2)]^2$ will always be denoted P^2.

One of the sources of disagreement among scientists has been just how to do the data analysis. The primary purpose of this dissertation is to provide an analysis of this aspect of gravitational radiation detection. We will examine all the steps in the detection process. Expecially we are interested in finding anything that might explain the positive results obtained by Weber when his data analysis is based on P^2, in contrast to the poor results obtained with $\dot{x}^2 + \dot{y}^2$. It is generally agreed by scientists that data analysis based on $\dot{x}^2 + \dot{y}^2$ is superior to that based on P^2 for pulse like signals of short duration compared to the time resolution of the detector. While we have not studied the equation of motion for the detector under theories other than General Relativity, it is hard to imagine any theory in which radiation would mainly affect the energy of the detector and not change the phase. This would be the only way gravity could make "P^2" superior to "$\dot{x}^2+\dot{y}^2$".

We will now review some of the major results of this dissertation. An improved method for data analysis, based on Bayesian techniques or more precisely on the evidence function (for details see chapter two), has been developed. This method provides both optimal filter properties and optimal methods for integrating the data. Employing a signal parameter variation technique, it provides information on the nature of the signal. In the two detector case information on "unwanted" or uncorrelated excitations is also provided. By evaluating the form of the evidence function for various types of signal hypotheses, one can search for cases where the evidence function contains data variables that more or less approximate P^2. Such a case has not been found. (Naturally one cannot try all conceivable types of signals, but the trend is clearly not in favor of P^2). On the other hand, it may be that for some types of signals $\dot{x}^2 + \dot{y}^2$ is even more mismatched than P^2. We have also found that if the data is recorded in the form $(\dot{x}, \dot{y})$, the quantization and other "data handling" noise is minimized, and that no information is lost when the data is transformed in this way.

To investigate further the problems with P^2 versus $x^2 + y^2$ analysis, a computer simulation of a Weber type detector has been made. The results of this shows that for most cases "$x^2 + y^2$" is superior to "P^2". Two cases are of special interest however. If the signal is a short large pulse, the two methods are about equally effective. (By a large pulse we mean a pulse that gives the detector an increase in energy, large than its thermal energy kT). This result is not so surprising since phase information is less important for a large pulse. Further if there is a frequency offset (usually caused by a slow drift in detector frequency, due to small changes in temperature) between the detector and the reference oscillator, the $\dot{x}^2 + \dot{y}^2$ method is strongly affected in a negative way (if the offset is not corrected for in the data analysis) especially for small signals or a stochastic burst of small signals, while the P^2 method remains practically unchanged. This result agrees with that obtained by G. Rydbeck and J. Weber in 1974. We must conclude that of all possibilities tried in this investigation only "Frequency drift" could account for making "P^2" superior to "$\dot{x}^2 + \dot{y}^2$".

In the following a brief account of the contents of each chapter is given. The first chapter studies the response of a solid media to a gravitational wave. The equations of motion are specialized to the case of a cylindrical detector and transformed into corotating phase space coordinates.

The uninteresting harmonic rotation is transformed away so that any change in the coordinates of the detector represents the presence of a force.

Bayes' equation and the evidence function are introduced in chapter two; these will provide the basis for our approach to data analysis. The approach allows a general definition of signal to noise ratio. Data analysis with the above method is studied in an example with one detector. It is seen to increase the signal to noise ratio by a factor of approximately two compared to the usual threshold technique. This result was for a special choice of signals, but other situations should give roughly the same result. Instead of threshold type analysis the method leads to a type of analysis in which contribution to the evidence-increase from each data point is a function of its amplitude.

Noise and filter theory is briefly reviewed in chapter three. It is shown by example how the evidence approach in fact describes both optimal filters and optimal data integration methods. We thus have a straight forward method (although at times lengthy and messy) for secondary filter design (assuming the data is stored in the "prefiltered" form $(\dot{x}, \dot{y})$) and data analysis under complicated signal hypotheses.

The fourth chapter studies a computer simulation of a pair of Weber type detectors, the results of which has already been mentioned.

All the detailed calculations and applications have been done in appendices. Below is an account of the problems that are considered in these.

APPENDIX A. Calculation of the signal to noise ratio, in the two dimentional case, for a given signal, with and without a definite phase.

APPENDIX B. The thermal fluctuations of the detector amplitude are studied, and an expression for the signal to noise ratio of a "bare" detector, in terms of the spectral power of the gravitational radiation, is derived.

APPENDIX C. The "dressed" detector is investigated. Preamplifier properties are considered, and a preamplifier temperature is defined in terms of its noise sources. Equivalent circuits are introduced as a convenient tool for discussing detector systems. Detector and preamplifier matching is discussed briefly. A detector with nonresonant pick-up system is studied, and an expression for the signal to noise ratio in terms of the spectral power of the gravitational radiation is given. The form of this expression allows the definition of a spectral signal to noise ratio, which when integrated gives the total signal to noise ratio. It is seen that the spectral signal to noise ratio splits up into two factors, of which one is the spectral signal to noise ratio of the "bare" detector, and the other a spectral quality factor of the dress. This quality factor is related to the signal to noise of a "kT" excitation" (a sudden excitation which, if initially the detector has zero energy, gives it the energy kT) a measure which is often used in discussing the quality of the "dress". The optimal filter for such excitations is derived, and it is found that this filter is invertible (i.e. it does not lead to a loss of information) and allows further filtering to match any type of signal. Next a detector with a resonant type of pick-up system is considered, and expressions for sensitivity and optimal filter are given. As expected, it is seen that this filter is more complicated than the optimal filter for a nonresonant pick-up system.

Thus, unless the resonant pick-up system has other advantages (such as providing a better match) it should be avoided.

APPENDIX D. The details of data analysis, along the lines suggested in this dissertation, are studied.

It is seen that there is an optimal form in which to record the data (i.e. the impact of quantization noise, which arises when the data is digitized, and other data handling noise, is minimized).

Some possible type of signals are considered, such as single random pulses with exponential distribution in energy or random stochastic bursts of pulses with a certain average energy. Expressions for integrating the data under a parametrized set of "random pulse, exponential energy distribution" signal hypotheses are derived (for both one and two detector systems). Optimization gives a set of parameter values that provide information on the signals. In the two detector case, these parameters also provide data on local, nonthermal excitations.

Further in the two detector case, the method checks for correlation without the use of time delay. This is not to suggest that time delay experiments are useless in any sense. Indeed, the time delay method is a useful complement to the above method, and is naturally included in the parameter variation approach, by extending the set of signal hypothesis to include signals that are relatively delayed between the two channels.

Results of the analysis of some of Weber's data tapes is presented.

APPENDIX E. Gravitational radiation detection with two or more detectors with synchronized reference oscillators is considered. Such systems can be used for "long baseline interferometric detection." It is seen that the interference, although it provides directional sensitivity, does not increase the "internal" signal to noise ratio of the system. It provides however a shield against unwanted "external" disturbances. Such a shield may be essential for the future generation of super sensitive detectors.

APPENDIX F. The details of the computer simulation experiment and the results (which have been mentioned previously) are studied.

APPENDIX G. A method for eliminating "bare" detector noise from the signal output is consered. This method turns out not to be very useful for resonant high Q detectors, since it also eliminated the resonance, which is needed to "overcome" sensor and preamplifier noise. It may however be a useful method when the "Earth" or similar objects are used as detectors. (The method really amounts to transforming a solid detector into a free mass system).

G. Rydbeck and J. Weber (1974) "Frequency drift in gravitational radiation detection experiments". Technical Report #75-026.

CHAPTER I

THE ANTENNA

THE RESPONSE OF AN ELASTIC MEDIUM TO A GRAVITATIONAL WAVE

This problem has been studied by many workers, notably Weber (1961), and later by Dyson (1968). It is also considered in M.T.W. (1973). A very general and thorough treatment has been given by Carter and Quintana (1972), followed by Carter (1973,A) and (1973,B), and also by Glass and Winnicourt (1972). We refer to the above articles for "in depth" studies of the subject and will mainly review here some of the relevant concepts and results. Regarding the possibility of using gravitational radiation detectors as a tool for testing relativistic theories of Gravity, we refer to the article by Eardly, Lee and Lightman, (1973) and just note that gravitational wave detection may be the only in practice feasible way to test certain theories of Gravity.

Consider now the equation of motion for the medium as given by Dyson:

(1.1)
$$\frac{\partial}{\partial t}(\rho \dot{Z}_j) = \frac{\partial}{\partial x_k}\left\{ C_{jkmn}\left(Z^{m,n} + \frac{1}{2}h^{mn}\right)\right\}$$

where ρ is the density of the medium $C_j{}^{kmn}$ relates the strain to the stress, $h^{mn} = g^{mn} - \eta^{mn}$, where g^{mn} is the space time metric and η^{mn} the flat Minkowski metric. (Note that in our notiona which agrees with that of M.T.W., the interaction term $1/2\ h^{mn}$, enters with a plus sign while Dyson has a minus sign). Equation (1.1) is the equation of motion for the elastic medium in the linearized theory of General Relativity. In the following discussion, we will consider the linearized theory from the geometric point of view (i.e. not the"spin 2" point of view where instead of geometry, measuring rods etc. are affected by the gravitational field). We note that the amplitude Z_j in (1.1) is what one might call a R.I.M. (relative to inertial motion) amplitude, defined as follows. Assume that before the wave arrives the space is completely "flat" and the medium is in equilibrium. Consider a point P of the medium, this point will trace out a world line $P(\mathcal{T})$. Before the wave arrives $P(\mathcal{T})$ will be a geodesic while as the wave passes by (and a while after) this will in general not be the case. Let's now extend the prewave geodesic to form a geodesic at all times, and call it $P_o(\mathcal{T})$ (i.e. $P_o(\mathcal{T})$ is the world line of a free test mass, that originally had the same position as the point P of the medium.) The amplitude Z_j is the separation vector between $P(\mathcal{T})$ and $P_o(\mathcal{T})$, or in short measures the separation between a masspoint bound by the medium and a fictious free test mass that originally had the same position as the bound mass point. As pointed out by Dyson the R.I.M. amplitude is thus exactly the quantity one is interested in when the experimental sensor is a seismometer or accelerometer. We note further that the T.T. (transverse traceless coordinate system (see M.T.W. pp. 945-952) is a very natural choice of coordinate system when one is working with R.I.M. amplitudes. In these coordinates $P_o(\mathcal{T})$ will just have a constant position (See M.T.W. page 952 excercise 35.5), if the system is initially chosen so as to be at rest relative to the medium. (Thus since free masspoints at rest relative to the system, remains at rest, one may call it an inertial coordinate system). With this "initial condition" the T.T. coordinate system is fixed at all times. In the T.T. system then, Z_j is just the displacement vector, $Z_j(P(\mathcal{T})) = x_j(P(\mathcal{T})) - x_{oj}$ where $x_{oj} = \text{Const.} = x_j(P_o(\mathcal{T}))$ and $x_j(P(\mathcal{T}))$ are the coordinates of the points $P_o(\mathcal{T})$ and $P(\mathcal{T})$. One should note that even if the amplitudes $Z_j(P(\mathcal{T}))$ are zero in part (or all) of the medium, it may still be "straining" (the distance between points of the medium vary in time). In this case a seismometer would not register a signal while a strainsensitive device would. Obviously, one cannot, for example, by placing one seismometer on the moon and one on the earth detect any changes in distance between the two (if the change is of geometric origin, and not a result of applied forces) while such a measurement can be done e.g. with a laser ranging device. For the above reasons it is also of interest to consider the equation of motion in terms of strain, $S_{ij}(P(T))$ of the medium. For small velocities (of the medium relative to the coordinate syste and small (linear) strains there is a simple relation between the strain of the medium and the amplitude

(1.2) $$S^{ij} = \tfrac{1}{2}\left(Z^{i,j} + Z^{j,i}\right) + \tfrac{1}{2} h^{ij}$$

(See also Glass and Winnicourt page 1938, eq. (4, 19)). We give the following derivation of relation (1.2). Consider the distance between two near points in the medium, and let Δx^i be the initial displacement vector between the two points and let $\Delta x^{i'}$ be the displacement vector at time t. We have:

$$\Delta x^{i'} = \Delta x^i + \frac{\partial \Delta x^{i'}}{\partial \Delta x^j} \Delta x^j = \Delta x^i + Z^i_{,j} \Delta x^j$$

With the metric $g_{ij} = \eta_{ij} + h_{ij}$ at time t we have:

$$|\Delta \vec{x}'|^2 - |\Delta \vec{x}|^2 = 2\left\{\tfrac{1}{2}\left(Z_{i,j} + Z_{j,i}\right) + \tfrac{1}{2} h_{ij}\right\} \Delta x^i \Delta x^j .$$

The term 1/2 $(Z_{i,j} + Z_{j,i}$ $) + 1/2\, h_{ij}$, corresponds to the usual definition of linear strain (See Landau L. and Lifschitz, E (1957) page).

One may think of (1.2) as being the sum of the strain $1/2(Z_{i,j} + Z_{j,i})$ of the medium relative to the coordinate system and the strain $1/2\ h_{ij}$ of the coordinate system relative to its original form. Making the approximation that ρ is constant, one may now rewrite equation (1.1) by taking its derivative $\frac{\partial}{\partial x^i}$ and symmetrizing with respect to i and j. We get;

(1.3) $$\rho \frac{\partial^2}{\partial t^2}(S_{ij} - h_{ij}) = \frac{1}{2}\left\{\left(\frac{\partial}{\partial x^j} C_{ikmn} + \frac{\partial}{\partial x^i} C_{jkmn}\right) S^{mn}\right\}^{,k}$$

For isotropic media one has $C_{jkmn} = \lambda\, \delta_{jk}\delta_{mn} + \mu\,(\delta_{jm}\delta_{kn} + \delta_{jn}\delta_{km})$
Following Dyson, we can rewrite (1,1) in transverse traceless coordinates as

(1.4) $$\frac{\partial}{\partial t}(\rho \dot{Z}_j) = \frac{\partial}{\partial x_j}(\lambda Z^m{}_{,m}) + \frac{\partial}{\partial x_k}\{\mu\,(Z_{j,k} + Z_{k,j})\} + \frac{\partial \mu}{\partial x_k} h_{jk}\,.$$

where $\mu = \rho s^2$ is the shear modulus $\lambda = \rho(v^{2} - s^2)$ the Lamé constant, and V and s are the compressive and shear wave velocities respectively.
In the same way

(1.5) $$\rho \frac{\partial^2}{\partial t^2}\left(S_{ij} - \frac{1}{2} h_{ij}\right) = \left\{\frac{\partial}{\partial x^i}\left(\lambda \frac{\partial}{\partial x^j}\right) + \frac{\partial}{\partial x^j}\left(\lambda \frac{\partial}{\partial x^i}\right)\right\} S^m{}_m +$$
$$+ \frac{\partial}{\partial x_k}\, \mu \left\{\frac{\partial S_{ik}}{\partial x^j} + \frac{\partial S_{jk}}{\partial x^i}\right\}\,.$$

We can see from (1.4) that if $\frac{\partial \mu}{\partial x_k} = 0$ (i.e. if the medium is infinite and isotropic) there will be no coupling to the R.I.M. amplitude, while (1.5) shows that the strain amplitude is still excited. If one considers the strain as composed of two parts $S_{ij} = S_{ij}^{(1)} + S_{ij}^{(2)}$ where $S_{ij}^{(1)} = \frac{1}{2} h_{ij}$ and $S_{ij}^{(2)} = \frac{1}{2}(Z_{i,j} + Z_{j,i})$ one may think of equation (1.5) as describing the motion of two different superposed strain-waves, one wave $S_{ij}^{(1)}$ (which is purely inertial, i.e. it does not give rise to accelerations) is coupled to and travelling with the gravitational wave, while $S_{ij}^{(2)}$ is purely non-inertial, and has as its source regions where $\frac{\partial \mu}{\partial x^k}$, differs from zero. One may take the point of view that the inertial waves $S_{ij}^{(1)}$ scatter on regions where $\frac{\partial \mu}{\partial x^k}$ differs from zero, thereby producing non inertial waves $S_{ij}^{(2)}$. This scattering will for example for a momogenous detector occur at the boundaries and may for certain frequencies lead to resonances, in which case $S_{ij} \simeq S_{ij}^{(2)}$. Thus near resonance it will in practice not matter whether an accelerometer or strain type of sensor is used. One may without too many complications include simple types of friction in (1.1) and (1.3). The simplest case is when there is an additional stress arising from friction, in such a way that it is proportional to the rate of change of strain. Including such friction is accomplished by letting C_{jkmn} in (1.1) and (1.3) be replaced by $C_{jkmn} + \eta_{jkmn}\frac{\partial}{\partial t}$ where η_{jkmn} has the same symmetry properties as C_{jkmn}, and for the isotropic case:

$$\eta_{jkmn} = \alpha\, \delta_{jk}\delta_{mn} + \beta\,(\delta_{jm}\delta_{kn} + \delta_{jn}\delta_{km}) .$$

(See Landau and Lifshits page 125).

Including such friction in for example, equation (1.1) we would get

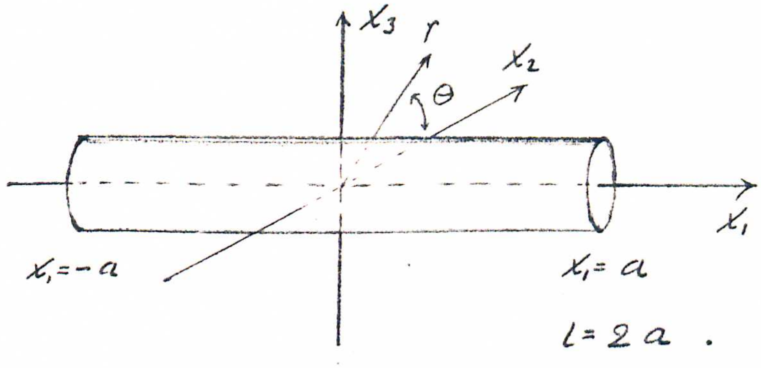
$$\frac{\partial}{\partial t}(\rho \dot{Z}_j) = \frac{\partial}{\partial x_k}\left\{\left(\eta_{jkmn}\frac{\partial}{\partial t} + C_{jkmn}\right)\left(Z^{m,n} + \frac{1}{2}h^{mn}\right)\right\} \tag{1.6}$$

It is noteworthy that even for purely inertial waves $(Z_j = 0)$ this friction (which may be caused by conversion of clastic energy to for example electromagnetic energy or heat) would in the isotropic case lead to dissipation of energy. In fact, we would in the isotropic have a dissipation per unit volume given by

THE FINITE CYLINDER

It is usually quite complicated to find the "exact" vibrational modes for a given elastic body. Even the "simple" case of longitudinal modes of a finite cylinder is complicated enough. In fact to my knowledge, no exact solution to this problem exists. With some approximations however, it is easy to make the problem manageable. Let the cylinder and the coordinate system be defined by the following figure

Fig. 1.2



We consider the equation of motion (1.5) for the (homogenous) medium

$$\varrho \ddot{S}_{ij} - \lambda \frac{\partial^2 S^m{}_m}{\partial x^i \partial x^j} - \mu \left\{ \frac{\partial^2 S_{jk}}{\partial x^i \partial x^k} + \frac{\partial^2 S_{ik}}{\partial x^j \partial x^k} \right\} = \frac{1}{2} h_{ij}$$

We are mainly interested in longitudinal oscillations, and as a first step we neglect modes with angular dependance (which is not an approximation, it just means that we assume bending modes etc. not to be excited). The first approximation is to assume S_{11} to be constant in r . We now observe that one can always write the strain $S_{11}(x_1, t)$ as an expansion over a set of functions $f_\nu(x_1)$ which is complete on the interval $-a \leq x_1 \leq a$, and gives $S_{11}(x_1, t)$ the correct boundary behaviour $S_{11}(\pm a, t) = 0$. We may thus choose the set of functions to be;

$\{\cos k_n x, \sin \tilde{k}_m x\}$ $n, m = 0, 1 \cdots \infty$, and where $k_n = \frac{\pi}{a}(\frac{1}{2}+n)$ and $k_m = \frac{\pi}{a} m$. One may thus write;

$$S_{11}(x,t) = \sum_{n=1}^{\infty} \left(\xi_n(t) \cos k_n x + \eta_n(t) \sin \tilde{k}_n x \right) .$$

We will now have to make a second approximation, namely that for each mode (ξ_n and η_n) one can replace, in the equation of motion, the coupling between the $S_{11}(x,t)$ component and other components of the strain tensor with an effective elastic constant Y_n. This is in general a "fairly good" approximation, except when there is a "beat" between two or more components of the strain tensor. (It could happen for example that the frequency of a particular longitudinal mode would be the same or close to that of a particular radial mode, in which case one would get a "beat" effect). Anyway with the suggested approximation, we get when the above expansion is inserted into the equation of motion,

$$(1.7) \quad \sum_n \left\{ \left(\ddot{\xi}_n(t) + \frac{Y_n}{\rho} k_n^2 \xi_n(t) \right) \cos k_n x + \left(\ddot{\eta}_n(t) + \frac{\tilde{Y}_n}{\rho} \tilde{k}_n^2 \eta_n(t) \right) \sin \tilde{k}_n x \right\} = \\ = \frac{1}{2} \ddot{h}_{11}(x,t) .$$

Y_n and $\tilde{Y}_n$ are functions of k_n and d, the diameter of the cylinder, i.e. $Y_n = Y_n(k_n, d)$ and $\tilde{Y}_n = \tilde{Y}_n(\tilde{k}_n, d)$. If the cylinder is thin compared to the wavelength, $d \ll \left(\frac{2\pi}{k_n}\right)$, $Y_n, \tilde{Y}_n \to Y$ (Youngs modulus of the material);

$$Y_n, \tilde{Y}_n \to Y = 2\mu\left(3 + 2\frac{\mu}{\lambda}\right)$$

In the opposite limit when the diameter of the cylinder is much larger than the wave-length, we have:

$$Y_n, \tilde{Y}_n \to \lambda + 2\mu \quad \text{as} \quad d \gg \left(\frac{2\pi}{k_n}\right) .$$

For reference see eq. Achenback J.D (19). The mode frequency is:

$$\omega_n = k_n \sqrt{\frac{Y_n}{\rho}} \quad \text{and} \quad \tilde{\omega}_n = \tilde{k}_n \sqrt{\frac{\tilde{Y}_n}{\rho}} .$$

One may now multiply (1.7) with $1/a \,\cos k_n x$ or $1/a \,\sin \tilde{k}_n x$ and integrate over the interval $\pm a$ to separate out the particular mode equations. If h_{11} is assumed to be constant along the cylinder the Sine modes (which we from now leave out) will vanish. For the cosine modes we get;

$$(1.8) \qquad \ddot{\xi}_n(t) + \omega_n^2 \xi_n(t) = \frac{1}{2} \ddot{h}_{11} \int_{-a}^{+a} dx \, \frac{\cos k_n x}{a} = \frac{2(-1)^n}{\pi(1+2n)} \ddot{h}_{11} ,$$

so that $S_{11}(x,t) = \sum_n \xi_n(t) \cos k_n x$.

The "change of length" amplitude, which we from now call the COL. amplitude, between zero and the point x , is obtained by integrating the strain from zero to x . We denote this amplitude by $u_n \sin k_n x$ and one has:

$$(1.9) \qquad \ddot{u}_n + \omega^2 u_n = \ddot{h}_{11} \frac{2(-1)^n}{k_n \pi(1+2n)} = \ddot{h}_{11} \frac{4a(-1)^n}{\pi^2(1+2n)^2}$$

so that the total COL. amplitude is $u(x,t) = \sum_n u_n(t) \sin k_n x$. Making a similar approximation as above, it is a straight forward matter to include damping. We assume that one for each mode (ξ_n and u_n), can replace the damping of the mode itself and the damping due to coupling to other damped components of the strain tensor, by an effective damping constant D_n . The damping term will thus be of the form; (See equation 1.6)

$$D_n \frac{\partial}{\partial t}\frac{\partial^2}{\partial x^2} \xi_n(t) \cos k_n x = -D_n k_n^2 \dot{\xi}_n(t) \cos k_n x .$$

Including k_n^2 in D_n , equation (1.8) with damping is given by;

$$\ddot{\xi}_n(t) + D_n \dot{\xi}_n(t) + \omega_n^2 \xi_n(t) = \frac{2(-1)^n}{\pi(1+2n)} \ddot{h}_{11} . \tag{1.10}$$

It is of interest to express the equations of motion also in terms of the R.I.M. amplitudes defined on page 1 . First we note that

$S_{11}(x,t) = S_{11}^{(2)}(x,t) + \frac{1}{2} h_{11}(t)$ where $S_{11} = \frac{\partial Z_1}{\partial x'}$ (see page 13).

We may again expand $S_{11}^{(2)}$ as

$$S_{11}^{(2)}(x,t) = \sum_n \psi_n(t) \cos k_n x . \text{ Thus } \psi_n(t) + h_{11} \frac{2(-1)^n}{\pi(1+2n)} = \xi_n(t) .$$

Inserting this in equation (1.8) gives

$$\ddot{\psi}_n(t) + \omega_n^2 \psi_n(t) = -h_{11} \frac{Y_n \pi(1+2n)}{2\rho a} (-1)^n$$

Clearly $\psi_n(t)$ differs from $\xi_n(t)$, when $h_{11} \neq 0$.

If $h_{11} = 0$ (e.g. after the passage of a pulse of gravitational radiation) however $\psi_n(t) = \xi_n(t)$. The R.I.M. amplitude $Z(x,t)$, (note that $Z(x,t) = \int_0^x dx \frac{\partial Z}{\partial x} = \int_0^x dx\, S_{11}^{(2)}$ may also be expanded as $Z(x,t) = \sum Z_n(t) \sin k_n x$.

It is then easily seen that the equations of motion in terms of $Z_n(t)$ are;

$$\ddot{Z}_n(t) + \omega_n^2 Z_n(t) = -h_{11} \frac{Y_n}{\rho a} (-1)^n . \tag{1.11}$$

We note again that Z_n will differ from U_n (See 1.9) only when $h_{11} \neq 0$. It is further clear that any non gravitational force $f_n(t)$ acting on the detector will enter (1.9) and (1.11) in an identical way. i.e.

$$\ddot{U}_n + \omega_n^2 U_n = \ddot{h}_{11}(t) \frac{4a(-1)^n}{\pi^2(1+2n)^2} + f_n(t)$$

$$\ddot{Z}_n + \omega_n^2 Z_n = -h_{11}(t) \frac{Y_n(-1)^n}{\rho a} + f_n(t) \ .$$

This peculiar difference between the R.I.M. and the C.O.L. amplitude (which would be non-existent for other than gravitational interactions) was noted by William R. Burke (1973) and utilized to cancel non gravitational forces (e.g. noise or other types of interfering forces). Subtracting the second equation from the first we get

$$\ddot{d}_n + \omega_n^2 d_n = \ddot{h}_{11} \frac{4a(-1)^n}{\pi^2(1+2n)^2} + h_{11} \frac{Y_n(-1)^n}{\rho a} \tag{1.12}$$

where $d = U_n - Z_n$. Clearly the solution to (1.12) is

$$d_n = h_{11} \frac{Y_n(-1)^n}{\rho a \omega_n^2} = h_{11} \frac{4a(-1)^n}{\pi^2(1+2n)^2} . \quad \text{Further} \tag{1.13}$$

$$d(a) = U(a) - Z(a) = \sum_{n=0}^{\infty} (-1)^n d_n = h_{11}\, a \sum_{n=0}^{\infty} \frac{4a}{\pi^2(1+2n)^2} = \frac{1}{2} h_{11}\, a.$$

Essentially thus the amplitude $d(a)$ is equivalent to the amplitude between two free masses at distance a from each other. Let's consider a specific detector system for which the above relations can be applied. Assume that the detector consists of a piezoelectric bar with accellerometers mounted on its ends. With the piezoelectric effect one can monitor the change of length of the bar and with the accelerometer the R.I.M. amplitude of its ends. (An accelerometer consists of an essentially free mass, placed at the end of the bar, plus a device to measure the relative motion of the two. A similar device, with a very high sensitivity, has been developed by the Stanford experimental relativity group). With proper normalization one thus has the two outputs, u_n and z_n .(not counting added wideband electronic noise). Thus according to (1.13) by taking the difference between the two output amplitudes u_n and z_n , we would obtain an output, completely free of any noise or disturbances of mechanical origin, acting on the piezoelectric bar (which seems almost like magic). It should be clear however that one does not in general get rid of noise forces or disturbances acting on (non detector parts of) sensors and amplifiers. Moreover, the resonance behaviour present in u_n and z_n is lost. For most detectors however the resonance is essential for "overcoming" wide band sensor and preamplifier noise. Thus it turns out (see appendix G) that "amplitude subtraction method" is not useful for resonant detectors, but may be useful for other types of gravitational radiation detection experiments.

PHASE SPACE COORDINATES FOR THE ANTENNA

We will now see how the dynamics of the detector can be formulated in terms of a phase space vector in a corotating coordinate frame. This is useful from two points of view, first it transforms away the noninteresting harmonic motion (what one is interested in is deviations from harmonic motion), second, most experimental detector systems employ a "lock-in" detector which in effect transforms the signal to the above phase space vector. This two dimensional "corotating state vector" is thus in practice the output quantity the experimenter deals with and which is later filtered and processed in the attempt to find a signal.

We will do our calculations for a Weber type antenna (cylinder). Most of the results (such as sensitivity, etc.) can be applied directly to other types of resonant antennas as well. In all these cases (e.g. dumbell, disk) we would get, just as above, a set of harmonic equations coupled to the gravitational wave field. The dumbell however turns out to be fairly uninteresting. Its detection efficiency is in general inferior to that of a cylinder. The disk should be at least as efficient as the cylinder but we have not done any calculations for this case.

We rewrite equation (1.9) as

$$(1.14) \quad \ddot{u}_n(t) + 2\gamma_n u_n(t) + \omega_n^2 u_n = -h_{11,00}\, C_n = a_n(t)$$

where γ_n is the inverse of the amplitude damping time, ω_n the frequency and $C_n = \dfrac{(-1)^n 4a}{\pi^2(1+2^n)2}$ is a coupling constant. We note that under the circumstances (plane wave transvers traceless coordinates) we may replace $h_{11,00}$ by $-2C^2 R_{1010}$ (see Misner, Thorne.K., Wheeler J., Gravitation, W.M. Freeman, San Francisco, 1973 , page 948).

The instantaneous energy of a mode is (Assuming $u_n(t) = u_n \sin\omega_n t$)

$$E_n = \frac{1}{2}\int_{-a}^{+a} dx\, \rho u_n^2 \omega_n^2 \cos^2\omega_n t \sin^2 k_n x + \frac{1}{2}\int_{-a}^{+a} dx\, u_n^2 Y_n k_n^2 \sin^2\omega t \cdot \cos^2 k_n x =$$

$$= \frac{a}{4}\{\rho\omega_n^2 + Y_n k_n^2\} u_n^2 .$$

We have: $\omega_n = k_n\sqrt{\frac{Y_n}{\rho}}$; $k_n = \frac{\pi}{a}\left(\frac{1}{2}+n\right)$

If we approximate $Y_n = Y_o$;

We can write $\omega_n = \omega_o(1+2n)$, we get $E_n(t) = \frac{1}{4}S2a\omega_n^2 u_n^2$, since $\rho 2a = m$ (the mass of the cylinder) we get

$$(1.15)\ E_n(t) = \frac{1}{4}m\omega_n^2\ u_n^2(t) = \frac{1}{4}m\omega_o^2(1+2n)^2\ u_n^2(t) =$$

$$= \frac{\pi^2 Y_o}{2a}\ (1+2n)^2\ u_n^2(t)$$

The solution of (1.1) is

$$u_n(t) = \int^t dt' \frac{1}{\omega_n}\ e^{-\gamma_n(t-t')}\ \sin\omega_n(t-t')\,a(t')$$

from which also

$$\frac{1}{\omega}\dot{u}_n(t) = \int^t dt'\frac{1}{\omega_n}\ e^{-\gamma(t-t')}(\cos\omega_n(t-t') - \frac{\gamma_n}{\omega_n}\sin\omega_n(t-t')\,a(t')$$

(From now we drop the "n" remembering that u etc. is really u_n but we let ω_n be , to differ from ω in fourier expansions).

The fundamental oscillation may be represented in "phase" space by the vector $\vec{u} = (\dot{u}, \omega_n u)$ or in complex notation; $u_c = \dot{u} + i\omega_n u =$

$$= \int^t dt'\ e^{-\gamma(t-t')}\left\{e^{i\omega_n(t-t')} - \frac{\gamma}{\omega_n}\ \sin\omega_n(t-t')\right\}a(t')$$

We observe that the signal is a real quantity. It is also important to note that a signal may change the amplitude as well as the phase of the rotating phase space vector. The simplest way to extract the actions of the forces acting on the cylinder is thus to go to a coordinate system rotating in phase space so that the expected harmonic rotation vanishes. In this way eventual changes of the phase space vector are due to signals or noise only.

We multiply the u above with $e^{i\omega_n t}$ and we get u in the rotating frame; Let $v = u\, e^{i\omega_n t}$;

$$v = \int^t dt'\, e^{-\gamma_n (t-t')} \left\{ e^{-i\omega_n t'} - \frac{\gamma_n}{\omega_n} e^{-i\omega_n t} \sin \omega_n (t-t') \right\} a(t')$$

In practice $\frac{\gamma_n}{\omega_n} \ll 1$ (In fact $\frac{\gamma_n}{\omega_n} < 0{,}5 \cdot 10^{-6}$ for most detectors) so to a very good approximation;

$$v = \int^t dt'\, e^{-\gamma_n (t-t')}\, e^{-i\omega_n t'}\, a(t').$$

We can now go back to the equation of motion in the corotating frame.

$$\dot{v} + \gamma_n v = e^{-i\omega_n t}\, a(t) \tag{1.16}$$

or in component form

$$\left\{ \begin{array}{l} \dot{v}_1(t) + \gamma_n v_1(t) = \cos \omega_n t\; a(t) \\ \\ \dot{v}_2(t) + \gamma_n v_2(t) = \sin \omega_n t\; a(t) \end{array} \right. \tag{1.17}$$

From (1.15) we have that the energy of the cylinder is

$$E = \frac{1}{4} m |v^2|. \tag{1.18}$$

We will now see what happens when we try to do this transformation in reality. We have

$$v_1 = \int_{-\infty}^{t} dt' \frac{1}{\omega_n} e^{-\gamma_n (t-t')} \cos \omega_n (t-t')\, a(t')$$

$$v_2 = \int_{-\infty}^{t} dt' \frac{1}{\omega_n} e^{-\gamma_n (t-t')} \sin \omega_n (t-t')\, a(t').$$

(Note that V_2 is the original output of the detector and that V_1 is obtained from V_2 by differentiation). We will at this point need a reference oscillator to give us $\cos\omega_n t$ and $\sin\omega_n t$ functions, which means that we actually will be given functions $\cos(\omega_n + \Delta\omega)t$ and $\sin(\omega_n + \Delta\omega)t$. We can thus multiply V_1 and V_2 with these functions in the following way

$$\begin{cases} V_1' = V_1 \cos(\omega_n + \Delta\omega)t + V_2 \sin(\omega_n + \Delta\omega)t \\ V_2' = V_1 \sin(\omega_n + \Delta\omega)t - V_2 \cos(\omega_n + \Delta\omega)t \end{cases}$$

Performing this multiplication we get

$$V_1' = \int_{-\infty}^{t} dt' \frac{1}{\omega_n} e^{-\gamma(t-t')} \cos(\omega_n t' - \Delta\omega t)\, a(t')$$

$$V_2' = \int_{-\infty}^{t} dt' \frac{1}{\omega_n} e^{-\gamma(t-t')} \sin(\omega_n t' - \Delta\omega t)\, a(t')$$

If the oscillator is stable enough so that $\Delta\omega$ does not change appreciably within what we expect to be the duration-time of the signal, we may continuously correct the oscillator by constructing the correlation - function $C_V(t)$ of V. (For definition see page 50).

If the oscillator is offset with $\Delta\omega$, $C_V t$ will oscillate with that frequency, which information we can weakly feed back to the oscillator. If the feedback is too strong however, we might kill eventual signals.

When deriving the equation of motion for the antenna, the only driving term included was that due to gravitational interaction, unfortunately, however, there are also "noise forces" acting on the antenna, arising from the interaction of the oscillating modes with the surrounding heat bath.

As we shall see, the coupling to the heat bath is essentially proportional to γ_h, which is why a number of groups are now trying to make detectors of materials such as quartz or sapphire, which both have extremely high Q.

We will now leave the subject of detectors for a while to study relevant probabilistic concepts, noise, etc. The study of the detector is continued in appendix B and C.

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CHAPTER II

CONCEPTS OF PROPABILITY .

SIGNAL TO NOISE, LIKELIHOOD RATIOS

BAYES EQUATION AND EVIDENCE

Let's first consider the most elementary signal plus noise situation, described by the equation (2.1) $d = s + n$, where we call d the data, s the signal and n the noise, which is here assumed to have a gaussian distribution, i.e. $P(n) = \frac{1}{\sqrt{2\pi\langle n^2\rangle}} \exp \frac{n^2}{2\langle n^2\rangle}$

In this case consider a single data point d. The ability for the observer to distinguish wether the data contains a signal or not depends on the signal to noise ratio, which is usually defined as

(2.2) $E_s/E_n = \frac{s^2}{\langle n^2\rangle}$ one can frequently see other definitions

such as $\frac{|S|}{\langle n^2\rangle^{1/2}}$ which definition one adopts is really immaterial, but we shall see that a generalization of definition (2.2) has a simple and direct interpretation in terms of Bayes equation, which will be introduced in a moment. If no signal is present in (2.1), the noise will *induce* a distribution in the data:

(2.3) $P_n(d) = \frac{1}{\sqrt{2\pi\langle n^2\rangle}} \exp - \frac{d^2}{2\langle n^2\rangle}$, if a signal is present, the signal + noise will induce a distribution in the data

$$(2.4) \qquad P_{n+s}(d) = \frac{1}{\sqrt{2\pi\langle n^2\rangle}} \exp - \frac{(d-s)^2}{2\langle n^2\rangle}$$

If we consider a specific data point d_i and the only thing about s that is unknown is if it is on or off, it seems intuitively clear that we should base our judgment about presence of a signal on the quantity

$$(2.5) \quad \mathcal{L}(s, d_i) = \frac{P_{n+s}(d_i)}{P_n(d_i)}$$

$\mathcal{L}(s,d)$ is called the likelihoodratio.

If s is not exactly known but has some distribution $P_{(s)}(s)$,

$$P_{n+(s)}(d) = \int ds \; P_{n+s} \; P_{(s)}(s)$$

will be resulting distribution in the data, and

$$(2.6) \quad \mathcal{L}_{(s)}(d) = \int ds \frac{P_{n+s}(d)}{P_n(d)} P_{(s)}(d)\, ds = \frac{P_{n+(s)}(d)}{P_n(d)}$$

We will now try to make the role of these quantities more precise..

It is not immediate how to generalize the concept of signal to noise ratio to the case where the data is distributed in a non-gaussian manner.

To do this generalization and understand the meaning of it we introduce the *Bayes* equation. (For a thorough introduction to this and related concepts see Tribus 1969,"Rational description and Designs, Pergamon Press, New York" and also Helstrom, 1968, "Statistical Theory of Signal Detection, Pergamon Press, New York.)

Let's consider an experiment, and further let's introduce the following notations and assumptions concerning this experiment.

We shall let C denote all the prior (before any data is examined) facts about the experiment; and in particular we let C contain the assumption:

(1) The data D in the experiment is produced by one of two possible underlying mechanisms, which we shall call G and g.

Let (D) be the set of all possible outcomes of the experiment, and further let $P(D|G)$ be the probability for the outcome D when G and C are known to be true, and similarly let $P(D|GC)$ be the probability for the outcome D when g and C are known to be true. $P(D|GC)$ and $P(D|gC)$ are called probability distribution functions of the variable D.

Similarly we let $P(G|C)$ be the probability that G is true when C is known to be true, and $P(g|C)$ the probability that g is true. Since according to (1) G and g are an exclusive and exhaustive set of hypotheses we have $P(G|C)+P(g|C) = 1$.

Further on we shall let $P(G|DL)$ denote the adjusted probability that G is true when C is known to be true and the outcome of the experiment is found to be D. $P(g|DC)$ is defined similarly.

Finally, we define $P(D|C)$ to be the probability for an outcome D when only C is known to be true. It follows that $P(D|C) = P(D|G)P(G|C)+P(D|gC)P(g|C)$

The above probabilities are related by Baye's equation;

$$(2.7) \qquad P(G|DC) = P(G|C)\,\frac{P(D|GC)}{P(D|C)}$$

which may also be written in terms of g, the denial of G

$$(2.8) \qquad P(g|DC) = P(g|C)\,\frac{P(D|GC)}{P(D|gC)}$$

By taking the ratio of these two equations, we get the odds form of Baye's equation:

$$O(G|DC) = O(G|C)\ \frac{P(D|GC)}{P(D|gC)} \tag{2.9}$$

where $O(G|C) = \dfrac{P(G|C)}{P(g|C)}$ is the odds in favor of the hypothesis G, when C is true. $O(G|DC)$ is defined similarly.

We shall not attempt to prove Baye's equation here, but refer the interested reader to Tribus (1969). It is obvious how to extend the above formalism to a case where the set of hypotheses has more than two elements. One may call this set $\{G_i\}$. n the case of a continuous set we may call it $\{G(x)\}$ or just $\{G\}$. Likewise the set $\{D\}$ may be continuous. In these cases one uses probability densities rather than the probabilities themselves. Since confusion concerning the use of Baye's equation can easily arise, we will study a particular example. Consider a bag containing 10 dice. It is known that 3 of these are dishonent (g) with a given distribution P(D|g C) in outcomes, when thrown, and that the rest (7) are honest (G). After n throws D is a set of "n" numbers between 1 and 6. The probability for the outcome D for an honest is just $(1/6)^n$. Now let a person blindly pick a die from the bag. Clearly the probability G that this is an honest die is 7/10, and that it is dishonest, 3/10. Assuming that the only way to find out if a die is honest or dishonest is to throw it and study the outcomes, we go ahead and do just that, updating the probabilities (P(G|C) and P(g|C) according to Baye's equation. In this example thus, the workings of the theory should be clear.

It has been said however that one of the problems with Baye's equation is that in a realistic experimental situation, (when we are not given one out of a known set of different possibilities) the prior probabilities do not make sense, since different experimenters will most probably adopt different prior p probabilities, i.e. there is no unique way of defining these. We maintain that this fact is not contradictory at all, since prior probabilities describe an experimenter's prior information concerning an experiment. This may certainly vary from person to person.

Consider another example with dice. This time we have 10 bags with 10 dice in each. 70 of the total number of dice are known to be honest, 30 to be dishonest with the same properties as the dishonest dice in the example above. Only this time the ratio of honest to dishonest dice in each individual bag is different. It is known that bag 1 has 9 honest, and 1 dishonest die, bag 2, 5 honest, 5 dishonest , etc. A die is now drawn from a bag, while two experimenters are watching. One of them sees from which bag the die is drawn while the other does not. Clearly the two experimenters will start out with different prior probabilities, assuming they are of the type, that only believes what they have seen with their own eyes. It is further clear that prior probabilities relates to their information concerning the drawn die, this information obviously may be different for different observers. We shall not here go into the technique of constructing prior probabilities in a more general situation, but refer the reader to Tribus (1967) page 119. Our analysis can in fact in most cases be accomplished without involving prior probabilities. It is however of importance to establish that the Bayesian methods are applicable to analysis of data in a general experimental situation. We shall therefore give some further arguments on behalf of Bayesian methods.

It is said that in addition to the problems mentioned above, in a real experimental situation the number possible hypotheses is infinite and that in fact the nature of most of the possible hypotheses cannot be even known, so how can we even talk about probabilities of hypotheses, when we don't even know what they are. All this is certainly true, but in spite of this, the problems can be handled, although it leads to some limitations concerning probabilities of the hypotheses that are considered. In physics it means that a theory can never be proven to be correct in the sense that if the theory is A, $1-P(A|DC)$ can be made less than ε. One can however disprove or exclude theories in the sense that if A is not satisfactory $P(A|DC)$ can be made less than ε by comparing A with a better theory. In other words it will turn out that we do not need to consider all possible theories at a time, but may consider just two or three or whatever we can manage.

Let's consider a specific example. Let G be a certain hypothesis concerning the nature of the excitations of a gravitational wave detector. Further let g be the hypothesis that all excitations are due to system noise sources only. Also we let H denote all other possibilities. As before C denotes all the information we have, such as noise properties of the detector, astrophysics, etc. We have: $P(G|C)+P(g|C)+P(H|C)=1$. Although we don't know all possible details about H it is clearly specified as being anything other than g or G. We have no knowledge at all about $P(D|HC)$. Thus, we want to exclude H from the discussion. This may be accomplished by considering the ratio of $P(G|DC)$ and $P(g|DC)$. From Baye's equation we have:

$$(2.10) \qquad \frac{P(G|DC)}{P(g|DC)} = \frac{P(G|C)}{P(g|C)} \; \frac{P(D|GC)}{P(D|gC)} .$$

It differs from (2.9) in that $P(G|C)+P(g|C) < 1$ in this case. Assume that an experimenter, based on his prior information gives the ratio $P(G|C)/P(g|C)$ the value 10^{-10}. After a year of experimenting he found $P(G|DC)/P(g|DC)$ to be 10^{10}. He can thus state that $P(g|DC)$ is less than 10^{-10}, but can say nothing about $P(G|DC)$. This is however not a severe limitation. If the experimenter can say that the probability that his signals are of thermal origin is less than 10^{-10} he should be in pretty good shape. Again, we may expand to include more than two hypotheses in the discussion. Let us consider the parametrized set of hypotheses $[G(E)]$. As before we let H be all other possible hypotheses. We have $P(h|DC) = \int dE\, P(G(E)|DC)$, where h is the denial of H. Further let's simplify our notation and just write E instead of G(E). We may now form the ratio:

(2.11) $$\frac{P(E|DC)\,dE}{P(h|DC)} = \frac{P(E|C)\,dE}{P(h|C)} \; \frac{P(D|EC)}{P(D|hC)}$$

Clearly all the quantities on the right hand side are well defined. The ratio $P(E|DC)$ $/P(h|DC)$ is thus well determined while $P(E|DC)$ and $P(h|DC)$ individually are unknown. Equation (2.11) may be written in a different way;

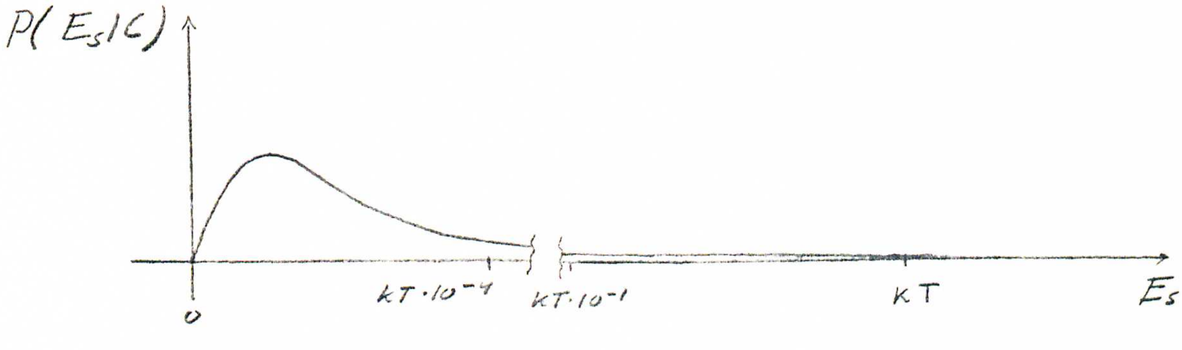
(2.13) $$P(E|DC)\,dE = P(E|hC)\,\frac{P(D|EC)}{P(D|hC)}\,dE$$

i.e. $P(E|DhC)$ is the probability that "$E \pm dE/2$" is true when hC is assumed to be true, or in other words, assuming that one of the hypotheses in the set [E] is true, $P(E|DhC)$ is the probability for the particular hypothesis E to be true. In the following we will include h in C to simplify the notation, but remember that we deal with probabilities "restricted" in the sense above.

Assume now that the experimenter decides to try a set of hypotheses corresponding to signal pulses with exponential distribution in energy, and an average pulse energy E_s. E_s can directly be considered as a hypotheses parameter with a range from zero to some given value. Further he assumes a density of pulses which we denote ϱ (e.g. on the average one out of every 3000 time intervals will contain a signal pulse). The signal hypotheses are thus determined by the parameters ϱ and E_s. The experimenter has certain prior knowledge about E_s and ϱ, he knows e.g. that E_s is not likely to be too large, etc.

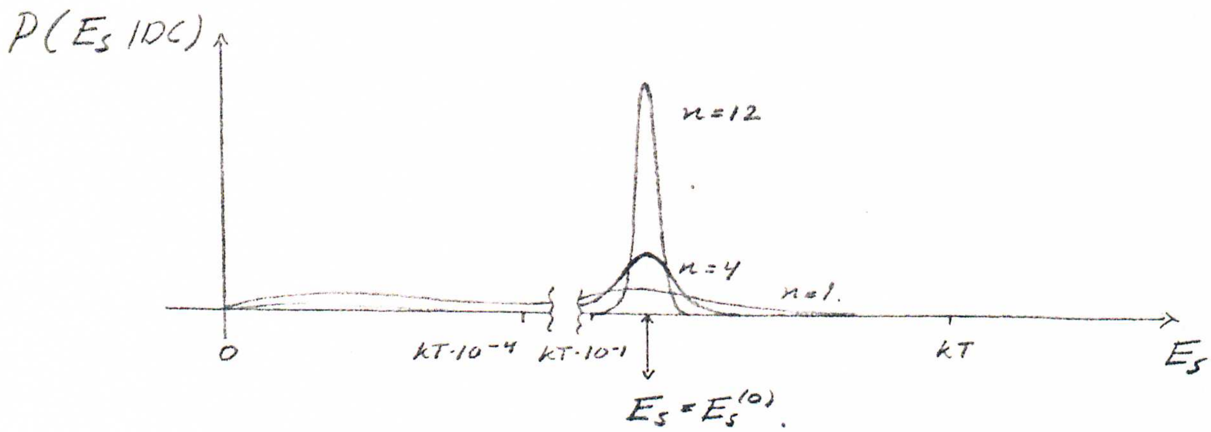
Applying the maximum entropy method with these conditions will give us the prior probabilities $P(E_s \varrho | C)$ (See e.g. Tribus page 440). We may arrive at a distribution like (For simplicity, we consider $P(E_s|C) = \int d\varrho \, P(E_s \varrho | C)$.)

Fig. 2.1.a



As new data D comes in, we use this data to change the prior distribution to $P(E_s|DC)$ and we may get something like

Fig 2.1. b.



where n is the number of months the experiment has been running. Surely the hypothesis $E_s^{(10)}$ will not correspond to reality in a detailed sense, (the true hypothesis may be an exponential with a bump at low energies, etc.), but of the considered hypotheses, it is the best approximation to reality.

In the usual way one may describe the uncertainty in the determination of E_s by the standard deviation

$$\Delta E_s = \int dE_s \, (E_s - E_s^{(0)})^2 \, P(E_s|DC)$$

Let's again consider equation (2.9). If we take the natural logarithm of this equation we get the evidence form of Baye's equation;

$$(2.13) \qquad ev(G|DC) = ev(G|C) + k \ln \frac{P(D|GC)}{P(D|gC)}$$

where $\quad ev(G|DC) = k \ln \frac{P(G|DC)}{P(g|DC)} \qquad$ etc.

(if $K=1$ the evidence is measured in napiers).

We will denote the increase in evidence by Δev, ie

$$\Delta ev(G|DC) = k \ln \frac{P(D|GC)}{P(D|gC)} .$$

Assuming that G is true, we may form the expected increase in evidence from the set of data D for this hypothesis.

$$(2.14) \qquad \langle \Delta ev(G|DC) \rangle^G = \int dD \, P(D|GC) \, k \ln \frac{P(D|GC)}{P(D|gC)} .$$

Where $\langle \ \rangle^G$ indicates that Δev is averaged over $P(D|GC)$.

We define

$$(2.15) \qquad Sn(G, g|C) = \langle \Delta ev(G|DC) \rangle^G .$$

In the case that G represents a signal hypothesis, which implies that D is a result of "signal + noise" activities, and g, the denial of G implies that D is a result of noise activity alone, one can see that Sn is a measure of how much the signal "stands out from the noise" (how much the signal increases the evidence for its own existence). Thus it seems that Sn should somehow be related to the "signal to noise ratio", and indeed, if we compute Sn for ordinary Gaussian cases (with $k=2$) it will be equal to the signal to noise ratio. For the example given on page 20 we get $E_s/E_n = Sn = S^2/\langle n^2 \rangle$.

We may thus identify Sn with the signal to noise ratio. As we shall see this makes it possible to uniquely compare different experimental situations, such as one signal per day, with a certain average energy, and say 10 signals per day with some other average energy. With the signal to noise ratio defined as above, we may compute the signal to noise ratio per day for these two cases, which will tell us which of the two, is the most favorable from an experimental point of view.

Usually, the meaning of the denial of (D is pure noise) is obvious and we may in this case drop g from the notation, and keep it only when confusion may otherwise arise. Likewise we may in most cases drop C.

To simplify the notation we may also replace G with some quantity that defines the signal, i.e. if G is defined by the power-spectrum $P(\omega)$ of a gravitational radiation pulse, we may write $Sn(P(\omega))$ or just $Sn(P)$, and in the case of a so called "kT excitation" we may just write $Sn(kT)$ (See page 120)

If we are referring to the signal to noise from just a part of the data, such as data from one day (say out of a month) or a frequency interval $\omega \pm \frac{d\omega}{2}$ we write:

$Sn(S/day)$ or $Sn(S/\omega)\,d\omega$. (Where S specifies the signal).

Further we simplify $P(D|HC)$ to $P_{n+s}(D)$ and $P(D|\bar{h}C)$ to $P_n(D)$. When evaluating (2.15) one has to be a bit careful. Let D represent a set of data $\{d_i\}$ $i=1,\cdots n$. In general, the probability distribution of d_i is a function of the signal and the noise, and also of other " nearby" data points. i.e $P_{n+s}(d_i) = P_{n+s}(d_i, \{d_j\}_{j\neq i})$. Only when the data points are independent in the sense that

$$\left\{\frac{\partial P_{n+s}(d_i)}{\partial d_j}\right\}_{j\neq i} = 0$$

does (2.15) split up in a sum so that

$$S_n(S/N) = \sum_{i=1}^{N} \int dd_i \, P_{n+s}(d_i) \, k \ln \frac{P_{n+s}(d_i)}{P_n(d_i)} ,$$

(And if S containes no dependence on i)

$$S_n(S/N) = N \int dd \, P_{n+s}(d) \, k \ln \frac{P_{n+s}(d)}{P_n(d)} .$$

We may say this in a different way, only if each new data point contains purely new information can we add up the signal to noise from each to get the total signal to noise ratio. The same holds true when we "add up" evidence. However, even if the raw data should happen to be "correlated" in the sense above there is usually a transformation that "untangles" or uncorrelates the data.

SIGNAL TO NOISE RATIO AND ENTROPY

We will present another aspect of signal to noise ratio in terms of entropy. A system with m states x_i has the entropy,

$$S = -k \sum_{i=1}^{m} P(x_i) \ln P(x_i), \text{ where } P(x_i) \text{ is the probability}$$

for the system to be in the state x_i. Let $n(x)$ be the density of states at the point x. We can write

$$\sum_{i=n}^{n'} P(x_i) \ln P(x_i) = \left(n(x_i) P(x_i) \ln P(x_i)\right) \Delta x_n,$$

where

$$x_n < x_i < x_{n'}, \quad \Delta x_n = x_{n'} - x_n \text{ and } |n'-n| \text{ is small}$$

enough so that $\dfrac{P(x_i) - P(x_i')}{P(x_i)} << 1; \; x_n < x_i, \; x_i' < x_{n'}.$

Thus we can write

$$S = -k \sum_{\nu=1}^{q} P(x_\nu) n(x_\nu) \ln P(x_\nu) \Delta x.$$

Now $P(x_\nu) n(x_\nu) = P(x)_{x=x_\nu}$, the probability density

So

$$S = -k \int dx \, P(x) \ln \frac{P(x)}{n(x)} \tag{2.16}$$

We stress at this point the necessity of having expressions such as (2.15)and (2.16), etc. invariant under variable transformation. A signal to noise ratio that depends on the way the data is presented would not make much sense, similarly with the entropy. Clearly, our expressions are invariant under such transformations. Had we given the signal to noise ratio as something like $S_{\kappa} = \frac{\int dx\, x\, P_{n+s}(x)}{(\int dx\, x^2 P_n(x))^{1/2}}$ it would not be invariant.

(For more information about this see E.T. Jaynes,page 201 in Statistical Physics, Brandeis Summer Institute, 1962, edited by K. W. Ford, W.A. Benjamin, New York).

Consider now a probability density $P_0(x) = \frac{n(x)}{N}$ where N is the total number of states. We can write

$$S = -k \int dx \left(P(x) \ln \frac{P(x)}{P_0(x)} - P(x) \ln N \right) =$$

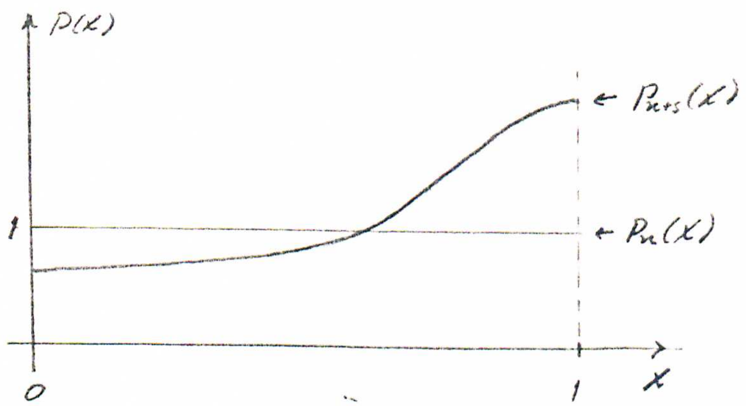
$$= -k \int dx \left(P(x) \ln \frac{P(x)}{P_0(x)} + P_0(x) \ln \frac{P_0(x)}{n(x)} \right) \text{ so}$$

$$S - S_0 = -k \int dx\, P(x) \ln \frac{P(x)}{P_0(x)}$$

Thus, we can identify a signal to noise ratio with $-(S_{n+s} - S_n)$, the negative of the entropy of the signal plus noise distribution minus the entropy of the noise distribution, if we define the noise distribution to have maximal possible entropy of all distributions, i.e. $P_n(x) = \frac{n(x)}{N}$. Note that even if $n(x) \to \infty$ so that S has a logarithmic infinity, the difference in entropy is independent of $n(x)$ and is finite.

We can make an interpretation of this result in terms of relative certainties. S_{n+s} is a measure of the uncertainty of the P_{n+s} distribution (See Tribus, 1969 , page 111) relative to the density $n(x)$ and S_n is the uncertainty of the $P_n(x)$ distribution relative to $n(x)$. In the same way we may say that $S_{n+s} - S_n$ is the uncertainty of the P_{n+s} distribution relative to the P_n distribution. If we define the negative of the uncertainty to be the certainty, $S_N = -(S_{n+s} - S_n)$ is the certainty of the P_{n+s} distribution relative to the P_n distribution. We illustrate with a graph where we make a variable transformation such that $n(x) \rightarrow n(x) = \text{constant}$, i.e. $P_n(x) = \text{constant}$ $0 \leq x \leq 1$

Fig 2.2



While $P_n(x)$ is minimally certain i.e. x can be expected to be found with equal probability anywhere in the interval 0 - 1, P_{n+s} exhibits more certainty. We may say that S_N is a measure of the certainty with which we can distinguish data distributed according to P_{n+s} distribution from data distributed according to the P_n distribution. We remark that there are different conventions about the signal to noise ratio. Our definition corresponds in the gaussian case to $(s,n) = \frac{E_S}{E_{Th}}$ in some literature one can find $(s,n) = \sqrt{\frac{E_S}{E_{Th}}}$ or $\ln \frac{E_S}{E_{Th}}$.

In fact, any increasing function f of the sn would do as a measure. For convenience we will sometimes use such a function for our signal to noise measure. We use quotation marks to indicate this i.e. "sn" $= f(sn)$.

CONCLUSIONS

The results of this chapter, suggests a unique optimal method for data analysis. We define this method (generally called hypothesis testing) by the following set of rules;

1. Form a (reduced) set of signal hypotheses $[G_i]$ (Let G_o represent the "noise only" hypothesis).

2. For each of the hypotheses G_i, $i = 1, 2$---n, compute the relative (to G_o) evidence increase

$$\Delta ev(G_i/(G_i+G_o)C) = k \ln \frac{P(G_i/DC)}{P(G_o/DC)}$$

3. The maximal such evidence increase (corresponding to a particular hypothesis G_{i_o}) is the measure of "success" in establishing the presence of a signal. The probability that G_i is true has increased by a factor exp Δev $(G_{i_o}/(G_{i_o}+G_o)C)$ $(G_{io}/(G_{io}+E_o)C)$ relative to the probability that G_o is true.

4. Of the possible signal properties considered the ones given by G_{io} are the most likely. The "restricted" probabilities for the truth of G_i, i=1,2,---n, and the uncertainty in determing the best hypothesis are specified by the procedure outlined on page and .

5. The signal to noise ratio for a given hypotheses G_i is given by the expectation value of the evidence increase.

$$Sn(G_i, G_o/C) = \langle \Delta ev(G_i/(G_i+G_o)C) \rangle = \int dD\, P(D/G_iC) \ln \frac{P(D/G_iC)}{P(D/G_oC)}.$$

COMMENTS

The above set of rules provides all that is needed for analysis of the outcome of an experiment, including filter design. For simple (one time parameter only) time invariant signal-hypotheses the final evidence increase will take on the form;

$$\Delta ev(G_i|(G_i+G_o)C) = \int_T dt\, F(G_i, C, \int_T dt' f(G_i, C, t-t') d(t')) .$$

where T is the duration of the experiment and d(t) is the raw output data. $f(G_i, C, t-t')$ is the optimal filter (i.e. it is matched to the hypothesis G_i). It will turn out that the properties of this filter will depend not only on the shape of the signal, but also on most other assumed signal properties, such as, definite or random arrival time of the signal pulses, definite pulse energy or pulse energy distributed according to some distribution, definite or random phase etc. Thus, the matched filter usually quoted in the literature (see e.g. Kafka 1975, page 87, and equation 5.24 of this dissertation) is matched to a short (relative to system time constants) input signal pulse of definite arrival time, phase and energy. This filter is thus not optimal for signal pulses with random arrival-times, random phase, and random (e.g. expential) distribution in energy. The effective signal to noise would still be close to optimal however, so in practice this filter would be quite adequate. Besides it would amount to a sizeable mess if one were to change the filter for every different hypothesis that is tested. It should also be noted that if one uses a threshold type detection procedure, strictly speaking, the filter should be matched to this procedure. The theory for such a filter would be quite involved however and one would expect that the resulting improvement in the effective signal to noise ratio would be small.

In order to apply our results to the analysis of an experiment, some basic noise theory is needed. We will study this in the next chapter.

Lastly, we note that one may take a somewhat different approach to the evidence concept and similar quantities, than what has been done here. We refer to a book by David Bridstone Osteyee and Irving John Good, 1974.

Tribus, M. 1969 Rational Descriptions and Designs (New York, Pergamon Press)

Helstrom, C.W. 1968 Statistical Theory of Signal Detection (New York, Pergamon Press.)

Jaynes E.T., 1962 Statistical Physics, Brandeis Summer Institute, 1962 edited by Ford K. W. (New York, W.A. Benjamin)

Kafka P. 1975 Optimal detection of Signals through Linear Devices with Thermal Noise Sources and Application to the Munich-Frascati Weber Type Gravitational Wave Detectors. Max Planck Institute fur Physic and Astrophysic

David Bridston Osteyee and Irving John Good, 1974 Information, Weight of Evidence, the Singularity between Probability Measures and Signal Detection. Lecture notes in mathematics. Edited by A. Dold, Heidelberg and B. Eckman, Zurich.

CHAPTER III

DEFINITION AND PROPERTIES OF A NORMALIZED WHITE NOISE SOURCE

We will in this section review some relevant concepts on noise. Consider a set $\{\alpha_i(t)\}$ of stochastic distributions in time, defined as follows:

Let the random variable $a_{iT}(t)$ be related to $\alpha_i(t)$ by

$$a_{iT}(t) = \frac{1}{\sqrt{T}} \int_{t-T/2}^{t+T/2} dt' \, \alpha_i(t') \quad ,$$

and let $P(a_T(t))$ be the probability distribution function of $a_{iT}(t)$. The set $\{\alpha_i(t)\}$ is then defined by the following conditions

3.1
$$\begin{cases} (1) \quad \lim\limits_{N \to \infty} \dfrac{1}{N} \sum\limits_{i=1}^{N} \alpha_i(t)\, \alpha_i(t') = \delta(t-t') \\[2em] (2) \quad P(a_T(t)) = \dfrac{1}{\sqrt{2\pi}} \exp - \dfrac{a_T^2(t)}{2} \end{cases} \quad .$$

As $N \to \infty$ it is assumed that all the members of the set are included. Since $a_T(t)$ is a Gaussian or normal variable, we may call $\{\alpha_i(t)\}$ a normal set of stochastic distributions. We assume this set to be the ensemble of all normalized white noise sources. For simplicity one may drop the "i" and write the above conditions in a simplified way as

3.2
$$\begin{cases} (1) \quad \langle \alpha(t)\alpha(t')\rangle = \delta(t-t') \\ (2) \quad \alpha(t) \ \text{normal}^{(1)} \end{cases} .$$

The bracket indicates ensemble average. One may generalize to a two dimensional or complex set of distributions. The corresponding definition is

3.3
$$\begin{cases} \langle \alpha(t)\alpha^*(t')\rangle = \delta(t-t') \\ \mathrm{Re}\ \alpha(t)\ ,\ \mathrm{Im}\ \alpha(t)\ ,\ \text{normal} . \end{cases}$$

It follows from 3.2 (2) that

3.4 $\langle \alpha(t)\rangle = 0$ (In the complex case one may write $\langle \alpha(t)\alpha(t')\rangle = 0$)

---

1 It is possible that the normality 3.2(2) would follows from 3.2.(1), a condition $\langle \alpha(t)\rangle = 0$ and the central limit theorem. We have not attempted any proof of this however, and leave the possibility as an open question.

Consider now the time average $\bar{\alpha}_T = \frac{1}{T}\int_{-T/2}^{T/2} dt\, \alpha(t)$ of $\alpha(t)$, and let $\bar{\alpha} = \bar{\alpha}_\infty$. Clearly $\langle \bar{\alpha}_T \rangle = 0$ for all T and $\langle \bar{\alpha}_T^2 \rangle = \frac{1}{T^2}\int_{-T/2}^{T/2} dt \int_{-T/2}^{T/2} dt'\, \delta(t-t') = 1/T$

Further it follows from the definition of $\alpha(t)$ that $\bar{\alpha}_T$ is a Gaussian variable, and thus that the probability distribution of $\bar{\alpha}_T$ is $P(\bar{\alpha}_T) = \sqrt{\frac{T}{2\pi}}\, \exp - \frac{T\bar{\alpha}_T^2}{2}$, and $P(\bar{\alpha}) = \delta(\bar{\alpha}) \Rightarrow \bar{\alpha} = 0$.
This result implies that the ensemble is ergodic, i.e. the time average of any of its members equals the ensemble average, or

$$(3.5) \qquad \bar{\alpha} = \langle \alpha(t) \rangle = 0 \ .$$

Note that this does not imply that other types of summations over all time are definite, e.g. the quantity $a_T = \frac{1}{\sqrt{T}}\int_{-T/2}^{T/2} dt\, \alpha(t)$ will remain finite when $T \to \infty$ and has for all time intervals T the Gaussian distribution

$$P(a_T) = \frac{1}{\sqrt{2\pi}} \exp - \tfrac{1}{2} a_T^2$$

Consider now the quantity $\beta_{\Delta t}(t) = \int_{t-\Delta t/2}^{t+\Delta t/2} dt'\, \alpha(t')$. We have

$$(3.6) \qquad \langle |\beta_{\Delta t}^2| \rangle = \int_{t-\Delta t/2}^{t+\Delta t/2} dt' \int_{t-\Delta t/2}^{t+\Delta t/2} dt'' \langle |\alpha(t')\alpha(t'')| \rangle = \int_{t-\Delta t/2}^{t+\Delta t/2} dt' = \Delta t, \text{ i.e. } \beta_{\Delta t} \text{ has}$$

the "random walk" property $\sqrt{\langle |\beta_{\Delta t}^2| \rangle} = \sqrt{\Delta t}$. Again from the definition of α (t) follows that $\beta_{\Delta t}$ has a Gaussian distribution, i.e. if $\alpha(t)$ is real

$$(3.7) \quad \begin{cases} P(\beta_{\Delta t})\, d\beta_{\Delta t} = \dfrac{1}{\sqrt{2\pi\langle \beta_{\Delta t}^2\rangle}} \exp - \dfrac{\beta_{\Delta t}^2}{2\langle \beta_{\Delta t}^2\rangle}\, d\beta_{\Delta t} \\ \\ \text{and if } \alpha(t) \text{ is complex,} \\ \\ P(\beta_{\Delta t})\, d\beta_{\Delta t} = \dfrac{1}{\pi\langle \beta_{\Delta t}\beta_{\Delta t}^*\rangle} \exp - \dfrac{\beta_{\Delta t}\beta_{\Delta t}^*}{\langle \beta_{\Delta t}\beta_{\Delta t}^*\rangle}\, d\beta_{\Delta t} \end{cases}$$

Where in (3.7) $d\beta_{\Delta t} = d\beta_{\Delta t}^{(1)} d\beta_{\Delta t}^{(2)}$ and $\beta_{\Delta t} = \beta_{\Delta t}^{(1)} + i\, \beta_{\Delta t}^{(2)}$.
It is clear that $\{\beta_{\Delta t}(t)\}$ constitutes a new ensemble, and that the properties of $\{\beta_{\Delta t}(t)\}$ reflects the properties of $\{\alpha(t)\}$, e.g. $\{\beta_{\Delta t}(t)\}$ must also be ergodic. From now we let the complex generalization be understood and include only some specific aspects thereof.
One may generalize the definition of $\beta_{\Delta t}$ in the following way.
Let $f(t-t')$ be a square integrable function, i.e. $\int_{-\infty}^{+\infty} dt' f^2(t-t') = N_f$,
and let $\beta_f = \int_{-\infty}^{+\infty} dt' f(t-t')\, \alpha(t')$
Clearly $\langle \beta_f^2 \rangle = N_f$, and

$$(3.8) \qquad P(\beta_f)\, d\beta_f = \frac{1}{\sqrt{2\pi \langle \beta_f^2 \rangle}} \exp - \frac{\beta_f^2}{2\langle \beta_f^2 \rangle}$$

The normalized correlation function $C_f(\Delta t)$ of $\beta_f(t)$ is defined,

$$(3.9) \qquad C_f(\Delta t) = \frac{\langle \beta_f(t)\, \beta_f(t+\Delta t) \rangle}{\langle \beta_f^2 \rangle} = \frac{\int_{-\infty}^{+\infty} dt' f(t'+\Delta t)\, f(t')}{\int_{-\infty}^{+\infty} dt' f^2(t')}$$

The ergodicity of $\beta_{\Delta t}(t)$ implies that we may also write $C_f(\Delta t)$ as a time average

$$(3.10) \qquad C_f(\Delta t) = \frac{\int_{-\infty}^{+\infty} dt'\, \beta_f(t')\, \beta_f(t'+\Delta t)}{\int_{-\infty}^{+\infty} dt'\, \beta_f^2(t')}$$

It is now a fairly easy matter to derive the joint probability distribution $P(\beta_f', \beta_f'' | \Delta t)$, where $\beta_f' = \beta_f(t)$ and $\beta_f'' = \beta_f(t+\Delta t)$. We have

$$(3.11) \qquad P(\beta_f', \beta_f'' | \Delta t)\, d\beta_f' d\beta_f'' = \frac{1}{2\pi \langle \beta_f^2 \rangle (1 - C_f^2(\Delta t))} \exp - \frac{\beta_f'^2 + \beta_f''^2 - 2C_f(\Delta t)\beta_f'\beta_f''}{2\langle \beta_f^2 \rangle (1 - C_f^2(\Delta t))}\, d\beta_f' d\beta_f''$$

This distribution may be obtained by considering the joint distribution of the two uncorrelated Gaussian quantities, $A = \frac{1}{\sqrt{2}}(\beta_f(t) + \beta_f(t+\Delta t))$ and $B = \frac{1}{\sqrt{2}} \cdot (\beta_f(t) - \beta_f(t+\Delta t))$ and then transform to the variables $\beta_f(t)$ and $\beta_f(t+\Delta t)$. For a different derivation see L. Kittel, 1958, Elementary Statistical mechanics, London, John Wiley and Sons, page 139. The conditional distribution $P(\beta_f''|\beta_f', \Delta t)$ that given $\beta_f(t) = \beta_f'$ one finds $\beta_f(t+\Delta t)$ at $\beta_f'' \pm \frac{1}{2} d\beta_f''$ follows immediately from (3.11) considering that $P(\beta_f', \beta_f''|\Delta t) = P(\beta_f''|\beta_f', \Delta t)\, P(\beta_f')$.

We have,

$$(3.12) \qquad P(\beta_f''|\beta_f', \Delta t) = \frac{1}{\sqrt{2\pi\langle\beta_f^2\rangle(1 - C_f^2(\Delta t))}} \exp - \frac{(\beta_f'' - C_f(\Delta t)\beta_f')^2}{2\langle\beta_f^2\rangle(1 - C_f^2(\Delta t))}$$

Let's now consider the fourier component $\alpha(\omega)$ of $\alpha(t)$. We have

$$\alpha(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dt\, \alpha(t)\, e^{-i\omega t} .$$

We note that $\alpha(\omega)$ has almost the same properties as $\alpha(t)$ (in the complex case exactly the same). We have, $\langle \alpha(\omega)\, \alpha^*(\omega')\rangle =$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} dt' \langle \alpha(t)\, \alpha^*(t')\rangle\, e^{-i\omega t} e^{i\omega' t'} =$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} dt\, e^{i(\omega - \omega')t} = \delta(\omega - \omega') .$$

In the same way we can find that if $\alpha(t)$ is real $\langle \alpha(\omega)\, \alpha(\omega')\rangle = \delta(\omega + \omega')$ (which also follows directly from above since $\alpha^*(\omega) = \alpha(-\omega)$) and if $\alpha(t)$ is complex $\langle \alpha(\omega)\alpha(\omega')\rangle = 0$. Clearly $\langle \alpha(\omega)\rangle$ is always zero. Thus we have,

$$(3.13\,a)\begin{cases}\langle \alpha(\omega)\,\alpha^*(\omega')\rangle = \delta(\omega-\omega') \\ \alpha^*(\omega) = \alpha(-\omega) \\ \mathrm{Re}\,\alpha(\omega),\ \mathrm{Im}\,\alpha(\omega)\ \text{normal} \end{cases},\ \text{if } \alpha(t) \text{ is real.}$$

$$(3.13\,b)\begin{cases}\langle \alpha(\omega)\,\alpha^*(\omega')\rangle = \delta(\omega-\omega') \\ \mathrm{Re}\,\alpha(\omega),\ \mathrm{Im}\,\alpha(\omega)\ \text{normal} \end{cases},\ \text{if } \alpha(t) \text{ is complex.}$$

As for $\alpha(t)$ we may define $\beta_{\Delta\omega}(\omega) = \int_{\omega-\Delta\omega/2}^{\omega+\Delta\omega/2} d\omega'\,\alpha(\omega')$. We have

$$(3.14)\qquad \langle|\beta^2_{\Delta\omega}(\omega)|\rangle = \Delta\omega \quad \text{or} \quad \sqrt{\langle|\beta^2_{\Delta\omega}|\rangle} = \sqrt{\Delta\omega}$$

Note the similarity between this "Nygvist" property and the "random walk" property $\sqrt{\langle\beta^2_{\Delta t}\rangle} = \sqrt{\Delta t}$ of white noise. In fact, regarding transformations of the kind above (i.e. fourier transformation, or any expansion in terms of a complete set of orthonormal functions) as just a change of basis in "function space" we may say that white noise components (or coordinates) from a statistical point of view "looks" the same in the $\{t\}$ basis as in the $\{\omega\}$ basis, and would in fact "look" the same in any orthonormal basis. One might say that white noise is absolutely nonpreferential, its components will be "white" in any orthonormal basis.

Just as $\beta_{\Delta t}(t)$, $\beta_{\Delta\omega}(\omega)$ is distributed with the Gaussian distribution

$$P(\beta_{\Delta\omega})\,d\beta_{\Delta\omega} = \frac{1}{\pi\langle\beta_{\Delta\omega}\beta^*_{\Delta\omega}\rangle}\exp-\frac{\beta_{\Delta\omega}\beta^*_{\Delta\omega}}{\langle\beta_{\Delta\omega}\beta^*_{\Delta\omega}\rangle}\,d\beta_{\Delta\omega}\,,\quad d\beta_{\Delta\omega} = d\beta^{(1)}_{\Delta\omega}\,d\beta^{(2)}_{\Delta\omega}\,,$$

Note that if $\alpha(t)$ is real $\beta^*_{\Delta\omega}(\omega) = \beta_{\Delta\omega}(-\omega)$. Again let $\alpha(t)$ be filtered by the time invariant filter $f(t-t')$ so $\beta_f(t) = \int_{-\infty}^{+\infty} dt'\, f(t-t')\,\alpha(t')$, or in fourier components, $\beta_f(\omega) = \sqrt{2\pi}\, f(\omega)\,\alpha(\omega)$. Since $\beta_f(t)$ has a "non flat" or non white spectrum it is usually called coulored stationary Gaussian noise, or simply coulored noise. (Non stationary noise can be obtained by filtering with a filter $f(t,t')$) All Gaussian coulored noise may be considered to originate from white noise filtered in the above manner. The spectral density of $\beta_f(t)$ at the frequency ω is defined as the spectral contribution to $\langle \beta_f^2(t)\rangle$ per unit frequency at the frequency ω.

$$(3.15)\quad D_f(\omega) = \lim_{\Delta\omega\to 0}\frac{1}{\Delta\omega}\int_{\Delta\omega}\frac{d\omega'}{\sqrt{2\pi}}\int_{\Delta\omega}\frac{d\omega''}{\sqrt{2\pi}}\sqrt{2\pi}\, f(\omega')\sqrt{2\pi}\, f(\omega'')\langle \alpha(\omega')\,\alpha^*(\omega'')\rangle = |f^2(\omega)|$$

It is easily seen that one can write the correlation function (3.8) in terms of the spectral density as

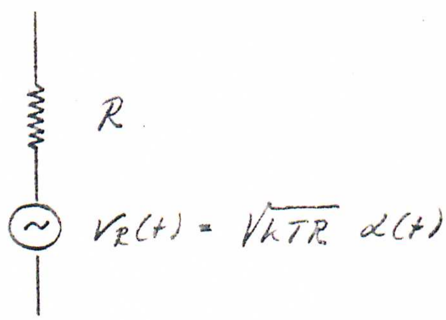
$$(3.16)\quad C_f(\Delta t) = \frac{\int_{-\infty}^{+\infty} dt\, f(t+\Delta t)\, f(t)}{\int_{-\infty}^{+\infty} dt\, |f^2(t)|} =$$

$$= \frac{\int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} d\omega \int_{-\infty}^{+\infty} d\omega'\, f(\omega)\, f^*(\omega')\, e^{i\{(\omega-\omega')t+\omega\Delta t\}}}{\int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} d\omega \int_{-\infty}^{+\infty} d\omega'\, f(\omega)\, f^*(\omega')\, e^{i(\omega-\omega')t}} =$$

$$= \frac{\int_{-\infty}^{+\infty} d\omega\, |f^2(\omega)|\, e^{i\omega\Delta t}}{\int_{-\infty}^{+\infty} d\omega\, |f^2(\omega)|} = \frac{\int_{-\infty}^{+\infty} d\omega\, D_f(\omega)\, e^{i\omega\Delta t}}{\int_{-\infty}^{+\infty} d\omega\, D_f(\omega)}$$

According to the Nyqvist theorem, a resitor in a circuit is an example of a white noise source. (Of course, this is only approximately true, since any real noise source will always have a finite correlation time. For a resistor it is however small enough to disregard in most practical circumstances.) One may thus replace a real resistor R with a "passive" resistor plus a voltage source (see figure 3.1) $V_R(t) = \sqrt{kTR}\;\alpha(t)$ or expressed in (normalized) fourier components $V_R(\omega) = \sqrt{kTR}\;\alpha(\omega)$. Thus the spectral density of the source $V_R(t)$ is $\frac{kTR}{2\pi}$ for all ω. (k is Boltzmann's constant and T the temperature).

FIG. 3.1



FILTERS AND MATCHED FILTERING TECHNIQUES

We will in this section review some basic theory for data filtering, mainly for the purpose of comparison with the "evidence" approach. For a thorough introduction to the theory of filters and matched filtering techniques the reader is referred to Helstroms book "Statistical Theory of Signal Detection."

A linear filter operating on a variable $x(t)$ can in general be defined by

$\bar{x}(t) = \int_{-\infty}^{+\infty} dt' f(t,t') x(t') dt'$ where $f(t,t')$ may be called a filterfunction and $\bar{x}(t)$ is the filtered variable.

We can write this in terms of fourier components,

$$\bar{x}(\omega) = \int_{-\infty}^{+\infty} d\omega' f(\omega,\omega') x(\omega')$$

where

$$f(\omega,\omega') = \int \frac{dt}{\sqrt{2\pi}} \int \frac{dt'}{\sqrt{2\pi}} f(t,t') e^{-i\omega t} e^{-i\omega' t'}$$

($f(\omega,\omega')$ is usually called the transfer function),

and $\check{\bar{x}}(\omega) = \frac{1}{\sqrt{2\pi}} \int dt\, \bar{x}(t)\, e^{i\omega t}$.

If the filter is time invariant, we have

$$f(t,t') = f(t-t'), \quad \text{and}$$

$$f(\omega,\omega') = \delta(\omega-\omega') \int_{-\infty}^{+\infty} d\tau\, f(\tau)\, e^{i\omega\tau} = \delta(\omega-\omega') \sqrt{2\pi}\, f(\omega)$$

so $\bar{x}(\omega) = \sqrt{2\pi}$

We will now see how one can choose the filter to maximize the signal to noise ratio. A filter that is designed in this way is usually called a matched filter, i.e. it is matched to a signal with certain known properties. We shall here only consider signals with definite properties, but the theory may be extended to include signals whose parameters are stochastic with a certain probability distribution (See Helstrom).

We shall however in the following section use the "evidence approach" to obtain some optimal filters, and will then consider an example of a stochastic signal. See also appendix where signal pulses of random phase, arrival time and exponentially distributed energy are considered .

SIGNALS IN WHITE NOISE

Consider now a variable $a(t)$ which is a sum of a signal $x(t)$ and white noise $n\alpha(t)$

(3.17) $$a(t) = x(t) + n\alpha(t)$$

We filter this signal

$$\overline{a(t)} = \int_{-\infty}^{+\infty} f(t-t')(x(t) + n\alpha(t))\,dt',$$ and form

the expectation value of $\bar{a}(t)^2$;

$$\langle \overline{a(t)}^2 \rangle = \left\{\int_{-\infty}^{+\infty} dt' f(t-t') x(t)\right\}^2 + 2\int_{-\infty}^{+\infty} dt' f(t-t') x(t') \cdot$$

$$\cdot \int_{-\infty}^{+\infty} dt' f(t-t') \langle n\alpha(t') \rangle + \left\langle \left\{\int_{-\infty} dt' f(t-t') n\alpha(t')\right\}^2 \right\rangle =$$

$$= \left\{\int_{-\infty}^{+\infty} dt' f(t-t') x(t')\right\}^2 + n^2 \int_{-\infty}^{+\infty} f^2(t-t')\, dt' .$$

The "signal to noise ratio" in this expression is

$$"Sn(x)" = \frac{\left\{\int_{-\infty}^{+\infty} dt' f(t-t') x(t')\right\}^2}{n^2 \int_{-\infty}^{+\infty} f^2(t-t')\, dt'}$$

and we can normalize the filter function

$$\int_{-\infty}^{+\infty} f^2(t-t')\, dt' = 1, \quad \text{so that}$$

$$"Sn(x)" = n^{-2}\left\{\int_{-\infty}^{+\infty} dt' f(t-t') x(t')\right\}^2 .$$

Let's now specify the typical pulse so that it has a form described by a normalized function $\tilde{x}(t)$; $\tilde{x}(t) = 0$; $t < -T$; $t > 0$ where T is the duration of the pulse. A pulse arriving at time $t = t_n$ can then be described by $x_n(t) = C_n \; \tilde{x}(t-(t_n+T))$, where C_n is the amplitude of the pulse and may vary in any fashion from pulse to pulse.

The signal to noise ratio of the n^{th} pulse is

$$"Sn(x/n)" = \frac{C_n^2}{n^2}\left\{\int_{-\infty}^{+\infty} dt' f(t-t')\, \tilde{x}(t'-(t_n+T))\right\}^2$$

It is now obvious that the optimal choice of the filter function is $f(t-t') = \tilde{x}^*(t'-t+\mathcal{T})$

where $\mathcal{T}$ is a time constant which time translates the output and can in the analysis be chosen freely. As long as all the pulses are translated in the same way it doen't matter. If we let $\mathcal{T}$ be zero, we will have the maximum $Sn(X|n)$; $Sn(X|n)_{max}$ at time $t = t_n + T$, just when the whole signal has arrived.

We have

$$Sn(X|n)_{max} = \frac{C_n^2}{n^2} \qquad \text{since} \int_{-\infty}^{+\infty} |\tilde{x}^2(t' - (t_n + T))| \, dt' = 1$$

or

(3.18) $$Sn(X|n)_{max} = n^{-2} \int_{-\infty} dt' \, x_n^2(t')$$

If terms of fourier components we have

$$f(\omega) = \check{X}(\omega)^* \quad ; \int d\omega \, |f^2(\omega)| = 1.$$

$$Sn(X|n) = n^{-2} C_n^2 \left\{ \int_{-\infty}^{+\infty} \frac{d\omega}{\sqrt{2\pi}} \, e^{i\omega(t - (t_n + T))} |\tilde{x}^2(\omega)| \right\}^2 .$$

We can also write

(3.19) $$Sn(X|n)_{max} = n^{-2} \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \, |x_n^2(\omega)| .$$

This is thus the signal to noise ratio if we know in advance the arrival time and the form of the signal.

SIGNALS IN COLORED NOISE

If we instead of white noise have colored noise, we have

(3.20) $a(t) = x(t) + N(t)$

where $N(t)$ can be thought of as being white noise filtered in some stationary way.

$N(t) = \int_{-\infty}^{+\infty} n(t-t')\alpha(t')\,dt'$. It is in this case more convenient to work in "ω" space. Here we can write (3.20) as

(3.21) $a(\omega) = x(\omega) + \sqrt{2\pi}\, n(\omega)\alpha(\omega)$

We can now transform $a(\omega)$ with the filter $(\sqrt{2\pi}\, n(\omega))^{-1}$,

$\dfrac{a(\omega)}{\sqrt{2\pi}\, n(\omega)} = \dfrac{x(\omega)}{\sqrt{2\pi}\, n(\omega)} + \alpha(\omega)$ so we have an expression which is a sum of a signal and white noise. We can then apply the rules for optimal filtering of a signal plus white noise.

Thus we have for an input $a(\omega) = x(\omega) + \sqrt{2\pi}\, n(\omega)\alpha(\omega)$ that the optimal transfer function is:

(3.22) $f(\omega) = \dfrac{\tilde{x}^*(\omega)}{2\pi|n^2(\omega)|}$ so $\bar{a}(\omega) = \dfrac{a(\omega)}{\sqrt{2\pi}}\,\dfrac{\tilde{x}^*(\omega)}{|n^2(\omega)|} \Rightarrow$

(3.23) $\bar{a}(\omega) = \dfrac{\tilde{x}^*(\omega)\,x(\omega)}{\sqrt{2\pi}\,|n^2(\omega)|} + \alpha(\omega)\,\dfrac{\tilde{x}^*(\omega)}{n(\omega)} = \bar{a}_s(\omega) + \bar{a}_n(\omega)$

and;

$$Sn(x|n) = \frac{2\left\{\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\,\frac{\tilde{x}^*(\omega)\,x_n(\omega)}{|n^2(\omega)|}\,e^{i\omega t}\right\}^2}{\int_{-\infty}^{+\infty} \frac{|\tilde{x}^2(\omega)|}{|n^2(\omega)|}\,\frac{d\omega}{2\pi}}$$

which we may write

$$Sn(x/n,t) = \frac{2\left\{\int_{-\infty}^{+\infty} dw \frac{|x_n^2(\omega)|}{|n^2(\omega)|} e^{i\omega(t-(t_n+T))}\right\}^2}{2\pi \int_{-\infty}^{+\infty} dw \frac{|x_n^2(\omega)|}{|n^2(\omega)|}}$$

and

(3.24) $$Sn(x/n)_{max} = 2\int_{-\infty}^{+\infty} dw \frac{|x_n^2(\omega)|}{2\pi|n^2(\omega)|}$$

We note here that when the data has been filtered with a matched filter the noise correlation function and the signal have the same form (and spectrum) i.e.

$$C_{\bar{N}}(\Delta t) = \langle \bar{N}(t), \bar{N}(t+\Delta t)\rangle = \int_{-\infty}^{+\infty} dw \frac{|\tilde{x}^2(\omega)|}{|n^2(\omega)|} e^{i\omega\Delta t}$$

and

$$\bar{x}(t) = \int_{-\infty}^{+\infty} \frac{dw}{2\pi} \frac{\tilde{x}^*(\omega)\, x(\omega)}{|n^2(\omega)|} e^{i\omega t} = \frac{C}{2\pi}\int_{-\infty}^{+\infty} dw \frac{|\tilde{x}^2(\omega)|}{|n^2(\omega)|} e^{i\omega t} =$$

$$= \frac{C}{2\pi} C_{\bar{N}}(t).$$

($x(\omega) = C\,\tilde{x}(\omega)$, see page 67).

The probability distribution of $\bar{a}(t)$ is

$$(3.25)\quad P(\bar{a}(t))\,da_1\,da_2 = \frac{1}{\pi\langle \bar{N}^2\rangle} \exp - \frac{(\bar{a}_1(t)-\bar{x}_1(t))^2+(\bar{a}_2(t)-\bar{x}_2(t))^2}{\langle \bar{N}^2\rangle}\,da_1\,da_2 .$$

(We remind that $\bar{a}$ is a complex quantity, $\bar{a} = \bar{a}_1 + i\,\bar{a}_2$).

We can write this in radial coordinates

$$\bar{a}_1 = r_a \cos\theta_a \qquad \bar{x}_1 = r_x \cos\theta_x$$

$$\bar{a}_2 = r_a \sin\theta_a \qquad \bar{x}_2 = r_x \sin\theta_x \quad \Rightarrow$$

$$P(r_a,\theta_a)\,dr_a\,d\theta_a = \frac{1}{\pi\langle \bar{N}^2\rangle}\left\{\exp - \frac{r_a^2+r_x^2-2r_a r_x \sin(\theta_a+\theta_x)}{\langle \bar{N}^2\rangle}\right\} r_a\,dr_a\,d\theta_a$$

$$\langle \bar{N}^2\rangle = \langle r_N^2\rangle = \langle N_1^2 + N_2^2\rangle$$

If θ_x (the phase of the signal) is unknown, we sum over θ_a and get

$$(3.26)\quad P(r_a) = \int_0^{2\pi} d\theta_a\, P(r_a,\theta_a) =$$

$$= \frac{2}{\langle \bar{N}^2\rangle} \exp\left(-\frac{r_a^2+r_x^2}{\langle \bar{N}^2\rangle}\right) I_0\left(2\frac{r_a r_x}{\langle \bar{N}^2\rangle}\right) r_a\,dr_a$$

where I_0 is the zeroth order Bessel function.

If we define $r_a^2 = E_a$, $r_x^2 = E_s$ and $\langle \bar{N}^2\rangle = E_{Th}$

we have

$$(3.27)\quad P(E_a)\,dE_a = \frac{1}{E_{Th}} \exp\left(-\frac{E_a+E_s}{E_{Th}}\right) I_0\left(2\frac{\sqrt{E_a E_s}}{E_{Th}}\right) dE_a$$

An asymptotic form for this is useful; we let $E_s \gg E_{Th}$

expand around E_s $\Rightarrow$

$$(3.28)\quad P(E_a)\,dE_a = (4\pi E_{Th} E_s)^{-1/2} \exp - \frac{(E_a-E_s)^2}{4E_{Th}E_s}\,dE_a$$

We will now as examples consider the particular form of the evidence increase for two cases, (1) a signal of known phase form and magnitude (the same case as was considered in "signals in coulored noise") (2) a stochastic signal.

It is assumed that the only possible hypotheses are; either a signal (t), or, no signal at all, We have,

If X is true $\qquad a(\omega) = X(\omega) + \sqrt{2\pi}\, n(\omega)\,\alpha(\omega)$

If X is not true $\qquad a(\omega) = \sqrt{2\pi}\, n(\omega)\,\alpha(\omega)$

If phase as well as form and magnitude is included in our hypothesis we should use the distribution (3.25) For the quantity

$$a_{\Delta\omega}(\omega) = \frac{1}{\Delta\omega}\int_{\omega}^{\omega+\Delta\omega} d\omega\, a(\omega) .$$

(Similarly with $X_{\Delta\omega}(\omega)$ and $n_{\Delta\omega}(\omega) = \frac{1}{\Delta\omega}\int_{\omega}^{\omega+\Delta\omega} d\omega\, n(\omega)\alpha(\omega)$)

we have

$$P_{s+n}(a_{\Delta\omega}(\omega)) = \frac{1}{\pi \langle n^2_{\Delta\omega}(\omega)\rangle 2\pi} \exp - \frac{|a_{\Delta\omega}(\omega) - X_{\Delta\omega}(\omega)|^2}{\langle |n^2_{\Delta\omega}(\omega)|\rangle 2\pi} \quad \text{and}$$

$$P_n(a_{\Delta\omega}(\omega)) = \frac{1}{\pi \langle n^2_{\Delta\omega}(\omega)\rangle 2\pi} \exp - \frac{|a_{\Delta\omega}(\omega)|^2}{\langle |n^2_{\Delta\omega}(\omega)|\rangle 2\pi} .$$

Now $\langle |n^2_{\Delta\omega}(\omega)|\rangle = \frac{1}{\Delta\omega}\, |n^2(\omega)|$ (compare)

and $a_{\Delta\omega}(\omega) \xrightarrow[\Delta\omega\to 0]{} a(\omega)$ etc., Thus we can write the increase in evidence as (with k = 2)

$$\Delta\, ev(X, a) = 2 \sum_{\substack{n=-\infty \\ \omega_n = n\cdot\Delta\omega \\ \Delta\omega\to 0}}^{+\infty} \ln \frac{\exp - \dfrac{|a_{\Delta\omega}(\omega_n) - X_{\Delta\omega}(\omega_n)|^2}{\dfrac{2\pi |n^2(\omega_n)|}{\Delta\omega}}}{\exp - \dfrac{|a^2_{\Delta\omega}(\omega_n)|}{\dfrac{2\pi |n^2(\omega_n)|}{\Delta\omega}}}$$

$$\Delta ev(x,a) = 2 \sum_{\substack{n=-\infty \\ \omega_n = n\cdot\Delta\omega \\ \Delta\omega \to 0}}^{+\infty} + \frac{a_{\Delta\omega}(\omega_n) x^*_{\Delta\omega}(\omega_n) + a^*_{\Delta\omega}(\omega_n) x_{\Delta\omega}(\omega_n) - |x^2_{\Delta\omega}(\omega_n)|}{2\pi |n^2(\omega_n)|} \Delta\omega =$$

$$= 2 \int_{-\infty}^{+\infty} d\omega \; \frac{a(\omega) x^*(\omega) + a^*(\omega) x(\omega) - |x^2(\omega)|}{2\pi |n^2(\omega)|} .$$

The implicit filter in this expression is similar to 3.22 but differs in that we are now also "filtering" with respect to the amplitude. The signal to noise ratio is

$$Sn(x) = \langle \Delta ev(x,a) \rangle_{x+n} = 2 \int_{-\infty}^{+\infty} d\omega \; \frac{2x^2(\omega) - x^2(\omega) + 2x(\omega) n(\omega) \langle \alpha(\omega) \rangle}{2\pi n^2(\omega)} =$$

$$= 2 \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \frac{x^2(\omega)}{n^2(\omega)} .$$

(2) Stochastic Signals

We will now investigate the case of random signals i.e. , the signal is defined by $s(t) = \int_{-\infty}^{+\infty} dt \; s(t-t') \alpha_s(t')$.

The data is thus

$$(3.29) \quad \begin{cases} a(t) = \int_{-\infty}^{+\infty} dt' \, s(t-t') \alpha_s(t') + \int_{-\infty}^{+\infty} dt' \, n(t-t') \alpha_n(t) \\ \frac{1}{\sqrt{2\pi}} a(\omega) = s(\omega) \alpha_s(\omega) + n(\omega) \alpha_n(\omega) \end{cases}$$ or

Since $a(t)$ is a sum of gaussian variables, it must itself be a gaussian variable, so its square must have an exponential distribution i.e. if

$$|a^2(t)| = E(t) = E_s(t) + E_n(t) \quad \text{and}$$

$$\langle E_s(t)\rangle = \int_{-\infty}^{+\infty} dt'\, s^2(t-t') \;;\; \langle E_n(t)\rangle = \int_{-\infty}^{+\infty} dt'\, n^2(t-t')$$

(Similarly for $a^2(\omega)$ etc.) we have

$$P_{n+s}(E(t)) = \frac{1}{\langle E_n\rangle + \langle E_s\rangle} \exp - \frac{E}{\langle E_n\rangle + \langle E_s\rangle}$$

$$P_n(E) = \frac{1}{\langle E_n\rangle} \exp - \frac{E}{\langle E_n\rangle}$$

It is easy to form the evidence in favor of "s" for a single data point $a(t_0)$, we have

$$\Delta ev(a(t_0), s) = 2 \ln \frac{P_{n+s}(E(t_0))}{P_n(E(t_0))} = \frac{E(t_0)\langle E_s\rangle}{\langle E_n\rangle\langle E_s + E_n\rangle} - \ln\left(1 + \frac{\langle E_s\rangle}{\langle E_n\rangle}\right)$$

This is not of much help however, because we are interested in the evidence from a strech of data say from $t=t_0$ to $t=t_0+T$. We cannot simply add up the evidence from each point since close points are correlated and thus contain redundant information. (See page 36)

Consider now $\frac{1}{\sqrt{2\pi}}\, a(\omega) = s(\omega)\alpha_s(\omega) + n(\omega)\alpha_n(\omega)$. We have $\frac{1}{2\pi}\langle a(\omega)a(\omega')\rangle = \delta(\omega-\omega')\{s^2(\omega) + n^2(\omega)\}$, thus there is no interdependent information among the fourier components $a(\omega)$.

As before we let $s_{\Delta\omega}(\omega) = \frac{1}{\Delta\omega}\int_{\Delta\omega} s(\omega)\alpha_s(\omega)\,d\omega$, similarly for $n_{\Delta\omega}(\omega)$ and $a_{\Delta\omega}(\omega)$, so that $\langle s^2_{\Delta\omega}(\omega)\rangle = \frac{1}{\Delta\omega}|s^2(\omega)|$ etc.

We then have:

$$
3.30)\begin{cases} P_{n+s}(|a^2_{\Delta\omega}(\omega)|) = \dfrac{\Delta\omega}{|n^2(\omega)+s^2(\omega)|2\pi}\exp -\dfrac{|a^2_{\Delta\omega}(\omega)|\Delta\omega}{|n^2(\omega)+s^2(\omega)|2\pi} \quad \text{and} \\[2ex] P_n(|a^2_{\Delta\omega}(\omega)|) = \dfrac{\Delta\omega}{|n^2(\omega)|2\pi}\exp -\dfrac{|a^2_{\Delta\omega}(\omega)|\Delta\omega}{|n^2(\omega)|2\pi} . \end{cases}
$$

We may thus write the total increase in evidence for "s" as (Note, $\langle a(\omega_i)a(\omega_j)\rangle = \delta_{ij}$)

$$
\Delta\, ev(s,a) = 2\sum_{\substack{\omega=\omega_i \\ \Delta\omega\to 0}} \frac{|a^2_{\Delta\omega}(\omega)|\,|s^2(\omega)|}{|n^2(\omega)|\,|n^2(\omega)+s^2(\omega)|2\pi}\Delta\omega - 2\sum_{\substack{\omega=\omega_i \\ \Delta\omega\to 0}} \ln\left(1+\frac{|s^2(\omega)|}{|n^2(\omega)|}\right)
$$

These terms may both be infinite, but nevertheless the expression makes sense in the following way.

We can write

$$
1 = \int_{\Delta\omega} d\omega\,\delta(\omega-\omega_i) = \int_{-\infty}^{+\infty} dt \int d\omega \frac{1}{2\pi}\exp i(\omega-\omega_i)t
$$

thus

$$
\Delta\, ev(s,a) = 2\int_{-\infty}^{+\infty} d\omega \frac{|a^2(\omega)|\,|s^2(\omega)|}{|n^2(\omega)|\,|n^2(\omega)+s^2(\omega)|2\pi} - 2\int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} d\omega \frac{1}{2\pi}\ln\left(1+\frac{|s^2(\omega)|}{|n^2(\omega)|}\right)
$$

We can write this as

$$
\Delta\, ev(s,a) = 2\int_{-\infty}^{+\infty} dt \int \frac{d\omega}{\sqrt{2\pi}} \int \frac{d\omega'}{\sqrt{2\pi}} \left\{\frac{a(\omega)\,s^*(\omega)\,e^{-i\omega t}}{\sqrt{2\pi}\,n^*(\omega)[|n^2(\omega)+s^2(\omega)|^{1/2}]}\right\}\left\{\frac{a^*(\omega')\,s(\omega')\,e^{i\omega' t}}{\sqrt{2\pi}\,n(\omega')[|n^2(\omega')+s^2(\omega')|^{1/2}]}\right\} -
$$

$$
- 2\int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} d\omega \frac{1}{2\pi}\ln\left(1+\frac{|s^2(\omega)|}{|n^2(\omega)|}\right) \qquad \text{or}
$$

$$(3.31)\quad \Delta ev(s,a) = 2\int_{-\infty}^{+\infty} dt\, |\bar{a}^2(t)| - 2\int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} d\omega \frac{1}{2\pi} \ln\left(1 + \frac{|s^2(\omega)|}{|n^2(\omega)|}\right)$$

where $$\bar{a}(t) = \int_{-\infty}^{+\infty} dt'\, f(t-t')\, a(t')$$

and $$f(t-t') = \int_{-\infty}^{+\infty} \frac{d\omega}{\sqrt{2\pi}} \frac{s^*(\omega)\, e^{i\omega(t-t')}}{\sqrt{2\pi}\, n^*(\omega)[(n^2(\omega)+s^2(\omega))^{1/2}]}$$ or

$$\bar{a}(\omega) = \frac{a(\omega)\, s^*(\omega)}{\sqrt{2\pi}\, n^*(\omega)[(n^2(\omega)+s^2(\omega))^{1/2}]} .$$

Thus we have that the optimal filter for $a(\omega)$ is[(+)]

$$(3.32)\quad f(\omega) = \frac{s^*(\omega)}{n^*(\omega)[(n^2(\omega)+s^2(\omega))^{1/2}]} .$$

(3.31) tells us that the new evidence from the strech of data

$\bar{a}(t_0)$ to $\bar{a}(t_0+\Delta t)$ is

$$\Delta ev(s,a/\Delta t) = 2\int_{t_0}^{t_0+\Delta t} dt\, |\bar{a}^2(t)| - \frac{2\Delta t}{2\pi}\int_{-\infty}^{+\infty} d\omega \ln\left(1 + \frac{|s^2(\omega)|}{|n^2(\omega)|}\right)$$

We may write this:

$$(3.33)\quad \Delta ev(s,a/\Delta t) = 2\int_{t_0}^{t_0+\Delta t} dt\, |\bar{a}^2(t)| - 2\frac{\Delta t}{\mu_0} \ln\left(1 + \frac{\langle \bar{s}^2(t)\rangle}{\langle \bar{n}^2(t)\rangle}\right)$$

(+) This formula corresponds to the formula for m(t,s) given by Helstrom on page 386.

where μ_0 is a time constant:

$$\mu_0 = \frac{2\pi \ln\left(1+\frac{\langle \bar{s}^2(t)\rangle}{\langle \bar{n}^2(t)\rangle}\right)}{\int_{-\infty}^{+\infty} d\omega \ln\left(1+\frac{|s^2(\omega)|}{|n^2(\omega)|}\right)} = \frac{2\pi \ln\left(1+\frac{\int_{-\infty}^{\infty} d\omega\, |f^2(\omega)\, s^2(\omega)|}{\int_{-\infty}^{\infty} d\omega\, |f^2(\omega)\, n^2(\omega)|}\right)}{\int_{-\infty}^{+\infty} d\omega \ln\left(1+\frac{|s^2(\omega)|}{|n^2(\omega)|}\right)}$$

taking the expectation value of Δev we get the signal to noise:

$$\text{(3.34)}\quad Sn(S|\Delta t) = \langle ev(S, a_{\Delta t})\rangle_S = 2\Delta t \left\{ \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \frac{|s^2(\omega)|}{|n^2(\omega)|} - \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \ln\left(1+\frac{|s^2(\omega)|}{|n^2(\omega)|}\right)\right\} 2\Delta t =$$

$$= 2\Delta t \left\{ \langle \bar{s}^2(t) \div \bar{n}^2(t)\rangle - \frac{1}{\mu_0} \ln\left(1+\frac{\langle \bar{s}^2(t)\rangle}{\langle \bar{n}^2(t)\rangle}\right)\right\}.$$

The derivation of (3.30) to (3.34) may seem a bit careless (with infinities, etc.) but a more careful derivation using a fourier series over a finite interval, then going to the limit of an infinite interval would give the same result.

Since in practice the output is multiplied with a gain factor one will have to normalize the factor $a^2(t)$ in 3.33. One way to do this is to measure the average output noise power (when no signal is present) n^2_{exp}, and then multiply $a^2(t)$ with the factor n^2/n^2_{exp}, where n^2 is the theoretical average noise power;

BURST OF STOCHASTIC SIGNALS

The case of bursts of random signals lasting for a finite time interval T could in principle be treated in a similar manner, but the resulting calculations would probably be formidable. One might guess that if $T \gg \mu$ is the average duration of the bursts that

$$E_T(t) = \int_{t-T}^{t} dt \; \bar{a}^2(t)$$ is approximately the correct output

variable to use. Based on the probability distribution of $E_T(t)$, we may then form the increase in evidence from $E_T(t)$.

We note here that the formulas for probability distributions of such quadratic integrals are quite complicated (See Helstrom page 372). We avoid these complications here and just note that if $T \gg \mu$, the distribution of E_T gets close to Gaussian i.e. $P_n(E_T) \sim \exp - \frac{(E_T - \langle E_T \rangle)^2}{2(\Delta E_T)^2}$

The results of this chapter are mainly important to design of sensors and electronics and as a basis for data analysis. We will come back to the first topic later when we analyze some specific detector designs and the second in the data analysis chapter.

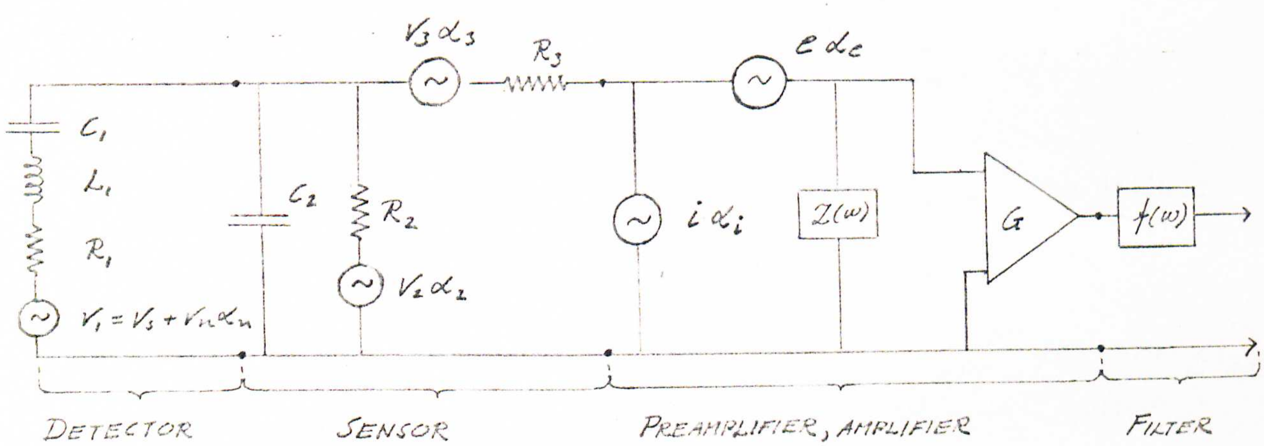
EVIDENCE AND GRAVITATIONAL RADIATION DETECTORS

Some conventions are useful in the discussion of detectors. We shall refer to a "bare" detector as just the antenna (e.g. Webers aluminum cylinder) without sensors and amplifiers. In referring to the sensitivity etc. of a "bare" detector, we mean the sensitivity such a detector would have if equipped with perfect (noiseless) sensors and amplifiers. The sensors and amplifiers alone shall be described as the "dress".

A dressed detector can conveniently be described in terms of an equivalent circuit (a method adopted by Weber since his first experiments).

This may be accomplished by constructing a circuit which has the same coupled differential equations as the "dressed detector. We give the typical Weber circuit as an example.

Figure 5.3



where V_s corresponds to the signal, (see page 126) V_n, V_2, V_3, e

and i are white noise sources. $V_n^2 = kT_1R_1$; $V_2^2 = kT_2R_2$; $V_3^2 = kT_3R_3$ (see page 72).

For such circuits we can in general write the output voltage as

$$V_{out}(\omega) = V_{s(out)}(\omega) + V_{n(out)}(\omega) = f(\omega)\{q_1(\omega)(V_s(\omega) + V_n \alpha_n(\omega)) + q_2 \alpha_2(\omega) + \cdots + q_n \alpha_n(\omega)\}$$

where α_n is a normalized white noise source, and $q_i(\omega)$ may be considered as propagators, propagating the signal or the noise from the source to the output.

Consider now a set of hypotheses $\{G_i\}$, and let G_o be the "noise only" hypothesis. Further let $V(t) = V_{out}(t)$ be the data from an experimental run. One may then write the evidence increase in favor of G_i relative to G_o as follows.

We define $$V_{\Delta\omega}(\omega_\nu) = \frac{1}{\Delta\omega}\int_{\omega_\nu - \Delta\omega/2}^{\omega_\nu + \Delta\omega/2} d\omega \, V(\omega) \quad .$$

Further let $P_i(V_{\Delta\omega}(\omega_\nu))$ be the probability distribution function for $V_{\Delta\omega}(\omega_\nu)$ under the hypothesis G_i. We have;

$$P_o(V_{\Delta\omega}(\omega)) = \frac{\Delta\omega}{2\pi\langle E(\omega)\rangle} \exp - \frac{|V_{\Delta\omega}^2(\omega)|\,\Delta\omega}{\langle E(\omega)\rangle}$$

where $$E(\omega) = |q_1^2(\omega)| + |q_2^2(\omega)| + \cdots + |q_n^2(\omega)| \,.$$

Now let the signal $V_{s_i\Delta\omega}(\omega)$ associated with G_i, be defined by some probability distribution $P_{s_i}(V_{s_i\Delta\omega}(\omega))$. We have,

$$(3.35) \quad P_i(V_{\Delta\omega}(\omega)) = \int d\, V_{s_i\Delta\omega}(\omega)\, P_{s_i}(V_{s_i\Delta\omega}(\omega_n)) \frac{\Delta\omega}{2\pi\langle E(\omega)\rangle} \exp - \frac{|V_{\Delta\omega}(\omega) - V_{s_i\Delta\omega}(\omega)|^2 \Delta\omega}{\langle E(\omega)\rangle}$$

The evidence increase is (Let $\omega_\nu = \nu \cdot \Delta\omega$)

$$(3.36) \quad \Delta ev(G_i|(G_i + G_o)c) = \lim_{\Delta\omega \to 0} \sum_{\nu=-\infty}^{+\infty} \Delta\omega \frac{k}{\Delta\omega} \ln \frac{P_i(V_{\Delta\omega}(\omega_\nu))}{P_o(V_{\Delta\omega}(\omega_\nu))} =$$

$$= \int_{-\infty}^{+\infty} d\omega \, k \ln F(\omega) \,, \quad \text{where,} \quad F(\omega) = \lim_{\Delta\omega \to 0} \frac{1}{\Delta\omega} \ln \frac{P_i(V_{\Delta\omega}(\omega))}{P_o(V_{\Delta\omega}(\omega))} \,.$$

The distribution (3.35) is, except in special cases (such as 3.30) not easy to calculate. In the more complex cases investigated in the appendix approximative methods have been used to evaluate (3.35).

CONCLUSION

In chapter two a theory for data analysis based on Bayesian techniques was introduced, and in this chapter we studied Gaussian noise theory which allows the application of the theory for data analysis to gravitational radiation detection experiments. It was seen that the theory integrates all the steps in the data processing, providing the linear filter and other data integrating functions.

We will in the appendices investigate the details of gravitational radiation detectors and study the specifics of the theory for data processing for such systems.

An optimal method for data recording is also proposed in appendix D, and the results of a strech of data, recorded and analyzed according to the methods suggested in this dissertation is also presented.

We give below a summary of the results of the appendices related to chapter two and three.

In appendix B, the thermal fluctuations of the detector are studied, and an expression for the "signal to noise" of a bare detector, in terms of the spectral power of the gravitational radiation is derived.

The"dressed" detector is investigated in appendix C. Preamplifier properties are considered, and a preamplifier temperature is defined in terms of its noise sources. Detector and preamplifier matching is discussed briefly. A detector with non-resonant sensor is studied and an expression for the signal to noise in terms of the spectral power of the gravitational radiation is given. The form of this expression allows the definition of a spectral signal to noise ratio, which when integrated gives the total signal to noise ratio. It is seen that the spectral signal to noise ratio splits up into two factors, of which one is the spectral signal to noise of the "bare" detector and the other a spectral quality factor of the "dress". This quality factor is related to a "kT" excitation" (a sudden excitation which, if initially the detector has zero energy, gives it the energy kT) which is often used in discussing the quality of the dress. The optimal filter for such excitations is derived and it is found that this filter is invertible (i.e. it does not lead to a loss of information) and allows further filtering to match any type of signal. Next a detector with a resonant type of sensor is considered, and expressions for sensitivity and optimal filter are given. As expected it is seen that this filter is more complicated than the optimal filter for a non resonant pick-up system. Thus unless the resonant sensor has other advantages (such as providing a good match to the preamplifier) it should be avoided.

In appendix D the detailed theory for data processing for Weber type detectors, under a realistic set of signal hypotheses is developed. The type of signals considered are random pulses, with a certain average frequency of occurrence, and with an exponential distribution in energy. In the two detector cases, two types of superposed signals are considered.

One type consists of pulses that are common to both detectors, i.e. these pulses will cause a correlation between the two detector-outputs. The other type of pulses are "local excitations" and will cause no correlation at all between the detector outputs.

The evidence for the best hypothesis allowing "common pulses" relative to the best hypothesis not allowing "common pulses" is computed. Finally, some preliminary results in this type of analysis of real data, is presented.

Helstrom C.W. 1963 <u>Statistical Theory of Signal Detection</u> (New York, Pergamon Press)

Kittel, L. 1958 <u>Elementary Statistical Mechanics</u> (London, John Wiley and Sons).

CHAPTER IV

COMPUTER SIMULATION OF A WEBER TYPE DETECTOR SYSTEM.

THEORY.

We will in this chapter consider a computer simulation of a pair of gravitational wave detectors. This study stands apart from other topics considered in the dissertation, and may be considered as a separate part. The reason of this study is the difficulty in understanding some of Webers experimental results. In analyzing his data with the well-known threshold-crossing-time delay technique, he gets positive results when the data analysis is based on the quantity P^2 (See page 79) but nothing or almost nothing when the analysis is based on $x^2 + y^2$ (See page 79). Since the object is to see if these results can be repeated with this simulation, we will use the usual threshold-crossing time delay technique in analyzing the simulated data.

The study could have been done analytically, but the calculations would be forbidding, and would have to be done separately for each type of signal considered, while once the computer model works, one can just put in any kind of signal and wait for the answer. The details of the results and the computer program is given in appendix E. We will here develop the generating formula for the simulated output. From appendix G we have two components of the detector output that can be written

$$(4.1) \qquad V_i(t) = \int_{-\infty}^{t} dt \; e^{-(t-t')/T} g_1\left(S_i(t') + n_1 \alpha_i^{(1)}(t')\right) + n_2 \alpha_i^{(2)}(t)$$

$$i = 1, 2$$

where $S_i(t)$ is the detector front end signal and $n_1 \alpha_i^{(1)}(t)$, the front end white noise (which when filtered by $e^{-(t-t')/\mathcal{J}}$ becomes narrowbanded) and $n_2\alpha_i^{(2)}(t')$ is the sensor and pre-amplifier white noise. The data $V_i(t)$ is integrally sampled to a new set of data $\{V_{i\nu}\}$ where $t_\nu - t_{\nu-1} = \Delta t$.

$$V_{i\nu} = \int_{t_{\nu-1}}^{t_\nu} dt\, V_i(t) = \int_{t_{\nu-1}}^{t_\nu} dt \left\{ \int_{-\infty}^{t} dt' \exp - \frac{(t-t')}{\mathcal{J}}\; g_1(S_i(t') + n_1\alpha_i^{(1)}(t')) + n_2\alpha_i^{(2)}(t) \right\}.$$

One may write this

$$V_{i\nu} = \int_{-\infty}^{t_\nu} dt \exp - \frac{t_\nu - t}{\mathcal{J}} \left\{ \int_{t-\Delta t}^{t} dt'\; g_1(S_i(t') + n_1\alpha_i^{(1)}(t')) + n_2\alpha_i^{(2)}(t') \right\} =$$

$$= \sum_{n=-\infty}^{\nu} \int_{t_{n-1}}^{t_n} dt \exp - \frac{t_\nu - t}{\mathcal{J}} \left\{ \int_{t-\Delta t}^{t} dt'\, g_1(S_i(t') - n_1\alpha_i^{(1)}(t')) + n_2\alpha_i^{(2)}(t') \right\}$$

which one may write

(4.2) $\quad V_{i\nu} = U_{i\nu} + n_2\beta_{i\nu}^{(2)}$ where

$$U_{i\nu} = \exp - \Delta t/\mathcal{J}\; U_{i\,\nu-1} + g_1 S_{i\nu} + g_1 n_1 \beta_{i\nu}^{(1)}, \text{ and } \beta_{i\nu}^{(2)} = \int_{t_{\nu-1}}^{t_\nu} dt\, \alpha_i^{(2)}(t).$$

further

$$S_{i\nu} = \int_{t_{\nu-1}}^{t_\nu} dt \exp - \frac{(t_\nu - t)}{\mathcal{J}} \int_{t-\Delta t}^{t} dt'\, S_i(t') \,, \qquad \text{and}$$

$$\beta_{i\nu}^{(1)} = \int_{t_{\nu-1}}^{t_\nu} dt \exp - \frac{(t_\nu - t)}{\mathcal{J}} \int_{t-\Delta t}^{t} dt'\, \alpha_i^{(1)}(t') \,.$$

Δt is chosen such that $\Delta t \ll T$. In this case one may approximate $\exp - \frac{(t_\nu - t)}{T} \approx 1$ in the integral.

The auto correlation function of $\beta_{i\nu}^{(1)}$ and $\beta_{i\nu}^{(2)}$ are

$$\langle \beta_{i\nu}^{(1)} \beta_{i\mu}^{(1)} \rangle = \begin{cases} 2/3 \, \Delta t^3, & \mu = \nu \\ 1/6 \, \Delta t^3, & \mu = \nu \pm 1 \\ 0 & , |\mu - \nu| > 1 \end{cases}$$

and

$$\langle \beta_{i\nu}^{(2)} \beta_{i\mu}^{(2)} \rangle = \Delta t \, \delta_{\mu\nu} \quad .$$

If Δt is chosen small enough $\beta_{i\nu}^{(1)}$ will have a correlation time that is much smaller than the correlation time T of the finally filtered output (i.e. $V_{i\nu}$ will be filtered by an optimal filter). The exact correlation time of $\beta_{i\nu}^{(1)}$ will then become irrelevant and can be chosen at will as long as it remains much smaller than T. (In the same way the actual correlation time of the white noise from a resistor is irrelevant in most applications). To preserve normalization however one must choose the new $\beta_{i\nu}^{(1)'}$ such that $\langle \beta_{i\nu} \sum_n \beta_{in} \rangle$ is conserved. We will thus replace the old $\beta_{i\nu}^{(1)}$ with $(\Delta t)^{3/2} \beta_{i\nu}^{'(1)}$ where $\beta_{i\nu}^{'(1)}$ is normal and $\langle \beta_{i\nu}^{'(1)} \beta_{i\mu}^{'(1)} \rangle = \delta_{\mu\nu}$. We also replace $\beta_{i\nu}^{(2)}$ with $(\Delta t)^{1/2} \beta_{i\nu}^{'(2)}$, where $\beta_{i\nu}^{'(2)}$ is normal and $\langle \beta_{i\nu}^{'(2)} \beta_{i\mu}^{'(2)} \rangle = \delta_{\mu\nu}$.

We can now write the generating equation

$$V_{i\nu} = U_{i\nu} + n_2 (\Delta t)^{1/2} \beta_{i\nu}^{(2)} \tag{4.3}$$

$$U_{i\nu} = \exp - \frac{\Delta t}{T} U_{i\nu-1} + g_1 S_{i\nu} + g_1 n_1 (\Delta t)^{3/2} \beta_{i\nu}^{(1)} ,$$

where $\beta_{i\nu}^{(1)}$ and

$\beta_{i\nu}^{(2)}$ are noraml uncorrelated random numbers with the average 1. $u_{i\nu}$ represents the "bare" detector output, and $n_2(\Delta t)^{\frac{1}{2}}\beta_{i\nu}^{(2)}$ the wideband noise that is added to the bare output by sensors and electronics. By comparing (5.1) and (5.2) one can see that

$$g_1 \cdot n_1 / g_2 \approx \frac{R_2}{L_1} \cdot \sqrt{\frac{R_1}{R_2}} = \frac{1}{L_1}\sqrt{R_1 \cdot R_2} \approx 0.5\,\mu = \frac{1}{2T},$$

where T is the resolution time constant for the detector. Renormalizing the above equation we have

(4.4) $$V_{i\nu} = u_{i\nu} + \frac{2T}{\Delta t}\beta_{i\nu}^{(2)},$$

$$u_{i\nu} = \exp - \Delta t/T \; u_{i\nu-1} + C \cdot S_{i\nu} + \beta_{i\nu}$$. (Where C is a constant).

We have ; $\langle u_1^2 + u_2^2 \rangle_{S_{i\nu}=0} \approx T/\Delta t$, which corresponds to the thermal energy of the detector. Thus a signal

(4.5) $$C\,S_{1\nu} = \sqrt{\frac{T}{\Delta t}}\sin\phi \cdot \delta_{\mu\nu}\,, \quad C\,S_{2\nu} = \sqrt{\frac{T}{\Delta t}}\cos\phi\;\delta_{\mu\nu}$$

arbitrary, is a "kT" excitation (See page).

The theory for the generation of the detector output is thus concluded. Necessary input parameters are, detector damping time, "dressed" detector resolution time, and signal parameters.

We give below an outline of the computer program. Pairs (i = 1,2) of random numbers $\{x_{i\nu}^{(j)}\}$ are generated by the computer for each detector (j=1,2). Unfortunately, the random number generator used can only produce random numbers with constant density in a given interval. A transformation is utilized to produce normal random numbers, $y_{i\nu}^{(j)}(x_{1\nu}^{(j)}, x_{2\nu}^{(j)})$ $i = 1, 2$ where i now represents the two phases. These numbers together with an eventual signal are fed into equation (4.4)* to produce the output $V_{i\nu}$. This output is filtered with an exponential filter $\exp - \frac{|t_\nu - t_\mu|}{T}$. From this filtered variable $\bar{V}_{i\nu}$ the quantities;

$$\dot{p}_\nu^{(j)2} \sim \left\{ \bar{V}_{1\nu}^{(j)2} + \bar{V}_{2\nu}^{(j)2} - \bar{V}_{1\nu-1}^{(j)2} - \bar{V}_{2\nu-1}^{(j)2} \right\}^2$$

and

$$(\dot{x}^2 + \dot{y}^2)_\nu^{(j)} = \left\{ (\bar{V}_{1\nu}^{(j)} - \bar{V}_{1\nu-1}^{(j)})^2 + (\bar{V}_{2\nu}^{(j)} - \bar{V}_{2\nu-1}^{(j)})^2 \right\}$$

are constructed. A time delay histogram of the "threshold" cross correlation function as a function of the delay in one channel is then constructed for a number of different thresholds for both $\dot{p}^2$ and $\dot{x}^2 + \dot{y}^2$.

*This equation can in fact be reduced further so that only one pair of random numbers is needed for every step. This was done here to save computer time.

RESULTS AND CONCLUSION

The results of the investigation shows that for most cases the $\dot{x}^2 + \dot{y}^2$ histogram is more sensitive to simultaneous signals in the two detectors, than in the P^2 histogram. Two cases are of special interest however. If the signal is a short large pulse (larger or equal to a "kT" excitation). This result is not so surprising since phase information is less important for a large pulse (See appendix). Further, if there is a frequency offset (usually caused by a slow drift in detector frequency due to small changes in temperature) between the detector and the reference oscillator, the $x^2 + y^2$ is strongly in a negative way (if the offset is not corrected for in the data analysis) especially for small signals or stochastic bursts of small signals, while the P^2 method remains practically unchanged.

The types of signals investigated are

(1) short, medium and long pulses on and off frequency (by "on frequency" we mean that the signal has a central frequency equal to resonance frequency of the detector)

(2) fast-slow frequency swept signals

(3) short-long bursts of stochastic signals

(4) all of the above with frequency offset in one channel.

We must conclude that of all the possibilities tried only frequency offset could account for making "P^2" superior to $x^2 + y^2$.

APPENDIX A

SIGNAL TO NOISE RATIO OF TWO DIMENSIONAL SIGNAL IN GAUSSIAN BACKGROUND

Consider a case where the data is two-dimensional

(A.1) $d = s + n \; ; \; d = d_1 + i d_2 \; ; \; s = s_1 + i s_2 \; ; \; n = n_1 + i n_2$

and the noise is gaussian i.e.

$\langle n n^* \rangle = 2 \langle n_1^2 \rangle = 2 \langle n_2^2 \rangle$ and

(A.2) $$P(n)\, dn_1\, dn_2 = \frac{1}{\pi \langle n n^* \rangle} \exp - \frac{n n^*}{\langle n n^* \rangle}\, dn_1\, dn_2 .$$

If s is completely known, we can multiply (A.1) by $s^* \Rightarrow$.

$ds^* = s s^* + n s^*$, since we now know that the signal part is real, we can write

$E = Re\, ds^* = s s^* + Re\, n s^*$. The R.M.S. deviation

of the noise term (which is now one dim. gaussian) is

$$\left(s_1^2 \langle n_1^2 \rangle + s_2^2 \langle n_2^2 \rangle \right)^{1/2} = \left(\tfrac{1}{2} s^2 \langle n n^* \rangle \right)^{1/2}, \text{ so}$$

$$Sn(E_s) = \frac{2 s^2}{\langle n n^* \rangle} = \frac{s^2}{\langle n_1^2 \rangle} = \frac{s^2}{\langle n_2^2 \rangle} = \frac{2 E_s}{E_{Th}}$$

It is easily verified that this agrees with the definition (25)

If the phase of s is random, we eliminate the phase by squaring;

(A.3) $$E = s s^* + s n^* + s^* n + n n^*$$

$$\langle E \rangle = s s^* + \langle n n^* \rangle \; ; \; E_{Th} = \langle n n^* \rangle ; \; E_s = s s^* .$$

The signal + noise distributions in (A.3) are no longer gaussian so we use (2.15) to find S_n.

We have (See page 71).

$$P_{n+s}(E) = \frac{1}{E_{Th}} \exp - \frac{E+E_s}{E_{Th}} I_0\left(2\frac{\sqrt{E_s E}}{E_{Th}}\right)$$

$$P_n(E) = \frac{1}{E_{Th}} \exp - \frac{E}{E_{Th}} \qquad \text{so}$$

$$\text{(A.4)} \quad S_n(E_s) = 2\int dE \frac{1}{E_{Th}} \exp - \frac{E+E_s}{E_{Th}} I_0\left(2\frac{\sqrt{E_s E}}{E_{Th}}\right) \cdot$$
$$\cdot \ln\left(\exp - \frac{E_s}{E_{Th}} I_0\left(2\frac{\sqrt{E_s E}}{E_{Th}}\right)\right).$$

We evaluate this in the limit of small and large signal energies E_s.

For small values of x;

$\ln I_0(x) \rightarrow \frac{1}{4}x^2 - \frac{1}{64}x^4$. For large values of x;

$\ln I_0(x) \rightarrow x - \frac{1}{2}\ln(2\pi x) + (8x)^{-1}$.

Thus for small values of E_s

$$\text{(A.5)} \quad S_n(E_s) = 2\int_0^\infty dE \frac{1}{E_{Th}} \exp - \frac{E+E_s}{E_{Th}} \exp \frac{E_s E}{E_{Th}^2}\left(\frac{E_s E}{E_{Th}^2} - \frac{E_s}{E_{Th}}\right) =$$

$$= \frac{2E_s^2}{E_{Th}(E_{Th} - E_s)} \qquad (E_s \ll E_{Th}).$$

For large values of E_s

$$S_n(E_s) = 2\int dE \frac{1}{E_{Th}} \exp - \frac{E+E_s}{E_{Th}} \exp \frac{2\sqrt{E_s E}}{E_{Th}} \left(4\pi \frac{\sqrt{E_s E}}{E_{Th}}\right)^{-1/2} \cdot$$

$$\cdot \left(2\frac{\sqrt{E_s E}}{E_{Th}} - \frac{E_s}{E_{Th}} - \frac{1}{2} \ln 4\pi \frac{\sqrt{E_s E}}{E_{Th}}\right) =$$

$$= 2\int \frac{dE}{E_{Th}} \exp - \frac{(\sqrt{E}-\sqrt{E_s})^2}{E_{Th}} \left(\frac{(E_s E)^{1/4}}{(\pi E_{Th})^{1/2}} - \frac{1}{2}\left(\frac{E}{\pi E_{Th}}\right)^{1/2}\right) =$$

$$= \int_0^\infty \frac{dE}{E_{Th}} \exp - \frac{(\sqrt{E}-\sqrt{E_s})^2}{E_{Th}} \frac{E_s^{1/2} - (E^{1/4}-E_s^{1/4})^2}{(\pi E_{Th})^{1/2}} \quad \text{let } x^2 = E$$

$$= \int_0^\infty \frac{2\,dx}{\langle x_{Th}^2\rangle} \exp - \frac{(x-x_s)^2}{\langle x_{Th}^2\rangle} \frac{x\{x_s - (x^{1/2}-x_s^{1/2})^2\}}{(\pi\langle x_{Th}^2\rangle)^{1/2}}$$

expand $(x^{1/2}-x_s^{1/2})^2$ around $x_s \Rightarrow (x^{1/2}-x_s^{1/2})^2 = \frac{(x-x_s)^2}{4x_s} \Rightarrow$

$$S_n(S) = \int_0^\infty \frac{dx}{\langle \pi x_{Th}^2\rangle^{1/2}} \exp - \frac{(x-x_s)^2}{\langle x_{Th}^2\rangle} \frac{2\{x x_s - (x_s+(x-x_s))\frac{1}{4}\frac{(x-x_s)^2}{x_s}}{\langle x_{Th}^2\rangle},$$

If the integral is extended to -infinity

$$S_n(S) \simeq \frac{2x_s^2 - \frac{1}{4}\langle x_{Th}^2\rangle - \frac{1}{2\sqrt{\pi}}\langle x_{Th}^2\rangle \frac{\langle x_{Th}^2\rangle^{1/2}}{x_s}}{\langle x_{Th}^2\rangle}$$

or in terms of energy

$$\text{(A.6)} \quad S_n(E_s) = \frac{2E_s - \frac{1}{4}E_{Th}\left(1 + \frac{1}{2\sqrt{\pi}}\frac{E_{Th}^{1/2}}{E_s^{1/2}}\right)}{E_{Th}} \approx \frac{2E_s - \frac{1}{4}E_{Th}}{E_{Th}}$$

We can conclude that for small values of E_S the loss of phase information essentially degrades the signal to noise ratio, while for large values of E_S, the loss hardly matters. Clearly, Sn is always an increasing function of $\dfrac{E_S}{E_{Th}}$ so we may for convenience use

$$"Sn" = \frac{2E_S}{E_{Th}} .$$

APPENDIX B

THE "BARE" DETECTOR

THERMAL FLUCTUATIONS OF THE CYLINDER AMPLITUDE

Consider again equation (1.16) with only a white noise source on the right hand side:

(B.1) $\dot{v} + \gamma v = a e^{i\omega_0 t} \alpha(t)$, where a can be determined from

energy considerations. For simplicity we may consider $e^{i\omega_0 t}\alpha(t)$ as a normalized white complex noise source, we have:

$$\langle (e^{i\omega_0 t}\alpha(t))(e^{i\omega_0 t'}\alpha(t'))^* \rangle = e^{i\omega_0(t-t')}\langle \alpha(t)\alpha(t')\rangle = \delta(t-t')$$

but

$$\langle (e^{i\omega_0 t}\alpha(t))(e^{i\omega_0 t'}\alpha(t'))\rangle = e^{i\omega_0(t+t')}\langle \alpha(t)\alpha(t')\rangle = e^{2i\omega_0 t}\delta(t-t')$$

However, with any time-averaging over Δt : $\frac{1}{\omega} \ll \Delta t \ll$ possible resolution times, the last expression will vanish. Thus the condition for complex white noise is approximately satisfied.

We may then write (B.1)

(B.2) $\dot{v} + \gamma v = a\,\alpha(t)$, $\alpha(t)$ complex.

(This is in fact a two dimensional "Langevin equation", see e.g. Reif, Statistical and thermal physics, page, 564).

a is determined by the condition:

$$\frac{1}{2} kT = E_{Th} = \frac{m}{4} \langle v(t)\, v(t^*) \rangle$$

The solution to (B.1) is

$$v = \int_{-\infty}^{t} dt'\, e^{-\gamma(t-t')}\, a\,\alpha(t'), \quad \text{so}$$

$$\langle |v^2| \rangle = \int_{-\infty}^{t} dt'\, e^{-2\gamma(t-t')}\, a^2 = \frac{a^2}{2\gamma} \Rightarrow a^2 = \langle |v^2| \rangle\, 2\gamma$$

$$\text{or} \quad a^2 = \frac{4kT\gamma}{m}\,; \qquad a = 2\sqrt{\frac{kT\gamma}{m}}$$

We can thus write (1.16)

$$\text{(B.3)} \qquad \dot{v} + \gamma v = e^{-i\omega_0 t} F(t) + 2\sqrt{\frac{kT\gamma}{m}}\, \alpha(t)$$

where $F(t) = \dfrac{8ac^2}{\pi^2 (1+2n)^2} R_{1100}(t)$.

Consider again the solution to (B.1) $v = \int_{-\infty}^{t} dt'\, e^{-\gamma(t-t')}\, a\,\alpha(t')$.

We have according to (3.8) that the normalized correlation function $C_v(\Delta t)$ for the amplitude V is

$$C_V(\Delta t) = \frac{\int_{-\infty}^{\min t;\, t+\Delta t} dt' \, e^{-\gamma(t+\Delta t-t')} \, e^{-\gamma(t-t')}}{\int_{-\infty}^{t} dt' \, e^{-2\gamma(t-t')}} = e^{-\gamma|\Delta t|}$$

The conditional probability density for V at time $t+\Delta t$, given that $V = V_0$ at time t is

$$(B.4)\quad P(V;V_0,\Delta t)\, dv_1\, dv_2 = \frac{1}{\pi\langle V^2\rangle(1-e^{-2\gamma\Delta t})} \exp - \frac{(V-V_0 e^{-\gamma\Delta t})^2}{\langle V^2\rangle(1-e^{-2\gamma\Delta t})} dv_1\, dv_2$$

$$\langle V^2\rangle = \langle V_1^{\,2} + V_2^{\,2}\rangle = \frac{E_{Th}}{m/2} = \frac{2kT}{m}$$

The probability desnity to find $V = V'$ at t and $V = V''$ at $t + \Delta t$ is

$$(B.5)\quad P(V', V''; \Delta t)\, ds'\, ds'' = \frac{1}{\pi^2\langle V^2\rangle^2(1-e^{-2\gamma\Delta t})} \cdot$$

$$\cdot \exp - \frac{V'^2 - 2V'V''e^{-\gamma\Delta t} + V''^2}{\langle V^2\rangle(1-e^{-2\gamma\Delta t})} ds'\, ds''$$

where $ds = dv_1\, dv_2$.

For reference see e.g. the Wax papers, page 22 to 27, and also Kittel Statistical Physics.

SENSITIVITY CONSIDERATIONS FOR THE "BARE" DETECTOR

(Where "bare" means that there is no wide-band noise from imperfect pick-up systems or amplifiers).

We first note that it is clear from (B.3) that (independently of the shape of the signal, long or short duration) the inverse of the damping time γ, the temperature T, and the inverse of the mass m^{-1} are all on equivalent footing in calculations of signal to noise. We make this observation since the situation has not always been made clear in the literature. (See e.g. Review by Press and Thorne "Gravitational Wave Astronomy, February 1972, Ann. Rev. Astron. Astrophys. 10, 335-374).

Since in the case of an ideal detector, the output is just a one to one transformation of the input, it is sufficient to analyze the input signal on the cylinder. This simplifies the procedure since the noise at the input is white.

From equation (B.3) we have the input signal (driving acceleration)

$$a(t) = \frac{8ac^2}{\pi^2(1+2n)^2} R(t)\, e^{-i\omega_0 t} + 2\sqrt{\frac{kT\gamma_n}{m}}\,\alpha(t), \text{ or}$$

in terms of fourier components:

$$a(\omega) = \frac{8ac^2}{\pi^2(1+2n)^2} R(\omega+\omega_0) + 2\sqrt{\frac{kT\gamma_n}{m}}\,\alpha(\omega)$$

For a linear detection procedure a(t) is filtered directly so that in the resulting output the signal to noise ratio gets optimal. As we know, (See page 57) the optimal choice for the filter transfer function will in this case be $f^*(\omega) \sim R(\omega+\omega_0)$, assuming that the spectrum of the signal is known.

The resulting output is

$$A(\omega) \sim a(\omega)\, R(\omega+\omega_0) = \frac{8ac^2}{\pi^2(1+2n)^2} R^2(\omega+\omega_0) + 2\sqrt{\frac{kT\gamma}{m}} \cdot$$

$$\cdot\, R^*(\omega+\omega_0)\,\alpha(\omega)$$

and the signal to noise ratio is

$$\text{(B.6)} \quad Sn(R|\omega) = \frac{2\cdot 16\, a^2 c^4 m}{\pi^4 kT\gamma_n(1+2n)^4} \int_{-\infty}^{+\infty} d\omega\, |R^2(\omega+\omega_0)|$$

We can relate this formula directly to the spectral power of the gravitational wave. From Gibbons and Hawking (1971) we have the power of the gravitational wave

$$P(t) = c^7(4\pi G)^{-1}\left\{\int_{-\infty}^{t} R(t)\,dt\right\}^2 \quad \text{or in terms of}$$

spectral power;

$$P(\omega) = c^7(4\pi G)^{-1}\frac{|R(\omega)|^2}{\omega^2} \quad \text{or}$$

$$|R^2(\omega)| = c^{-7}(4\pi G)\,\omega^2 P(\omega)$$

So we can write

$$S(P|\omega) = \frac{32\, m a^2 (4\pi G)}{\pi^4 kT\gamma_n c^3(1+2n)^4} \int_{-\infty}^{+\infty} \omega^2 P(\omega)\, d\omega \,.$$

Or if we use the relation $a^2 = \frac{\pi^2}{4} \frac{V_s^2}{\omega_0^2}$

where V_s is the speed of sound in the cylinder.

$$(B.7) \quad S_n(P/\omega) = \frac{8 m V_s^2 (4\pi G)}{\pi^2 k T \gamma_n C^3 (1+2n)^2} \int_{-\infty}^{+\infty} \frac{\omega^2}{\omega_n^2} P(\omega)$$

If $P(\omega)$ is centered around ω_s and narrowbanded enough so that $\int_{-\infty}^{+\infty} \omega^2 P(\omega)\, d\omega \simeq \omega_s^2 \int_{-\infty}^{+\infty} P(\omega)\, d\omega = \omega_s^2 \mathcal{E}_s$, where $\mathcal{E}_s$ is the total energy of the pulse, we have

$$(B.8) \quad S_n(E_s) = \frac{8 m v_s^2 (4\pi G)}{\pi^2 k T \gamma_n C^3} \frac{\omega_s^2}{\omega_n^2 (1+2n)^2} \mathcal{E}_s .$$

REMARKS

We will later see that even the best of todays pick-up systems and amplifiers will narrow down the bandwidth of the detector to a few Hertz around ω_n (See e.g. page 115) Thus the factor ω_s^2/ω_n^2 is practically 1. This cancels the apparent improvement of a long cylinder as a detector (i.e. a long cylinder makes ω_0 smaller, which apparently increases the signal to noise ratio, but since ω_s is in practice selected to the same frequency ω_0, the improvement is cancelled).

From (B.8) we can also see that the signal to noise for higher modes goes down as $(1 + 2\ )^{-2}$, in addition the damping time γ_n^{-1} will usually decrease with higher n .

Note in (B7) that the signal to noise is proportional to γ^{-1} independently of the duration of the signal. This will not be so, when the limiting effects of electronics is taken into account, in which case the signal to noise ratio for pulses shorter than the resolution time is proportional to the square root of γ^{-1} while for long pulses it is still directly proportional to γ^{-1}.

Antenna parameters where improvements in sensitivity can be made are V_s (the speed of sound), γ^{-1} (the damping time) and T (the temperature).

Beryllium is probably the material with the highest speed of sound of all (about 2.5 times that of aluminum). It's quite likely to have a large Q, or damping time, as well. Unfortunately, however, this metal is extremely poisonous.

Detectors kept at temperatures as low as a few millidegrees are presently under development, but it may still take years before these are in "working condition." Low temperature detectors made of a single crystal of e.g. quartz or sapphire may also lead to a substantial improvement in sensitivity. It has been reported that such crystals may have a mechanical Q of 10^{18}, while the highest measured such Q is about 10^{9}, while present detectors have a Q of about 10^{5}. Technically, there is thus room for substantial improvements in sensitivity.

We give below sensitivity limits for some types of resonant detectors. Note that these are "bare" detector sensitivities, i.e. they are optimal limits in sensitivity. We use (B.8) with $\omega_s = \omega_o$. We have

$G = 6.7 \ 10^{-8} \ cm^3/gsec^2$

$c = 3 \ 10^{10}$ cm/sec

$k = 1.4 \ 10^{-6}$ ergs/degree

Further for aluminum $v_s = 6\cdot 10^5$ cm/sec. With these numbers we get, demanding $Sn(\mathcal{E}_s) \geq 1$, that $\mathcal{E}_s \geq 7\cdot 10^9 \ \mathrm{ergs}\cdot\frac{\mathrm{gm}}{{}^\circ\mathrm{K}\,\mathrm{sec}\,\mathrm{cm}^2}\,\frac{T\cdot\gamma}{m}$.
For an ordinary aluminum detector at room temperature $\gamma = 2{,}5\cdot 10^{-2}\ \mathrm{sec}^{-1}$, $T = 3\cdot 10^2\ {}^\circ\mathrm{K}$, $m = 10^6\ \mathrm{gm}$ $\Rightarrow$

$\mathcal{E}_s = 5\cdot 10^4\ \mathrm{ergs/cm}^2$. for a similar 3 millidegree detector;
$\mathcal{E}_s = 0.5\ \mathrm{ergs/cm}^2$.

For the ultimate detector, a one ton quartz or sapphire crystal at 3 millidegrees, which we optimistically give a Q of 10^{18}, we get

$\mathcal{E}_s = 10^{-13}\ \mathrm{ergs/cm}^2$.

This last limit is pretty small, but apparently not small enough to make a controlled emission-detection experiment an easy matter. Using approximate formulas for power generation (M.T.W. page 979) it seems in general hard to generate much more than 10^{-20} ergs/sec.

Gibbons, G.W. and Hawking S.W. 1971 Phys. Rev. D Volume 4, Number 8
15 October 1971

M.T.W. See page 19.

Wax, N. (1954) Selected papers on noise and stochastic processes (New York, Dover)

Kittel, See page 74.

APPENDIX C

THE "DRESSED" DETECTOR

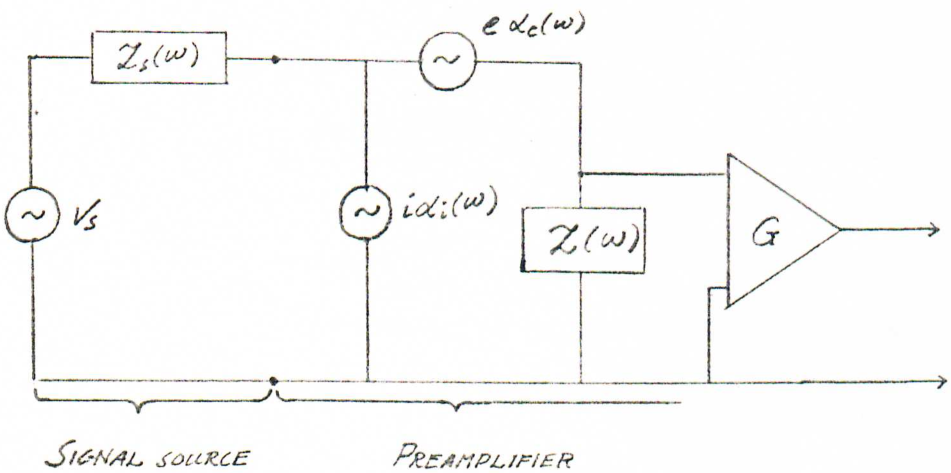
PRE-AMPLIFIER AND FILTER

The "dressed detector will in our case be the antenna, a pick-up or sensor device (which in some cases is built into the antenna, e.g. a quartz antenna, where the piezo-electric effect may serve as a sensor) an amplifier and a "final electronic filter." The data analysis system is really also an integral part of the detector system, but we will treat this part in a separate chapter. In fact it is up to the "experimenter" how much of the "data handling" or filtering should be done electronically and how much should be done by the computer. It can easily be shown that any final filter effect $f(\omega)$ can be accomplished in two steps by first filtering with $f_1(\omega)$ and then with $f_2(\omega)$ such that $f(\omega) = f_1(\omega) f_2(\omega)$. Thus as long as $f_1(\omega)$ is not zero for any finite interval $\Delta\omega$, where $f(\omega) \neq 0$, we can always filter with a second filter to get the filter effect we want. However, between the filters there is always a little amount of wide-band noise added (assume e.g. that the first filter is the final electronic filter and that the second filtering is done on tape recorded data by a computer, to match a certain type of signal). As is shown later (page 153) the most economic (tape saving way) to limit the impact of this noise addition (i.e. limit the loss of information) is to prefilter the signal to a form approximately that of $(\dot{x}, \dot{y})$ which is then digitized. In other words an (x,y) recording requires higher resolution (more bits) in the tape recording than does $(\dot{x},\dot{y})$. It is shown on page that for a Weber type detector $(\dot{x},\dot{y})$ is equivalent to (x,y) in terms of information content (i.e. the transformation $(\dot{x}, \dot{y}) \rightarrow (x, y)$ is invertible) Further, it is true in general that there is never a loss of signal information in going from (x, y) to $(\dot{x}, \dot{y})$, since if the input signal is $s(t)$ in form, the optimally filtered output for $s(t)$ is

$$(\bar{x}(t), \bar{y}(t)) = \int_{-\infty}^{+\infty} dt' \, s(t-t') \, (\dot{x}(t'), \dot{y}(t))$$

The preamplifier can in most cases be described by the following circuit:

Figure C.1



e and i are white noise sources, which may in some cases have some correlation. From (3.24) and fig. (C.1) we get the spectral signal to noise ratio

$$Sn(s/\omega) =$$

$$= \frac{V_s^2(\omega) \left| \frac{Z(\omega)}{Z_s(\omega) + Z(\omega)} \right|^2}{e^2 \left| \frac{Z(\omega)}{Z_s(\omega) + Z(\omega)} \right|^2 + 2\,RE\left\{ \frac{Z^2(\omega) Z_s(\omega)}{(Z_s(\omega) + Z(\omega))^2} \left(i^2 e^2\right)^{1/2} C_{i,e} \right\} + i^2 \left| \frac{Z(\omega) Z_s(\omega)}{Z_s(\omega) + Z(\omega)} \right|^2}$$

where $C_{i,e}$ is a correlation coefficient: $C_{i,e}(\omega)\,\delta(\omega-\omega') = \langle \alpha_e(\omega)\,\alpha_i(\omega')\rangle$

We can write this:

$$Sn(S/\omega) = \frac{V_S^2(\omega)}{e^2 + 2i\cdot e\,|C_{i,e}|\,\cos\Theta\,|Z_s(\omega)| + i^2|Z_s(\omega)|^2}$$

Where ;

$$\cos\Theta = \frac{RE\left\{\frac{Z^2(\omega)Z_s(\omega)}{(Z_s(\omega)+Z(\omega))^2}C_{i,e}\right\}}{\left|\frac{Z^2(\omega)Z_s(\omega)}{(Z_s(\omega)+Z(\omega))^2}C_{i,e}\right|}$$

If we imagine that we would be able to match the system (say by using an ideal transformer, or adjusting the impedance of the source so that it keeps the same maximal power output), we would have a variable factor n in front of V_S and n^2 in front of $Z_S(\omega)$ (n corresponding to the ratio of number of turns on the input and output of the transformer).

This would lead to a signal to noise ratio

$$Sn(S/\omega) = \frac{n^2 V_S^2(\omega)}{e^2 + 2n^2 i\cdot e\,|C_{i,e}(\omega)|\,\cos\Theta(\omega)\,|Z_S(\omega)| + n^4 i^2 |Z_S(\omega)|^2}$$

Maximizing with respect to n gives $n^2 = \dfrac{e}{i|Z_S(\omega)|} \Rightarrow$

$$Sn(S/\omega) = \frac{V_S^2(\omega)/|Z_S(\omega)|}{2i\cdot e\,(1+|C_{i,e}(\omega)|\,\cos\Theta(\omega))}$$

It is clear that if we don't have a transformer we should try to design the source or the pre-amplifier so that $|Z_S(\omega)| = e/i$.

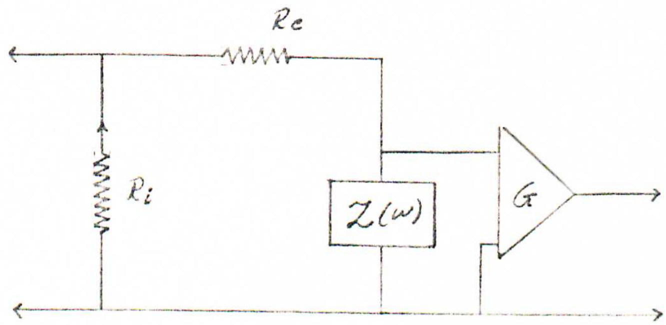
We may describe the quality of the preamplifier by the spectral power density

$$P(\omega) = 2i\cdot e(1+C_{i;e}).$$

$i\cdot e$ has energy units and may be given in terms of temperature. A good F.E.T. amplifier may have a temperature of $\sim 10^{-3,5}C^{o}$.

Sometimes, instead of having equivalent noise sources e and i, they are replaced by equivalent resistors. That is instead of the current source ia_i one puts in a noise producing resistor R_i and instead of ea_e another noise producing resistor R_e. This gives the equivalent circuit:

FIG. C.2.



Assuming R_i to be large we have

$$i^2 = \frac{kT}{R_i} \quad \text{and} \quad e^2 = kTR_e \qquad \text{so}$$

$$e\cdot i = kT\sqrt{\frac{R_e}{R_i}} \qquad \text{or in temperature units;}$$

$$T_e = T\sqrt{\frac{R_e}{R_i}}$$

These resistors R_i and R_e however should usually not be considered to have anything to do with the input impedance of the preamplifier (which may be quite different). In calculations, one should therefore not include the "passive" effect of these resistors.

As was pointed out by J.P. Richard, the fact that the quality of an amplifier, can be described by a temperature, implies that although the employment of several amplifiers in parallel can be useful as a technique for changing the imput impedance, there will be no gain in quality.

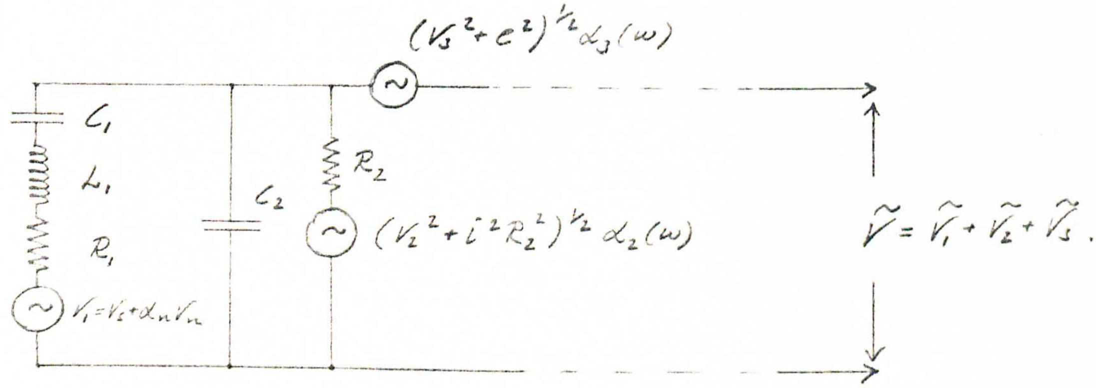
DETECTOR WITH A NONRESONANT PICK-UP SYSTEM

We can now proceed to calculate the "spectral signal to noise ratio" for a "Weber" type detector, which is later expressed in terms of the spectral power of the gravitational wave. This shows that the "dressed" detector has a certain limited bandwidth (whereas the "bare" detector is not bandwidth limited). The limitation in bandwidth leads to a low sensitivity to pulses of gravitational radiation with a short duration time, which is demonstrated by calculating the signal to noise of a minimum uncertainty wave-packet with a central frequency ω_0, total energy $\mathcal{E}$ per unit area and duration Δt.

We also calculate the signal to noise for a delta function type excitation, where the strength of the excitation is measured in terms of the average thermal energy of the detector. The signal to noise ratio $Sn(kT)$ corresponding to suddenly changing the amplitude by an amount corresponding to its thermal root mean square, is in general useful as a quality factor of a "dressed" detector.

Let's now consider the typical "Weber type" equivalent circuit.

If we let $Z(\omega)$ in Fig. C.3 be infinite (which is usually approximately true) we can simplify the circuit to Fig. C.4



We have omitted the filter. $\tilde{V}_1$, $\tilde{V}_2$ and $\tilde{V}_3$ are the contributions to $\tilde{V}$ from V_1, v_2 and v_3 respectively. We have

$$\tilde{V}_1(\omega) = \frac{V_1(\omega)\dfrac{R_2}{i\omega C_2 R_2 + 1}}{R_1 + i\omega L_1 + \dfrac{1}{i\omega C_1} + \dfrac{1}{i\omega C_2 + \frac{1}{R_2}}}$$

For any "good" sensor $\frac{1}{R_2} << i\omega C_2 \Rightarrow$

$$\frac{1}{i\omega C_2 + \frac{1}{R_2}} \approx \frac{1}{i\omega C_2} + \frac{1}{\omega_2{}^2 C_2{}^2 R_2{}^2} \Rightarrow$$

$$\tilde{V}_1(\omega) = \frac{V_1(\omega)\dfrac{R_2}{i\omega C_2 R_2 + 1}}{R_1 + \dfrac{1}{\omega^2 C_2{}^2 R_2} + i\omega L_1 + \dfrac{1}{i\omega C_2} + \dfrac{1}{i\omega C_1}}$$

Let $\frac{1}{i\omega C} = \frac{1}{i\omega C_1} + \frac{1}{i\omega C_2}$. Since the detector is very narrow-banded (which will be verified later) we may for nonresonant terms replace ω by ω_0 where ω_0 is the resonant frequency. We may thus let

$R = R_1 + \frac{R_2}{1+Q^2}$ and we get

$$\tilde{V}_1(\omega) = \frac{V_1(\omega)\,\dfrac{i\omega C_2 R_2^2 + R_2}{1+Q_2^2}}{R + i\omega L_1 + \dfrac{1}{i\omega C}} =$$

$$= \frac{V_1(\omega)\,\dfrac{i\omega R_2 - \omega^2 C_2 R_2^2}{1+Q_2^2}}{-L_1 C_2\left\{\left(\omega - i\dfrac{R}{2L_1}\right)^2 - \left(\dfrac{1}{L_1 C} - \left(\dfrac{R}{2L_1}\right)^2\right)\right\}}.$$

Let $\frac{R}{2L_1} = \gamma$, $\omega_0^2 = \frac{1}{L_1 C} - \gamma^2$ and $Q_2 = \omega_0 C_2 R_2 \Rightarrow$

$$\tilde{V}_1(\omega) = \frac{V_1(\omega)\,\dfrac{i\omega R_2 C_2 - \omega^2 C_2^2 R_2^2}{1+Q_2^2}}{-L_1 C_2\left\{(\omega - i\gamma)^2 - \omega_0^2\right\}} = \frac{V_1(\omega)\,\dfrac{i\omega R_2 C_2 - \omega^2 C_2^2 R_2^2}{1+Q_2^2}}{-L_1 C_2(\omega+\omega_0-i\gamma)(\omega-\omega_0-i\gamma)} =$$

$$= \frac{V_1(\omega)\,\dfrac{i\omega R_2 C_2 - \omega^2 C_2^2 R_2^2}{1+Q_2^2}}{2L_1 C_2 \omega_0}\left\{\frac{1}{(\omega+\omega_0-i\gamma)} - \frac{1}{(\omega-\omega_0-i\gamma)}\right\}.$$

We may thus write

$$\tilde{V}_1(t) = \frac{1}{2L_1C_2\omega_0}\int_{-\infty}^{+\infty}\frac{d\omega}{\sqrt{2\pi}}\left\{\frac{V_1(\omega)\,\dfrac{i\omega R_2C_2 - Q_2^2}{1+Q_2^2}}{(\omega+\omega_0-i\gamma)} - \frac{V_1(\omega)\,\dfrac{i\omega R_2C_2 - Q_2^2}{1+Q_2^2}}{(\omega-\omega_0-i\gamma)}\right\}e^{i\omega t}.$$

Let $\omega \Rightarrow -\omega$ in the first integral, since $V_1(t)$ is real

$V_1(-\omega) = V_1^*(\omega) \Rightarrow$

$$\tilde{V}_1(t) = \frac{-1}{2L_1C_2\omega_0}\int_{-\infty}^{+\infty}\frac{d\omega}{\sqrt{2\pi}}\left\{\frac{V_1(\omega)\,\dfrac{i\omega R_2C_2 - Q_2^2}{1+Q_2^2}\,e^{i\omega t}}{\omega-\omega_0-i\gamma} + \right.$$

$$\left. + \left(\frac{V_1(\omega)\,\dfrac{i\omega R_2C_2 - Q_2^2}{1+Q_2^2}\,e^{i\omega t}}{\omega-\omega_0-i\gamma}\right)^*\right\}.$$

Thus $$\tilde{V}_1(t) = \frac{-1}{L_1C_2\omega_0}\,\mathrm{Re}\int_{-\infty}^{+\infty}\frac{d\omega}{\sqrt{2\pi}}\,\frac{V_1(\omega)\,\dfrac{i\omega R_2C_2 - Q_2^2}{1+Q_2^2}\,e^{i\omega t}}{\omega-\omega_0-i\gamma}$$ or

$$\tilde{V}_1(t) = \frac{1}{L_1C_2\omega_0}\,\mathrm{Im}\int_{-\infty}^{+\infty}\frac{d\omega}{\sqrt{2\pi}}\,\frac{-iV_1(\omega)\,\dfrac{i\omega R_2C_2 - Q_2^2}{1+Q_2^2}\,e^{i\omega t}}{\omega-\omega_0-i\gamma}$$ and

$$\frac{1}{\omega_0}\dot{\tilde{V}}_1(t) = \frac{1}{L_1C_2\omega_0}\,\mathrm{Re}\int_{-\infty}^{+\infty}\frac{d\omega}{\sqrt{2\pi}}\,\frac{-iV_1(\omega)\,\dfrac{i\omega R_2C_2 - Q_2^2}{1+Q_2^2}\,\omega\,e^{i\omega t}}{(\omega-\omega_0-i\gamma)\,\omega_0}.$$

Thus we may let the phase space state vector of the output be represented by (similarly to the procedure on pages 15-16)

$$\bar{V}_1(t) = \frac{1}{\omega_0}\dot{\tilde{V}}_1(t) + i\tilde{V}_1(t) = \int_{-\infty}^{+\infty}\frac{d\omega}{\sqrt{2\pi}}\,\frac{V_1(\omega)\frac{iQ_2}{1+iQ_2}\,e^{i\omega t}}{L_1 i\omega_0 C_2(\omega-\omega_0-i\gamma)}$$

so that $\tilde{V}_1(t) = \mathrm{Im}\,\bar{V}_1(t)$.

We may now go to a corotating coordinate system by multiplying with $e^{-i\omega_0 t}$ (let $V_1(t)e^{-i\omega_0 t} \to V_1(t)$) $\Rightarrow$

$$(C.1)\qquad \bar{V}_1(t) = \int_{-\infty}^{+\infty}\frac{d\omega_c}{\sqrt{2\pi}}\,\frac{V_1(\omega_0+\omega_c)\,iQ_2\,e^{i\omega_c t}}{L_1 i\omega_0 C_2(1+iQ_2)(\omega_c-i\gamma)} =$$

$$= \int_{-\infty}^{+\infty}\frac{d\omega}{\sqrt{2\pi}}\,\frac{V_1(\omega_0+\omega_c)\,R_2\,e^{i\omega_c t}}{L_1(1+Q_2)(\omega_c-i\gamma)}$$

where $\omega_c = \omega-\omega_0$ is the corotating frequency.

For $\tilde{V}_2$ we have:

$$\tilde{V}_2(\omega) = \frac{V_2(\omega)\dfrac{1}{i\omega C_2+\dfrac{1}{i\omega L_1+R_1-\dfrac{1}{i\omega C_1}}}}{R_2+\dfrac{1}{i\omega C_2+\dfrac{1}{i\omega L_1+R_1+\dfrac{1}{i\omega C_1}}}} = \frac{V_2(\omega)}{R_2(i\omega C_2+\frac{1}{R_2}+\dfrac{1}{i\omega L_1+R_1+\frac{1}{i\omega C_1}}} =$$

$$= \frac{V_2(\omega)(i\omega L_1+R_1+\frac{1}{i\omega C_1}}{R_2(i\omega C_2+\frac{1}{R_2})(i\omega L_1+R_1+\frac{1}{i\omega C_1}+\dfrac{1}{i\omega C_2+\frac{1}{R_2}}} \Rightarrow$$

by similarity with V_1 we get

$$\tilde{V}_2(\omega)$$

$$= \frac{V_2(\omega)\, i\omega C_2 \left(i\omega L_1 + R_1 + \frac{1}{i\omega C_1}\right)}{2L_1 C_2 \omega_0 (1 + i\omega R_2 C_2)} \left\{ \frac{1}{(\omega+\omega_0 - i\gamma)} - \frac{1}{(\omega - \omega_0 - i\gamma)} \right\} =$$

$$= \frac{V_2(\omega)\left(-\omega^2 + i\omega \frac{R_1}{L_1} + \omega_0'^2\right)}{2\omega_0 (1 + i\omega R_2 C_2)} \left\{ \frac{1}{(\omega+\omega_0 - i\gamma)} - \frac{1}{(\omega-\omega_0 - i\gamma)} \right\}$$

so

$$\tilde{V}_2(t) = Re \int_{-\infty}^{+\infty} \frac{d\omega}{\sqrt{2\pi}} \frac{V_2(\omega)\left(+\omega^2 - i\omega \frac{R_1}{L_1} - \omega_0'^2\right)}{\omega_0 (1 + i\omega R_2 C_2)} \left\{ \frac{e^{i\omega t}}{(\omega - \omega_0 - i\gamma)} \right\}$$

In the same way as for $\tilde{V}_1$ we get the corotating state vector by multiplying with $e^{-i\omega_0 t}$.

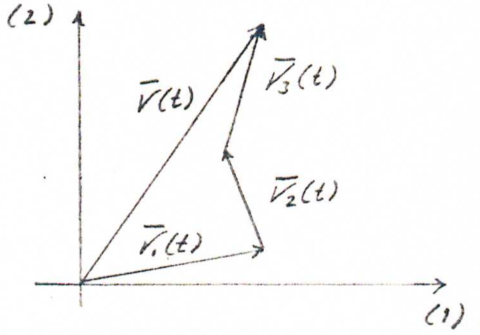
$$(C.2) \qquad \bar{V}_2(t) = \int_{-\omega_0}^{+\infty} \frac{d\omega_c}{\sqrt{2\pi}} \frac{iV_2(\omega)(\omega^2 - 2i\gamma_1\omega - \omega_1'^2)\, e^{i\omega_c t}}{\omega_0 (1 + i\omega R_2 C_2)(\omega_c - i\gamma)}$$

and;

$$(C.3) \qquad \bar{V}_3(t) = \int_{-\omega_0}^{+\infty} \frac{d\omega_c}{\sqrt{2\pi}}\, i V_3(\omega_0 + \omega_c) \frac{(\omega_c + 2\omega_0 - i\gamma)(\omega_c - i\gamma)}{\omega_0 (\omega_c - i\gamma)}$$

The sum of the (two dimensional) state-vectors $\bar{V}_1$, $\bar{V}_2$ and $\bar{V}_3$ is the total output state vector $\bar{V}$, which can be represented in a two dimensional graph

Fig. C.5



where $\bar{V}(t)$ changes with time due to the combined action of signal (from source one) and noise (from sources 1, 2 and 3).

Consider now $V_3(\omega+\omega_c) = v_3 \alpha_3(\omega_0+\omega_c)$, we have

$$\langle \alpha_3(\omega_0+\omega_c)\, \alpha_3^*(\omega_0+\omega_c')\rangle = \delta(\omega_c-\omega_c') \quad \text{and}$$

$$\langle \alpha_3(\omega_0+\omega_c)\, \alpha_3(\omega_0+\omega_c')\rangle = \delta(2\omega_0+\omega_c+\omega_c'),$$ since α_3 is real.

But we can see that for any situation where we are limited in bandwidth to $|\omega_c| \ll \omega_0$. $\alpha_0(\omega_0+\omega_c)$ will behave as a complex $\alpha(\omega_c)$ defined by 3.13. The same holds true for the noise sources V_2 and V_n. With this approach, all integrals should be taken from $-\omega_0$ to ∞.

We write the factor $(\omega^2 - 2i\gamma_1 - \omega_1'^2)$ in (C2) as

$$\{(\omega - i\gamma_1)^2 - (\omega_1'^2 - \gamma_1^2)\} = (\omega - i\gamma_1 + \omega_1)(\omega - i\gamma_1 - \omega_1)$$

where $\omega_1^2 = \omega_1'^2 - \gamma_1^2$. We may approximate $\omega - i\gamma_1 + \omega_1$, with $2\omega_0$. We can then write (C.2)

$$V_2(t) = \int_{-\infty}^{+\infty} \frac{d\omega_c}{\sqrt{2\pi}} \frac{iV_2(\omega)\, 2\omega_0(\omega-\omega_1) - i\gamma_1)\, e^{i\omega_c t}}{\omega_0(1+i\omega_0 R_2 C_2)(\omega_c - i\gamma)} =$$

$$= \int_{-\infty}^{+\infty} \frac{d\omega_c}{\sqrt{2\pi}} \frac{iV_2(\omega)(\omega_c + \Delta\omega_2 - i\gamma_1)\, e^{i\omega_c t}}{(1+i\omega_0 R_2 C_2)(\omega_c - i\gamma)}$$

Let $Q_2 = \omega_2 R_2 C_2$

where $\Delta\omega_2 = \sqrt{\frac{1}{L_1}}\left\{\sqrt{\frac{1}{C_1}+\frac{1}{C_2}} - \sqrt{\frac{1}{C_1}}\right. \approx \frac{1}{2}\omega_0 \frac{C_1}{C_2}$.

Doing a similar approximation for $\overline{V}_3(t)$,
we can write the noise power as

$$|v_{on}^2(\omega)| = \frac{v_1^2(\omega) R_2^2}{L_1^2(1+Q_2^2)(\omega_c^2+\gamma^2)} + \frac{4v_2^2(\omega_c) L_1^2\{(\omega_c+\Delta\omega_2)^2+\gamma_1^2\}}{L_1^2(1+Q_2^2)(\omega_c^2+\gamma^2)} +$$

$$+ \frac{4v_3^2(\omega_c) L_1^2(1+Q_2^2)(\omega_c^2+\gamma^2)}{L_1^2(1+Q_2^2)(\omega_c^2+\gamma^2)} .$$

(3.24) gives us the signal to noise ratio;

$$S_n(S) = \int_{-\infty}^{+\infty} d\omega \frac{2\, v_S^2(\omega_c) R_2^2}{v_n^2(\omega_c) R_2^2 + 4 v_2^2(\omega_c) L_1^2 \{(\omega_c) + \Delta\omega_2)^2 + \gamma_1^2\} + V_3^2(\omega_c) L_1^2 (1+Q_2^2)(\omega_c^2 + \gamma^2)}$$

$$= \int_{-\infty}^{+\infty} d\omega \frac{2\, v_S^2(\omega_c) R_2^2}{4 L_1^2 (v_2^2 + (1+Q_2^2) v_3^2) \left\{ \frac{v_n^2 R_2^2 + 4\gamma_1^2 v_2^2 L_1^2 + 4\gamma^2 v_3^2 L_1^2 (1+Q_2^2)}{4 (v_2^2 + (1+Q_2^2) v_3^2) L_1^2} + \cdots \right.} \cdots$$

$$\cdots \overline{\cdots + \frac{\Delta\omega_2^2\, v_2^2 v_3^2 (1+Q_2^2)}{(v_2^2 + (1+Q_2^2) v_3^2)^2} + \left(\omega_c + \frac{\Delta\omega_2 v_2^2}{v_2^2 + (1+Q_2^2) v_3^2} \right)^2 \Bigg\}}$$

(C.4)
$$\left\{ \begin{aligned} &\text{Let } \Delta\omega = \Delta\omega_2 \frac{v_2^2}{v_2^2 + (1+Q_2^2) v_3^2} \quad \text{and} \\ &\mu^2 = \frac{v_n^2 R_2^2 \frac{1}{4} + v_2^2 \frac{1}{4} R_1^2 + v_3^2 L_1^2 (1+Q_2^2)(\gamma^2 + \Delta\omega_2 \Delta\omega)}{(v_2^2 + (1+Q_2^2) v_3^2) L_1^2} \\ &\beta^2 = \frac{(\frac{1}{2} v_n R_2)^2}{L_1^2 (v_2^2 + v_3^2 (1+Q_2^2))} . \end{aligned} \right.$$

We have

$$(C.5)\quad v_{on}^2(\omega_c) = 4\left(v_3^2(\omega_c) + \frac{v_2^2(\omega_c)}{1+Q_2^2}\right) \frac{(\omega_c+\Delta\omega)^2+\mu_n^2}{\omega_c^2+\gamma^2}$$

and

$$(C.6)\quad \begin{cases} Sn(S) = \displaystyle\int_{-\infty}^{+\infty} d\omega_c \; 2 \, \frac{V_S^2(\omega_c)}{v_n^2(\omega_c)} \, \frac{\beta^2}{\mu^2+(\omega_c+\Delta\omega)^2} \\ \text{or in terms of the "spectral" signal to noise ratio} \\ Sn(S|\omega_c) = 2 \, \dfrac{V_S^2(\omega_c)}{v_n^2(\omega_c)} \, \dfrac{\beta^2}{\mu^2+(\omega_c+\Delta\omega)^2} \end{cases}$$

We can see that for a flat signal ($V_S(\omega_c)$ = constant) the spectral signal to noise has maximum at $\omega_c = -\Delta\omega$ (A fact first pointed out by Kafka). For a detector with a high Q and a sensor with not so high Q we would expect $\Delta\omega$ to become important, we have approximately

$$(C.7)\quad \frac{\Delta\omega}{\mu} \le \sqrt{\frac{Q_1 C_1}{Q_2 C_2}}$$

(Note that μ represents the effective bandwidth of the dressed detector).

Usually $\Delta\omega \ll \mu$ however.

The matched filter is

$$(C.8) \qquad f(\omega) \sim \frac{\bar{v}_s^*(\omega_c)}{|\bar{v}_{on}^2(\omega_c)|} \sim \frac{v_s^*(\omega_c)(\omega_c + i\gamma)}{\mu^2 + (\omega_c + \Delta\omega)^2}$$

Note that we have calculated the signal to noise ratio for signals of known amplitude, phase, shape and arrival time. If the phase is random we may use (2.19) to find $S_n(S)$, if the (amplitude)^2 is known only up to an exponential distribution we may use (3.28) to find $S_n(S)$ ($S_n(S|E_0) = S_n(S) - \ln(1 + S_n(S))$) and if also the arrival time is random we may use (2.25).

SPECTRAL SENSITIVITY IN TERMS OF SPECTRAL POWER OF THE GRAVITATIONAL RADIATION

To determine $V_s(\omega)$ in terms of the Riemann tensor, we compare the mechanical and the "equivalent circuit" equation for the detector. The mechanical equation is: (from (1.9))

$$(C.9) \quad \ddot{\xi}(t) + 2\gamma\dot{\xi}(t) + \omega_n^2 \xi(t) = \frac{(-1)^n 4a}{\pi^2(1+2n)} 2c^2 R_{1010}(t)$$

and the "equivalent circuit" equation

$$(C.10) \quad \ddot{q}(t) + 2\gamma\dot{q}(t) + \omega_n^2 q(t) = \frac{V_s(t)}{L_1}$$

(where q is the charge on the capacitor C_1)

The energy at time t in the detector is

$$E_n(t) = \frac{m}{4}\left(\left(\frac{\partial\xi}{\partial t}\right)^2 + \omega_n^2 \xi^2\right) ,$$

and in the circuit

$$E_n(t) = \frac{L_1}{2}\left(\left(\frac{\partial q}{\partial t}\right)^2 + \omega_n^2 q^2\right) ,$$

thus we may identify

$\sqrt{\frac{m}{2}}\, \xi(t) \equiv \sqrt{L_1}\, q(t)$. Replacing $\xi(t)$ in the equation (C.9) with $\sqrt{\frac{2L_1}{m}}\, q(t)$ $\Rightarrow$

$$(C.11) \quad \ddot{q}(t) + 2\gamma \dot{q}(t) + \omega_n^{\,2} q(t) = \sqrt{\frac{m}{2L_1}}\, \frac{(-1)^n\, 8a}{\pi^2(1+2n)}\, c^2 R_{1010}(t)$$

Comparing with (C.9) we see that

$$\frac{V_s(t)}{L_1} = \sqrt{\frac{m}{2L_1}}\, \frac{8ac^2(-1)^n}{\pi^2(1+2n)^2}\, R_{1010}(t) \qquad \text{or}$$

$$V_s(\omega) = \sqrt{\frac{mL_1}{2}}\, \frac{8ac^2(-1)^n}{\pi^2(1+2n)^2}\, R_{1010}(t)$$

with the relation $\omega_0^{\,2} = V_s^{\,2}\, \frac{\pi^2}{4a^2}$ or $a = \frac{\pi}{2}\, \frac{V_s}{\omega_0}$

where V_s is the speed of sound in the cylinder, we get

$$V_s(\omega) = \sqrt{\frac{mL_1}{2}}\ \frac{4V_s{}^2c^2(-1)^n}{\pi\omega_0(1+2n)^2}\ R_{1010}(\omega) \qquad \text{and}$$

$$|V_s{}^2(\omega)| = \frac{8mL_1V_s{}^2c^4}{\pi^2(1+2n)^2\omega_n{}^2}\ |R^2_{1010}(\omega)|.$$

Again we use the relation for the spectral power flux of gravitational radiation:

$$P(\omega) = c^7(4\pi G)^{-1}\ \frac{|R^2_{1010}(\omega)|}{\omega^2} \qquad \text{or}$$

(Note that we are considering only one direction of polarization)

$$|R^2_{1010}(\omega)| = c^{-7}(4\pi G)\,\omega^2 P(\omega)$$

we get,

$$\text{(C.12)} \qquad |V_s{}^2(\omega)| = \frac{8mL_1V_s{}^2(4\pi G)}{\pi^2(1+2n)^2c^3}\ \frac{\omega^2}{\omega_n^2}\ P(\omega)$$

Together with (5.6) this gives $\quad$ (Let $P(\omega_c) = P(\omega_0+\omega_c)$)

$$Sn(P\,|\,\omega_c)_n = 2\,\frac{8mL_1V_s{}^2(4\pi G)}{\pi^2(1+2n)^2c^3v_n{}^2}\ \frac{\omega^2}{\omega_n^2}\ P(\omega_c)\,\frac{\beta_n{}^2}{\mu_n{}^2+(\omega_c+\Delta\omega_n)^2}$$

Now $v_n^2 = kT_1R_1$ (See page 64) and $\gamma_n = \frac{R_1}{2L_1} \Rightarrow$

$$Sn(P/\omega_c)_n = \frac{8mV_s^2(4\pi G)}{\pi^2 kT_1 \gamma_n (1+2n)^2 c^3} \frac{\omega^2}{\omega_n^2} P(\omega_c) \frac{\beta_n^2}{\mu_n^2 + (\omega_c + \Delta\omega_n)^2}$$

We can see that if $\beta_n \to \infty$ $(\Rightarrow \beta_n/\mu_n \to 1)$ the dressed detector gets an infinite bandwidth and we regain (4.7). Note however that γ_n will be different from the γ_n of a corresponding "clean" detector i.e. putting crystals or some other pick-up device on the detector will in general degrade the "Q" of the detector, $(\gamma = \frac{\omega_0}{2Q})$.

To make this difference obvious we may define $\gamma_n^{(b)}$ and $Q_n^{(b)}$ to be the γ_n and Q_n for the "bare" detector.

Further let's define

$$\mathcal{E}_n = \frac{\pi^2 kT_1 \gamma_n^{(b)} (1+2n)^2 c^3}{8mV_s^2(4\pi G)} \quad \text{so that}$$

$$(C.13) \quad Sn(P/\omega_c) = \mathcal{E}_n^{-1} \frac{\omega^2}{\omega_n^2} P(\omega_c) \frac{\gamma_n^{(b)} \beta_n^2}{\gamma_n(\mu_n^2 + (\omega_c + \Delta\omega)^2)}$$

From this we may define the "spectral signal to noise per unit power flux", a quantity which is independent of the signal and defined purely by the quality of the detector system.

$$(C.14) \quad Sn(u/\omega_c) = \mathcal{E}_n^{-1} \frac{\omega^2}{\omega_n^2} \frac{\gamma_n^{(b)} \beta_n^2}{\gamma_n(\mu_n^2 + (\omega_c + \Delta\omega)^2)} \quad \text{so that}$$

$$Sn(P) = \int_{-\infty}^{+\infty} d\omega_c \; Sn(u/\omega_c)\, P(\omega_c).$$

Further $Sn(u/\omega_c)$ is a "product" of the quality or "spectral signal to noise per unit power flux" of the "bare" detector, $Sn^{(b)}(u/\omega_c) = \mathcal{E}_n^{-1} \frac{\omega^2}{\omega_n^2}$

and the quality of the dress, $qd(\omega_c) = \dfrac{\gamma_n^{(b)} \beta_n^2}{\gamma_n(\mu_n^2 + (\omega_c + \Delta\omega)^2)}$,

(which is always less than one). Note further that $qd(\omega_c)$ gives the detector an effective bandwidth, defined by

$$\omega_c = -\Delta\omega_n \pm \mu_n, \quad \text{or} \quad \omega = (\omega_n - \Delta\omega_n) \pm \mu_n.$$

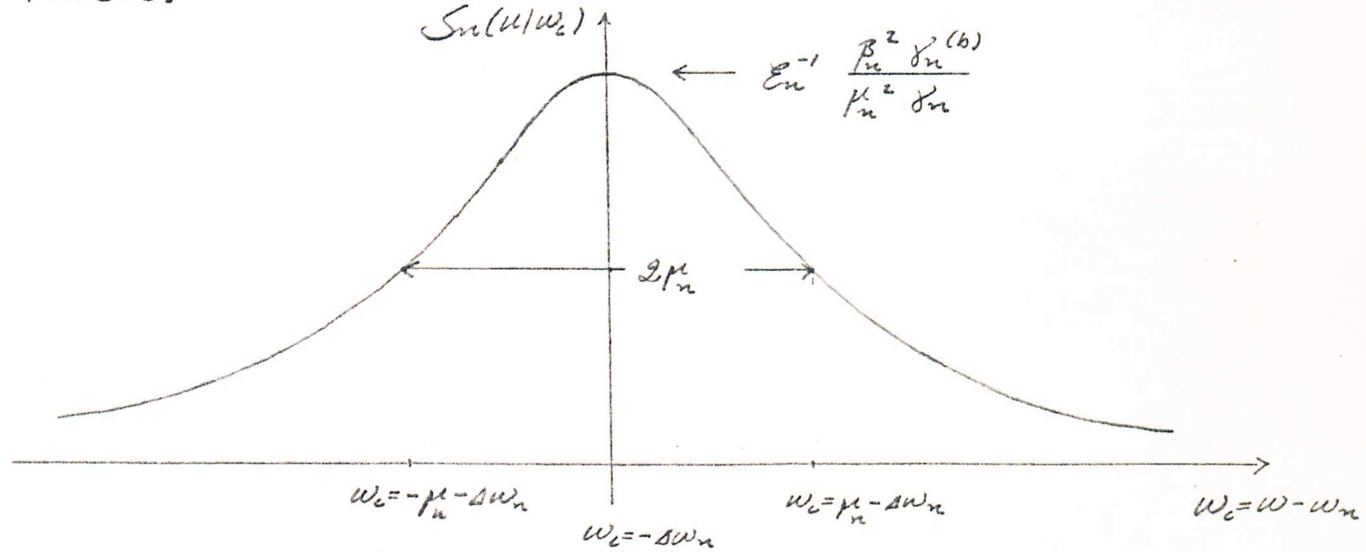
We may write $qd(\omega_c)$ as $qd(\omega_c) = \dfrac{\beta_n^2}{\gamma_n \mu_n} \dfrac{\gamma_n^{(b)} \mu_n}{\mu_n^2 + (\omega_c + \Delta\omega)^2}$,

and as we shall see in a moment (page 116) $\dfrac{\beta_n^2}{\gamma_n \mu_n}$ is the signal to noise for a so called "kT excitation", $Sn(kT) = \dfrac{\beta_n^2}{\gamma_n \mu_n}$.

For a "flat" power spectrum (let for simplicity $P_{flat}(\omega_c) = P_0 \dfrac{\omega_n^2}{\omega^2}$), we have, $Sn(P_0) = \mathcal{E}_n^{-1} P_0 \pi \gamma_n^{(b)} Sn(kT)$. Thus in the case $P(\omega_c)$ is flat, $Sn(kT)$ measures the performance of the "dress".

We give beneath a typical graph of $S_n(u/\omega_c)$ versus ω_c.

Fig. C. 6.



As typical values we give μ_0, $\Delta\omega_0$ and β_0 for Weber's 1973 set up. $(\omega_0 = 1660\ Hz)$ $\mu_0 = 1{,}7\ sec^{-1} \approx 0{,}27\ Hz$, $\Delta\omega \approx 0{,}8\ sec^{-1} \approx 0{,}13\ Hz$, and $\beta_0 = 1{,}1\ sec^{-1} \approx 0{,}17\ Hz$.

For a room temperature detector of Weber type $\mathcal{E}_0$ is approximately $5 \cdot 10^4\ ergs/cm^2$.

We will now consider how sensitive the detector is to a minimum uncertainty wave packet, with the frequency width $\Delta\omega$ (or duration $\Delta t = \frac{1}{\Delta\omega}$) centered around $\omega_0 - \Delta\omega_n$. We can write the radiation power per unit area and bandwidth as

$$P(\omega) = \mathcal{E} \frac{1}{\sqrt{2\pi}\,\Delta\omega} \exp - \frac{\omega^2}{2(\Delta\omega)^2} \quad (\text{where } \omega = \omega_c - \Delta\omega_n).$$

$\mathcal{E}$ is the total energy that passes through the detector per unit area.

The signal to noise is given by the integral

$$S_n(\mathcal{E}) = \frac{\mathcal{E}}{\mathcal{E}_n} \int_{-\infty}^{+\infty} \frac{d\omega}{\sqrt{2\pi}} \frac{\gamma_n^{(b)} \beta_n^2}{\gamma_n \Delta\omega(\mu_n^2 + \omega^2)} \exp - \frac{\omega^2}{2(\Delta\omega)^2}$$

Solving $\mathcal{E}$ for $S_n(\mathcal{E}) = 1$ we get

$$\mathcal{E} = \mathcal{E}_n \left\{ \int_{-\infty}^{+\infty} \frac{d\omega}{\sqrt{2\pi}} \frac{\beta_n^2}{\Delta\omega(\mu_n^2 + \omega^2)} \exp - \frac{\omega^2}{2(\Delta\omega)^2} \right\}^{-1}$$

which we may rewrite as:

$$\mathcal{E} = \mathcal{E}_n \, {\mu_n^2}/{\beta_n^2} \, f({\mu_n}/{\Delta\omega}) \quad \text{where}$$

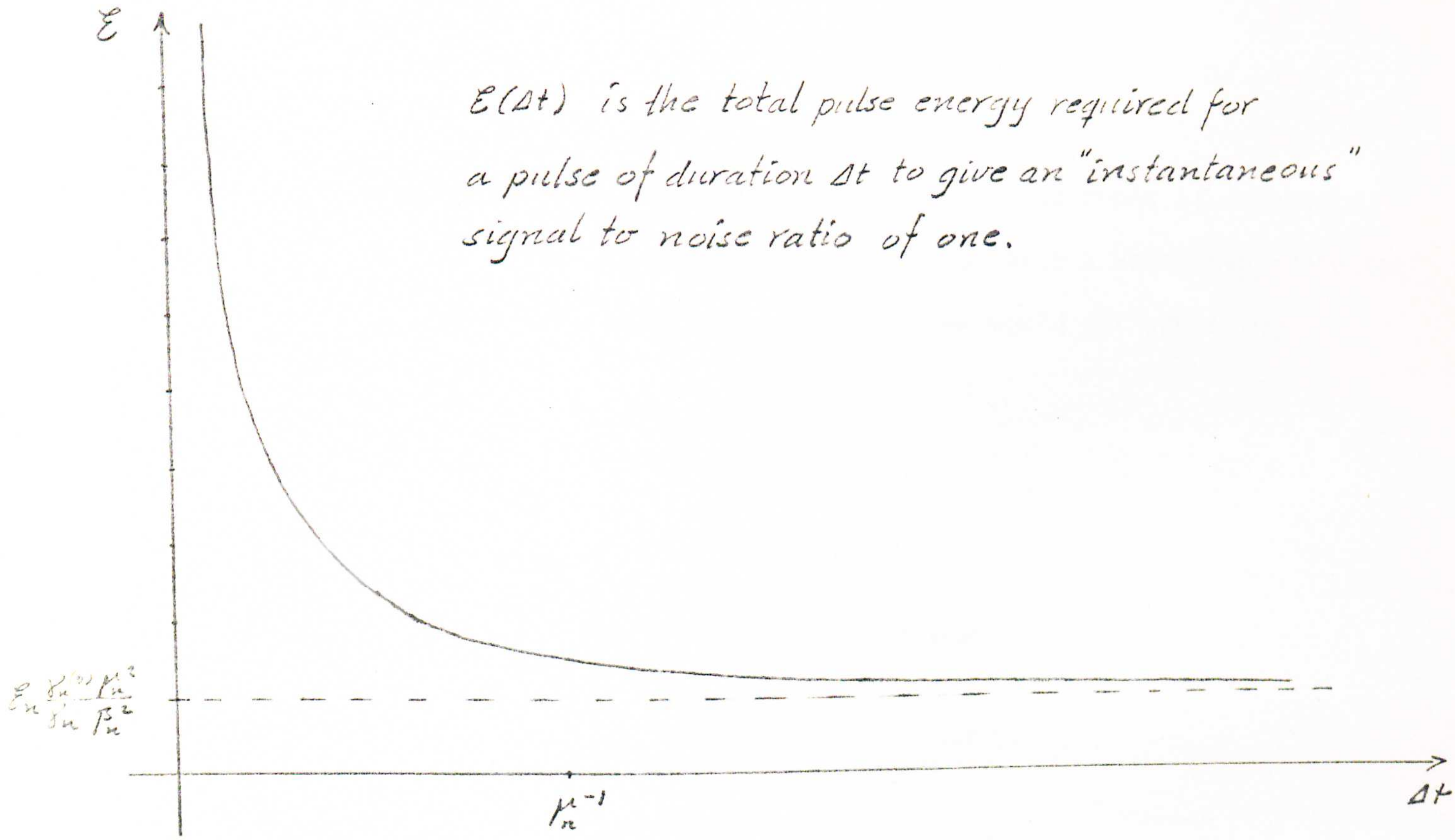
$$f({\mu_n}/{\Delta\omega}) = \left\{ {\mu_n}/{\Delta\omega} \exp \frac{\mu_n^2}{2\Delta\omega^2} \int_{\mu_n/\Delta\omega}^{\infty} \frac{dx}{\sqrt{2\pi}} \exp - \frac{x^2}{2} \right\}^{-1}.$$

We give a graph of $\mathcal{E} = \mathcal{E}_n \dfrac{\gamma_n^{(b)} \mu_n^2}{\gamma_n \beta_n^2} f(\mu_n \cdot \Delta t)$, where $\Delta t = \dfrac{1}{\Delta\omega}$.

Again we give as typical values μ_0^{-1} (the effective resolution time) and $\mathcal{E}_0 \, \mu_0^2/\beta_0^2$ for Weber's 1973 set up. $\mu_0^{-1} \approx 0.6$ sec. and

$$\mathcal{E}_0 \frac{\gamma_n^{(b)} \mu_n^2}{\gamma_n \beta_n^2} \approx 5 \cdot 10^4 \text{ ergs/cm}^2.$$

FIG. C.7



We note from fig. C.6 and C.7, that the detector system has a built in bandwidth or minimal "resolution time constant" which has nothing to do with how the signal later is actually filtered.

The matched filter function for the kind of signal would be

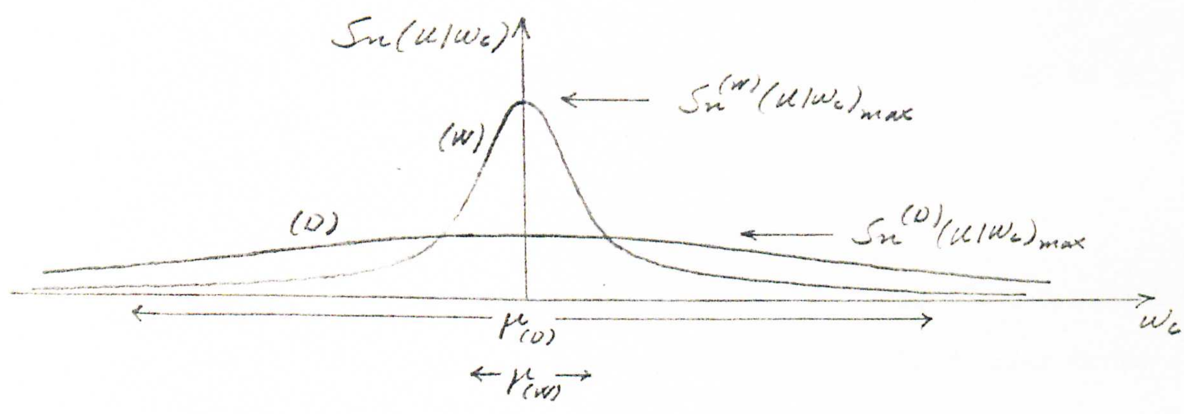
$$f(\omega) \sim \frac{\omega_c + i\gamma_n}{\mu_n^2 + (\omega_c + \Delta\omega_n)^2} \exp - \frac{(\omega_c + \Delta\omega)^2}{(\Delta\omega)^2} \quad \text{or}$$

$$f(t-t') \sim \int_{-\infty}^{+\infty} \frac{d\omega_c}{\sqrt{2\pi}} \frac{\omega_c + i\gamma_n}{\mu_n^2 + (\omega_c + \Delta\omega_n)^2} \exp - \frac{(\omega_c + \Delta\omega)^2}{(\Delta\omega)^2} e^{i\omega_c (t-t')} .$$

If we change reference to $\omega_n + \Delta\omega_n$ we get rid of the oscillating terms in $f(t-t')$.

One may now compare the overall performance of different types of detectors with graphs like (8) and (9). If one for example compares a Weber type detector (W) with a Drever type* detector (D), we would get something like

FIG. C.8



The graph is not correctly scaled, in fact $\dfrac{Sn^{(W)}(n/\omega_c)_{max}}{Sn^{(D)}(n/\omega_c)_{max}} \approx 45.$

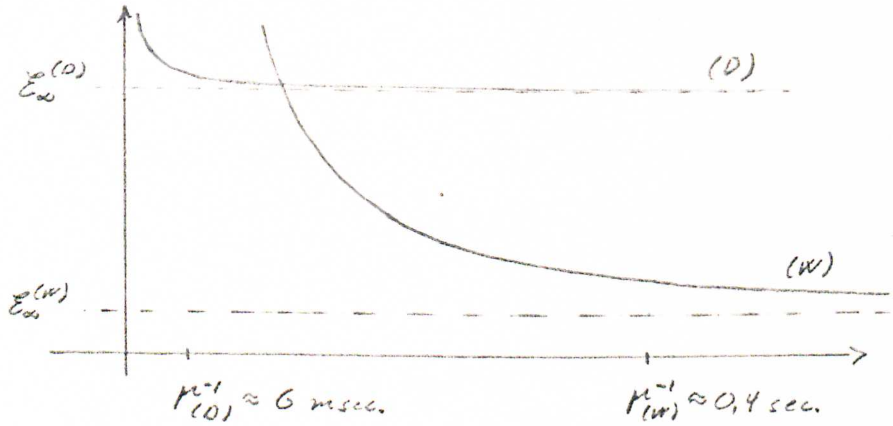
(*With Drever type, we mean a detector with a comparatively strong coupling and low Q).

(Making the detector masses and the central frequency the same) and

$$\frac{\mu_{(D)}}{\mu_{(W)}} \approx 67.$$

Or in terms of a graph like that in Fig. 9.

Fig C.9



$$\mathcal{E}_\infty^{(D)} / \mathcal{E}_\infty^{(W)} \approx 45.$$

(Note that the values used are probably not "up to date" and used to provide for an example only).

Sensitivity to a delta function type excitation

According to an argument given by Gibbons and Hawking a gravitational radiation pulse can never be delta function like in the Riemann tensor, since for finite energies, we must have $\int_{-\infty}^{+\infty} dt\, R_{1010}(t) = 0$ (See the formula for the power of gravitational radiation, and the reference given on page 89). Still if $R_{1010}(t)$ just consists of a few cycles in the frequency region ω_n, it will appear as a delta function-like excitation of the detector with respect to the resolution time μ_0 (We can see from Fig.C.7, however that detectable excitations of this type are costly in terms of radiation energy). Further the signal to noise ratio for a sudden kT excitation (i.e. an excitation such that $\frac{1}{4} m |\Delta V^2| = kT$, where V is defined by (1.16) or in other words such that $\Delta E = kT$ if $E_{initial} = 0$) is an often used measure for the quality of the detector.

We have from (C.6)

$$S_n(s/n, \omega_c) = 2 \frac{V_s^2(\omega_0 + \omega_c)}{v_n^2} \frac{\beta_n^2}{\mu_n^2 + (\omega_c + \Delta\omega)^2} .$$

(Let $V_s(\omega_0 + \omega_c) = V_s(\omega_c)$).

Since we assume a delta function type signal we have

$$V_s(\omega_c) = \frac{c}{\sqrt{2\pi}} \quad (\text{so that } V(t) = \int \frac{d\omega_c}{2\pi} \, c \exp i\omega_c(t - t_0) = c\,\delta(t - t_0))$$

where we want to choose c so that we get a kT excitation.

In terms of the current of the equivalent circuit, the energy of the cylinder is $E = \frac{1}{2} L I^2$ and the current is obtained by solving (C.10) with $I = \dot{q}$

$$I = \int_{-\infty}^{t} dt' \exp - \gamma_n(t - t') \exp i\omega_n t' \frac{\dot{V}(t')}{L\omega_n} =$$

$$= \int_{-\infty}^{t} dt' \exp(-\gamma_n(t - t') + i\omega_n t') \frac{c \frac{d}{dt} \delta(t' - t_0)}{L_1 \omega_n} =$$

$$= \int_{-\infty}^{t} dt'(\gamma_n - i\omega_n) \frac{c\,\delta(t'-t_0)}{L\omega_n} \exp(-\gamma_n(t-t') + i\omega_n t') = \frac{c}{L_1}\left(-i + \frac{\gamma_n}{\omega_n}\right) e^{i\omega_n t_0} e^{-\gamma_n (t-t_0)}$$
$$t > t_0$$

(Dropping the small $\frac{\gamma_n}{\omega_n}$) we have

$$|I^2| = \frac{c^2}{L_1^2}, \quad \text{with } \tfrac{1}{2} L_1 |I^2| = kT \quad \text{we have} \quad c = \sqrt{2kTL_1}.$$

So we have $V_s(\omega_0 + \omega_c) = \sqrt{\frac{kTL_1}{\pi}}$, with $v_n^2 = kTR_1$

$$S_n(kT) = \frac{2L_1}{\pi R_1} \int_{-\infty}^{+\infty} d\omega_c \frac{\beta_n^2}{\mu_n^2 + (\omega_c + \Delta\omega_n)^2}$$

$$= \frac{\beta_n^2}{\gamma_{1n}\mu_n} \int_{-\infty}^{+\infty} d\omega_c \frac{\mu_n}{\pi(\mu_n^2 + (\omega_c + \Delta\omega_n)^2)} \Longrightarrow$$

$$\text{(C.15)} \quad S_n(kT) = \frac{\beta_n^2}{\gamma_{1n}\mu_n}$$

where $\gamma_{1n} = \frac{R_1}{2L_1}$

(Note that $S_n(kT)$ is the "instantaneous" signal to noise ratio, of a completely defined signal.)

We will now see how this signal to noise ratio is related to the so called $\bar{\beta Q}$. For the case where $R_3 = 0$, and perfect amplifier $i = 0,\ e = 0$, we have

$$\mu^2 = \frac{R_1 R_2^2 + R_1^2 R_2}{4 R_2 L_1^2} \qquad \text{and}$$

$$\beta^2 = \frac{R_1 R_2^2}{4 R_2 L_1^2} \qquad \Rightarrow$$

$$(C.16) \qquad S_n(kT) = \sqrt{\frac{R_2}{R_1(1 + R_1/R_2)}} \approx \sqrt{\frac{R_2}{R_1}} \, .$$

This may be written in a different way. With $Q_1 = \dfrac{1}{\omega_0 R_1 C_1}$ and $Q_2 = \omega_0 R_2 C_2$ we have

$$(C.17) \qquad S_n(kT) \approx \sqrt{Q_1 Q_2 \frac{C_1}{C_2}} \, .$$

The term $\frac{C_1}{C_2} Q_1$ is usually called the $\bar{\beta} Q$ (we give this β a bar to make it differ from the β above) where $\bar{\beta} = C_1/C_2$ is a measure of the strength of the coupling between the detector and the sensor. Note though that this signal to noise ratio really does not depend on $\bar{\beta}$, but only on R_1 and R_2. When the amplifier or R_3 contributes noise however the $\bar{\beta} = C_1/C_2$ may be important. We give some other useful forms of (C.16) If $R_1 << R_2$ we have $\mu^2 \approx \beta^2$, thus from (C.16)

$$(C.18) \qquad S_n(kT) = \frac{\mu_n}{\gamma_{in}} = \frac{\tau_n}{\Delta t_n}$$

where τ_n is the amplitude damping time of the detector and Δt_n is the time resolution (of the detector system).

We may also express (C.16) in terms of a "detectability limit" of the pulse energy.

Clearly we have $(R_1 << R_2)$

$$S_n(\epsilon kT) = \epsilon \sqrt{\frac{R_2}{R_1}}$$

i.e. the signal to noise is linear in the pulse strength ϵ (measured in units of kT). We may define the lower limit of the energy of detectable pulse by ϵ_0 such that $S_n(\epsilon_0) = 1 \Rightarrow$

$$(C.19) \qquad \epsilon_0 = \sqrt{\frac{R_1}{R_2}} = \frac{\Delta t_n}{\tau_n} .$$

Or in terms of amplitude α measured in units of the root mean square of the thermal amplitude $\langle x^2 + y^2 \rangle^{1/2}$

$$(C.20)\quad \alpha_0 = \left(\frac{R_2}{R_1}\right)^{1/4} = \left(\frac{\Delta t_n}{J_n}\right)^{1/2} .$$

Lastly we note that we may express the resolution time as

$$\Delta t_n = \frac{1}{\omega_0}\sqrt{\frac{Q_1}{\beta Q_2}} \;.$$

To see the role of C_2 we investigate the more complex situation when a preamplifier with a given e' and i', is included. Let's further assume that we have the ability to match our system perfectly. This means essentially that we can let $e' \rightarrow n \cdot e$ and $i' \rightarrow \frac{1}{n} i$ where n can be chosen to optimize the signal to noise. Assume further that the system has the same temperature all over. This means that

$v_n^2 = R_1 kT$, $v_2^2 = R_2 kT + \frac{i'^2 R_2^2}{n^2}$ (see figure 6) and $v_3^2 = n^2 e^2$.

To simplify the notation let e and i be given in terms of temperature so that $e^2 = n^2 \alpha kT$ and $i'^2 = \frac{1}{n^2} \alpha kT$ (which gives the preamplifier the temperature $\alpha kT\ k^\circ$, note also that n^2 now has the dimension ohm). Further we simplify μ^2 in (5.4) by excluding the γ_1^2 and γ^2 terms (which are much smaller than the other terms.)

We thus have

$$
\text{(C.21)}\quad
\left\{
\begin{aligned}
&\Delta\omega_2 = \frac{1}{2}\,\omega_0\,\frac{C_1}{C_2}\,, \quad \text{and} \quad \Delta\omega = \Delta\omega_2\,\frac{R_2 + \frac{R_2^{\,2}}{n^2}\alpha}{R_2 + \frac{R_2^{\,2}}{n^2}\alpha + n^2\alpha(\omega_0^{\,2}C_2^{\,2}R_2^{\,2})}\\[2ex]
&\mu_n^{\,2} = \frac{R_1 R_2^{\,2} + n^2\alpha R_2^{\,2}\,\dfrac{R_2 + \frac{R_2^{\,2}}{n^2}\alpha}{R_2 + \frac{R_2^{\,2}}{n^2}\alpha + n^2\alpha(\omega_0^{\,2}C_2^{\,2}R_2^{\,2})}}{4\left(R_2 + \frac{R_2^{\,2}}{n^2}\alpha + n^2\alpha(\omega_0^{\,2}C_2^{\,2}R_2^{\,2})\right)L_1^{\,2}}\\[2ex]
&\beta_n^{\,2} = \frac{R_1 R_2^{\,2}}{4\left(R_2 + \frac{R_2^{\,2}}{n^2}\alpha + n^2\alpha(\omega_0^{\,2}C_2^{\,2}R_2^{\,2})\right)L_1^{\,2}}
\end{aligned}
\right.
$$

and

$$
\text{(C.22)}\quad S_n(kT,n) = \frac{R_2^{\,2}}{\left\{\left(R_1R_2^{\,2} + n^2\alpha R_2^{\,2}\,\dfrac{R_2 + \frac{R_2^{\,2}}{n^2}\alpha}{R_2 + \frac{R_2^{\,2}}{n^2}\alpha + n^2\alpha\,\omega_0^{\,2}C_2^{\,2}R_2^{\,2}}\right)\left(R_2 + \frac{R_2^{\,2}}{n^2}\alpha + n^2\alpha\,\omega_0^{\,2}C_2^{\,2}R_2^{\,2}\right)\right\}^{1/2}} =
$$

$$
= \frac{\sqrt{R_2}}{\left\{R_1 + \alpha^2 R_2 + n^2\alpha\left(1 + \omega_0^{\,2}C_2^{\,2}R_1R_2\right) + \frac{\alpha}{n^2}R_1R_2\right\}^{1/2}}
$$

In terms of resistances R_1, R_2 and R_e (see page 91) this would be (with n = 1)

$$S_n(kT) = \frac{\sqrt{R_2}}{\left\{R_1 + \left(1 + Q_2^2 \frac{R_e}{R_2}\right) + R_e\right\}^{1/2}}$$

$S_n(kT, n)$ may now be maximized with respect to n, which is realized by choosing $n^2 = \sqrt{\frac{R_1 R_2}{1 + \omega_0^2 C_2^2 R_1 R_2}}$

So

(C.23) $$S_n(kT, n)_{max} = \left(\frac{\beta^2}{\gamma_1 \mu}\right)_{max} = \left\{R_1/R_2 + \alpha^2 + 2\alpha\sqrt{R_1/R_2 + \omega_0^2 C_2^2 R_1^2}\right\}^{-1/2}$$

Or in terms of sensitivity (compare 5.19)

$$\epsilon_0 = \left\{R_1/R_2 + \alpha^2 + 2\alpha\sqrt{R_1/R_2 + \omega_0^2 C_2^2 R_1^2}\right\}^{1/2}$$

In the limit of "a perfect sensor" i.e $R_2 \to \infty$, $C_2 \to 0$, we have; $\epsilon_0 \to \alpha$, i.e. the sensitivity equals the preamplifier-temperature.

For a given R_1/R_2 we can find from the above expression if and how much it pays to reduce C_2 and α (and vice versa). It does not pay much to reduce C_2 beyond $C_2 < \frac{1}{\sqrt{R_1 R_2 \omega_0^2}}$ and α beyond

$$\alpha < \alpha_0 \sqrt{2\, R_1/R_2 + \omega_0^2 C_2^2 R_1^2} - \sqrt{R_1/R_2 - \omega_0^2 C_2^2 R_1^2} .$$

(if we have a reasonably matched system). We may reexpress the condition for C_2 as

$$1 < C_{20}/C_2 = \sqrt{\frac{Q_1 C_1}{Q_2 C_2}} \qquad ; \quad C_{20} = \frac{1}{\sqrt{\omega_0^2 R_1 R_2}}$$

We note further that the match derived here (imput impedance of amplifier $= n^2 = \sqrt{\frac{R_1 \cdot R_2}{1 + \omega_0^2 C_2^2 R_1 R_2}}$ is optimal only for delta function excitations. In a Hertz type experiment the quantity β^2/μ^2 should be optimized (See fig. 5.6). (If the signal is monocromatic with frequency ω_0).

Again we use the Weber 1973 set up for typical numbers.

$$R_1/R_2 = 2{,}17 \cdot 10^{-4}$$

$$C_2 = 1 \cdot 10^{-8}$$

$$C_{20} = \frac{1}{\sqrt{R_1 \cdot R_2\, \omega_0^2}} = 1{,}3 \cdot 10^{-8}$$

Optical input impedance of preamplifier;

$$n_0^2 = \sqrt{\frac{R_1 \cdot R_2}{1 + \omega_0^2 C_2^2 R_1 \cdot R_2}} = 5{,}4 \cdot 10^2\, \Omega$$

$$\omega_0^2 C_2^2 R_1^2 = 12{,}3\ 10^{-4}$$

$$\alpha_0 = 2{,}3 \cdot 10^{-3}$$

$$Sn(kT)_{\alpha=0} = 136$$

$$Sn(kT)_{\alpha=10^{-2,5}} = 100$$

We can see that one can improve slightly by reducing C_2 but not much if $\alpha = 10^{-2,5}$, Sn would go from 100 to 114 if $C_2 \to 0$.

We will now determine the matched filter for the delta-function signal. From (5.1) we have the output signal spectrum

$$V_s(\omega) \sim \frac{1}{\omega_c - i\gamma}$$

and the noise power spectrum from (5.5)

$$|V_{on}^2(\omega)| \sim \frac{(\omega_c + \Delta\omega)^2 + \mu_n^2}{\omega_c^2 + \gamma_n^2}$$

so

$$f(\omega_c) \sim \frac{\omega_c^2 + \gamma_n^2}{(\omega_c + i\gamma_n)\{(\omega_c + \Delta\omega)^2 + \mu_n^2\}} = \frac{\omega_c - i\gamma_n}{\{(\omega_c + \Delta\omega_n)^2 + \mu_n^2\}}$$

We change reference to $\omega_n - \Delta\omega_n \Rightarrow \omega_c' = \omega_c + \Delta\omega_n \Rightarrow$

$$f(\omega_c') \sim \frac{\omega_c' - \Delta\omega_n - i\gamma_n}{(\omega_c'^2 + \mu_n^2)}$$

and

$$f(t-t') \sim \int_{-\infty}^{+\infty} d\omega_c \, \frac{\omega_c' - \Delta\omega_n - i\gamma_n}{(\omega_c'^2 + \mu_n^2)} \, e^{i\omega_c'(t-t')} \Rightarrow$$

$$f(t-t') = S(t'-t)(\mu_n + \gamma_n - i\Delta\omega_n)e^{-\mu_n(t-t')} -$$

$$- S(t-t')(\mu_n - \gamma_n + i\Delta\omega_n)e^{-\mu_n(t'-t)}$$

where $S(t-t')$ is the step-function. Evidently this filter mixes the components of the signal. We will try to make the situation clear by the following.

Let $f_1(t-t') = S(t'-t)\, e^{-\mu_n(t'-t)} - S(t-t')\, e^{-\mu_n(t-t')}$ and

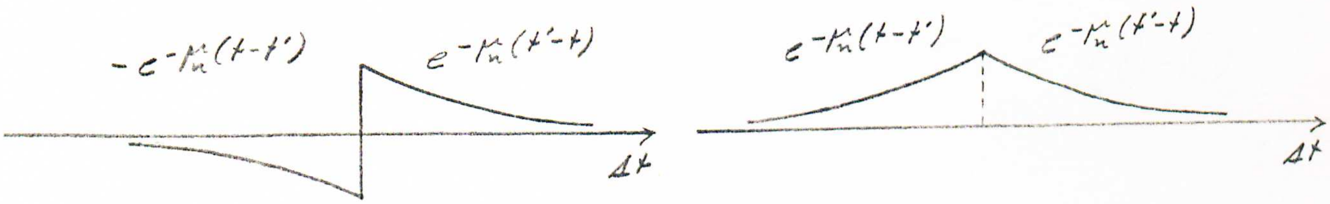
$f_2(t-t') = S(t-t')\, e^{-\mu_n(t'-t)} + S(t-t')\, e^{-\mu_n(t-t')}$, so that

$$f(t-t') \sim \mu_n f_1(t-t') + (\gamma_n - i\Delta\omega_n)\, f_2(t-t') \qquad \text{(C.24)}$$

f_1 and f_2 can be represented by the following two graphs

Fig 5.10

$\Delta t = t' - t.$



$f_1(t-t')$ $\qquad$ $f_2(t-t')$

As we can see $f_1(t-t')$ acts like a derivation averaged over the time μ_n^{-1} while $f_2(t-t')$ just averages over the time μ_n^{-1}.

(In fact $f_1(t-t') \sim \frac{\partial}{\partial t} f_2(t-t')$ so that $\int dt' f_1(t-t')\, g(t') \sim$
$\sim \frac{\partial}{\partial t} \int dt' f_2(t-t')\, g(t')$.)

With

$$\bar{d}(t) = \bar{V}_1(t) + \bar{V}_2(t) + \bar{V}_3(t) \quad , \quad \bar{d} = d_{(1)} + i d_{(2)} \quad \text{and}$$

the filtered data $\bar{d}$ denoted by $\bar{d}_f$ we have

$$d_{(1)f}(t) = \int_{-\infty}^{+\infty} dt' \left(\gamma_n f_1(t-t') + \gamma_n' f_2(t-t')\right) d_{(1)} + \Delta\omega_n f_2(t-t')\, d_{(2)}$$

$$d_{(2)f}(t) = \int_{-\infty}^{+\infty} dt' - \Delta\omega_n f_2(t-t')\, d_{(1)} + \left(\gamma_n f_1(t-t') + \gamma_n' f_2(t-t')\right) d_{(2)}$$

The form of the signal part of the output in the primed coordinate system is easily obtained;

$$\bar{V}_{sf}(t) \sim \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \frac{C}{(\omega_c'^2 + \mu_n^2)}\, e^{i\omega_n (t-t_0)} = C f_2(t-t_0)$$

where t_0 is the arrival time of the pulse and C some complex constant. $f_2(\Delta t)$ is also the correlation function of the output noise . (See page 82).

In practice, the filter $f(t-t')$ may be a bit complicated and one usually uses a filter approximating $f_1(t-t')$ with the reference ω_n .
We will see how much signal to noise is lost by this procedure. We have

$$f_1(\omega) \sim \frac{\omega_c - i\gamma}{\omega_c^2 + \mu_n^2}$$

so the filtred signal is from (C.1)

$$(V_{sf})_{max} = \int \frac{d\omega}{\sqrt{2\pi}} \frac{C\, i Q_2}{(1+iQ_2)\, i\omega_0 C_2 (\omega_c^2 + \mu_n^2)}$$

and from (C.5)

$$|\overline{V_{onf}^2}| = \int \frac{d\omega_c}{2\pi} 4\left(\overline{V_3^2} + \frac{\overline{V_2^2}}{1+Q_2^2}\right) \frac{(\omega_c + \Delta\omega_n)^2 + \mu_n^2}{(\omega_c^2 + \mu_n^2)^2}$$

or

$$|\overline{V_{onf}^2}| = \int \frac{d\omega_c}{2\pi} 4\left(\overline{V_3^2} + \frac{\overline{V_2^2}}{1+Q_2^2}\right) \left(\frac{1}{\omega_c^2 + \mu_n^2} + \frac{\Delta\omega^2}{(\omega_c^2 + \mu_n^2)^2}\right)$$

Now

$$\int_{-\infty}^{+\infty} d\omega_c \frac{1}{(\omega_c^2 + \mu_n^2)^2} = \left[\frac{\omega_c}{2\mu_n^2(\omega_c^2 + \mu_n^2)}\right]_{-\infty}^{+\infty} + \frac{1}{2\mu_n^2} \int_{-\infty}^{+\infty} d\omega \frac{1}{\omega_c^2 + \mu_n^2} \Rightarrow$$

$$\overline{V_{onf}^2} = \int \frac{d\omega_c}{2\pi} \left(\overline{V_3^2} + \frac{\overline{V_2^2}}{1+Q_2^2}\right) \frac{1}{\omega_c^2 + \mu_n^2} \left(1 + \frac{\Delta\omega^2}{2\mu_n^2}\right)$$

Since the original signal to noise is

$$Sn(kT) = \frac{\left|\int \frac{d\omega}{\sqrt{2\pi}} \frac{C\, i Q_2}{(1+iQ_2)\, L\, i\omega_0 C_2 (\omega_c^2 + \mu_n^2)}\right|^2}{\int \frac{d\omega}{2\pi} 4\left(\overline{V_3^2} + \frac{\overline{V_2^2}}{1+Q_2^2}\right) \frac{1}{\omega_c^2 + \mu_n^2}}$$

we get, calling the new signal to noise ratio $Sn^e(kT)$ (effective signal to noise ratio)

$$\text{(C.25)} \quad Sn^e(kT) = SnkT \left(1 + \frac{\Delta\omega^2}{2\mu_n^2}\right)^{-1}$$

Im most cases $\Delta\omega \ll \mu_n$ so the procedure is o.k. In a case however where we have a high Q antenna, and a not so high Q pick-up system with fairly strong coupling, we may have to be more careful. If $R_3 = 0$, $i = 0$, $c = 0$ we have

$$\frac{\Delta\omega_n}{\mu_n} \approx \frac{Q_1 \bar{\beta}}{Q_2} \qquad \text{where } \bar{\beta} = \frac{c_1}{c_2} \quad .$$

It is obvious that if the output data $\bar{V}(t) = (V_1(t), V_2(t))$ is put on tape after being filtered in the above manner it can be refiltered by the computer to match other types of signals as well. In fact if we would like to filter with respect to signal pulses with the envelope $S(t)$, and random arrival times, we should just filter the output

$\bar{V}(t)$ with $S(t-t')$

so that

$$(C.26) \quad \bar{V}_{new}(t) = \int_{-\infty}^{+\infty} dt\, \bar{V}(t)\, S(t-t') \; .$$

We note in connection with this that it is frequently implied in the literature that filtering necessarily leads to loss of information, this is however in general not so, it is the addition of noise to the signal that causes loss of information (of course we can deliberately choose a filter such that it wastes information. This would not be a very good idea though).

It is easily seen that for example an ordinary RC filter is invertible (i.e. there is a one to one map from the imput to the output) and as we shall see the filter $(C.24)$ is also invertible.

We have

$$f^{-1}(\omega_c) = \frac{\omega_c^2 + \mu_n^2}{\omega_c - \Delta\omega_n - i\,\gamma_n}$$

This looks a bit suspicious at first sight ($f^{-1}(\omega)$ does not seem to converge), but $f^{-1}(\omega)$ is in fact just trying to tell us that it is a differential type of filter. We have

$$f^{-1}(t-t') \sim \int \frac{d\omega_c}{\sqrt{2\pi}} \frac{(\omega_c^2+\mu_n^2)\, e^{i\omega_c(t-t')}}{\omega_c - \Delta\omega_n - i\delta_n} =$$

$$= \int \frac{d\omega_c}{\sqrt{2\pi}} \frac{\mu_n^2\, e^{i\omega(t-t')}}{\omega_c - \Delta\omega_n - i\delta_n} + \frac{d}{dt}\int \frac{d\omega_c}{\sqrt{2\pi}} \frac{\omega_c\, e^{i\omega(t-t')}}{i(\omega_c - \Delta\omega_n - i\delta_n)} =$$

$$= \sqrt{2\pi}\, i\mu_n^2\, e^{-(i\Delta\omega_n+\delta_n)(t-t')} S(t-t') + \frac{d}{dt}\int \frac{d\omega_c}{\sqrt{2\pi}} \left(-i - \frac{i(\Delta\omega_n + i\delta_n)}{\omega_c - \Delta\omega_n - i\delta_n}\right) e^{i\omega_c(t-t')} =$$

$$= \sqrt{2\pi}\, i\mu_n^2\, e^{-(i\Delta\omega_n+\delta_n)(t-t')} S(t-t') + \frac{d}{dt}\left\{-iS(t-t') + \sqrt{2\pi}(\Delta\omega_n + i\delta_n)\, e^{-i(\Delta\omega_n+\delta_n)} S(t-t')\right\} \sim$$

where $S(t-t')$ is the step function.

$$\sim \sqrt{2\pi}\left\{\mu_n^2 - i\frac{\partial}{\partial t}(\Delta\omega_n + \delta_n)\right\} e^{-(i\Delta\omega_n+\delta_n)(t-t')} S(t-t') - \frac{\partial}{\partial t} S(t-t').$$

Thus if the filtered output is $\overline{V_s}(t) = \int_{-\infty}^{+\infty} dt'\, f(t-t')\, V_s(t')$

we have $V_s(t) = \int_{-\infty}^{+\infty} dt'\, f^{-1}(t-t')\, \overline{V_s}(t) =$

$$= \sqrt{2\pi}\left(\mu_n^2 + (\Delta\omega_n + i\delta_n)^2\right) \int_{-\infty}^{t} dt'\, e^{-(i\Delta\omega_n+\delta_n)(t-t')}\, \overline{V_s}(t') +$$

$$+ \sqrt{2\pi}(\Delta\omega_n + i\delta_n)\, \overline{V_s}(t) - \frac{\partial}{\partial t}\overline{V_s}(t).$$

This is of course, a rather complicated expression, and is not intended for any practical use (one is never interested in a totally unfiltered amplitude, since it contains wide band noise), but only to show that there is no information loss connected with the filter (C.24). Rather in practice one would use the relation (C.26) to look for signals other than of delta-function type. We note however that if we have output data d filtered by a combination of f_1 and f_2 (such as 5.24)

$$\dot{d}(t) = \int_{-\infty}^{+\infty} dt' \left(A\, f_1(t-t') + B\, f_2(t-t')\right) d(t')$$

then the quantity

$$\bar{d}(t) = \int_{-\infty}^{+\infty} dt'\, B\, f_2(t-t')\, d(t')$$

(which is just the smoothly filtred amplitudes) is related to $\dot{d}$ by a simple "non-differentiating" filter operation. We have

$$\dot{d}(t) = \int dt' \left(\frac{A}{\mu}\frac{d}{dt} f_2(t-t') + B\, f_2(t-t')\right) d(t') \quad \text{or}$$

$$\dot{d}(t) = \left(\frac{A}{\mu B}\frac{d}{dt}\bar{d} + \bar{d}\right) \quad \Rightarrow$$

$$\bar{d}(t) = \int_{-\infty}^{t} dt' \frac{\mu B}{A} \exp -(t-t')\frac{\mu B}{A} \cdot \dot{d}(t') .$$

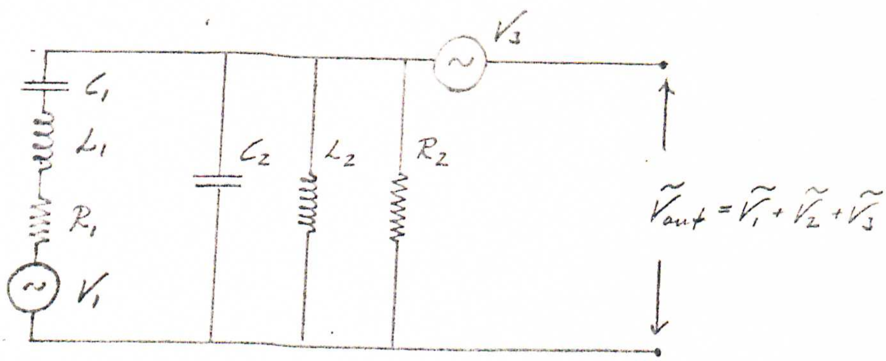
Thus $\bar{d}(t)$ is related to $\dot{d}(t)$ by a simple exponential filter if A and B are real. (Note that we use $\dot{}$ as a symbol for a "differentiating filter operation" and not for a pure differentiation).

DETECTOR WITH A RESONANT PICK-UP SYSTEM

It is possible to reduce the negative effect of the capacitor C_2 by introducing a parallel inductance L_2. We will briefly investigate such a system for the case when the free resonating frequencies of the (1) and (2) part of the circuit are equal.

The equivalent circuit is

Fig. C.11



The only difference from the circuit on page 94 is that a loss-free inductor (L_2) is added to resonate with the capacitor C_2. This gives us two coupled harmonic oscillators, the cylinder and the pick-up system, and leads to a complete transfer of the stored energy, from one to the other, with a transfer time (or period) $t_c = \frac{4\pi}{\omega_0}\sqrt{C_2/C_1}$

The inverse of the transfer time is a measure of the coupling between the systems.

This case could be handled similarly to the nonresonant case, but we will instead use an approximative method that is accurate enough for our purpose.

We have

$$\tilde{V}_1(\omega) = \frac{V_1(\omega)\,\dfrac{1}{i\omega C_2 + \frac{1}{R_2} + \frac{1}{i\omega L_2}}}{R_1 + i\omega L_1 + \frac{1}{i\omega C_1} + \dfrac{1}{i\omega C_2 + \frac{1}{R_2} + \frac{1}{i\omega L_2}}} =$$

$$= \frac{V_1(\omega)}{\left(R_1 + i\omega L_1 + \frac{1}{i\omega C_1}\right)\left(i\omega C_2 + \frac{1}{R_2} + \frac{1}{i\omega L_2}\right) + 1}$$

Now let $\omega = \omega_0 + \omega_c$ where $\omega_0^2 = \frac{1}{L_1 C_1} = \frac{1}{L_2 C_2}$

Consider $R_1 + i(\omega_0 + \omega_c) L_1 + \frac{1}{iC_1(\omega_0 + \omega_c)}$. If the system is narrowbanded so that "$\omega_c << \omega_0$" we may expand so that we get

$$R_1 + i(\omega_0 + \omega_c) L_1 + \frac{1}{i\omega_0 C_1} - \frac{1}{i\omega_0^2 C_1}\omega_c + i\omega_c L_1 = R_1 + 2i\omega_c L_1$$

Doing the same for the second term we get

$$\tilde{V}_1(\omega) = \frac{V_1(\omega)}{(2i\omega_c L_1 + R_1)(2i\omega_c C_2 + \frac{1}{R_2}) + 1} =$$

$$= - \frac{V_1(\omega)\,\bar{\omega}_s^2}{(\omega_c + \omega_s - i\gamma)(\omega_c - \omega_s - i\gamma)}$$

where $\omega_s^2 = \dfrac{1 + R_1/R_2}{4L_1C_2}$; $\bar{\omega}_s^2 = \dfrac{1}{4L_1C_2}$; Since R_1/R_2 is small we drop the bar on ω_s from here. $\gamma = \frac{1}{2}\left(\frac{1}{R_2C_2} + \frac{R_1}{L_1}\right)$.

Similarly we get for $\tilde{V}_2$

$$\tilde{V}_2(\omega) = \frac{V_2(\omega)\, \frac{2L_1}{R_2}\,(i\omega_c + \gamma_1)\,\omega_s^2}{(\omega_c + \omega_s - i\gamma)(\omega_c - \omega_s - i\gamma)}$$

and

$$\tilde{V}_3(\omega) = \frac{V_3(\omega)\,(\omega_c^2 - \omega_s^2 - \gamma^2 - 2i\omega_c\gamma)}{(\omega_c + \omega_s - i\gamma)(\omega_c - \omega_s - i\gamma)} .$$

And the spectral noise power output

$$|V_{on}^2(\omega)| = \frac{|V_n^2(\omega_c)|\,\omega_s^4 + |V_2^2(\omega_c)|\,\omega_s^4\,\frac{4L_1^2}{R_2^2}(\omega_c^2 + \gamma_1^2) + |V_3^2(\omega_c)|\left((\omega_c^2 - \omega_s^2 - \gamma^2)^2 + 4\omega_c^2\gamma^2\right)}{|(\omega_c + \omega_s - i\gamma)^2|\,|(\omega_c - \omega_s - i\gamma)^2|}$$

And the spectral signal power output

$$|\tilde{V}_s^2(\omega_c)| = \frac{|V_s^2(\omega_c)|\,\omega_s^4}{|(\omega_c + \omega_s - i\gamma)^2|\,|(\omega_c - \omega_s - i\gamma)^2|}$$

We have:

$$\text{(C.27)}\quad S_n(S/\omega_c) = 2\,\frac{|\tilde{V}_s^{2}(\omega_c)|}{|\tilde{V}_{on}^{2}(\omega_c)|} = 2\,\frac{|V_s^{2}(\omega_c)|}{|V_n^{2}(\omega_c)|}\;\frac{\beta^2}{\mu^2\left(1+\omega_c^2/\mu^2+\omega_c^4/\eta^4\right)}$$

where

$$\text{(C.28)}\quad \begin{cases} \beta^2 = \omega_s^2\,\dfrac{V_n^{2}}{V_2^{2}\,2\,\dfrac{L_1}{R_2^2C_2} - V_3^{2}\,2\left(1-\delta^2/\omega_s^2\right)} \\[2ex] \mu^2 = \omega_s^2\,\dfrac{V_n^{2}+V_2^{2}\,R_1^2/R_2^2+V_3^{2}\left(1+\delta^2/\omega_s^2\right)}{V_2^{2}\,\dfrac{2L_1}{R_2^2C_2} - V_3^{2}\,2\left(1-\delta^2/\omega_s^2\right)} \\[2ex] \eta^2 = \omega_s^2\sqrt{\dfrac{V_n^{2}+V_2^{2}\,R_1^2/R_2^2+V_3^{2}\left(1+\delta^2/\omega_s^2\right)}{V_3^{2}}} \end{cases}$$

These $\beta_i s$ and $\mu_i s$ are similar to the ones given by (5.4)

To see how much we have gained we take a look at the $S_n(kT)$. In most cases one would expect $\eta \gg \mu$. In terms of R_1, R_2 and R_c we have, $\frac{\mu^2}{\eta^2} \approx \frac{Q_2}{2Q_1}\sqrt{\frac{R_c}{R_1}\left(1+\frac{R_c}{R_1}\right)}$ usually $R_c \ll R_1$ and $Q_2 \ll Q_1$, if this is the case η won't contribute when (C.27) is integrated, and we get similarly to (C.22)

$$(C.29) \qquad S_n^{(\lambda)}(kT) = \frac{\sqrt{R_2}}{\left\{R_1 + \alpha^2 R_2 + n^2\alpha + \frac{\alpha}{n^2} R_1 R_2\right\}^{1/2}}$$

In terms of R_1 R_2 and R_c we have approximately $S_n(kT) = \sqrt{\frac{R_2}{R_1+R_c}}$

Comparing this with (5.22) we see that in effect C_2 has been cancelled by the parallel inductor. If η^2 is of the order of μ^2 or larger, the situation gets more complex. $S_n(kT)$ may be evaluated exactly;

$$S_n^{\lambda}(kT) = \begin{cases} \dfrac{\beta^2}{8\mu} \dfrac{\sqrt{1+\sqrt{1-4\frac{\mu^4}{\eta^4}}} - \sqrt{1-\sqrt{1-4\frac{\mu^4}{\eta^4}}}}{\sqrt{2\left(1-4\frac{\mu^4}{\eta^4}\right)}} & ;\ \eta^4 \geqslant 4\mu^2 \\[2ex] \dfrac{\beta^2}{8\mu^2}\sqrt{\dfrac{\mu^2\eta^2}{2\mu^4+\eta^2}} & ;\ \mu^4 \geqslant \frac{1}{4}\eta^4 \end{cases}$$

and we would have to compare $S_n^{(\lambda)}(kT)$ with $S_n(kT)$ of (C.15), from case to case to see if it pays to "resonate" with an inductor.

One should also note that the filter process in general gets more complicated when we use a resonating sensor.

The filter function matched to a short pulse is

$$f(t-t') \sim \int_{-\infty}^{+\infty} \frac{d\omega}{\sqrt{2\pi}} \frac{e^{i\omega_c(t-t')}}{\left(1 + \omega_c^2/\mu^2 + \omega_c^4/\eta^4\right)} \Rightarrow$$

(C.30) $$f(t-t') = S(t-t')\left\{ \frac{e^{-(t-t')\mu_1}(\mu_1 - \gamma)}{\mu_1 \eta^2 \sqrt{\frac{\eta^4}{4\mu^4} - 1}} - \frac{e^{-(t-t')\mu_2}(\mu_2 - \gamma)}{\mu_2 \eta^2 \sqrt{\frac{\eta^4}{4\mu^4} - 1}} \right\} -$$

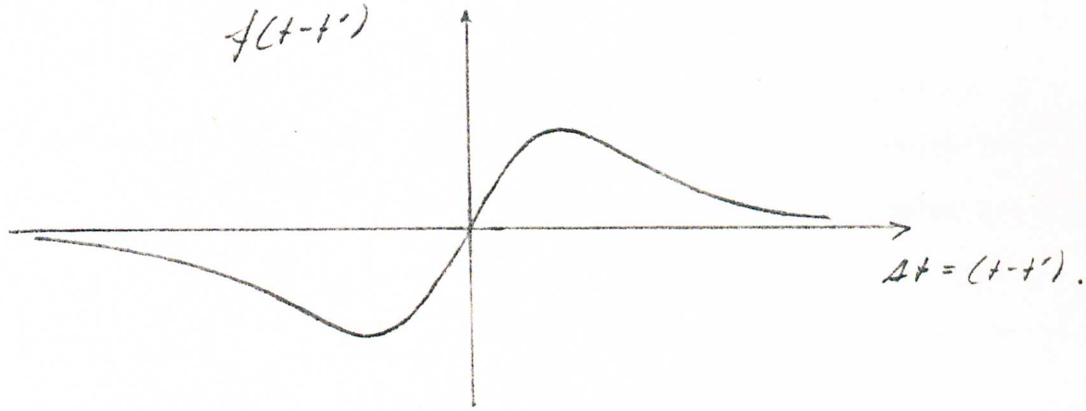
$$- S(t'-t)\left\{ \frac{e^{-(t'-t)\mu_1}(\mu_1 + \gamma)}{\mu_1 \eta^2 \sqrt{\frac{\eta^4}{4\mu^4} - 1}} - \frac{e^{-(t'-t)\mu_2}(\mu_2 + \gamma)}{\mu_2 \eta^2 \sqrt{\frac{\eta^4}{4\mu^4} - 1}} \right\}$$

where $$\mu_1 = \sqrt{\frac{\eta^4}{2\mu^2} + \eta^2 \sqrt{\frac{\eta^4}{4\mu^4} - 1}}$$ and

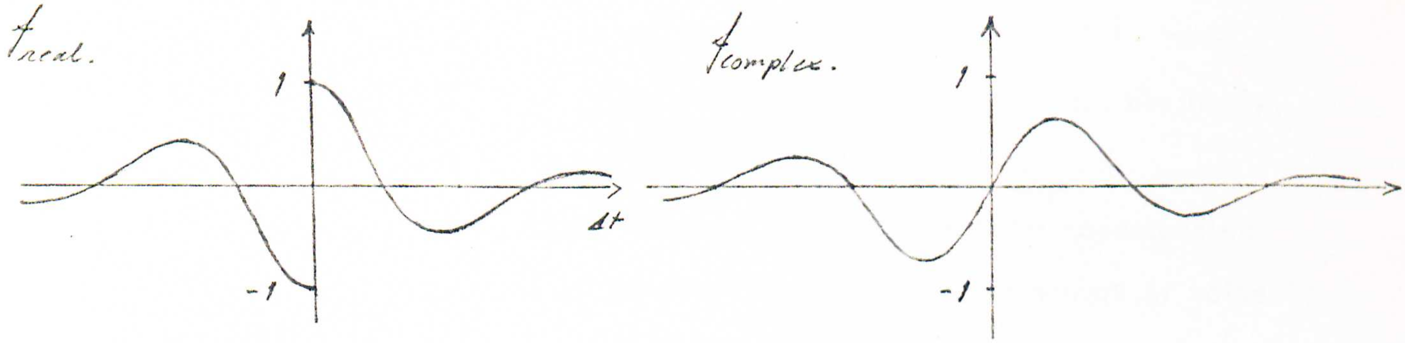
$$\mu_2 = \sqrt{\frac{\eta^4}{2\mu^2} - \eta^2 \sqrt{\frac{\eta^4}{4\mu^4} - 1}}$$

For $\eta^4 \geq 4\mu^4$, and neglecting the small symmetric γ part $f(t-t')$ is an antisymmetric function with a typical shape

Fig.C.12



If $4\mu^4 \geq \eta^4$, f splits up in a damped oscillating complex and real part,



Note that $f_{complex}$, is component mixing.

APPENDIX D

DATA ANALYSIS

RECORDING THE DATA

The problem we will consider here is, in which form should the data be recorded. The raw data is given to us in the form $(x(t), y(t))$, the output amplitudes of the detector. There are many ways in which this data could be transformed without any loss of information. Some examples are: a one to one variable transformation g; $(x'(t), y'(t)) = \bar{g}(x(t), y(t))$, a fourier transformation, or "invertible time invariant filtering" f

$$(\bar{x}(t), \bar{y}(t)) = \int dt' \, f(t-t') \, (x(t'), y(t'))$$

The word filter is somewhat misleading since it suggests that it reduces the information content of the data. This would only be the case if the filter is non invertible, e.g. if the filter is defined by $f(\omega)$ and $f(\omega)$ is zero for a finite interval $\Delta\omega$, then $f(\omega)$ would be non invertible. Obviously, we should not filter the data with such a filter.

Although as we said above, there are many arbitrary ways of representing the data, it is important to realize however that if a certain amount of noise is later added to the data, the form of the data (or the way it is represented) matters a lot, in terms of information loss. (The situation is similar with the detector itself, on the front end we have signal and white noise activity, this input is transformed by the detector in a one to one manner to the vibrational amplitude, in fact the forces acting on the detector are easily given in terms of the output vibrational amplitude $x(t)$) as;

$$F(t) = S(t) + n_0 \alpha(t) = m(\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x)$$

The "bare" detector thus acts as a one to one time invariant filter on the input $S(t) + n_0 \alpha(t)$, which transformation in no way limits the bandwidth (Part of the input $n_0 \alpha(t)$ of course, degrades our information about the signal in a nonbandwidth limiting way since $\alpha(t)$ is white and affects all frequencies equally). The bandwidth limiting effect (and thus a severe loss of information for signals of short duration) comes about when wide-band noise from sensors and preamplifier is added to the output).

Consider first the data in form of the amplitudes $(\dot{x}, \dot{y})$ (where $(\dot{x}, \dot{y})$ corresponds to an output given by the amplitudes (x, y) filtered by a filter f_1 optimally matched to a delta function type input signal). Note that $(\dot{x}, \dot{y})$ will always be an invertible transformation of (x, y), as long as the noise sources are not zero or infinite for a finite interval (a special case we need not consider here).

As can easily be seen a second filter f_2 to make the combined filter match our imput signal $S(t)$ is given by $f_2(t-t') = S^*(t-t')$, so that the new output is $(\bar{x}, \bar{y}) = \int dt' f_2(t-t') (\dot{x}, \dot{y})$.
This filter f_2, is "at the worst" a delta function in which case the output $(\bar{x}, \bar{y}) = (\dot{x}, \dot{y})$ and in all other cases a "smoothly varying function, with no derivation effects (except if we have signals "wilder" than delta functions, e.g. derivations of delta functions, or signals far of resonance, as we will see this does not alter the basic point of this argument however). Unfortunately, any recording of data in digitized or quantizized form will introduce "quantization noise".

The quantization noise is essentially white and may be described by $q\alpha(t_i)$, where $\alpha(t_i)$ is defined by $\langle \alpha(t_i), \alpha^*(t_j) \rangle = \delta_{ij}$, and q is a magnitude factor. (Compare with the definition of $\alpha(t)$ page 58.) If we include this noise in the recorded data we have

$$(\bar{\dot{x}}(t_i), \bar{\dot{y}}(t_i)) = \sum_j f_2(t_i - t_j)\{(\dot{x}(t_j), \dot{y}(t_j)) + q(\alpha_1(t_j), \alpha_2(t_j))\}$$

where $(\alpha_1(t_j), \alpha_2(t_j)) = \alpha(t_j)$ (α is a vector)

and f_2 is a filter operating on the digitized recorded data. Now if we make the quantization procedure "good enough" so that $q^2 \ll \langle(\dot{x}_{noise}, \dot{y}_{noise})^2\rangle$ and the quantization time or time increment at least as short as the correlation time of $(\dot{x}_{noise}, \dot{y}_{noise})$ then the"integrating" filter f_2 (which is smooth and without any strong derivation effect) will always tend to decrease the $q\alpha(t_i)$ (since it's white) compared to $(\dot{x}_{noise}, \dot{y}_{noise})$, in other words there is no risk that the filter f_2 could make $q\alpha(t_i)$ blow up.

Consider next recording a different choice of variables for example the plain output amplitudes (x, y). With quantization noise added, we get $(x(t_i), y(t_i)) + q\alpha(t_i)$, a subsequent filter operation gives

$$(\bar{x}(t_i), \bar{y}(t_i)) = \sum_j f(t_i - t_j)\{(x(t_j), y(t_j)) + q(\alpha_1(t_j), \alpha_2(t_j))\}$$

Again we choose the quantization procedure such that $q^2 \ll \langle(x_{noise}, y_{noise})^2\rangle$ As long as we are only interested in the amplitudes or any smoothly filtered amplitudes there are no problems. But note what happens if we want to obtain quantities like $(\dot{x}(t), \dot{y}(t))$, then $f(t_i - t_j)$ becomes a "derivation type filter." $(x(t_i), y(t_i))$ is a slowly varying quantity with a long correlation time and $\alpha(t_i)$ is a small but fast varying quantity. Thus any (time) derivation on $(x(t_i), y(t_i)) + g(\alpha_1(t_i), \alpha_2(t_i))$ will make $g(\alpha_1(t_i), \alpha_2(t_i))$ blow up compared to $(x(t_i), y(t_i))$. We may thus conclude that the best form in which to record the data is a form such that in any subsequent processing, we will never need to apply any"strongly differentiating" type of filter.

One may restate the above considerations in terms of a more precisely formulated problem, or question. Let (D.1) $d(t) = \chi(t) + N(t)$,
be a continuous time series, where N(t) is stationary colored Gaussian noise, and χ (t) is a signal, which is assumed to be known in a statistical sense, e.g. let a particular signal be defined by a set of parameters $\{a_\nu\}$.
A signal χ (t) may then be defined by the probabilities $P_\nu(a_\nu)$.

The problem, or questions one would like to answer are the following:

Given that one is allowed to record m bits of information per unit time, which is the best way (minimal information loss) to record the data, and what are the limits on m if one allows only a small fraction of the signal information to be lost in the recording process.

As far as I know this problem has not been solved, and further it appears to be far from trivial.

In practice one may of course be satisfied with some approximative answers. We will have to consider a simplified version of the problem, and even in this case we can at the present give only conjectured answers.

We consider the following questions:

I. Given that one is allowed $n = 1/\Delta t$ data points (real numbers) per unit time in a first reduction of $d(t)$, in which way is the data best reduced,

$$\{ d(t) \} \overset{?}{\rightarrow} \{ d_i \} \quad \text{(n number of } d_i\text{'s} \quad \text{per unit time).}$$

II. What is the minimum n required if only a small fraction of the original information is allowed to be lost in the above data reduction.

III. Given that one is allowed m possible numbers in recording d_i, in which way is d_i best reduced to m numbers.

IV. What is the minimum m required if only a small fraction of the original information is allowed to be lost in the recording.

CONJECTURED ANSWERS

I. We shall take as a guiding principle for question one that the reduced data points d_i should be uncorrelated (when no signal is present) i.e. $\langle d_i\, d_j \rangle = \delta_{ij}$, so that every point d_i contains independent information.* One way of accomplishing this is the following; transform (D.1) with a filter $f(t-t')$ so that the noise goes white i.e.

$$\bar{d}(t) = \int dt' f(t-t')\, d(t') = \bar{x}(t) + \alpha(t).$$

(where $\alpha(t)$ is a white noise source) and form data points d_i by integrating $d(t)$ for the time Δt,

(D.2) $$d_i = \int_{t_i-\Delta t}^{t_i} dt\, \bar{d}(t).$$

*e.g. since the detector amplitude $x(t)$ has a long correlation time τ, $x(t+\Delta t)$ has relatively little new information compared to $x(t)$, if $\Delta t \ll \tau$, while $\dot{x}(t)$ has a short correlation time so that most of $\dot{x}(t+\Delta t)$ is "new compared to $x(t)$.

II. The minimum n required is given by $n \gg \mu$, where μ^{-1} is the resolution time of the detector. The resulting resolution time is approximately $\Delta T = \frac{1}{n} + \frac{1}{\mu}$.

III. I am not sure about the answer to this. In actual recordings, the linear amplitude has been used, with a cutoff, $\pm d_{max}$.

IV. With the approach taken in II one clearly must demand $m \gg \frac{2 d_{max}}{\langle d_i^2 \rangle}$, and d_{max} should be compatible with the largest types of signals expected.

Although, at the moment, I to IV are only conjectured approximate "thumbrules", they have when applied in practice led to considerable increase in resolution.

One has of course also to consider the amount of extra complication (computer time) these rules might lead to. For example if the data is not optimally filtered when recorded, it will have to be filtered later by the computer, which might considerably increase the amount of computer time one has to spend.

Consider now a specific example of a data recording where we compare the "(x, y)" method and the "$(\dot{x}, \dot{y})$" method.

Let the system parameters be the following:

The damping time of the detector is $\tau = 40$ sec,

the resolution time $\Delta t = 0{,}04$ sec and the data is recorded with six bit resolution and 0.04 sec. time increment. Further let the data span 12 days.

With no signal x has the distribution

$$P(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp - \frac{x^2}{2\sigma^2}, \quad \text{where} \quad \sigma^2 = \langle x^2 \rangle = \tfrac{1}{2} \langle x^2 + y^2 \rangle$$

and similarly for y . Let the recorded data range be $-x_0 \, ; \, +x_0$.
This means that the number of times this range will be exceeded by the input data is rougly of the order

$$n = \frac{12 \text{ days}}{T \text{ seconds}} \left\{ 2 \int_{x_0}^{\infty} \frac{dx}{\sqrt{2\pi}\,\sigma} \exp - \frac{x^2}{2\sigma^2} + 2 \int_{x_0}^{\infty} \frac{dy}{\sqrt{2\pi}\,\sigma} \exp - \frac{y^2}{2\sigma^2} \right\} =$$

$$= 26 \cdot 10^3 \cdot 4 \int_{a_0}^{\infty} \frac{da}{\sqrt{2\pi}} \exp - \frac{a^2}{2}$$

where $a = \frac{x}{\sigma}$, $a_0 = \frac{x_0}{\sigma}$.

(for $a_0 \gg 1$ we may expand) $\Rightarrow$

$$n \approx 10^5 \frac{1}{\sqrt{2\pi}} \exp - \frac{a_0^2}{2} \left(\frac{1}{a_0} - \frac{1}{a_0^3} + \cdots \right)$$

with $a_0 = 4{,}5$ we get $n \approx 0{,}4$. The range $-4{,}5$ to $+4{,}5$ is thus a bit tight so let's use the range -5 to $+5$ (in terms of a).
Now from page , before recording we have a resolution

$$\alpha = \frac{(\Delta x^2 + \Delta y^2)^{1/2}}{(x^2 + y^2)^{1/2}} = \left(\frac{\Delta t}{T} \right)^{1/2} = \left(\frac{0{,}04}{40} \right)^{1/2} \approx 0{,}03$$

but after recording the resolution is $\frac{10}{64} = 0{,}4$ (6 bits means the range gets split up in 6 points). Obviously this leads to a fairly large loss of information in the recording.
Now let's consider doing the same recording with the $(\dot{x}, \dot{y})$ method.
We have

$$P(\dot{x}) = \frac{1}{\sqrt{2\pi}\,\dot{\sigma}} \exp - \frac{\dot{x}^2}{2\dot{\sigma}^2}\,,$$ where $\dot{\sigma} = \langle \dot{x}^2 \rangle = \frac{1}{2}\langle \dot{x}^2 + \dot{y}^2 \rangle$.

(and similarly for $\dot{y}$). The resolution in these variables

$$\Delta\dot{x}, \Delta\dot{y} \geq \dot{\sigma} \quad \text{or} \quad \dot{a}_x = \left(\frac{\Delta\dot{x}}{\dot{\sigma}}\right) \geq 1 .$$

Again we define a range $-\dot{x}_0, +\dot{x}_0$, and the number of times this range will be exceeded by the input data is (the correlation time of $(\dot{x}, \dot{y})$ is 0.04 sec, which is also the resolution time of the detector, See page ,

$$n = \frac{12 \text{ days}}{0{,}04 \text{ sec}}\; 4 \int_{\dot{a}_0}^{\infty} \frac{1}{\sqrt{2\pi}} \exp - \frac{\dot{a}^2}{2} \approx 4 \cdot 10^7 \exp - \frac{\dot{a}_0^2}{2}\left(\frac{1}{\dot{a}_0} - \frac{1}{\dot{a}_0^3} + \cdots\right)$$

where $\dot{a}_0 = \frac{\dot{x}_0}{\dot{\sigma}}$.

Now let $\dot{a}_0 = 10 \Rightarrow$

$n \approx 10^{-15}$. Thus we may use the range $-10 < \dot{a} < 10$ with essentially no risk of exceeding this range (due to noise activity) and it will not be exceeded by signals with a signal to noise ratio if less than 10^2. If we expect even stronger signals we might decide to use the range $-20 < \dot{a} < 20$. (not exceeded by $S_n(s) < 400$). In the first case we have a recording resolution of $\Delta\dot{a}_x, \Delta\dot{a}_y \geq \frac{20}{63} = 0{,}32$, in the second case $\Delta\dot{a}_x, \Delta\dot{a}_y \geq 0{,}62$. The original resolution being $\Delta\dot{a}_x, \Delta\dot{a}_y \geq 1$, this recording will give only a small loss of information.

SIGNALS

We have very little information so far about the nature of eventual signals. One of the few things we can say is that if there are pulse-like signals, each pulse is not likely to have more energy than what corresponds to a kT excitation of a Weber type detector. Further we can say in general that the only property of the output data that any type of signal has to affect is the energy distribution. Aside from this, the signal may have special properties that can show up in the data (if the detector system is sensitive to this property).

The most nonspecialized signal of all would be a Gaussian white noise type of signal. (Only one parameter, average power is needed to specify this signal). The three degree C° background radiation would for example be such a signal (unfortunately too weak to detect). The only effect this type of signal would have on the output is an increase in its average energy.

We will now consider some special properties a signal might have which can be used to increase its detectability.

We define a pulse as a signal which is preceeded by "zero signal" and followed by "zero signal" for at least the time τ, where τ is some time constant (one may set $\tau = \mu^{-1}$ the resolution time of the detector, but we let τ be unspecified for the time being). We give beneath a list of possible types of signals that we think may be relevant in connection with gravitational radiation.

We may classify the signals by three numbers (k,i,j) in the following way.

k = 1 Pulses consisting of stochastic signals

k = 2 Pulses consisting of an exponentially damped stochastic signals

k = 3 Pulses of special form with exponential distribution in energy

k = 4 Pulses of special form and specific energy (must be considered fairly unlikely though)

i = 1 random arrival times

i = 2 periodic arrival times or other types of periodicity

j = 1 non directional signals

j = 2 directional signals

The above definitions are fairly broad, i.e. the stockastic signals may be narrow-banded or broadbanded, on or off frequency. One may for example imagine a stockastic signal as arising from a black hole capturing a massive star which by tidal forces is broken into a number of smaller parts, which in a short time spirals into the black hole. One would imagine that this would create a fairly broadbanded stockastic signal centered around some frequency. In fact it seems not unlikely that it would damp out in an exponential manner. It seems to us that the exponential function is the most likely form of the signal pulse energy distribution. Certainly, the occurrance of constant energy signal pulses appears to be quite unlikely.

We also note that if there were signals with periodic arrival times, (compare pulsars) for the same energy they would have a much larger signal to noise ratio than random signals.

We will next consider detection of signals of type (3,1,1) and (3,1,2).

OUTPUT DATA DISTRIBUTION FOR A DETECTOR EXCITED BY SIGNAL PULSES WITH RANDOM ARRIVAL TIME AND EXPONENTIAL DISTRIBUTION IN ENERGY

We shall consider a reduced set of hypotheses, $(E_i \; T_j)$ $i=1,--n, j=1---m$, of "short" signal pulses with random arrival time, and exponential distribution in energy, where E_i is the average pulse energy and T_j is the average time between arrivals of pulses. One might also consider generalizing the hypotheses to include different stochastially distributed pulse shapes. We shall, however, limit this investigation to "short" pulses (pulses that are short compared to the detector resolution time).

We will have to take an approximative approach in deriving the evidence function. Let the detector resolution time be ΔT. We assume that the detector output should be filtered with a filter matched to a delta function input signal. The points $\{x(t_\nu)\}, t_\nu - t_{\nu-1} = \Delta T$ of the filtered output are approximately uncorrelated. The probability for a signal pulse to arrive in the interval $t_\nu \pm \frac{\Delta T}{2}$ is $\Delta T / T_j$. The distribution of $x(t_\nu)$ if a signal $S(t, t_\nu)$ is present is

$$P(x(t_\nu)) = \frac{1}{\pi \langle E_n \rangle} \exp - \frac{|x(t_\nu) - S(t, t_\nu)|^2}{\langle E_n \rangle}$$

where $\langle E_n \rangle$ is the "noise only" expectation value of $x^2(t)$.

$\langle E_n \rangle = \langle x^2(t) \rangle = \langle x_1^2(t) + x_2^2(t) \rangle$,

and $S(t_\nu, t)$ is the signal output at t_ν, if the signal input pulse arrives at t.

$S(t, t_\nu)$ is a 1two component signal, $S = (S_1, S_2)$.
Since the energy of the signal has an exponential distribution it follows that the corresponding amplitude components S_1 and S_2 have normal distributions, with $\langle S_1^2(t,t_\nu) + S_2^2(t,t_\nu)\rangle = E(t,t_\nu)$. For a Weber-type detector

$$E(t,t_\nu) = E(t_\nu,t_\nu)\exp - \frac{2|t-t_\nu|}{\Delta t} \qquad t_\nu - \frac{\Delta T}{2} < t < t_\nu + \frac{\Delta T}{2} .$$

We shall approximate $E(t,t_\nu)$ to be constant E_i in the interval and we let

$$E_s = E(t_\nu,t_\nu)\frac{1}{\Delta T}\int_{t_\nu - \Delta T/2}^{t_\nu + \Delta T/2} dt \exp - \frac{2|t-t_\nu|}{\Delta T} = E(t_\nu,t_\nu)(1-e^{-1}) .$$

Since $S_1(t,t_\nu)$ and $S_2(t,t_\nu)$ are normal, it follows:

$$P_{n+s}(X(t_\nu)) = \int dS_1 \int dS_2\, P(S_1,S_2)\frac{1}{\pi\langle E_n\rangle}\exp - \frac{|X(t_\nu) - S(t,t_\nu)|^2}{\langle E_n\rangle} =$$

$$= \frac{1}{\langle E(t,t_\nu) + E_n\rangle}\exp - \frac{|X(t_\nu)|^2}{\langle E(t,t_\nu) + E_n\rangle} \propto \frac{1}{\langle E_s + E_n\rangle}\exp - \frac{|X(t_\nu)|^2}{\langle E_s + E_n\rangle}$$

The signal probability in the interval $t_\nu \pm \Delta T$ is $\Delta T/T_j$

With $E_s = E_i$, we get

(D.3) $$P_{n+(i,j)}(X_\nu(t)) \approx \left(1 - \frac{\Delta T}{T_j}\right)\frac{1}{\langle E_n\rangle}\exp - \frac{|X(t_\nu)|^2}{\langle E_n\rangle} + \frac{\Delta T}{T_j}\frac{1}{\langle E_n + E_i\rangle}\exp - \frac{|X^2(t_\nu)|}{\langle E_n + E_i\rangle}$$

where $n+(i,j)$ denotes noise + signal (E_i, T_j).
We will later joint output distributions and evidence for a two detector system but will first consider detection with a single detector.

DETECTION OF RANDOM PULSES WITH A SINGLE DETECTOR

Since the data points x_i are approximately uncorrelated they will give a total contribution to the evidence;

$$\Delta ev(E_i, T_j) = \ln \prod_\nu \frac{P_{n+(i,j)}(x(t_\nu))}{P_n(x(t_\nu))} =$$

$$= \sum_\nu \left\{ \left(1 - \frac{\Delta T}{T_j}\right) + \frac{\Delta T}{T_j} \frac{\langle E_n \rangle}{\langle E_n + E_i \rangle} \exp \frac{\langle E_i \rangle / x^2(t_\nu) /}{\langle E_n + E_j \rangle} \right\}$$

One might want to have sample points more close than ΔT . Since points closer than ΔT are correlated one will have to correct with a factor $\Delta t / \Delta T$, where ΔT is the new sampling interval;

(D.4) $$\Delta ev(E_i, T_j) = \frac{1}{\Delta T} \sum_\nu \Delta T \left\{ \left(1 - \frac{\Delta T}{T_j}\right) + \frac{\Delta T}{T_j} \frac{\langle E_n \rangle}{\langle E_n + E_i \rangle} \exp - \frac{\langle E_i \rangle / x^2(t_\nu) /}{\langle E_n + E_i \rangle}\right.$$

The signal to noise ratio for the hypothesis (i,j) is given by (See

(D.5) $$Sn(i,j/T) = \frac{T}{\Delta T} \int dx \, P_{n+(i,j)}(x^2) \, 2 \log \frac{P_{n+(i,j)}(x^2)}{P_n(x^2)} \, .$$

where T is the total observation time. We shall now evaluate (D.5) for some parameter values, and compare with the effective signal to noise ratio for threshold detection, and later consider the typical form of (D.4). Let $x^2 = E$.

Writing $P_{n+(i,j)}$ in terms of P_{n+s_i} and P_n we have:

$$(D.6)\ S_n(E_i,T_j/T) = 2\frac{T}{\Delta T}\int dE\left\{\left(1-\frac{\Delta T}{T_j}\right)P_n(E) + \frac{\Delta T}{T_j}P_{n+s_i}(E)\right\}\cdot$$

$$\cdot\ \ln\left\{\left(1-\frac{\Delta T}{T_j}\right)+\frac{\Delta T}{T_j}\frac{P_{n+s_i}(E)}{P_n(E)}\right\}$$

We have $P_n(E) = \frac{1}{E_n}\exp-\frac{E}{E_n}$, $P_{n+s_i}(E) = \frac{1}{E_n+E_i}\exp-\frac{E}{E_n+E_i}$.

Let $\frac{\Delta T}{T_j} = A$ (Note that typically A is very small; $A < 10^{-4}$ for a couple of pulses per day).

$$(D.7)\ S_n(i,j/T) = \frac{2T}{\Delta T}\int_0^\infty dE\left\{\frac{(1-A)}{E_n}\exp-\frac{E}{E_n}+\frac{A}{E_n+E_s}\exp-\frac{E}{E_n+E_s}\right\}\cdot$$

$$\cdot\ \ln\left\{(1-A)+\frac{AE_n}{(E_n+E_i)}\exp+\frac{EE_i}{E_n(E_n+E_i)}\right\}.$$

Since this integral resists evaluation (in terms of simple functions) we will settle for an approximate (better than 5%) evaluation. We will use the approximation $\ln(1+x) = x - 0{,}31x^2$ which is accurate to at least 5% if $0 < x < 1$. We split the integral in two parts.

$$I_1 = \int_0^{E_0} dE\left\{\frac{(1-A)}{E_n}\exp-\frac{E}{E_n}+\frac{A}{E_n+E_i}\exp-\frac{E}{E_n+E_i}\right\}\left\{\ln(1-A)+\ln\left(1+\frac{AE_n}{(1-A)(E_n+E_i)}\exp\frac{EE_i}{E_n(E_n+E_i)}\right)\right\}$$

$$I_2 = \int_{E_0}^\infty dE\left\{\frac{1-A}{E_n}\exp-\frac{E}{E_n}+\frac{A}{E_n+E_i}\exp-\frac{E}{E_n+E_i}\right\}\left\{\ln\frac{AE_n}{E_n+E_i}+\ln\left(1+\frac{(1-A)(E_n+E_i)}{AE_n}\right.\right.\cdot$$

$$\left.\left.\cdot\ \exp-\frac{EE_i}{E_n(E_n+E_i)}\right)\right\}$$

where $E_0 = E_n\frac{(E_i+E_n)}{E_i}\ln\frac{(E_i+E_n)(1-A)}{AE_n}$. We note that our approach is good only if $E_0 > 0, \Rightarrow A < \frac{1+E_i/E_n}{2+E_i/E_n}$.

A = 1 means that the signal is on all the time, for this case we can easily evaluate $(D.7)$ exactly.

4.

Let $\frac{E_i}{E_n} = a$. We then have

$$(D.8)\quad S_n(a,A/T) = \frac{2T}{T_j}\left\{\left(\frac{\frac{A}{1-A}}{1+a}\right)^{1/a}\left\{a\left(1+\frac{1}{(1+a)^2}\right)-1+\frac{0{,}69}{1+2a}-\frac{0{,}31}{1+3a}+\right.\right.$$

$$\left.\left. + \frac{0{,}69}{a-1}\left(1-\left(\frac{\frac{A}{1-A}}{1+a}\right)^{1-1/a}\right)-\frac{0{,}31}{2a-1}\left(1-\left(\frac{\frac{A}{1-A}}{1+a}\right)^{2-1/a}\right)\right\}-\right.$$

$$\left. - 0{,}69A - 38A^2\right\}.$$

We repeat the meaning of the symbols; T_i is the average time between pulses. A is the factional time occupied by pulses (e.g. 1 sec. in 10^4 sec., etc.)

a is $\frac{E_i}{E_n}$ (i.e. a is related to the signal to noise ratio for a single pulse with known arrival time t in fact $S_n(a,t)$ single pulse = $2(a - {}_n(1 + a))$.

In the limits $a \longrightarrow 0$ and $a \longrightarrow \infty$, the signal to noise ratio is

$$S_n(a,A/n) \underset{a\to\infty}{\to} 2n(a-\ln(a+1)) \quad , \text{where } n = \frac{T}{T_j}.$$

(D.9)

$$S_n(a,A/n) \underset{a\to 0}{\to} nA\{1{,}38 - 0{,}48A\}a^2$$

If $A = 1$ the integral 5) is trivial to evaluate;

$$S_n(a,A{=}1/T) = 2\frac{T}{\Delta T}(a-\ln(a+1)); \qquad \text{and}$$

$$S_n(a,A{=}1/T) \underset{a\to 0}{=} \frac{T}{\Delta T}a^2 .$$

We will now compare the above "optimal" signal to noise ratio with the "effective" signal to noise resulting from a threshold type data processing. Threshold detection corresponds to reducing the data to binary form, e.g. $\{E_n\} \rightarrow \{i_n\}$ = 0,1, where = 0 of and = 1 f $E_n > E_o$, where E_o is the threshold. We have

$$\overline{P}_{n+s}(i=0) = \int_0^{E_o} dE\, \overline{P}_{n+s}(E) \quad , \quad \overline{P}_{n+s}(i=1) = \int_{E_o}^{\infty} dE\, \overline{P}_{n+s}(E)$$

$$P_n(i=0) = \int_0^{E_o} dE\, P_n(E) \quad , \quad P_n(i=1) = \int_{E_o}^{\infty} dE\, P$$

Thus

$$\overline{P}_{n+s}(0) = (1-A)\left(1-\exp-\frac{E_o}{E_n}\right) + A\left(1-\exp-\frac{E_o}{E_n+E_s}\right)$$

$$\overline{P}_{n+s}(1) = (1-A)\exp-\frac{E_o}{E_n} + A\exp-\frac{E_o}{E_n+E_s}$$

$$P_n(0) = \left(1-\exp-\frac{E_o}{E_n}\right)$$

$$P_n(1) = \exp-\frac{E_o}{E_n} \quad .$$

Thus the evidence increase from m data points $\{i_\nu\}$ for the signal hypothesis is

$$\Delta ev(S,A/m) = \sum_{\nu=1}^{m} \ln \frac{\overline{P}_{n+s}(i_\nu)}{P_n(i_\nu)} =$$

$$= q(E_o)\ln\left\{(1-A) + A\frac{\left(1-\exp-\frac{E_o}{E_n+E_s}\right)}{\left(1-\exp-\frac{E_o}{E_n+E_s}\right)}\right\} + r(E_o)\ln\left\{(1-A)+A\exp\frac{E_oE_s}{E_n(E_n+E_s)}\right\}$$

where $q(E_o)$ is the number of data points below the threshold , and $r(E_o)$ the number above. E_o may now be varied to give maximal evidence increase. Next we consider the effective signal to noise ratio

(D.10) $$S_n^{eb}(S,A/m) = 2m\left\{\left((1-A)\left(1-\exp-\frac{E_o}{E_n}\right)+A\left(1-\exp-\frac{E_o}{E_n+E_s}\right)\right.\right.\cdot$$

$$\ln\left((1-A) - A\frac{\left(1-\exp-\frac{E_o}{E_n+E_s}\right)}{\left(1-\exp-\frac{E_o}{E_n}\right)}\right) + \left((1+A)\exp-\frac{E_o}{E_n} + A\exp-\frac{E_o}{E_n+E_s}\right)\ln\left((1-A)+A\exp\frac{E_o E_s}{E_n(E_n+E_s)}\right)\Bigg\}_{\max E_o},$$

where "eb" denotes "effective binary". The maximum of S_n^{eb} cannot be found analytically. Over the next couple of pages we give some computer evaluated graphs of $S_n(S, A/m)$ and $S_n^{eb}(S, A/m)$. The graphs are evaluated for $m = 10^6$ (approximately the number of time intervals during a 24 hour period for a Weber type detector. Instead of E_s (average energy per signal pulse) we use the variable $\alpha = m \cdot A \frac{E_s}{E_n}$, the total average pulse energy for m time intervals, in units of average noise energy. We remind that A is the fraction of time intervals occupied by pulses. It is seen from the graphs that slightly more than half of the signal to noise is lost with the threshold process.

The threshold approach stems of course from considerations of individual signals. We would then be concerned with the problem of assigning each individual time interval a signal or no signal (yes or no) stamp. To do this we would really have to know the prior probability for a signal (which we so far don't know much about) so that we could use Bayes' equation to find the probability for a signal hypothesis to be true for the individual time interval in question, and then based on this probability say yes or no.

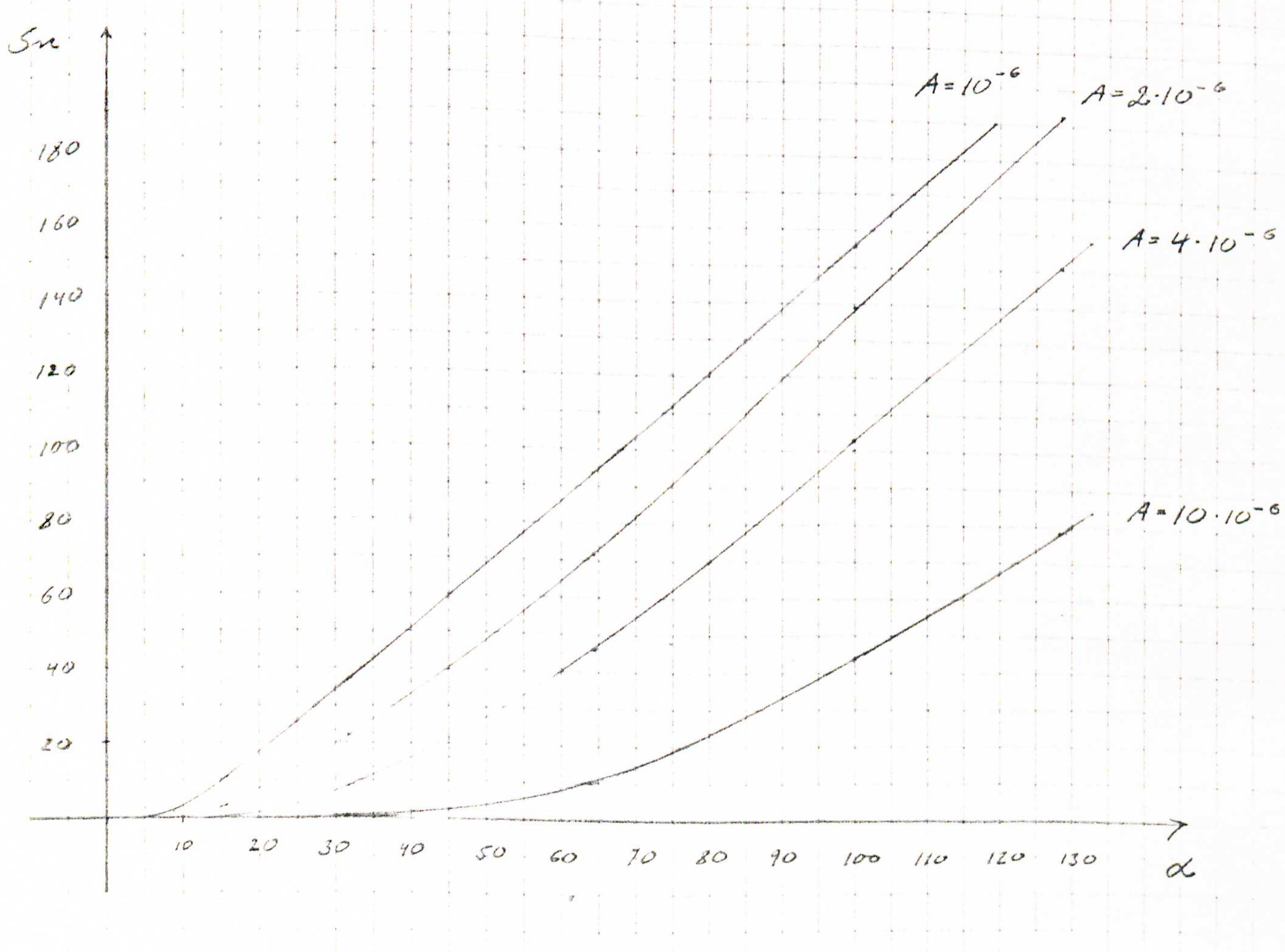
Since we are mainly (so far) concerned with the problem of gravitational waves or no gravitational waves (and not if there is an individual gravitational wave in the n^{th} time interval) it seems more appropriate to "gather evidence" for (or against) the gravitational wave hypothesis as such. Once we have gathered say 99 napiers of evidence (to satisfy even the most severe skeptic) for say an hypothesis like (E_i , T_j) we may decide that the hypothesis is true or rather that the no signal hypothesis is untrue. This decision will then define a prior probability $P_{(j)}$ for an individual time interval to contain a signal, defined by the distribution

$$P_i(E) = \frac{1}{\langle E_i \rangle} \exp - \frac{E}{\langle E_i \rangle} \;,$$

namely $P_{(j)} = \frac{\Delta T}{T_j}$. Thus it seems natural first to convince ourselves that there are gravitational waves and then as the next step worry about wether individual time intervals contain a signal or not.

Graph of the signal to noise ratio $Sn(\alpha, A/T_{day})$.
α is the total pulse energy per day, in units of average noise energy output ($\alpha = n \cdot a$, $a = E_s/E_{noise}$, where E_s is the average signal energy, E_{noise} the average noise energy and n the number of signals per day), and A is the fraction of time occupied by the signal pulses. $A = n \frac{\Delta t}{T_{day}}$, where Δt is the data recording time unit.

FIG. D.3.



THE EFFECTIVE SIGNAL TO NOISE RATIO $Sn^e(\alpha, A/T_{day})$, FOR THRESHOLD ANALYSIS OF THE DATA. α is again the total pulse energy per day in units of average noise energy output, and A is the fraction of time units occupied by signal pulses.

FIG. D.4

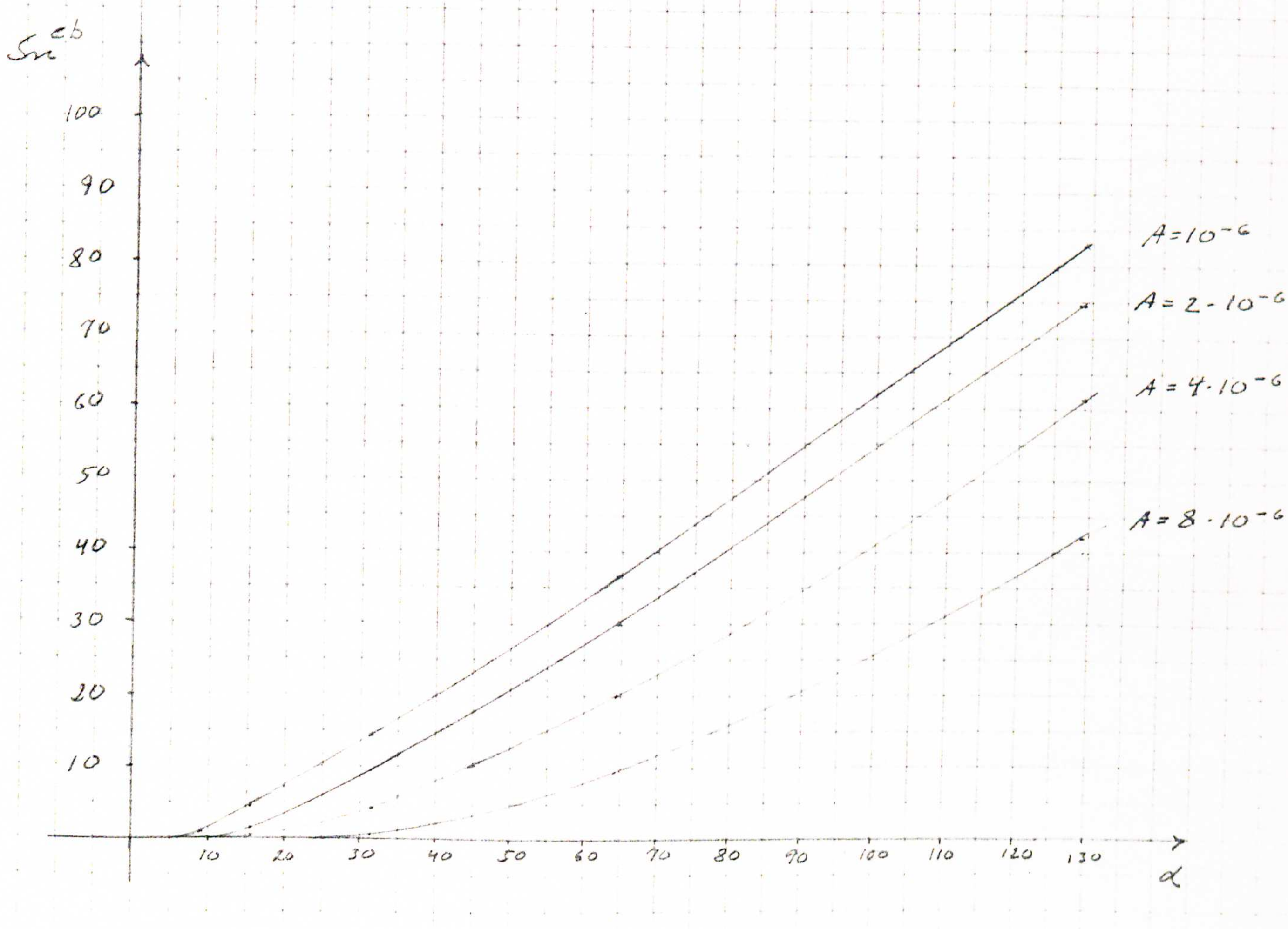
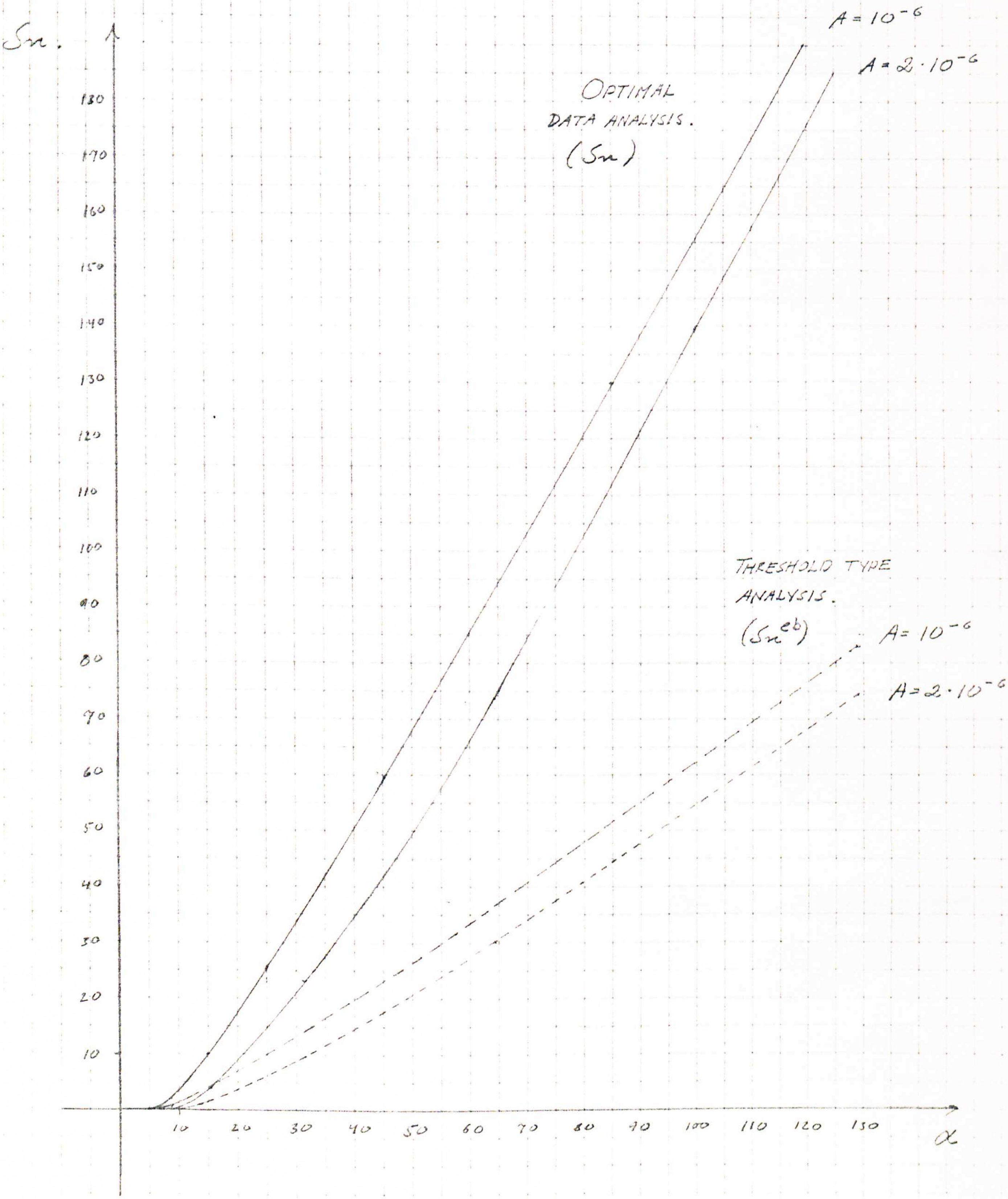


FIG. D. 5



Sn.
$A = 10^{-6}$
$A = 2 \cdot 10^{-6}$
OPTIMAL
DATA ANALYSIS.
(Sn)
THRESHOLD TYPE
ANALYSIS.
(Sn^{eb})
$A = 10^{-6}$
$A = 2 \cdot 10^{-6}$
180
170
160
150
140
130
120
110
100
90
80
70
60
50
40
30
20
10
10
20
30
40
50
60
70
80
90
100
110
120
130
α

The analysis of the data from a "single" detector is accomplished by evaluating (D.2) for the chosen set of parameters, and find those which give the maximal evidence increase. The parameter variation process may be fairly time consuming. Once a "good" hypothesis is found one can do away with the variation and just keep "gathering" evidence for this hypothesis. This could be done in an extremely simple way. In fact it can be done directly on the filtered output of the detector in a recursive manner. For each new data point we get some new evidence so that

$$\Delta ev(E_i, T_j / n) = \Delta ev(E_i, T_j) + k \ln \frac{P_{n+(i,j)}(E(t_n))}{P_n(E(t_n))}$$

where $\Delta ev(- / n)$ denotes the evidence from the data points $E(t_0)$ to $E(t_n)$.

On the next page, we give a graph of the functions

$$\ln \frac{P_{n+(i,j)}(E)}{P_n(E)} \quad \text{and} \quad P_{n+(i,j)}(E) \ln \frac{P_{n+(i,j)}(E)}{P_n(E)} .$$

the total surface of which is the signal to noise per time Δt (resolution time). Note that although $\Delta ev(E/E_n)$ appears to cut off for energies lower than $\sim E/E_n = 10$, this part of the curve is still important, due to the large number of outputs E/E_n in this range (as demonstrated by Figure 2a).

Fig. D.6 a.

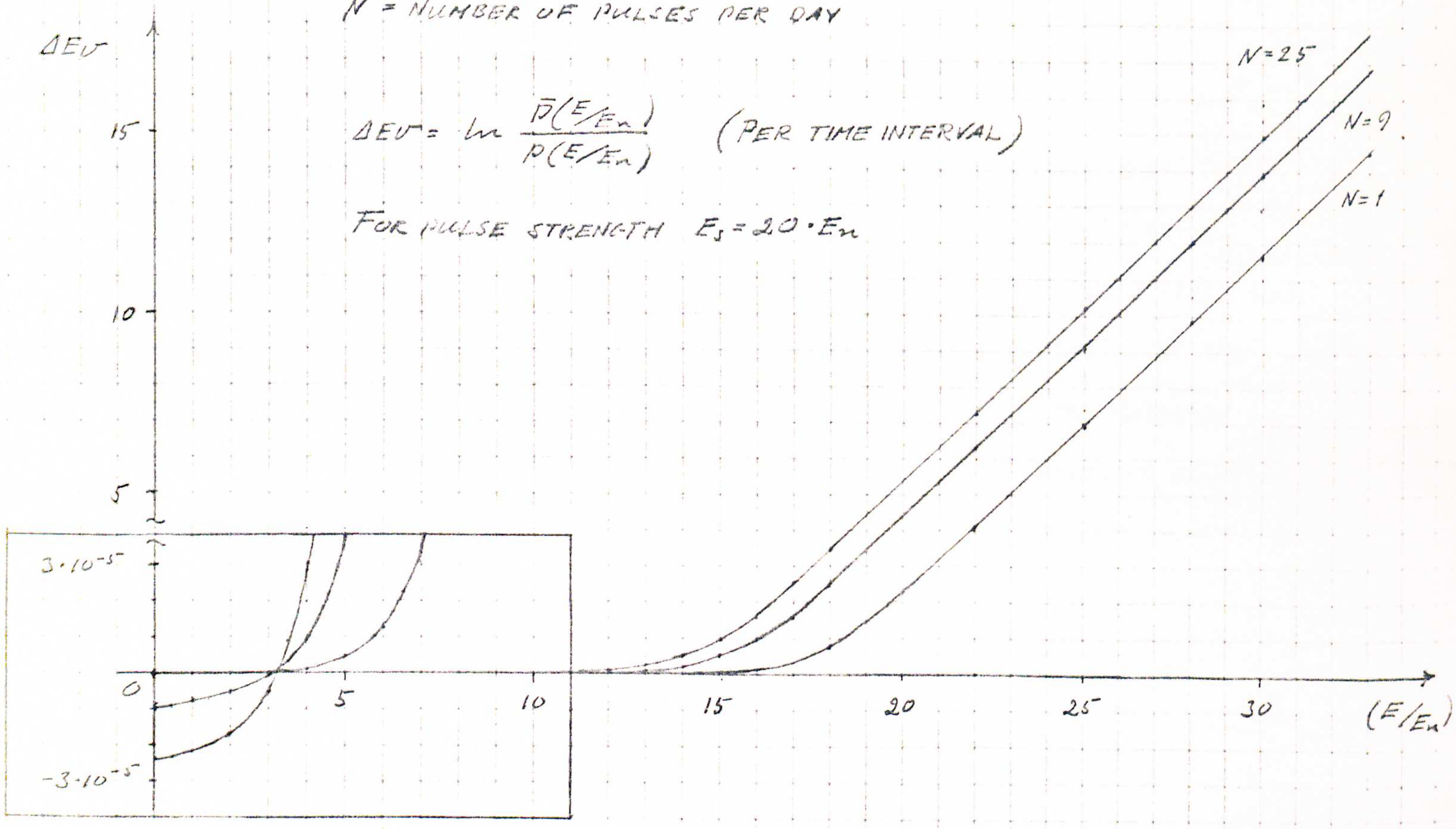
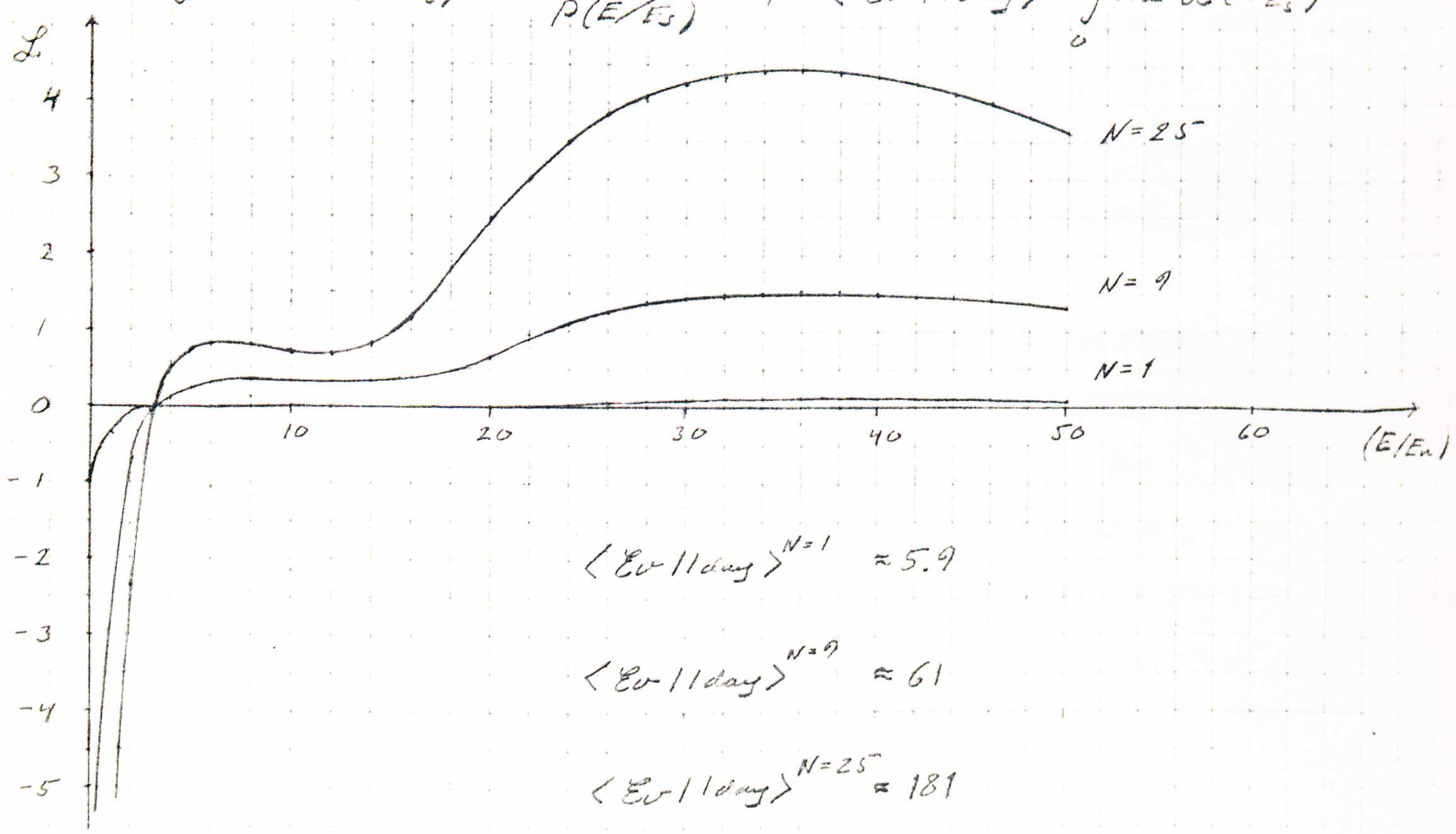


Fig. D.6 b.

$$\mathcal{L} = 10^6\, \bar{P}(E/E_s) \ln \frac{\bar{P}(E/E_s)}{P(E/E_s)} \quad ; \quad \langle Ev\,/\,1\,day \rangle = \int_0^\infty dE\, \mathcal{L}(E/E_s)$$



TWO DETECTOR COINCIDENCE EXPERIMENTS

One of the problems with the "single detector" experiment is that there is no way of telling wether the signals are gravitational radiation pulses or local disturbances of some sort. We can of course try to limit such distrubances by improving the acoustic and electromagnetic isolation of the detector, but this gets increasingly hard, the more sensitive the detector is.

The best way out of this problem is to use two or more detectors at well separated sites.

Consider the output data from two such detectors

$$\begin{cases} d^{(1)} = s + s^{(1)} + n^{(1)} \\ d^{(2)} = s' + s^{(2)} + n^{(2)} \end{cases}$$

Where $d^{(1)}$ and $d^{(2)}$ both have been filtered to match the same signal. $s^{(1)}$ and $s^{(2)}$ are excitations due to local disturbances and are assumed to be completely uncorrelated. s and s' is the same signal occuring in both detectors. We will consider the case when the reference oscillators of the two detectors are unsyncronized, the effect being that s and s' have uncorrelated phases.

The first problem we encounter is what type of distribution we should use for the pulses $s^{(1)}$ and $s^{(2)}$. With a limited amount of information we should choose the distribution with the least constraints, i.e. we let $s^{(1)}$ and $s^{(2)}$ have exponential distribution in energy. Clearly $s^{(1)}$ and $s^{(2)}$ could have different density and average energy, but for simplicity we assume that they are the same and are defined by the density $\mathcal{S}_\ell$ and average energy Ξ_ℓ . As before we choose the exponential distribution for the signal pulses defined by the density ϱ_s and average energy $\langle E_s \rangle$. We make the approximation $P(s)\cdot P(s_1)=0$ i.e. the propability for these pulses to occur at the same time is zero. The distribution of the data, the moment a signal pulse arrives is in the two channels

$$(1) \qquad P^{(1)}_{n+s}(\bar{d}^{(1)}) = \frac{1}{\pi \langle E_n \rangle} \exp - \frac{(\bar{d}_1 - \bar{S})^2}{\langle E_n \rangle}$$

$$(2) \qquad P^{(2)}_{n+s}(\bar{d}^{(2)}) = \frac{1}{\pi \langle E_n \rangle} \exp - \frac{(\bar{d}_2 - \bar{S}')^2}{\langle E_n \rangle} \qquad \langle E_n \rangle = \langle n^{(1)2} \rangle = \langle n^{(2)2} \rangle$$

(We have for simplicity assumed that the two detectors have the same average noise energy).

$\bar{S}$ and $\bar{S}'$ have the same magnitude but random phase correlation.

Now let $\bar{d}^{(2)} \rightarrow \bar{d}^{(2)'}$; $|\bar{d}_2| = |\bar{d}_2'|$ and

$$\Theta(\bar{d}^{(2)'}, \bar{S}) = \Theta(\bar{d}^{(2)}, \bar{S}') \Rightarrow (\bar{d}^{(2)} - \bar{S}')^2 = (\bar{d}^{(2)'} - \bar{S})^2 \Rightarrow$$

$$(D.11) \qquad P^{(2)}_{n+s}(\bar{d}^{(2)'}) = \frac{1}{2\pi \langle E_n \rangle} \exp - \frac{(\bar{d}^{(2)'} - \bar{S})^2}{\langle E_n \rangle}$$

Now $\bar{S}$ has the distribution $P(\bar{S})\, dS_x dS_y = \frac{1}{\pi \langle E_s \rangle} \exp - \frac{\bar{S}^2}{\langle E_s \rangle}\, dS_x dS_y$

x and y being the two phase directions. hereafter we denote $dS_x dS_y$ by $d\bar{S}$.

Thus the distribution $P_{n+(s)}(\bar{d}^{(1)}, \bar{d}^{(2)})$ is

$$P_{n+(s)}(\bar{d}^{(1)}, \bar{d}^{(2)'}) = \int_{-\infty}^{+\infty} d\bar{S} \frac{1}{(2\pi \langle E_n \rangle)^2\, \pi \langle E_s \rangle} \exp - \left\{ \frac{(\bar{d}_1^{(1)} - \bar{S})^2 + (\bar{d}^{(2)'} - \bar{S})^2}{\langle E_n \rangle} + \frac{\bar{S}^2}{\langle E_s \rangle} \right\} =$$

$$= \int_{-\infty}^{+\infty} d\xi \, \frac{\langle E_n+2E_s\rangle}{\pi\langle E_n\rangle\langle E_s\rangle} \exp - \frac{\langle E_n+2E_s\rangle\left(\xi - \frac{\langle E_s\rangle}{\langle E_n+2E_s\rangle}\left(\bar{d}^{(1)}+\bar{d}^{(2)'}\right)^2\right)}{\langle E_n\rangle\langle E_s\rangle} .$$

$$\cdot \frac{1}{\pi^2\langle E_n\rangle\langle E_n+2E_s\rangle} \exp - \frac{\bar{d}^{(1)2}+\bar{d}^{(2)'2} - \frac{\langle E_s\rangle}{\langle E_n+2E_s\rangle}\left(\bar{d}^{(1)}+\bar{d}^{(2)'}\right)^2}{\langle E_n\rangle}$$

We have $\bar{d}^{(1)2} = d^{(1)2}$ $(d^{(1)} = |\bar{d}^{(1)}|)$, $\bar{d}^{(2)'2} = \bar{d}^{(2)2} = d^{(2)2}$ $(d^{(2)} = |\bar{d}^{(2)}|)$ and $\bar{d}^{(1)} \cdot \bar{d}^{(2)} = d^{(1)} d^{(2)} \cos\theta_{12} \Rightarrow$

$$(D.12) \quad P_{n+s}\left(\bar{d}^{(1)}, \bar{d}^{(2)}\right) = \frac{1}{\pi^2\langle E_n\rangle\langle E_n+2E_s\rangle} \exp - \frac{d^{(1)2}+d^{(2)2} - \frac{\langle E_s\rangle}{\langle E_n+2E_s\rangle}\left(d^{(1)2}+d^{(2)2}+2d^{(1)}d^{(2)}\cos\theta_{12}\right)}{\langle E_n\rangle} .$$

Now the angle θ_{12} between $\bar{d}^{(1)}$ and $\bar{d}^{(2)'}$ is not measurable (since the phases of the reference oscillators are unsyncronized). The angle θ_1 between $\bar{d}_1$ and the x axis is not important, so we integrate over θ_{12} and θ_1 $\Rightarrow$

$$(D.13) \quad P_{n+s}\left(d^{(1)}, d^{(2)}\right) = \frac{4 \, d^{(1)} d^{(2)}}{\langle E_n\rangle\langle E_n+2E_s\rangle} \exp - \frac{\left(1 - \frac{\langle E_s\rangle}{\langle E_n+2E_s\rangle}\right)\left(d^{(1)2}+d^{(2)2}\right)}{\langle E_n\rangle} \, I_0\left(\frac{2\langle E_s\rangle d^{(1)} d^{(2)}}{\langle E_n\rangle\langle E_n+2E_s\rangle}\right)$$

Where Io is the zeroth order Bessel function. We will use the approximation for large argument;

$$I_0(x) \sim \frac{1}{\sqrt{2\pi x}} \exp\left(x + \frac{1}{8x}\right) \sim \frac{1}{\sqrt{2\pi x}} \exp x .$$

$$(D.14)\quad P_{n+(s)}(d^{(1)}, d^{(2)}) \approx \left(\frac{\pi d^{(1)} d^{(2)}}{\langle E_s\rangle\langle E_n\rangle\langle E_n+2E_s\rangle}\right)^{1/2} \exp - \frac{\langle E_n\rangle(d^{(1)^2}+d^{(2)^2})+\langle E_s\rangle(d^{(1)}-d^{(2)})^2}{\langle E_n\rangle\langle E_n+2E_s\rangle}$$

$$\text{if}\qquad d^{(1)}\cdot d^{(2)} \gg \frac{\langle E_n+2E_s\rangle\langle E_n\rangle}{2\langle E_s\rangle}$$

We may express this in terms of energy;

$$(D.15)\quad P_{n+(s)}(E^{(1)}, E^{(2)}) = \left(\frac{\pi}{\sqrt{E^{(1)}E^{(2)}}\langle E_n\rangle\langle E_s\rangle\langle E_n+2E_s\rangle}\right)^{1/2} \exp - \frac{\langle E_n\rangle(E^{(1)}+E^{(2)})+\langle E_s\rangle(\sqrt{E^{(1)}}-\sqrt{E^{(2)}})^2}{\langle E_n\rangle\langle E_n+2E_s\rangle}$$

$$\text{if}\qquad \sqrt{E^{(1)}}\sqrt{E^{(2)}} \gg \frac{\langle E_n+2E_s\rangle\langle E_n\rangle}{2\langle E_s\rangle}$$

We can see that the term $(\sqrt{E^{(1)}}-\sqrt{E^{(2)}})^2$ is a "coincidence factor" which gives preference to energies which are of equal order in the two channels. The distribution of the data at the moment of disturbances $S^{(1)}$ and $S^{(2)}$ are:

$$(D.16)\quad \begin{cases} P^{(1)}_{n+(s(1))}(E^{(1)}) = \dfrac{1}{\langle E_n+E_i\rangle} \exp - \dfrac{E^{(1)}}{\langle E_n+E_i\rangle} \\[2ex] P_{n+(s(2))}(E^{(2)}) = \dfrac{1}{\langle E_n+E_i\rangle} \exp - \dfrac{E^{(2)}}{\langle E_n+E_i\rangle} \end{cases}$$

We can now write the total joint distribution;

$$(D.17)\quad \bar{P}_{n+(S_i)+(S)}(E^{(1)}, E^{(2)}) = (1-2S_i-S_s)\frac{1}{\langle E_n\rangle^2} \exp - \frac{E^{(1)}+E^{(2)}}{\langle E_n\rangle} +$$

$$\frac{\mathcal{S}_L}{\langle E_n\rangle\langle E_n+E_L\rangle}\exp-\frac{E^{(1)}+E^{(2)}}{\langle E_n+E_L\rangle}\cdot\left\{\exp-\frac{\langle E_L\rangle E^{(1)}}{\langle E_n\rangle\langle E_n+E_L\rangle}+\exp-\frac{\langle E_L\rangle E^{(2)}}{\langle E_n\rangle\langle E_n+E_L\rangle}\right\}+$$

$$+\ \mathcal{S}_s\,2\pi\left(\frac{\pi}{\sqrt{E^{(1)}E^{(2)}}\langle E_s\rangle\langle E_n\rangle\langle E_n+2E_s\rangle}\right)^{1/2}\exp-\frac{\langle E_n\rangle(E^{(1)}+E^{(2)})+\langle E_s\rangle(\sqrt{E^{(1)}}-\sqrt{E^{(2)}})^2}{\langle E_n\rangle\langle E_n+2E_s\rangle}$$

$$\text{if } \sqrt{E^{(1)}}\sqrt{E^{(2)}} \gg \frac{\langle E_n+2E_s\rangle\langle E_n\rangle}{2\langle E_s\rangle}$$

We have dropped all terms of order $\mathcal{S}^2$. We will also need the distribution of noise plus disturbances $\mathcal{S}_L$, but no signals, which is given from (D.15) by making $\mathcal{S}_s=0$. We will now have to consider three different sets of hypothesis,

$H_0(E_n)$ (The data contains noise only)

$H_L(E_n;\mathcal{S}_L,E_L)$ (The data contains noise and $\mathcal{S}_L$ only)

$H_s(E_n;\mathcal{S}_L,E_L;\mathcal{S}_s,E_s)$ (The data contains noise, $\mathcal{S}_L$ and $\mathcal{S}$)

Where the last two are really parametrized sets of hypothesis with parameters $\mathcal{S}_L, E_L$ respective $\mathcal{S}_L, E_L, \mathcal{S}_s$ and E_s.

First let's consider H_s and H_0 where we let H_0 be the denial of H_s. Let $\Delta ev(H_s|\{E_i\}(H_s+H_0))$ be the increase in evidence in favor of H_s. From a single point at $t=t_{i_0}$ we have

$$(D.18)\quad \Delta ev(H_s|E_{i_0}(H_s+H_0)) = k\ln\frac{\overline{P}_{n+(S_L)+(S)}(E_{i_0}^{(1)},E_{i_0}^{(2)})}{P_n(E_{i_0}^{(1)},E_{i_0}^{(2)})}$$

, and totally;*

$$(D.19)\quad \Delta ev(H_s|\{E_{i}\}(H_s+H_0)) = \frac{k}{\Delta T}\sum_i \Delta t\ln\frac{\overline{P}_{n+(S_L)+(S)}(E_{i}^{(1)},E_{i}^{(2)})}{P_n(E_{i}^{(1)},E_{i}^{(2)})},$$

where ΔT is the detector resolution time.

*Again this is an approximative expression of the same kind as (D.2).

By varying ρ_L, E_L, ρ_S and E_S we can find the specific hypothesis H_S^o that has the maximum increase in evidence. We let the corresponding parameters be defined by $\rho_L = \rho_L^o$, $E_L = E_L^o$, $\rho_S = \rho_S^o$ and $E_S = E_S^o$. (We note that this procedure not only specifies the nature of the signals, number and average energy, but also the nature of the local disturbances, thereby telling us if we should worry further about improving the isolation or not). Let's consider a parameter variation for say two Weber type detectors with $S_n(kT) = 50$. The following would certainly be an adequate set of trial values

$$n_s \text{ and } n_L = 0.1 \cdot 2^n \quad ; \quad n = 1, 2 \cdots 10 \qquad \left(\rho_S = \frac{n_s}{\mu_s T}, \rho_L = \frac{n_L}{\mu T} \right.$$
$$E_S \text{ and } E_L = 5 \cdot n \quad ; \quad n = 1, 2 \cdots 10 . \qquad \left. \text{see page} \right) .$$

Next we consider the hypotheses H_S^o and H_L^o where H_L^o is defined by $\rho_L = \rho_L^o$ and $E_L = E_L^o$. H_L^o is now considered to be the denial of H_S^o. The increase in evidence in favor of H_S^o is from a single data point at $t = t_{i_o}$.

$$(D.20) \qquad \Delta ev\left(H_S^o | E_{i_o}(H_S^o + H_L^o)\right) = k \ln \frac{\bar{P}_{n+(S_L)+(S)}(E_{i_o}^{(1)}, E_{i_o}^{(2)})}{P_n(E_{i_o}^{(1)}, E_{i_o}^{(2)})} \; ,$$

and totally:

$$(D.21) \qquad \Delta ev\left(H_S^o | \{E_i\}(H_S^o + H_L^o)\right) = \sum_i k \ln \frac{\bar{P}_{n+(S_L)+(S)}(E_i^{(1)}, E_i^{(2)})}{P_n(E_i^{(1)}, E_i^{(2)})} \; ,$$

This is the final expression for the evidence increase in favor of a "coincident" signal hypothesis. Note that this gives us all information we need and it is not necessary to make time delays. A time delay experiment would amount to testing, for example, the set of hypotheses

which are all similar to the hypothesis except that the arrival time of the signal pulse in the second detector is delayed by the time compared to the first.

One may then consider the function

$$(D.22) \quad f(\nu) = \Delta\, ev\left(H_s^{\circ}(\nu\cdot\Delta t) / \{E_i^{(1)}, E_i^{(2)}\}\left(H_s^{\circ}(\nu\cdot\Delta t) + H_o\right)\right) =$$

$$= \frac{k}{\Delta T} \sum_i \Delta t \ln \frac{\bar{P}_{n+(S_1)+(S)}(E_i^{(1)}, E_{i+\nu}^{(2)})}{P_n(E_i^{(1)}, E_i^{(2)})}$$

We remind that Δt is the time increment in the data recording. $f(\nu)$, $\nu = 0, \pm 1, \pm 2$ -- defines a "time delay histogram", which provides a very useful way of double checking the results. Because of the limitations of original set of hypotheses (they may not be sophisticated enough to differ between a "bug signal" and a real signal) this kind of check is necessary at least until one is sure that there are no "bugs" anywhere in the detector system, or in the data analysis programs.

ANALYSIS OF SOME DATA FROM THE MARYLAND ARGONNE DETECTORS

We will give a brief account of how the data analysis of some Weber tapes were analyzed.

Let, $E_\mu^{(1)} = (\dot{x}^2(t_\mu) + \dot{y}^2(t_\mu))^{(1)}$ and

$E_\mu^{(2)} = (\dot{x}^2(t_\mu) + \dot{y}^2(t_\mu))^{(2)}$, where (1) and (2) are channel notations (e.g. $E_\mu^{(1)}$ may be the Argonne signal and $E_\mu^{(2)}$ the Maryland signal).

The data, $(E_\mu^{(1)}, E_\mu^{(2)})$, $\mu = 1, \cdots, n$ (μ denotes time t_ν) is stored in 6 bits, i.e. $E_\mu^{(\nu)} = 0, 1 - - - -$ 63, on the data tape. The first step was to transform the data into a "frequency" matrix N(i,j), i,j=0...63, where N(i,j) is the number of times the pair ($E_\mu^{(1)}$, $E_\mu^{(2)}$) took on the value (i,j), when μ goes through all its possible values (All time information is thus lost in this procedure). The individual frequency distributions can easily be obtained from N(ij);

$$n^{(1)}(i) = \sum_j N(i,j) \quad ; \quad n^{(2)}(j) = \sum_i N(i,j) \; .$$

We also define the averages.

$$\bar{i} = \frac{\sum_i i\, n^{(1)}(i)}{\sum_i n^{(1)}(i)} \quad ; \quad \bar{j} = \frac{\sum_j j\, n^{(2)}(j)}{\sum_j n^{(2)}(j)} \; .$$

If we assume that the number of excitations is very small (the fractional time that the detector is excited is very small), one may identify

$$\bar{i} = \langle E_n^{(1)} \rangle \quad , \quad \bar{j} = \langle E_n^{(2)} \rangle \; .$$

We rewrite the distributions D.13 and D.16.

(D.13) $\Rightarrow$

$$P_{n+a_s}(i,j) = \frac{1}{(1+2a_s)} \exp\left\{-\left(1-\frac{a_s}{1+2a_s}\right)\left(\frac{i}{\bar{i}}+\frac{j}{\bar{j}}\right)\right\} \cdot$$

$$\cdot I_0\left(2\frac{a_s}{1+2a_s}\sqrt{\frac{i\,j}{\bar{i}\,\bar{j}}}\right)\frac{1}{\bar{i}\,\bar{j}} \,,$$

where $a_s = E_s/E_n$.

(D.16) $\Rightarrow$

$$P_{n+(a_L)}(i,j) = \frac{1}{2(1+a_L)}\left\{\exp\left(-\frac{i}{\bar{i}}-\frac{j}{\bar{j}(1+a_L)}\right)+\exp\left(-\frac{i}{\bar{i}(1+a_L)}-\frac{j}{\bar{j}}\right)\right\}\frac{1}{\bar{i}\,\bar{j}}$$

where $a_L = E_L/E_n$. The "noise only" distribution is,

$$P_n(i,j) = \left\{\exp-\left(\frac{i}{\bar{i}}+\frac{j}{\bar{j}}\right)\right\}\frac{1}{\bar{i}\,\bar{j}}$$

From these we construct the new probability densities;

$$\bar{P}_1(i,j\,|\,\rho_s, a_s, \rho_L, a_L) = (1-\rho_L-\rho_s)\,P_n(i,j) +$$

$$\rho_L\,P_{n+a_L}(i,j) + \rho_s\,P_{n+a_s}(i,j) \quad \text{and}$$

$$\bar{P}_2(i,j\,|\,\rho_L, a_L) = (1-\rho_L)\,P_n(i,j) + \rho_L\,P_{n+a_L}(i,j)\,.$$

The evidence increase can now be given in terms of these probabilities.

(D.23) $\Delta ev_1(\rho_s, a_s, \rho_L, a_L\,|\,C\{i,j\}) =$

$$= \frac{2\Delta t}{\Delta T}\sum_{i,j} N(i,j)\ln\frac{\bar{P}_1(i,j\,|\,\rho_s, a_s, \rho_L, a_L)}{P_n(i,j)}$$

and

(D.24) $$\Delta ev_2(\varsigma_s, a_s, \varsigma_i, a_i \mid C\{i,j\}) =$$

$$= \frac{2\Delta t}{\Delta T} \sum_{i,j} N(i,j) \ln \frac{P_1(i,j \mid \varsigma_s, a_s, \varsigma_i, a_i)}{P_2(i,j \mid \varsigma_i, a_i)} .$$

corresponding to the formulas D.17 and D.19. The advantage with the matrix approach lies entirely in computation time. Computing (D.23) or (D.24) with a P.D.P. 11...takes about 30 sec. while the same computation done point by point from a three-day tape would take hours or days.

In our calculations a "scrambled" matrix M(i,j) is constructed along with N(ij), for the purpose of comparison. M(ij) is constructed in the following way; $\{(E_\mu^{(1)}, E_\mu^{(2)})\}_{\mu=0,1 \cdots n}$ is the set of output data pairs. This set is transformed into the set $\{(E_\mu^{(1)}, E_{\nu_\mu}^{(2)})\}$ where $\{\nu_\mu\}$ is the same set of numbers as $\{\mu\}$, but the order has been randomized. The matrix M(i,j) is then constructed from $\{(E_\mu^{(1)}, E_{\nu_\mu}^{(2)})\}$ just as N(i,j) was constructed from $\{(E_\mu^{(1)}, E_\mu^{(2)})\}$. Clearly M(i,j) should give no evidence for a common signal in the two channels. The computer program produces; $\Delta ev_1(N)$, $ev_1(M)$ and $\Delta ev_2(N)$ for a given set of parameter values.

A second computer program was constructed to produce a time delay histogram. It gives $\Delta ev_2(\nu)$ where one channel is delayed with $\nu \cdot \Delta t$ sec, and $\nu = 1, 2, \cdots\cdots n$. The above matrix approach was not used this time since a) it would require one matrix (64 X 64) for each time delay, b) no parameter variation is performed. Instead the computation was done point by point, with the aid of a matrix, (D.25) $$f(i,j) = \ln \frac{P_1(i,j|\rho_s, a_s, \rho_\iota, a_\iota)}{P_2(i,j|\rho_\iota, a_\iota)} \quad , i,j = 0, \cdots 63 .$$

and

(D.26) $$\Delta\, ev_2\,(\rho_s, a_s, \rho_\iota, a_\iota | C\{E_\mu^{(1)}, E_{\mu+\nu}^{(2)}\}) =$$

$$= \sum_\mu f(E_\mu^{(1)}, E_{\mu+\nu}^{(2)}) \quad .$$

This approach has one advantage in that the program is extremely versatile. If the matrix $f(i,j)$ is chosen, instead of as above, to be $f(i,j) = 1$ if $i > i_o$ and $j > j_o$, and $f(i,j) = 0$ otherwise, a "binary cross correlation function" is obtained. If $f(i,j) = i \cdot j$ the ordinary crosscorrelation with time delay is obtained. In the same way outocorrelation functions are easily computed.

Unfortunately, however, the computation time with this method is fairly long. In order to cut the computation time integer numbers were used which in turn led to certain errors (which may decrease, but not increase, the central peak in the time delay histogram). We can at this time only present a few preliminary results. Presently, we have only eight "good" tapes where the data is recorded the new way, i.e. in the form $(\dot{x}, \dot{y})$.(Good means that when these tapes were recorded, both detectors and recording equipment worked satisfactorily). These tapes were initally filtered with a simple three point filter, with weights (1,2,1). The filtered output is then squared, and $\dot{x}^2 + \dot{y}^2$ is formed and recorded on a new tape. It was then found that two out of eight tapes showed signs of containing a signal. One of these tapes was analyzed in six hour periods and it was found that only two of these periods contained "signal activity."

We will not for the time being speculate about possible origins for these "signals." The above mentioned filter is simple, but not optimal. A new computer program* which allows filtering over eleven points (note that on these tapes the "sampling time" is 0.1 sec., and optimal filter time ranges from 0.3 to 0.8 sec) has recently been completed. The programs, and some other aspects of the new approach, such as the initial electronic filter, are not yet working to satisfaction. We will therefore only include some preliminary results for "weighted" (as defined by (D.26)) and "binary" crosscorrelation of the two detectors (Argonne and Maryland). Although we are at this point not sure that the "Weighted" crosscorrelation is "optimally" weighted, it is still a valid procedure. In all the cases the weight function f(i,j) was matched to signals with exponential distribution in energy. It is noteworthy that even though the testpulses were constant in energy, the weighted crosscorrelation performed better than the binary crosscorrelation. More complete results of the investigation will be forthcoming in a special technical report.

*Special thanks to Bruce Webster, who wrote all these programs. His ability with P.D.P. 11 assembly language, allowed the construction of very fast programs.

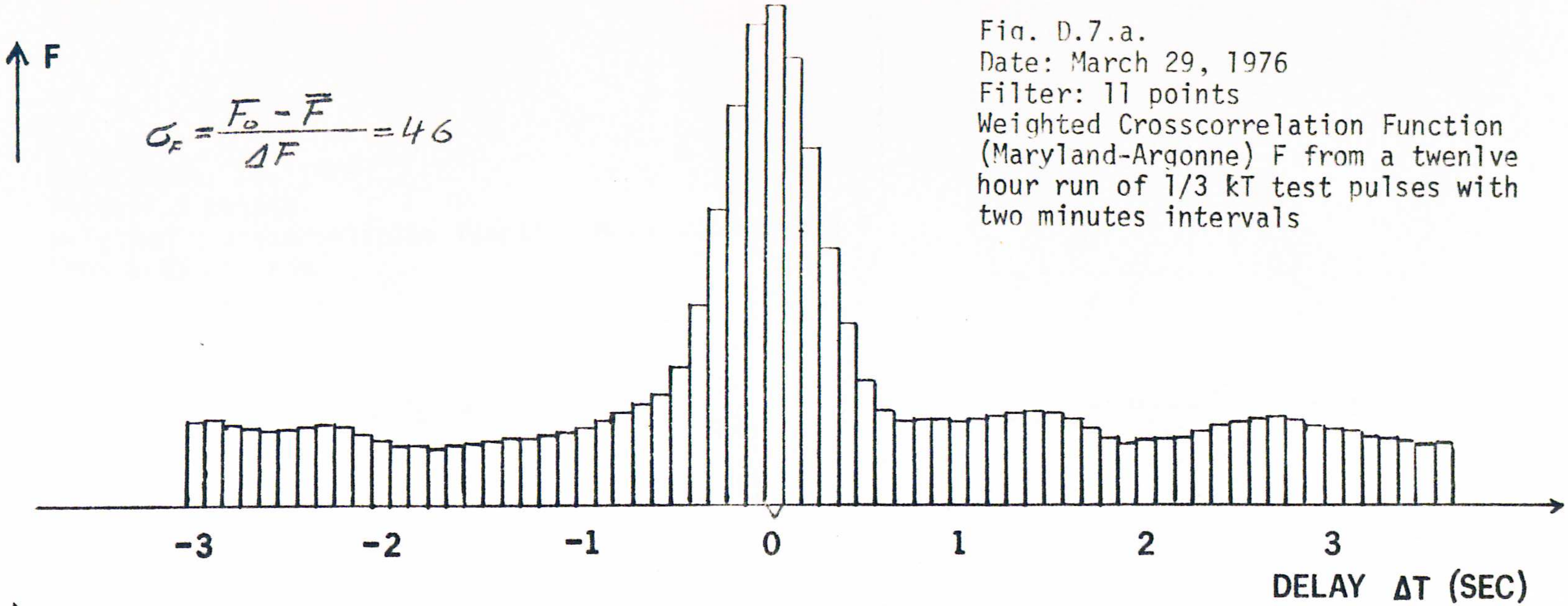


Fig. D.7.a.
Date: March 29, 1976
Filter: 11 points
Weighted Crosscorrelation Function (Maryland-Argonne) F from a twenlve hour run of 1/3 kT test pulses with two minutes intervals

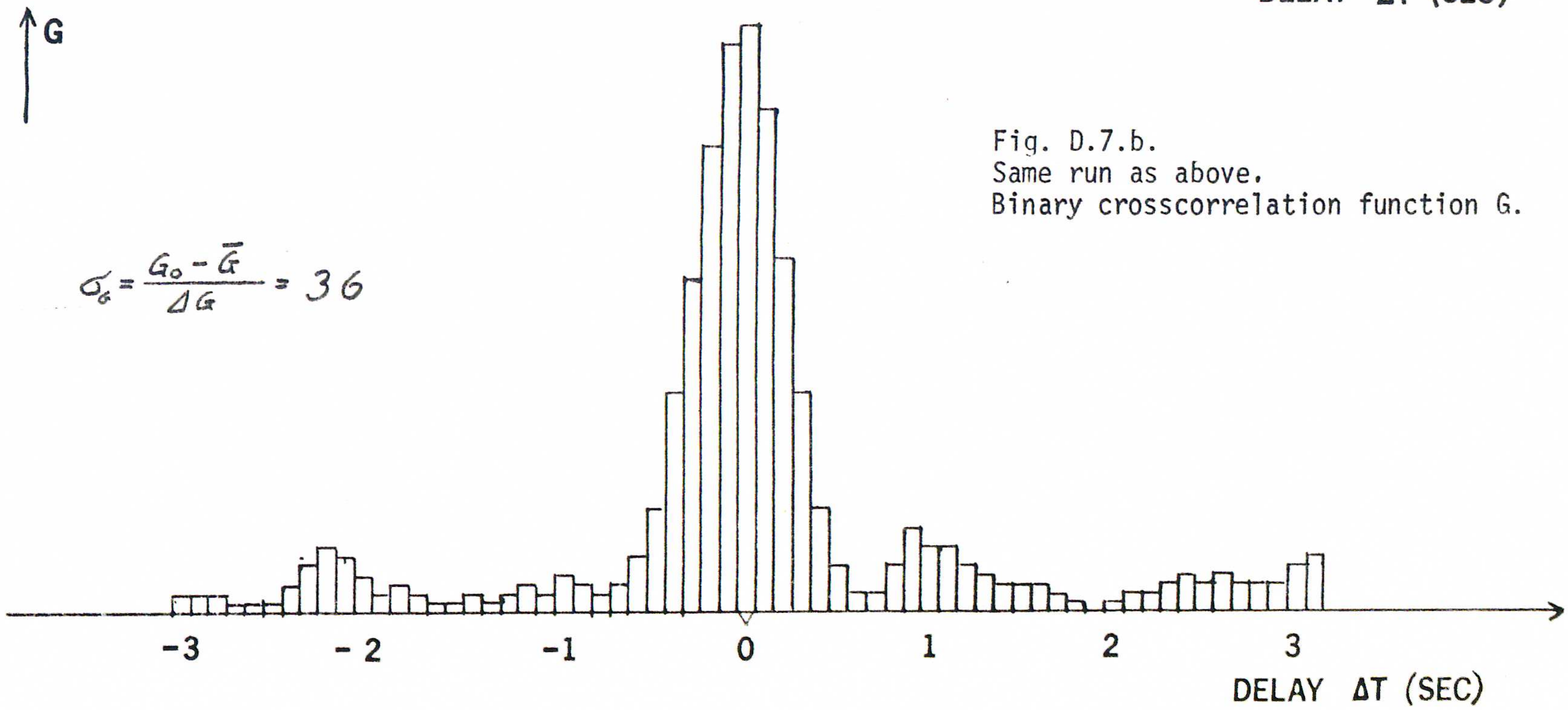


Fig. D.7.b.
Same run as above.
Binary crosscorrelation function G.

1750

Fig. D.8
Date: Feb., 18, 1976
Filter: 3 points
Weighted crosscorrelation function Maryland-Argonne F
From a 65 hr run

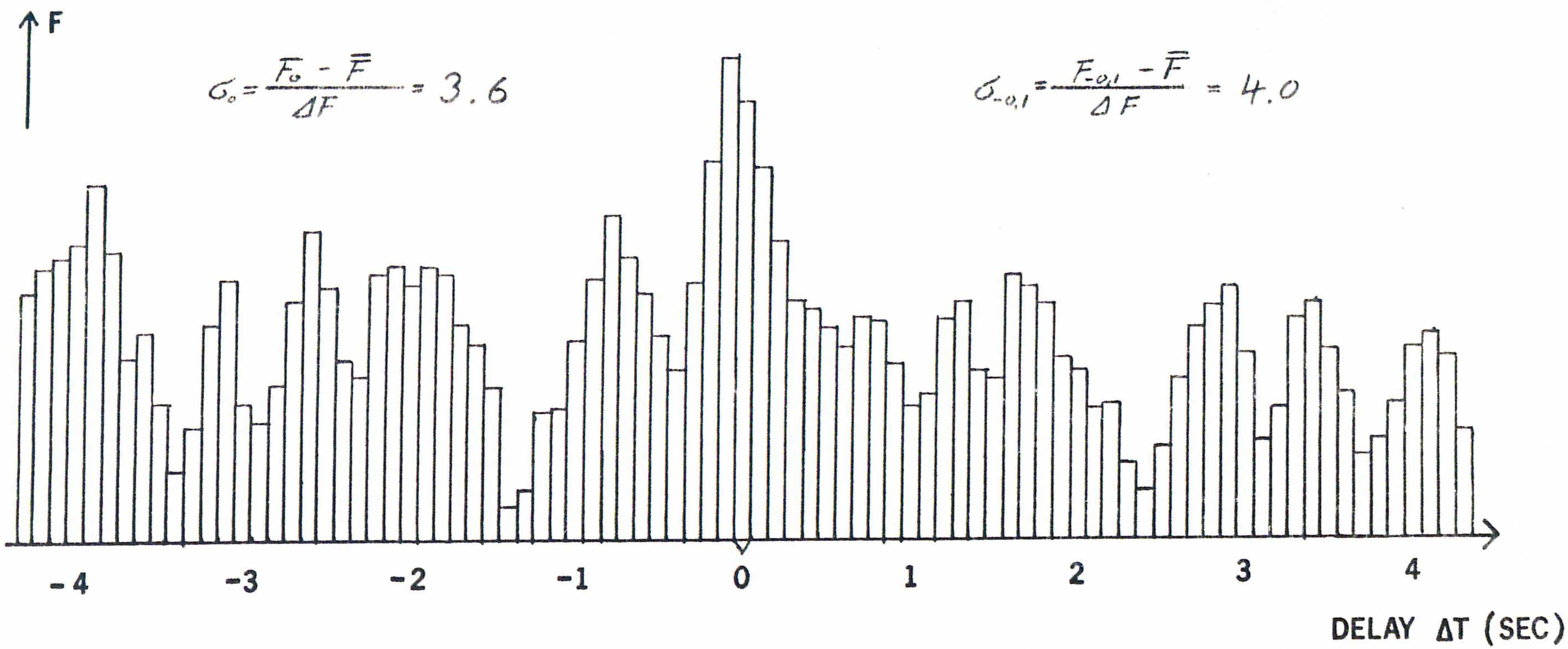
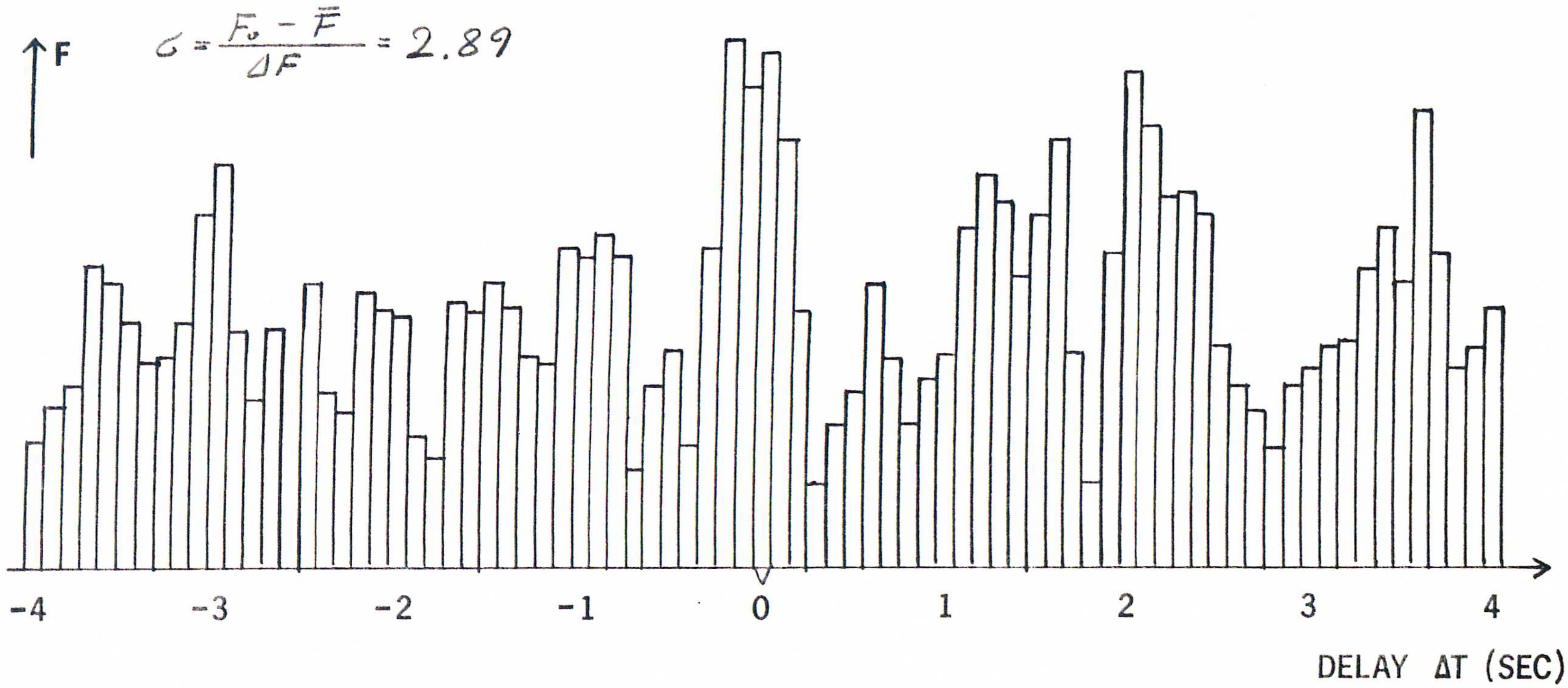


Fig. D.9
Date: Apr. 1, 1976
Filter: 3 points
Weighted Crosscorrelation Function
(Argonne Maryland) F from a 65 hrs run



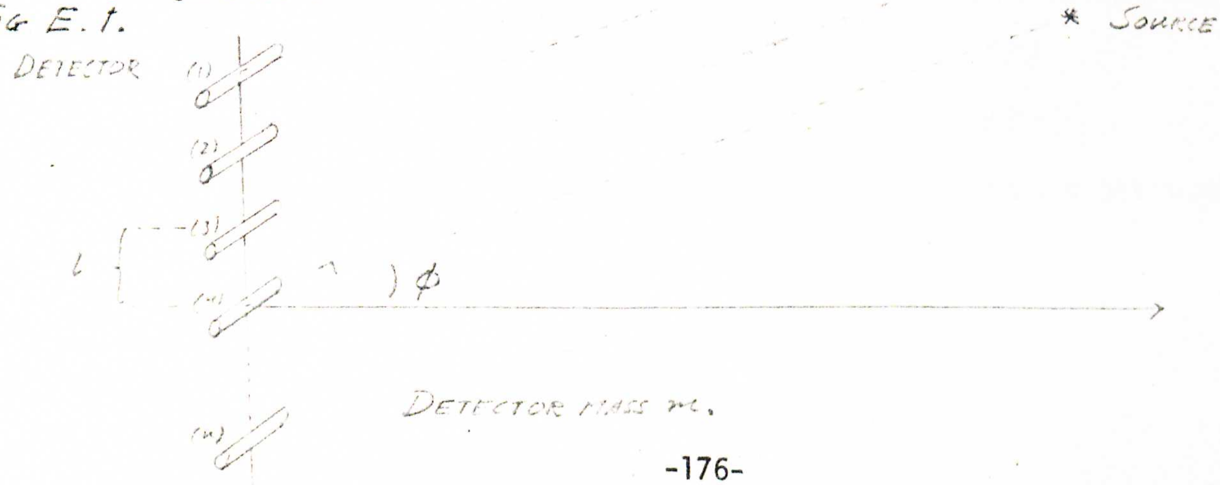
APPENDIX E

GRAVITATIONAL RADIATION DETECTION WITH TWO OR MORE DETECTORS WITH SYNCHRONIZED REFERENCE OSCILLATORS

We will in this appendix consider the possibility of performing experiments which are sensitive to the direction of propagation of the gravitational wave. (In the following we call this s.d.p). Certainly since a single detector by itself has an absorption crossection which varies as $\cos^4\theta$ where θ is the angle between the direction of the incident signal and a normal to the detector, it will be "weakly" s.d.p. (There is also a polarization dependence $\cos 2\phi$ where ϕ is the angle between the polarization direction and the detector, for simplicity we consider the case when $\phi = 0$). As we shall see one may by using several synchronized detectors obtain an "interference" between the outputs which will cancel, except when the detector system is looking in the "right" direction. It turns out however that the maximal signal to noise ratio (that is the signal to noise ratio when we "look" in the direction of the source) is no greater using n distant "interfering" detectors than n nearby (noninterfering) detectors or one detector with n times the mass of one of the n detectors. which both have the same signal to noise ratio in all directions. Thus it seems that before starting an interferometric experiment, one should use a non-interferometric set up to find out if there is any detectable gravitational radiation, and as a next step worry about pinpointing the source with interferometry.

Let's now consider n detectors separated by a distance l and arranged as in the figure below .

FIG. E.1.



The *time* delay for signals reaching the ν;*th* detector relative to the first is

$\Delta t_\nu = \nu \, l/c \, \sin\phi$ where c is the speed of light *or*

$\Delta t_\nu = \nu \Delta t$ where $\Delta t = l/c \, \sin\phi$.

Consider now the input signals $S^{(1)}(t)$ *to* $S^{(n)}(t)$

We have $S^{(1)}(t) = S(t)$ and $S^{(\nu)}(t) = S(t - \nu\Delta t)$. The fourier expansion of these signals are

$$S^{(1)}(t) = \int_{-\infty}^{+\infty} \frac{d\omega}{\sqrt{2\pi}} S(\omega) e^{i\omega t}$$

- - - - - - - - - - - - -

$$S^{(\nu)}(t) = \int_{-\infty}^{+\infty} \frac{d\omega}{\sqrt{2\pi}} S(\omega) e^{i\omega(t - \nu\Delta t)}$$

Now let's consider the phase space vector (in a *corotating frame*) defined by these signals (i.e. multiply by $e^{-i\omega t}$) we have

$$S_c^{(1)}(t) = \int_{-\infty}^{+\infty} \frac{d\omega_c}{\sqrt{2\pi}} S(\omega_0 + \omega_c) e^{i\omega_c t}$$

- - - - - -

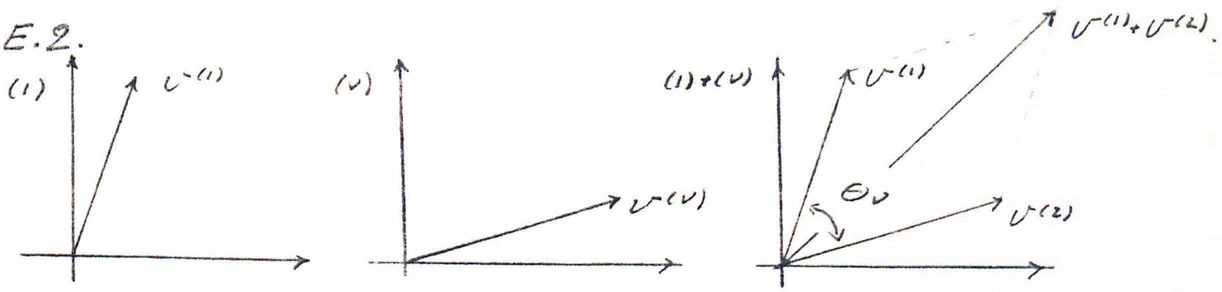
$$S_c^{(\nu)}(t) = e^{-i\omega_0 \Delta t} \int_{-\infty}^{+\infty} \frac{d\omega_c}{\sqrt{2\pi}} S(\omega_0 + \omega_c) e^{i\omega_c(t - \nu\Delta t)}$$

where the $S_c^{(\nu)}$'s are now comple . $S_c^{(\nu)}(t) = S_1^{(\nu)}(t) + i\, S_2^{(\nu)}(t)$

Thus we can see that the time delay not only shows up as a time delay of $S_c^{(\nu)}$ relative to $S_c^{(1)}$ but also as a phase lag; $\theta_\nu = \omega_0 \nu \Delta t$

If we at the arrival time of the signal display the outputs $U^{(1)}(t)$ and $U^{(2)}(t)$ in a corotating phase space diagram, we should get something like:

FIG. E.2.



Clearly the sum $\left|\sum_\nu v^{(\nu)} e^{i\nu\alpha}\right|$ has maximum for $\nu\alpha = \theta_\nu$.

We should thus construct this quantity from the output and find its maximum by varying α.

Consider now the output data of the detectors (signals plus noise)

$$d^{(1)} = s^{(1)} + n^{(1)} \qquad \text{where} \qquad s^{(1)}(t) = s(t)$$
$$d^{(2)} = s^{(2)} + n^{(2)} \qquad\qquad s^{(2)}(t) = s(t+\Delta t)\, e^{-i\omega_0 \Delta t}$$
$$- - - - - - - - - \qquad\qquad - - - - - - - - - - -$$
$$d^{(n)} = s^{(n)} + n^{(n)} \,. \qquad\qquad s^{(n)}(t) = s(t+n\Delta t)\, e^{-i\omega_0 n \Delta t}$$

Let's now introduce the hypothesis that there are incident signals with exponential distribution in energy, with an average energy $\langle E_s \rangle$ and with density (frequency at occurence) S_s . Let's further include in the hypothesis that the signal is time delayed by $\nu\Delta t'$, and has a phase lag of $\theta_\nu = \nu\Delta t'\omega_0$ in the $\nu_{,th}$ detector. We denote this hypothesis by $H_s(E_s, S_s, \Delta t')$.

Time delaying the data above and multiplying with the phase factor $e^{i\nu\omega_0\Delta t'}$ we get

$$d^{(1)}(t) = s(t) + n^{(1)}$$
$$d^{(2)}(t+\Delta t)\, e^{i\omega_0 \Delta t} = s(t)\, e^{i\omega_0(\Delta t - \Delta t')} + n^{(2)}$$
$$- - - - - - - - - - - - -$$
$$d^{(n)}(t+n\Delta t)\, e^{i\omega_0 \nu \Delta t} = s(t)\, e^{i\omega_0 n(\Delta t - \Delta t')} + n^{(n)}$$

(Note that we have for simplicity left out local disturbances (E_L, S_L)).

We omitted the factors multiplying the noise, since statistically it is invariant under time translation and phase changes. For simplicity we may now let

$$d^{(\nu)}(t+\nu\Delta t)\, e^{i\omega_0 \nu\Delta t} = d^{(\nu)}(t,\Delta t) = d^{(\nu)} .$$

At the time of signal incidence we have that joint probability distribution for $d^{(1)}, d^{(2)}, \ldots, d^{(n)}$ is (can be derived similarly to (D.12))

$$P_{n+(s)}(d^{(\nu)}) = \frac{1}{(2\pi\langle E_n\rangle)^{n-1}\, 2\pi(n\langle E_s\rangle+\langle E_n\rangle)}\, \exp - \frac{\sum |d^{(\nu)}|^2 - \frac{\langle E_s\rangle}{n\langle E_s\rangle+\langle E_n\rangle}\left|\sum d^{(\nu)}\right|^2}{\langle E_n\rangle}$$

The time averaged distribution is

$$\bar{P}_{n+(s)}(d^{(\nu)}) = (1-S_s)\,\frac{\exp - \frac{\sum |d^{(\nu)}|^2}{\langle E_n\rangle}}{(2\pi\langle E_n\rangle)^{n}} + \frac{S_s}{(2\pi\langle E_n\rangle)^{n-1}\, 2\pi(n\langle E_s\rangle+\langle E_n\rangle)}\, \exp - \frac{\sum |d^{(\nu)}|^2 + \frac{\langle E_s\rangle \left|\sum d^{(\nu)}\right|^2}{n\langle E_s\rangle+\langle E_n\rangle}}{\langle E_n\rangle}$$

and thus the evidence for the hypothesis $H_s(E_s, S_s, \Delta t)$ versus a pure noise hypothesis is

$$\Delta ev(E_s, S_s, \Delta t \,|\, d_i^{(\nu)}) = \sum_i \ln\left(1 + S_s\left(\frac{1}{(2\pi\langle E_n\rangle)^{n-1}\, 2\pi(n\langle E_s\rangle+\langle E_n\rangle)}\, \exp - \frac{\langle E_s\rangle \left|\sum d_i^{(\nu)}\right|^2}{\langle E_n\rangle (n\langle E_s\rangle+\langle E_n\rangle)} - 1\right)\right)$$

where $\{d_i^{(\nu)} = d_i^{(\nu)}(t,\Delta t)\}$ is the time and phase delayed output data of the n detectors at time t_i, $i=1,2\ldots N$.

This expression shows that with phase sensitive detection a system of n nearby detectors (i.e. always interfering constructively) each with mass m, will have the same sensitivity as one detector with mass n m (if local nonthermal excitations are not considered). It is further clear that only in the case of maximal constructive interference will a system of n detectors, with masses m, have a sensitivity which equals that of a single detector with mass n m. In other words it does not pay in terms of sensitivity to set up a multidetector long baseline interferometric system (while it may of course pay greatly in terms of scinetific information). It may be however, that for a future generation of supersensitive detectors, possibly made of high Q saphire, and cooled to millidegrees, local excitations from external sources will be the main source of uncertainty in the detection of pulses. In this case a system of "long base line" interfering detectors may be much superior to the usual two- detector system.

Let's now assume that Ξ_s and $\mathcal{S}_s$ are known by the experimenter (from previous non directional experiments). The probability distribution for θ, the phase shift between two detectors, assuming that the prior distribution in θ is constant over $-\pi$ to $+\pi$, is

$$P(\theta | D\, E_s \mathcal{S}_s C) = \frac{P(D|\theta E_s \mathcal{S}_s C)}{\int d\theta\, P(D|\theta E_s \mathcal{S}_s C)} \quad , \text{ or}$$

(6.14) $P(\theta | D E_s S_s c) =$

$$= \frac{\prod_{i=1}^{m} \left\{ (1-S_s) \frac{\exp - \frac{\sum |d_i^{(\nu)}|^2}{\langle E_n \rangle}}{(2\pi \langle E_n \rangle)^2} + S_s \frac{\exp - \frac{\sum |d_i^{(\nu)}|^2 - \frac{\langle E_s \rangle}{(n\langle E_s \rangle + \langle E_n \rangle)} |\sum_\nu d_i^{(\nu)} e^{i\nu\theta'}|^2}{\langle E_n \rangle}}{(2\pi \langle E_n \rangle)^{n-1} 2\pi (n\langle E_s \rangle + \langle E_n \rangle)} \right\}}{\int d\theta' \left(\text{------} \; '' \; \text{------} \right)} =$$

$$= \frac{\prod_{i=1}^{m} \left\{ 1 + \frac{S_s \langle E_n \rangle}{(1-S_s)(n\langle E_s \rangle + \langle E_n \rangle)} \exp \frac{\langle E_s \rangle |\sum d_i^{(\nu)} e^{i\nu\theta'}|^2}{\langle E_n \rangle (n\langle E_s \rangle + \langle E_n \rangle)} \right\}}{\int d\theta' \left(\text{------} \; '' \; \text{------} \right)}$$

The dependence of this distribution on the data points and the directional hypothesis may be understood by considering the term

$$E = |\sum_\nu d_i^{(\nu)} e^{i\nu\theta'}|^2 = |\sum_\nu (n_i^{(\nu)} + S_i^{(\nu)}) e^{i\nu\theta'}|^2 =$$

$$= |\sum_\nu n_i^{(\nu)} e^{i\nu\theta'} + S_i \, e^{i\nu(\theta'-\theta)}|^2 =$$

$$= |\sum_\nu n_i^{(\nu)} e^{i\nu\theta'}|^2 + 2\,\mathrm{RE}\left((\sum n_i^{(\nu)} e^{i\nu\theta'})(\sum_\nu S_i \, e^{i\nu(\theta'-\theta)}) \right) +$$

$$+ |\sum_\nu S_i \, e^{i\nu(\theta'-\theta)}|^2 .$$

Clearly $P(\theta)$ is large if E is large, which is mainly a function of the interference term $I = |\sum_\nu S_i \, e^{i\nu(\theta'-\theta)}|^2$.

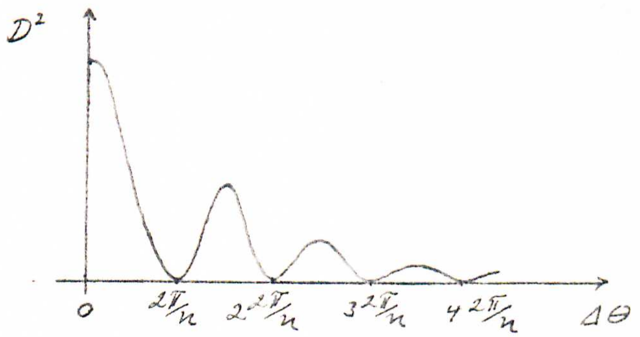
We have in the above assumed $S(t) = S(t + \nu\Delta t)$ which is a valid approximation for most of today's detectors (since they have a time resolution $\mu^{-1} \gg 4 \cdot 10^{-2}$, and for earthbound detectors $\nu \cdot \Delta t$ will always be less than $d_{earth}/c = 4 \cdot 10^{-2}$ sec.

With the same technique, that is used when adding up vectors in diffraction theory (sec. e.g. Feineman (1963), Lectures on Physics Vol. I page 30-31) we get

$$I = |S^2(t)| \frac{\sin^2 \frac{n\omega_0(\Delta t - \Delta t')}{2}}{\sin^2 \frac{\omega_0(\Delta t - \Delta t')}{2}}$$

Graph of I versus $\Delta\theta = (\theta' - \theta)$

Fig. E.3.



At $\alpha = 2\pi$ we will have another maximal peak. If we wish to have only one maximal peak corresponding to a particular direction, we must choose L such that $\omega_0(\pm\Delta t) < 2\pi$ or $\omega_0 \Delta t_{max} = 2\pi \Rightarrow L/c = 2\pi/\omega_0$. In a crude sense we can see from the graph that the phase resolution is about $2\pi/n$.

(the actual resolution depends of course on the signal to noise ratio) so that

$\omega_0 L/c\,(\sin\phi - \sin\phi') \leq \frac{2\pi}{n}$ with $L/c = 2\pi/\omega_0$ we have $\sin\phi - \sin\phi' < \frac{1}{n}$

or $2\cos\frac{\phi+\phi'}{2}\sin\frac{\phi-\phi'}{2} \approx (\phi-\phi')\cos\phi < \frac{1}{n}$,

or $\Delta\phi = \phi - \phi' < \frac{1}{n\cos\phi}$.

If $\phi \approx 0$ we have $\Delta\phi \leq \frac{1}{n}$ where $\Delta\phi$ is a "crude" measure if the directional uncertainty.

If the time resolution could be improved significantly and the signals are pulse-like or have an incoherent form, so that they add up coherently only when they are added with the correct time delay, one may have L larger than $c\frac{2\pi}{\omega_0}$. The "wrong" maximal peaks will then be reduced, because of the "wrong" corresponding time delay. This approach would however not be possible with todays' detectors.

APPENDIX F

COMPUTER SIMULATION OF A WEBER TYPE DETECTOR

We will in this section study a special type of signal analysis based on the quantity $\dot{P}^2 = \{\frac{\partial}{\partial t}(x^2+y^2)\}^2$, where x and y are the smoothly filtered output amplitudes. The reason for the interest in $\dot{P}^2$ is that the Maryland-Argonne detector system more or less consistently shows presence of signals when this method is used, while $\dot{x}^2 + \dot{y}^2$ does not in general give a significant result. One thingis immediately clear. When x and y are squared and added, phase information is lost. Thus unless the signal is of such a nature that it has a tendency not to change the phase of the detector but only its energy (a situation which must be considered highly unlikely) the $\dot{P}^2$ method is necessarily not optimal. It could still however, for some reasonable types of signals be "better" than $\dot{x}^2 + \dot{y}^2$, which is matched to a delta function type signal input. In general if $\dot{P}^2$ is "good" one would expect the similar quantity $(x^2 + y^2)(\dot{x}^2+\dot{y}^2)$ to be better since this quantity also registers changes in phase. (Note that $P^2 = x^2\dot{x}^2+y^2\dot{y}^2-2xy\dot{x}\dot{y}$). One may, by using the "evidence method" (See page try various types of signals and see if a quantity like $\dot{P}^2$ or similar comes up in the final result. For any simpler kind of signal no such terms have been found. For more complex situations like bursts of stochastic signals the computations get very involved. It could of course be that in some cases neither $\dot{P}^2$ nor $\dot{x}^2+\dot{y}^2$ is a "good" variable, but that P^2 still is better than $\dot{x}^2+\dot{y}^2$. We therefore decided to study the situation with a computer simulation of a pair of Weber type detectors. First some observations.

It is easily seen (from optimal filter theory) that for very narrowbanded signals on frequency (i.e. long lasting coherent signals with a central frequency equalling the detector resonance frequency) optimal data analysis will be based on the sum of the squares of the smoothly filtered output amplitudes, i.e $\bar{x}^2 + \bar{y}^2$ (where details of the smooth filter depends on the exact shape of the signals). This type of signal ought thus to be fairly mismatched to $\dot{x}^2 + \dot{y}^2$, and perhaps better matched to $\dot{P}^2$ which is essentially a product of $(x^2 + y^2)$ and $(\dot{x}^2 + \dot{y}^2)$, and perhaps even more so if the signal is altered to have slight ripples, and thus "excite" both $x^2 + y^2$ and $\dot{x}^2 + \dot{y}^2$. These types of signals must be considered unlikely however. It may not be as unlikely though, with a signal that sweeps slowly enough through the resonance of the detector to act like a long coherent signal, of the kind above. Trials with these types of signals did not come out in favour of P^2 however. If the sweep was slow enough (0.05 rad/sec^2), $\dot{P}^2$ and $\dot{x}^2 + \dot{y}^2$ were about equally efficient, but the time delay histogram had a far larger spread around zero than in reality, and for faster sweep $\dot{x}^2 + \dot{y}^2$ was more efficient. Fairly long bursts of stochastic pulses seems promising on similar grounds as above, but "$\dot{x}^2 + \dot{y}^2$" proved in general to be at least slightly more effective also in this case.

Of these cases all except one, which was equally "good" for $\dot{P}^2$ and $\dot{x}^2 + \dot{y}^2$, turned out in favor of $\dot{x}^2 + \dot{y}^2$. The one exception was for rare (not necessary) short pulses of large magnitude (larger than or equal to a "kT" pulse). This result is not surprising since for large pulses phase changes becomes less important. One situation however was found when "$\dot{p}^2$" was superior to "$\dot{x}^2 + \dot{y}^2$". In this and all other cases, the two detectors were defined by a 40 second damping time and a 0.5 second optimal filter time. When a frequency offset of 0.5 radians per second (between the detector and the reference oscillator) was introduced in one channel, the efficiency of the $\dot{x}^2 + \dot{y}^2$ method decreased sharply to become inferior to the $\dot{p}^2$ method. Curiously, the $\dot{p}^2$ method even appeared to improve slightly. The effect was most pronounced for small signals or a stochastic burst of (small) pulses. For 0.3 radians the two methods appeared about equal. These results agree essentially with what was found with analytical methods by G. Rydbeck and J. Weber in 1974. In this paper it was found that the detector will drift about 0.1 radians per second in frequency for every 0.04°C drift in temperature. It follows that if the frequency drift is not corrected for in the data analysis, a temperature change of more than 0.12 °C will cause the $\dot{x}^2 + \dot{y}^2$ method to become inferior to the $\dot{p}^2$ method. It requires a fairly sophisticated temperature control system to keep variations in temperature within this limit. We must thus conclude that of all possibilities tried in this investigation only frequency drift could account for making "$\dot{p}^2$" superior to "$\dot{x}^2+\dot{y}^2$".

Over the next couple of pages we give some time-delay histograms showing the results of the more interesting cases. (Optimal thresholds are chosen for these histograms). As an example we also provide a computer printout of the program output which is a set of (numberical) time delay histograms for different thresholds. A printout of the computer program itself is also given.

The "thermal" rate of coincident threshold crossings N is given on the graphs. N is calculated as the average number of crossings from 6 to 50 seconds delay. SD(N) is the standard deviation of $\overline{N}$ and σ is defined as

$$\sigma = \frac{N_0 - \overline{N}}{SD(\overline{N})} ,$$

where N_0 is the number of coincident threshold crossings at zero delay.

Note on Fig. F3. On account of the large pulse energy and the relatively short experimental runtime, the delayed rate of crossings is relatively small. The program provided three pulses of energy 2.2; 0.76 and 0.47 kT, in 8 hours experimental run-time. More realistically, we should have had one such pulse in 24 to 48 hours or more. Since however the computer (PDP-11) runtime was about two and a half times longer than the experimental runtime, this was not possible to do (as I was not the only user of the computer).

FIG. F.1 TIME DELAY HISTOGRAM

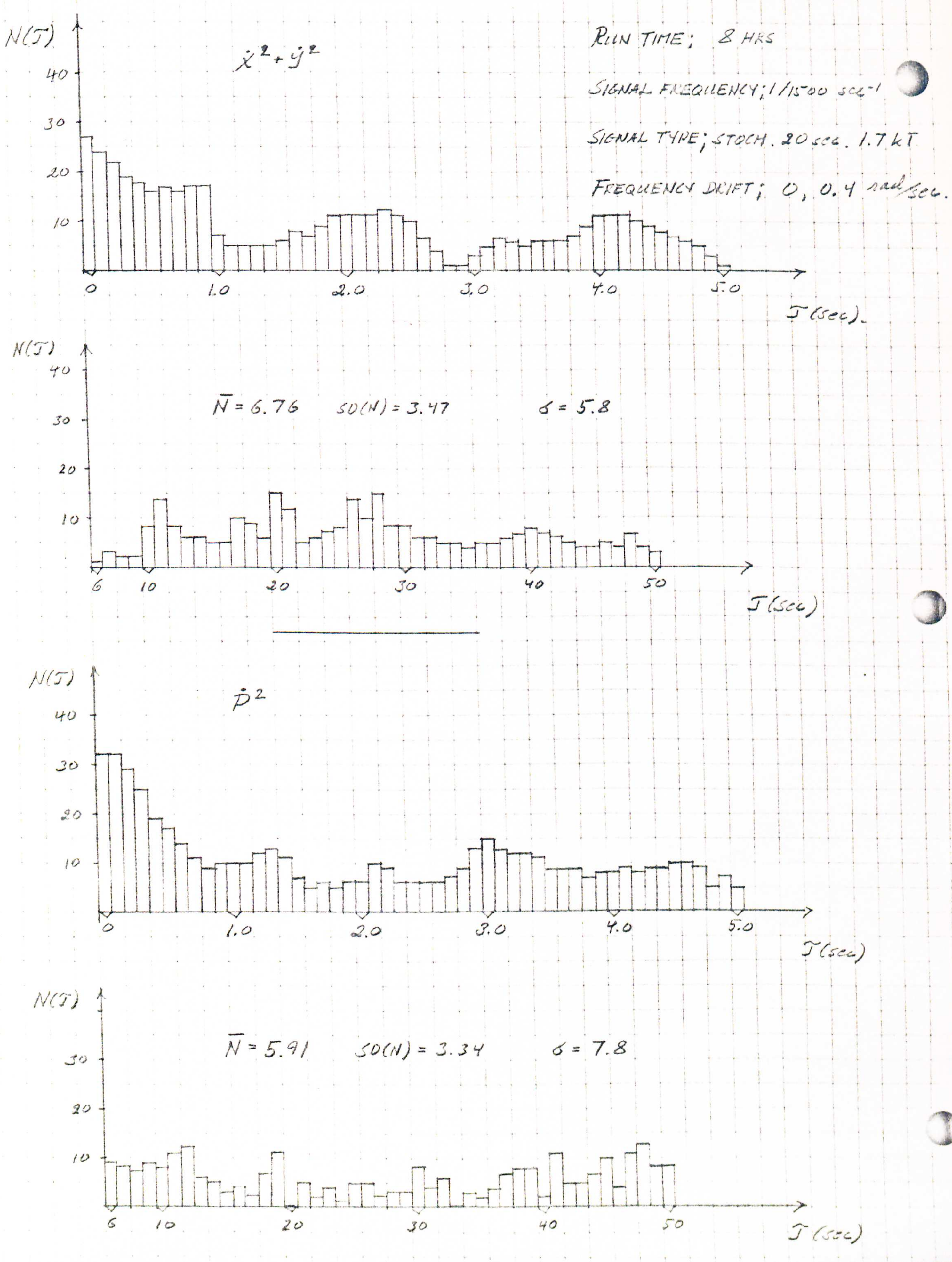


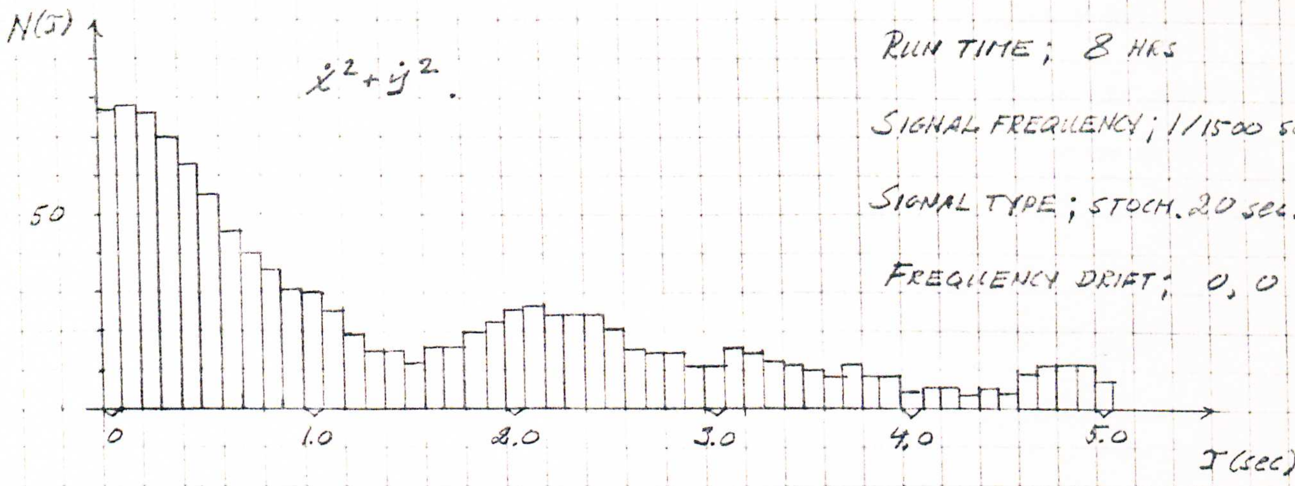
FIG. F.2 TIME DELAY HISTOGRAM

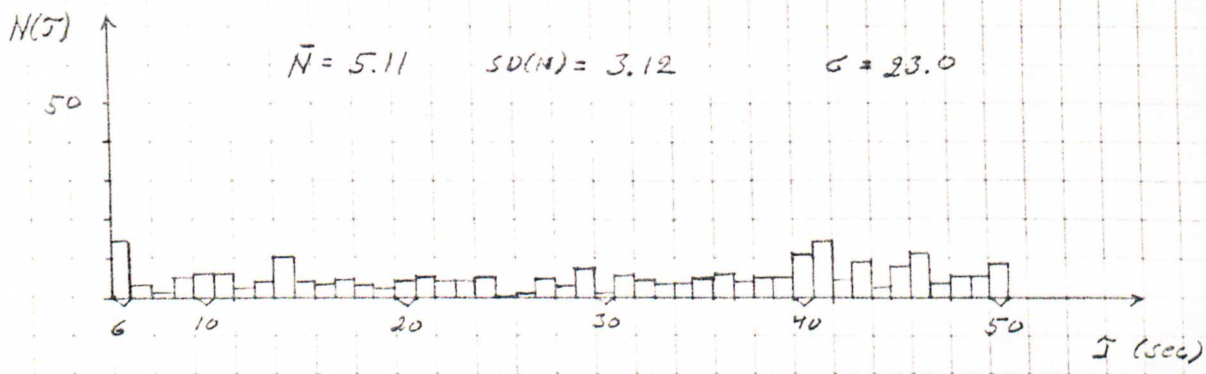
RUN TIME; 8 HRS

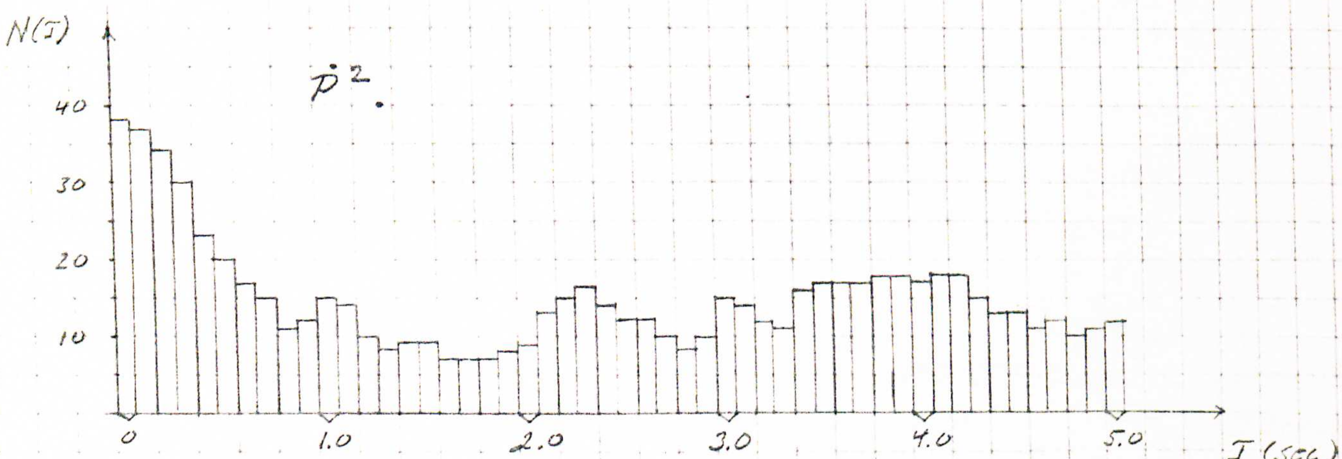
SIGNAL FREQUENCY; 1/1500 sec^{-1}

SIGNAL TYPE; STOCH. 20 sec. 1.7 kT

FREQUENCY DRIFT; 0, 0 rad/sec.







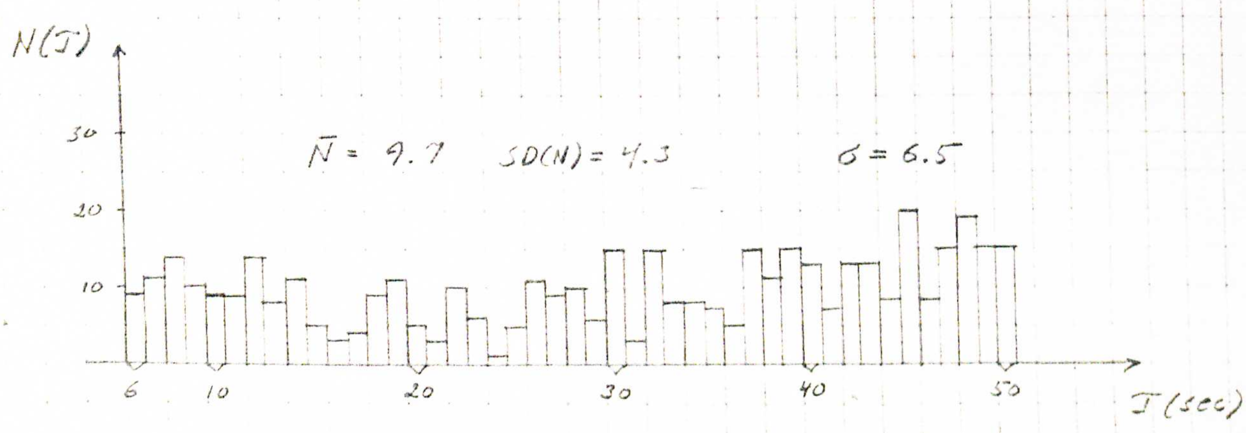


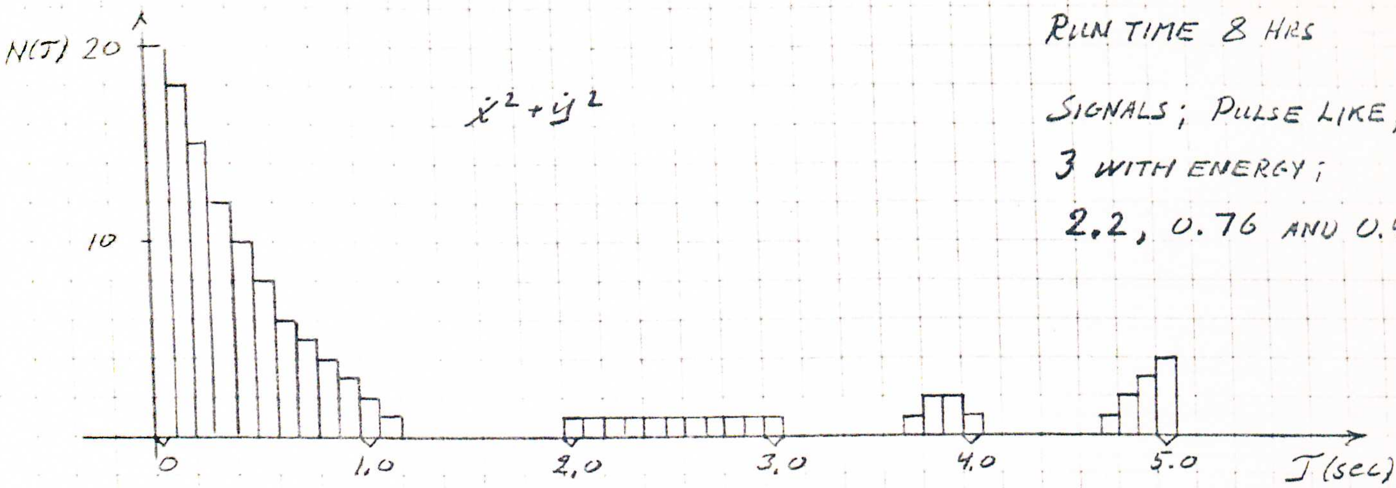
Fig F.3

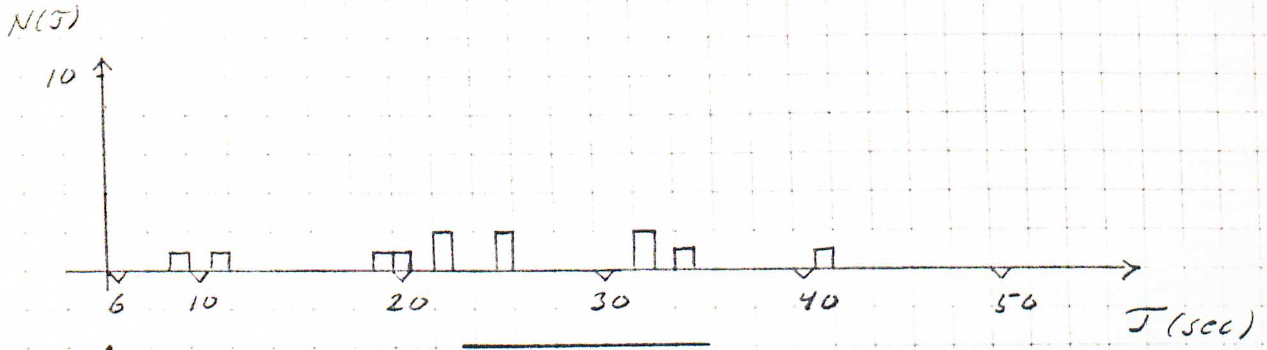
RUN TIME 8 HRS

SIGNALS; PULSE LIKE,

3 WITH ENERGY;

2.2, 0.76 AND 0.47 kT.





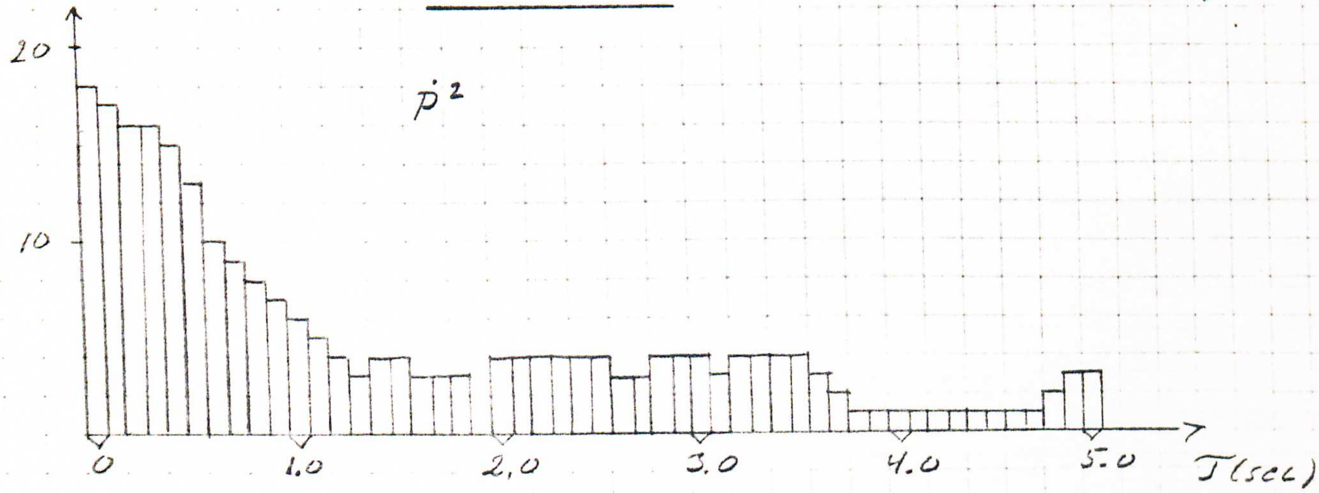


Fig. F. 4 a

RUN

GUS2 07-NOV-75 BASIC V01-05

TO PRINT YDOTSQ ETC ONLY SET H=1 TIME DELAY ONLY H=2 BOTH H=3 H=?2
FOR PRESET CONSTANTS SET C9=1 FOR SELECTION SET C9=2 C9=?2
GIVE TIMES IN SECONDS ENERGIES IN UNITS OF KT
GIVE DAMPING TIME T1=?40
GIVE RESOLUTION TIME T2=?0.5
GIVE AVERAGE TIME BETWEEN SIGNALS T3=?1500
SIGNAL TYPE 1 EXP DAMPED STOCH GIVE AVE EXC ENERGY AND
DECAY TIME E4,T4=?0,1
SIGNAL TYPE 2 PULSELIKE EXP DIST GIVE AVE EXC ENERGY E5=?0
SIGNAL TYPE 3 FREQUENCY SWEEPING PAST RESONANCE GIVE OPTIMAL
SIGNAL TO NOISE AND RATE OF SWEEP(RAD/SEC2) E6,D6=?6,0.12
DIFFERENT TYPES OF SIGNALS WILL ARRIVE SIMULTANEOUSLY
AND BE SUPERPOSED ON EACH OTHER
FREQUENCY OFFSET W0 IN CH 1 GIVE W0 IN RAD./SEC. W0=?0
GIVE TOTAL RUN TIME IN HOURS T6=?8

```
EXC          0  0  40
EXC          0  13 20
EXC          0  22 1
EXC          0  57 22
HRS 1
EXC          1  26 4
EXC          1  44 5
EXC          1  58 45
HRS 2
EXC          2  5  6
EXC          2  34 7
HRS 3
EXC          3  0  49
EXC          3  11 29
HRS 4
EXC          4  13 32
HRS 5
HRS 6
EXC          6  48 20
HRS 7
EXC          7  25 2
TIME         7           59          59
```

COINCIDENT THRESHOLD CROSSINGS FOR YDOTSQ+YDOTSQ WITH
TIME DELAY AND DIFFERENT THRESHOLDS
THRESHOLD LEVEL ->
TIME DELAY(SEC)
!
V

0	644	219	93	40	17	4	2	0	0	0	0
.1	633	212	90	33	13	4	1	0	0	0	0
.2	621	203	88	29	11	3	0	0	0	0	0
.3	615	198	87	27	9	2	0	0	0	0	0
.4	609	190	80	26	6	1	0	0	0	0	0
.5	596	189	77	21	5	0	0	0	0	0	0
.6	592	179	71	23	6	0	0	0	0	0	0
.7	585	190	69	22	4	0	0	0	0	0	0
.8	579	183	61	20	3	0	0	0	0	0	0
.9	567	173	63	19	1	0	0	0	0	0	0
1	552	171	63	14	1	0	0	0	0	0	0
1.1	577	172	61	12	1	0	0	0	0	0	0
1.2	580	171	52	14	0	0	0	0	0	0	0
1.3	569	161	50	13	2	0	0	0	0	0	0
1.4	572	160	44	12	1	0	0	0	0	0	0
1.5	565	169	49	14	0	0	0	0	0	0	0
1.6	583	174	48	10	1	0	0	0	0	0	0
1.7	624	183	47	5	0	0	0	0	0	0	0
1.8	635	190	50	7	0	0	0	0	0	0	0
1.9	635	195	53	6	0	0	0	0	0	0	0
2	639	201	55	9	0	0	0	0	0	0	0
2.1	620	217	51	11	2	1	0	0	0	0	0
2.2	622	215	53	19	3	1	0	0	0	0	0
2.3	595	206	53	15	3	0	0	0	0	0	0
2.4	600	203	54	20	4	0	0	0	0	0	0
2.5	599	202	51	22	3	0	0	0	0	0	0
2.6	581	197	55	19	2	0	0	0	0	0	0
2.7	595	187	54	19	2	0	0	0	0	0	0
2.8	570	169	52	18	1	0	0	0	0	0	0
2.9	547	159	50	17	2	0	0	0	0	0	0
3	530	156	47	19	3	0	0	0	0	0	0
3.1	521	152	44	13	3	1	1	0	0	0	0
3.2	524	161	45	11	4	2	0	0	0	0	0
3.3	523	152	43	12	3	2	1	0	0	0	0
3.4	525	152	43	9	2	2	0	0	0	0	0
3.5	532	143	37	8	1	1	0	0	0	0	0
3.6	557	153	37	7	0	0	0	0	0	0	0
3.7	554	145	41	9	0	0	0	0	0	0	0
3.8	557	159	45	9	1	0	0	0	0	0	0
3.9	559	170	47	8	2	0	0	0	0	0	0
4	557	161	43	13	4	1	0	0	0	0	0
4.1	545	163	53	15	6	2	1	1	0	0	0
4.2	541	167	53	19	9	2	1	1	0	0	0
4.3	554	176	52	22	9	2	1	1	0	0	0
4.4	556	175	58	23	10	3	1	1	0	0	0
4.5	574	182	66	28	14	4	2	1	0	0	0
4.6	584	183	63	25	11	3	1	1	0	0	0
4.7	611	183	66	23	11	3	1	0	0	0	0
4.8	552	191	73	18	11	3	1	1	0	0	0
4.9	555	191	74	19	12	1	1	0	0	0	0

Fig. F. 4 b.

6	497	130	37	7	0	0	0	0	0	0	0
7	455	112	29	4	1	0	0	0	0	0	0
8	435	125	31	13	4	0	0	0	0	0	0
9	407	116	32	9	2	0	0	0	0	0	0
10	386	102	24	7	3	0	0	0	0	0	0
11	365	67	19	6	2	0	0	0	0	0	0
12	344	51	6	0	0	0	0	0	0	0	0
13	329	54	9	0	0	0	0	0	0	0	0
14	282	47	5	0	0	0	0	0	0	0	0
15	245	36	4	0	0	0	0	0	0	0	0
16	251	31	2	0	0	0	0	0	0	0	0
17	261	28	4	1	1	0	0	0	0	0	0
18	231	23	2	0	0	0	0	0	0	0	0
19	219	34	3	0	0	0	0	0	0	0	0
20	223	19	0	0	0	0	0	0	0	0	0
21	217	12	1	0	0	0	0	0	0	0	0
22	204	13	0	0	0	0	0	0	0	0	0
23	201	21	2	0	0	0	0	0	0	0	0
24	226	22	1	0	0	0	0	0	0	0	0
25	216	23	3	2	0	0	0	0	0	0	0
26	236	31	1	1	0	0	0	0	0	0	0
27	199	20	0	0	0	0	0	0	0	0	0
28	229	29	3	0	0	0	0	0	0	0	0
29	213	20	0	0	0	0	0	0	0	0	0
30	216	24	0	0	0	0	0	0	0	0	0
31	187	16	2	0	0	0	0	0	0	0	0
32	197	21	1	0	0	0	0	0	0	0	0
33	206	21	3	0	0	0	0	0	0	0	0
34	224	21	1	0	0	0	0	0	0	0	0
35	223	25	4	0	0	0	0	0	0	0	0
36	189	16	1	0	0	0	0	0	0	0	0
37	196	27	4	0	0	0	0	0	0	0	0
38	221	37	9	0	0	0	0	0	0	0	0
39	224	26	3	0	0	0	0	0	0	0	0
40	239	38	7	0	0	0	0	0	0	0	0
41	215	28	2	0	0	0	0	0	0	0	0
42	214	22	5	0	0	0	0	0	0	0	0
43	237	26	7	2	0	0	0	0	0	0	0
44	193	22	1	1	0	0	0	0	0	0	0
45	220	38	6	1	0	0	0	0	0	0	0
46	195	30	5	0	0	0	0	0	0	0	0
47	188	19	0	0	0	0	0	0	0	0	0
48	203	27	5	1	0	0	0	0	0	0	0
49	215	29	5	1	0	0	0	0	0	0	0
50	155	17	2	0	0	0	0	0	0	0	0

SAME AS ABOVE FOR FOOTSQ
THRESHOLD LEVEL ->
TIME DELAY(SEC)
I
V

0	661	278	141	59	26	10	1	0	0	0	0
.1	653	275	135	60	24	10	1	0	0	0	0
.2	619	270	131	55	22	9	1	0	0	0	0
.3	619	267	130	55	21	7	2	0	0	0	0
.4	618	264	132	57	20	5	2	0	0	0	0
.5	622	260	128	57	19	4	1	0	0	0	0
.6	622	259	128	55	19	5	1	0	0	0	0
.7	613	249	126	57	19	5	1	0	0	0	0
.8	599	243	123	52	19	5	1	0	0	0	0
.9	607	243	120	45	17	4	0	0	0	0	0
1	599	252	121	42	15	4	0	0	0	0	0
1.1	601	252	117	40	14	3	1	0	0	0	0
1.2	593	253	122	40	15	5	0	0	0	0	0
1.3	597	259	122	39	15	5	0	0	0	0	0
1.4	525	262	119	37	14	4	0	0	0	0	0
1.5	539	258	120	35	11	5	0	0	0	0	0
1.6	567	264	123	35	10	4	0	0	0	0	0
1.7	566	282	124	38	14	5	0	0	0	0	0
1.8	578	276	121	38	14	7	0	0	0	0	0
1.9	575	263	118	40	15	5	0	0	0	0	0
2	554	262	115	41	15	5	1	0	0	0	0
2.1	539	255	116	39	16	5	0	0	0	0	0
2.2	513	254	115	41	15	4	0	0	0	0	0
2.3	503	253	106	35	13	4	1	0	0	0	0
2.4	505	244	93	31	10	2	0	0	0	0	0
2.5	615	243	93	30	9	1	0	0	0	0	0
2.6	629	240	88	27	7	1	0	0	0	0	0
2.7	613	225	84	25	5	1	0	0	0	0	0
2.8	595	213	73	24	5	2	0	0	0	0	0
2.9	579	201	77	28	5	1	0	0	0	0	0
3	577	195	77	31	5	1	0	0	0	0	0
3.1	574	204	79	31	10	1	0	0	0	0	0
3.2	551	205	83	31	11	1	0	0	0	0	0
3.3	557	205	83	34	11	1	0	0	0	0	0
3.4	558	200	84	38	14	1	0	0	0	0	0
3.5	555	201	88	42	15	2	0	0	0	0	0
3.6	515	201	85	41	11	1	0	0	0	0	0
3.7	540	202	83	40	13	1	0	0	0	0	0
3.8	557	199	91	40	17	1	0	0	0	0	0
3.9	557	194	90	37	15	4	0	0	0	0	0
4	551	200	97	31	15	4	0	0	0	0	0

Fig. F. 4c.

4	553	200	87	31	15	4	0	0	0	0	0
4.1	542	191	83	31	12	5	0	0	0	0	0
4.2	531	182	81	30	11	4	0	0	0	0	0
4.3	517	176	72	28	10	4	0	0	0	0	0
4.4	521	179	67	23	10	4	0	0	0	0	0
4.5	527	181	68	27	9	4	0	0	0	0	0
4.6	519	183	74	27	8	3	0	0	0	0	0
4.7	511	179	75	28	7	3	0	0	0	0	0
4.8	507	171	72	30	10	4	0	0	0	0	0
4.9	505	165	73	31	13	4	0	0	0	0	0
5	507	170	75	33	14	5	0	0	0	0	0
6	478	165	62	21	8	1	0	0	0	0	0
7	488	153	62	21	11	4	0	0	0	0	0
8	524	166	60	14	7	3	0	0	0	0	0
9	457	124	35	8	2	2	0	0	0	0	0
10	428	126	47	13	8	4	0	0	0	0	0
11	422	117	33	8	3	1	0	0	0	0	0
12	436	111	38	11	5	1	0	0	0	0	0
13	412	103	27	11	4	2	1	0	0	0	0
14	360	79	20	0	0	0	0	0	0	0	0
15	333	83	20	7	1	0	0	0	0	0	0
16	365	86	18	3	0	0	0	0	0	0	0
17	323	72	14	0	0	0	0	0	0	0	0
18	290	62	11	4	0	0	0	0	0	0	0
19	302	58	10	3	0	0	0	0	0	0	0
20	349	53	7	0	0	0	0	0	0	0	0
21	325	71	14	3	0	0	0	0	0	0	0
22	326	55	10	0	0	0	0	0	0	0	0
23	326	76	12	1	0	0	0	0	0	0	0
24	356	72	23	7	0	0	0	0	0	0	0
25	335	63	13	4	0	0	0	0	0	0	0
26	317	57	14	2	0	0	0	0	0	0	0
27	334	77	18	2	1	0	0	0	0	0	0
28	351	89	19	4	0	0	0	0	0	0	0
29	342	76	12	1	0	0	0	0	0	0	0
30	358	78	17	2	0	0	0	0	0	0	0
31	395	94	20	3	0	0	0	0	0	0	0
32	321	64	17	1	0	0	0	0	0	0	0
33	304	47	6	0	0	0	0	0	0	0	0
34	331	69	14	3	0	0	0	0	0	0	0
35	369	80	21	1	0	0	0	0	0	0	0
36	329	59	7	1	0	0	0	0	0	0	0
37	310	65	11	0	0	0	0	0	0	0	0
38	319	62	11	0	0	0	0	0	0	0	0
39	357	70	10	0	0	0	0	0	0	0	0
40	316	58	10	2	0	0	0	0	0	0	0
41	331	63	6	1	0	0	0	0	0	0	0
42	343	56	9	1	0	0	0	0	0	0	0
43	356	73	17	0	0	0	0	0	0	0	0
44	341	92	30	5	0	0	0	0	0	0	0
45	343	68	18	5	1	0	0	0	0	0	0
46	323	71	19	9	3	0	0	0	0	0	0
47	333	71	25	6	1	0	0	0	0	0	0
48	333	68	14	6	3	0	0	0	0	0	0
49	302	60	16	7	4	0	0	0	0	0	0
50	317	70	14	4	0	0	0	0	0	0	0

SLUT

READY

7.5

Fig. 7.4

FIG. F.5 a

```
READY

LIST

GUS2        BASIC V01-05

1 PRINT "TO PRINT XDOTSQ ETC ONLY SET H=1,TIME DELAY ONLY,H=2,BOTH,H=3,H=";
2 INPUT H
3 PRINT "FOR PRESET CONSTANTS SET C0=1,FOR SELECTION SET C0=2,C0=";
4 INPUT C0
5 IF C0=2GO TO 10
6 T1=40\T2=.5\T3=400\E4=.5\T4=20\E5=.5\T6=.25
7 T3=1
9 GO TO 140
10 PRINT "GIVE TIMES IN SECONDS ,ENERGIES IN UNITS OF KT"
20 PRINT "GIVE DAMPING TIME, T1=";
30 INPUT T1
40 PRINT "GIVE RESOLUTION TIME, T2=";
50 INPUT T2
60 PRINT "GIVE AVERAGE TIME BETWEEN SIGNALS, T3="
70 INPUT T3
80 PRINT "SIGNAL TYPE 1 EXP.DAMPED STOCH.GIVE AVE.EXC.ENERGY AND"
81 PRINT "DECAY TIME,E4,T4=";
82 INPUT E4,T4
85 PRINT "SIGNAL TYPE 2 PULSELIKE EXP.DIST GIVE AVE.EXC ENERGY E5=";
86 INPUT E5
90 PRINT "SIGNAL TYPE 3 FREQUENCY SWEEPING PAST RESONANCE GIVE OPTIMAL
91 PRINT "SIGNAL TO NOISE AND RATE OF SWEEP(RAD/SEC2),E6,D6=";
92 INPUT E6,D6
100 PRINT "DIFFERENT TYPES OF SIGNALS WILL ARRIVE SIMULTANEOUSLY
101.PRINT "AND BE SUPERPOSED ON EACH OTHER"
114 PRINT "FREQUENCY OFFSET W0 IN CH 1 GIVE W0 IN RAD/SEC, W0=";
115 INPUT W0
120 PRINT "GIVE TOTAL RUN TIME IN HOURS,T6=";
130 INPUT T6
140 DIM N(500),M(500)
142 DIM S(10,100),V(10,100)
150 T6=3600*T6
160 T7=.2*T2
162 D0=E4+E5
170 N=T6
175 W0=W0*T7
180 D1=EXP(-T7/T1)
190 D2=EXP(-.2)
200 D3=EXP(.2)
210 D4=D3^10
215 D5=SQR(T1*T2)
220 C0=SQR(.2*T2/T1)
222 C0=C0*D4
225 D7=SQR(E6*T1*T2)*T2*D6*C0
227 D6=T7*D6
230 C1=E4*T1/T4
240 C2=1.25*T4/T2
250 C3=10*T2/T3
260 C4=T1/T2
270 C5=.02*C4
280 C6=.2*T2/T1
290 C7=1+E4*.2*T1/T4+.5*E5*T1/T2
292 C7=C7+E6
295 C8=T1/(.2*T2)*E5
297 C9=C7*(1+E4+.5*E6)
300 I=I+1
302 T0=INT(I*T7)
304 IF T0<3600 GO TO 310
306 I=0\T9=T9+1\H0=60+T9-3600*T9
308 IF T9<3600 GO TO 310 \T9=0\T8=T8+1\PRINT "HRS";T9
310 J=J+1
312 H9=H0+T0
320 E=0\S1=0\S2=0
330 IF J>400GO TO 335 \IF S=0GO TO 390 \GO TO 347
335 J=0\E=C9
340 IF RND(0)<C3GO TO 345 \S=0\GO TO 390
345 S=1
346 PRINT "EXC",T8,T9,T0
347 IF D0=0GO TO 352
349 V=ABS(200-J)
350 R3=SQR(-LOG(1-RND(0))*(E+C1*EXP(-V/C2)))*C0
355 W3=6.28318*RND(0)
360 S1=R3*SIN(W3)\S2=R3*COS(W3)
361 IF E6=0GO TO 390
362 W5=D6*(J-200)\W6=T7*W5
365 S1=S1+D7/(1+.1666*W5*W5)*SIN((200-J)*W6)
367 S2=S2+D7/(1+.1666*W5*W5)*COS((200-J)*W6)
390 R1=SQR(-LOG(1-RND(0)))*C0
400 R2=SQR(-LOG(1-RND(0)))*C0
410 W1=6.28318*RND(0)
```

Fig. F.5 b.

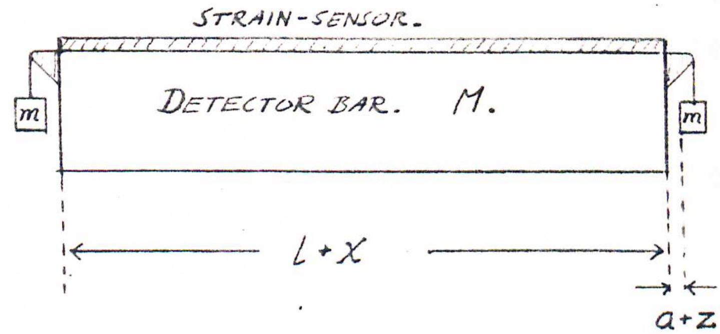
```
420 W2=6.28318*RND(0)
425 W4=W4+W0
427 W4=W4-6.28318*INT(W4/6.28318)
430 G1=R1*SIN(W1)
440 G2=R1*COS(W1)
450 G3=R2*SIN(W2)
460 G4=R2*COS(W2)
470 A1=G1+S1+D1*A1
480 A2=G2+S2+D1*A2
482 B3=G3+S1+D1*B3
484 B4=G4+S2+D1*B4
486 M1=COS(W4)\M2=SIN(W4)
490 A3=B3*M1-B4*M2
500 A4=B3*M2+B4*M1
510 X0=0\Y0=0\Z0=0\U0=0
520 A=X(0)+D2*A\B=Y(0)+D2*B\C=Z(0)+D2*C\D=U(0)+D2*D
530 FOR K=0 TO 9 STEP 1
540 X(K)=X(K+1)*D3\X0=X0+X(K)\Y(K)=Y(K+1)*D3\Y0=Y0+Y(K)
550 Z(K)=Z(K+1)*D3\Z0=Z0+Z(K)\U(K)=U(K+1)*D3\U0=U0+U(K)
560 NEXT K
570 X(10)=A1+D5*G1\X0=X0+X(10)\Y(10)=A2+D5*G2\Y0=Y0+Y(10)
580 Z(10)=A3+D5*G3\Z0=Z0+Z(10)\U(10)=A4+D5*G4\U0=U0+U(10)
670 H1=K1
680 K1=X0+A
690 H2=K2
700 K2=Y0+B
710 H3=K3
720 K3=Z0+C
730 H4=K4
740 K4=U0+D
750 L1=K1-H1\L2=K2-H2\L3=K3-H3\L4=K4-H4
760 O1=(L1*L1+L2*L2)*C4
770 O2=(L3*L3+L4*L4)*C4
780 O3=(K1*K1+K2*K2-H1*H1-H2*H2)\O3=O3*O3*C5
790 O4=(K3*K3+K4*K4-H3*H3-H4*H4)\O4=O4*O4*C5
792 IF H9>4000 TO 799
793 IF H=200 TO 300
794 PRINT I,O1,O3\IF H=100 TO 300
799 IF H=100 TO 1040
800 N(Q)=O2\M(Q)=O4
810 FOR R=0 TO 10 STEP 1
820 R1=4.3+.1*C7*R
822 R3=(.6+.1*R+.02*R*R)*C3
830 IF O1>R1GO TO 840\IF O3<R3GO TO 920
840 FOR P=0 TO 95 STEP 1\P1=P
842 IF P<51GO TO 850\P1=50+10*(P-50)
850 Q=Q-P1+501*INT((500-Q+P1)/501)
860 IF O1<R1GO TO 880
870 IF N(Q)<R1GO TO 880\S(R,P)=S(R,P)+1
880 IF O3<R3GO TO 900
890 IF M(Q)<R3GO TO 900\V(R,P)=V(R,P)+1
900 NEXT P
910 NEXT R
920 Q=Q+1
922 IF Q<501GO TO 950\Q=0
950 IF H9<NGO TO 300
951 PRINT "TIME",T9,T8,T0
952 PRINT "COINCIDENT THRESHOLD CROSSINGS FOR XDOTSQ+YDOTSQ WITH "
953 PRINT "TIME DELAY AND DIFFERENT THRESHOLDS"
954 PRINT "THRESHOLD LEVEL ->"
955 PRINT "TIME DELAY(SEC)"
956 PRINT "!"
957 PRINT "V"
960 FOR I1=0 TO 95 STEP 1
962 T=T7*I1\IF I1<51GO TO 964 \T=T7*(50+10*(I1-50))
964 PRINT T,
970 FOR I1=0 TO 10 STEP 1
972 PRINT S(I1,I1),
974 NEXT I1\PRINT \NEXT I1
980 PRINT "SAME AS ABOVE FOR RDOTSQ"
993 PRINT "THRESHOLD LEVEL ->"
994 PRINT "TIME DELAY(SEC)"
995 PRINT "!"
996 PRINT "V"
1000 FOR I2=0 TO 95 STEP 1
1002 T=T7*I2\IF I2<51GO TO 1004 \T=T7*(50+10*(I2-50))
1004 PRINT T,
1010 FOR I2=0 TO 10 STEP 1
1020 PRINT V(I2,I2),
1030 NEXT I2\PRINT \NEXT I2
1040 PRINT "SLUT"

READY
```

APPENDIX G

CANCELLATION OF DETECTOR NOISE

This method is also discussed on page 11. Consider a detector equipped with a strain sensitive device and also with accelerometers on each end, as described by the figure below.

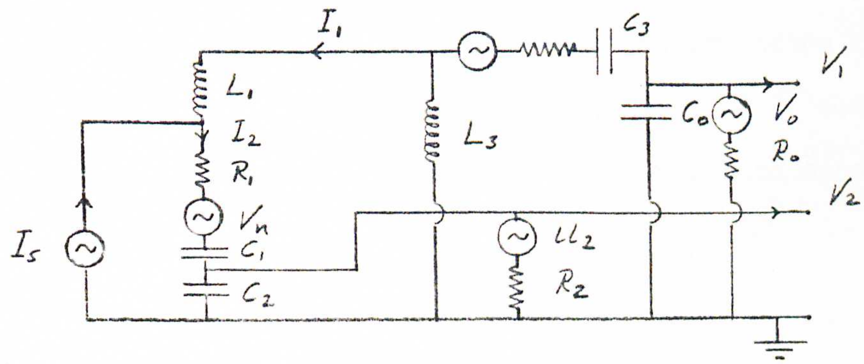
Fig. G.1



l and a are equilibrium distances.

The distance $a + z$ may be measured by a capacitor, and the strain sensor (which measures changes in l + x) may be a P.Z.T. crystal. It can be shown that the equivalent circuit for this detector is the following

Fig. G.2



where I, representes "non inertial motion" and I_2 "stretch motion" of the detector, I_s represents the gravitational interaction, and

$q_s = \int dt\, I_s = C_n^{ij} h_{ij}$, where C_n^{ij} is a coupling constant, and h_{ij} the metric perturbation. L, R, and C represent the effective mass, the damping (with noise source V_n) and the elasticity of the detector.

C_2 and R_2 (with noise source u_n) represents the strain gauge, and L_3, R_3 and C_3, the mass (m), the damping (with noise source V_3) and the harmonic force (which keeps m in place) of the accelerometer. C represents the device measuring the distance (a + z) between the detector endface and the mass m, and (with V_0) the associated noise. As will be seen, the sensitivity using the method of amplitude subtraction (as outlined on page) which cancels the detector noise from the combined output is in most practical cases far inferior to the ordinary method. The two methods could be used in combination, but the resulting increase in sensitivity is minimal. The reason for this is that the "subtraction method",not only subtracts the detector noise but also cancels the resonating behaviour of the outputs which is needed to "overcome" the wideband noise sources u_2 and V_0 . Only if the noise contribution from u_2 and V_0 are extremely small compared to the detector noise would the method pay off. We now give a brief calculation comparing the sensitivity of the two methods for the detector described by figure 7.G, and the equivalent circuit above. We simplify by assuming that we have a good accelerometer so that L_3 , C_3 and R_3 are infinite.

We have

$$V_1(\omega) = \frac{\dfrac{I_s(\omega)\,\dfrac{\frac{1}{i\omega C_3}}{i\omega L_1 + \frac{1}{i\omega C_3} + R_3}}{\dfrac{1}{R_1 + \frac{1}{i\omega C}} + \dfrac{1}{i\omega L_1 + \frac{1}{i\omega C_3} + R_3}} + \frac{V_n(\omega)\,\frac{1}{i\omega C_3}}{\frac{1}{i\omega C} + i\omega L_1 + R_1 + \frac{1}{i\omega C_3} + R_3} +$$

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}$$

$$+ \frac{U_2(\omega)\left\{\dfrac{1}{i\omega C_2 + \dfrac{1}{\frac{1}{i\omega C_1} + R_1 + i\omega L_1 + \frac{1}{i\omega C_3} + R_3}}\right\}\left\{\dfrac{1}{\frac{1}{i\omega C_1} + R_1 + i\omega L_1 + \frac{1}{i\omega C_3} + R_3}\right\}}{\dfrac{1}{i\omega C_2 + \dfrac{1}{\frac{1}{i\omega C_1} + R_1 + i\omega L_1 + \frac{1}{i\omega C_3} + R_3}}} -$$

$$- \frac{V_3(\omega)\,\frac{1}{i\omega C_3}}{\frac{1}{i\omega C} + R_1 + i\omega L_1 + R_3 + \frac{1}{i\omega C_3}} =$$

$$= \frac{I_s(\omega)\left(R_1 + \frac{1}{i\omega C}\right)}{i\omega C_3\left(i\omega L_1 + \frac{1}{i\omega C} + \frac{1}{i\omega C_3} + R_1 + R_3\right)} + \frac{V(\omega)}{i\omega C_3\left(i\omega L_1 + \frac{1}{i\omega C} + \frac{1}{i\omega C_3} + R_1 + R_3\right)} +$$

$$+ \frac{U_2(\omega)}{i\omega C_3\,(1 + i\omega R_2 C_2)\left(i\omega L_1 + \frac{1}{i\omega C} + \frac{1}{i\omega C_3} + R_1 + R_3\right)}$$

$$- \frac{V_3(\omega)}{i\omega C_3\left(i\omega L_1 + \frac{1}{i\omega C} + \frac{1}{i\omega C_3} + R_1 + R_3\right)} \quad .$$

$$V_L(\omega) = \frac{I_s(\omega)\,\dfrac{\dfrac{1}{i\omega C_2 + \frac{1}{R_2}}}{\left(\dfrac{1}{i\omega C_1} + \dfrac{1}{i\omega C_2 + \frac{1}{R_2}}\right)}}{\left(\dfrac{1}{R_1 + \frac{1}{i\omega C}} + \dfrac{1}{i\omega L_1 + \frac{1}{i\omega C_3} + R_3}\right)} + \frac{V_n(\omega)\,\dfrac{1}{i\omega C_2 + \frac{1}{R_2}}}{\left(i\omega L_1 + \frac{1}{i\omega C} + \frac{1}{i\omega C_3} + R_1 + R_3\right)} +$$

$$+ \frac{U_2(\omega)\,\dfrac{1}{i\omega C_2 + \dfrac{1}{\frac{1}{i\omega C_1} + i\omega L_1 + \frac{1}{i\omega C_3} + R_1 + R_3}}}{R_2 + \dfrac{1}{i\omega C_2 + \dfrac{1}{\frac{1}{i\omega C_1} + i\omega L_1 + \frac{1}{i\omega C_3} + R_1 + R_3}}} + \frac{V_3(\omega)\,\dfrac{1}{i\omega C_2 + \frac{1}{R_2}}}{\left(i\omega L_1 + \frac{1}{i\omega C} + \frac{1}{i\omega C_3} + R_1 + R_3\right)} =$$

$$= \frac{I_s(\omega)\left(R_1 + \frac{1}{i\omega C}\right)\left(i\omega L_1 + \frac{1}{i\omega C_3} + R_3\right)\left(\dfrac{1}{i\omega C_2 + \frac{1}{R_2}}\right)}{\left(\dfrac{1}{i\omega C_1} + \dfrac{1}{i\omega C_2 + \frac{1}{R_2}}\right)\left(i\omega L_1 + \frac{1}{i\omega C} + \frac{1}{i\omega C_3} + R_1 + R_3\right)} -$$

$$- \frac{V_n(\omega)\,\dfrac{1}{i\omega C_2 + \frac{1}{R_2}}}{i\omega L_1 + \frac{1}{i\omega C} + \frac{1}{i\omega C_3} + R_1 + R_3} + \frac{U_2(\omega)\left(i\omega L_1 + \frac{1}{i\omega C_1} + \frac{1}{i\omega C_3} + R_1 + R_3\right)}{R_2\left(i\omega C_2 + \frac{1}{R_2}\right)\left(i\omega L_1 + \frac{1}{i\omega C} + \frac{1}{i\omega C_3} + R_1 + R_3\right)} +$$

$$+ \frac{V_3(\omega)}{\left(i\omega L_1 + \frac{1}{i\omega C} + \frac{1}{i\omega C_3} + R_1 + R_3\right)} .$$

Now define

$$d(\omega) = i\omega C_3\left(\frac{1}{i\omega C_2 + \frac{1}{R_2}}\right) V_1(\omega) + V_2(\omega) \Rightarrow$$

$$d(\omega) = \frac{1}{1+i\omega C_2 R_2}\left\{\frac{I_s(\omega)\left(R_1 + \frac{1}{i\omega C}\right) R_2}{\left(\frac{1}{i\omega C_1} + \frac{1}{i\omega C_2 + \frac{1}{R_2}}\right)} + u_2(\omega)\right\} .$$

As we can see both V_n and V_3 have disappeared, and also the resonance "boosting" $I_s(\omega)$. Let's now calculate the spectral signal to noise in the two cases; we consider first the spectral signal to noise of $V_2(\omega)$. Since $\langle V_3^2(\omega)\rangle$ is much less than $\langle V_n^2(\omega)\rangle$ for any real detector we neglect this term. We have $\langle V_n^2(\omega)\rangle = kTR$, and $\langle u_2^2(\omega)\rangle = kTR_2$. Further we approximate $R_1 + \frac{1}{i\omega C} \approx \frac{1}{i\omega C}$,

and $\left(\frac{1}{i\omega C_1} + \frac{1}{i\omega C_2 + \frac{1}{R_2}}\right) \approx \frac{1}{i\omega C_1} + \frac{1}{i\omega C_2} = \frac{1}{i\omega C}$

both of which are extremely good approximations in a real situation. Thus we have:

$$S_n(V_2|\omega) = \frac{\left|i\omega L_1 + \frac{1}{i\omega C_3} + R_3\right|^2 I_s(\omega)}{\left\{\frac{|u_2^2(\omega)|}{R_2^2}\left|i\omega L_1 + \frac{1}{i\omega C_1} + \frac{1}{i\omega C_3} + R_1 + R_3\right|^2 + V_n^2(\omega)\right\}} =$$

$$= \frac{I_s^2(\omega)\left|i\omega L_1 + \frac{1}{i\omega C_3} + R_3\right|^2 R_2}{kT\left\{\left|i\omega L_1 + \frac{1}{i\omega C_1} + \frac{1}{i\omega C_3} + R_1 + R_3\right|^2 + R_1 \cdot R_2\right\}} \approx$$

$$\approx \frac{I_s^2(\omega)\,\omega^4 L_1^2 R_2}{kT\left\{L_1^2(\omega^2 - \omega_0^2)^2 + \omega^2(R_1^2 + R_1 \cdot R_2)\right\}} .$$

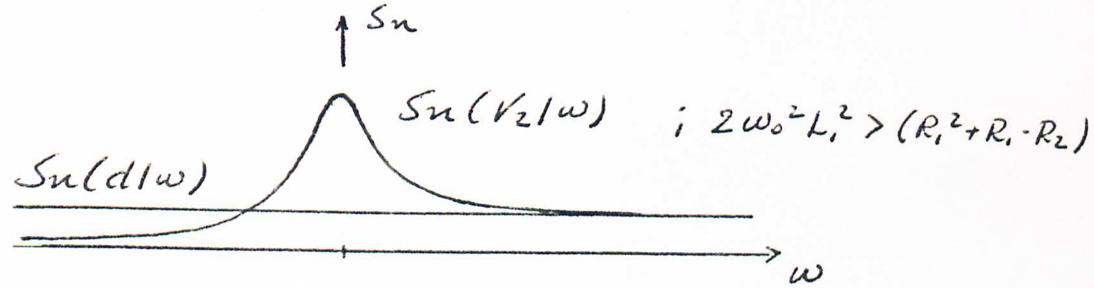
The spectral signal to noise of $d(\omega)$ is,

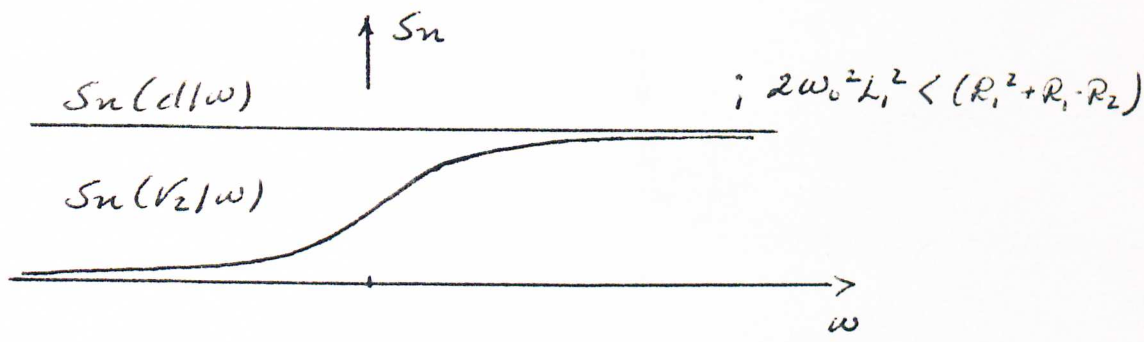
$$Sn(d/\omega) = \frac{I_s^2(\omega)\, R_2}{kT}$$

$Sn(V_2/\omega)$ has maximum for $$\omega^2 = \frac{2\,\omega_0^4\, L_1^2}{2\,\omega_0^2 L_1^2 - (R_1^2 + R_1 \cdot R_2)}$$

Thus if $(R_1^2 + R_1 \cdot R_2) > 2\omega_0^2 L_1^2$, $Sn(V_2/\omega)$ has no maximum (which is not likely to occur in a real situation). We give a typical plot of $Sn(d/\omega)$ and $Sn(V_2/\omega)$

Fig. G.3





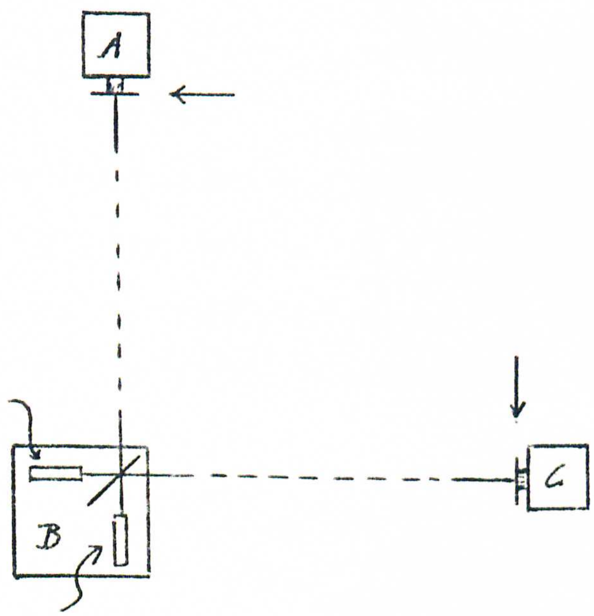
(Note that the scaling is arbitrary).

For the Weber 1973 set-up $\frac{S_n(V_2/\omega)_{max}}{S_n(d/\omega)} = 5 \cdot 10^6$.

Clearly the subtraction method is not useful in practice for such detectors, and is not likely to be for any efficient (high Q) detector.

The scheme may be useful in other types of gravitational radiation experiments however. Consider for example (a Forward type) Laser interferometry experiment over large distances as described by the figure below. One would probably want to set up such an experiment in a desert, and use evacuated tubes for the laser rays, to avoid convective disturbances on the ray.

Figure G.4



It is essential for this experiment that the masses, A,B and C are completely free, so that vibrations in the earth's crust do not interfere with the experiment. This may be technically hard to accomplish. To get around this problem one may mount a seismometer on each of the masses to monitor "non free" motion, and subtract any such motion from the motion registered by the interferometer. It may in fact be technically even simpler not to have the masses free, but have them mounted solidly on the earth, which would be possible with the above scheme.

The sensitivity of this detector depends mainly on the resolution of the interferometer. It has been reported (T.J. Sejnovski 1974) that Forwards strain sensitivity is given by (with our notation)

$$\Delta l(\omega)/l = \alpha(\omega)\, 2\cdot 10^{-16}\, (2\pi\, \mathrm{rad/sec})^{-1/2}$$

over a bandwidth, 1,3 to 20 kHZ. (l) between A and B and Band C in his experiment is approximately 2.5 meters. In the experiment described above this length could be increased by at least a factor of 10^4. Thus in this case $\Delta l(\omega)/l = \alpha(\omega)\cdot 2\cdot 10^{-20}\,(2\pi\, \mathrm{rad/sec})^{-1/2}$. $\Delta l(\omega)/l$ due to a gravitational wave is $\Delta l(\omega)/l = \frac{1}{2} h_{11}(\omega)$ (where h_{11} is the deviation from the flat metric, along the direction of the detector arms). Thus the sensitivity limit in h_{11} is $h_{11} \leq \alpha(\omega)\, 4\cdot 10^{-20}\,(2\pi\, \mathrm{rad/sec})^{-1/2}$. Multiplying with $h_{11}(\omega)$ (in the manner described by optimal filtering), squaring and taking expectation value, we get

$$\left\{\int d\omega\, h_{11}^2(\omega)\right\}^2 \geq \int d\omega\, h_{11}^2(\omega)\cdot 16\cdot 10^{-40}\,(2\pi\, \mathrm{rad/sec})^{-1},$$

or

$$\int d\omega\, h_{11}^2(\omega) \geq 16/2\pi\; 10^{-40}\ \mathrm{sec}.$$

Now the spectral power per/cm^2 of the gravitational wave is related to h_{11} (ω) as

$$P(\omega) = c^3/4\,(4\pi G)^{-1}\, h_{11}^2(\omega)\,\omega^2,\ \text{thus}$$

$$\int d\omega\, P(\omega)/\omega^2 \geq \frac{c^3\,16}{2\pi \cdot 4\,(4\pi G)}\, 10^{-40}\ \text{ergs sec}/cm^2 \simeq 2\cdot 10^{-3}\ \text{ergs sec}/cm^2$$

For a pulse centered around 1.3 kHz, the total Energy per cm^2 must thus be at least $E = (2\ )^2(1.3)^2\ 10^3$ ergs/cm^2 = 3 10^4 ergs/cm^2 for the pulse to be seen by the detector. This compares favorably with todays room temperature detectors. The "bare" detector (Weber type) sensitivity is about $5\cdot 10^4$ ergs/cm^2. This sensitivity however is appreciably reduced for the "dressed" detector if the radiation is wide banded. It is further possible that improved interferometry techniques will be developed in the future.

T.J. Sejnowski (1974) Physics Today, page 40 , January 1974.

NOTATIONS AND SYMBOLS

a	Half length of detector
$\alpha(t)$	White noise
$\beta(t)$	Filtered white noise (colored noise)
E	Square of amplitude quantity such as x^2, $\dot{x}^2$, $\dot{x}^2+\dot{y}^2$, etc.
$f(t-t')$	Filter function
h_{ij}	metric perturbations
R_{ijkl}	Riemann curvature tensor
$\dot{x}$, $\dot{y}$	$\dot{x}$, $\dot{y}$ etc. usually means x, y optimally filtered with respect to a delta function input sigmal (approximates $\partial x/\partial t$, $\partial y/\partial t$).
$\tilde{\nu}$	If ν is a source voltage, $\tilde{\nu}$ is the corresponding final detector output voltage
$\vec{x}$	A state vector usually in complex notation is denoted by an arrow-bar.
$\bar{x}$	Filtered variable x
γ	Inverse detector damping time
μ	Inverse detector resolution time
V_s	Speed of sound in detector material
S_n	Signal to noise ratio
Q	Quality factor

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"New Gravitational Radiation Experiments" Weber, Gretz, Lee, Rydbeck, Trimble and Steppel. Phys. Rev. Letters 31, 779,1973.

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