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Ergodicity of M-dimensional Random Walks and Random Access Systems

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Abstract

We apply a set of recent results by Malysev and Mensikov, concerning necessary or sufficient conditions for ergodicity of constrained M-dimensional random walks to the problem ofstability of M coupled queueing systems that describe a system of M buffered terminals accessing a common channel by means of the slotted ALOHA protocol. We obtain a neccesary and sufficient condition for the stability of such a system. Although the condition does not yield a descriptive characterization of the stability region, it allows a reduction of the stability problem of a M–user system to the determination of the steady state distribution of a $(M-1)$–user system. The plausibility of a recent conjecture concerning the stability of this system is also discussed.

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1 Introduction

Some systems of multiple access used in radio and/or local area networks are modeled by Markov Chains in the positive sector of M–dimensional space, whose transitions allow them to coincide with special forms of M–dimensional random walks. As one of the problems of interest in these multiple access systems is the determination of conditions for the stability (or ergodicity) of the underlying Markov Chain, it is natural that the problem of ergodicity of general random walks is directly relevant.

In this paper we focus on the discrete time slotted ALOHA protocol, operating with M buffered terminals over the collision channel. The model of this system is essentially the same with the one proposed by Tsybakov and Mikhailov in [1]. We consider a set of M users, accessing a common error-free channel. Time is assumed to be slotted and each user, i, generates packets for transmission in the channel according to a Bernoulli process with rate λ_i.[1] During each slot user i attempts to transmit the first packet in his queue with probability p_i, provided that he has a nonempty queue. The usual assumptions of slotted ALOHA are made, concerning successful transmissions and collisions (see [2,3]). In particular if exactly one packet is transmitted during any given slot, then this packet is perfectly received by all users (successful transmission), while, if more than one packets are transmitted during any given slot, then they are all lost and must be retransmitted during a later slot (collision).

This system is modeled by a random walk with state $\mathbf{Q}(t) = (Q_1(t), \ldots, Q_M(t))$ where $Q_i(t)$ denotes the queue size (number of packets) in the buffer of terminal i, just prior to slot t. From time slot to time slot, $\mathbf{Q}$ executes transitions in the grid of points whose coordinates take non–negative values and the probabilities of which

[1] In [1] a general arrival pattern is assumed. The restriction to Bernoulli arrivals here is not of crucial importance and is made fo rthe sake o notational simplicity.

are given easily in terms of the packet arrival rates λ_i and packet transmission probabilities p_i, $i = 1, \ldots, M$, see [1].

Our goal in this paper is to derive a necessary *and* sufficient condition for ergodicity of the random walk $\mathbf{Q}(t)$, that describes the M–user slotted ALOHA system. In particular, we will prove that there exists a set of "well defined" drifts δ_i (where $i = 1, \ldots, M$) such that

$$\delta_i < \lambda_i \quad \forall\, i \implies \mathbf{Q}(t) \quad \text{is ergodic}$$

$$\delta_i > \lambda_i \quad \text{for some } i \implies \mathbf{Q}(t) \quad \text{is non-ergodic}$$

To prove this result we use the notion of persistent user, introduced by Rao and Ephremides in [4] and a necessary condition for ergodicity of general Random Walks (expressed in terms of existence of Lyapunov functions) due to Malysev and Mensikov, [5].

The paper is organized as follows: In the next section, 2, we introduce the necessary notation and draw the connections to the notation used in [5]. The objective of this section is to give a proper physical interpretation to the abstract tools used in [5] and use this interpretation to draw some interesting properties of them, that will lead to the main results. The necessary and suffincient condition for the ergodicity of the random walk $\mathbf{Q}(t)$ is derived in section 3.

2 Notation

We consider a system slightly different than the one described in the introduction. In particular we assume two disjoint sets of users, $N_M = \{\ 1, \ldots, M\ \}$ and L, of cardinalities M and l respectively. Each user $i \in N_M$ behaves as described in the introduction, namely he transmits a packet in every slot with probability p_i, provided that he has a nonemty queue, (typical user). Each user $i \in L$ transmits always with probability p_i independently of whether he has a nonempty queue or not (persistent user). The concept of persistent users was introduced in [4]. We will denote this system $calS_M(L)$.

We want to study the random walk $calS_M(L)$, in Z_+^M, describing the queues of the M users $i \in N_M$, in the overall system of $M + l$ users, the l of which attempt transmission regardless of their queue status and are of no direct interest to us.

Consider now a set $A \subset N_M$. For this set, define the faces of Z_+^M and $\mathcal{R}_+^M$, respectively denoted by $\mathcal{B}_M(A)$ and $\Phi_M(A)$, as follows

$$\mathcal{B}_M(A) \triangleq \{\ \mathbf{z} = (z_1, \ldots, z_M) \in Z_+^M\ :\ z_i > 0\ \ \forall i \in A \ \text{ and } \ z_i = 0\ \ \forall i \notin A\ \} \quad (1)$$

$$\Phi_M(A) \triangleq \{\ \mathbf{r} = (r_1, \ldots, r_M) \in \mathcal{R}_+^M\ :\ r_i > 0\ \ \forall i \in A \ \text{ and } \ r_i = 0\ \ \forall i \notin A\ \} \quad (2)$$

Observe that the face $\mathcal{B}_M(A)$ corresponds to the set of states with the common property that in each of them the queues of users $i \notin A$ are empty while the queues of users $i \in A$ are nonempty. As a consequence, for any given $A \subset N_M$ the face $\mathcal{B}_M(A)$ constitutes a homogeneous subset of the state space, in the sense that

$$\forall \ \mathbf{x}^1 , \mathbf{x}^2 \ \in \ \mathcal{B}_M(A) \quad \text{and} \quad \forall \ \mathbf{y} \ : \ \Pr\{ \ \mathbf{q}(n+1) = \mathbf{x}^1 + \mathbf{y} \ / \ \mathbf{q}(n) = \mathbf{x}^1 \ \} =$$

$$\Pr\{ \ \mathbf{q}(n+1) = \mathbf{x}^2 + \mathbf{y} \ / \ \mathbf{q}(n) = \mathbf{x}^2 \ \}$$

The existence of different homogeneous phases $\mathcal{B}_M(A)$, each with countable states renders the analysis of the random walk $calS_M(L)$ a very difficult problem. To bypass this difficulty we consider a number of auxilliary systems. In particular, with any set $A_k \subset N_M$, having k elements, we associate an auxilliary system $\hat{\mathcal{S}}_M(A_k)$, that is identical to $calS_M(L)$, except that the users $i \in \hat{\mathcal{S}}_M(A_k)$ have been removed from the set of typical users, N_M, to the set of persistent users, L. There are three important observations that can be made regarding the auxilliary systems $\hat{\mathcal{S}}_M(A_k)$:

1. Each system $\hat{\mathcal{S}}_M(A_k)$ dominates the initial system $calS_M(L)$. (For a proof see [4].)
2. Each system $\hat{\mathcal{S}}_M(A_k)$ is of actually a system of the form $\mathcal{S}_{M-k}(L \bigcup A_k)$ (i. e. a system with a set $N_M - A_k$ of $M-k$ typical users and a set $L \bigcup A_k$ of $l+k$ persistent users).
3. Each system $\hat{\mathcal{S}}_M(A_k)$ is described by an induced $(M-k)$–dimensional random walk $\mathcal{I}_M(A_k)$, that is actually the projection of the initial random walk on a $(M-k)$–dimensional hyperplane $\mathcal{C}_M(A_k)$, orthogonal to $\mathcal{B}_M(A_k)$ at a point $\mathbf{x} \in \Phi_M(A_k)$. We denote the state space of the $M-k$–dimensional random walk $\mathcal{I}_M(A_k)$ by $\Psi(A_k)$.

We say that the face $\mathcal{B}_M(A_k)$ is ergodic if the induced random walk $\mathcal{I}_M(A_k)$ is ergodic. In this case we denote the steady state probability distribution of $\mathcal{I}_M(A)$ by $\pi_A(\mathbf{x})$ (where $\mathbf{x} \in \Psi(A)$).

following probability distribution

$$\pi_{A_k}(E) \triangleq \lim_{t \to \infty} \Pr\{\mathbf{q}(t) \in \Psi_M(A_k) \ : \ q_i(t) = 0 \ \forall \, i \notin E \, , \, q_i(t) \geq 1 \ \forall i \ \in \ E\}$$

We have

$$\pi_{A_k}(E) = \sum_{\mathbf{q} \in \Psi_M(A_k) \bigcap \Phi_M(E)} \pi_{A_k}(\mathbf{q}) \tag{3}$$

If $\mathcal{I}_M(A_k)$ is nonergodic then

$$\pi_{A_k}(\mathbf{x}) = 0 \quad \forall \mathbf{x} \in \psi_M(A_k)$$

However in any case

$$\pi_A(E) \not\equiv 0$$

and

$$\sum_{E \subset N_M - A} \pi_A(E) = 1$$

Note that $\pi_A(E)$ (for $E \in N_M - A$) is the probability that in $\mathcal{I}_M(A)$ the users $i \in E$ have a nonempty buffer, and that the users $i \notin E$ have an empty buffer. A very useful property is that these probabilities do not become identically zero when the random walk $\mathcal{I}_M(A)$ is nonergodic.

We will use the distribution $\pi_A(E)$ in order to define a set of drift vectors, that will play a key role in the determining the stability of the random walk $calS_M(L)$. Let $A \subset N_M$ and $\mathbf{r} \in \mathcal{B}_M(A)$, where $\mathcal{B}_M(A)$ is an ergodic face. Consider the vector

$$\mathbf{v}(\mathbf{r}) = (v_1(\mathbf{r}), \ldots, v_M(\mathbf{r}))$$

defined by

$$\text{if } A \neq N_M, \quad v_i(\mathbf{r}) \triangleq \begin{cases} \bar{D}_i(A) & \text{if } i \in A \\ 0 & \text{if } i \notin A \end{cases}$$

$$\text{if } A = N_M, \quad \mathbf{v}(\mathbf{r}) \triangleq \mathbf{D}(\mathbf{r})$$

where

$$\bar{D}_i(A) \triangleq \sum_{\mathbf{x} \in \psi_M(A)} \pi_A(\mathbf{x}) D_i(\mathbf{x}) \qquad (\text{ for } i \in A)$$

is the expectation of the i-th component of $\mathbf{D}(\mathbf{x})$ (for $i \in A$) with respect to the probability distribution $\pi_A(\mathbf{x})$, i. e. the steady state probability distribution of the induced random walk $\mathcal{I}_M(A)$.

An important observation is that the mean jump vector $D(\mathbf{x})$, associated with the random walk $S_M(L)$, is independent of $\mathbf{x}$ as far as $\mathbf{x}$ belongs to a given face $\Phi_M(A)$. Henceforth we will denote the mean jump vector as $D(A)$ in order to show that it depends on the face in which $\mathbf{x}$ belongs (which in turn is completely determined by the set A) rather than on $\mathbf{x}$ itself. Then we can express the components $\bar{D}_i(A)$ as

$$\bar{D}_i(A) = \sum_{E \subset N_M - A} \pi_A(E) D_i(E) \qquad (\text{for } i \in A) \tag{4}$$

The same property is carried to the vector $\mathbf{v}(\mathbf{r})$, for which we will henceforth use the notation

$$\mathbf{v}(A) = (v_1(A), \ldots, v_M(A))$$

We recall that, using the new notation, the vector $\mathbf{v}(A) = (v_1(A), \ldots, v_M(A))$ is defined (for $A \neq N_M$) as:

$$v_i(A) = \begin{cases} \bar{D}_i(A) & \text{if } i \in A \\ 0 & \text{if } i \notin A \end{cases}$$

The vectors $\mathbf{v}(A)$ can be interpreted as drifts for the queues of users $i \in A$ in a system $\mathcal{S}_{M^-}(L \bigcup A)$. The entire set of these vectors may determine the nonergodicity of the random walk $calS_M(L)$. In particular, it was proved in [5] that a random walk of the form of $calS_M(L)$ is transient if there exist $\delta, b, t > 0$, a nonempty set $T \subset \mathcal{R}_+^M$ and a function $f(\cdot) : \mathcal{R}_+^M \to \mathcal{R}$ that satisfy the following four conditions:

$$\begin{array}{lcl}
(T.1): & f(\mathbf{x}) \geq 0 & \forall \mathbf{x} \in \mathcal{R}_+^M \\
(T.2): & f(\mathbf{x}) - f(\mathbf{y}) \leq b\|\mathbf{x}-\mathbf{y}\| & \forall \mathbf{x},\mathbf{y} \in \mathcal{R}_+^M \\
(T.3): & f(\mathbf{x}) \geq t & \forall \mathbf{x} \in T \\
 & f(\mathbf{x}) < t & \forall \mathbf{x} \in \mathcal{R}_+^M - T \\
(T.4): & f(\mathbf{r} + \mathbf{v}(A)) - f(\mathbf{r}) > \delta & \forall \mathbf{r} \in \mathcal{B}_M(A)
\end{array}$$

where (T. 4) holds for all $A \subset N_M$, for which $\mathcal{B}_M(A)$ is an ergodic face as well as for $A = N_M$.

These conditions together are sufficient for transience (and their negation necessary for ergodicity).

3 A Necessary and Sufficient Condition for Ergodicity

We can now proceed to prove the necessary and sufficient condition for ergodicity of $calS_M(L)$. We start by recursively defining a nested set of M conditions, denoted by $C.1, \ldots, C.M$, as follows:

$$(C.1): \ \exists\, i_1 \in A_M \triangleq N_M \ \text{ such that } \ v_{i_1}(A_M) < 0$$

For $k \in \{1, \ldots, M-1\}$, if conditions $(C.1)$ up to $(C.k)$ are satisfied with $i_1, \ldots, i_k$ (distinct), then $(C.(k+1))$ is defined as:

$$(C.k+1): \ \exists\, i_{k+1} \in A_{M-k} \triangleq N_M - \{i_1, \ldots, i_k\} \ \text{ such that } \ v_{i_{k+1}}(A_{M-k}) < 0$$

The interpretation of these conditions is as follows: Starting with condition (C. 1) and by the use of the vector $\mathbf{v}(N_M)$ the existence of a stable user in a system consisting entirely of persistent users is verified. As soon as one such user is found he is moved from the set of persistent users to the set of the typical ones. The procedure continues by examining, in the new system, the existence of a persistent user that is stable and removing him, if one exists, to the set of typical users.

The following Theorem 1 provides a necessary and sufficient condition for ergodicity of the M-user slotted ALOHA system and the underllying random walk $calS_M(L)$,

Theorem 1 *The M-dimensional random walk $S_M(L)$ is ergodic iff there exists a sequence of M distinct integers $i_1, \ldots, i_M \in N_M$ such that for $k = 1, \ldots, M$ condition (C. k) is satisfied with i_k.*

The sufficiency of the above condition is intuitively clear. It merely states that the M-user slotted ALOHA system is stable if each user is stable in a dominant system of the form $\hat{S}_M(A_k)$. A formal proof is given in Appendix A1.

On the other hand, the necessity of the above condition is pretty counter-intuitive, stating that the M-user slotted ALOHA system is unstable unless we can find a dominant system of the form $\hat{\mathcal{S}}_M(A_k)$, that is stable. A formal proof for it is given in Appendix A2.

The following two corollaries provide alternative forms for Theorem 1.

Corollary 1 *The M–dimensional random walk $S_M(L)$ is ergodic iff there exists an $i \in N_M$ such that the $(M-1)$–dimensional random walk $\mathcal{I}_M(\{i\})$ is ergodic and, furthermore, such that $v_i(\{i\}) < 0$.*

To prove Corollary 1 we make use of Lemma 2, proved in Appendix A. Observe that, if its condition is satisfied, then

- By the ergodicity of $\mathcal{I}_M(\{i\})$ we have that there exists a sequence of distinct integers $i_1, \ldots, i_{M-1}$ where $\{i_1, \ldots, i_{M-1}\} = N_M - \{i\}$ such that for all $n = 1, \ldots, M-1$, condition $(C.n)$ is satisfied with i_n.
- By $v_i(\{i\}) < 0$ we have that condition $(C.M)$ is satisfied with $i_M = i$.

Then the hypothesis of Lemma 2 (see Appendix A) is satisfied and the random walk $calS_M(L)$ is ergodic.

If, on the other hand, there exists $i \in N_M$ such that $\mathcal{I}_M(\{i\})$ is ergodic but $v_i(\{i\}) > 0$, then, as explained above, there exists a sequence of distinct integers $i_1, \ldots, i_{M-1}$, where $\{i_1, \ldots, i_{M-1}\} = N_M - \{i\}$, such that for $n = 1, \ldots, M-1$ condition $(C.n)$ is satisfied with i_n. However condition $(C.M)$ is not satisfied with the unique element of the set $N_M - \{i_1, \ldots, i_{M_1}\} = \{i\}$. Then, following Lemma 1, we conclude that $calS_M(L)$ is nonergodic.

If now there exists no $i \in N_M$ such that $\mathcal{I}_M(\{i\})$ is ergodic then there exists some $k < M-1$ and a sequence of distinct integers $\{i_1, \ldots, i_k\}$ such that for $n = 1, \ldots, k$ condition $(C.n)$ is satisfied with i_n but there exists no $j \in N_M - \{i_1, \ldots, i_k\}$ such that condition $(C.k+1)$ is satisfied with j. Again, following Lemma 1, we conclude that $calS_M(L)$ is nonergodic.

The proof is completed.

The above theorem can be interpreted as follows: If we can find a system that dominates $S_M(L)$ and that is ergodic, then, obviously, $S_M(L)$ is also ergodic. If on the other hand all systems dominant to $S_M(L)$ are nonergodic then so is $S_M(L)$.

Corollary 2 *The M–dimensional random walk $S_M(L)$ is ergodic if*

$$v_i(\{i\}) < 0 \quad , \quad \forall\, i \in N_M$$

while it is unstable if

$$v_i(\{i\}) < 0 \quad , \quad \textit{for some } i \in N_M$$

Proof The proof is straightforward by observing that if (14) is true then all component queues are stable, which implies that $\mathcal{S}_M(L)$ is ergodic.
On the other hand, if for some i we have that

$$v_i(\{i\}) > 0$$

then there can be no sequence $i_1, \ldots, i_M$ such that (C. n) is satisfied with i_n, since this would contradict the observation made earlier (section 3).

Corollary 2 establishes the fact that there exists a set of well defined "drift" vectors $\delta_i \triangleq v_i(\{i\})$ that completely determine the stability of the random walk $calS_M(L)$. Unfortunately, computing these drifts constitutes, in general, a formidable problem, since they are expressed in terms of the stationary distribution of $(M-1)$-dimensional random walks. Still they can be used to obtain the following necassary condition for stability of the M-user slotted ALOHA system, that already improves the one given in [5].

Corollary 3 *The M–user ALOHA system with arrival rate vector λ and transmission probability vector $\mathbf{p}$ is ergodic only if there exist i_1, i_2 satisfying:*

$$\lambda_{i_1} < p_{i_1} \prod_{j \neq i_1} (1 - p_j)$$

and

$$\lambda_{i_2} < p_{i_2} \prod_{j \neq i_1, i_2} (1 - p_j) - \frac{p_{i_2}}{1 - p_{i_2}} \lambda_{i_1}$$

The above corollary is a straightforward application of Theorem 1 and its proof is not presented here.

A Appendix

A.1 Sufficiency of Theorem 1

Here we prove the sufficiency of Theorem 1. We use induction on M.
For $M = 2$ the result is well known (see [5]).
Assume that it holds for $M - 1$. We will prove it for M. To do so we assume that the condition in Theorem 1 is satisfied and we will show that $calS_M(L)$ is ergodic.

Observe that for all sets $A_1 \subset N_M$; $\mathcal{I}_M(A_1)$ is equivalent to $\mathcal{S}_{M-1}(L \bigcup A_1)$. Then by the induction hypothesis and by the hypothesis that

$$\exists\, i_1, \ldots, i_M : \{ i_1, \ldots, i_M \} = N_M \ni (C.k) \text{ is satisfied with } i_k \text{ for } k = 1, \ldots, M,$$

we conclude that the hypothesis of Lemma 2 is satisfied for $\mathcal{S}_{M-1}(L \bigcup \{ i_M \})$, hence $\mathcal{S}_{M-1}(L \bigcup \{ i_M \})$ is ergodic.

Consider now the dominant system with M users $1, \ldots, M$ in which:

- user i_M transmits with probability p_M even when having an empty buffer
- users i ($\forall i \neq i_M$) transmit normally
- there is an arbitrary number of users $i \in L$ that transmit always with probability p_i.

We consider the ergodicity of the system of users $1, \ldots, M$. This system is described by an M–dimensional Markov Chain. Ergodicity of $\mathcal{I}_M(\{ i_M \})$ implies that all users $i \neq i_M$ are stable in the sense of Definition 1. Furthermore, i_M has average arrival rate smaller than its average service rate, hence (see [4]) it is also stable. Then by Proposition 1, this system is ergodic and, since it dominates $calS_M(L)$, $calS_M(L)$ is also ergodic.

The proof is completed.

A.2 Necessity of Theorem 1

Here we provide a proof for the necessity part of Theorem 1. We start by proving couple of statements that will be needed in our proof.

Lemma 1 *Assume that there exists a sequence of $k-1$ distinct integers $i_1, \ldots, i_{k-1}$ ($i_n \in N_M$ for $n = 1, \ldots, k-1$) such that $(C.n)$ is satisfied with i_n, for $n = 1, \ldots, k-1$. Assume furthermore that $(C.k)$ is satisfied with both $i_k, i_{k+1} \in N_M - \{ i_1, \ldots, i_{k-1} \}$. Then for $n = 1, \ldots, k+1$, condition $(C.n)$ is satisfied with i_n.*

Proof We use dominant systems. In particular we define a system that dominates $calS_M(L)$ and consists of

- users $i \in L$ and $i \in \{ i_1, \ldots, i_{k-1} \}$ who behave as in $calS_M(L)$.
- users $i \in N_M - \{ i_1, \ldots, i_{k-1} \}$ who transmit in every slot with probability p_i regardless of their status.

By our hypothesis, as well as by Lemma 2 (Appendix A), we conclude that the random walk that describes the queues of users $i_1, \ldots, i_{k-1}$ is ergodic. Hence users $i_1, \ldots, i_{k-1}$ are all stable. Furthermore both users i_k and i_{k+1} have average service rate larger than their average arrival rate, hence they are also stable. This implies that the random walk that describes the queues of users $i_1, \ldots, i_{k+1}$ is ergodic.

If in the above system we modify the status of user i_k, i. e. force him to behave as in $calS_M(L)$ (that is to transmit only when he has a non-empty buffer) we obtain a new system, obviously dominated by the one described in the beginning of the proof. Then the random walk that describes the queues of users $i_1, \ldots, i_{k+1}$ is ergodic, hence user i_{k+1} is stable. Since user i_{k+1} always transmits, his stability implies that his average service rate is larger than his average arrival rate. The proof is completed.

The following corollary folows directly from Lemma 1.

Corollary 4 *Assume that*

> *there exists sequence of distinct integers* $i_1, \ldots, i_k \in N_M$ *such that for* $n = 1, \ldots, k$ *condition* $(C.n)$ *is satisfied with* i_n
> *condition* $(C.(k+1))$ *is not satisfied with any* $i \in N_M - \{i_1, \ldots, i_k\}$.

Then, for all $i \in \{1, \ldots, k\}$, *condition* $(C.n)$ *is not satisfied with any* $i \in N_M - \{i_1, \ldots, i_k\}$.

We proceed now with the proof of the necessity in Theorem 1. Observe first that, for the extreme case $k = 0$, the necessity of the condition in Theorem 1 becomes trivially true. We will omit this case and consider only the case $k \geq 1$.

Proof of (only-if) part of Theorem 1 We use induction on M . For $M = 2$ the assertion of Lemma 1 is well known (see for example [3, 4, 8]). Assume now that the assertion of Lemma 1 is valid for $\hat{M} = 2, \ldots, M-1$ and for all $k = 1, \ldots, \hat{M}-1$, where we use the symbol $\hat{M}$ as a running index in place of M (as it was used in Lemma 1) in order not to confuse it with the value of M in this proof. We prove then that it is also valid for $\hat{M} = M$ and for all $k = 1, \ldots, M-1$. To do so we assume that there exists a sequence of distinct integers $i_1, \ldots, i_k$ where $\{i_1, \ldots, i_k\} \subset N_M$, such that for $n = 1, \ldots, k$ condition $(C.n)$ is satisfied with i_n. We assume furthermore that condition $(C.k+1)$ is not satisfied with any $i \in N_M - \{i_1, \ldots, i_k\}$. We will show then that $calS_M(L)$ is nonergodic.

In the following we will use the notation:

$$\begin{aligned} J_k &\triangleq \{i_1, \ldots, i_k\} \\ J_k^c &\triangleq N_M - J_k \end{aligned}$$

We prove the nonergodicity of $calS_M(L)$ in 2 steps:

Step 1: We show that for all $A_T \subset N_M$ (with T elements, where $T < M$), $\mathcal{I}_M(A_T)$ is nonergodic unless

$$J_k^c \subset A_T \tag{5}$$

To do so we define

$$A_T^c \triangleq N_M - A_T = \{m_1, \ldots, m_{M-T}\}$$

Observe that, unless equation (5) is valid,

$$A_T^c \bigcap J_k^c \neq \emptyset$$

Thus assume that

$$A_T^c \bigcap J_k^c = \{m_{R+1}, \ldots, m_{M-T}\}$$

or, equivalently,

$$\left.\begin{aligned} m_1, \ldots, m_R &\notin J_k^c \\ m_{R+1}, \ldots, m_{M-T} &\in J_k^c \end{aligned}\right\} \tag{6}$$

The fact that the induced random walk $\mathcal{I}_M(A_T)$ is equivalent to a random walk of the form $\mathcal{S}_{M-T}(L \bigcup A_T)$ and that $M - T < M$, justify the use of the induction hypothesis for the completion of Step 1.

Observe that if (6) holds, then at most R conditions (i. e. (C. 1), ..., (C. R)) can be satisfied. The reason is that, even if there exist distinct $j_1, \dots, j_R$, ($\{j_1, \dots, j_R\} = \{m_1, \dots, m_R\}$) such that for $n = 1, \dots, R$, (C. n) is satisfied with j_n, condition (C. (R + 1)) will not be satisfied with any $j \in \{m_1, \dots, m_{M-T}\} - \{j_1, \dots, j_R\} = \{m_{R+1}, \dots, m_{M-T}\}$. (This is a direct consequence of the observation immediately preceeding Lemma 1; namely that for all $n = 1, \dots, k+1$, hence also for $n = R + 1 \leq k + 1$, condition $(C.n)$, is not satisfied with any $j \in J_k^c \supset \{m_{R+1}, \dots, m_{M-T}\}$.)

Thus, if (6) is valid, then there exists a $\hat{k} < M - T$ such that for $n = 1, \dots, \hat{k}$, condition $(C.n)$ is satisfied with j_n (where j_n 's are distinct) but condition $(C.\hat{k}+1)$ is not satisfied with any $j \in \{m_1, \dots, m_{M-T}\} - \{\ j_1, \dots, j_{\hat{k}}\ \}$. Then by the induction hypothesis the walk $\mathcal{S}_{M-T}(L \bigcup A_T)$, is nonergodic, and hence $\mathcal{I}_M(A_T)$ is nonergodic either.

This completes Step 1. Notice that we do not claim that whenever (5) is valid $\mathcal{I}_M(A_T)$ will be ergodic. We are only interested in excluding from the set of ergodic faces, the faces $\mathcal{B}_M(A_T)$ for which A_T does not satisfy (5).

Step 2: We apply the results in [4], as described in section 2, to prove nonergodicity of $calS_M(L)$.

We define $f(\cdot) : \mathcal{R}_+^M \to \mathcal{R}$ by

$$f(\mathbf{r}) \triangleq \sum_{i \in J_k^c} |r_i| \tag{7}$$

Obviously $f(\mathbf{r})$ satisfies conditions (T. 1) - (T. 3).

We want to show that $f(\mathbf{r})$ also satisfies condition (T. 4). By the result obtained in Step 1, we are only interested in subsets A_T —with T elements— for which equation (5) holds (i. e. $J_k^c \in A_T$). Then by equation (5) we have that

$$T \geq M - k$$

or, equivalently

$$M - T \leq k \tag{8}$$

Observe now that:

for any $n = 1, \dots, k+1$, condition (C. n) is not satisfied with any $j \in J_k^c$;

Hence

$$v_i(A_T) > 0 \quad , \quad \forall \ \ i \in J_k^c \subset A_T \tag{9}$$

Furthermore by equation (6) we have that

$$r_i > 0 \quad , \quad \forall \ \ i \in J_k^c \subset A_T \ \ ; \qquad \mathbf{r} \in \Phi_M(A_T) \tag{10}$$

Equations (9) and (10) imply that

$$|r_i + v_i(A_T)| = r_i + v_i(A_T) \quad , \quad \forall \quad i \in J_k^c \subset A_T \quad , \quad \mathbf{r} \in \Phi_M(A_T)$$

Hence we have that

$$\begin{aligned} &\forall \; \mathbf{r} \in \Phi_M(A) \supset J_k^c \; : \\ &f(\mathbf{r} + \mathbf{v}(A)) - f(\mathbf{r}) = \textstyle\sum_{i \in J_k^c}(\, |r_i(A_T) + v_i(A_T)| - |r_i(A_T)|) = \\ &\qquad\qquad = \textstyle\sum_{i \in J_k^c}(\, r_i + v_i(A_T) - r_i \,) \end{aligned}$$

or, equivalently,

$$\forall A \ni J_k^c \subset A : \; f(\mathbf{r} + \mathbf{v}(A)) - f(\mathbf{r}) \geq \delta \, \forall \, \mathbf{r} \in \mathcal{B}_M(A) \tag{11}$$

where

$$\delta \triangleq \min_{A: J_k^c \subset A} \{ \sum_{i \in J_k^c} v_i^2(A) \} \tag{12}$$

Since, now

$$\sum_{i \in J_k^c} v_i(A) > 0 \quad , \quad \forall A \ni J_k^c \subset A$$

and since (12) is a maximization problem over a finite set we have that

$$\delta > 0 \tag{13}$$

Then by equations (11) and (13) we have that

$$\begin{aligned} \exists \, \epsilon \; (\epsilon = \delta/2) \quad \ni \quad \forall \; A \supset J_k^c \; \text{for which} \quad & \mathcal{B}_M(A) \text{ is ergodic } : \\ & f(\mathbf{r} + \mathbf{v}(A)) - f(\mathbf{r}) \; > \; \epsilon > 0 \, \forall \, \mathbf{r} \; \in \; \mathcal{B}_M(A) \end{aligned}$$

Hence $f(\mathbf{r})$ satisfies also condition (T. 4) and $calS_M(L)$ is nonergodic. This concludes Step 2 as well as the proof of Lemma 1.

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