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Homogenization and Control of Lattice Structures

by

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Abstract: Under certain natural conditions the dynamics of large, low mass lattice structures with a regular infrastructure are well approximated by the dynamics of continua, e.g., trusses may be modeled by beam equations. Using a technique from the mathematics of asymptotic analysis called *homogenization,* we show how such approximations may be derived in a systematic way which avoids errors made using "direct" averaging methods. We also develop a model for the combined problem of homogenization and control of vibrations in lattice structures and present some preliminary analysis of this problem.

Key Words: Homogenization, asymptotic analysis, flexible structures, vibration control.

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Contents

1. Homogenization of regular structures

It is now generally accepted that large, low mass lattice structures, e.g., trusses, are natural for space applications.[1] Their large size and repetitive infrastructure require special techniques for structural analysis to cope with the large number of degrees of freedom. Approximations of such systems by continua provide a simple means for comparing structural characteristics of lattices with different configurations, and they are effective in representing macroscopic vibrational modes and structural response due to temperature and load inputs. Our approach to the construction of such models is based on a technique for asymptotic analysis called *homogenization*. It has been widely used in mathematical physics for the treatment of composite systems like porous media for which one wishes to have an effective approximating system with parameters which are constant across the structure.[2] Before developing the general features of the method and applying it to the treatment of lattice structures, we shall make a few remarks on other work on continuum models which has appeared in the recent structural mechanics literature.

Noor, et. al. (1978) use an energy method to derive a continuum approximation for trusses with triangular cross sections in which the modal displacements of the truss are related to a linearly varying displacement field for an equivalent bar. Plates with a lattice infrastructure are also treated. In (Dean and Tauber 1959) and (Renton 1969) exact analytical expressions for the solutions of trusses under load were derived using finite difference calculus. By expressing the difference operators in terms of Taylor's series Renton (1970) was able to derive continuum approximations to the finite difference equations resulting in expressions for equivalent plate stiffnesses, for example. In a recent paper Renton (1984) used this approach to give equivalent beam properties for trusses, which complements the earlier work of Noor, Anderson and Greene (1978) and Nayfeh and Hefzy (1978). (See also (Anderson 1981).)

In most cases a continuum model is associated with the original (lattice) structure by averaging the parameters of the lattice over some natural volume (e.g., of a cell of the structure)

[1]See, for example, (Aswani 1982), (Juang 1984), (Mikulus 1978), (Renton 1970), and (Taylor and Balakrishnan 1984), and the references therein.

[2]See, for example, the papers of Larsen (1975 1976), Keller (1977), and the reports of Babuska (1975) for applications and discussions of design techniques.

and identifying the averaged parameter value (mass density, stress tensor, etc.) with the corresponding distributed parameter in the continuum model. A specific form for the continuum model is postulated at the outset of the analysis; e.g., a truss with lattice structure will be approximated by a beam, with the beam dynamical representation assumed in advance. While this approach has an appealing directness and simplicity, it has some problems.

First, it is very easy to construct an example in which the approximate model obtained by averaging the parameters over a cell is not a correct approximation to the system behavior. This is done in the next section.[3] Second, one cannot use this procedure to obtain "corrections" to the approximation based on higher order terms in an expansion, which may sometimes be done in an asymptotic analysis. These terms can be used to describe the microscopic behavior (e.g., local stresses) in the structure. Third, the averaging method (averaging the parameters over space) does not apply in a straightforward way to systems with a random structure, since the appropriate averaging procedure may not be obvious.[4] Fourth, the method cannot be naturally imbedded in an optimization procedure; and controls and state estimates based on the averaged model may not be accurate reflections of controls and state estimates derived in the course of a unified optimization - averaging procedure. In particular, the method does not provide a systematic way of estimating the degree of suboptimality of controls and state estimates computed from the idealized model.

In this work we use a totally different technique called *homogenization* from the mathematical theory of asymptotic analysis to approximate the dynamics of structures with a repeating cellular structure. Homogenization produces the distributed model as a consequence of an asymptotic analysis carried out on a rescaled version of the physical system model.

Unlike the averaging method, homogenization can be used in combination with optimization procedures; and it can yield systematic estimates for the degree of suboptimality of controls and estimators derived from idealized models. While our results are stated in terms of simple struc-

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[3]See the numerical experiments in (Bourgat 1978).

[4]Homogenization methods do apply to systems with a randomly heterogeneous structure, see (Papanicolaou and Varadhan 1982) and (Kunnemann 1983). We shall not treat such systems in this paper.

tures, they demonstrate the feasibility of the method; and they suggest its potential in the analysis of structures of realistic complexity.

In section 2 we give an example derived from (Bensoussan, Lions, and Papanicolaou 1978) illustrating some of the subtleties of homogenization, particularly in the context of control problems. In section 3 we derive a homogenized representation for the dynamics of a lattice structure undergoing transverse deflections. We show that the behavior of the lattice is well approximated by the Timenshenko beam equation; and we show that this equation arises naturally as the limit of the lattice dynamics when the density of the lattice structure goes to infinity in a well defined way. The problem of vibration control of a lattice is posed and discussed in section 4. In section 5 we derive a diffusion approximation for the thermal conductivity of a one-dimensional lattice structure. This property is useful in analyzing new materials for large space structures. An operational calculus for homogenization is sketched in the Appendix.

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2. A one-dimensional example

From (Bensoussan, Lions, and Papanicolaou 1978) we have the following example:

$$-\frac{d}{dx}\left[a^{\epsilon}(x) \frac{du^{\epsilon}(x)}{dx} \right] = f(x), \quad x \in (x_0, x_1) \tag{2.1}$$

$$u^{\epsilon}(x_0) = 0 = u^{\epsilon}(x_1)$$

where $a^{\epsilon}(x) \triangleq a(x/\epsilon)$, and $a(y)$ is periodic in y with period Y_0, $a(y) \geq \alpha > 0$. It is simple to show that

$$|| u^{\epsilon} ||_{H^1}^2 \triangleq \int_{x_0}^{x_1} | u^{\epsilon}(x) |^2 + \left| \frac{du^{\epsilon}(x)}{dx} \right|^2 dx \leq c \tag{2.2}$$

and so, $u^\epsilon \rightarrow u$ weakly in the Hilbert space H^1.[5] Moreover,

$$a^\epsilon \rightarrow M(a) \triangleq \frac{1}{Y_0} \int_0^{Y_0} a(y)\, dy \tag{2.3}$$

and it is natural to suppose that $u^\epsilon \rightarrow u$ with the limit defined by

$$- \frac{d}{dx} \left[M(a) \frac{d}{dx} u(x) \right] = f(x), \quad x \in (x_0, x_1) \tag{2.4}$$

$$u(x_0) = u(x_1)$$

This is untrue in general (Bensoussan, Lions, and Papanicolaou 1978, pp. 8-10). The correct limit is given by

$$- \frac{d}{dx} \left[\overline{a} \frac{d}{dx} u(x) \right] = f(x), \quad x \in (x_0, x_1) \tag{2.5}$$

$$u(x_0) = u(x_1)$$

with

$$\overline{a} \triangleq \left[M\left(\frac{1}{a} \right) \right]^{-1} \tag{2.6}$$

In general, $M(a) > \overline{a}$; and so, the error in identifying the limit, (2.4) versus (2.5), is fundamental.

The system (2.4) corresponds to averaging the parameter $a^\epsilon(x)$ over a natural *cell;* a procedure similar to that used in the past to define continuum models for lattice structures. As (2.5) shows, the actual averaging process can be more subtle than one might expect, even for simple problems.

2.1. Homogenization of the example

To see how (2.5) arises, we can use the method of *multiple scales* which applies to a variety of problems. Suppose

[5] Here $H^1 \triangleq \{ u \in L^2(x_0, x_1) : \|u\|_{H^1} < \infty \}$

$$u^\epsilon(x) = u^\epsilon(x, \frac{x}{\epsilon}) = u_0(x, \frac{x}{\epsilon}) + \epsilon\, u_1(x, \frac{x}{\epsilon}) + \cdots \tag{2.7}$$

that is, we suppose that u^ϵ depends on the slow scale x and the fast scale $y \triangleq x/\epsilon$; and we adopt an *ansatz* which reflects this dependence. Using the identity

$$\frac{d}{dx}\,[\,u\,(x, \frac{x}{\epsilon})\,] = \frac{\partial u}{\partial x} + \frac{1}{\epsilon}\,\frac{\partial u}{\partial y}, \quad y = \frac{x}{\epsilon} \tag{2.8}$$

then (2.1) may be rewritten as

$$-\,(\frac{\partial}{\partial x} + \frac{1}{\epsilon}\,\frac{\partial}{\partial y})\cdot\{\ a\,(y)\,(\frac{\partial}{\partial x} + \frac{1}{\epsilon}\,\frac{\partial}{\partial y})\ [\,u_0 + \epsilon\, u_1 + \cdots\,]\,\} = f \tag{2.9}$$

Simplifying and equating coefficients of like powers of ϵ, we find first that

$$-\,\frac{1}{\epsilon^2}\,\frac{\partial}{\partial y}\,[\,a\,(y)\,\frac{\partial}{\partial y}\,u_0\,] = 0. \tag{2.10}$$

The assumptions on $a(y)$ imply

$$u_0(x, y) = u_0(x) \tag{2.11}$$

i.e., no y-dependence. The coefficients of ϵ^{-1} satisfy

$$\{\,\frac{\partial}{\partial y}\,[\,a\,(y)\,\frac{\partial}{\partial x}\,u_0\,] + \frac{\partial}{\partial x}\,[\,a\,(y)\,\frac{\partial}{\partial y}\,u_0\,] - \frac{\partial}{\partial y}\,[\,a\,(y)\,\frac{\partial}{\partial y}\,u_1\,]\,\} = 0 \tag{2.12}$$

or

$$\frac{\partial}{\partial y}\,[\,a\,(y)\,\frac{\partial}{\partial y}\,u_1\,] = -\,\frac{\partial a}{\partial y}\,\frac{\partial u_0}{\partial x} \tag{2.13}$$

If we look for u_1 in the form

$$u_1(x, y) = -\,\chi(y)\,\frac{\partial u_0}{\partial x} + \hat{u}_1(x)\,, \tag{2.14}$$

then the *corrector* $\chi(y)$ must satisfy

$$-\,\frac{d}{dy}\,[\,a\,(y)\,\frac{d}{dy}\,\chi(y)\,] = -\,\frac{da}{dy} \tag{2.15}$$

and be periodic. That is,

$$a(y)\frac{d\chi}{dy} = a(y) + c \tag{2.16}$$

which has a periodic solution (unique up to an additive constant in y) if and only if

$$\frac{1}{Y_0}\int_0^{Y_0}\left[1 + \frac{c}{a(y)}\right]dy = 0 \tag{2.17}$$

which implies

$$c = -\left[M\left(\frac{1}{a}\right)\right]^{-1} \triangleq \overline{a} \tag{2.18}$$

We obtain an equation for $u_0(x)$ from the solvability condition for $u_2(x,y)$. Equating the coefficients of ϵ^0 in the expansion, we have

$$-\frac{\partial}{\partial y}\left[a(y)\frac{\partial}{\partial y}u_2\right] - \frac{\partial}{\partial y}\left[a(y)\frac{\partial}{\partial x}u_1\right] \tag{2.19}$$

$$-\frac{\partial}{\partial x}\left[a(y)\frac{\partial}{\partial y}u_1\right] - \frac{\partial}{\partial x}\left[a(y)\frac{\partial}{\partial x}u_0\right] = f(x)$$

This has a solution $u_2(x,y)$, periodic in y if and only if

$$\left\{\frac{1}{Y_0}\int_0^{Y_0}\left[a(y) + \frac{\partial}{\partial y}\left[a(y)\chi(y)\right] - a(y)\frac{\partial}{\partial y}\chi(y)\right]dy\right\}\cdot\frac{d^2u_0(x)}{dx^2} \tag{2.20}$$

$$+ f(x) = 0$$

where we have used (2.14). The integral of the second term is zero, since it is the integral of the derivative of a periodic function over one period. Using (2.16) and (2.18), (2.20) reduces to

$$-\overline{a}\frac{d^2u_0}{dx^2} + f(x) = 0 \tag{2.21}$$

(plus the boundary conditions) which is (2.5) (2.6).

2.2. Control and homogenization of the one dimensional system

One of the simplest stochastic control problems associated with the preceding system is defined by the Hamilton - Jacobi - Bellman equation

$$- a\left(\frac{x}{\epsilon}\right) \frac{d^2 u^\epsilon}{dx^2} - \frac{1}{\epsilon} b\left(\frac{x}{\epsilon}\right) \frac{du^\epsilon}{dx}$$

$$= \inf_{v \in \mathbb{R}} \left[\frac{1}{2} v^2 + g(x,y)\, v \frac{du^\epsilon}{dx} - c\, u^\epsilon \right] \tag{2.22}$$

$$x \in O, \quad u^\epsilon(x) = 0 \text{ on } \Gamma \triangleq \partial O$$

where O is an open interval in $\mathbb{R}$, and each function $a(y)$, $b(y)$, and $g(x,y)$ is periodic in y with period Y_0. We assume that $a(y) \geq \alpha > 0$ and that $c > 0$, and that the controls v take values in $\mathbb{R}$.

This Bellman equation corresponds to the stochastic control problem

$$u^\epsilon(x) = \inf_{v(\cdot)} J^\epsilon[v(\cdot)]$$

$$J^\epsilon[v(\cdot)] = E_x \left\{ \int_0^{\tau_x} l\left(x^\epsilon, \frac{x^\epsilon}{\epsilon}, v\right) \left[e^{-\int_0^t c\left(x^\epsilon, \frac{x^\epsilon}{\epsilon}, v\right) ds} \right] dt \right\} \tag{2.23}$$

$$dx^\epsilon(t) = \sigma\left(x^\epsilon, \frac{x^\epsilon}{\epsilon}\right) dw(t) + \frac{1}{\epsilon} b\left(x^\epsilon, \frac{x^\epsilon}{\epsilon}\right) dt + G\left(x^\epsilon, \frac{x^\epsilon}{\epsilon}, v\right) dt$$

$$x^\epsilon(0) = x \in O, \quad t \geq 0.$$

with $\sigma^2(x,y) \triangleq a(y)$, $b(x,y) = b(y)$, $G(x,y,v) = g(x,y)v$, $l(x,y,v) = \frac{1}{2} v^2$, and $c(x,y,v) = c$, a constant in (2.22). Each function in (2.23) is assumed to be periodic in y with period one. We are interested in the behavior of the optimal cost and control law for (2.22) in the limit as $\epsilon \to 0$. The stochastic control problem (2.23) was treated in (Bensoussan, Boccardo, and Murat 1984); the analysis here uses different arguments which emphasize the computational aspects of the system.

Evaluating the infimum in (2.22), we have the nonlinear system

$$a\left(\frac{x}{\epsilon}\right) u^\epsilon_{xx} + \frac{1}{\epsilon} b\left(\frac{x}{\epsilon}\right) u^\epsilon_x - c u^\epsilon_x - \frac{1}{2} g^2\left(x, \frac{x}{\epsilon}\right) (u^\epsilon_x)^2 = 0 \tag{2.24}$$

$$x \in O, \quad u^\epsilon(x)\,\big|_\Gamma = 0.$$

The analysis of the control problem involves homogenization of this system.

Let

$$A_1 = a(y)\,\partial_{yy} + b(y)\,\partial_y \tag{2.25}$$

with its formal adjoint defined by

$$A_1^* = \partial_y\,[\,a(y)\,\partial_y \,\bullet\,] - \partial_y\,[(b(y) - a_y(y))\,\bullet\,]. \tag{2.26}$$

The problem

$$\begin{gathered} A_1^*\, m = 0, \quad y \to m(y) \quad \text{periodic} \\ m > 0, \quad \int_Y m(y)\, dy = 1 \end{gathered} \tag{2.27}$$

has a unique solution $m(\bullet)$ on $Y = S^0$, the unit circle, with

$$0 < \underline{m} \le m(y) \le \overline{m} < \infty. \tag{2.28}$$

So $m(\bullet)$ is a density on Y. We *assume* that $b(\bullet)$ is *centered*

$$\int_Y m(y)\, b(y)\, dy = 0. \tag{2.29}$$

As a consequence the system

$$\begin{gathered} A_1\, \chi(y) = b(y) \\ y \to \chi(y) \ \text{ periodic}, \quad \int_Y \chi(y)\, dy = 0 \end{gathered} \tag{2.30}$$

has a well defined solution. $\chi(\bullet)$ is the *corrector* associated with the problem.

As before we look for u^ϵ in the form

$$u^\epsilon(x) = u^\epsilon(x, y) = u_0(x, y) + \epsilon\, u_1(x, y) + \cdots, \quad y \triangleq x/\epsilon \tag{2.31}$$

and we use

$$\begin{gathered} \partial_x\, \phi(x, y) = \phi_x(x, y) + \frac{1}{\epsilon}\, \phi_y(x, y), \quad y = x/\epsilon \\ \partial_{xx}\, \phi(x, y) = \phi_{xx}(x, y) + \frac{2}{\epsilon}\, \phi_{xy}(x, y) + \phi_{yy}(x, y). \end{gathered} \tag{2.32}$$

Substituting in (2.24), we have

$$a(y)[u_{0xx} + \frac{2}{\epsilon} u_{0xy} + \frac{1}{\epsilon^2} u_{0yy}] + a(y)[\epsilon u_{1xx} + 2 u_{1xy} + \frac{1}{\epsilon} u_{1yy}]$$

$$+ a(y)[\epsilon^2 u_{2xx} + 2\epsilon u_{2xy} + u_{2yy}]$$

$$+ \frac{1}{\epsilon} b(y)[u_{0x} + \frac{1}{\epsilon} u_{0y}] + \frac{1}{\epsilon} b(y)[\epsilon u_{1x} + u_{1y}] \tag{2.33}$$

$$+ \frac{1}{\epsilon} b(y)[\epsilon^2 u_{2x} + \epsilon u_{2y}] - c[u_0 + \epsilon u_1 + \epsilon^2 u_2]$$

$$- \frac{1}{2} g^2(x,y)[(u_{0x} + \epsilon u_{1x} + \epsilon^2 u_{2x}) + \frac{1}{\epsilon}(u_{0y} + \epsilon u_{1y} + \epsilon^2 u_{2y})]^2 = O(\epsilon^2)$$

The last term is

$$- \frac{1}{2} g^2(x,y)[(\frac{1}{\epsilon^2} u_{0y}^2 + \frac{2}{\epsilon} u_{0x} u_{0y} + 2u_{0x} u_{1y} + u_{0x}^2) \tag{2.34}$$

$$+ \epsilon(2u_{0x} u_{1x} + 2u_{1x} u_{1y} + 2u_{1y} u_{2y} + 2u_{0x} u_{2y})] + O(\epsilon^2)$$

Equating coefficients of like powers of ϵ, we obtain

$$(\epsilon^{-2}) \quad a(y) u_{0yy} + b(y) u_{0y} - \frac{1}{2} g^2(x,y) u_{0y}^2 = 0 \tag{2.35a}$$

$$(\epsilon^{-1}) \quad a(y) u_{1yy} + b(y) u_{1y} + 2a(y) u_{0xx} \tag{2.35b}$$

$$+ b(y) u_{0x} - g^2(x,y) u_{0x} u_{0y} = 0$$

$$(\epsilon^0) \quad a(y) u_{2yy} + b(y) u_{2y} + 2a(y) u_{1xy} + b(y) u_{1x} \tag{2.35c}$$

$$+ a(y) u_{0xx} - cu_0 - \frac{1}{2} g^2(x,y) u_{0x}^2 - g^2(x,y) u_{0x} u_{1x} = 0.$$

Choosing $u_0(x,y) = u_0(x)$, which must be justified, satisfies (2.35a). We can then solve (2.35b) by choosing

$$u_1(x,y) = -\chi(y) u_{0x}(x) + \hat{u}_1(x). \tag{2.36}$$

Equation (2.35c) has a solution for $u_2(x,y)$ if

$$\int_Y m(y) \{ -2a(y)\chi_y u_{0xx} - b(y)\chi_y u_{0x} + a(y) u_{0xx} \tag{2.37}$$

$$- cu_0 - \frac{1}{2} g^2(x,y)(1 - 2\chi_y) u_{0x}^2 \} dy = 0.$$

This gives an equation for $u_0(x)$

$$q \ u_{0xx} - c \ u_0 - \frac{1}{2} \Gamma \ u_{0x}^2 = 0 \tag{2.38}$$

where

$$q \triangleq \int_Y m(y) \{ a(y) [1 - 2\chi_y(y)] - \chi(y) \, b(y) \} \, dy \tag{2.39}$$

$$\Gamma \triangleq \int_Y m(y) \, g^2(x,y) [1 - 2\chi_y(y)] \, dy.$$

Remark. From the definition of A_1 and the corrector $\chi(y)$ we have

$$\int_Y m(y) \, b(y) \, \chi(y) \, dy = \int_Y [a(y) \chi_{yy} + b(y) \chi_y] \chi(y) \, m(y) \, dy \tag{2.40}$$

$$= \int_Y \chi(y) \, \partial_{yy} [a(y) \chi(y) \, m(y)] \, dy - \int_Y \chi(y) \, \partial_y [b(y) \chi(y) \, m(y)] \, dy$$

Also, using (2.30),

$$\int_Y m(y) \, b(y) \, \chi(y) \, dy = \int_Y \chi(y) \, a(y) \, m(y) \, \chi_{yy}(y) \, dy \tag{2.41}$$

$$- \int_Y \chi(y) \, b(y) \, m(y) \, \chi_y \, dy + 2 \int_Y \chi(y) \, \chi_y \, \partial_y [a(y) \, m(y)] \, dy$$

Adding these two expressions, we have

$$2 \int_Y m(y) \, b(y) \, \chi(y) \, dy$$

$$= 2 \int_Y \chi(y) \, a(y) \, m(y) \, \chi_{yy} \, dy + 2 \int_Y \chi(y) \, \chi_y \, [a \ m]_y \, dy \tag{2.42}$$

$$= -2 \int_Y \partial_y [\chi \ a \ m] \chi_y \, dy + 2 \int_Y \chi(y) \, \chi_y \, [a \ m]_y \, dy - 2 \int_Y \chi_y \, a(y) \, m(y) \, \chi_y \, dy$$

Thus, q may be rewritten as

$$q = \int_Y m(y) \{ a(y) [1 - 2 \chi_y + \chi_y^2] \} \, dy \tag{2.43}$$

$$= \int_Y m(y) \, a(y) [1 - \chi_y]^2 \, dy$$

and clearly $q \geq 0$.

The term q in (2.39) summarizes the effects of the averaging process on the uncontrolled system. The homogenization process interacts with the control system through the term Γ, whose form would be difficult to "guess" from simple averaging procedures.

3. Continuum Model for a Simple Structural Mechanical System

3.1. Problem definition

Consider the truss shown in Figure 1 (undergoing an exaggerated deformation). We shall assume that the truss has a regular (e.g., triangular) cross-section and no "interlacing" supports. We assume that the displacements of the system are "small" in the sense that no components in the system buckle. We are interested in describing the dynamical behavior of the system when the number of cells (a unit between two (triangular) cross sections) is large; that is, in the limit as

$$\epsilon \triangleq l/L \to 0. \tag{3.1}$$

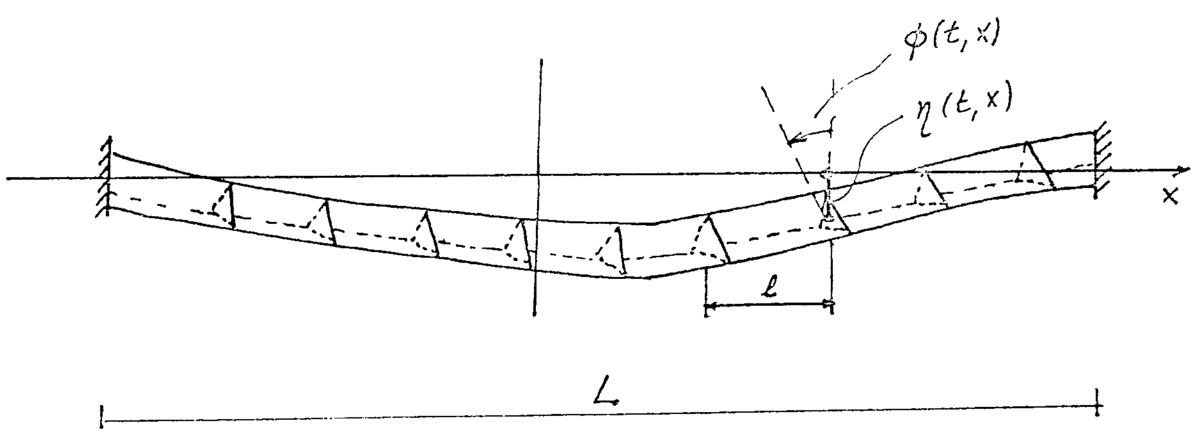

Figure 1. Deformed truss with regular cross-section.

We shall make several assumptions to simplify the analysis. First, we shall assume that the triangular sections are essentially rigid, and that all mobility of the system derives from the flexibility of the members connecting the triangular components. Second, we shall ignore damping and frictional effects in the system. Third, we shall confine attention to small transverse displacements $\eta(t,x)$ and small in plane rotations $\phi(t,x)$ as indicated in Figure 1, ignoring longitudinal and out of plane motions and torsional twisting. Fourth, we shall assume that the mass of the triangular cross members dominates the mass of the interconnecting links.

Systems of this type have been considered in several papers including (Noor et al. 1978) (Nayfeh and Hefzy 1978) (Anderson 1981) and (Renton 1984). In those papers a continuum beam model was hypothesized and effective values for the continuum system parameters were computed by averaging the associated parameters of the discrete system. Our approach to the problem is based on homogenization-asymptotic analysis and is quite different.

The assumptions simplify the problem substantially, by suppressing the geometric structure of the truss. We can retain this structure by writing dynamical equations for the nodal displacements of the truss members. For triangular cross sections nine parameters describe the displacements of each sectional element. The analysis which follows may be carried over to this case, but the algebraic complexity prevents a clear presentation of the main ideas. As suggested in (Noor et al. 1978) one should use a symbolic manipulation program like MACSYMA to carry out the complete details of the calculations. We shall take up this problem on another occasion; for now we shall treat the highly simplified problem which, as we shall see, leads to the one dimensional Timoshenko beam.

We shall begin by reformulating the system in terms of a discrete element model as suggested in (Crandal et al. 1980); see Figure 2. In this model we follow the displacement $\eta_i(t)$ and rotation $\phi_i(t)$ of the i^{th} mass M. The bending springs (k_b^i) tend to keep the system straight by keeping the masses parallel and the shearing springs (k_s^i) tend to keep the masses perpendicular to the connecting links. We assume small displacements and rotations so the approximations

$$\sin \phi_i(t) \cong \phi_i(t) \tag{3.2}$$

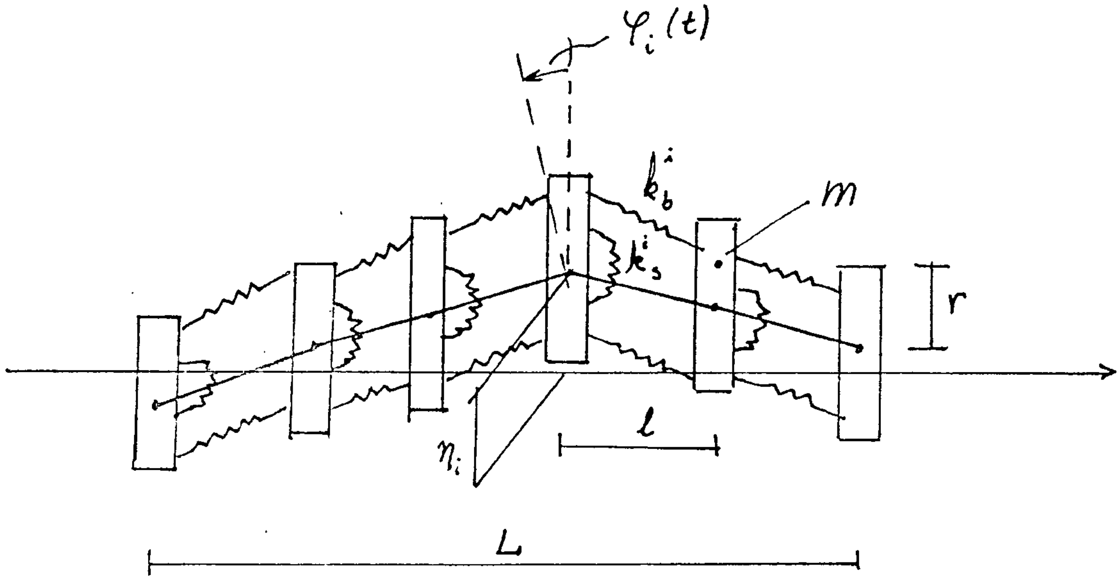

Figure 2. A lumped parameter model of the simplified truss system

$$\tan^{-1} [\, \eta_i(t)/l \,] \cong \eta_i(t)/l$$

are valid.

In this case the (approximate) equations of motion of the i^{th} mass are[6]

$$\frac{d^2\phi_i}{dt^2} \cong \frac{1}{r}\, k_s^i \left\{ \left[\frac{\eta_{i+1}(t) - \eta_i(t)}{l} \right] - \phi_i(t) \right\} \tag{3.3a}$$

$$+ \; S_l^{-} \left\{ k_b^i \left[\frac{\phi_{i+1}(t) - \phi_i(t)}{l} \right] \right\}$$

$$\frac{d^2\eta_i}{dt^2} = S_l^i \left\{ k_s^i \left[\frac{\eta_i(t) - \eta_{i+1}(t)}{l} \right] - \phi_i(t) \right\} \tag{3.3b}$$

where we have normalized $M = 1$ and defined

[6]The spring constants depend on i since they represent the restorative forces of flexed bars, bent by different amounts.)

$$S_l^- \eta_i \triangleq \frac{1}{l} [\eta_{i-1} - \eta_i] \tag{3.4}$$

and similarly for $S_l^- \phi_i$.

To proceed, we shall introduce the nondimensional variable $\epsilon = l/L$ and rewrite the system (3.3) as

$$r \frac{d^2\phi_i^\epsilon}{dt^2} = \frac{1}{r} K_s^i \{ \nabla^{\epsilon +} \eta_i^\epsilon(t) - \phi_i^\epsilon \} + \nabla^{\epsilon +} \{ K_b^i \nabla^{\epsilon -} \phi_i^\epsilon(t) \} \tag{3.5}$$

$$\frac{d^2\eta_i^\epsilon}{dt^2} = -\nabla^{\epsilon -} \{ K_s^i [\nabla^{\epsilon +} \eta_i^\epsilon(t) - \phi_i^\epsilon(t)] \}$$

where

$$K_s^i = k_s^i L, \quad K_b^i = k_b^i L \tag{3.6}$$

$$\nabla^{\epsilon +} \eta_i = \frac{1}{\epsilon} [\eta_{i+1} - \eta_i], \quad \nabla^{\epsilon -} \eta_i = \frac{1}{\epsilon} [\eta_i - \eta_{i-1}].$$

Normalizing $L = 1$, we associate a position $x \in [-\frac{1}{2}, \frac{1}{2}]$ with each mass; and we introduce the notation

$$\eta(t, x_i) = \eta_i(t), \quad \phi(t, x_i) = \phi_i(t). \tag{3.7}$$

Having normalized $L = 1$, we have $\epsilon = l$ and $x_{i+1} = x_i + l = x_i + \epsilon$. Let $\tilde{Z} = \{x_i\}$ be the set of all points in the system. In this notation

$$(\nabla^{\epsilon +} \eta)(t, x) = \frac{1}{\epsilon} [\eta(t, x + \epsilon) - \eta(t, x)] \tag{3.8}$$

$$(\nabla^{\epsilon -} \eta)(t, x) = \frac{1}{\epsilon} [\eta(t, x) - \eta(t, x - \epsilon)], \quad x \in \tilde{Z}$$

and the system is

$$\frac{d^2\phi^\epsilon(t, x_i)}{dt^2} = K_s(x_i) \{ \nabla^{\epsilon +} \eta^\epsilon(t, x_i) - \phi^\epsilon(t, x_i) \}$$

$$+ r \nabla^{\epsilon +} \{ K_b(x_i) \nabla^{\epsilon -} \phi^\epsilon(t, x_i) \} \tag{3.9}$$

$$\frac{d^2\eta^\epsilon(t, x_i)}{dt^2} = -\nabla^{\epsilon -} \{ K_s(x_i) [\nabla^{\epsilon +} \eta^\epsilon(t, x_i) - \phi^\epsilon(t, x_i)] \}, \quad x \in \tilde{Z}$$

The scaling of (3.9) may be interpreted in the following way: Formally, at least, the right sides of both terms in (3.9) are $O(\epsilon^{-2})$. This implies that the time variations are taking place in the "fast time scale" $\tau = t/\epsilon$. Also, the spatial variations are taking place in the "microscopic scale" x which varies in ϵ-increments (e.g., $x_{i+1} = x_i + \epsilon$). Introducing the macroscopic scale $z = \epsilon x$, and the slow time scale $\sigma = \epsilon \tau$, we may rescale (3.9) and observe its dynamical evolution on the large space-time scale on which macroscopic events (e.g., "distributed phenomena") take place.

Rewritten in this spatial scale, the system becomes

$$\frac{d^2\phi^\epsilon(t,\frac{z_i}{\epsilon})}{dt^2} = \frac{1}{\epsilon} K_s(\frac{z_i}{\epsilon}) \{ \delta^{\epsilon +} \eta(t,\frac{z_i}{\epsilon}) - \phi^\epsilon(t,\frac{z_i}{\epsilon}) \}$$

$$+ \frac{1}{\epsilon^2} \delta^{\epsilon +} \{ r K_b(\frac{z_i}{\epsilon}) \delta^{\epsilon -} \phi^\epsilon(t,\frac{z_i}{\epsilon}) \} \tag{3.10a}$$

$$\frac{d^2\eta^\epsilon(t,\frac{z_i}{\epsilon})}{dt^2} = \frac{1}{\epsilon^2} \delta^{\epsilon -} \{ K_s(\frac{z_i}{\epsilon}) [\delta^{\epsilon +} \eta(t,\frac{z_i}{\epsilon}) - \epsilon \phi^\epsilon(t,\frac{z_i}{\epsilon})] \} \tag{3.10b}$$

where

$$\delta^{\epsilon\pm} = \epsilon \nabla^{\epsilon\pm} = O(1) \text{ in } \epsilon. \tag{3.11}$$

The essential mathematical problem is to analyze the solutions ϕ^ϵ, η^ϵ of (3.10) in the limit as $\epsilon \to 0$.

3.2. Mathematical analysis

To proceed, we shall generalize the problem (3.10) slightly by allowing K_s and K_b to depend on z as well as z/ϵ. This permits the restoring forces in the model system to depend on the large scale shape of the structure as well as on local deformations. We use the method of multiple scales; that is, we look for solutions of (3.10) in the form

$$\eta^\epsilon(t) = \eta^\epsilon(t,z,y), \quad \phi^\epsilon(t) = \phi^\epsilon(t,z,y), \quad y = \frac{z}{\epsilon} \tag{3.12}$$

and we have

$$K_s = K_s(z,y), \quad K_b = K_b(z,y), \quad y = \frac{z}{\epsilon} \tag{3.13}$$

On smooth functions $\psi(z,\frac{z}{\epsilon})$ the operators $\delta^{\epsilon\pm}$ satisfy

$$(\delta^{\epsilon+}\psi)(z,y) = \psi(z+\epsilon, y+1) - \psi(z,y)$$

$$= \psi(z,y+1) - \psi(z,y) + \psi(z+\epsilon, y+1) - \psi(z,y+1) \tag{3.14a}$$

$$= (S^{+}\psi)(z,y) + \epsilon\frac{\partial\psi}{\partial z}(z,y+1) + \frac{1}{2}\epsilon^2\frac{\partial^2\psi}{\partial z^2}(z,y+1) + O(\epsilon^3)$$

$$(\delta^{\epsilon-}\psi)(z,y) = \psi(z,y) - \psi(z-\epsilon, y-1)$$

$$= \psi(z,y) - \psi(z,y-1) + \psi(z,y-1) - \psi(z-\epsilon,y-1)$$

$$= (S^{-}\psi)(z,y) - \epsilon\frac{\partial\psi}{\partial z}(z,y+1) + \frac{1}{2}\epsilon^2\frac{\partial^2\psi}{\partial z^2}(z,y-1) + O(\epsilon^3) \tag{3.14b}$$

We assume that ϕ^ϵ and η^ϵ may be represented by

$$\phi^\epsilon(t,z,y) = \phi_0(t,z) + \epsilon\,\phi_1(t,z,y) + \cdots \tag{3.15}$$

$$\eta^\epsilon(t,z,y) = \eta_0(t,z) + \epsilon\,\eta_1(t,z,y) + \cdots$$

and substituting (3.15) in (3.10) and using (3.13) (3.14), we arrive at a sequence of equations for (ϕ_0, η_0), (ϕ_1, η_1),... by equating the coefficients of like powers of ϵ.

Starting with ϵ^{-2}, ϵ^{-1}, ϵ^{0}, $\cdots$, we have

$$\frac{1}{\epsilon^2} S^{+}[\,r\,K(z,y)\,S^{-}\phi_0(t,z)\,] = 0 \tag{3.16}$$

which is trivially true from (3.14b) (3.15). The same term involving $\eta_0(t,x)$ from (3.10b) is trivially satisfied by the assumption (3.15). Continuing

$$\frac{1}{\epsilon}[\,S^{+}\{\,rK_b(z,y)\,S^{-}\phi_1(t,z,y)\,\} \tag{3.17}$$

$$+\;K_s(z,y)\{\,S^{+}\eta_0(t,z) - \phi_0(t,z)\,\}\,] = 0$$

which may be solved by using the corrector $\chi_\phi(z,y)$ and taking

$$\phi_1(t,z,y) = \chi_\phi(z,y)\,\phi_0(t,z) \tag{3.18}$$

with

$$S^+\{\, rK_b(z,y)\, S^- \chi_\phi(z,y)\,\} = K_s(z,y) \tag{3.19}$$

If we regard z as a parameter in (3.19), then there exists a solution χ_ϕ, unique up to an additive constant, if $K_b(z,\cdot)$, $K_s(z,\cdot)$ are periodic in y, if there exist constants A and B so that

$$0 < A \le K_b(z,y) \le B < \infty \tag{3.20}$$

and if the average of $K_s(z,\cdot)$ is zero

$$\frac{1}{L}\int_{-L/2}^{L/2} K_s(s,y)\, dy = 0 \tag{3.21}$$

which holds if the system is pinned at the ends as indicated in Figure 2. Let us *assume* that (3.20) (3.21) hold, and

$$0 < A \le K_s(z,y) \le B < \infty \tag{3.22}$$

(which we shall need shortly).

Considering (3.10b), the $O(\epsilon^{-1})$ term in the asymptotic expansion is

$$\frac{1}{\epsilon}\,[\, S^-\{\, K_s(z,y)\,(S^+\eta_1(t,z,y) - \phi_0(t,z))\,\}\,] = 0\,. \tag{3.23}$$

Again we introduce the corrector $\chi_\eta(z,y)$, and take η_1 in the form

$$\eta_1(t,z,y) = \chi_\eta(z,y)\,\phi_0(t,z) \tag{3.24}$$

which gives the equation for the corrector

$$S^-\{\, K_s(z,y)\,[\, S^+\chi_\eta(z,y) - 1\,]\,\} = 0 \tag{3.25}$$

or

$$S^-\{\, K_s(z,y)\, S^+\chi_\eta(z,y)\,\} = K_s(z,y) - K_s(z,y-1) \tag{3.26}$$

By hypothesis the right side in (3.26) is periodic in y and has zero average (3.21). Hence, (3.26) has a periodic solution, unique to an additive constant.

Continuing, the $O(\epsilon^0)$ term in (3.10a) is

$$\begin{aligned}
&S^+ \{rK_b(z,y)\, S^- \phi_2(t,z,y)\} + K_s(z,y)\,[\, S^+ \eta_1(t,z,y) - \phi_1(t,z,y)\,] \\
&+ K_s(z,y) \frac{\partial \eta_0}{\partial z}(t,z) + S^+ \{rK_b(z,y) \frac{\partial}{\partial z} \phi_1(t,z,y)\} \\
&+ S^+ \{rK_b(z,y) \frac{\partial^2}{\partial z^2} \phi_0(t,z,y)\} + \frac{\partial}{\partial z} \{rK_b(z,y+1)\} \frac{\partial}{\partial z} \phi_0(t,z) \\
&+ \frac{\partial^2}{\partial z^2} \{ rK_b(z,y+1)\} \phi_0(t,z) - \frac{\partial^2 \phi_0}{\partial t^2} = 0 .
\end{aligned} \tag{3.27}$$

This should be regarded as an equation for ϕ_2 as a function of y with (t,z) as parameters. In this sense the solvability condition is as before, the average of the sum of all terms on the left in (3.27), except the first, should be zero. We must choose ϕ_0 so that this in fact occurs; and that defines the *limiting system.*

Using the correctors (3.18) (3.24), we must have

$$\begin{aligned}
&\underset{(y)}{\text{Average}} \{\frac{\partial^2 \phi_0}{\partial t^2} - \frac{\partial^2 \phi_0}{\partial z^2} [\, S^+ (rK_b(z,y)) + S^+ (rK_b(z,y)\, \chi_\phi(z,y))\,] \\
&- \frac{\partial \phi_0}{\partial z} [\, \frac{\partial}{\partial z} (rK_b(z,y+1))\,] - \frac{\partial \eta_0}{\partial z} K_s(z,y) \\
&- \phi_0 [\, \frac{\partial^2}{\partial z^2} (rK_b(z,y+1)) + S^+ (rK_b(z,y) \frac{\partial}{\partial z} \chi_\phi(z,y)) \\
&+ K_s(z,y)\,(S^+ \chi_\eta(z,y) - \chi_\phi(z,y))\,]\,\} = 0
\end{aligned} \tag{3.28}$$

Defining the functions $EI(z)$, $G(z)$ by the associated averages in (3.28), the averaged equation is

$$\frac{\partial^2 \phi_0}{\partial t^2} = \frac{\partial}{\partial z} (EI(z) \frac{\partial \phi_0}{\partial z}) + G(z) \frac{\partial \eta_0}{\partial z} - H(z)\, \phi_0 \tag{3.29}$$

which is the angular component of the Timoshenko beam system (Crandall et al. 1980 p. 348).

Arguing in a similar fashion, we can derive the equation for the macroscopic approximation displacement of the lattice system in terms of the "equivalent" displacement $\eta_0(t,z)$ in the Timoshenko beam system

$$\frac{\partial^2 \eta_0}{\partial t_2} = \frac{\partial}{\partial z} [\, N(z) (\frac{\partial \eta_0}{\partial z} - \phi_0(t,z))\,] \tag{3.30}$$

3.3. Summary

We have shown that a simplified model of the dynamics of the truss with rigid cross sectional area may be well approximated by the Timoshenko beam model in the limit as the number of cells (proportional to L/l) becomes large. The continuum beam model emerges naturally in the analysis, as a consequence of the periodicity and the scaling.

To compute the approximate continuum model, one must solve (3.19) and (3.26) (numerically) for the correctors and then compute the parameters in (3.29) (3.30) by numerically averaging the quantities in (3.28) (and its analog for (3.10b)) which involve the correctors and the data of the problem.

4. Homogenization and Stabilizing Control of Lattice Structures

In this section we show that the process of deriving effective "continuum" approximations to complex systems may be developed in the context of optimal control designs for those systems. This procedure is more effective than the procedure of first deriving homogeneous/continuum approximations for the structure, designing a control algorithm for the idealized model, and then adapting the algorithm to the physical model. In fact, separation of optimization and asymptotic analysis can lead to incorrect algorithms or ineffective approximations, particularly in control problems where nonlinear analysis (e.g., of the Bellman dynamic programming equation) is required.

We shall apply the combined homogenization - optimization procedure described in section 1 (based on (Bensoussan, Boccardo, and Murat 1984)) to the problem of controlling the dynamics of lattice structures like the truss structure analyzed in the previous section. We shall only formulate a prototype problem of this type and discuss its essential features.

Consider the model for the lattice structure analyzed in section 3 with control actuators added. The truss shown in Figure 1 is again constrained to move in the plane and torsional motion is excluded to simplify the model and confine attention to the basic ideas. Now, however, we include a finite number of actuators acting to cause transverse motions. The truss with actuator forces indicated by arrows is shown in Figure 3. The corresponding discrete element model is shown in Figure 4.

Suppose that the physical actuators act along the local normal to the truss midline as shown in the figures, and that the forces are small so that linear approximations to transcendental functions (e.g., $\sin \phi_i \cong \phi_i$, etc.) are valid. Then the controlled equations of motion of the discrete element system are (recall equation (3.5))

$$r \frac{d^2 \phi_i^\epsilon}{dt^2} = \frac{1}{r} K_s^i \{ \nabla^{\epsilon +} \eta_i^\epsilon(t) - \phi_i^\epsilon \} + \nabla^{\epsilon +} \{ K_b^i \nabla^{\epsilon -} \phi_i^\epsilon(t) \} \tag{4.1}$$

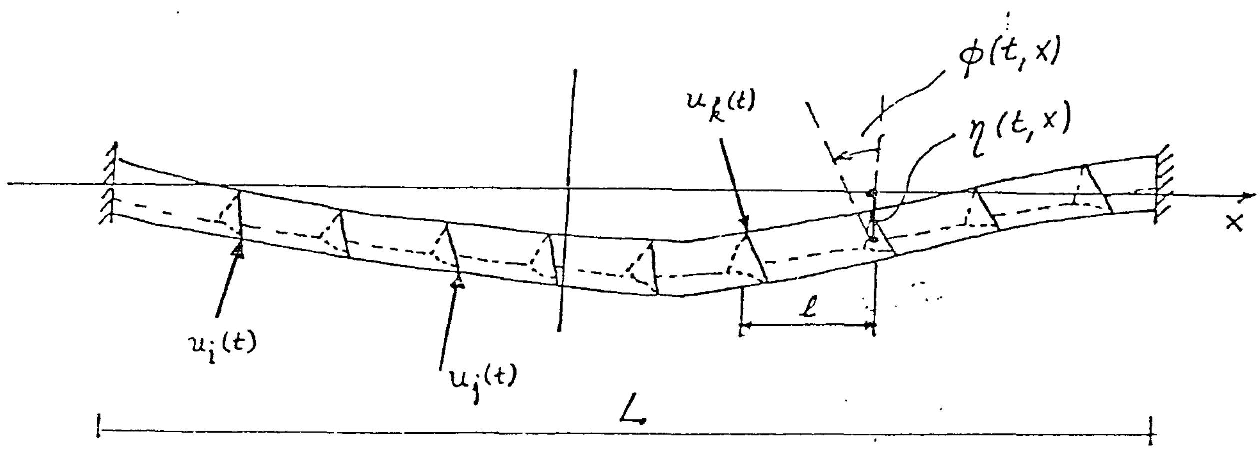

Figure 3. Truss with transverse actuator forces.

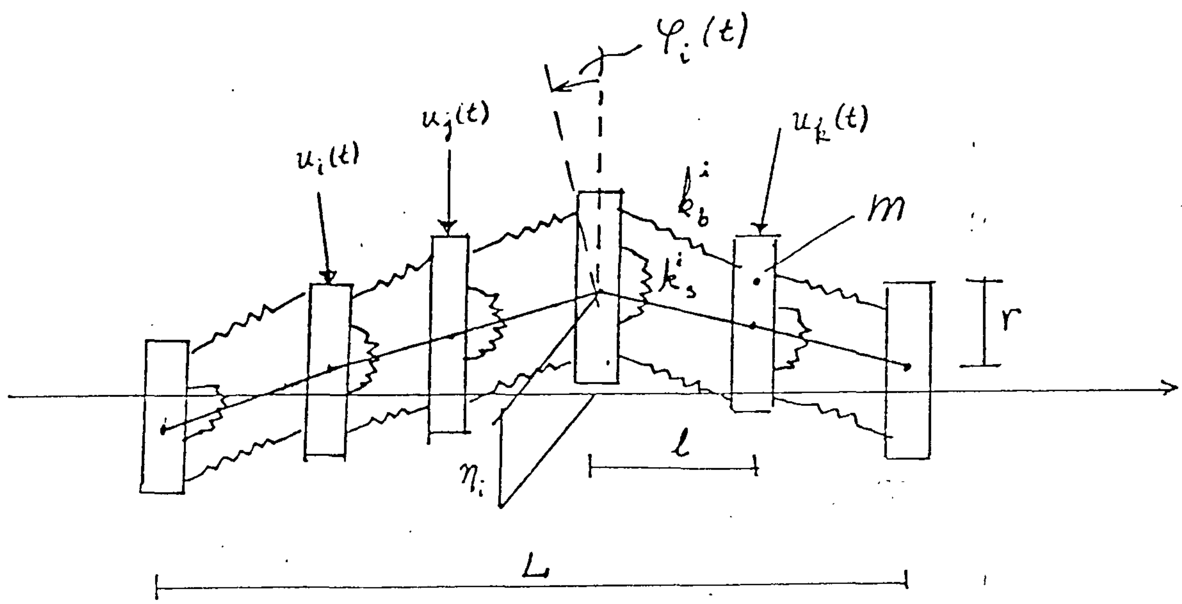

Figure 4. Discrete element model of the controlled truss.

$$\frac{d^2\eta_i^\epsilon}{dt^2} = -\nabla^{\epsilon-}\{K_s^i[\nabla^{\epsilon+}\eta_i^\epsilon(t) - \phi_i^\epsilon(t)]\} + \sum_{j=1}^{m}\delta(i,i_j)\,u_j(t)$$

where the notation in (3.6) has been used,

$$\delta(i,j) = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \tag{4.2}$$

and i_j, $j = 1,\ldots,m$ are the locations of the actuators. Hence, if $\delta(i,i_j) = 0$ for all $j = 1,\ldots,m$ there is no actuator located at the i^{th} point which corresponds to the physical point $x \in [0,L]$. The number m of actuators is given at the outset and does not, of course, vary with the scaling.

The control problem is to select the actuator forces as functions of the displacements and velocities of components of the structure to damp out motions of the structure. Measurements would typically be available from a finite number of sensors located along the structure. We shall not elaborate on this component of the model, and shall instead assume that the entire state can be measured. To achieve the stabilization, we shall associate a cost functional with the system (4.1). Let

$$u(t) = [\, u_1(t), \ldots, u_m(t)\,]^T \tag{4.3}$$

be the vector of control forces, and

$$\begin{aligned} J^\gamma[\, u(\cdot)\,] = \int_0^\infty \sum_{i=1}^N \{\, & a_i [\, \phi_i^\epsilon(t)\,]^2 + b_i [\, \eta_i^\epsilon(t)\,]^2 \\ & + \alpha_i \, [\, \dot\phi_i^\epsilon(t)\,]^2 + \beta_i \, [\, \dot\eta_i^\epsilon(t)\,]^2 \\ & + \sum_{j=1}^m \delta(i, i_j)\, u^2(t) \,\}\; e^{-\gamma t}\, dt \end{aligned} \tag{4.4}$$

where (a_i, b_i) and (α_i, β_i) are non-negative weights. Formally, the control problem is to select $\delta(i, i_j)\, u_j(t)$, $i = 1,\ldots,N$, $j = 1,\ldots,m$ to achieve

$$\inf_{u(\cdot)} J^\gamma[\, u(\cdot)\,] \tag{4.5}$$

subject to (4.1) (4.2) and the appropriate boundary conditions. The case $\gamma \to 0$ corresponds to stabilization by feedback.

The analysis of this control problem is based on the scaling used in section 3, equations (3.5) - (3.11). Let $\tau = t/\epsilon$ be the fast time scale, then

$$\begin{aligned} J^\gamma[\, u(\cdot)\,] = \int_0^\infty \epsilon \sum_{i=1}^N \{\, & a_i \, [\, \hat\phi_i^\epsilon(\tau)\,]^2 + b_i \, [\, \hat\eta_i^\epsilon(\tau)\,]^2 \\ & + \alpha_i \, \epsilon^2 [\, \dot{\hat\phi}_{i_\epsilon}(\tau)\,]^2 + \beta_i \, \epsilon^2 [\, \dot{\hat\eta}_i^\epsilon(\tau)\,]^2 \\ & + \sum_{j=1}^m \delta(i, i_j)\, u^2(\tau) \,\}\; d^{-\epsilon\gamma\tau}\, d\tau. \end{aligned} \tag{4.6}$$

with $\hat\phi_i^\epsilon(\tau) = \phi_i^\epsilon(\epsilon\tau)$, etc.

Let $(\phi, \dot\phi, \eta, \dot\eta)$ be the state vector of the system (4.1) with $\phi = [\, \phi_1, \cdots, \phi_N\,]^T$ and similarly for the other terms. Let $V = V^{\epsilon,\gamma}(\phi, \dot\phi, \eta, \dot\eta)$ be the optimal value function for the problem (4.1) (4.6). Then the Bellman problem associated with (4.1) (4.6) is

$$\epsilon \sum_{i=1}^{N} [\,\dot{\phi} V_{\phi_i} + \dot{\eta}_i\, V_{\eta_i}\,]$$

$$+ \epsilon \sum_{i=1}^{N} \{ \frac{1}{r^2} K_s^i [\nabla^{\epsilon +} \eta_i - \phi_i] + \frac{1}{r} \nabla^{\epsilon +} [K_b^i \nabla^{\epsilon -} \phi_i] \} V_{\dot{\phi}_i}$$

$$+ \epsilon \sum_{i=1}^{N} \{ -\nabla^{\epsilon -} [K_s^i (\nabla^{\epsilon -} \eta_i - \phi_i)] \} V_{\dot{\eta}_i} \tag{4.7}$$

$$\min_{u_j \in U_{ad}} \{ \epsilon \sum_{i=1}^{N} \sum_{j=1}^{m} [\, \delta(i, i_j)\, u_j\, V_{\dot{\eta}_i} + \delta(i, i_j)\, u_j^2] \}$$

$$+ \epsilon \sum_{i=1}^{N} [\, a_i\, \phi_i^2 + b_i\, \eta_i^2 + \epsilon^2 (\alpha_i\, \dot{\phi}^2 + \beta_i\, \dot{\eta}_i^2)] - \epsilon \gamma V = 0 \, .$$

REMARKS:

(1) Note that the minimization in (4.7) is well defined if the admissible range of the control forces is convex since the performance measure has been assumed to be quadratic in the control variables $\delta(i, i_j)\, u_j$.

(2) Since we have not included the effects of noise in the model, the state equations are deterministic and the Bellman equation (4.7) is a first order system. To "regularize" the analysis, at least along the lines followed in conventional homogenization analysis, it is useful to include the effects of noise in the model and exploit the resulting coercivity properties in the asymptotic analysis.

(3) If we introduce the macroscopic spatial scale $z = \epsilon x$, the mesh $\{x_i\}$, and the variables

$$\phi(t, z_i) = \phi_i^\epsilon(t), \quad \dot{\phi}(t, z_i) = \dot{\phi}_i^\epsilon(t), \quad \text{etc.} \tag{4.8}$$

then the sums may be regarded as Riemann approximations to integrals over the macroscopic spatial scale z. The asymptotic analysis of (4.7) with this interpretation defines the mathematical problem constituting simultaneous homogenization - optimization for this case.

5. Effective conductivity of a periodic lattice

In this section we consider a version of a heat conduction problem treated by Kunnemann. Simple expressions for thermal properties of composite materials, have been derived in the past using homogenization techniques. The derivation of effective conductivities for discrete structures is useful for assessing the behavior of such structures in variable environmental conditions.

5.1. Problem definition

Let $Z = \{0, \pm 1, \pm 2, \cdots\}$ and $Z^d = Z \times \cdots \times Z$ (d times) be a d-dimensional lattice. Let $\epsilon > 0$ be a number small relative to 1. We want to describe the effective conduction of thermal energy on the ϵ-spaced lattice ϵZ^d. Let $e_i = (0,0,\ldots 0,1,0,\ldots 0)^T$ with 1 in the i^{th} position, $i = 1,2,\ldots,d$. If x is a point in ϵZ^d, then $x \pm \epsilon e_i$, $1 \leq i \leq d$, are the nearest neighbors of x. Let $a_{\pm}(x)$, $x \in Z^d$, $1 \leq i \leq d$, be the two functions defined on the lattice, and *assume*

$$a_i(x) \triangleq a_{+}(x) = a_{i-}(x + e_i), \quad x \in Z^d, \quad 1 \leq i \leq d \tag{5.1a}$$

$$0 < A \leq a_i(x) \leq B < \infty, \quad \forall\, x \in Z^d, \quad 1 \leq i \leq d \tag{5.1b}$$

$$a_i(x) \text{ is periodic with period } l \geq 1 \tag{5.1c}$$

in each direction,[7] $1 \leq i \leq d$.

Next let

$$a_{i\pm}^{\epsilon}(x) = a_{i\pm}\left(\frac{x}{\epsilon}\right), \quad x \in \epsilon Z^d, \quad 1 \leq i \leq d. \tag{5.2}$$

Equation (5.1b) means that the conduction process is reversible and that the conductivity $a_i(x)$ is a "bond conductivity," i.e., independent of the direction in which the bond $(x, x + e_i)$ is used by the process. Equation (5.2) means that the configuration of bond conductivities $a_{i\pm}^{\epsilon}(\cdot)$ on ϵZ^d is simply $a_{i\pm}(\cdot)$ on ϵZ^d "viewed from a distance." Assumption (5.1c) imposes a regularity condition on the physics of the conduction process. An assumption like this is essential for existence of a limit as $\epsilon \to 0$. In one dimension the situation is illustrated in Figure 5a,b. A system similar to this with random bond conductivities was treated by Kunnemann (1983) by

[7]The period may be different in different directions.

imposing some ergodicity properties on the bond conductivities.

One can associate with this system a random (jump) process

$$\{ X^{\epsilon}(t, x),\ t \geq 0,\ x \in \epsilon Z^{d} \}$$

on the ϵ-spaced lattice.[8] In effect, as $\epsilon \to 0$, $\{ X^{\epsilon} \}$ converges to a Brownian motion on the lattice; and the main result of the analysis is an expression for the diffusion matrix $Q \triangleq [\, q_{ij} \,;\ i, j = 1,2,\ldots,d \,]$ of this process. This matrix describes the macroscopic diffusion of

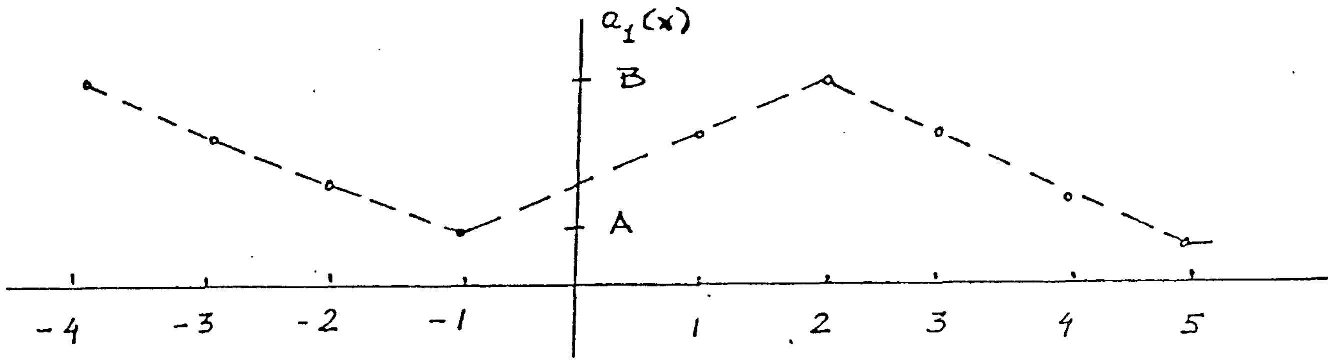

Figure 5a. Conductivity on unscaled lattice with period $l = 6$.

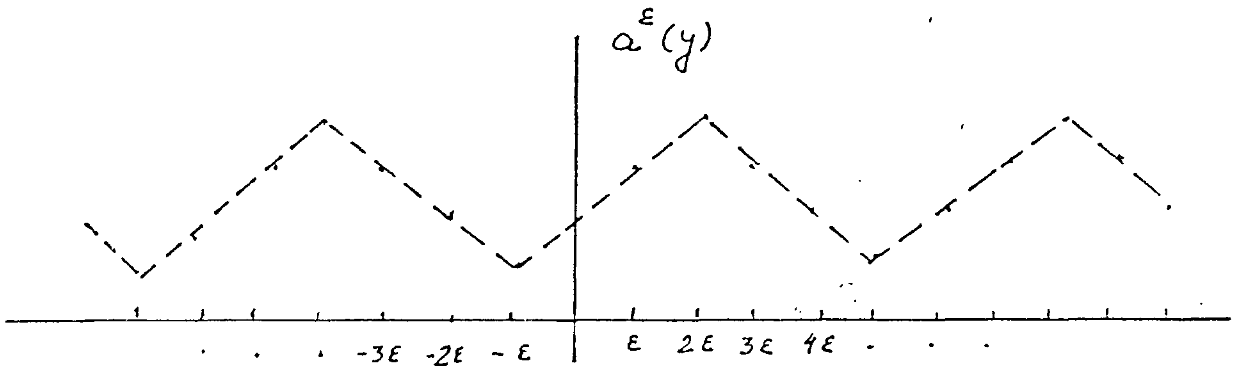

Figure 5b. Conductivity on ϵ-scaled lattice, $y = \epsilon x$, $x \in Z$, period $\epsilon l = 6\epsilon$.

[8]Definition of this process is not necessary for the analysis, but it bolsters the intuition.

thermal energy in the system. It is the effective conductivity.

We shall carry out the asymptotic analysis of this system in the limit as $\epsilon \rightarrow 0$ using homogenization. Let

$$(\nabla_i^{\epsilon -} u)(x) \triangleq \frac{1}{\epsilon}[u(x - \epsilon e_i) - u(x)] \tag{5.3a}$$

$$(\nabla_i^{\epsilon +} u)(x) \triangleq \frac{1}{\epsilon}[u(x + \epsilon e_i) - u(x)] \tag{5.3b}$$

$$x \in \epsilon Z^d, \quad 1 \leq i \leq d,$$

for any u square summable on ϵZ^d or square integrable on $\mathbb{R}^d$ with e_i the i^{th} natural basis vector in $\mathbb{R}^d$. Then

$$\begin{aligned} \frac{\partial u^\epsilon(t,x)}{\partial t} &= -\sum_{i=1}^{d} \nabla_i^{\epsilon -}[a_i(\frac{x}{\epsilon})\nabla_i^{\epsilon +} u^\epsilon(t,x)] \\ &\triangleq L^\epsilon u^\epsilon(t,x) \end{aligned} \tag{5.4}$$

is the diffusion equation on the ϵ-spaced lattice with density $u^\epsilon(t,x)$ and conductivity $a_i(x/\epsilon)$. We are interested in an effective parameter representation of the thermal conduction process as $\epsilon \rightarrow 0$.

Remark: Although probabilistic methods are not required in the analysis, the associated probabilistic framework has a great deal of intuitive appeal. The operator L^ϵ may be identified as the infinitesimal generator of a pure jump process $X^\epsilon(s)$ in the "slow" time scale $s \triangleq \epsilon^2 t$; (Breiman 1968). Moreover, L^ϵ is selfadjoint on ϵZ^d with the inner product

$$(f,g) \triangleq \sum_{x \in Z^d} f(x)g(x). \tag{5.5}$$

Hence, the backward and forward equations for the process $X^\epsilon(s)$ are, respectively,

$$\begin{aligned} \frac{\partial p^\epsilon(y,t \mid x)}{\partial t} &= [L^\epsilon p^\epsilon(y,t \mid \cdot)](x) \\ \frac{\partial p^\epsilon(y,t \mid x)}{\partial t} &= [L^\epsilon p^\epsilon(\cdot,t \mid x)](y) \end{aligned} \tag{5.6}$$

So the process is "symmetric" in the sense of Markov processes (Breiman 1968).

The asymptotic analysis of (5.4), when interpreted in this context, means that as the bond lattice is contracted by ϵ and time is sped up by ϵ^{-2}, the jump process $\{ X^{\epsilon}(s) \}$ approaches a diffusion process with diffusion matrix Q. In other words, on the microscopic scale thermal energy is transmitted through the lattice by a jump process; but when viewed on a macroscopic scale the energy appears to diffuse throughout the lattice. The microscopic physics are described in (Kirkpatrick 1973) and (Kittel 1976). The approximation developed below for a periodic lattice is similar to the one developed by Kunnemann for a random lattice. This similarity demonstrates the robustness of the method, and the limted dependence of the macroscopic properties of the medium on the details of the microscopic variations of the structure.

Because the basic problem (5.4) is "parabolic," we can introduce the probabilistic mechanism and make use of it in the analysis. In the "hyperbolic," structural mechanical problems we treated before this device is not available.

5.2. Asymptotic analysis-homogenization

The essential mathematical step is to show strong convergence of the semigroup of L^{ϵ}, say

$$T^{\epsilon}(t) \triangleq e^{L^{\epsilon}t} \xrightarrow[\epsilon \to 0]{} T(t) \triangleq e^{Lt} \tag{5.7}$$

and to identify the limiting operator

$$L = \sum_{i,j=1}^{d} q_{ij} \frac{\partial^2}{\partial x_i \partial x_j}. \tag{5.8}$$

This is accomplished by proving convergence of the resolvents

$$\text{for } \alpha > 0, \ [-L^{\epsilon} + \alpha]^{-1} \xrightarrow[\epsilon \to 0]{} [-L + \alpha]^{-1} \tag{5.9}$$

That is, if f is a given function and

$$\begin{aligned} u^{\epsilon}(\cdot) &\triangleq [-L^{\epsilon} + \alpha]^{-1} f \\ u(\cdot) &\triangleq [-L + \alpha]^{-1} f \end{aligned} \tag{5.10}$$

then $u^{\epsilon} \to u$ in an appropiate sense.

The method of multiple scales will be used to compute the limit. Because the conductivities $a_i(x)$ in (5.4) do not depend on time, we may work directly with L^ϵ rather than the parabolic PDE (5.4) (cf. (Bensoussan, Lions, and Papanicolaou 1978) Remark 1.6, p. 242). The method of multiple scales is convenient because it is a systematic way of arriving at the "right answers" - something which is not always simple in this analysis.

Bearing in mind (5.10), we consider

$$(L^\epsilon u^\epsilon)(x) = f(x) \tag{5.11}$$

with $u^\epsilon(x)$ in the form

$$u^\epsilon(x) = u_0(x, \frac{x}{\epsilon}) + \epsilon\, u_1(x, \frac{x}{\epsilon}) + \epsilon^2 u_2(x, \frac{x}{\epsilon}) + \cdots \tag{5.12}$$

with the functions $u_j(x, y)$ periodic in $y \in \epsilon Z^d$ for every $j = 0, 1, \ldots$. (As it turns out the boundary conditions are somewhat irrelevant to the construction of "right answers.") To present the computations in a simple form, it is convenient to introduce $y = x/\epsilon$, to treat x and y as independent variables, and to replace y by x/ϵ at the end.

Recall the operators $\nabla_i^{\epsilon\pm}$ from (5.4). Applied to a smooth function $u = u(x, x/\epsilon)$, we have

$$(\nabla_i^{\epsilon-} u)(x, y) = \frac{1}{\epsilon}[\, u(x - \epsilon e_i, y - e_i) - u(x, y)\,] \tag{5.13}$$

$$= \frac{1}{\epsilon}[\, u(x, y - e_i) - u(x, y)\,] + \frac{1}{\epsilon}[\, u(x - \epsilon e_i, y - e_i) - u(x, y - e_i)\,]$$

$$= \frac{1}{\epsilon}(\nabla_i^- u)(x, y) - \frac{\partial u}{\partial x_i}(x, y - e_i) + \epsilon\, \frac{1}{2}\frac{\partial^2 u}{\partial x_i^2}(x, y - e_i) + O(\epsilon^2)$$

where on functions $\phi = \phi(y)$

$$(\nabla_i^- \phi)(y) = \phi(y - e_i) - \phi(y) \tag{5.14}$$

Defining

$$(\nabla_i^+ \phi)(y) = \phi(y + e_i) - \phi(y) \tag{5.15}$$

we also have

$$(\nabla_i^{\epsilon +} u)(x,y) = \frac{1}{\epsilon}(\nabla_i^{+} u)(x,y) + \frac{\partial u}{\partial x_i}(x,y+e_i) \tag{5.16}$$

$$+ \epsilon \frac{1}{2} \frac{\partial^2 u}{\partial x_i^2}(x,y+e_i) + O(\epsilon^2).$$

Now we substitute (5.12) into (5.11) and use the rules (5.14) (5.15). Equating coefficients of like powers of ϵ, this leads to a sequence of equations for u_0, u_1, $\cdots$. Specifically, (using the summation convention)

$$\begin{aligned}
(L^{\epsilon} u^{\epsilon})(x,y) &= -\nabla_i^{\epsilon -}[a_i(y)\nabla_i^{\epsilon +} u^{\epsilon}] \\
&= \frac{1}{\epsilon^2}\nabla_i^{-}[a_i(y)\nabla_i^{+} u_0(x,y)] - \nabla_i^{\epsilon -}[a_i(y)\frac{\partial u_0}{\partial x_i}(x,y+e_i)] \\
&\quad - \frac{1}{2}\epsilon\nabla_i^{\epsilon -}[a_i(y)\frac{\partial^2 u_0}{\partial x_i^2}(x,y+e_i)] + O(\epsilon) \\
&\quad - \frac{1}{\epsilon}\nabla_i^{-}[a_i(y)\nabla_i^{+} u_1(x,y)] \\
&\quad - \epsilon\nabla_i^{\epsilon -}[a_i(y)\frac{\partial u_1}{\partial x_i}(x,y+e_i)] + O(\epsilon) \\
&\quad - \nabla_i^{-}[a_i(y)\nabla_i^{+} u_2(x,y)] + O(\epsilon) = f(x)
\end{aligned} \tag{5.17}$$

That is, labeling each term by its order in ϵ

$$(\epsilon^{-2}) \quad \nabla_i^{-}[a_i(y)\nabla_i^{+} u_0] = 0 \tag{5.18}$$

$$(\epsilon^{-1}) \quad \epsilon\nabla_i^{\epsilon -}[a_i(y)\frac{\partial u_0}{\partial x_i}(x,y+e_i)] + \nabla_i^{-}[a_i(y)\nabla_i^{+} u_1(x,y)] = 0 \tag{5.19}$$

and (recall $\epsilon\nabla_i^{\epsilon\pm}$ is $O(1)$ in ϵ)

$$(\epsilon^{0}) \quad \frac{1}{2}\epsilon\nabla_i^{\epsilon -}[a_i(y)\frac{\partial^2 u_0}{\partial x_i^2}(x,y+e_i)] - \epsilon\nabla_i^{\epsilon -}[a_i(y)\frac{\partial u_1}{\partial x_i}(x,y+e_i)] \tag{5.20}$$

$$- \nabla_i^{-}[a_i(y)\nabla_i^{+} u_2(x,y)] = f(x)$$

From (5.18) we have

$$a_i(y-e_i)[u_0(x,y) - u_0(x,y-e_i)] \tag{5.21}$$

$$- a_i(y)[u_0(x,y+e_i) - u_0(x,y)] = 0$$

If we take $u_0(x,y) = u_0(x)$, this is trivially true; and (5.19) simplifies to

$$\epsilon \nabla_i^{\epsilon -} [a_i(y) \frac{\partial u_0}{\partial x_i}(x)] + \nabla_i^{-} [a_i(y) \nabla_i^{+} u_1(x,y)] = 0 . \tag{5.22}$$

At this point we introduce "correctors." That is, we assume

$$u_1(x,y) = \sum_{k=1}^{d} \chi_k(y) \frac{\partial u_0}{\partial x_k} + \hat{u}_1(x) \tag{5.23}$$

with $\chi_k(\cdot)$ the correctors. Using this in (5.22), we have (again using the summation convention)

$$\nabla_i^{-} [a_i(y) \nabla_i^{+} \chi_k(y)] \frac{\partial u_0}{\partial x_k} + [a_k(y - e_k) - a_k(y)] \frac{\partial u_0}{\partial x_k} = 0 \tag{5.24}$$

If we take $\chi_k(y)$ as the solution of

$$\nabla_i^{-} [a_i(y) \nabla_i^{+} \chi_k(y)] + [a_k(y - e_k) - a_k(y)] = 0 \tag{5.25}$$

(we have to verify the well-posedness of (5.25)), then (5.24) is satisfied. (The term $\hat{u}_1(x)$ is determined (formally) from the $O(\epsilon)$ term in the system (5.12) (5.17).)

Regarding the well-posedness of (5.25), note that

$$\nabla_i^{-} [a_i(y) \nabla_i^{+} \phi(y)] = \psi(y) \tag{5.26}$$

has a *periodic* solution on ϵZ which is unique up to an additive constant iff the average of the function $\psi(y)$ over a period (ϵl) is zero; i.e.,

$$\bar{\psi} \triangleq \frac{1}{l} \sum_{k=1}^{l} \psi(y + ke_n) = 0, \quad n = 1,2,\ldots,d. \tag{5.27}$$

This condition clearly holds in (5.25), and so, $\chi_k(y)$ is well defined (up to an additive constant).

We shall determine the equation for $u_0(x)$ by using (5.23) (5.25) in (5.20). Using the Kronecker delta function δ_{ik}, we have

$$\begin{aligned}
&\frac{1}{2} \epsilon \nabla_i^{\epsilon -} [a_i(y) \delta_{ik}] \frac{\partial^2 u_0}{\partial x_i \partial x_k} - \epsilon \epsilon \nabla_i^{\epsilon -} [a_i(y) \chi_k(y + e_i)] \frac{\partial^2 u_0}{\partial x_i \partial x_k} = f(x) \\
&= \{ \frac{1}{2} \nabla_i^{-} [a_i(y) \delta_{ik}] - \nabla_i^{-} [a_i(y) \nabla_i^{+} \chi_k] \} \frac{\partial^2 u_0}{\partial x_i \partial x_k} \\
&\quad - \nabla_i^{-} [a_i(y) \chi_k(y)] \frac{\partial^2 u_0}{\partial x_i \partial x_k} - \nabla_i^{-} [a_i(y) \nabla_i^{+} u_2] = f(x)
\end{aligned} \tag{5.28}$$

The term in braces is zero from (5.25). To obtain the solvability condition (5.27) for u_2 in (5.28), we introduce the average

$$\frac{1}{2} q_{ik} \triangleq \text{symmetric part} \{ \overline{- \nabla_i^- [a_i(y) \chi_k(y)]} \} \tag{5.29}$$

Then solvability of (5.28) for u_2 gives the equation

$$\frac{1}{2} \sum_{i,k=1}^{d} q_{ik} \frac{\partial^2 u_0}{\partial x_i \partial x_k} = f(x) . \tag{5.30}$$

And this is the diffusion equation which defines the limiting behavior of the system (5.11) in the macroscopic x-scale in the limit as $\epsilon \to 0$.

We can justify the asymptotic analysis by using energy estimates or probabilistic methods as in (Bensoussan, Lions, and Papanicolaou 1978). (See also Kunnemann 1983).) We shall omit this analysis here.

5.3. Summary

Returning to the original problem (5.4) for the evolution of thermal energy on a microscopic scale, we have shown that the thermal density $u^{\epsilon}(t,x) \to u_0(t,x)$ as $\epsilon \to 0$ (in an appropriate norm) where

$$\frac{\partial u_0}{\partial t} = \frac{1}{2} \sum_{i,j=1}^{d} q_{ij} \frac{\partial u_0}{\partial x_i \partial x_j} \tag{5.31}$$

with

$$q_{ij} = - \frac{1}{l} \sum_{k=1}^{l} \{ \nabla_i^+ [a_i(y) \chi_k(y)] + \nabla_k^- [a_i(y) \chi_i(y)] \} \tag{5.32}$$

with the correctors χ_k, $k = 1,2,\ldots,d$, given by

$$\sum_{i=1}^{d} \nabla_i^- [a_i(y) \nabla_i^+ \chi_i(y)] = - [a_k(y - e_k) - a_k(y)] \tag{5.33}$$

$$k = 1,2, \cdots, d$$

To compute the limiting "homogenized" model (5.31), one must solve the system (5.33) (numerically) and then evaluate the average (5.32).

The fact that the original problem (5.4) is "parabolic" (i.e., it describes a jump random process), enables us to exploit the associated probabilistic structure to anticipate and structure the analysis. In this way we can anticipate that the limit problem will involve a diffusion process. In fact, the arguments used are entirely analytical[9] and the limiting diffusion (5.31) is constructed in a systematic way. It is not postulated.

Appendix: An operational calculus for homogenization

The techniques used to treat the one dimensional example are based on an *operational calculus* for homogenization developed in (Bensoussan, Lions, and Papanicolaou 1978). It is worthwhile to outline the essential features of this operational system.

Consider the problem (using the convention that repeated indices are summed)

$$A^{\epsilon} u^{\epsilon} = -\frac{\partial}{\partial x_i}\left[a_{ij}\left(\frac{x}{\epsilon}\right) \frac{\partial u^{\epsilon}}{\partial x_j} \right] + a_0\left(\frac{x}{\epsilon}\right) u^{\epsilon} = f(x) \tag{1}$$

$$x \in \mathrm{O}, \quad u^{\epsilon}(x) \mid_{\partial \mathrm{O}} = \phi$$

where O is a smooth domain in $\mathbb{R}^n$, $\partial \mathrm{O}$ is its boundary; and we assume

$$[\, a_{ij}(y) \,] > 0 \quad \text{(positive definite matrix)}$$

$$a_0 \geq \alpha > 0 \tag{2}$$

$$a_{ij}(\,\cdot\,), \quad a_0(\,\cdot\,) \quad \text{periodic with period 1.}$$

We are interested in approximating u^{ϵ} in the limit as $\epsilon \to 0+$.

As in the example, we introduce $y = x/\epsilon$ and the *ansatz*

$$u^{\epsilon}(x) = u^{\epsilon}(x, \frac{x}{\epsilon}) = u_0(x, \frac{x}{\epsilon}) + \epsilon\, u_1(x, \frac{x}{\epsilon}) + \cdots \tag{3}$$

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[9] Probabilistic arguments can be used (Bensoussan, Lions, and Papanicolaou 1978, Chapter 3); and they have some advantages.

Making the change of coordinates

$$\frac{\partial}{\partial x_i} \to \frac{\partial}{\partial x_i} + \frac{1}{\epsilon}\frac{\partial}{\partial y_i}, \quad y_i = \frac{x_i}{\epsilon} \tag{4}$$

we have

$$A^{\epsilon} = \frac{1}{\epsilon^2} A_1 + \frac{1}{\epsilon} A_2 + A_3 \tag{4}$$

with

$$\begin{aligned} A_1 &= -\frac{\partial}{\partial y_i}\left[a_{ij}(y) \frac{\partial}{\partial y_j} \right] \\ A_2 &= -\frac{\partial}{\partial y_i}\left[a_{ij}(y) \frac{\partial}{\partial x_j} \right] - \frac{\partial}{\partial x_i}\left[a_{ij}(y) \frac{\partial}{\partial y_j} \right] \\ A_3 &= -\frac{\partial}{\partial x_i}\left[a_{ij}(y) \frac{\partial}{\partial x_j} \right] + a_0(y) \end{aligned} \tag{5}$$

Substituting (3) and (5) into (1) and equating coefficients of like powers of ϵ, we have

$$\begin{aligned} &(\epsilon^{-2}) \quad A_1 u_0 = 0 \\ &(\epsilon^{-1}) \quad A_1 u_1 + A_2 u_0 = 0 \\ &(\epsilon^{0}) \quad A_1 u_2 + A_2 u_1 + A_3 u_0 = f \\ &(\epsilon^{1}) \quad A_1 u_3 + A_2 u_2 + A_3 u_1 = 0 \end{aligned} \tag{6}$$

Solvability of this system depends on the operator A_1. The assumptions in (2) mean that the system

$$A_1 \phi(y) = g(y), \quad y \in Y \tag{7}$$

where Y is the unit torus in $\mathbb{R}^n$ and $g(\cdot)$ is periodic, has a solution unique up to an additive constant if and only if

$$\int_Y g(y)\, dy = 0 \tag{8}$$

Assumption (2) means the null space of A_1 is spanned by the constant function; and (8) means the data $g(\cdot)$ is orthogonal to the null space of A_1; i.e., a Fredholm Alternative holds for A_1.

Hence, $A_1 u_0 = 0$ implies

$$u_0(x,y) = u_0(x). \tag{9}$$

Also,

$$0 = A_1 u_1 + A_2 u_0 \tag{10}$$

$$= \frac{\partial}{\partial y_i}\left[a_{ij}(y) \frac{\partial u_1(x,y)}{\partial y_j} \right] - \frac{\partial}{\partial y_i}\left[a_{ij}(y) \frac{\partial u_0(x)}{\partial x_j} \right]$$

may be solved by taking

$$u_1(x,y) = - \chi^j(y) \frac{\partial}{\partial x_j} u_0(x) + \hat{u}_1(x) \tag{11}$$

where the *corrector* $\chi^j(y)$ satisfies

$$\frac{\partial}{\partial y_i}\left[a_{ij}(y) \frac{\partial}{\partial y_j} \chi^i(y) \right] = - \frac{\partial}{\partial y_i}\left[a_{ij}(y) \right] \tag{12}$$

The right side of (12) satisfies the centering condition (8); and so, (12) has a periodic solution, unique up to an additive constant.

Continuing in the same fashion, the third equation in (6) has a solution u_2 if

$$\int_Y \left[A_2 u_1 + A_3 u_0 - f \right] dy = 0. \tag{13}$$

Substituting for u_1 from (11) (12) and A_3 from (5), the solvability condition (13) reduces to

$$- q_{ij} \frac{\partial^2 u_0(x)}{\partial x_i \partial x_j} + M[a_0]\, u_0(x) = f(x) \tag{14}$$

$$u_0(x)\,|_\Gamma = \phi$$

where

$$q_{ij} = M[a_{ij}] - M\left[a_{ik} \frac{\partial \chi^j}{\partial y_k} \right]. \tag{15}$$

The role of the *corrector* in the approximation is clear from this. A simple argument similar to the one used for the one dimensional case in section 2 shows that the matrix $\{ q_{ij} \}$ is non-negative definite.

References

(1) M. Anderson 1981, Buckling of periodic lattice structures, AIAA Journal, 19, pp. 782-788.

(2) M. Aswani 1982, Development of an Analytical Model for Large Space Structures, DTIC Report ADA 119349.

(3) I. Babuska 1975, Homogenization and its applications. Mathematical and computational problems, Technical Note BN-821, Institute for Fluid Dynamics and Applied Mathematics, University of Maryland, College Park.

(4) A. Bensoussan 1979, Homogenization theory, Conf. del Seminaria de Math. dell'Univ. de Bari, No. 158.

(5) A. Bensoussan, L. Boccardo, and F. Murat 1984, Homogenization of nonlinear elliptic equations with operators not in divergence form, preprint.

(6) A. Bensoussan, J.L. Lions, and G.C. Papanicolaou 1978, Asymptotic Analysis for Periodic Structures, North Holland, Amsterdam.

(7) G.L. Blankenship 1979, Asymptotic analysis in mathematical physics and control theory: Some problems with common features, Richerchi di Automatica, vol. 10, pp. 265-315.

(8) J.F. Bourgat 1978, Numerical Experiments of the Homogenization Method for Operators with Periodic Coefficients, INRIA Report No. 277.

(9) L. Breiman 1968, Probability, Addison-Wesley, Reading, Mass.

(10) S.H. Crandall, et al. 1979, Dynamics of Mechanical and Electromechanical Systems, McGraw-Hill, New York.

(11) D.L. Dean and S. Tauber 1959, Solutions for one dimensional lattice structures, J. Eng. Mech. Div., ASCE, 85, pp. 31-41.

(12) J.-N. Juang 1984, Optimal design of a passive vibration absorber for a truss beam, J. Guidance, 7, pp. 733-739.

(13) S. Kirkpatrick 1973, Perculation and conduction, Rev. Mod. Phys., 45, pp. 574-588.

(14) C. Kittel 1976, Introduction to Solid State Physics, Wiley, New York.

(15) J.B. Keller 1977, Effective behavior of heterogeneous media, in Statistical Mechanics and Statistical Methods in Theory and Application, U. Landman, ed., Plenum, New York, pp. 613-644.

(16) R. Kunnemann 1983, The diffusion limit for reversible jump processes on Z^d with ergodic random bond conductivities, Commun. Math. Phys., 90, pp. 27-68.

(17) E. Larsen 1975 1976, Neutron transport and diffusion in inhomogeneous media, I, J. Math. Phys., vol. 16, pp. 1421-1427; II, Nucl. Sci. Eng., vol. 60, pp. 357-368.

(18) M.M. Mikulus 1978, Structural efficiency of long lightly loaded truss and isogrid columns for space application, NASA TM 78687.

(19) A. Nayfeh and M.S. Hefzy 1978, Continuum modeling of three- dimensional truss-like space structures, AIAA Journal, 16, pp. 779-787.

(20) A.K. Noor, M.S. Anderson, and W.H. Greene 1978, Continuum models for beam- and plate-like lattice structures, AIAA Journal, 16, pp. 1219-1228.

(21) G.C. Papanicolaou and S.R.S. Varadhan 1979, Boundary value problems with rapidly oscillating random coefficients, Proceedings of Conference on Random Fields, Hungary, North Holland, Amsterdam.

(22) J.D. Renton 1984, The beam-like behavior of space trusses, AIAA Journal, 22, pp. 273-280.

(23) J.D. Renton 1973, Buckling of long, regular trusses, Int. J. Mech. Sci., 9, pp. 1489-1500.

(24) J.D. Renton 1970, General properties of space grids, Int. J. Mech. Sci., 12, pp. 801-810.

(25) J.D. Renton 1969, Behavior of Howe, Pratt and Warren trusses, J. Struc. Div., ASCE., 95, pp. 185-202.

(26) L.W. Taylor, Jr. and V. Balakrishnan 1984, A mathematical problem and a spacecraft control laboratory experiment (SCOLE) used to evaluate control laws for flexible spacecraft, NASA/IEEE design challenge, Spacecraft Control Branch, NASA Langley Research Center, Hampton, VA.