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ESTIMATION OF THE RATE OF A DOUBLY-STOCHASTIC TIME-SPACE POISSON PROCESS

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Abstract

We consider the problem of estimating the rate of a doubly-stochastic, time-space Poisson process when the observations are restricted to a region $D \subseteq \mathbb{R}^2$. In the general case, we obtain a representation of the minimum mean-square-error (MMSE) estimate in terms of the conditional characteristic function of an underlying state process. In the case $D = \mathbb{R}^2$, we extend a known result to compute the MMSE estimate explicitly. For a special form of the rate process, a well-defined integral equation is presented which defines the *linear* MMSE estimate of the rate.

Key Words: doubly-stochastic, time-space Poisson process, MMSE estimate, linear MMSE estimate, likelihood ratio.

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I. Introduction

We consider a doubly-stochastic, time-space Poisson process $\mathbf{N}^0$ with intensity function $\lambda(t, r) = f(t, r - H(t)x_t)$, where $t > 0$ and $r \in \mathbb{R}^2$. Here, f is a known, deterministic function; $x_t \in \mathbb{R}^n$ is the solution of an Ito stochastic differential equation, and $H(t)$ is a known, deterministic, $\mathbb{R}^{2\times n}$-valued function. The process $\mathbf{N}^0$ under consideration counts events which occur in all of $\mathbb{R}^2$; however, suppose that only those events which occur within a region $D \subseteq \mathbb{R}^2$ can be observed. We wish to compute minimum mean-square-error (MMSE) estimates of $\lambda(t, r)$, given our limited observations. In the general case, $D \neq \mathbb{R}^2$, we obtain a representation of these estimates in terms of the conditional characteristic function of x_t. When $D = \mathbb{R}^2$, and $f(t, r) = e^{-\frac{1}{2}r'R(t)^{-1}r}$, for some deterministic matrix $R(t)$, we extend a result of Rhodes and Snyder [1] to compute the MMSE estimate of $\lambda(t, r)$ explicitly. We also consider *linear* estimates of $\lambda(t, r)$ for the same choice of f when $D \neq \mathbb{R}^2$. These filtering problems are frequently encountered in optical communication systems [2, 3], particularly in the context of hypothesis-testing; this issue is discussed in Section V.

II. Probabilistic Setting

Let $\mathcal{B}^2$ denote the Borel subsets of $\mathbb{R}^2$. Next, if I is any interval of $\mathbb{R}$, let $\mathcal{B}(I)$ denote the Borel subsets of I. We define $\mathcal{B}(I)\otimes\mathcal{B}^2$ to be the smallest σ-field containing all sets of the form $E \times A$, such that $E \in \mathcal{B}(I)$ and $A \in \mathcal{B}^2$. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space on which we let

$$\mathbf{N}^0 = \{ N(B) : B \in \mathcal{B}(0,\infty)\otimes\mathcal{B}^2 \},$$

be a time-space point process. Sometimes, $\mathbf{N}^0$ is called a random point field or a random measure. Here, this means that with each $B \in \mathcal{B}(0,\infty)\otimes\mathcal{B}^2$, we associate a nonnegative, integer-valued random variable, $N(B) = N(\omega, B)$; in addition, for each $\omega \in \Omega$, $N(\omega, \bullet)$ is assumed to be an integer-valued measure on $\mathcal{B}(0,\infty)\otimes\mathcal{B}^2$. We let $\mathcal{F}_t$ represent the times and locations at which points have occurred up to and including time t. More precisely, let

$\mathbb{F}_0$ denote the trivial σ-field, and for $t > 0$, set

$$\mathbb{F}_t = \sigma\{ N(B) : B \in \mathbb{B}(0,t] \otimes \mathbb{B}^2 \} .$$

Now, let D be a Borel subset of $\mathbb{R}^2$. We take $\mathbb{G}_0$ to be the trivial σ-field, and for $t > 0$, we set

$$\mathbb{G}_t = \sigma\{ N(B \bigcap \{ (0,\infty) \times D \}) : B \in \mathbb{B}(0,t] \otimes \mathbb{B}^2 \} .$$

Note that $\mathbb{G}_t$ represents the history of the point process restricted to the region D, up to time t. We shall refer to $\mathbb{G}_t$ as our "observations up to time t." On the same probability space, $(\Omega , \mathbb{F} , \mathbf{P})$, let X be an n-dimensional Gaussian random vector with known mean, m, and known, positive-definite covariance, S. Let $\{ v_t , t \geq 0 \}$ be a standard Wiener process independent of X. We let the n-dimensional process $\{ x_t , t \geq 0 \}$ be the solution to the Ito stochastic differential equation

$$dx_t = F(t) x_t \, dt + V(t) dv_t \ ; \quad x_0 = X . \tag{1}$$

Here F and V are known matrices with appropriate dimensions. We also assume that F and V are piecewise-continuous so that a unique solution of (1) exists (see Davis [4], pp. 108-111). Let

$$\mathbb{X}_0 \triangleq \sigma\{ x_s , 0 \leq s < \infty \} .$$

For $t > 0$, let $\mathbb{X}_t$ denote the smallest σ-field containing $\mathbb{F}_t \bigcup \mathbb{X}_0$. We write this symbolically as

$$\mathbb{X}_t \triangleq \mathbb{F}_t \vee \mathbb{X}_0 \ ; \quad t > 0.$$

We shall assume that N^0 is an $\{\mathbb{X}_t\}$-doubly-stochastic, time-space Poisson process, with $\mathbb{X}_0$-measurable intensity (see Bremaud [5], pp. 21-23 and 233-238)

$$\lambda(t , r) = f\,(t , r - H(t) x_t) ,$$

where $t \in (0,\infty)$, $r \in \mathbb{R}^2$, and x_t is defined by (1). Assume that $H : (0,\infty) \to \mathbb{R}^{2 \times n}$ and $f : (0,\infty) \times \mathbb{R}^2 \to (0,\infty)$ are deterministic and known. We further assume that the function

$$\mu(t) \triangleq \int_{\mathbb{R}^2} f(t, r)\, dr \tag{2}$$

is finite for all $t < \infty$. This means that for each $t \geq 0$, the process

$$\mathbf{N}^t \triangleq \{ N(B) : B \in \mathcal{B}(t, \infty) \otimes \mathcal{B}^2 \}$$

is a Poisson random field under the measure $\mathbf{P}(\bullet \mid \boldsymbol{X}_t)$, with rate $\lambda(s, r)$, where $s \in (t, \infty)$, and $r \in \mathbb{R}^2$. This implies the following. First, for $B \in \mathcal{B}(0,\infty) \otimes \mathcal{B}^2$, let $\Lambda(B) \triangleq \int_B \lambda(s, r)\, dr\ ds$; then if $B \in \mathcal{B}(t, \infty) \otimes \mathcal{B}^2$ and n is an arbitrary, nonnegative integer,

$$\mathbf{P}(N(B) = n \mid \boldsymbol{X}_t) = \frac{\Lambda(B)^n}{n!} e^{-\Lambda(B)}, \tag{3}$$

and hence, for $\theta \in \mathbb{R}$,

$$\mathbf{E}[e^{j\theta N(B)} \mid \boldsymbol{X}_t] = \exp[(e^{j\theta} - 1)\, \Lambda(B)]. \tag{4}$$

The second implication is that if B_1 and B_2 are disjoint sets in $\mathcal{B}(t, \infty) \otimes \mathcal{B}^2$, then the random variables $N(B_1)$ and $N(B_2)$ are independent under the measure $\mathbf{P}(\bullet \mid \boldsymbol{X}_t)$.

Notation. We let $N_0 \equiv 0$ and for $t > 0$, $N_t \triangleq N((0,t] \times D)$.

III. Nonlinear Filtering Results

We first establish some notation in order to state our results more compactly. Let $P_t(x)$, $x \in \mathbb{R}^n$, denote the (regular) conditional probability of x_t given $\mathcal{G}_t$. Let $\psi_t(\eta)$, $\eta \in \mathbb{R}^n$, denote the conditional characteristic function of x_t given $\mathcal{G}_t$:

$$\psi_t(\eta) \triangleq \mathbf{E}[e^{j\eta' x_t} \mid \mathcal{G}_t] = \int_{\mathbb{R}^n} e^{j\eta' x}\, dP_t(x); \quad \eta \in \mathbb{R}^n.$$

Next, let

$$\hat{\lambda}(t, r) \triangleq \mathbf{E}[\lambda(t, r) \mid \mathcal{G}_t] = \mathbf{E}[f(t, r - H(t)x_t) \mid \mathcal{G}_t],$$

and

$$\hat{l}(t, \theta) \triangleq \int_{\mathbb{R}^2} \hat{\lambda}(t, r)\, e^{j\theta' r}\, dr \ ; \quad \theta \in \mathbb{R}^2.$$

We also set

$$F(t, \theta) \triangleq \int_{\mathbb{R}^2} f(t, r) e^{j\theta' r}\, dr \ .$$

Theorem 1. *Under the foregoing assumptions,*

$$\hat{l}(t, \theta) = F(t, \theta)\, \psi_t(\, H(t)'\theta\,) \ .$$

Proof. Observe that

$$\begin{aligned}\hat{l}(t, \theta) &= \int_{\mathbb{R}^2} \mathrm{E}\,[\, f(t, r - H(t)x_t) \mid \mathcal{G}_t\,] e^{j\theta' r}\, dr \\ &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^n} f(t, r - H(t)x)\, dP_t(x)\, e^{j\theta' r}\, dr \ .\end{aligned}$$

By Fubini's Theorem,

$$\begin{aligned}\hat{l}(t, \theta) &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^2} f(t, r - H(t)x)\, e^{j\theta' r}\, dr\ dP_t(x) \\ &= F(t, \theta) \int_{\mathbb{R}^n} e^{j\theta' H(t) x}\, dP_t(x) \\ &= F(t, \theta) \int_{\mathbb{R}^n} e^{j(H(t)'\theta)' x}\, dP_t(x) \\ &= F(t, \theta)\, \psi_t(\, H(t)'\theta\,).\end{aligned}$$

QED

Theorem 2. *If* $D = \mathbb{R}^2$, *and if*

$$f(t, r) = e^{-\frac{1}{2} r' R(t)^{-1} r} \ , \tag{5}$$

for some deterministic, positive-definite matrix $R(t)$, *then*

$$\hat{\lambda}(t,r) \triangleq \mathbf{E}[\lambda(t,r) \mid \mathcal{G}_t]$$
$$= \mathbf{E}[f(t, r - H(t)x_t) \mid \mathcal{G}_t]$$
$$= \frac{\sqrt{\det R(t)}}{\sqrt{\det Q_t}} \exp[-\frac{1}{2}(r - H(t)\hat{x}_t)' \; Q_t^{-1} (r - H(t)\hat{x}_t)],$$

where

$$\hat{x}_t \triangleq \mathbf{E}[x_t \mid \mathcal{G}_t],$$
$$\hat{\Sigma}_t \triangleq \mathbf{E}[(x_t - \hat{x}_t)(x_t - \hat{x}_t)' \mid \mathcal{G}_t] > 0, \quad \mathbf{P}-\text{a.s.},$$
$$Q_t \triangleq H(t)\hat{\Sigma}_t H(t)' + R(t),$$

and

$$d\hat{x}_t = F(t)\hat{x}_t \, dt + \int_{\mathbb{R}^2} \hat{\Sigma}_{t-} H(t-)' \; Q_{t-}^{-1} \; (r - H(t-)\hat{x}_{t-}) \, N(dt \times dr) \, ; \quad \hat{x}_0 = m \, , \tag{6}$$

$$d\hat{\Sigma}_t = F(t)\hat{\Sigma}_t \, dt + \hat{\Sigma}_t F(t)' \, dt + V(t)V(t)' \, dt - \hat{\Sigma}_{t-} H(t-)' \; Q_{t-}^{-1} \; H(t-)\hat{\Sigma}_{t-} N(dt \times \mathbb{R}^2) \, ; \quad \hat{\Sigma}_0 = S. \tag{7}$$

Proof. First, since $D = \mathbb{R}^2$, $\mathcal{G}_t = \mathcal{F}_t$. Next, in [1] it is proved that the conditional density of x_t given $\mathcal{F}_t$ is *Gaussian* with conditional mean $\hat{x}_t$ and conditional covariance $\hat{\Sigma}_t$ (which is positive definite almost surely because of the assumption that S is positive definite) satisfying (6) and (7) above. So,

$$\psi_t(\eta) = e^{j\eta'\hat{x}_t - \frac{1}{2}\eta'\hat{\Sigma}_t\eta}.$$

Next, from equation (5), it follows that

$$F(t, \theta) = 2\pi \sqrt{\det R(t)} \; e^{-\frac{1}{2}\theta' R(t)\theta}.$$

Hence, by Theorem 1,

$$\hat{l}(t, \theta) = 2\pi \sqrt{\det R(t)} \; e^{j\theta' H(t)\hat{x}_t - \theta' Q_t \theta}.$$

Taking inverse Fourier transforms, we see by inspection that

$$\hat{\lambda}(t, r) = \frac{\sqrt{\det R(t)}}{\sqrt{\det Q_t}} \exp[-\frac{1}{2}(r - H(t)\hat{x}_t)' \; Q_t^{-1} \, (r - H(t)\hat{x}_t)] \, .$$

QED

When $D \neq \mathbb{R}^2$, or equation (5) does not hold, $\psi_t(\eta)$ is, in general, not known. This has led us to consider *linear* estimates of $\lambda(t, r)$. We discuss this in the next section.

IV. Linear Filtering Results

We call $\hat{\lambda}_L(t, r)$ a *linear* estimate of $\lambda(t, r)$ given $\mathcal{G}_t$, if $\hat{\lambda}_L$ can be written in the form

$$\hat{\lambda}_L(t, r) = \int_0^t \int_D h(t, r; \tau, \rho) \, [\, N(d\tau \times d\rho) - \overline{\lambda}(\tau, \rho) \, d\tau \, d\rho \,] + h_0(t, r) \, , \qquad (8)$$

where h and h_0 are deterministic, and $\overline{\lambda}(t, r) \triangleq \mathbf{E}\,[\,\lambda(t, r)\,]$. We wish to choose h and h_0 to minimize

$$\mathbf{E}\,[\; |\lambda(t, r) - \hat{\lambda}_L(t, r)|^2 \,] \, . \qquad (9)$$

Lemma 1. (Grandell [6]). *Let* $\hat{\lambda}_L(t, r)$ *be given by* (8). *Under the conditions outlined in Section II, the quantity in* (9) *will be minimized if* $h_0(t, r) = \overline{\lambda}(t, r)$, *and if* h *satisfies*

$$\Gamma(t, r; \tau, \rho) = \int_0^t \int_D h(t, r; \sigma, \varsigma)\Gamma(\sigma, \varsigma; \tau, \rho) \, d\varsigma \, d\sigma + h(t, r; \tau, \rho)\overline{\lambda}(\tau, \rho) \, , \qquad (10)$$

where

$$\Gamma(t, r; \tau, \rho) \triangleq \mathbf{cov}\,[\,\lambda(t, r), \lambda(\tau, \rho)\,] \, .$$

With Lemma 1 in mind, we state our Theorem 3.

Theorem 3. *If* $f(t, r)$ *is given by* (5), *and the conditions outlined in Section II hold, then*

$$\overline{\lambda}(t\,,\,r\,) = \frac{\sqrt{\det\, R\,(t\,)}}{\sqrt{\det\, Q\,(t\,)}}\exp[\,-\,\frac{1}{2}(r\,-H\,(t\,)\overline{x}\,(t\,))'\;\; Q\,(t\,)^{-1}\,(r\,-H\,(t\,)\overline{x}\,(t\,))\,]\;, \tag{11}$$

where

$$\overline{x}(t\,)\;\triangleq\;\mathbf{E}\,[x_t\,]\;,$$

$$\Sigma(t\,)\;\triangleq\;\mathbf{cov}\;[x_t\,]\;,$$

$$Q\,(t\,)\;\triangleq\;H\,(t\,)\Sigma(t\,)H\,(t\,)'\;\;+\;R\,(t\,).$$

Furthermore,

$$\Gamma(t\,,\,r\,;\,\tau,\,\rho)\,+\;\overline{\lambda}(t\,,\,r\,)\overline{\lambda}(\tau,\,\rho) = \sqrt{\frac{\det\, R\,(t\,)\;\det\, R\,(\tau)}{\det\, Q\,(t\,,\,\tau)}}\;\times$$

$$\exp[\,-\,\frac{1}{2}(\begin{bmatrix} r \\ \rho \end{bmatrix} - \begin{bmatrix} H\,(t\,) & 0 \\ 0 & H\,(\tau) \end{bmatrix}\begin{bmatrix} \overline{x}\,(t\,) \\ \overline{x}\,(\tau) \end{bmatrix})'\;\;Q\,(t\,,\,\tau)^{-1}\;(\begin{bmatrix} r \\ \rho \end{bmatrix} - \begin{bmatrix} H\,(t\,) & 0 \\ 0 & H\,(\tau) \end{bmatrix}\begin{bmatrix} \overline{x}\,(t\,) \\ \overline{x}\,(\tau) \end{bmatrix})\,], \tag{12}$$

where

$$\Sigma(t\,,\,\tau)\;\triangleq\;\mathbf{cov}\;\;[\,x_t\,,\,x_\tau\,]\;,$$

and

$$Q\,(t\,,\,\tau)\;\triangleq\;\begin{bmatrix} Q\,(t\,) & H\,(t\,)\Sigma(t\,,\,\tau)H\,(\tau)' \\ H\,(\tau)\Sigma(\tau,\,t\,)H\,(t\,)' & Q\,(\tau) \end{bmatrix}.$$

Proof. For completeness, we make the following observations. Recall that

$$dx_t\;=F\,(t\,)x_t\,dt\;+\;V(t\,)dv_t\;\;;\quad x_0 = X. \tag{13}$$

Let $\Phi(t_2,\,t_1)$ be the transition matrix corresponding to $F\,(t\,)$. Then

$$\overline{x}\,(t\,) = \Phi(t\,,\,0)m\;\;, \tag{14}$$

and

$$\Sigma(t,\tau) = \Phi(t,0)S\,\Phi(\tau,0)' + \int_0^{\min(t,\tau)} \Phi(t,s)\,V(s)V(s)'\,\Phi(\tau,s)'\,ds\ .$$

Note that $\Sigma(t) = \Sigma(t,t)$.

To compute $\overline{\lambda}(t,r) = \mathbf{E}[\lambda(t,r)]$, observe that x_t is *Gaussian* with mean $\overline{x}(t)$ and covariance $\Sigma(t)$. By considering the proofs of Theorem 1 and Theorem 2, equation (11) is immediate.

The computation of (12) is similar, but requires some judicious preliminary arithmetic. First, observe that $\Gamma(t,r;\tau,\rho) + \overline{\lambda}(t,r)\overline{\lambda}(\tau,\rho)$ is just another way of writing $\mathbf{E}[\lambda(t,r)\lambda(\tau,\rho)]$. Next, rewrite $\lambda(t,r)\lambda(\tau,\rho)$ as

$$\exp[-\frac{1}{2}(\begin{bmatrix} r \\ \rho \end{bmatrix} - \begin{bmatrix} H(t) & 0 \\ 0 & H(\tau) \end{bmatrix}\begin{bmatrix} x_t \\ x_\tau \end{bmatrix})' \begin{bmatrix} R(t)^{-1} & 0 \\ 0 & R(\tau)^{-1} \end{bmatrix} (\begin{bmatrix} r \\ \rho \end{bmatrix} - \begin{bmatrix} H(t) & 0 \\ 0 & H(\tau) \end{bmatrix}\begin{bmatrix} x_t \\ x_\tau \end{bmatrix})],$$

which is equal to

$$\exp[-\frac{1}{2}(\begin{bmatrix} r \\ \rho \end{bmatrix} - \begin{bmatrix} H(t) & 0 \\ 0 & H(\tau) \end{bmatrix}\begin{bmatrix} x_t \\ x_\tau \end{bmatrix})' \begin{bmatrix} R(t) & 0 \\ 0 & R(\tau) \end{bmatrix}^{-1} (\begin{bmatrix} r \\ \rho \end{bmatrix} - \begin{bmatrix} H(t) & 0 \\ 0 & H(\tau) \end{bmatrix}\begin{bmatrix} x_t \\ x_\tau \end{bmatrix})]. \quad (15)$$

Because $\{ x_t, t \geq 0 \}$ is a Gaussian process, $\begin{bmatrix} x_t \\ x_\tau \end{bmatrix}$ is a Gaussian random vector with mean, $\begin{bmatrix} \overline{x}(t) \\ \overline{x}(\tau) \end{bmatrix}$, and covariance $\begin{bmatrix} \Sigma(t) & \Sigma(t,\tau) \\ \Sigma(\tau,t) & \Sigma(\tau) \end{bmatrix}$. By the same reasoning used to deduce (11), (12) also follows.

QED

Remark. In equation (10), if we regard t and r as fixed, and divide through by $\overline{\lambda}(\tau,\rho)$, then the result has the form of the Fredholm equation

$$g = Bh + h\ ,$$

for known function g, known operator B, and unknown function h.

V. Discussion

The filtering problems considered above often arise in the design and implementation of receivers for optical communication systems. Typically, a binary message source is used by a transmitter to select the modulation of the intensity of a laser beam in accordance with whether a "0" or a "1" is to be sent. The laser beam travels to a receiver and strikes its photodetector. We assume that the laser beam has an intensity profile of the form

$$\nu_i(t) f(t, r) ; \quad i = 0, 1.$$

Here, $\nu_i(t)$ is a known, deterministic function, where $i = 0$ or 1 has been selected by the transmitter.

We model the surface of the receiver's photodetector as $\mathbb{R}^2$. If the receiver, for example, is subject to vibrations, the center of the spot of laser light may wander randomly over the photodetector surface [2]. We assume, as in [2], that the center of the spot of laser light is given by $H(t)x_t \in \mathbb{R}^2$. The output of photoelectrons from the photodetector is modeled by the process $\mathbf{N}^0$, with stochastic intensity now given by

$$\lambda_i(t, r) = \nu_i(t) f(t, r - H(t)x_t) . \tag{16}$$

Of course, an actual photodetector does not have an infinite photosensitive surface. We account for this fact by assuming that only those photoelectrons which occur in a region $D \subseteq \mathbb{R}^2$ are observed. For example, in this setting, D might be a square or a circle centered at the origin. After observing photoelectrons occurring in D during some time interval $[0, T]$, a decision as to whether a "0" or a "1" was sent has to be made based on one of the estimates $\hat{\lambda}_i(t, r)$ or $\hat{\lambda}_{i,L}(t, r)$. As an example of a decoding scheme, we could use the likelihood ratio test

$$L_T \underset{H_0}{\overset{H_1}{\gtrless}} 1,$$

to make the decision, using the minimum probability of error cost criterion and assuming equiprobable hypotheses (see Snyder [3], section 2.5). The likelihood ratio, L_T, is given by (see Snyder [3], pp. 471-476)

$$L_T = \frac{\prod_{j=1}^{N_T} \hat{\lambda}_1(t_j, r_j) \ \exp[-\int_0^T \int_D \hat{\lambda}_1(s, r) \, dr \ ds]}{\prod_{j=1}^{N_T} \hat{\lambda}_0(t_j, r_j) \ \exp[-\int_0^T \int_D \hat{\lambda}_0(s, r) \, dr \ ds]}, \tag{17}$$

where t_j and r_j are respectively the time and the location of the jth photoevent *in the region* D, and we adopt the convention that when $N_T = 0$, the factors preceeding *exp* in equation (17) are taken to be unity. Here, of course,

$$\hat{\lambda}_i(t, r) \triangleq \mathbf{E}[\lambda_i(t, r) \mid \mathcal{G}_t]; \quad i = 0, 1.$$

Now, using (16), (17) simplifies to

$$L_T = \prod_{j=1}^{N_T} \frac{\nu_1(t_j)}{\nu_0(t_j)} \ \exp[-\int_0^T \int_D \hat{\lambda}_1(s, r) - \hat{\lambda}_0(s, r) \, dr \ ds]. \tag{19}$$

In the general case, $D \neq \mathbb{R}^2$, $\hat{\lambda}_i(t, r)$ is not known, and hence, L_T cannot be computed. However, when $D = \mathbb{R}^2$, it turns out that we do not need to know $\hat{\lambda}_i(t, r)$ in order to compute L_T. Observe that if $D = \mathbb{R}^2$, then

$$\begin{aligned}
\int_D \hat{\lambda}_1(s, r) - \hat{\lambda}_0(s, r) \, dr &= \mathbf{E}[\int_{\mathbb{R}^2} \lambda_1(s, r) - \lambda_0(s, r) \, dr \mid \mathcal{G}_s] \\
&= \mathbf{E}[(\nu_1(s) - \nu_0(s)) \int_{\mathbb{R}^2} f(s, r - H(s) x_s) \, dr \mid \mathcal{G}_s] \\
&= \mathbf{E}[(\nu_1(s) - \nu_0(s)) \mu(s) \mid \mathcal{G}_s] \qquad (20) \\
&= \mu(s) [\nu_1(s) - \nu_0(s)].
\end{aligned}$$

In equation (20) we used the fact that for all $r_0 \in \mathbb{R}^2$,

$$\mu(s) \triangleq \int_{\mathbb{R}^2} f(s, r) \, dr = \int_{\mathbb{R}^2} f(s, r - r_0) \, dr.$$

Thus, when $D = \mathbb{R}^2$, (19) becomes

$$L_T = \prod_{j=1}^{N_T} \frac{\nu_1(t_j)}{\nu_0(t_j)} \exp[- \int_0^T \mu(s)[\nu_1(s) - \nu_0(s)] \, ds] . \tag{21}$$

With (21) in mind, consider the following theorem.

Theorem 4. *The random field*

$$\mathbf{M}^t \triangleq \{ N(E \times \mathbb{R}^2) : E \in \mathbb{B}(t, \infty) \} ,$$

is independent of the σ-field $\boldsymbol{\mathcal{X}}_t$.

Proof. To prove that $\mathbf{M}^t$ is independent of $\boldsymbol{\mathcal{X}}_t$, it is sufficient to show that the conditional characteristic function of $N(E \times \mathbb{R}^2)$ is deterministic for $E \in \mathbb{B}(t, \infty)$. Now, it follows immediately from the assumption that $\mathbf{N}^0$ is an $\{\boldsymbol{\mathcal{X}}_t\}$-doubly-stochastic, time-space Poisson process, that for $\theta \in \mathbf{R}$,

$$\begin{aligned} \mathbf{E} [e^{j\theta N(E \times \mathbb{R}^2)} \mid \boldsymbol{\mathcal{X}}_t] &= \exp[(e^{j\theta} - 1) \int_E \int_{\mathbb{R}^2} \lambda_i(s, r) \, dr \, ds] \\ &= \exp[(e^{j\theta} - 1) \int_E \nu_i(s) \int_{\mathbb{R}^2} f(s, r - H(s)x_s) \, dr \, ds] \\ &= \exp[(e^{j\theta} - 1) \int_E \nu_i(s) \mu(s) \, ds] . \end{aligned}$$

Hence $\mathbf{M}^t$ is independent of $\boldsymbol{\mathcal{X}}_t$.

QED

It follows from equation (21) and Theorem 4 that for all $t \geq 0$, the random variable L_t is independent of the σ-field $\boldsymbol{\mathcal{X}}_t$.

If we replace equation (1) by

$$dx_t = F(t)x_t \, dt + G(t)u_t \, dt + V(t)dv_t \; ; \quad x_0 = X , \tag{22}$$

where $\{ u_t, t \geq 0 \}$ is predictable with respect to $\{ \boldsymbol{\mathcal{C}}_t, t \geq 0 \}$ and $G(t)$ is a known matrix with appropriate dimensions, then most of the above results hold with only minor modifications. The term $G(t)u_t$ in (22) is interpreted as a control signal driven by the output of the photodetector. Since $H(t)x_t$ represents the center of the spot of laser light striking the receiver, one might try to use $G(t)u_t$ to drive x_t to the origin. This problem is addressed in [1]. If (1) is replaced by (22), Theorem 1 is unchanged. Theorem 2 still holds except that equation (6) must be replaced by

$$d\hat{x}_t = F(t)\hat{x}_t\,dt + G(t)u_t\,dt$$
$$+ \int_{\mathbb{R}^2} \hat{\Sigma}_{t-}H(t-)'\ Q_{t-}^{-1}\ (r - H(t-)\hat{x}_{t-})\ N(dt \times dr)\ ; \quad \hat{x}_0 = m\ .$$

Lemma 1 is unchanged, and if $u_t = u(t)$ for some deterministic control $\{ u(t),\ t \geq 0 \}$, then Theorem 3 holds; of course, (13) becomes (22) and (14) is replaced by

$$\bar{x}(t) = \Phi(t,0)m + \int_0^t \Phi(t,s)G(s)u(s)\,ds\ .$$

In addition, the results of the preceding paragraphs of Section V, including Theorem 4, are unchanged by substituting equation (22) for equation (1). Note also that since $\boldsymbol{G}_t \subseteq \boldsymbol{X}_t$, and L_t is independent of $\boldsymbol{X}_t$ when $D = \mathbb{R}^2$, it follows that L_T is independent of the control law $\{ u_t,\ 0 \leq t \leq T \}$ when $D = \mathbb{R}^2$. This implies that the probability of a decoding error corresponding to the likelihood ratio test preceding equation (17) is not a function of the control law $\{ u_t,\ 0 \leq t \leq T \}$ when $D = \mathbb{R}^2$. In this sense, *all controls are optimal,* when $D = \mathbb{R}^2$. In general, when $D \neq \mathbb{R}^2$, this is not to be expected.

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