

ABSTRACT

Title of Dissertation: **SIGNAL LEVEL STATISTICS
IN A NETWORK OF CABLES:
THEORY**

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We present the theoretical framework required to describe the statistics of microwave networks that serve to model quantum graphs. The networks are characterized by impedance and admittance matrices relating the voltages and currents at the network's ports. As we show, these matrices can be calculated in a number of ways. Normal modes of the network are characterized by a discrete set of wavenumbers corresponding to the propagation constants on the network's bonds for which the determinant of the admittance matrix vanishes. The distribution of the spacings between adjacent eigenmode wavenumbers is found to depend on the nature of the way bonds are connected at nodes. The critical quantity is the reflection coefficient presented at a node to a wave on a bond. As the reflection coefficient increases, the spacing distribution changes from one characteristic of the spacing of eigenvalues of a GOE matrix to a Poisson distribution. The effect of loss is studied, and the scaling of the variance of the impedance values on network size, degree distribution, and other parameters is characterized. We attempted to find universal

scaling relations for the distribution of impedance values for networks of different sizes. Finally, we compare the distribution of impedance values predicted by the model with those measured in a network of cables.

SIGNAL LEVEL STATISTICS IN A NETWORK OF CABLES: THEORY

by

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Chapter 1: Introduction

Quantum graphs, networks of nodes connected by bonds that support waves, have been extensively studied experimentally and theoretically for the last decades. Due to their simplicity, they are useful for analyzing and understanding various quantum systems. The simplicity of the quantum graphs comes from the one-dimensional nature of the system, the waves propagate on one-dimensional bonds. Experimentally the bonds can be realized by coaxial cables where EM waves propagate, and the nodes correspond to junctions where the waves interfere and interact. In practice, the length of each bond is substantially larger than its width, and in this limit propagation of the waves are constrained in a single transversal mode which propagates along the bond.

The quantum graph model was introduced by Linus Pauling in 1936 to describe free electron behavior in an organic molecule. Since then this model has been used in variety of wave systems [4–7] including quantum wires [8, 9], mesoscopic quantum systems [10], electromagnetic waveguide networks [11, 12], and others. The model carries the name "quantum" due to the fact that the system can be described by a one-dimensional Schrödinger equation, or similar wave equation. Therefore, quantum graphs, like wave systems in two and three dimensions, have a discrete set of eigenmodes and frequency eigenvalues.

The spectral properties of quantum systems in 2D and 3D domains have properties dependent on their dynamics in the classical limit. These differences appear readily in the distributions

of spacings between adjacent eigenvalues. If its corresponding classical dynamics are integrable, then the quantum system has one type of spectral statistics. If its corresponding classical dynamics are chaotic then the quantum system has a second type of spectral statistics. For integrable system the spacing distributions tend to be described by a Poisson probability distribution function. For non-integrable (chaotic) systems the spacing distribution is that of the eigenvalues of a random matrix drawn from the Gaussian Orthogonal Ensemble (GOE) if the system has time reversal invariance and drawn from the Gaussian Unitary Ensemble (GUE) if time reversal is broken.

Although the waves in a quantum graph propagate on one dimensional bonds, the system is complicated by the topology of the graph. Multiple bonds of different lengths meet at nodes. There may be multiple paths connecting pairs of nodes. Thus, it is natural to explore whether the same spectral properties that apply to 2D and 3D enclosures also apply to quantum graphs. In fact, modes of a quantum graph can feature the characteristics of waves in 2D and 3D enclosures. Similar to 2D and 3D wave systems, quantum graphs can be divided into three categories: orthogonal, unitary and symplectic. The spectral properties of quantum graphs with time-reversal invariance are similar to those of random matrices generated from the Gaussian Orthogonal Ensembles (GOE). Such systems are realized experimentally by networks of transmission lines connected at nodes characterized by symmetric scattering matrices. Without time-reversal invariance, quantum graphs have features similar to random matrices generated from the Gaussian Unitary Ensemble (GUE). Such networks are experimentally produced by inserting circulators onto transmission cables. Quantum graphs with Gaussian Symplectic Ensembles (GSE) spectrum can be approximated by creating graphs with symmetric pathways and circulators.

While quantum graph eigenmodes share features with 2D and 3D counterparts, a study [\[13\]](#)

has shown that the statistics of quantum graph eigenmodes deviate from the RMT predictions, which well-describe 2D and 3D cavity eigenmodes. This is due to two main factors. One mainly appears in the experimental setup, and another one is found in both experimental measurements and theoretical calculations. The first one is the incomplete spectra that are caused by the absorption of the energy levels by the cables. Thus, we have missing energy levels that cause deviations from the RMT predictions. The second factor is wave reflection at the ends of bonds that causes wave intensity to retrace its path and become partially trapped on a bond. Waves in a 2D or 3D cavity typically strike the surface of the cavity at an angle and are specularly reflected. The full wave intensity continues its trajectory. For cases in which the ray trajectories are chaotic, each eigenmode can then be viewed as a random superposition of plane waves. In contrast, modes of a quantum graph can be localized in a small portion of the graph. These modes are nongeneric due to the fact that they do not perceive the chaotic nature of the underlying classical dynamics, which occurs from scattering at all the nodes. These non-universal contributions lead to significant deviations, especially for the long-range spectral properties, which can be investigated using number variance [14] and power spectrum [15–17]. Here it should be noted that in experiments systems with time reversal invariance have more significant deviations from RMT than those without time reversal invariance. Higher deviations happen due to the fact that in experiments circulators are used as nodes, which hinders backscattering. Therefore, fewer waves are trapped on bonds.

The statistics of the spacings between adjacent mode wave numbers, or the spacings of the corresponding frequencies and energy levels, are studied extensively in various papers [13, 18–21]. As has been mentioned, the distribution of spacings of mode wave numbers in integrable systems is described by the Poisson probability distribution function. The Poisson distribution describes the spacing between a sequence of random numbers that are independent and uniformly

distributed on an interval. The Poisson distribution of spacings is realized in a quantum graph when the nodes are fully reflecting. In this case each node reflects all the wave amplitude incident on a bond, and none is transmitted to the other bonds shared by that node. This happens when we set Dirichlet boundary conditions at all of the nodes and consider bonds of varying, random lengths. As a result, waves are trapped on their corresponding bonds, bonds do not communicate with other bonds. The total wavenumber spectrum is thus a superposition of independent spectra with uniform spacing distributions. From here, we can conclude that the probability distribution function of the level spacings of the mode wave numbers depend on the reflection coefficient of the nodes.

Neumann boundary conditions apply when the wave functions on each bond connected to a particular node have the same values, and the sum of the derivatives of the wave functions at each node vanish. The signal reflection coefficient in this case is $2/n - 1$, where n is the number of bonds connected at the node. For $n = 3$ the magnitude of the reflection coefficient is $1/3$, and a very small amount of signal power is reflected, thus, giving the waves the ability to travel inside the network and giving the system more universal wave chaotic properties. A system with Neumann boundary conditions and with three bonds connected at each node is well-described by RMT. In this thesis we will introduce a boundary condition that allows us to vary the magnitude of the reflection coefficient between 1 and $1/3$, and observe the transition between GOE spacing statistics and Poisson statistics. We can also compare the spacing statistics of a graph with this continuously variable reflection coefficient with graphs in which a fraction of nodes has reflection coefficient magnitude $1/3$ and the remainder have reflection coefficient unity. All these ideas will be discussed in more detail in the next chapter.

In addition to the spacing distribution of wavenumbers, there have been studies [19] of other

characteristics of wavenumbers, for example, the properties of the counting function, $N(k)$, that describes how many wavenumbers for a particular graph are less than k . The counting function is a monotonically increasing function of k and is approximately $N(k) = kL_T/\pi$ for large k . Here L_T is the total length of the graph.

Less well studied are the statistical properties of the graph wave functions. We have investigated the node wavefunction statistics and corresponding wave function amplitudes on the bonds. For wavefunctions in 2D and 3D domains the statistics are described by the “random plane wave hypothesis”. This results in the wavefunctions behaving as zero-mean Gaussian random variables with a universal two-point correlation function. We find that as a graph grows in size the statistics of the node voltages approach the “random plane wave hypothesis”. The numerical calculations and corresponding results will be discussed in the next chapter.

In this thesis, the statistics of the impedance matrix of the system is the central area of the study. In the literature, one can find studies of the scattering matrix [3,22] and the Wigner reaction matrix [23] (sometimes called the K matrix [24]). Both of these are related to the impedance matrix. In wave chaotic systems, the impedance matrix is most often presented in normalized form. The normalization is made using the real part of the radiation impedance matrix. The radiation impedance matrix is diagonal and the elements are the real part of the radiation resistance of each port. This is the real part of the impedance seen at each port and which determines the prompt reflection for a wave incident on the port. It is very convenient to use because, in experiments, it can be measured very well, and in addition, it can be easily computed in theoretical calculations. The radiation impedance can be conceptualized in the following way. If other boundaries in the system are perfectly absorbing or are moved to infinity, then the impedance we are calculating or measuring at the port is the radiation impedance. The radiation impedance is a N by N complex-

valued matrix. The elements of this matrix depend weakly on wave frequency change, and thus, they have non-statistical values. The diagonal elements of the radiation impedance matrix describe the coupling between the port and the system. However, non-diagonal elements give us information on how different ports communicate with each other [25].

The statistics of the normalized impedance matrix are described quite well by the random coupling model (RCM) [26], which is based on RMT. Here we should note that, as we already mentioned, other characteristics of the wave chaotic systems are described by RMT. There are several studies [4, 27, 28] that compare experimental measurements of either Wigner's reaction K matrix or the impedance matrix of the microwave networks to the theoretical calculations based on RCM or RMT [26, 29]. In these works, they showed that experimental and numerical results of the diagonal elements of impedance or K matrices are in good overall agreement systems with the time-reversal symmetry. The study [4] also shows that non-diagonal elements from experimental measurements and theoretical calculations also match each other quite well. However, in theoretical calculations based on RCM, the loss parameter should be different from the one used in calculations for diagonal elements. This means that when we have experimental measurements, two loss parameters are needed in theoretical calculations. One is needed to match the diagonal elements, and another one is needed to match the non-diagonal elements. In addition, the loss parameter used for diagonal elements is always higher than the one used for non-diagonal elements.

Now let us summarize our results. We have developed a theoretical framework to calculate the characteristics of the microwave network, including the impedance matrix. We have developed three different approaches to the calculation of the impedance matrix. One of the approaches was using summed contributions from the different paths a signal may take in traversing the

graph. This approach allows us to calculate the contributions only coming from the short-orbits and, therefore, enables us to calculate the impedance matrix without short-orbit contributions. Finally, we compared results from experimental measurements of the impedance matrix and the theoretical calculations based on the predictions of RCM with the numerical calculations using our theoretical framework. On the one hand, we have found that our theoretical framework is in good agreement with the experimental results and, therefore, accurately describes the impedance statistics of the microwave network.

On the other hand, during the comparison of the results of the impedance matrices to each other, one coming from the numerical calculations from our theoretical framework and another coming from the theoretical calculations based on RCM, we found out that to match these results, we needed two different loss parameter values in the calculations based on RCM. This is quite similar to the studies [4, 27, 28] mentioned above, where we compared the experimental measurements and the theoretical results from RCM. This is not surprising due to the fact that our numerical calculations were matching to the experimental measurements.

Chapter 2: Graph Topology

In this chapter, we will discuss the topology of the graph. First, we define a graph, the kinds of graphs, and the simplifications we consider in the thesis. After that, we will define the metric graph and then investigate the theoretical framework that exists on a metric graph. This chapter mainly follows the book by G. Berkolaiko and P. Kuchment [30], and the review paper by S. Gnuzmann and U. Smilansky [5].

2.1 Graph Definition

A graph is defined as a number of bonds (or edges) connected to a number of nodes (or vertices). Let's denote it as $G = G(N, B)$, where N and B are the numbers of nodes and bonds it contains correspondingly. N and B can be defined as sets of corresponding elements, $N = \{n_i\}, B = \{b_i\}$. The way nodes and bonds are distributed in the graph is described by the connectivity matrix (or adjacency matrix) C_{ij} . C_{ij} is $N \times N$ matrix and represents how many bonds connect one node to another; thus, it is generally defined as

$$C_{ij} = \begin{cases} m & \text{if } i \neq j, \text{ and node } n_i \text{ is connected to node } n_j \text{ by } m \text{ bonds.} \\ 2m & \text{if } i = j \text{ and } n_i \text{ is connected to itself with } m \text{ loops.} \\ 0 & \text{if } i \text{ and } j \text{ are not connected by any bond.} \end{cases} \quad (2.1)$$

To further characterize the graph we have to mention some of its main features. These are the valency (or the degree of the nodes), the neighborhood, and the boundary. The valency tells us how many bonds connect to a particular node. We can denote it as v_i corresponding to each node, and to calculate it using the connectivity matrix, we can write the following equation

$$v_i = \sum_{j=1}^N C_{ij} . \quad (2.2)$$

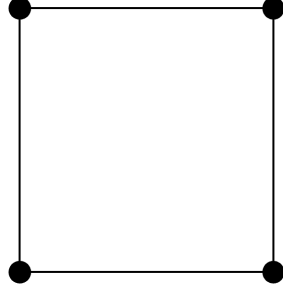
The neighborhood, denoted as Γ_i , is defined for each node n_i , and it consists of the list of which other nodes connect directly with node n_i . In other words, all the nodes with at least one connected bond to node n_i are in the neighborhood of Γ_i . The last feature mentioned above is the boundary, defined as the collection of the nodes surrounding the subgraph. The boundary contains every node in the neighborhood of any node contained in the subgraph, where subgraph is defined as a graph whose nodes and bonds are subsets of another graph.

The total number of bonds can be easily written in terms of the connectivity matrix

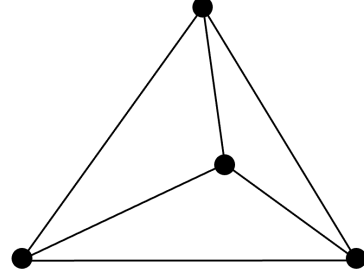
$$B = \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N C_{ij} \quad (2.3)$$

Let's now define connected graphs. Graphs are called connected if we are not able to divide a graph into two or more subgraphs such that any node from one subgraph does not connect through a bond to a node from another subgraph. In this thesis, we always assume that we have a connected graph. Graphs can be divided into categories in terms of how they are connected. Some of the most well-known categories are simple graphs, v -regular graphs, tree graphs, etc.

Simple graphs, as the name suggests, are graphs with no loops and without multiple bonds connecting any two nodes. The connectivity matrix will have zeros at the diagonal, and the non-



(a) Quadrangular graph



(b) Tetrahedron graph

Figure 2.1: Simple examples of v -regular graph. (a) shows quadrangular graph which has 4 nodes with valency 2. It is also an example of ring graph with 4 nodes. (b) shows tetrahedron graph with 4 nodes with valency 3. It is also an example of complete graph.

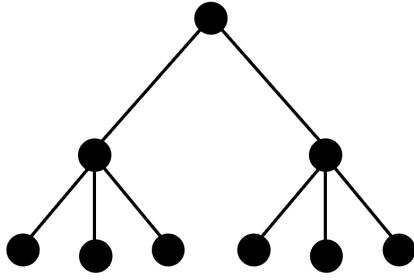
diagonal elements will be either zero or one. In terms of a connectivity matrix, it can be written as

$$C_{ij} = \begin{cases} 1 & \text{if } n_i \text{ and } n_j \text{ are connected.} \\ 0 & \text{if } n_i \text{ and } n_j \text{ are not connected.} \end{cases} \quad (2.4)$$

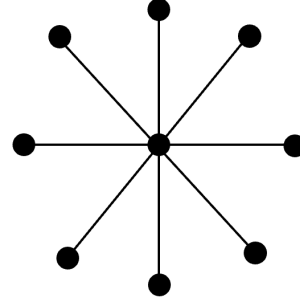
Simple graphs are the most commonly used graphs in quantum graph theory. This comes from the fact that simple graphs can be created from any graph by adding more nodes into the bonds without changing the spectral properties of the system.

The v -regular graphs, as the name suggests, are the graphs with the same valency for every node. One of the good examples is the ring graph, which has four nodes, each with a valency of two. Furthermore, if all the nodes are connected to each other then we have a subcategory of v -regular graphs called a complete graph. A common example of complete graphs is a tetrahedron graph, which has four nodes, and the valency is three for each node. Figure 2.1 shows the examples of a quadrangular graph and a tetrahedron graph.

Tree graphs are the graphs having no cycle in them. A cycle is defined as having a sequence



(a) Tree graph



(b) Star graph

Figure 2.2: Simple examples of tree graphs. (a) shows one of the common case of the tree graph which has 9 nodes. (b) shows star graph with 9 nodes and 8 bonds. It is a subcategory of tree graphs.

of nodes that are connected to each other in a way that you can return to a starting point without retracing your path on a bond. One of the interesting subgroups of graphs is star graphs, which have one node with valency the same as the total bond number, $v_i = B$. Other nodes have a valency one, $v_i = 1$. Figure 2.2 shows the example of a typical tree graph and a star graph.

As mentioned bonds can be directive, which we can use to define the trajectory inside the graph. Recall, a directed bond; as its name suggests, permits information to travel only in one direction between the two nodes it connects. A notation we use for a directed bond is $b = [i, j]$, meaning that you can travel from i to j . In addition, a notation for any function, f , defined on the same bond is f_{ij} . If one directed bond ends on a node which is a starting point for the second directed bond, this means that the second bond follows the first one. A trajectory is a path on the graph starting from one node and ending on another; this type of object can be described by a sequence of directed bonds from where i^{th} directed bond, b_i , follows b_{i-1} , and b_{i+1} follows b_i . The topological length of the trajectory is the same as the number of bonds it contains; thus, it might be anything from zero to infinity. A periodic orbit is a trajectory that ends on the same node from where it starts. The simplest form of a periodic orbit is called a primitive periodic

orbit, which is the trajectory that can not be written with two or more of the same periodic orbits. Furthermore, closed trajectories are defined similarly to periodic orbits, and we can use them interchangeably.

2.2 Schrodinger's Equation on a metric graph

In the previous section, we talked about graphs as abstract objects. The nodes were connected by the bonds mathematically and did not convey any physical properties. Let's now define metric graphs to be graphs where the bonds connecting the nodes are assigned lengths. A graph is said to be a metric graph [30] if the following conditions are satisfied:

1. Each bond must have a positive length, $L_{ij} > 0$.
2. The length of the bond is the same for both directions, $L_{ij} = L_{ji}$.
3. A coordinate x_{ij} defined on the bond satisfies, $0 \leq x_{ij} \leq L_{ji}$.
4. Coordinates on the same bond with reversal indices have the following relation, $x_{ji} = L_{ij} - x_{ij}$.

To consider metric graphs as quantum graphs, we should be able to apply a quantum operator to the graph, and primarily, a self-adjoint Hamiltonian operator is used. The simplest form we can use is the negative second order derivative that is applied to the bond

$$f(x) \rightarrow -\frac{d^2 f}{dx^2} . \quad (2.5)$$

Where, x is the coordinate on the bond, $0 \leq x \leq L$.

A more general way of writing operator on the graph is to use one-dimensional Schrödinger's operator with potential $V(x)$.

$$f(x) \rightarrow -\frac{d^2 f}{dx^2} + V(x)f(x) \quad (2.6)$$

It should be said that the two operators defined here assume the bond has no preferred direction. This is due to the lack of a first-order derivation operator, d/dx , in the equation. However, if we take one of the most general forms of the Schrödinger's operator that includes both a scalar potential, $V(x)$, and vector-like potential, $A(x)$, then the direction of the graphs plays an important role.

$$f(x) \rightarrow \left(\frac{1}{i} \frac{d}{dx} + A(x) \right)^2 f(x) + V(x)f(x) \quad (2.7)$$

It is important to notice that magnetic potential, $A(x)$, here is a vector field and it changes sign according to the direction of x , $A_{[i,j]}(x_{[i,j]}) = -A_{[j,i]}(x_{[j,i]})$, recalling the fact that $x_{[j,i]} = L_{[i,j]} - x_{[i,j]}$. We have to make important assumptions about the electric and magnetic potentials. The electric potential is non-negative and smooth on the whole bond length, $V(x) \geq 0$ and $0 \leq x \leq L_b$. Similarly, the magnetic potential is non-negative; however, it is mostly considered a constant function over the length of the bond, $A(x) = A$ and $A \geq 0$. One of the quantum graphs' basic solutions is the eigenfunctions of the closed system. We assume that there are no external energy sources, which is the general case in most literature. This assumption translates as non-existence of electric potential, $V(x) = 0$. To find the wave function of this particular case, we should solve the following equation

$$\left(\frac{1}{i} \frac{d}{dx} + A\right)^2 \psi(x) = k^2 \psi(x) \quad (2.8)$$

$$\psi(x) = e^{-iAx} (c_1 e^{ikx} + c_2 e^{-ikx}) \quad (2.9)$$

Here, k is the wave vector, $\psi(x)$ is the wave function solution of the equation, c_1 and c_2 are arbitrary constants. The form of the solution, $\psi(x)$, shows that existence of magnetic potential, A , breaks the symmetry under the reflection of x , thus, $\psi(x) \neq \psi(-x)$. Furthermore, time reversal invariance is broken because a complex conjugate of $\psi(x)$, $\psi^*(x)$, is not the solution of the differential equation.

If we define the wave function solutions on the nodes as ϕ_i for node i , then we have the following boundary conditions on the bond $[i, j]$

$$\psi_{ij}(x) = \begin{cases} \phi_i, & x = 0. \\ \phi_j, & x = L_{ij}. \end{cases} \quad (2.10)$$

These boundary conditions uniquely define the solution of the Equation (2.9) given by (2.10). Thus, c_1 and c_2 can be written in terms of ϕ_i and ϕ_j in the following way

$$c_1 = \frac{\phi_i e^{iA_{ij}L_{ij}} - \phi_j e^{-ikL_{ij}}}{2i \sin(kL_{ij})}$$

$$c_2 = \frac{\phi_i e^{ikL_{ij}} - \phi_j e^{iA_{ij}L_{ij}}}{2i \sin(kL_{ij})}$$

this leads to the final form of the solution, given by (1.9), in terms of boundary values of the wave

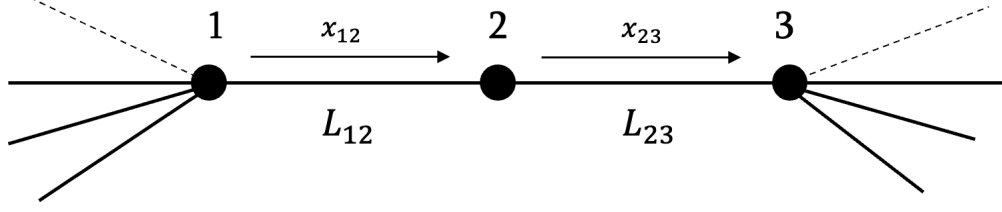


Figure 2.3: Example of a part of a graph with the node with valency 2.

function. Thus we have

$$\psi_{ij}(x) = \frac{e^{-iA_{ij}x}}{\sin(kL_{ij})} \left(\phi_j e^{iA_{ij}L_{ij}} \sin(kx) + \phi_i \sin(k(L_{ij} - x)) \right) . \quad (2.11)$$

We have already ensured wave functions satisfy boundary conditions on the nodes. However, we have not shown yet that these boundary conditions guarantee continuity and current conservation on each node. The continuity means that every wave function on the bonds connecting to the same node should have the same value on that node. The current conservation requires that the current entering the node from the connected bonds is described by the boundary condition on the node. To derive the current equation, we start from the Schrödinger's equation on the bond, and for simplicity, let's look into the case with the node that has valency two. This case is shown in figure 2.3. We are using a common assumption of having a delta potential (or delta coupling) at the node. We focus on node 2, and we want to write the current equation on that. We have already created a rule for how coordinates are defined on the bond. Basically, we have following coordinates x_{12}, x_{21}, x_{23} and x_{32} , and there are the following relations between them, $x_{12} = L_{12} - x_{21}$ and $x_{23} = L_{23} - x_{32}$. Because we have two different bonds, and they have their coordinate system, it will be hard to write Schrödinger's equation on both bonds together. So, let's introduce a new coordinate y , and assign a value 0 at node 2, a value $-L_{12}$ at node 1, and a value L_{23} at node 3, as shown in Figure 2.4. Now we can write Schrödinger's equation with

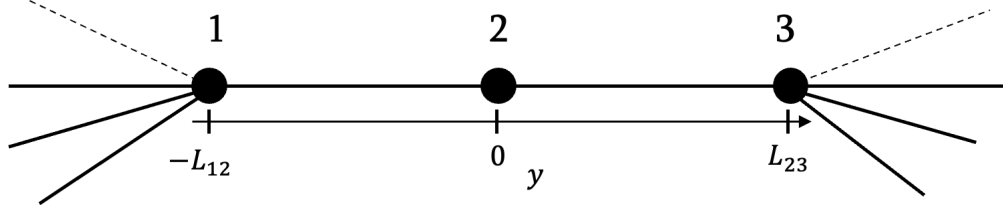


Figure 2.4: Example of a part of a graph with the node with valency 2. Using y coordinate.

respect of y coordinate.

$$\left(\frac{1}{i} \frac{d}{dy} + A(y) \right)^2 \psi(y) + \lambda \delta(y) \psi(y) = E \psi(y) \quad (2.12)$$

Let's simplify it and then integrate from $y = -\epsilon$ to $y = +\epsilon$, which means integrating around node 2. Here, we should note that $A(y) = A_{12}$ if $-L_{12} < y < 0$ and $A(y) = A_{23}$ if $0 < y < L_{23}$

$$\lim_{\epsilon \rightarrow 0^+} \int_{-\epsilon}^{+\epsilon} \left(-\frac{d^2 \psi(y)}{dy^2} - i A(y) \frac{d\psi(y)}{dy} + A^2(y) \psi(y) + \lambda \delta(y) \psi(y) - E \psi(y) \right) dy = 0 \quad (2.13)$$

After simple transformations we end up with the following equation

$$\lim_{\epsilon \rightarrow 0^+} \left(\frac{d}{dy} + i A(y) \right) \psi(y) \Big|_{-\epsilon}^{+\epsilon} = \lambda \psi(0) \quad (2.14)$$

Now let's translate the y coordinate into the x coordinates. For $y < 0$ we can use either x_{12} or x_{21} , and similarly, for $y > 0$ we can use either x_{23} or x_{32} . We can assign the primary coordinates with the following rule to remove the confusion between these options. If the first index is lower than the second, then it is the primary coordinate, and we can simply call it x . This assignment creates a rule that specifies conditions for $x = 0$ and $x = L_{ij}$. Which can be described in the

following way

$$x|_{\text{at node } i} = \begin{cases} 0, & \text{if } i < j. \\ L_{ij}, & \text{if } i > j. \end{cases} \quad (2.15)$$

Similarly, $A(y)$ becomes $A_{12}(x)$ and $A_{23}(x)$ if $y < 0$ and $y > 0$, correspondingly. Finally, the translated form of equation (2.14) from y coordinate to x coordinate reads the following way

$$\left(\frac{d}{dx} + iA_{23}(x) \right) \psi_{23}(x) \Big|_{x=0} - \left(\frac{d}{dx} + iA_{12}(x) \right) \psi_{12}(x) \Big|_{x=L_{12}} = \lambda \psi_{23}(0) \quad (2.16)$$

Here it should be noted that $\psi_{23}(0) = \psi_{12}(L_{12})$ and it is defined as ϕ_2 . Equation (2.16) can be easily generalized. Thus, the final and general form of the current equation reads the following way

$$\sum_{i < j} \left(\frac{d}{dx} + iA_{ij} \right) \psi_{ij}(x) \Big|_{x=0} - \sum_{i > j} \left(\frac{d}{dx} + iA_{ji} \right) \psi_{ji}(x) \Big|_{x=L_{ij}} = \lambda_i \phi_i. \quad (2.17)$$

Here, every λ is defined on the nodes and determined by the boundary condition on each node.

Now we have the wave function equation given by (2.11) and the current equation on each node given by (2.17); thus, we can use them to create an equation that describes the whole graph. Let's substitute ψ_{ij} and ψ_{ji} by their expressions from (2.11) inside (2.17), and calculate piece by piece,

$$\left(\frac{d}{dx} + iA_{ij} \right) \psi_{ij}(x) \Big|_{x=0} = -\phi_i k \cot(kL_{ij}) + \phi_j \frac{ke^{iA_{ij}L_{ij}}}{\sin(kL_{ij})}, \quad (2.18)$$

and

$$\left(\frac{d}{dx} + iA_{ji} \right) \psi_{ji}(x) \Big|_{x=L_{ij}} = \phi_i k \cot(kL_{ij}) - \phi_j \frac{ke^{iA_{ij}L_{ij}}}{\sin(kL_{ij})}. \quad (2.19)$$

Here we used the fact that $A_{ji} = -A_{ij}$, and $L_{ji} = L_{ij}$. So, as a result we have,

$$\left(-\frac{\lambda_i}{k} - \sum_{j \neq i} \cot(kL_{ij}) \right) \phi_i + \sum_{j \neq i} \frac{e^{iA_{ij}L_{ij}}}{\sin(kL_{ij})} \phi_j = 0 \quad (2.20)$$

This is a set of linear equations for ϕ 's and can be written in a matrix form

$$\sum_j h_{ji}(k) \phi_j = 0 \quad (2.21)$$

where,

$$h_{ii}(k) = -\frac{\lambda_i}{k} - \sum_{j \neq i} \cot(kL_{ij}) , \quad (2.22)$$

and

$$h_{ji}(k) = \sum_{j \neq i} \frac{e^{iA_{ij}L_{ij}}}{\sin(kL_{ij})} . \quad (2.23)$$

Equation (1.21) has trivial solution by setting all the ϕ 's to zeros. However, we are only interested in non-trivial solution that is

$$\det(h_{ij}(k)) = 0 . \quad (2.24)$$

Equation (2.24) is called secular equation [5], and its solution gives us the eigenspectrum of wave vectors, k 's, of the system.

Chapter 3: Random Coupling Model

Wave chaos refers to the study of the systems in which high-frequency (or short-wavelength) waves propagate in irregular enclosures. Since the waves have a shorter wavelength compared to the geometrical sizes of the surrounding enclosure, they act like a particle that reflects and bounces around in the enclosure. The geometrical characteristics of the system can cause the interaction with the high-frequency wave, which can be represented as a particle, to be a chaotic process. Wave chaos is actively studied in many different fields [31–38]. Because chaotic systems are extremely sensitive to the initial conditions and the characteristics of the enclosure, direct calculations of the wave fields take a long time and large resources. Therefore, a statistical approach is a good alternative to describe the system and the solutions of the corresponding processes. Such an approach has been developed based on random matrix theory (RMT) [39, 40] and is known as the random coupling model (RCM) [26].

Random matrix theory has been used extensively over the years and has successfully predicted the statistical properties of the wave chaotic systems. [29, 41–46]. The statistics depend on two main features. The first one is the time symmetry of the system, meaning that if the system has time-reversal symmetry, it is described by GOE (Gaussian Orthogonal Ensemble), and if time reversal symmetry is broken, it is described by GUE (Gaussian Unitary Ensemble). The second one is the characteristics of the loss in the system.

The random coupling model has been successfully used to study microwave electromagnetic waves coupled with the chaotic systems. [26,47–58]. Commonly, microwaves are coupled with systems having several ports. In practice, the ports can be anything that can receive or send an electromagnetic wave signal, for example, some sensors or antennas. Due to the success of the random coupling model over the years, new fields have started adapting this model. People are studying systems with both regular and chaotic ray dynamics and coupled chaotic systems. At first, the random coupling model was used mainly for linear systems; however, now, the random coupling model is used in nonlinear systems [59–69]. Nonlinear systems have more complex settings; therefore, it will modify this model to address nonlinear chaotic systems’ problems correctly. One thing to note for nonlinear systems is that the universal properties of the system do not work as a linear superposition of each system.

Wave chaotic systems have non-universal properties that alter the universal fluctuations of the undergoing processes. The most common non-universal features are short-orbit effects and the coupling between the ports and the electromagnetic waves. These non-universal features are addressed quite well by the random coupling model. We can divide this model into two parts. The first describes an integrable (or deterministic) process, which contains a non-statistical part of the system. The second one describes a non-integrable (or non-deterministic) process with universal statistics, which is assumed to be predicted by the random matrix theory. In other words, the eigenvalues of large random matrices are used as the values of the characteristic elements of the system. The random coupling model mainly defines the impedance matrix of the system. To capture the non-universal contributions from the impedance matrix, we can average the impedance matrix over an ensemble of realizations that are easily doable experimentally and numerically. Once we capture non-universal properties, we can remove them from the data, and the remain-

ing data should match the predictions of the random matrix theory. This technique is used in various dimensional (1D, 2D, 3D) wave chaotic systems [50, 51, 70]. Other than fluctuations of the impedance matrix, the random coupling model also describes fluctuations of the scattering matrix, which is closely related to the impedance matrix. The relation between the impedance and the scattering matrices is the following

$$S = Z_0^{\frac{1}{2}}(Z + Z_0)^{-1}(Z - Z_0)Z_0^{-\frac{1}{2}}, \quad (3.1)$$

where S is the scattering matrix, Z_0 is the matrix describing system specific features and Z is the impedance matrix of the system. Each element in the matrix is complex, and (i, j) member of the matrix contains the value of impedance between i^{th} and j^{th} ports. According to the random coupling model, the impedance matrix contains the universally fluctuating impedance ξ_Z (also known as normalized impedance matrix), and the impedance matrix equation is given in the following way

$$Z = i \text{Im}[Z_{avg}] + (\text{Re}[Z_{avg}])^{\frac{1}{2}} \xi_Z (\text{Re}[Z_{avg}])^{\frac{1}{2}}, \quad (3.2)$$

where

$$Z_{avg} = \text{Re}[Z_{avg}] + i \text{Im}[Z_{avg}] \quad (3.3)$$

is the average impedance of the system, which is averaged over the ensembles of realization. Z_{avg} contains information on the specific system features it is realized on. It is non-statistical, system-specific, and can be estimated if we know the radiation impedance and the geometrical properties of the system. In other words, Z_{avg} contains information on non-universal systems properties, such as the radiation impedance and the short orbit effects. The radiation impedance

can be easily measured experimentally and numerically. The diagonal elements of the radiation impedance matrix give us information about the coupling between ports and the chaotic system. These elements represent the impedance of the system at the ports in the case where the signal waves injected to the port do not come back; they either are absorbed by the system or the system is so large that the waves never return to the ports. In the experiments on quantum graphs, the radiation impedance can be measured by setting high loss on the bonds (either taking relatively long or relatively highly lossy bonds). In the 2D-3D cavities, the walls of the cavities can be covered with the perfect absorbers. We assume the ports are perfectly coupled and communicate with each other by passing waves through the chaotic system. Therefore, the radiation impedance is a diagonal matrix where the diagonal elements correspond to the radiation impedance of each port. The real part of the radiation impedance acts as radiation resistance because it essentially represents the ability of far-field radiation of the ports. The imaginary part of radiation impedance represents the near-field energy of the ports. As already mentioned above, short orbit effects are the paths of the waves that are not ergodic and do not travel around the whole system. Therefore, short orbit contributions are system-specific non-universal features.

The random coupling model (RCM) proposes the equation for the universally fluctuating impedance matrix, it is given in the following form

$$\xi_{Z,i,j} = \frac{i}{\pi} \sum_{m=1}^M \frac{W_{i,m} W_{j,m}}{i\alpha - \lambda_m}, \quad (3.4)$$

where $\xi_{Z,i,j}$ is the impedance between the port i and port j , M is the number of mode in the system, and $i = \sqrt{-1}$. The element $W_{i,m}$ in the equation gives the value of the coupling between i^{th} port and m^{th} eigenmode. In addition, $W_{i,m}$ and $W_{j,m}$ are assumed to be independent and identi-

cally distributed while having Gaussian probability distribution with zero mean and unit variance. These assumptions come from the hypothesis known as the random plane wave hypothesis (or the Berry hypothesis). The idea is that eigenmodes behave as a superposition of random plane waves, due to the superposition, we have a sum of the random values that are independent and identically distributed; thus, using the central limit theorem, the corresponding summed value should have Gaussian probability distribution. The term α is a loss parameter and determines the statistics of the universal fluctuations of the impedance matrix or corresponding scattering matrix of the system. According to the random coupling model, the loss parameter α is given by

$$\alpha = \frac{k^2}{\Delta k_n^2 Q}, \quad (3.5)$$

where k is the wave vector, Q is the quality factor of the chaotic cavity, $\Delta k_n^2 = \langle k_{n+1}^2 - k_n^2 \rangle \approx 2k_n \Delta k_n$, and here Δk_n is the mean level spacing of the mode wave vectors.

The final term in the equation that we have yet to discuss is λ_m is the m^{th} eigenvalue of the large random matrix, whose statistics are given by the random matrix theory. In the case of time-reversal symmetry, the random matrix should be from the Gaussian Orthogonal Ensemble (GOE). If the symmetry is broken, the random matrix should be from the Gaussian Unitary Ensemble (GUE). The probability distribution function of the eigenvalues of the matrix coming from the Gaussian Orthogonal Ensemble (GOE) is described by Wigner's semicircle. Therefore, the probability distribution function for λ is Wigner's semicircle, too; however, according to the random coupling model, the average spacing between the λ_m and λ_{m+1} should be unity. Therefore, we can numerically generate the distribution using the steps from this study [26]. Here are the key steps:

1. Create $5M \times 5M$ symmetric matrix which is a member of the Gaussian Orthogonal Ensemble (GOE).
2. Find the eigenvalues of the chosen matrix.
3. Normalize the middle M eigenvalues so that the spacings between them become unity.

Now let us discuss the ideas behind each step.

In step 1, let us explain more details about the Gaussian Orthogonal Ensemble (GOE). It is a collection of symmetric matrices whose diagonal elements have unit variance and zero mean, and non-diagonal elements have $1/2$ variance and zero mean. This is because the symmetric matrices are given by

$$H = \frac{A + A^T}{2},$$

where H is the member of the Gaussian Orthogonal Ensemble (GOE), A^T is a transpose of A matrix, and A is a matrix whose each element is independent and identically distributed random variable has a gaussian probability distribution function with zero mean and unit variance. Therefore, on the diagonal element of H matrix, we simply have diagonal elements of A and A^T (both A and A^T have the same diagonal elements). However, the non-diagonal elements are the sum of the two independent and identically distributed random variables divided by two. That is why non-diagonal elements of matrix H have a zero mean and $1/2$ variance

In step 2, we find the eigenvalues because we assume that they describe the λ parameter in the equation.

In step 3, if we choose the M number large enough (for example $M \geq 200$), the eigenvalues will not have uniform spacings and will be distributed as Wigner's semicircle law. The mean

spacing distribution will be given by

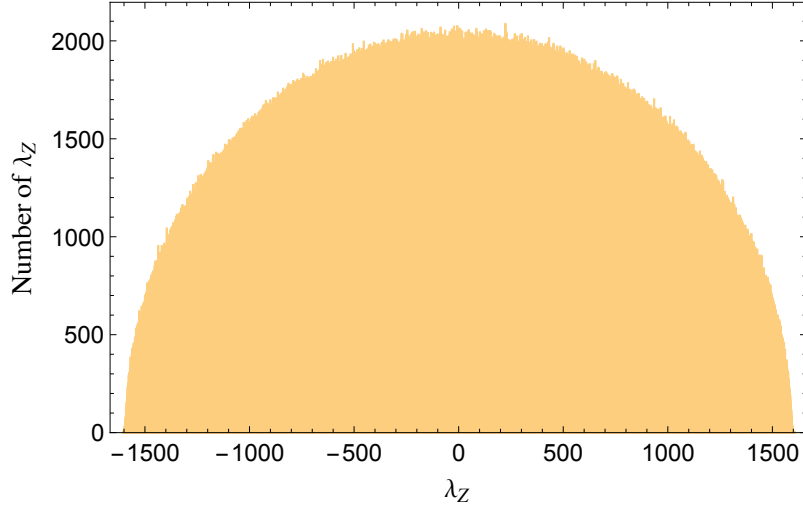
$$\Delta\lambda = \frac{\pi}{\sqrt{10M - \lambda^2}}. \quad (3.6)$$

If we multiply our eigenvalues, λ , on $\sqrt{10M}/\pi$, then the distribution of the mean spacings of the eigenvalue, λ , became

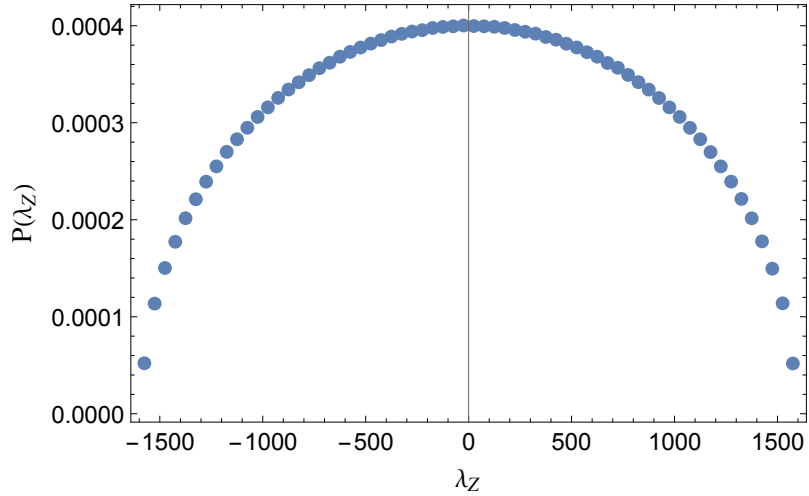
$$\Delta\lambda = \frac{1}{\sqrt{1 - \frac{\lambda^2}{10M}}}. \quad (3.7)$$

Here we want to make the mean spacing of the eigenvalues, λ , equal unity. This will happen if $\lambda \ll 10M$. Since eigenvalues are distributed symmetrically around 0, the smallest eigenvalues are in the middle of this distribution. That's why we choose the middle M eigenvalues to normalize by multiplying on $\sqrt{10M}/\pi$, and as a result, we have spacing that is very close to unity. The numerical results following these steps are shown in Fig. 3.1, and we see that Wigner's semicircle law very nicely describes the probability distribution of λ_Z , having unit spacing for normalized eigenvalues, $\lambda \rightarrow \lambda\sqrt{10M}/\pi$.

Now we have discussed how to numerically find every term except α from the equation of the normalized impedance matrix (3.4). The α parameter is found experimentally by comparing the statistics of the normalized impedance matrix, ξ_Z . We can measure the system impedance matrix and corresponding deterministic features in the experiments. Therefore, we can extract the data of a universally fluctuating impedance matrix. After that, we can change α parameter in equation (3.4), where other terms are generated numerically. In the end, we say the system has a loss parameter α for which experimentally and numerically generated statistics of ξ match each other. Note that these calculations are for the two port systems; therefore, we have 8 different



(a) Histogram of the λ_Z parameter from the equation (3.4).



(b) Probability distribution function of the λ_Z parameter from the equation (3.4).

Figure 3.1: We generated the statistics of the λ_Z parameter using the random coupling model and the steps discussed in this chapter. $M = 500$, and we used 2000 different realizations to generate the sample data for the eigenvalues of the matrices from the Gaussian Orthogonal Ensemble (GOE). Afterward, eigenvalues were normalized by multiplying on $\sqrt{10M}/\pi$.

statistics to compare, which are imaginary and real parts of the 2×2 matrix. However, we don't really need 8 different statistics because the statistics of ξ_{11} and ξ_{12} are the same, and also, the statistics of ξ_{12} and ξ_{21} are the same. Thus, finally we have left the following 4 statistics to compare, $\text{Re}[\xi_{11}]$, $\text{Im}[\xi_{11}]$, $\text{Re}[\xi_{12}]$, and $\text{Im}[\xi_{12}]$. Finally, one more detail should be checked, and that is if the one α parameter makes all 4 of the statistics from the experimental or the theoretical assessment match the random coupling model predictions simultaneously. The studies [26, 52] show that in 2D systems, we have the agreement between the experimental measurements and the random coupling model predictions. In other words, all 4 of the statistics match each other for one α parameter. However, the studies [4] on the quantum graphs show that we need two α parameters to have a match. One α for diagonal elements and another for non-diagonal elements.

Chapter 4: Transmission Lines

In this chapter, we will discuss the theory of transmission lines. Transmission lines are one of the main realizations of quantum graphs, and understanding them is crucial. Transmission lines are used to guide high-frequency electromagnetic waves in a contained manner. A transmission line consists of two conductors arranged in a parallel that carrying current in opposite directions. The necessity of transmission lines comes from the fact that electric cables not arranged in this way will radiate away energy. If the wavelength of the electromagnetic signal is smaller than the distance between a conductor and its return path, the cable acts as an antenna. In addition, high-frequency waves tend to reflect at the connectors and joints and, afterward, travel back toward the source; therefore, power signals do not reach the destination. These are the main reasons why ordinary cables are only used to carry low-frequency waves. A couple of examples are utility powers and audio signals.

Transmission lines are used to carry high-frequency waves. The term transmission line applies to cables when the cables are long enough that the wave nature of the transmission must be taken into account. This applies especially to the cables transmitting high-frequency electromagnetic waves. Due to the short wavelength, the wave phenomena arise over a very short distance. However, at first, the theory was created to describe a very long telegraph line, in particular underwater telegraph cables. Transmission lines' main purpose is to connect wave transmitters or

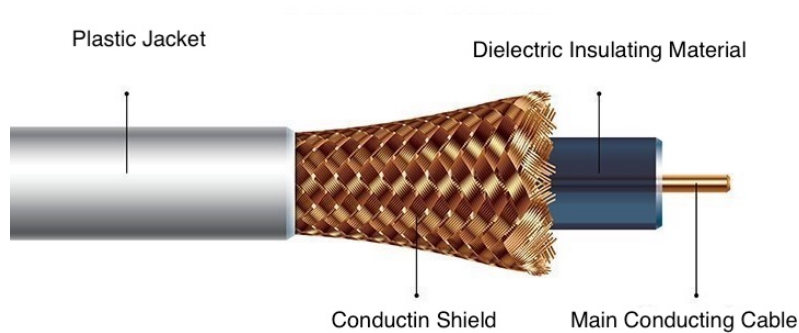


Figure 4.1: [1] Graphical view of the coaxial cable. It contains a main conducting cable, a dielectric insulating material, a conducting shield and a plastic jacket.

receivers to the antennas. Therefore, the use cases are to distribute cable television signals, network connections of computers, etc. Transmission lines are created to conduct electromagnetic waves with minimal power loss and reflections. In order to diminish the reflections as much as possible, the transmission lines have a uniform cross-sectional area along the line that ensures constant impedance as a function of length, which prevents reflections along the cable. Types of transmission lines include coaxial cables, parallel lines, etc.

The coaxial cable, shown in Figure 4.1, is a transmission line with a main conducting cable. Around the conducting cable is a dielectric insulating material, and around it is a conducting shield. Finally, this whole construction is protected by an outer plastic jacket. All elements are concentric, which is why it is called coaxial cable. They share a geometric axis. Coaxial cables are mainly used to carry radio frequency signals, for example, cable television signals, ethernet connection signals, etc. The advantage of coaxial cables compared to others is that the perfect coaxial cable keeps all the electromagnetic energy inside the cable, between the main conducting cable and the conducting shield. Thus, it can be installed nearby metal constructions without any power losses and interaction with outer objects.



Figure 4.2: [2] Graphical view of the twin-lead transmission line. It contains two parallel conducting cables held together by plastic band.

Parallel lines can be of different types. However, the twin-lead transmission line is the most common, as shown in Figure 4.2. Twin-lead is two parallel conducting cables held apart by a plastic band. Distance between the parallel conducting cables should be kept the same. Otherwise, we have reflected waves going back and, therefore, signal loss. Twin-lead transmission lines are mainly used to connect radio transmitters to their antennas. Their main advantage over the other transmission lines is their low signal loss.

Now let us discuss the physical properties of the transmission line. In order to see how the signal is transmitted through the cable, first, look at the arbitrarily small segment, δx , of the transmission line, which commonly is presented as a lumped element circuit, given in Figure 4.3. The properties such as R' , L' , G' , and C' are resistance, inductance, conductance, and capacitance per unit length, correspondingly. $I(x)$ and $V(x)$ are the values of the current and the voltage at the position x correspondingly. On the left-hand side of the arbitrarily small transmission line segment, we choose to be the position x , and therefore, the current is $I(x)$, and the voltage is $V(x)$. On the right-hand side, we choose the position $x + \delta x$; therefore, the current is $I(x + \delta x)$, and the voltage is $V(x + \delta x)$.

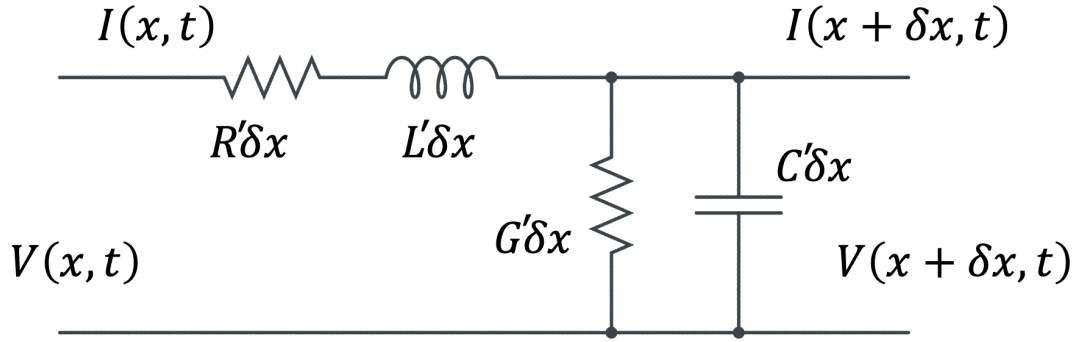


Figure 4.3: A model of a arbitrarily small segment of the transmission line.

Now let us apply Kirchhoff's circuit laws on the transmission line segment shown in Figure 4.3. The first Kirchhoff's law we use the total current entering a node is zero. The second Kirchhoff law we use is the sum of all the voltages around a loop is zero. As a result, applying the current law to the node where the capacitor and inductor meet produces

$$I(x) - G'\delta x V(x + \delta x, t) - C'\delta x \frac{\delta V(x + \delta x, t)}{\delta t} - I(x + \delta x, t) = 0 . \quad (4.1)$$

Applying the voltage law to a loop around all of Figure 4.3 produces

$$V(x, t) - R'\delta x I(x, t) - L'\delta x \frac{\delta I(x, t)}{\delta t} - V(x + \delta x, t) = 0 . \quad (4.2)$$

After simple modifications this equations can be written as

$$\frac{I(x + \delta x, t) - I(x)}{\delta x} = -G'V(x + \delta x, t) - C'\frac{\delta V(x + \delta x, t)}{\delta t} , \quad (4.3)$$

and

$$\frac{V(x + \delta x, t) - V(x, t)}{\delta x} = -R'I(x, t) - L'\frac{\delta I(x, t)}{\delta t} . \quad (4.4)$$

If we take the limits both of the sides, such that $\delta x \rightarrow 0$ and $\delta t \rightarrow 0$, we end up with the following expressions

$$\frac{\partial I(x, t)}{\partial x} = -G'V(x, t) - C'\frac{\partial V(x, t)}{\partial t} , \quad (4.5)$$

and

$$\frac{\partial V(x, t)}{\partial x} = -R'I(x, t) - L'\frac{\partial I(x, t)}{\partial t} . \quad (4.6)$$

These equations (4.5) and (4.6) are called the Telegrapher's equation [71–74], and they are the time-domain form of the transmission line. We can further transform these equations to have each with only one dependent variable, either $V(x, t)$ or $I(x, t)$. To do these transformations, at first, we should apply $\partial/\partial t$ on equation (4.5) and $\partial/\partial x$ to equation (4.6), and afterward, substitute $\partial^2 I(x, t)/\partial t \partial x$ inside the equation (4.6) by the left-hand side of modified equation (4.5). Afterward, we should do the same things and vice-versa, and finally, we end up with the following expressions

$$\frac{\partial^2 V(x, t)}{\partial^2 x} - L'C'\frac{\partial^2 V(x, t)}{\partial^2 t} = (R'C' + G'L')\frac{\partial V(x, t)}{\partial t} + G'R'V(x, t) , \quad (4.7)$$

and

$$\frac{\partial^2 I(x, t)}{\partial^2 x} - L'C'\frac{\partial^2 I(x, t)}{\partial^2 t} = (R'C' + G'L')\frac{\partial I(x, t)}{\partial t} + G'R'I(x, t) . \quad (4.8)$$

These equations, (4.7) and (4.8), are identical except for the dependent variable. Therefore, the solutions for $V(x, t)$ and $I(x, t)$ will be different up to multiplication on a constant.

The Telegrapher's equations (4.5) and (4.6) can be simplified for the sinusoidal steady-state condition. While having cosine-based phasors. The simplified results are as follows

$$\frac{d\hat{V}(x)}{dx} = -(R' - i\omega L')\hat{I}(x) , \quad (4.9)$$

and

$$\frac{d\hat{I}(x)}{dx} = -(G' - i\omega C')\hat{V}(x) . \quad (4.10)$$

We can modify this equation to have each with only one dependent variable. Apply d/dx to equations (4.9) and (4.10), and afterward, substitute $dI(x)/dx$ and $dV(x)/dx$ into modified equations from the unmodified equations. In phasor forms, the voltage is written as $V(x, t) = \text{Re} \left(\hat{V}(x)e^{-i\omega t} \right)$, and the current is written as $I(x, t) = \text{Re} \left(\hat{I}(x)e^{-i\omega t} \right)$. As a result, the modified equations are given by

$$\frac{d^2\hat{V}(x)}{dx^2} - \gamma^2\hat{V}(x) = 0 , \quad (4.11)$$

and

$$\frac{d^2\hat{I}(x)}{dx^2} - \gamma^2\hat{I}(x) = 0 , \quad (4.12)$$

where

$$\gamma = \gamma_R + i\gamma_I = \sqrt{(R' - i\omega L')(G' - i\omega C')} \quad (4.13)$$

is a complex propagation constant. The real part, γ_R , is the attenuation constant that represents how quickly the propagating wave gets dumped. The imaginary part, γ_I is the phase constant.

The solutions of equations (4.11) and (4.12) are

$$\hat{V}(x) = V_+e^{\gamma x} + V_-e^{-\gamma x} , \quad (4.14)$$

and

$$\hat{I}(x) = I_+e^{\gamma x} + I_-e^{-\gamma x} . \quad (4.15)$$

These solutions represent propagating voltage and current waves, respectively.

We derived the theoretical framework and the Telegrapher's equations for the transmission lines using a model of the arbitrarily small segment, shown in Figure 4.3. Furthermore, we simplified the Telegrapher's equation and, afterward, found the solutions of the equations for the simplified sinusoidal steady-state condition.

Chapter 5: Network Model

In this section we present our model of a network of transmission lines. We will consider both the normal modes of the network (the eigenmodes) and the case of the network driven by an injected signal. We will then show how these two situations are related. In the case of the normal modes, we will discuss eigenmodes of the propagating wave vectors. In addition, we study the probability distribution functions of the voltage values at the nodes and the probability distribution functions of the traveling voltage wave amplitudes corresponding to normal modes. Furthermore, we will introduce several ways of calculating the matrices that describe the network.

The graph presented in Figure 5.1 serves as our network model. It is made up of transmission lines (the bonds) and junctions (the nodes) where the transmission lines join. We use a simple model that treats the graph as metric graph [5]; thus, the transmission lines has assigned the lengths. Furthermore, we assume that all the bonds have a characteristics impedance Z_0 , which, for simplicity, has the same value for every bond. Our simple model sets a single voltage value V_i to each node- i . In addition, each node have an impedance Z_i to the ground. All the nodes can be used as a port to receive an external signal, and therefore, current signal entering to the node- i travels to the ground through the impedance Z_i .

Now we start formulating our theoretical framework. Let us start from the basic parts of the graph, and look at two nodes i and j connected by a bond with the characteristic impedance

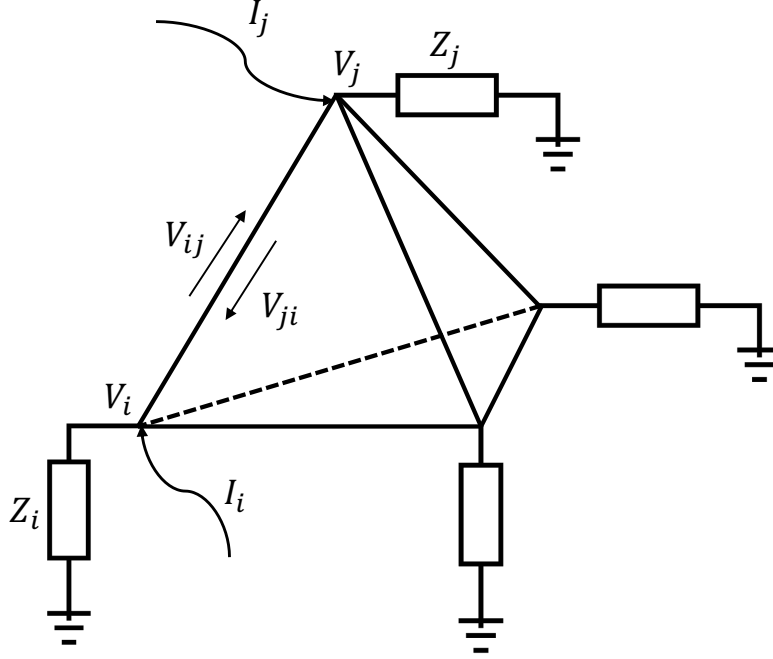


Figure 5.1: A schematic representation of a tetrahedral network with 4 nodes and 6 bonds. It is used as a representation of our network model.

Z_0 and the length of L_{ij} as shown on Figure 5.2. We have assumed that the voltages and currents on the transmission lines satisfy the frequency domain version of the Telegraphers equations

$$ikV(x) = Z_0 \frac{\partial I(x)}{\partial x} \quad (5.1a)$$

$$ikZ_0 I(x) = \frac{\partial V(x)}{\partial x} . \quad (5.1b)$$

Here $k = \omega/v$ is a propagation constant, where ω is the angular frequency, and v is the propagation speed. In the future, we will discuss the dumping rate on the bonds, and that case our propagation constant will be represented as $k = \omega/v + ik_{\text{im}}$ where k_{im} is a damping (or loss) rate.

We will consider both frequency independent and frequency dependent loss rates.

The electromagnetic waves can travel in both directions along each transmission line with

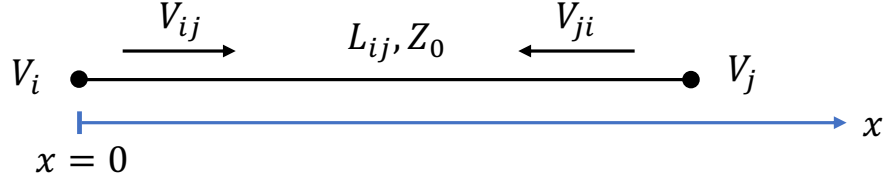


Figure 5.2: A model of two nodes i and j connected by one bond with the length of L_{ij} and characteristic impedance Z_0 . V_i and V_j are the voltage values on the nodes for node- i and node- j , correspondingly. V_{ij} and V_{ji} are the amplitudes of the traveling waves through the bond. V_{ij} travels from node- i to node- j , and V_{ji} vice-versa.

the propagation constant k . For simplicity, we assume all the transmission line properties are identical from bond to bond except for their lengths $L_{ij} = L_{ji}$. The traveling voltage waves from node- i to node- j have an amplitude labeled as V_{ij} . We assume that the voltage wave has the amplitude V_{ij} when it leaves the node- i . In order to connect the voltage values on the nodes and the amplitudes of the traveling voltage waves, we need to solve the Telegraphers equations (5.1a) and (5.1b). To accomplish that we can substitute $I(z)$ in the equation (5.1a) from the equation (5.1b), and we get the following

$$\frac{\partial^2 V(x)}{\partial x^2} + k^2 V(x) = 0 . \quad (5.2)$$

The solution on this equation is the traveling wave given by

$$V(x) = V_{ij} e^{ikx} + V_{ji} e^{-ik(x-L_{ij})} . \quad (5.3)$$

Here we should note that the voltage a distance z' from the node- j can be found either by replacement $z \rightarrow L_{ij} - z'$ or by the interchange of indices i and j . Using the equation (5.3) we can express the node voltage in terms of the wave amplitudes V_{ij} .

$$V_i = V_{ij} + V_{ji}e^{ikL_{ij}} \quad (5.4a)$$

$$V_j = V_{ji} + V_{ij}e^{ikL_{ij}} . \quad (5.4b)$$

From these equations we can derive expression for the traveling voltage wave amplitudes in terms of the node voltages,

$$V_{ij} = \frac{i(e^{ikL_{ij}}V_i - V_j)}{2\sin(kL_{ij})} \quad (5.5a)$$

$$V_{ji} = \frac{i(e^{ikL_{ij}}V_j - V_i)}{2\sin(kL_{ij})} . \quad (5.5b)$$

Now let us derive similar expression for the current waves traveling along the transmission line. In order to have every equation related to each other, we should express the current wave in terms of voltage waves. Substituting $V(z)$ from the equation (5.3) inside the equation (5.1b) results in the following

$$I(x) = \frac{1}{Z_0} (V_{ij}e^{ikx} - V_{ji}e^{-ik(x-L_{ij})}) , \quad (5.6)$$

where Z_0 is the characteristic impedance of the transmission line. This expression represents the current flowing on the line in the $+x$ direction. Thus, the current leaving node- i in the direction of node- j is

$$I_{i \rightarrow j} = \frac{1}{Z_0} (V_{ij} - V_{ji}e^{ikL_{ij}}) = \frac{i}{Z_0 \sin(kL_{ij})} (V_i \cos(kL_{ij}) - V_j) , \quad (5.7)$$

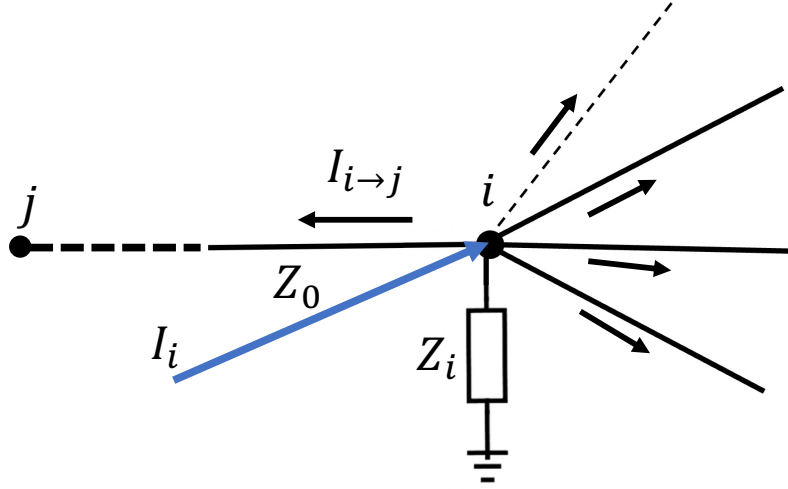


Figure 5.3: A schematic model of the node i connected node- j and other nodes. I_i is an injected current, $I_{i \rightarrow j}$ is the current leaving from node- i to node- j . The arrows point out the current leaving toward any connected node. Z_0 is the characteristic impedance of the bond, and Z_i is the impedance to the ground at node- i .

where we have used the equations (5.5a) and (5.5b) to express the voltage wave amplitudes in terms of the node voltages.

We now apply Kirchhoff's current law to node- i . The schematic view of the nodes are shown on Figure 5.3. We imagine there is a current source injecting a current I_i into node- i . This current must balance all the other currents leaving node- i , either through the transmission line bonds or through the impedance Z_i to ground,

$$I_i = \sum_{j \neq i} I_{i \rightarrow j} + \frac{V_i}{Z_i} = V_i \left(\frac{1}{Z_i} + \frac{i}{Z_0} \sum_{j \neq i} \frac{\cos(kL_{ij})}{\sin(kL_{ij})} \right) - \frac{i}{Z_0} \sum_{j \neq i} \frac{V_j}{\sin(kL_{ij})}. \quad (5.8)$$

Here it is understood that the sum over j is only over those nodes that share a bond with

node- i . Relation (7) can be cast in the form of an admittance matrix,

$$I_i = \sum_j Y_{ij}(k) V_j , \quad (5.9)$$

where

$$Y_{ii} = \left(\frac{1}{Z_i} + \frac{i}{Z_0} \sum_{j \neq i} \frac{\cos(kL_{ij})}{\sin(kL_{ij})} \right) , \quad (5.10a)$$

and

$$Y_{ij} = -\frac{i}{Z_0 \sin(kL_{ij})} , \quad j \neq i . \quad (5.10b)$$

The inverse of this admittance matrix is an impedance matrix,

$$V_i = \sum_j Z_{ij}(k) I_j . \quad (5.11)$$

In the next sections we will investigate the properties of these matrices for networks of varying complexity. Before doing that we will derive an expression for the impedance matrix in terms of the normal modes of a network.

Normal modes of the system are found for values of the propagation constant $k = k_n$ for which the node voltages are nonzero in the absence of injected current. This corresponds to setting the determinant of the admittance matrix to zero,

$$\det(Y_{ij}(k_n)) = 0 . \quad (5.12)$$

The discrete propagation constants k_n will be real in the absence of losses, $k_{\text{im}} = 0$ and $\text{Re}(Z_i) = 0$. Further, if the load impedance is independent of frequency, $dZ_i/d\omega = 0$, all mode energy is

stored on the bonds. In this case we expect the average spacing in wave numbers between modes to be determined by the total length of the bonds, $\langle k_{n+1} - k_n \rangle = \pi/L_T$, where $L_T = \sum_{ij} L_{ij}$. This relation will be checked when we solve various network realizations numerically.

The energy stored on a bond is the sum of the electric and magnetic field energy of the waves on the transmission line constituting the bond,

$$U_{ij}^{(n)} = \frac{1}{v} \int_0^{L_{ij}} dx \left(Z_0^{-1} \frac{|V^{(n)}(x)|^2}{2} + Z_0 \frac{|I^{(n)}(x)|^2}{2} \right) = \frac{2L_{ij}|V_{ij}^{(n)}|^2}{vZ_0}. \quad (5.13)$$

Here we have made use of the fact that the network is time reversal symmetric, and as a consequence $|V_{ij}^{(n)}| = |V_{ji}^{(n)}|$. Also, we now use a superscript (n) to denote a voltage or current value associated with the normal mode of the un-driven network having propagation constant k_n . When we solve for the modes of the network we will normalize the node voltages such that each mode has the same total energy, $U_T^{(n)} = \sum_{ij} U_{ij}^{(n)}$.

There is a relation between the normal modes and the impedance matrix, which for a driven system $I_j \neq 0$ is defined for all values of propagation constant k (see appendix A).

$$Z_{ij} = \sum_n \frac{iV_j^{(n)}V_i^{(n)*}}{(k - k_n)vU_T^{(n)}}, \quad (5.14)$$

Here $V_i^{(n)}$ is the voltage amplitude at node- i for mode n , $U_T^{(n)} = \sum_{ij} U_{ij}^{(n)}$, where $U_{ij}^{(n)}$ is given by equation (5.13). It should be noted for this system that the sum is over modes with both positive and negative values of k_n and that the node voltages for the modes are real. This gives the elements of impedance matrix the property that for real k they are imaginary and odd functions of k . Further, if we give $k = \omega/v$ a small positive imaginary part the real parts of the diagonal

elements of the impedance matrix are positive as demanded by causality. This expression will be compared with direct inversion of the admittance matrix, equations (5.10a) and (5.10b), in the next section.

There is also a third way to calculate the impedance matrix based on summing contributions from the different paths a signal may take in traversing the graph. Specifically, we can express elements of the impedance matrix in the following way. For diagonal elements we write

$$Z_{ii,path} = Z_{in,i} \left[1 + \sum_{paths} \prod_{Bonds} \sigma_b e^{i\theta_b} \right], \quad (5.15)$$

and for off-diagonal elements we write

$$Z_{ij,path} = Z_{in,i} \sum_{paths} \prod_{Bonds} \sigma_b e^{i\theta_b}. \quad (5.16)$$

Here the sum over paths is a sum over all paths starting and ending at node- i in the case of the diagonal elements and starting at node- i and ending at node- j in the case of the off-diagonal elements. The products are over the bonds that constitute the steps in the paths. On each bond a phase factor $\theta_b = kL_b$ is accumulated, where L_b is the length of the bond. The factors σ_b that are also accumulated are determined by how the path is followed from one bond to the next. If the signal is reflected at the end of a bond and then retraces the same bond the factor is given by $\sigma_b = \rho_b$ where ρ_b is the voltage reflection coefficient at the node where the reflection occurs. If the signal passes through the node to a different bond than $\sigma_b = 1 + \rho_b$, which is the voltage transmission coefficient. The voltage reflection coefficient depends on the number of bonds connected to a node and the impedance Z_j that is connected to ground at that node. Let us

define the equivalent impedance to ground of a node which is connected to N_j bonds,

$$Z_{eq,j}^{-1} = Z_j^{-1} + (N_j - 1)Z_0^{-1} . \quad (5.17)$$

This is the impedance seen by a wave incident on one bond due to the load and the other bonds. It is the parallel combination of the load to ground Z_j and the $N_j - 1$ other bonds. The voltage reflection coefficient is then

$$\rho_b = \frac{Z_{eq,j} - Z_0}{Z_{eq,j} + Z_0} . \quad (5.18)$$

When a path ends on a node which is considered to be a port, then the factor $\sigma_b = 1 + \rho_b$ is applied as if the signal were passing to another bond. The input impedance at the node is the parallel combination of the load impedance Z_j and the N_j transmission lines forming the bonds,

$$Z_{in,j}^{-1} = Z_j^{-1} + N_j Z_0^{-1} = Z_{eq,j}^{-1} + Z_0^{-1} . \quad (5.19)$$

It can be shown that with these definitions each term in the sum for off-diagonal impedance elements satisfies the reciprocity condition $Z_{ij} = Z_{ji}$.

In principle, the sum over paths gives the exact value of an element of the impedance matrix only when an infinite number of paths is considered. However, useful approximations to the value of an impedance element can be obtained with a finite number of paths in two cases. The first case is if the network has loss, either in the transmission lines or the loads to ground. In this case signals are attenuated fast enough as they propagate so that the size of the individual terms in the sum over longer and longer paths decreases faster than the number of such paths increases. The second case is the one in which the sliding window average using the Lorentzian weighting

function is sought, rather than the precise value at a given frequency. Averaging over a window of frequencies (or wavenumbers) of width Δk , in this way, is equivalent to adding loss $k_{\text{im}} = \Delta k$ and causes the contribution of paths of length L to decrease as $\exp(-\Delta k L)$.

Chapter 6: Numerical Analysis of the Graphs

In the previous section we discussed the theoretical framework necessary to study network graphs. Now, we numerically analyze different graphs. The set of graphs we choose are the graphs with 4 nodes and 3 bonds per node, with 6 nodes and 3 bonds per node, with 8 nodes and 3 bonds per node, with 8 nodes and 5 bonds per node, and with 8 nodes and 7 bonds per node. We examine the properties of the eigenfunctions and the eigenvalues for these graphs. Recall the eigenvalues are the set of propagation constants k_n for which Equation (5.9) has solutions with all injected currents I_i set to zero. For each propagation constant k_n we find a set of node voltages, which we normalize according to Equation (5.13).

For each graph we have choose the length of the bonds randomly in the range of 1 to 5, $1 \leq L_{ij} \leq 5$. We also make sure that the average length of the bonds is 3, that helps us to fix the mean and total length in the similarly structured graph, with the same number of nodes and number of bonds per node (or valency for each node). To create better statistics for each physical feature, we use five different realizations. The lengths for each realization for each graphs are given by the tables as follows:

- The bonds lengths of the graph with $N = 4$, $B = 3$ are shown on Table 6.1,
- The bonds lengths of the graph with $N = 6$, $B = 3$ are shown on Table 6.2,

- The bonds lengths of the graph with $N = 8, B = 3$ are shown on Table 6.3,
- The bonds lengths of the graph with $N = 8, B = 5$ are shown on Table 6.4,
- The bonds lengths of the graph with $N = 8, B = 7$ are shown on Table 6.5,

We should also set other physical features of the graph in order to start to numerically analyze the statistics of the graph. For now, we will set the impedance to ground at each node to infinity, and the characteristic impedance on the bonds to 1.

N = 4, B = 3	
Realization 1	1.304, 3.588, 2.305, 2.155, 4.082, 4.566
Realization 2	4.150, 4.932, 3.030, 3.569, 1.084, 1.235
Realization 3	4.413, 1.787, 2.187, 1.188, 4.333, 4.092
Realization 4	4.598, 3.256, 3.034, 1.462, 2.153, 3.497
Realization 5	4.271, 3.005, 2.368, 1.814, 4.620, 1.921

Table 6.1: The set of bond lengths for each realization in the graph with $N = 4, B = 3$.

N = 6, B = 3	
Realization 1	3.038, 2.731, 1.184, 4.718, 3.265, 1.959, 4.603, 2.628, 2.873
Realization 2	1.167, 4.370, 4.683, 1.071, 2.706, 3.493, 1.109, 4.788, 3.614
Realization 3	1.674, 3.079, 3.256, 4.368, 3.979, 2.247, 1.699, 2.273, 4.426
Realization 4	2.509, 4.246, 1.884, 4.113, 1.244, 1.429, 4.421, 4.211, 2.942
Realization 5	4.284, 1.496, 3.544, 3.191, 3.997, 3.888, 2.661, 2.227, 1.713

Table 6.2: The set of bond lengths for each realization in the graph with $N = 6, B = 3$.

N = 8, B = 3	
Realization 1	3.748, 4.094, 3.669, 4.937, 1.869, 2.342, 1.938, 4.366, 1.721, 2.425, 3.732, 1.159
Realization 2	3.810, 4.368, 2.034, 1.301, 1.304, 1.525, 3.516, 4.942, 3.545, 4.194, 3.500, 1.962
Realization 3	4.517, 2.101, 2.094, 3.979, 3.256, 1.618, 1.397, 4.078, 4.186, 3.402, 2.013, 3.359
Realization 4	2.444, 3.185, 1.229, 4.451, 2.008, 3.157, 4.012, 4.021, 1.674, 2.232, 3.947, 3.642
Realization 5	3.089, 2.297, 4.019, 2.718, 3.610, 2.605, 3.936, 2.450, 2.561, 3.415, 2.998, 2.302

Table 6.3: The set of bond lengths for each realizations in the graph with $N = 8, B = 3$.

N = 8, B = 5	
Realization 1	2.869, 1.958, 1.214, 1.206, 2.444, 2.188, 3.844, 2.967, 2.212, 3.765, 2.305, 4.072, 4.270, 3.297, 4.320, 3.006, 2.306, 4.250, 2.803, 4.703
Realization 2	1.677, 1.366, 2.001, 4.664, 1.199, 4.794, 1.924, 1.392, 3.825, 4.223, 3.374, 3.523, 4.586, 2.023, 2.591, 3.231, 2.628, 4.090, 4.681, 2.208
Realization 3	1.229, 2.722, 4.889, 3.714, 1.846, 3.178, 2.424, 3.038, 3.924, 3.919, 2.486, 3.887, 4.583, 3.409, 2.114, 3.359, 2.208, 2.194, 3.468, 1.409
Realization 4	2.711, 3.883, 3.127, 2.792, 3.343, 1.614, 1.123, 1.975, 3.165, 2.007, 3.004, 3.666, 3.918, 3.744, 4.566, 4.586, 3.918, 1.550, 3.411, 1.897
Realization 5	4.188, 4.020, 4.402, 2.461, 4.489, 3.050, 3.388, 2.686, 1.964, 4.561, 3.258, 1.309, 2.831, 3.529, 1.867, 4.921, 1.362, 1.943, 2.321, 1.449

Table 6.4: The set of bond lengths for each realizations in the graph with $N = 8, B = 5$.

Now we have set all the physical properties of the graph, and we can calculate corresponding statistics. We start with the statistics of mode wave numbers. In spectral theory Weyl's law [75, 76] describes the asymptotic behavior of eigenvalues of the Laplacian operator. In our case, both Schrödinger's and Telegrapher's equation are similar to Laplacian operator, and we

N = 8, B = 7	
Realization 1	4.883, 1.125, 1.334, 1.848, 3.435, 1.947, 2.294, 3.295, 2.834, 2.428, 2.949, 1.746, 3.850, 4.702, 2.275, 1.013, 4.861, 2.344, 3.534, 3.986, 2.769, 2.145, 4.641, 3.052, 2.241, 4.127, 4.185, 4.157
Realization 2	3.692, 1.186, 4.111, 1.339, 2.439, 1.581, 1.733, 3.673, 4.500, 1.365, 1.920, 4.240, 3.376, 4.437, 4.550, 2.943, 1.131, 3.571, 1.659, 4.087, 3.712, 4.828, 1.718, 3.159, 3.427, 1.684, 3.816, 4.125
Realization 3	2.411, 2.685, 3.060, 3.771, 2.067, 2.171, 3.610, 4.627, 2.651, 1.964, 2.907, 4.336, 2.144, 1.536, 1.496, 3.068, 4.288, 1.149, 1.937, 4.997, 1.293, 2.684, 3.842, 4.891, 1.187, 4.816, 4.348, 4.063
Realization 4	4.629, 1.433, 3.773, 3.149, 1.469, 3.171, 2.429, 1.077, 4.702, 4.742, 3.312, 4.911, 1.205, 3.417, 3.837, 4.479, 1.263, 3.819, 3.719, 3.132, 1.261, 1.632, 3.648, 3.369, 1.154, 3.181, 1.909, 4.179
Realization 5	3.164, 2.352, 2.025, 4.445, 2.012, 1.080, 1.517, 2.283, 1.092, 3.548, 4.960, 2.099, 4.545, 1.938, 4.170, 4.152, 2.335, 3.381, 4.164, 2.358, 3.437, 3.539, 4.677, 1.805, 2.300, 3.813, 2.159, 4.649

Table 6.5: The set of bond lengths for each realizations in the graph with $N = 8, B = 7$.

expect that the mode wave number will obey to the Weyl's law, which is given by

$$N(k) = \frac{k \sum L_{ij}}{\pi}, \quad (6.1)$$

where $N(k)$ is the counting function, and it shows how many normal mode wave numbers are less than k . k is wave number, and $\sum L_{ij}$ is the total length of the graph. Using Equation (6.1) we can conclude that on average normal mode wave number dependance on mode number should follow this line

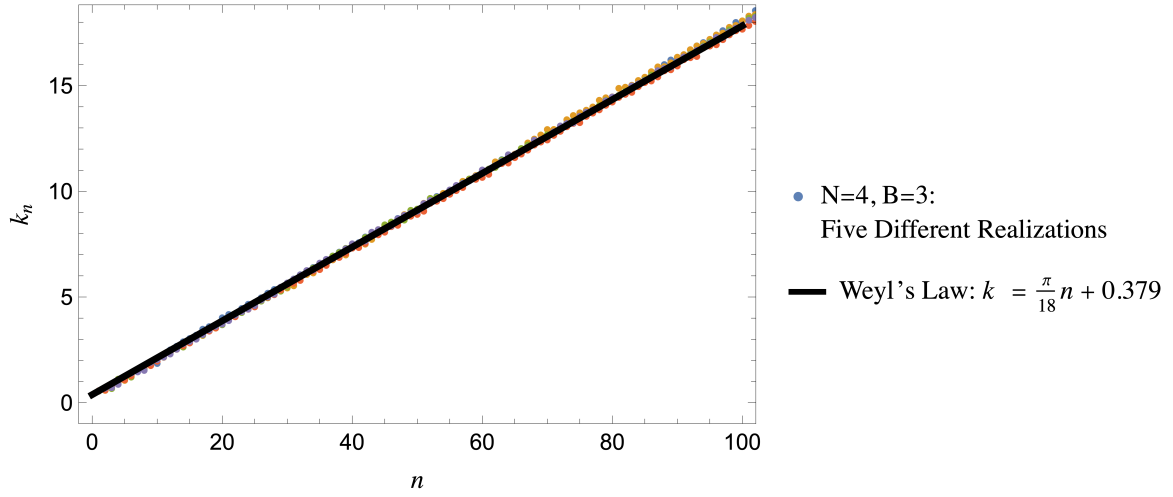
$$k = \frac{\pi}{\sum L_{ij}} n + A, \quad (6.2)$$

where k_n is the wave number corresponding to n -th mode, n is the mode number, $\sum L_{ij}$ is the

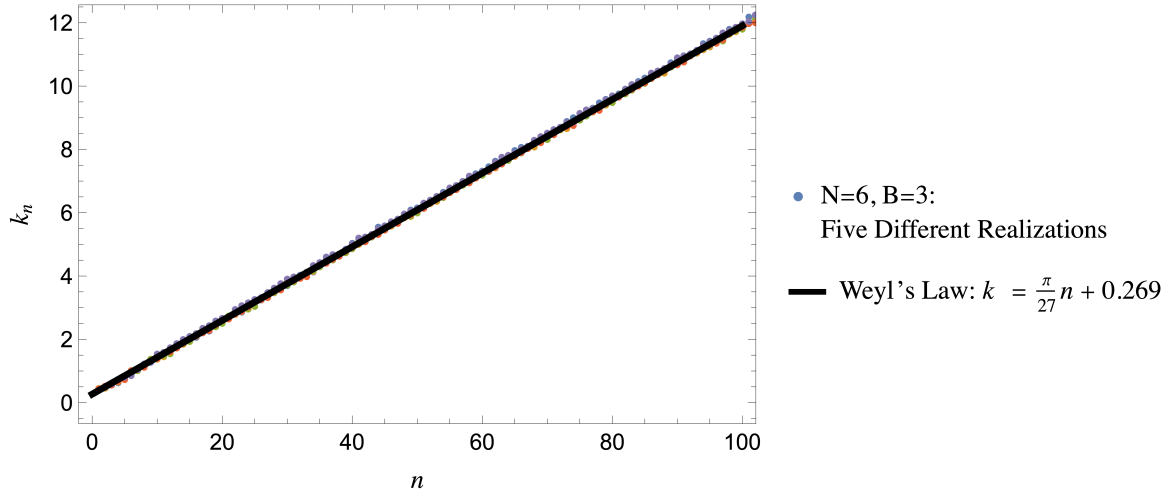
total length of the graph, and A is the intercept. We have checked how the mode wave numbers agree with Weyl's law, and the result is shown on Fig. 6.1.

Knowing the mode wave numbers of the graphs, we can then construct histograms for the corresponding normalized node voltages. The node voltages are normalized by dividing by the square root of total energy, $V_i/\sqrt{U_T^{(n)}}$, where $U_T^{(n)}$ is the total energy, from the Equation (5.14). In the case of total energy being unity for every mode, the normalized node voltages are just the regular voltage values of the nodes. Furthermore, we normalize histograms to make them probability density functions, in other words we set the integrals of the functions to unity. The results are displayed in Figure 6.2. We have constructed the probability distribution functions for different size graphs, and we expect a Gaussian to be a good fit based on experience with eigenfunction of so-called ray-chaotic cavities in 2 and 3 dimensions. Here the statistics of values of the eigenfunctions are Gaussian as a consequence of the random plane wave hypothesis. That is, the field at any point in the enclosed domain can be viewed as a random superposition of a large number of plane waves with random phases. We can clearly see that our expectation is more accurate as the size of the graphs gets larger. We have plotted them compared to their best Gaussian fits. The fitting is done by minimizing the root mean square difference between the two functions, the numerical data and the Gaussian fit. The resulting minimum root mean square values are shown on Table 6.6., where we can see that it gets smaller as the graph gets larger. For example moving from the graph with $N = 4, B = 3$ to the graph with $N = 8, B = 7$, the root means square value dropped more than 18 times. Therefore, we can anticipate that the node voltage statistics for relatively large size graphs will have very good agreement with Gaussian probability distribution.

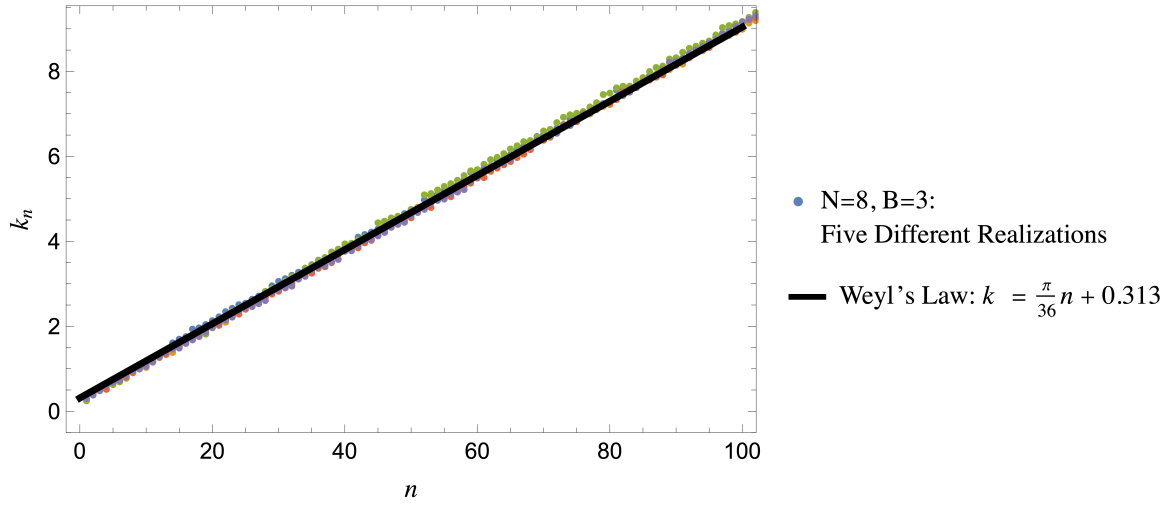
Other than the node voltage statistics, we can calculate the statistics of the traveling voltage



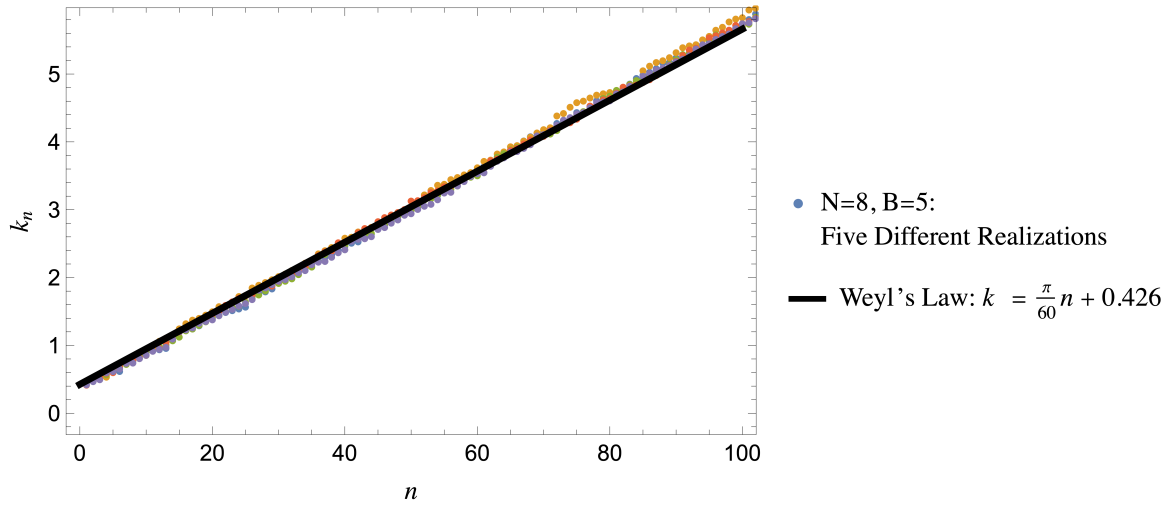
(a) The mode wave number values (colored dots) of the graph with $N = 4, B = 3$ compared to Weyl's law (black solid line) given by Equation 6.1.



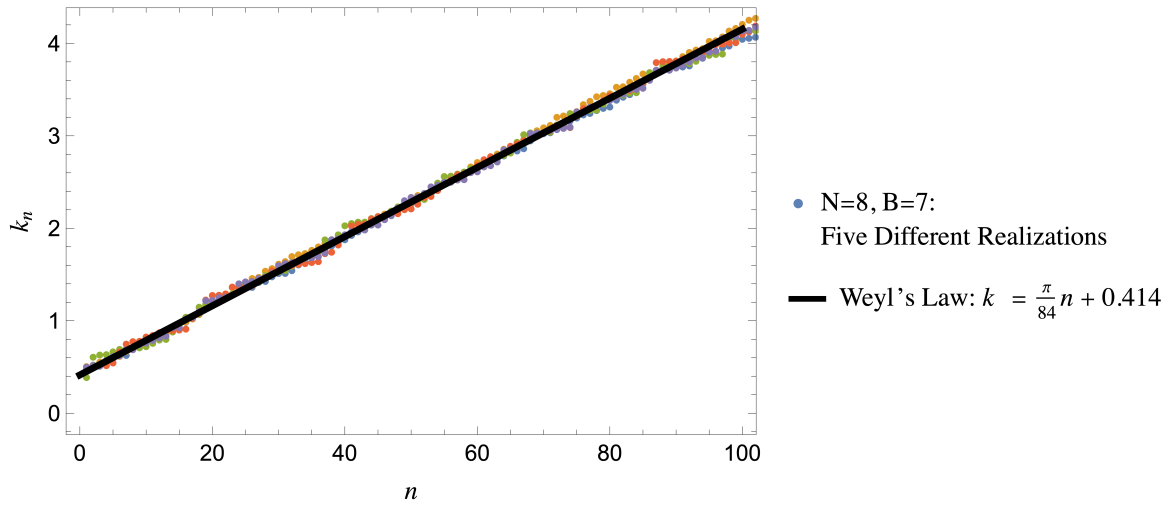
(b) The mode wave number values (colored dots) of the graph with $N = 6, B = 3$ compared to Weyl's law (black solid line) given by Equation 6.1.



(c) The mode wave number values (colored dots) of the graph with $N = 8, B = 3$ compared to Weyl's law (black solid line) given by Equation 6.1.

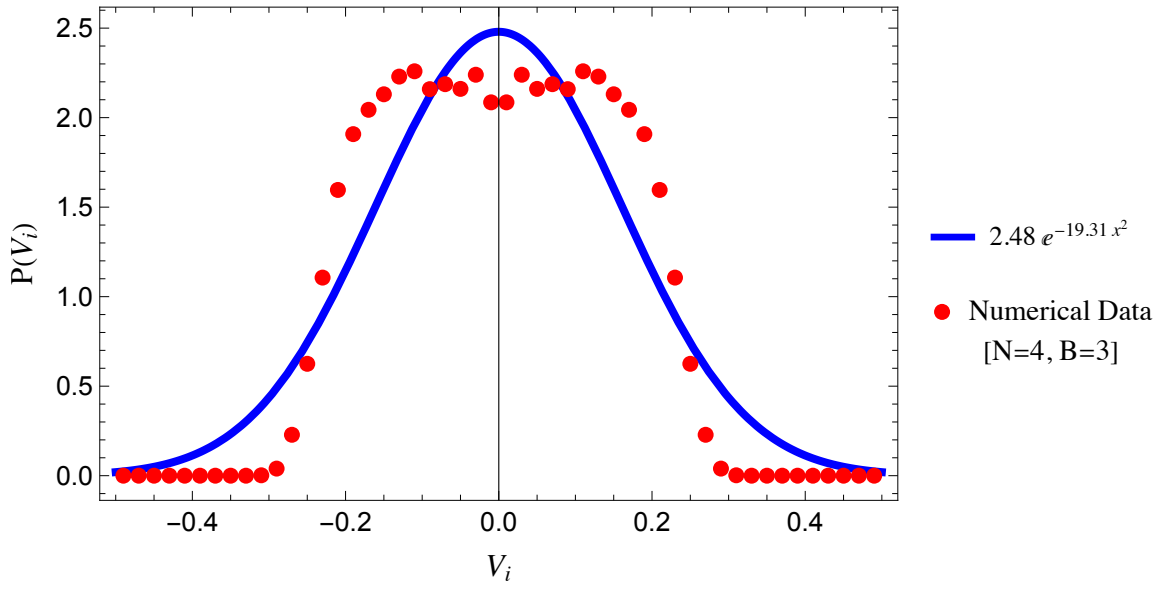


(d) The mode wave number values (colored dots) of the graph with $N = 8, B = 5$ compared to Weyl's law (black solid line) given by Equation 6.1.

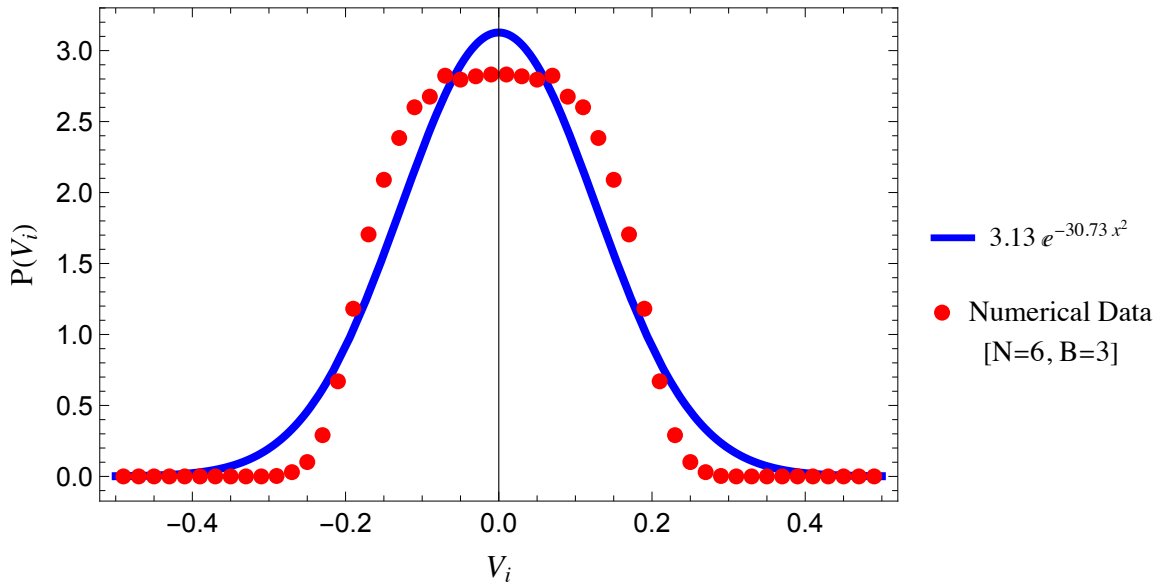


(e) The mode wave number values (colored dots) of the graph with $N = 8, B = 7$ compared to Weyl's law (black solid line) given by Equation 6.1.

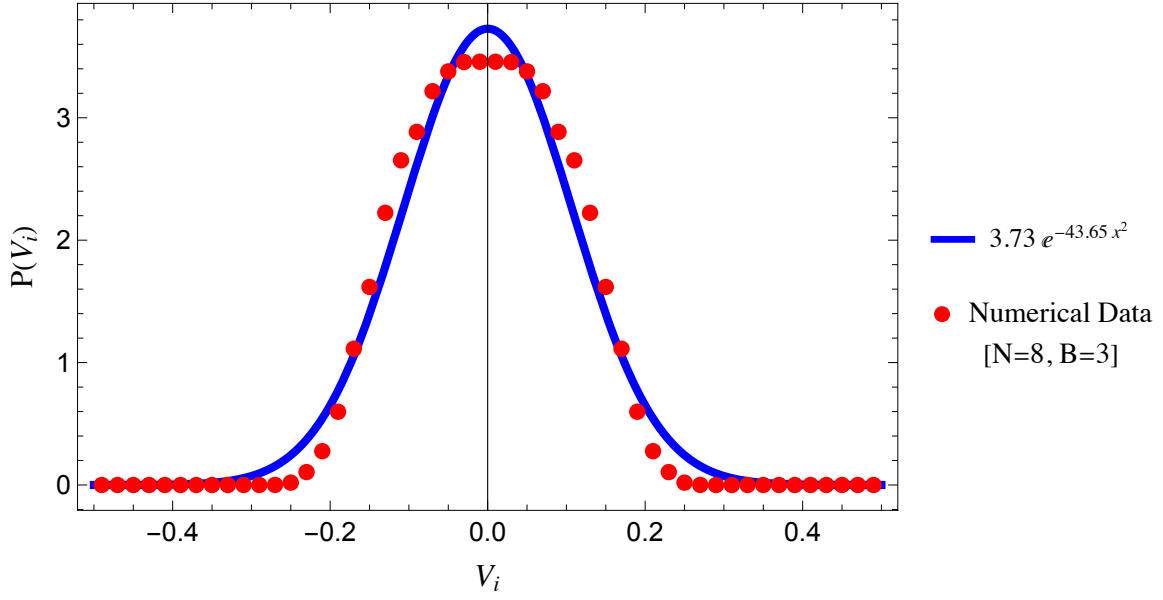
Figure 6.1: Comparing Weyl's law (black solid line) to the values of mode wave numbers (colored dots) for different size graphs. In all of the cases intersection to y -axis is different, and never equals to zero.



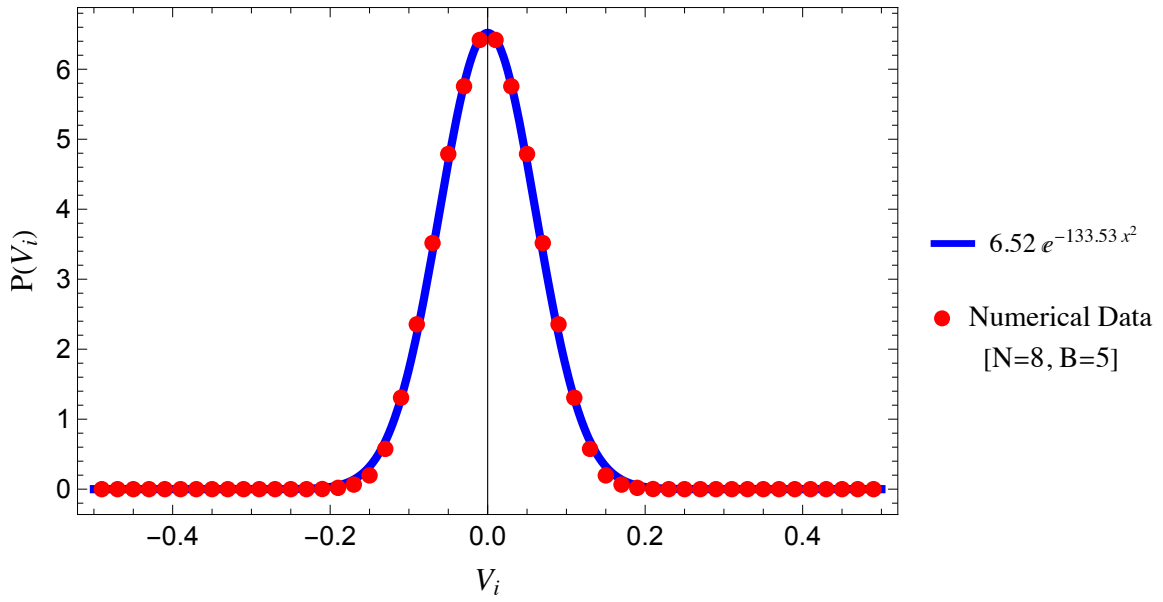
(a) Probability distribution functions of the (Red dots) of the voltage values at the nodes of the graph with 4 nodes and 3 bonds per node plotted versus its Gaussian fit (Blue line).



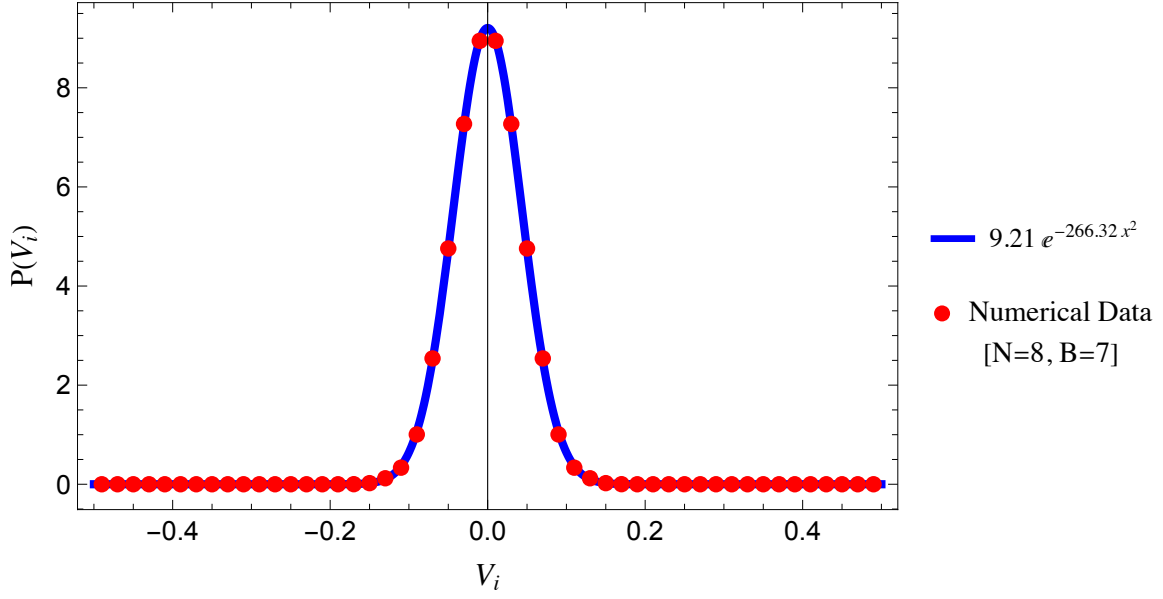
(b) Probability distribution function (Red dots) of the voltage values at the nodes of the graph with 6 nodes and 3 bonds per node plotted versus its Gaussian fit (Blue line).



(c) Probability distribution function (Red dots) of the voltage values at the nodes of the graph with 8 nodes and 3 bonds per node plotted versus its Gaussian fit (Blue line).



(d) Probability distribution function (Red dots) of the voltage values at the nodes of the graph with 8 nodes and 5 bonds per node plotted versus its Gaussian fit (Blue line).



(e) Probability distribution function (Red dots) of the voltage values at the nodes of the graph with 8 nodes and 7 bonds per plotted versus and its Gaussian fit (Blue line).

Figure 6.2: Probability distribution functions (Red dots) of the voltage values at the nodes of different size graphs plotted versus its Gaussian fit (Blue line). Gaussian fittings are done by minimizing the root mean square value of the differences between numerical data and the Gaussian.

Graph	N = 4, B = 3	N = 6, B = 3	N = 8, B = 3	N = 8, B = 5	N = 8, B = 7
$\sqrt{\frac{1}{n} \sum (x_i - y_i)^2}$	0.328	0.253	0.191	0.059	0.018

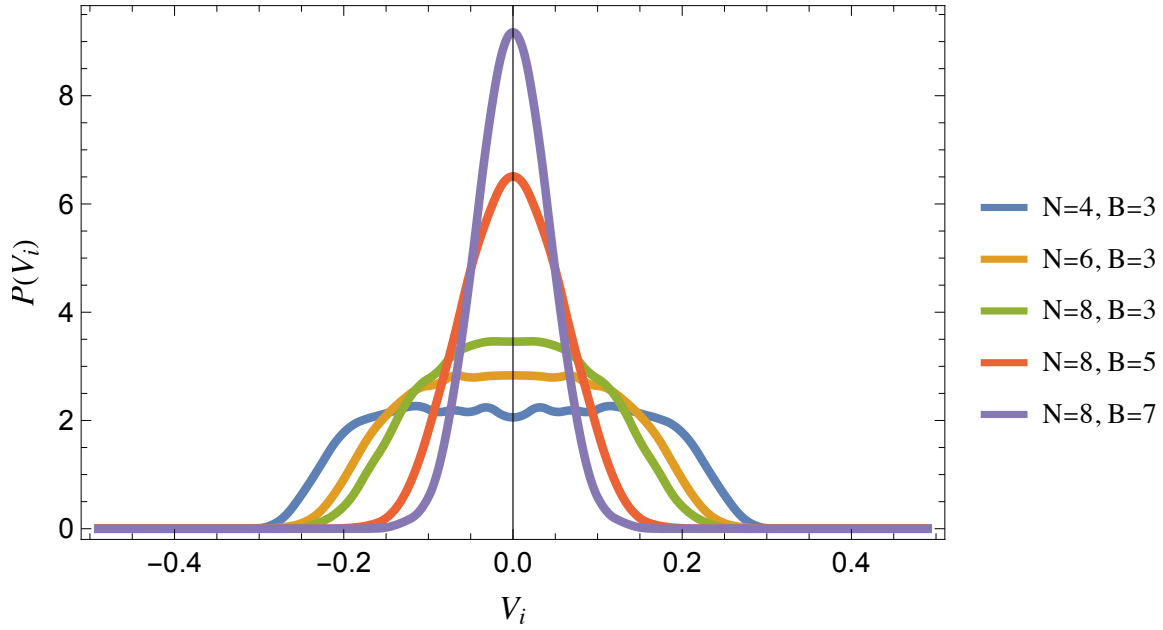
Table 6.6: The root mean square fitting values between the probability distribution function of node voltages and its Gaussian fit shown on Fig. 6.2.

wave amplitudes. Using Equations (5.5a) and (5.5b), the corresponding statistics are shown on Figure 6.3 (b). As for the node voltages, the amplitudes here are normalized by dividing on the square root of total energy, $V_{ij}/\sqrt{U_T^{(n)}}$. On Figure 6.3 (b), we can see that as the graph gets larger the probability of having smaller traveling voltage wave amplitudes increases. This is not surprising due to the fact that the total energy is distributed over more bonds. Similarly, Figure 6.3 (a) shows that the larger the graph the smaller the values of the normalized node voltages.

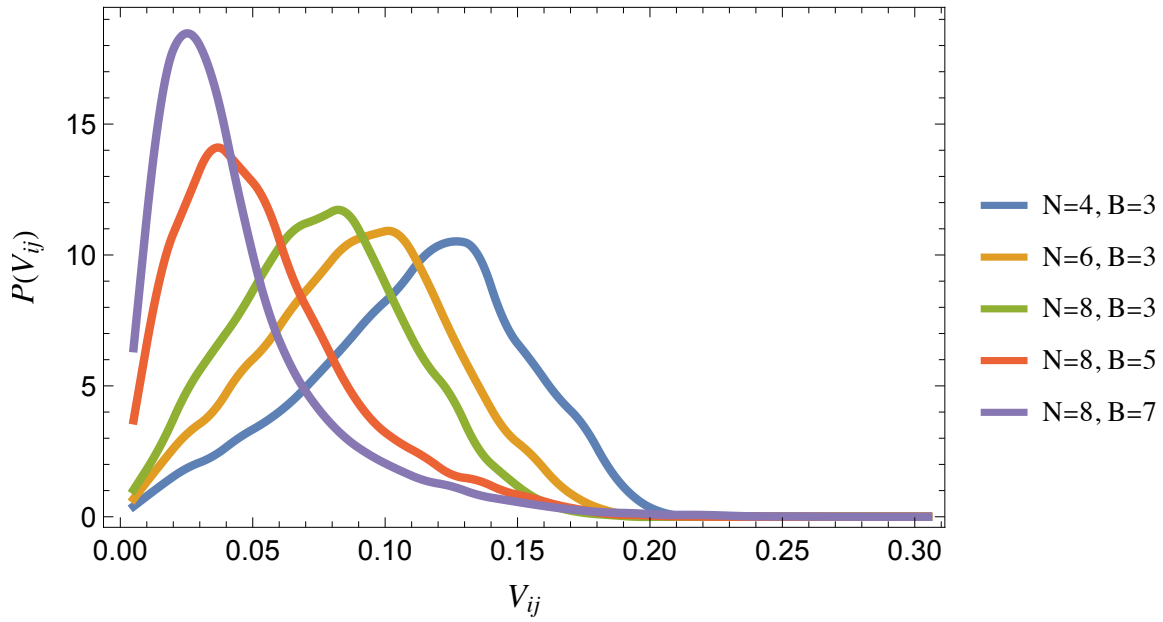
Now we can start investigating the level spacing distributions of the wave numbers. We have shown that the mode wave numbers follow Weyl's law, given by Equation (6.1). The spacings between the mode wave numbers are defined as $k_{n+1} - k_n$, and thus, the notation for means spacing is $\langle k_{n+1} - k_n \rangle$. From the Equation (6.1) we can conclude that mean spacing of the mode wave numbers are given by

$$\langle k_{n+1} - k_n \rangle = \frac{\pi}{\sum L_{ij}}. \quad (6.3)$$

Therefore, the mean spacing of eigenvalues k_n was determined by the total length of the bonds. We now define the normalized spacing $s_n = (k_{n+1} - k_n) / \langle k_{n+1} - k_n \rangle$, we make histograms of the values s_n . For this study we add a reactance $Z_i = iX_i$ to ground at each node, and we construct histograms for a number of values of the reactance.



(a) Probability distribution function of the voltage values at the nodes for different size graphs.



(b) Probability distribution functions of the traveling wave amplitudes in different size graphs.

Figure 6.3: Comparison of (a) the statistics of the values of node voltages and (b) the statistics of the amplitudes of the traveling voltage waves. The larger total number of bonds is, and therefore, total length of the graph, the higher the probabilities of smaller voltage values either on the nodes or on the bonds.

We varied reflection coefficients on the nodes in two different ways. On the one hand, we changed reflection coefficients simultaneously on all nodes and at the same time, keep the value the same across the nodes, displayed in Figure 6.4 (a). The main effect of the reactance, X_i , is to increase the reflection coefficient for a wave incident on a node according to Equation (5.18). We note that for the lowest value of reflection coefficient the spacing distribution is well approximated by the spacing distribution of eigenvalues of a random matrix drawn from the Gaussian Orthogonal Ensemble, $P(s) = (\pi s/2) \exp(-\pi s/4)$. This is the distribution expected for eigenvalues of a wave chaotic cavity. We note from Figure 6.4 (a), as the reflection coefficient at a node increases the shape of the histogram changes, approaching a Poisson distribution $P(s) = \exp(-s)$ at the highest reflection value. This can be understood as follows. When the reflection coefficient becomes large adjacent bonds become isolated from each other. The result is that the eigenvalues are determined by quantizing the individual bonds. The eigenvalues then fall uniformly distributed along the real k -line and there is no level repulsion as exhibited in a wave chaotic cavity, or as exhibited in a graph with strong coupling between bonds. On the other hand, we set Dirichlet boundary conditions on some bonds and Neumann boundary conditions on the others, displayed in Figure 6.4 (b), where the average reflection coefficient for each node is calculated by

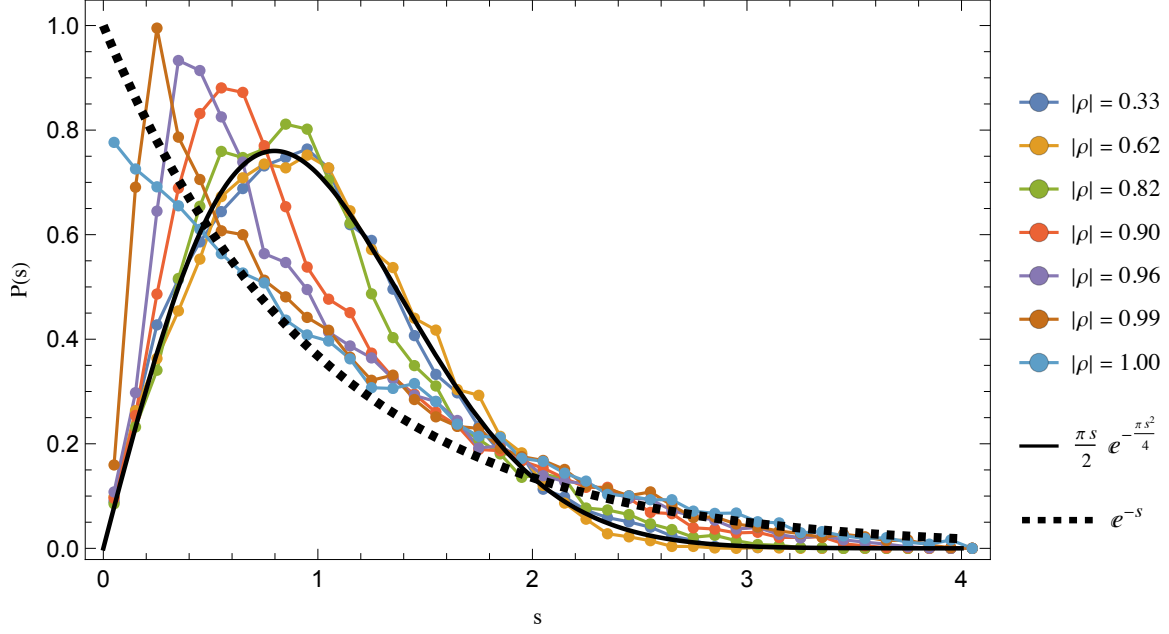
$$|\rho_{avg}| = \sqrt{\frac{1}{9} + \frac{8}{9} \frac{N_D}{N}}, \quad (6.4)$$

where N_D is the number of Dirichlet nodes used in the graph, and N is the total number of the nodes. We wanted to check if the distributions look alike when the average reflection coefficient on the nodes are the same. As we see from Figures 6.4(a) and 6.4(b), the distribution functions are similar when the reflection coefficient is less than 0.75; however, starting from 0.75, when we

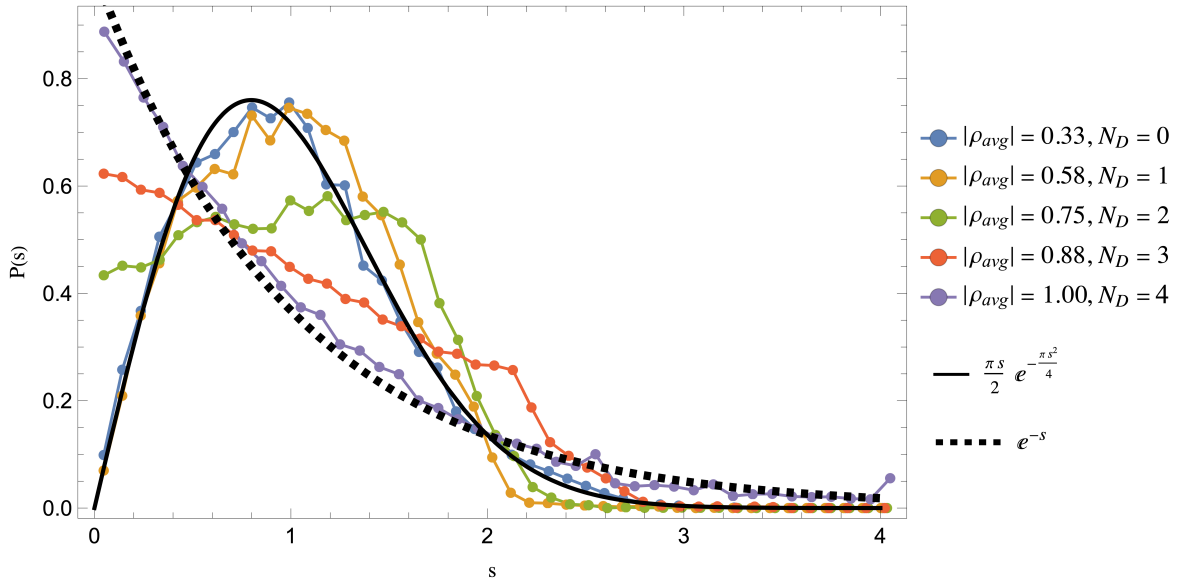
use Dirichlet nodes the distributions changes and goes towards Poisson distribution faster than it does when we use the same reflection coefficients on each node. Therefore, we don't have one to one similarities between these two use cases of reflection coefficients shown on Figure 6.4(a) and 6.4(b).

In Chapter 5, we discussed that the impedance matrix can be calculated by inverting the admittance matrix or by representing the impedance matrix in terms of modes of the undriven network using Equation (5.14). We now check that here. In principle, using Equation (5.14) requires summing over all normal modes of the system. As this is not practical we must truncate the sum. From the form of Equation (5.14) we see that the representation of the impedance matrix will be accurate only for k -values in the middle of the range of the eigenvalues k_n of the modes retained in the sum. In Figure 6.5, we show a comparison of the inverse of the admittance matrix and Equation (5.14) for two different size graphs, one with $N = 4, B = 3$ and another with $N = 8, B = 5$. For the graph with $N = 4, B = 3$, k -values are shown in three different ranges. In the middle, $433 < k < 436$, the left and the right edges of used values of the mode wave numbers, $1 < k < 4$ and $869 < k < 872$, correspondingly. For the graph with $N = 8, B = 5$, k -values are also shown in three different ranges. In the middle, $64 < k < 65$, the left and the right edges of used values of the mode wave numbers, $1 < k < 2$ and $128 < k < 129$, correspondingly. In the sum in Equation (5.14), for the graph with $N = 4, B = 3$ we have retained 5040 modes spanning the range of eigenvalues, $0 < k < 873$. Furthermore, for the graph with $N = 16, B = 3$ we have retained 2413 modes spanning the range of eigenvalues, $0 < k < 131$. We see that on the edges of the used mode wave number values, the agreement is not good. However, in the middle we have quite good agreement for both of the graphs.

A second comparison can be made between the inverse of the admittance matrix and the

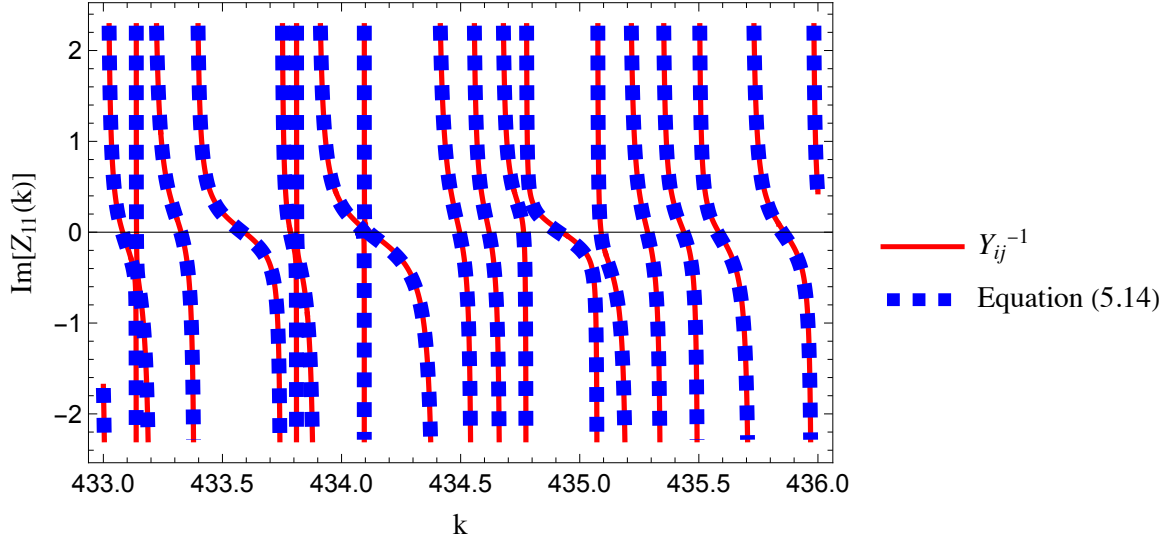


(a) Nearest neighbor spacing distribution for different reflection coefficients on the node. All the node have the same reflection coefficient.

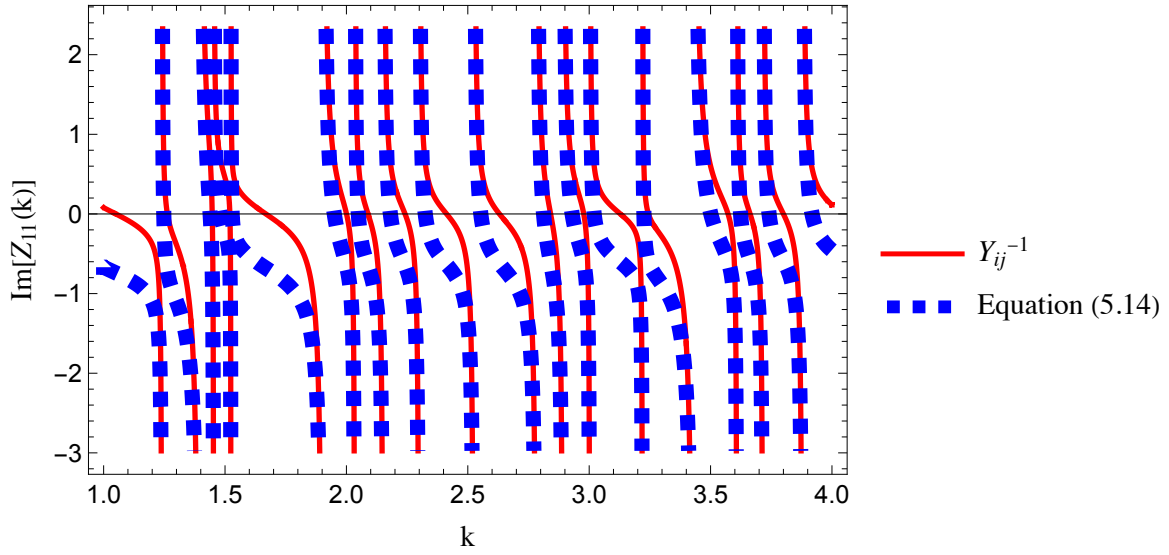


(b) Nearest neighbor spacing distribution for different reflection coefficients on the node. We use either Dirichlet or Neumann boundary conditions on each node. N_D defines how many Dirichlet boundary conditions were used. ρ is averaged reflection coefficient inside the system.

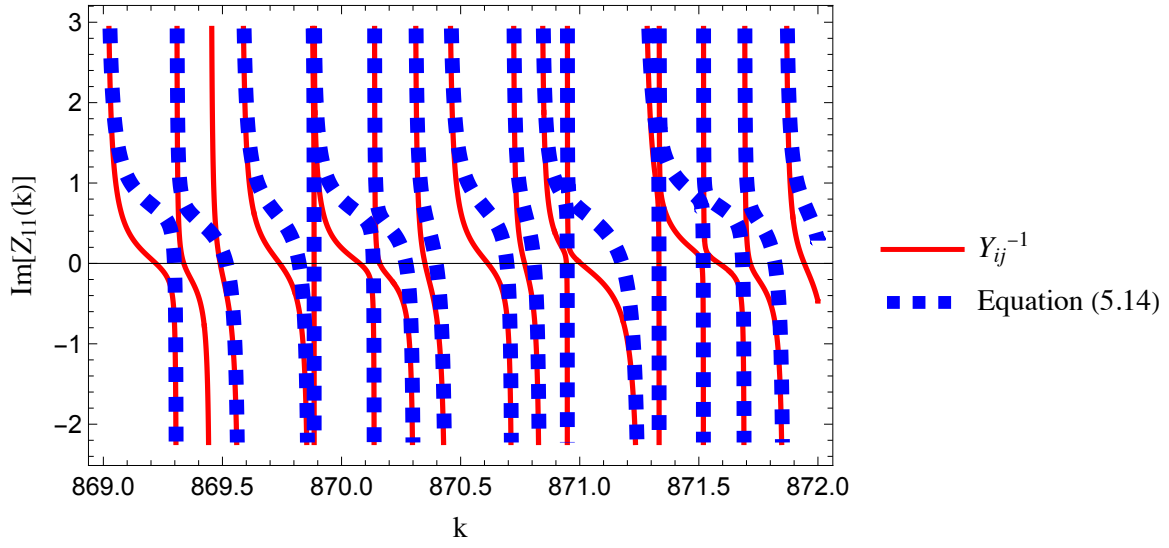
Figure 6.4: Nearest neighbor spacing distribution of normal mode wave numbers in a graph with $N = 4, B = 3$, where $s_n = (k_{n+1} - k_n) / \langle k_{n+1} - k_n \rangle$. Shown for different reflection coefficients of the node. We have used different boundary conditions in (a) and (b).



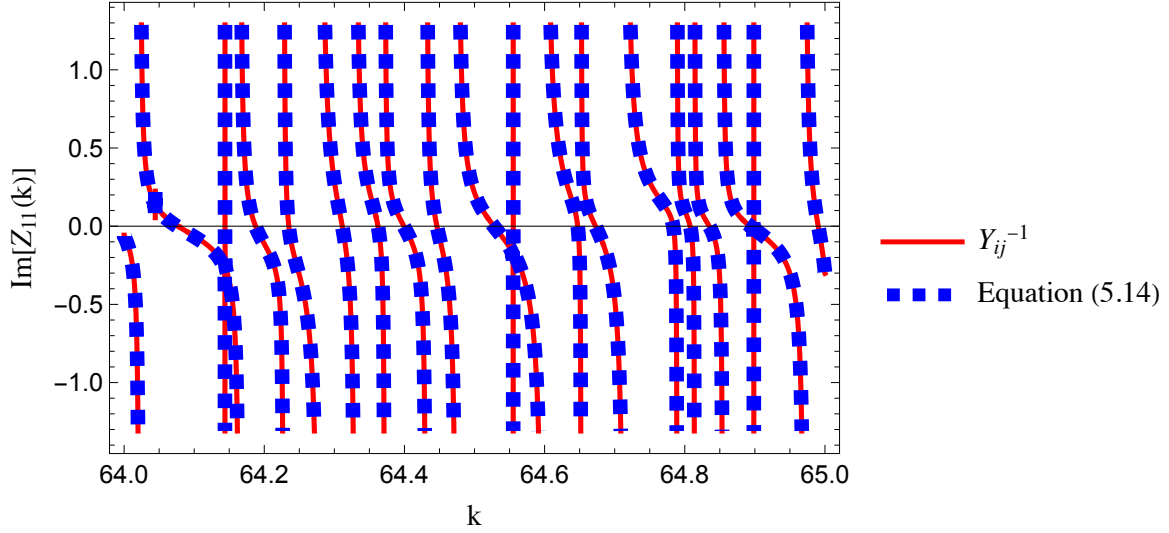
(a) Imaginary part of the diagonal elements of the impedance matrix with $N = 4, B = 3$. Middle part of the used mode wave numbers.



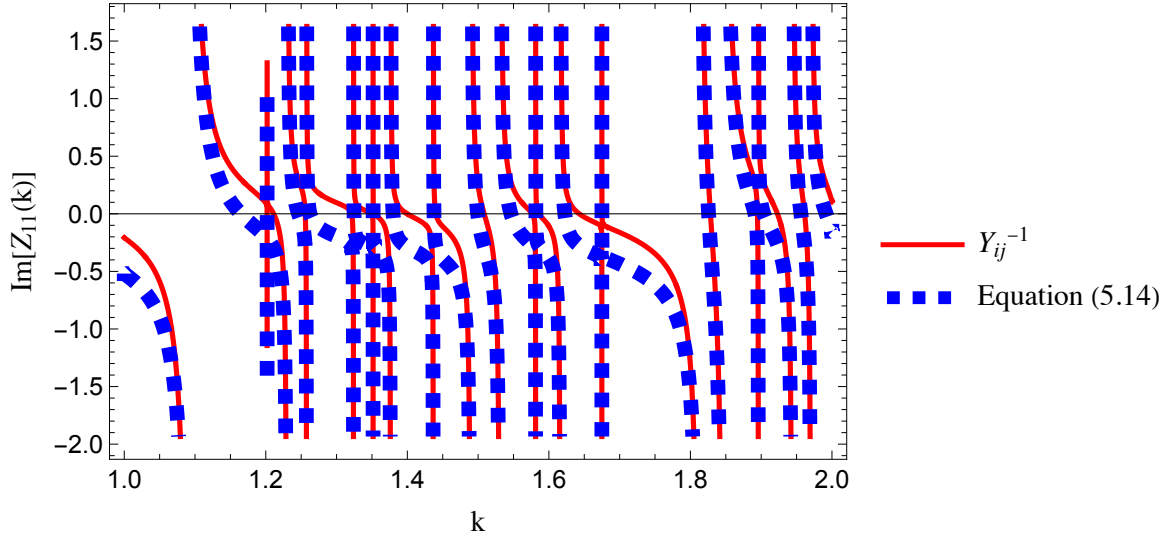
(b) Imaginary part of the diagonal elements of the impedance matrix with $N = 4, B = 3$. Left part of the used mode wave numbers.



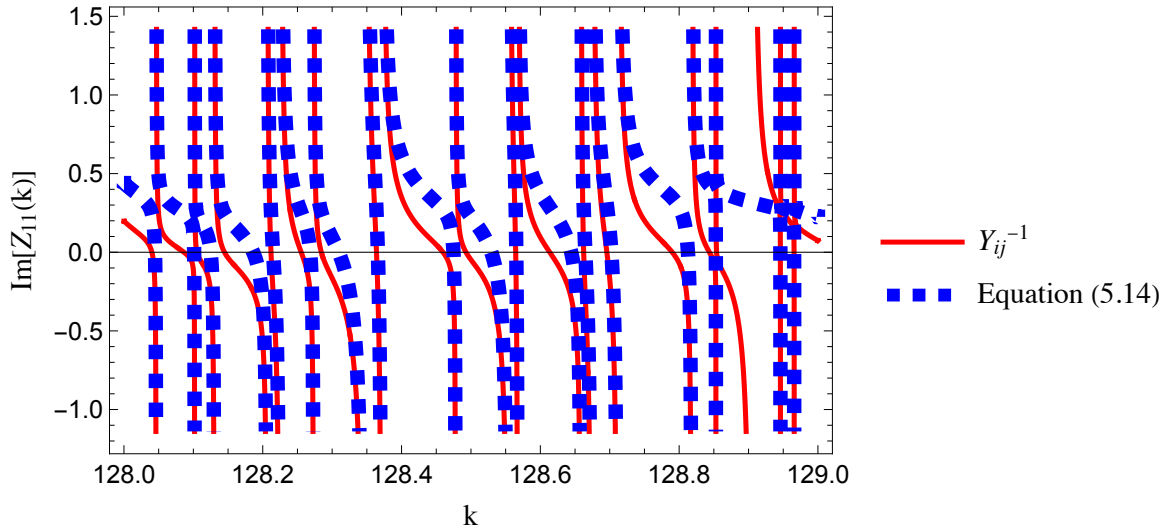
(c) Imaginary part of the diagonal elements of the impedance matrix with $N = 4, B = 3$. Right part of the used mode wave numbers.



(d) Imaginary part of the diagonal elements of the impedance matrix with $N = 8, B = 5$. Middle part of the used mode wave numbers.



(e) Imaginary part of the diagonal elements of the impedance matrix with $N = 8, B = 5$. Left part of the used mode wave numbers.

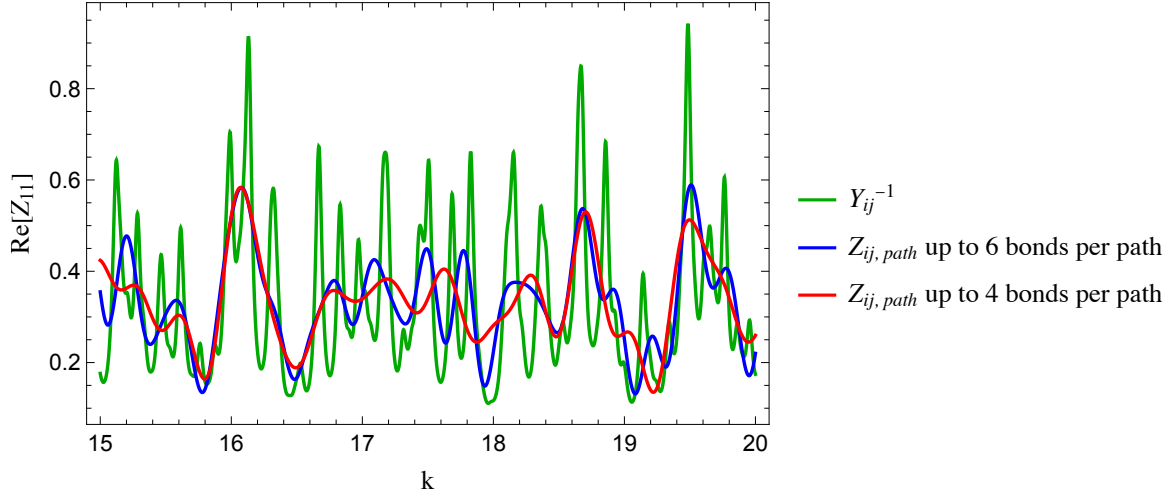


(f) Imaginary part of the diagonal elements of the impedance matrix with $N = 8, B = 5$. Right part of the used mode wave numbers.

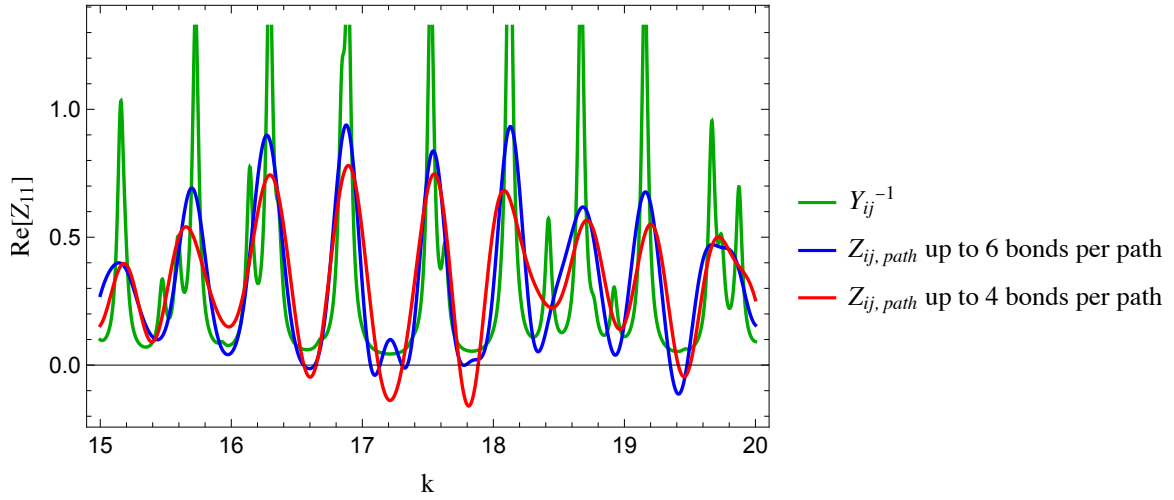
Figure 6.5: Imaginary part of the diagonal elements of the impedance matrix for two different graphs, one with $N = 4, B = 3$ and another with $N = 8, B = 5$. The impedance values were determined two ways: by inverting the admittance matrix (Equation 9(a) and 9(b)) (red line) and by sum over modes (Equation 5.14) (blue line).

summed contributions from the different paths a signal may take in traversing the graph, Equations (5.15) and (5.16). Similar to the case of Equation (5.14) the infinite sums in Equations (5.15) and (5.16) will have to be truncated. In Figure 6.6, we show comparisons between the inverse of the admittance matrix Equations (5.10a) and (5.10b) and the path sums Equations (5.15) and (5.16) when the path sums have been truncated for lengths over six bonds and also, when the path sums have been truncated for lengths over four bonds. Impedance values are shown as functions of the real part of k for two different values of k_{im} , and for two different graphs. Recall that evaluating the impedance with a complex k is equivalent to evaluating it for real k using a sliding Lorentzian window average for which the imaginary part of k defines the width of the window. Figure 6.6 shows that for small imaginary $k_{\text{im}} = 0.03$ the truncated path sums only reproduce the coarse variations of the impedance. However, with larger $k_{\text{im}} = 0.3$, the truncated sums agree quite well with inverse of the admittance matrix. This is to be expected because for the two graphs the average lengths of a bond are $\langle L_{ij} \rangle = 3.00$, for both graphs. Evaluating the matrix with an imaginary k is equivalent to adding a spatial damping to the waves propagating on the bonds. For both of the graphs in the case of using paths up to 6 bonds we find $6k_{\text{im}}\langle L_{ij} \rangle = 0.54$, and with $k_{\text{im}} = 0.3$ we find $6k_{\text{im}}\langle L_{ij} \rangle = 5.4$. In addition, in the case of using paths up to 4 bonds we find $4k_{\text{im}}\langle L_{ij} \rangle = 0.36$, and with $k_{\text{im}} = 0.3$ we find $4k_{\text{im}}\langle L_{ij} \rangle = 3.6$. Thus, the contributions from 6 and 4 paths are sufficient for the $k_{\text{im}} = 0.3$ case, but not in the $k_{\text{im}} = 0.03$ case.

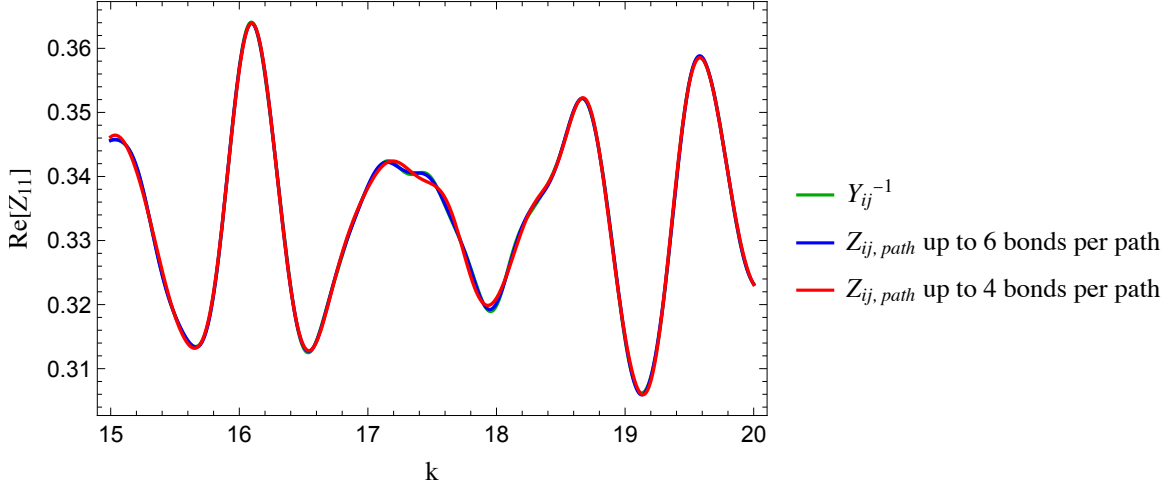
We now investigate the distribution of impedance values for networks of varying size. We focus on the variance of the imaginary part of the elements of networks with $N = 4, 8$, and 16 nodes. We also consider two values of the number of bonds per node (the degree) $B = 3$ and $B = 5$. We choose to characterize the imaginary part of the impedance as opposed to the real part as it has zero mean and a pdf of even symmetry. The statistics of the real part of the diagonal



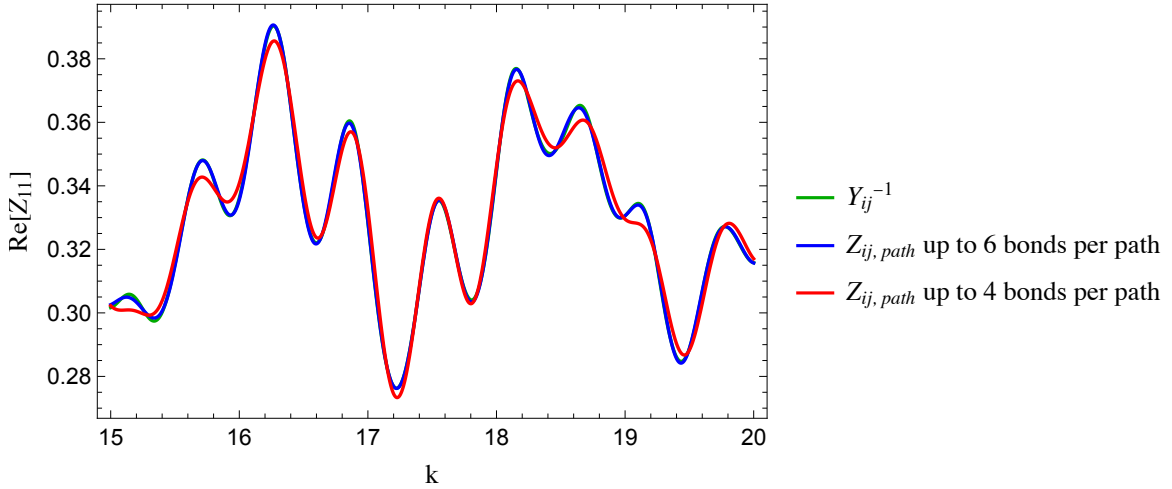
(a) Comparison between Y_{ij}^{-1} and $Z_{ij,path}$ for the graph with $N = 16$, $B = 3$, when $k_{im} = 0.03$.



(b) Comparison between Y_{ij}^{-1} and $Z_{ij,path}$ for the graph with $N = 4$, $B = 3$, when $k_{im} = 0.03$.



(c) Comparison between Y_{ij}^{-1} and $Z_{ij,path}$ for the graph with $N = 16$, $B = 3$, when $k_{im} = 0.3$.



(d) Comparison between Y_{ij}^{-1} and $Z_{ij,path}$ for the graph with $N = 4$, $B = 3$, when $k_{im} = 0.3$.

Figure 6.6: Comparison between the inverse of the admittance matrix, Y_{ij}^{-1} , and summed contributions from the different paths a signal may take in traversing the graph, $Z_{ij,path}$. The are results from two different graphs, one with $N = 4$, $B = 3$, and another with $N = 16$, $B = 3$. $Z_{ij,path}$ is used in two cases, one with up to 4 bonds per path, and another with up to 6 bonds per path. Furthermore, we use two different losses, $k_{im} = 0.03$ and $k_{im} = 0.3$.

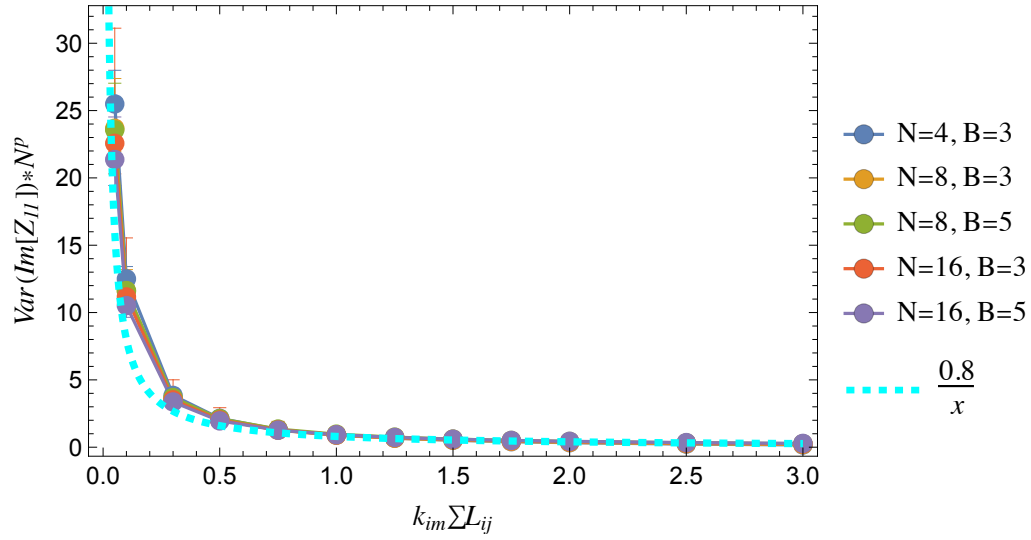
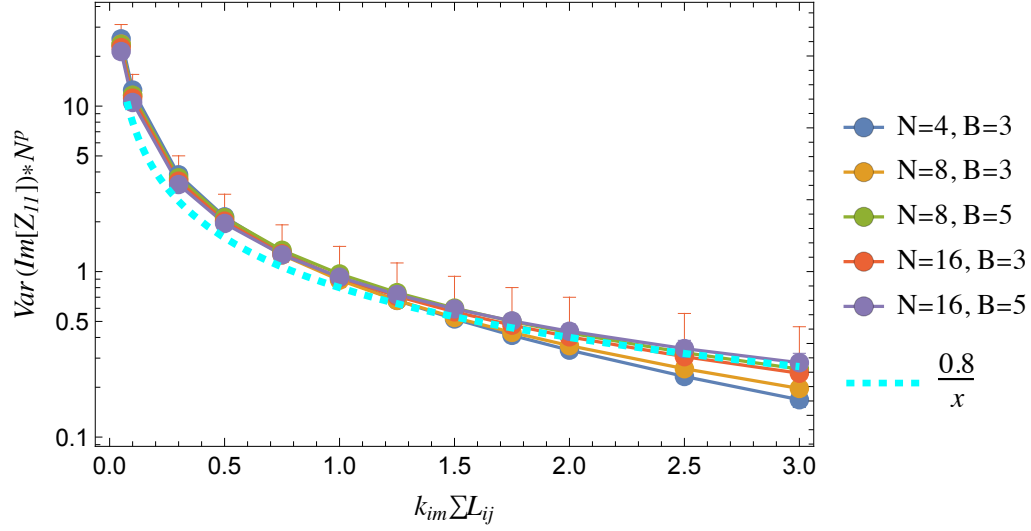
elements are complicated by the fact that the real part is positive definite and has a nonzero mean. The mean value of the real part of a diagonal element is independent of the loss rate. When the loss rate is small the distribution of values of the real part of a diagonal element becomes skewed in the sense that there is a large probability of a small real value and a small probability of a large value.

We computed the imaginary part of the diagonal and non-diagonal impedance elements of 20 realizations of networks of varying size, each for 49980 values of k , and with varying loss rate k_{im} . We then attempted to find universal scaling relations that characterized the variance. The variance of the imaginary part of diagonal elements scaled by N^p , $\text{Var}(\text{Im}(Z_{11}))N^p$, where p is adjusted to make the curves fall on top each other is plotted in Figure 6.7(a). We find $p = -0.3$ for graphs with $B = 3$ and $p = -0.4$ for graphs with $B = 5$. The variance of the imaginary part of non-diagonal elements, each averaged over 49980 values of k , versus $k_{\text{im}} \sum L_{ij}$ is plotted Figure 6.7(b), respectively. Here $\sum L_{ij}$ is the total length of a graph. We find that scaling by N^p for diagonal, and by plotting versus $k_{\text{im}} \sum L_{ij}$ results from networks of different sizes and degrees fall on top of each other. This is true for both large and small values of damping as indicated in the figure and its inset. The scaling with $k_{\text{im}} \sum L_{ij}$ is similar to that which is obtained for 2 and 3 dimensional cavities.

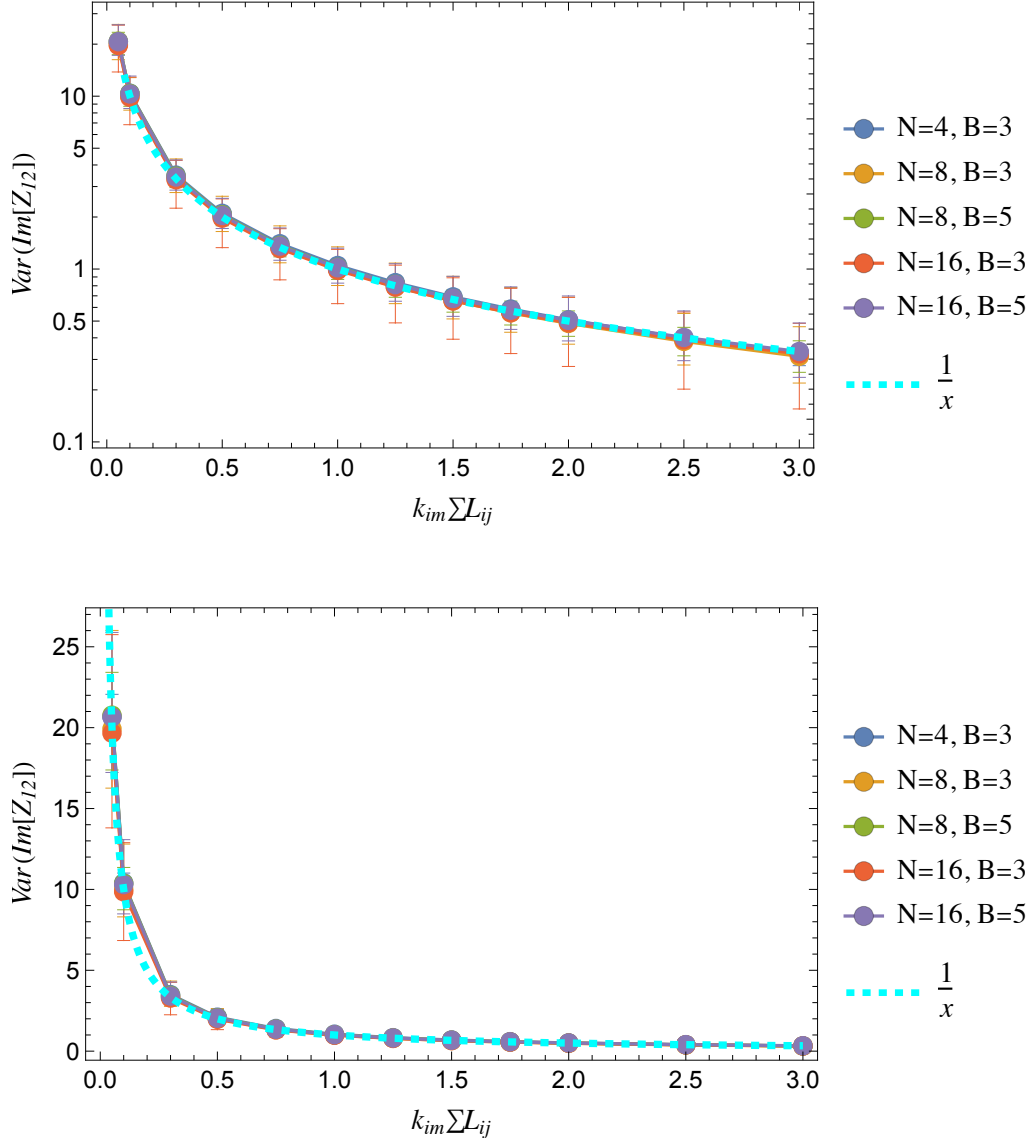
The expected scaling of the variance of the diagonal and off-diagonal elements of the impedance matrix based on the RCM are $2/(\pi\alpha)$ and $1/(\pi\alpha)$ respectively. Here $\alpha = k_{\text{im}}/\Delta k$, where $\Delta k = \pi/\sum L_{ij}$ for the case of graphs. Thus, based on the RCM, one expects the variance of the diagonal elements to scale as $2/(k_{\text{im}} \sum L_{ij})$ and the variance of the off diagonal elements to scale as $1/(k_{\text{im}} \sum L_{ij})$. As can be seen in Figure 6.7(b) the variance of the off-diagonal elements agree with the RCM scaling while the diagonal elements show deviation.

Thus, these networks exhibit the an expected dependence on $k_{\text{im}} \sum L_{ij}$, where $\sum L_{ij}$ is the total length of the bonds.

We also examined how the actual probability distribution functions compare to each other when we have the universal scaling relations that characterizes their variances. In Figure 6.7 we can see that for $k_{\text{im}} \sum L_{ij} = 1.5$ the variance of diagonal elements of the impedance matrix scaled by N^p fall on top of each other, and also, the same happens to the variances of non-diagonal elements. To exclude the scaling factor that is used for diagonal elements of impedance matrix, we compared the probability distribution functions of diagonal elements of impedance matrix scaled by $1/\sqrt{N^p}$. Eventually, we found out that as long as their variances are very close, their probability distributions does not fall on top each other. The results are provided on the Figure 6.8.

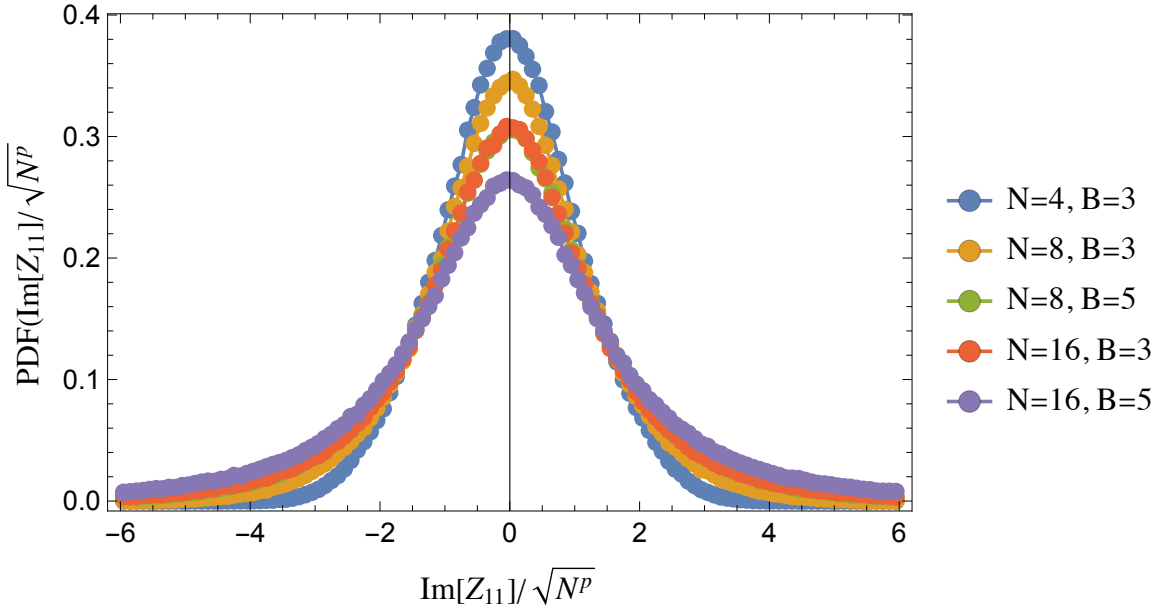


(a) The variance of imaginary part of diagonal elements of impedance matrix scaled by N^p , where $p = -0.3$ for graphs with $B = 3$ and $p = -0.4$ for graphs with $B = 5$. Fitting is done via $f(x) = \frac{0.8}{x}$ function.

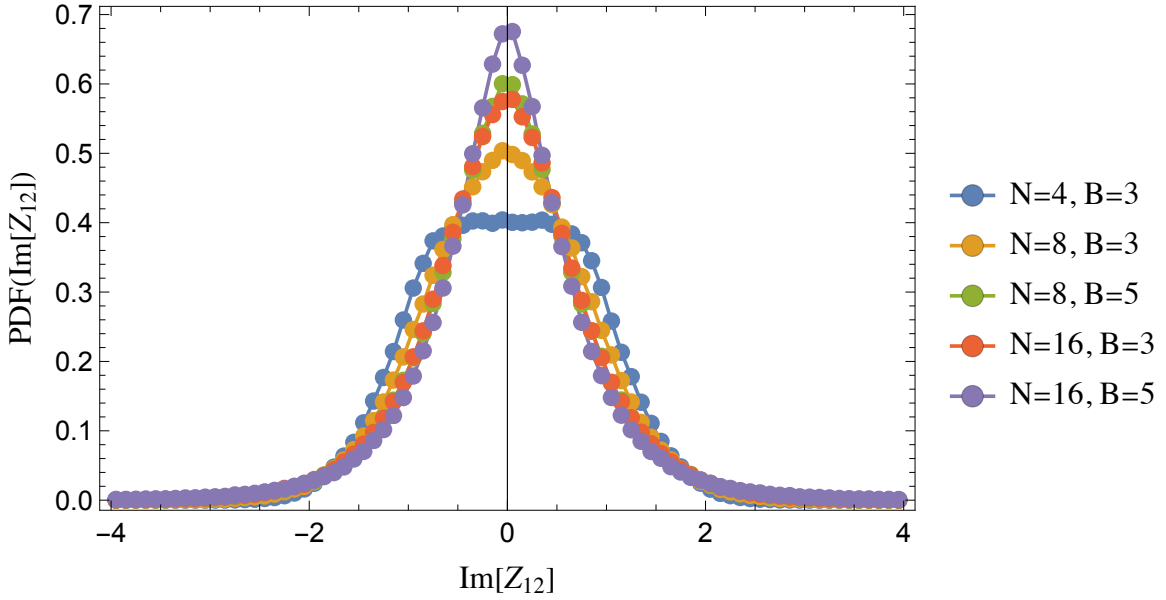


(b) The non-diagonal elements plotted versus $k_{\text{im}} \sum L_{ij}$. Fitting is done via $f(x) = \frac{1}{x}$ function.

Figure 6.7: (a) The variance of imaginary part of diagonal elements of impedance matrix scaled by N^p , where $p = -0.3$ for graphs with $B = 3$ and $p = -0.4$ for graphs with $B = 5$, and (b) non-diagonal elements plotted versus $k_{\text{im}} \sum L_{ij}$. We have also plotted a fitting functions, to see better what functions do the variances of the elements of the impedance matrix follow. There are 5 different sized graphs, each curve represents the result averaged over 20 different realizations, and error bars on each plot point represents the range in which the values were varying during all realizations.



(a) The probability distribution functions of imaginary part of diagonal elements of impedance matrix scaled by $1/\sqrt{N^p}$, where $p = -0.3$ for graphs with $B = 3$ and $p = -0.4$ for graphs with $B = 5$.



(b) The probability distribution functions of imaginary part of the non-diagonal elements.

Figure 6.8: The probability distribution functions of imaginary part of (a) diagonal elements of impedance matrix scaled by $1/\sqrt{N^p}$, where $p = -0.3$ for graphs with $B = 3$ and $p = -0.4$ for graphs with $B = 5$, and (b) non-diagonal elements. There are 5 different sized graphs, each curve represents the result averaged over 20 different realizations.

Chapter 7: Comparison with measured impedances

We now compare predicted impedance statistics with those measured on a network of cables. The experimental configuration is described extensively in Refs [3, 4, 22, 70, 77, 78]. It consists of a tetrahedral network ($N = 4, B = 3$) of 50 Ohm coaxial cables of varying lengths. These cables are connected at nodes using T-junctions. The experimental setup is shown on Figure 7.1

Comparisons between our simulations and measured [3] impedance values are displayed in Figure 7.3. For these comparisons, we simulated a tetrahedral graph ($N = 4, B = 3$) for which we choose bond lengths that correspond to those in the experiment. The bond lengths are shown on the Table 7.1. There are 9 different sized cables, and therefore 84 different realizations for the tetrahedron graph ($N = 4, B = 3$). In the experimental setup, the nodes are not grounded through an impedance, thus, we set Z_i to infinity in the first numerical calculations.

N = 4, B = 3	
Bond Lengths	0.31400, 0.38880, 0.42310, 0.45600, 0.49510, 0.53160, 0.56720, 0.63840, 0.7108

Table 7.1: The set of bond lengths used for experimental measurements and, correspondingly, for theoretical numerical calculations.

Network Analyzer

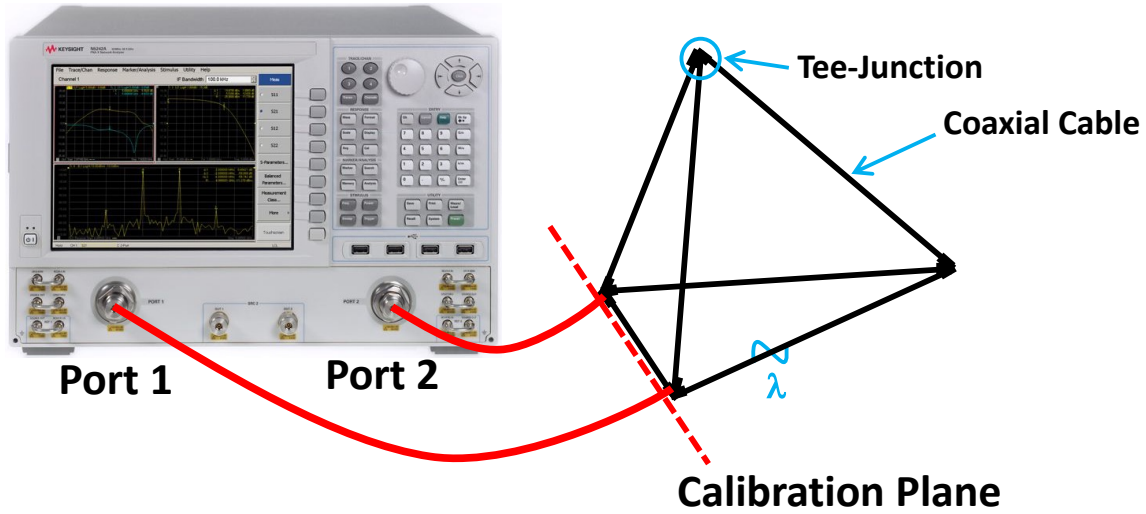


Figure 7.1: A schematic representation of the experimental setup [3] for the tetrahedral network with 4 nodes and 6 bonds. Tee-Junctions are used as nodes, coaxial cables are used as bonds, and network analyzer is connected to different ports (nodes) in order to measure signal going in and going out.

To compare numerical results with that measured [3], we need to determine an attenuation rate k_{im} on the bonds. The calculated and measured attenuation rate on the coaxial cables is presented in Appendix B of Ref. [3] as a function of frequency. The authors of Ref. [3] found that the attenuation varies with frequency, the result is shown in the figure 7.2. They calculated the attenuation using three different approach: direct measurement (by connecting the network analyzer at the ends of the coaxial cable, and checking how much the signal is dumped), fitting results to complex time delays, and direct modeling using Equation (B1) from [3].

We can conclude from this study [3] that we have data in five ranges: 3-6 GHz, 6-9 GHz, 9-12 GHz, 12-15 GHz and 15-18 GHz, in which the variation of the attenuation rate with frequency is approximately linear. The details about used k_{im} is given on Table 7.2. Results for impedance values in the form of histograms are shown in Figure 7.3 for all the frequency ranges. There

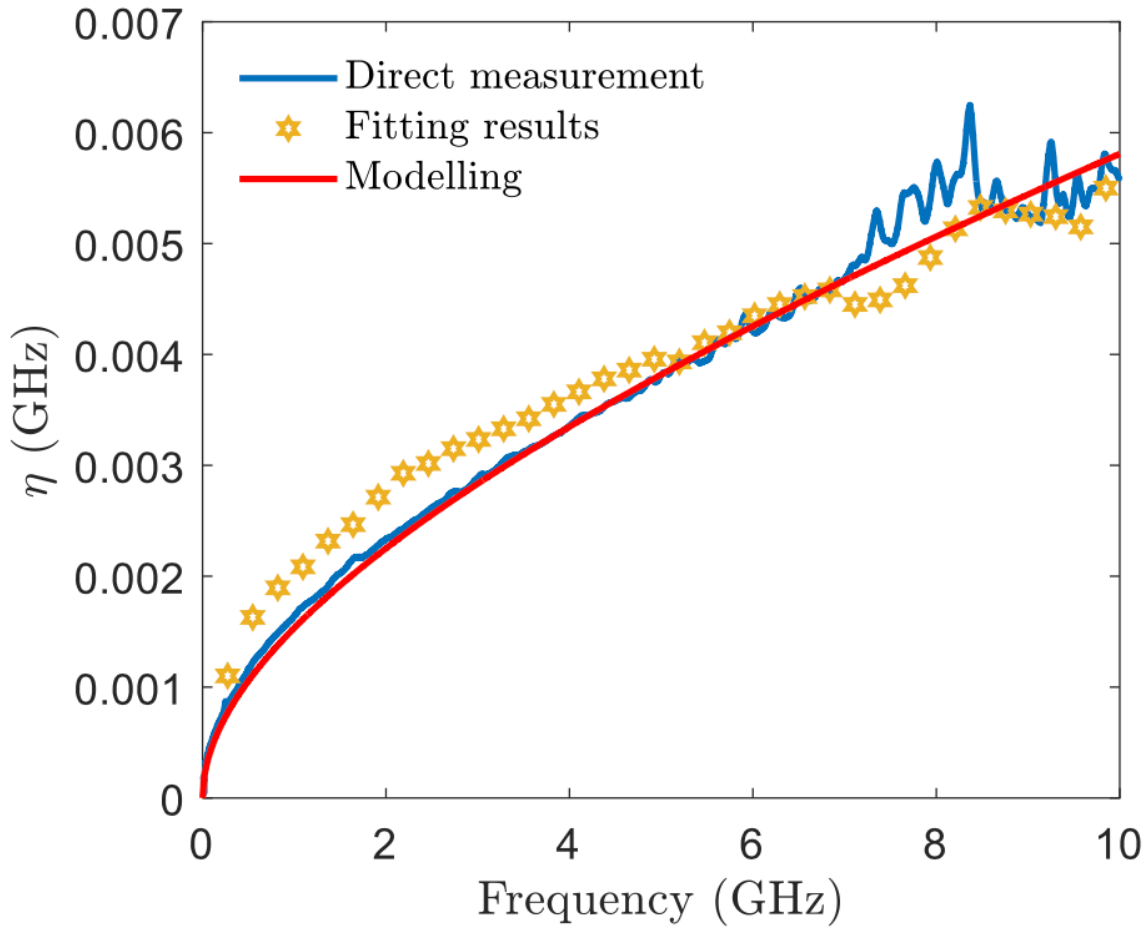


Figure 7.2: This figure is from the study [3]. Comparison of three different ways to determine the attenuation: direct measurement, fitting results to complex time delays, and direct modeling using Equation (B1) from [3]

are four panels for the real and imaginary parts of the diagonal and off-diagonal elements of the impedance matrix. For each case, there are 84 different realizations of the network using bonds of different lengths, and for each realization, the data is generated by changing wave number value (essentially the frequency).

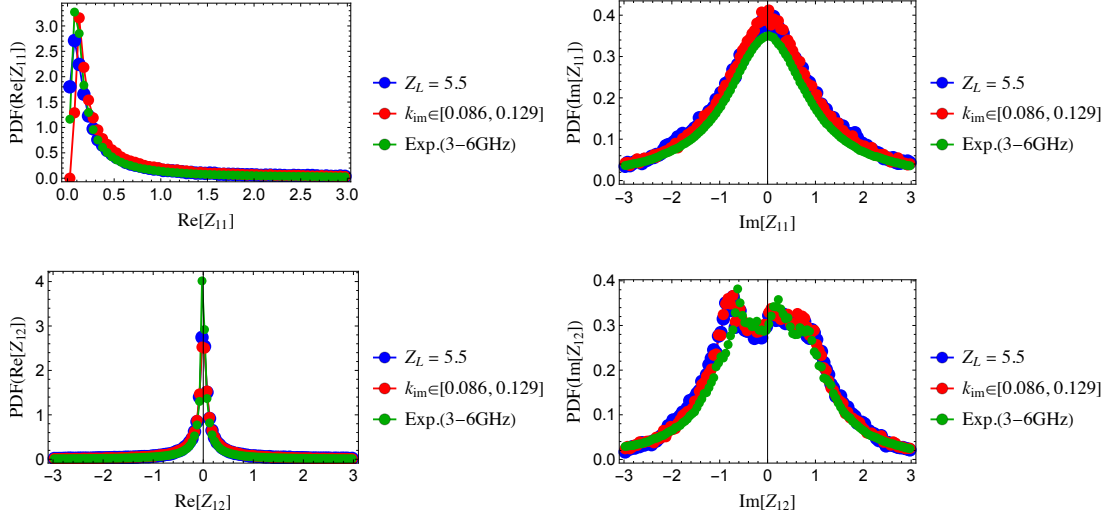
To investigate the dependence on the type of loss, we have added to Figure 7.3 calculated impedance values assuming the cables are lossless, but instead, there are real impedance elements connecting the nodes to the ground. The value of this impedance was adjusted to match the other

N = 4, B = 3	
3-6 GHz	$0.086m^{-1} < k_{\text{im}} < 0.129m^{-1}$
6-9 GHz	$0.129m^{-1} < k_{\text{im}} < 0.165m^{-1}$
9-12 GHz	$0.165m^{-1} < k_{\text{im}} < 0.198m^{-1}$
12-15 GHz	$0.198m^{-1} < k_{\text{im}} < 0.228m^{-1}$
15-18 GHz	$0.228m^{-1} < k_{\text{im}} < 0.256m^{-1}$

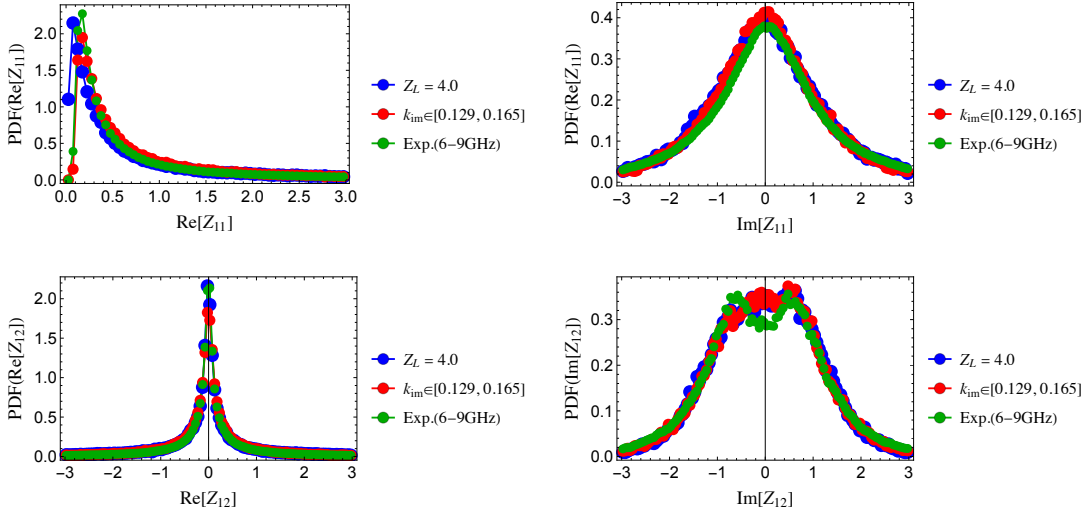
Table 7.2: Values of the used frequency ranges and corresponding k_{im} that changes linearly.

histograms, $Z_i = 5.5$ for $f = 3\text{-}6$ GHz, $Z_i = 4.0$ for $f = 6\text{-}9$ GHz, $Z_i = 2.9$ for $f = 9\text{-}12$ GHz, $Z_i = 2.6$ for $f = 12\text{-}15$ GHz and $Z_i = 1.9$ for $f = 15\text{-}18$ GHz. Note the lossy element connects a node to ground, so a larger value of Z_i corresponds to lower loss. As can be seen, the three types of histograms (experiment, distributed loss simulated, and localized loss simulated) are quite similar. This includes the multiple peaks in the $\text{Im}(Z_{12})$ histograms, which are due to direct paths from port 1 to port 2.

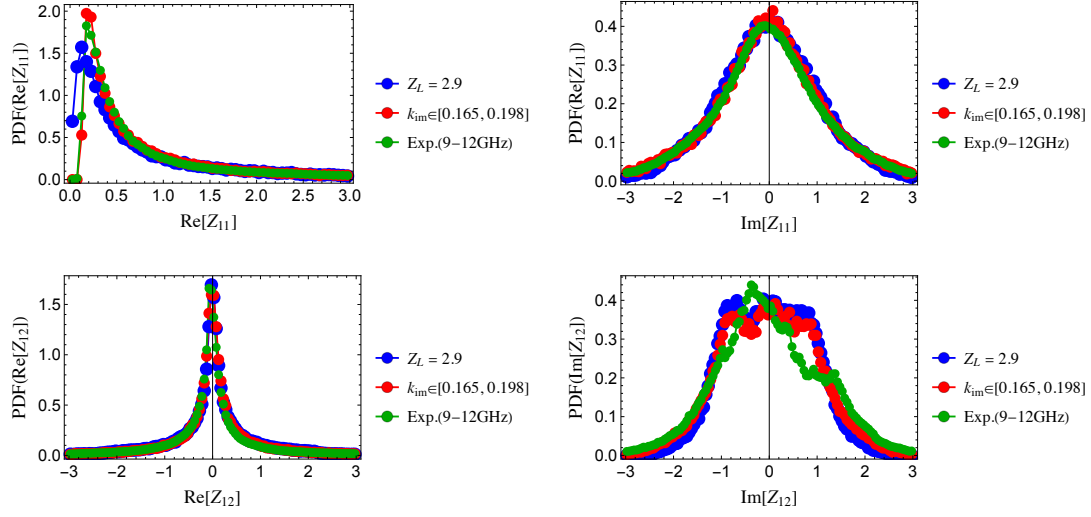
Previously [4], the impedance statistics of this configuration were compared with predictions of the RCM. It was found that the statistics of the non-diagonal and diagonal elements of the impedance matrix required fitting with different loss factors to obtain agreement. Generally, the diagonal elements required higher loss than the off-diagonal elements to obtain a fit. Our purpose here is to determine whether this is a feature of these networks reproduced by our simple model. What we have found is that the histograms of the diagonal and non-diagonal elements are matched to the experimental histograms by the same loss parameter. Thus, the network is not modeled by the RCM. In Section I, we have described three different ways to compute the impedance matrix. We found the most straightforward way to calculate the impedance matrix



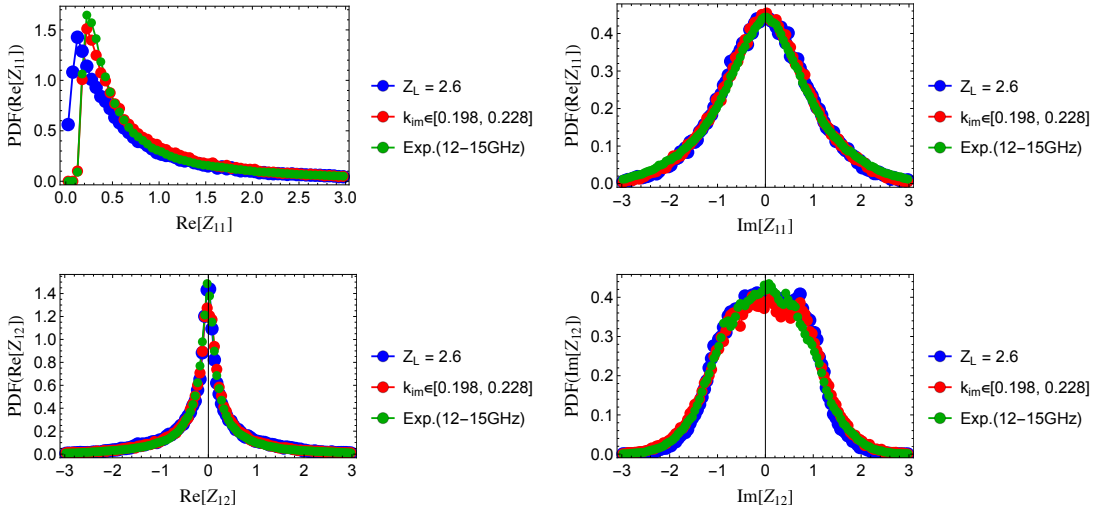
(a) Comparison between histograms of numerically generated impedance matrix elements and experimentally measured values for an $N = 4, B = 3$ graph. $Z_i = 5.5$ and $k_{\text{im}} = 0.0$ (Blue), $Z_i = \infty$ and $k_{\text{im}} \in [0.086 - 0.129]$ changing linearly (Red). Experimental data (Green) was produced for the frequency range of 3 – 6 GHz.



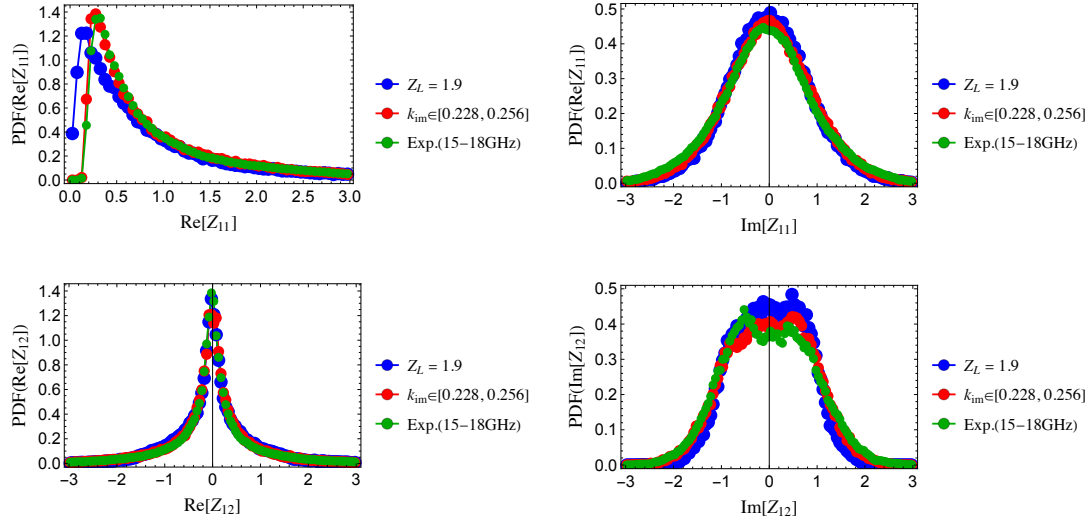
(b) Comparison between histograms of numerically generated impedance matrix elements and experimentally measured values for an $N = 4, B = 3$ graph. $Z_i = 4.0$ and $k_{\text{im}} = 0.0$ (Blue), $Z_i = \infty$ and $k_{\text{im}} \in [0.129 - 0.165]$ changing linearly (Red). Experimental data (Green) was produced for the frequency range of 6 – 9 GHz.



(c) Comparison between histograms of numerically generated impedance matrix elements and experimentally measured values for an $N = 4, B = 3$ graph. $Z_i = 2.6$ and $k_{im} = 0.0$ (Blue), $Z_i = \infty$ and $k_{im} \in [0.165, 0.198]$ changing linearly (Red). Experimental data (Green) was produced for the frequency range of 9 – 12 GHz.



(d) Comparison between histograms of numerically generated impedance matrix elements and experimentally measured values for an $N = 4, B = 3$ graph. $Z_i = 2.6$ and $k_{im} = 0.0$ (Blue), $Z_i = \infty$ and $k_{im} \in [0.198, 0.228]$ changing linearly (Red). Experimental data (Green) was produced for the frequency range of 12 – 15 GHz.



(e) Comparison between histograms of numerically generated impedance matrix elements and experimentally measured values for an $N = 4, B = 3$ graph. $Z_i = 1.9$ and $k_{\text{im}} = 0.0$ (Blue), $Z_i = \infty$ and $k_{\text{im}} \in [0.228, 0.256]$ changing linearly (Red). Experimental data (Green) was produced for the frequency range of 15 – 18 GHz.

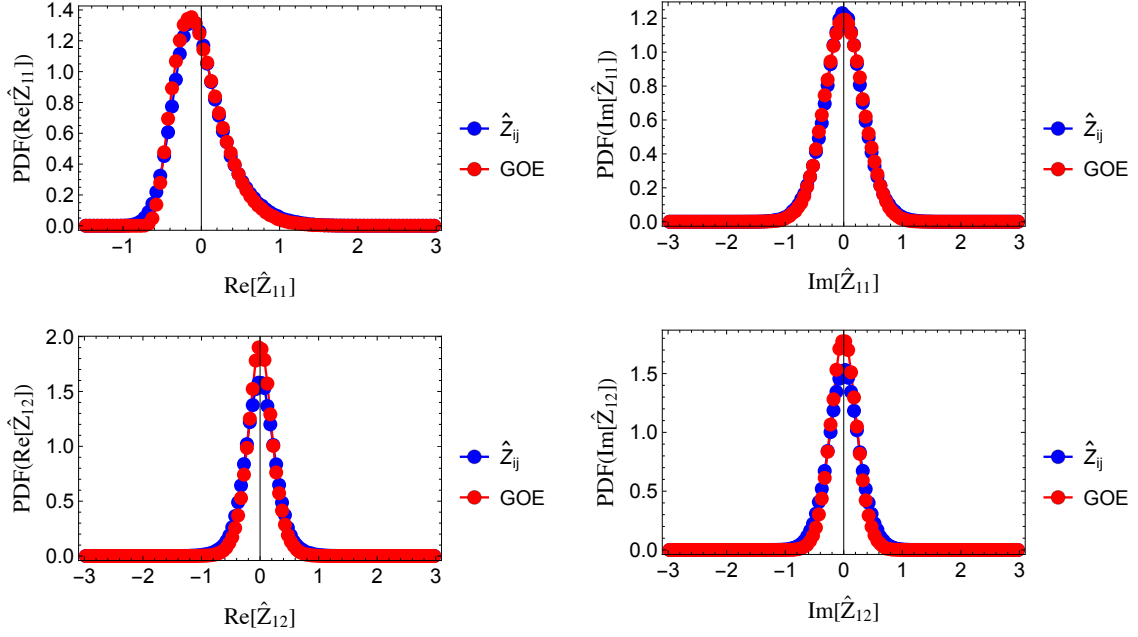
Figure 7.3: Comparison between histograms of numerically generated impedance matrix elements and experimentally measured values for an $N = 4, B = 3$ graph. Loss is modeled in two different ways: using Equations (5.10a) and (5.10b) with $k_{\text{im}} = 0.0$ (Blue) and with $Z_i = \infty$ (Red). Experimental data (Green) was produced for the frequency range given on each subfigure, and corresponding k_{im} changes linearly.

accurately is to invert the admittance matrix given by Equations (5.10a) and (5.10b), which results in Equation (5.11). In general, the random coupling model (RCM) describes the systems without short orbit [47] contributions. Therefore, to compare our results to RCM, we should ensure that we have excluded short-orbit contributions from the calculations. For that purpose, we can recall that we have discussed how to calculate the impedance matrix based on summing contributions from the different paths, $Z_{ij,path}$, and was described by Equation (5.15) and Equation (5.16). However, if we use a finite number of paths in $Z_{ij,path}$, we can generate only the short orbit contributions in the impedance matrix. Afterward, we can subtract these contributions from the impedance matrix calculated using Equation (5.11), which results in the redefined impedance matrix without short orbit contributions

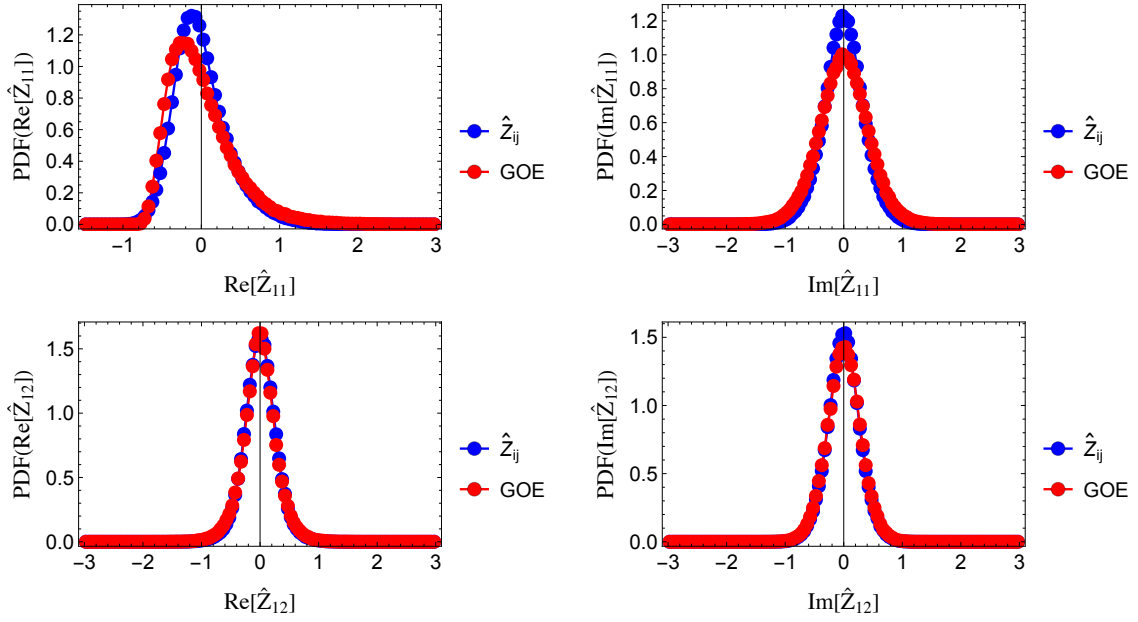
$$\hat{Z}_{ij}(k) = (Y(k))_{ij}^{-1} - Z_{ij,path}(k) . \quad (7.1)$$

We generated data for two different networks, $(N = 4, B = 3)$ and $(N = 16, B = 3)$. We set a fixed loss rate, $k_{im} = 0.03$, for both of the cases and varied α , the loss parameter in RCM. We found out that the statistics of the redefined impedance matrix for the $(N = 4, B = 3)$ network, calculated by Equation (7.1), and RCM have the best agreement when $\alpha = 1.20$ and $\alpha = 1.00$, for diagonal and non-diagonal elements respectively. The corresponding results for the $(N = 16, B = 3)$ network are $\alpha = 3.00$ and $\alpha = 1.90$, for diagonal and non-diagonal elements, respectively. The results are shown in Figure 7.4

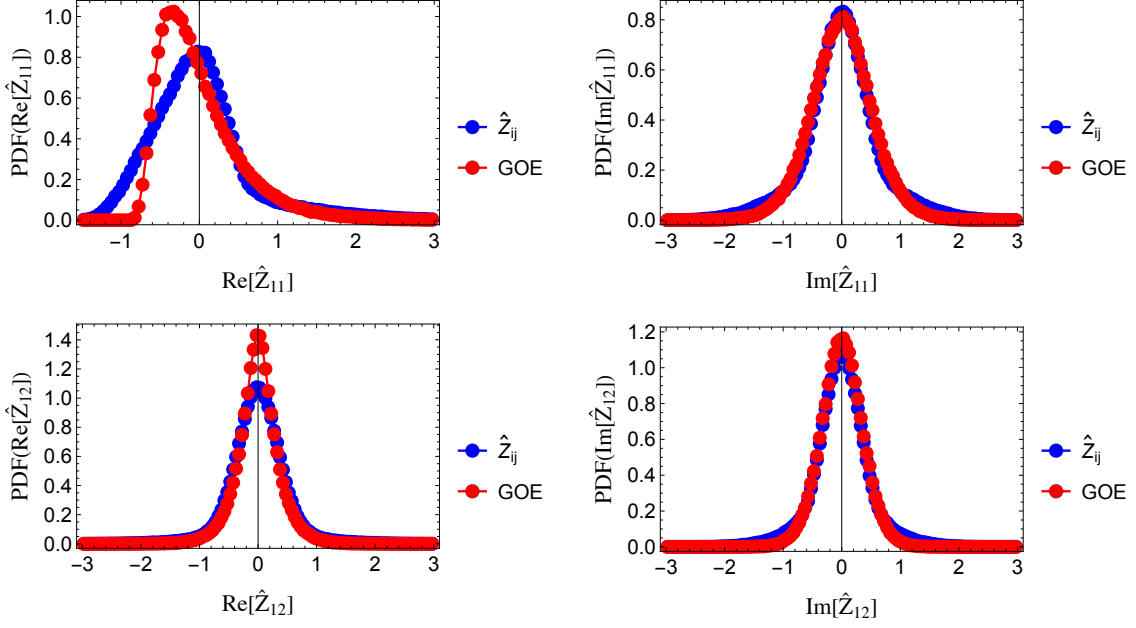
Our findings are consistent with those described in [4] based on measurements. The predicted distributions of values of the elements of the impedance matrix based on our model agree with those measured. However, as in [4], the distributions of impedance values from our model,



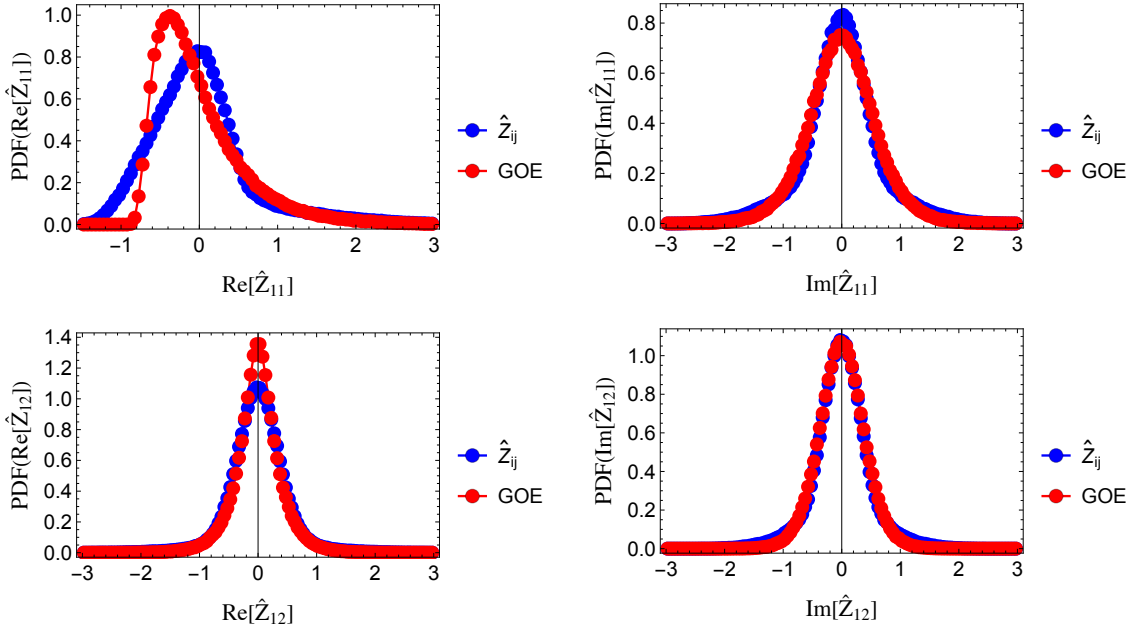
(a) Comparing impedance matrix pdf elements for the graph with $N = 16$, $B = 3$ (Red), calculated by Equation (7.1), to RCM (Blue). $k_{\text{im}} = 0.03$ and $\alpha = 3.00$.



(b) Comparing impedance matrix pdf elements for the graph with $N = 4$, $B = 3$ (Red), calculated by Equation (7.1), to RCM (Blue). $k_{\text{im}} = 0.03$ and $\alpha = 1.90$.



(c) Comparing impedance matrix pdf elements for the graph with $N = 16$, $B = 3$ (Red), calculated by Equation (7.1), to RCM (Blue). $k_{\text{im}} = 0.03$ and $\alpha = 1.20$.



(d) Comparing impedance matrix pdf elements for the graph with $N = 4$, $B = 3$ (Red), calculated by Equation (7.1), to RCM (Blue). $k_{\text{im}} = 0.03$ and $\alpha = 1.00$.

Figure 7.4: Comparing impedance matrix pdf elements, calculated by Equation (7.1), to RCM. Equation (7.1) (Red) has been used for two different graphs, one with $N = 4$, $B = 3$ and another with $N = 16$, $B = 3$. We have used two different k_{im} for each cases, $k_{\text{im}} = 0.03$. For RCM (Blue) data generations, we used α values that was closest to the numerical data calculated by Equation (7.1).

when fit to the RCM, require different values of loss to fit the diagonal and off-diagonal elements. A possible cause for this discrepancy is the non-vanishing reflection seen by a wave propagating on a bond and incident on a node. Given our model of the nodes, the smallest magnitude of the voltage reflection coefficient can be $1/3$. This reflection is absent in 2D and 3D cavities for which good agreement with the RCM is attained. We have found that increasing this reflection coefficient causes the mode eigenfrequency spacing statistics to transition from a GOE-like distribution to a Poisson-like distribution. Thus, the small reflection may be responsible for the deviations from RCM statistics. A recent study [79, 80] has modeled the nodes using a scattering matrix based on a Fourier decomposition of the elements. Such a representation allows for the reflection to be eliminated, and the authors find agreement with RCM statistics. Implementing such a scattering matrix in an experiment is a challenge. Various matching techniques can be considered. However, these are usually effective only over a limited range of frequencies, making generating a large ensemble of impedance values difficult.

Chapter 8: Conclusion

The main goal of the first four chapters of the thesis is to put together the knowledge and the studies that preceded the work that have been done in this thesis. The detailed description of our work is shown from chapters five to chapter seven.

In chapter one, we have described the work that has been done over the years in the field of Quantum Graphs. A quantum graph is a network of nodes connected by bonds that support waves. Linus Pauling was the first person to propose the idea of a quantum graph, and after that, quantum graphs were extensively studied [4–12].

In chapter two, we follow the book by G. Berkolaiko, and P. Kuchment [30], and the review paper by S. Gnuzmann and U. Smilansky [5] to present the vital theoretical framework that was developed in their works. First, we defined the graph and showed examples of the most common graphs. Then, we wrote Schrödinger’s one-dimensional equation on the metric graph, and after setting boundary conditions at the ends of the bonds, we found the solutions, the wave functions, given by Equation (2.11). Finally, we derive the secular equation [5] that describes the system’s eigenvalues.

In chapters three and four, we have described the importance and use cases of the random coupling model and transmission lines.

From chapter five to chapter seven, we described our work. At first, we created a theoretical

framework that proposed how to calculate the normal modes of the system. We also presented three different approaches to calculating the impedance matrix: one that is precise, one that is based on summing over the normal modes of the graph, and one that is based on summing over paths through the graph. We compared these approaches and determined the circumstances under which they agreed.. As a result, we have obtained several interesting results. We found that by changing the values of the reflection coefficients at the nodes, we can change the statistics of the distribution of normal mode wave number spacings. This is shown in Figure 6.4. We have shown that the results of the developed framework adequately reproduce the experimental results, see Figure 7.1. Finally, we showed that there is a universal formula for the dependence of the size of the fluctuations of $\text{Im}(Z_{11})$ on propagation loss rate and the number of nodes in the graph. The dependence is given in Figure 6.7.

Finally, I want to discuss what can be done in the future to continue this study. One of the main results in this thesis is the scaling rule of variances of the diagonal elements of the impedance matrix shown on Figure 6.7 (a). We have found out that when we multiply the variances by N^p and plot versus $k_{\text{im}} \sum L_{ij}$, the curves fall on top each other. Where $p = -0.3$ for graph with $B = 3$, and $p = -0.4$ for graphs with $B = 5$. This phenomena can be studied further for the graphs with different B . There might some kind of universal relation between B and p .

Another idea that can be continued in the future is the dependence of the spacing distributions of the wave numbers on the boundary conditions at the nodes. We have discussed the results when we have Dirichlet and Neumann boundary conditions. However, these studies [79,80] show that using Fourier boundary condition the reflection coefficient at the nodes can go to zero. Having a zero reflection coefficient at the nodes cannot be done by Dirichlet and Neumann boundary conditions. Therefore, Fourier boundary conditions open new possibilities to study statistics of

the spacing distributions of the wave numbers.

Lastly, I want to mention our fascinating result, when we compared statistics of the elements of impedance matrices with the experimentally measured ones. We had a great match between them, shown on Figure 7.1. However, this was done only for tetrahedron graph, which has $N = 4$ and $B = 3$. In future, there can be comparison between different size graphs to check if this properties still hold.

Appendix A: Appendix

To find the relation between normal modes and impedance matrix, which for a driven system $I_j \neq 0$ is defined for all values of propagation constant k , we first multiply the Telegrapher's equations by the conjugate of a mode voltage and current respectively, integrate in z over each bond, add the two equations, and sum over bonds,

$$i(k - k_n) \sum_{ij} \int_0^{L_{ij}} dx (Z_0^{-1} V^{(n)*} V(x) + Z_0 I^{(n)*}(x) I(x)) = \sum_{ij} (V^{(n)*} I(x) + V(x) I^{(n)*}) \Big|_0^{L_{ij}} \quad (\text{A.1})$$

The right-hand side involves the evaluation of voltages and currents at nodes, and can be converted into a sum over nodes. Each node will enter twice, first as the lower limit, ($z = 0$) and second as the upper limit, ($z = L_{ij}$). The current variables have opposite meanings in these cases. At $z = 0$ the current variables represent the currents leaving a node, and at $z = L_{ij}$ they represent the currents entering a node. Thus we write both in terms of the currents leaving node- i in the direction of node- j

$$\sum_{ij} (V^{(n)*} I(x) + V(x) I^{(n)*}) \Big|_0^{L_{ij}} = -2 \left(\sum_{nodes-i} V_i^{(n)*} \sum_{ij} I_{i \rightarrow j} + \sum_{nodes-i} V_i \sum_{ij} I_{i \rightarrow j}^{(n)*} \right).$$

The value of the sum of the currents leaving a node will depend on whether the current variables

apply to a normal mode or a driven solution. In the case of a normal mode with index m ,

$$\sum_{ij} I_{i \rightarrow j}^{(m)} + \frac{V_i^{(m)}}{Z_i} = 0 . \quad (\text{A.2})$$

Thus, if we consider the loss free case $Z_i^* = -Z_i$ the two terms cancel. As a result

$$i(k - k_n) \sum_{ij} \int_0^{L_{ij}} dx \left(Z_0^{-1} V^{(n)*}(x) V^{(m)}(x) + Z_0 I^{(n)*}(x) I^{(m)}(x) \right) = 0 , \quad (\text{A.3})$$

and we conclude that the modes represent an orthogonal basis. In the case of a driven solution,

$$\sum_{ij} I_{i \rightarrow j}^{(m)} + \frac{V_i^{(m)}}{Z_i} = I_i , \quad (\text{A.4})$$

and we obtain

$$i(k - k_n) \sum_{ij} \int_0^{L_{ij}} dx \left(Z_0^{-1} V^{(n)*} V(x) + Z_0 I^{(n)*}(x) I(x) \right) = -2 \left(\sum_{nodes-i} V_i^{(n)*} I_i \right) . \quad (\text{A.5})$$

We now expand the voltages and currents on the transmission lines in a superposition of voltages and currents corresponding to normal modes,

$$(V(x), I(x)) = \sum_m C_m (V^{(m)}(x), I^{(m)}(x)) . \quad (\text{A.6})$$

Using the orthogonality property we find for the coefficients C_n ,

$$i(k - k_n) C_n = \frac{-2 \left(\sum_{nodes-i} V_i^{(n)*} I_i \right)}{\sum_{ij} \int_0^{L_{ij}} dx \left(Z_0^{-1} |V^{(n)}|^2 + Z_0 |I^{(n)}|^2 \right)} \quad (\text{A.7})$$

Consequently, we can express the voltage at a node in the driven case in terms of the impedance matrix, which intern is expressed in terms of the normal modes.

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